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OUTER APPROXIMATION METHOD FOR THE MINIMUM MAXIMAL FLOW PROBLEM

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Abstract The minimum maximal flow problem is the problem of minimizing the flow value on the set of maximal flows of a given network. The optimal value indicates how inefficiently the network can be utilized in the presence of some uncontrollability. After extending the gap function characterizing the set of maximal flows, we reformulate the problem as a D.C. optimization problem, and then propose an outer approximation algorithm. The algorithm, based on the idea of ε-optimal solution and local search technique, terminates after finitely many iterations with the optimal value of the problem.

Keywords: Network flow, minimum maximal flow, optimization over the efficient set, D.C. optimization, outer approximation, global optimization.

1. Introduction

We are given a network $(V, s, t, E, \boldsymbol{c})$, where V is the set of $m+2$ nodes containing the source node s and the sink node t, E is the set of n arcs and $\boldsymbol{c}$ is the n-dimensional column vector whose hth element c_h is the capacity of arc h. The set of *feasible flows*, denoted by X, is given by

$$X = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} = \boldsymbol{0},\ \boldsymbol{0} \leqq \boldsymbol{x} \leqq \boldsymbol{c} \,\}, \tag{1.1}$$

where $m \times n$ matrix A is the incidence matrix whose (v, h) element a_{vh} is

$$a_{vh} = \begin{cases} +1 & \text{if arc } h \text{ leaves node } v \\ -1 & \text{if arc } h \text{ enters node } v \\ 0 & \text{otherwise.} \end{cases}$$

The well-known conventional *maximum flow problem* is

$$\left| \begin{array}{ll} \max\limits_{\boldsymbol{x}} & \boldsymbol{d}\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in X, \end{array} \right.$$

where $\boldsymbol{d}$ is the n-dimensional row vector whose hth element is

$$d_h = \begin{cases} +1 & \text{if arc } h \text{ leaves source } s \\ -1 & \text{if arc } h \text{ enters source } s \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1.1 (minimum maximal flow problem) *A vector $\boldsymbol{x} \in X$ is said to be a maximal flow if there is no $\boldsymbol{y} \in X$ such that $\boldsymbol{y} \geqq \boldsymbol{x}$ and $\boldsymbol{y} \neq \boldsymbol{x}$. We use X_M to denote the set of maximal flows, i.e.,*

$$X_M = \{\, \boldsymbol{x} \in X \mid \text{ there is no } \boldsymbol{y} \in X \text{ such that } \boldsymbol{y} \geqq \boldsymbol{x} \text{ and } \boldsymbol{y} \neq \boldsymbol{x} \,\}. \tag{1.2}$$

A minimum maximal flow problem, abbreviated to (mmF)*, is defined as*

$$(mmF) \quad \left| \begin{array}{ll} \min\limits_{\boldsymbol{x}} & \boldsymbol{dx} \\ s.t. & \boldsymbol{x} \in X_M. \end{array} \right.$$

The purpose of this paper is to propose an algorithm for (mmF), which is based on the outer approximation method (OA method for short) for a D.C. optimization problem. Note that D.C. stands for difference of two convex sets (or functions), which will be defined in Section 3.

Below is our motivation to consider (mmF). When we attempt to solve a maximum flow problem on condition that we should not be allowed to decrease arc flows, we often fail to obtain the maximum flow and are obliged to put up with a maximal flow. Under this restricted controllability, the minimum flow value attained by a maximal flow, i.e., the optimal value of (mmF), indicates how inefficiently the network can be utilized. Figure 1 highlights the difference between maximum flow and minimum maximal flow. For network (a), both are 3. On the other hand, for network (b), the minimum maximal flow value reduces to 2 while the maximum flow value remains 3. The minimum maximal flow value does not monotonically increase as the capacities grow.

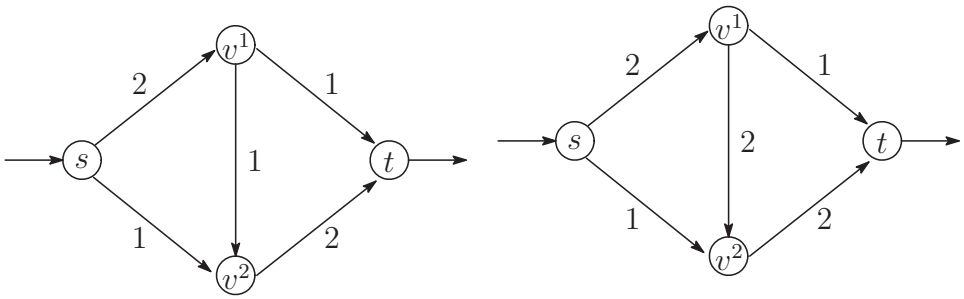


Figure 1: Maximum flow vs. minimum maximal flow

Shi-Yamamoto [24] first raised (mmF) and proposed an algorithm. Several algorithms for (mmF) combining local search and global optimization technique have been proposed in e.g., Gotoh-Thoai-Yamamoto [15] and Shigeno-Takahashi-Yamamoto [25]. An approach of D.C. optimization is found in Muu-Shi [18]. The difficulty of (mmF) is mainly due to the nonconvexity of X_M. Indeed, (mmF) embraces the minimum maximal matching problem, which is $\mathcal{NP}$-hard (see e.g., Garay-Johnson [14]).

It is readily seen that (mmF) is a special case of the optimization problem over the efficient set of a multi objective optimization, which was first studied by Philip [20]. Applying a well-known result of multi objective optimization, X_M is characterized as follows: The point $\bar{\boldsymbol{x}}$ is in X_M if and only if there exists $\boldsymbol{\lambda} \in \mathbb{R}_{p++}$ such that $\bar{\boldsymbol{x}}$ is an optimal solution of

$$(SC(\boldsymbol{\lambda})) \quad \left| \begin{array}{ll} \max\limits_{\boldsymbol{x}} & \boldsymbol{\lambda x} \\ \text{s.t.} & \boldsymbol{x} \in X. \end{array} \right.$$

Therefore we can easily obtain a point $\boldsymbol{x} \in X_M$ by solving $(SC(\boldsymbol{\lambda}))$ for an arbitrarily chosen $\boldsymbol{\lambda} \in \mathbb{R}_{p++}$. Furthermore, for a sufficiently large $M > 0$ the following set Λ can substitute

for $\mathbb{R}_{p++}$ above:

$$\Lambda = \{\, \boldsymbol{\lambda} \in \mathbb{R}_{p++} \mid \boldsymbol{\lambda} \geqq \boldsymbol{e},\ \boldsymbol{\lambda}\mathbf{1} = M \,\}. \tag{1.3}$$

Shigeno-Takahashi-Yamamoto [25] showed that n^2 suffices for M defining Λ of (1.3) for (mmF). It is also known and easily seen by applying the parametric optimization technique for $(SC(\boldsymbol{\lambda}))$ that X_M is a connected union of several faces of X. About the optimization problem over the efficient set, the reader should refer to e.g., White [31], Sawaragi-Nakayama-Tanino [22], Steuer [26] and Yamamoto [33]. For solution methods, see Benson [4–6], Bolintineanu [7], Ecker-Song [11], Fülöp [13], Dauer-Fosnaugh [10], Thach-Konno-Yokota [27], Sayin [23], Phong-Tuyen [21], Thoai [28], Muu-Luc [17], An-Tao-Thoai [3] and An-Tao-Muu [1, 2].

For simplicity we assume throughout this paper that the given network satisfies the following three assumptions.

Assumption 1.2

(i) *Each capacity takes a positive integer, i.e.,* $c_h \in \mathbb{Z}$ *and* $c_h > 0$ *for each* $h \in E$.

(ii) *There is some point* $\boldsymbol{x} \in X$ *such that* $\boldsymbol{x} \neq \mathbf{0}$.

(iii) *There is no t-s-path.*

Note that Assumption 1.2 (i) ensures the integrality of vertices of X as well as the optimal value of (mmF). Note also that $\mathbf{0} \notin X_M$ by Assumption 1.2 (ii), and $\min\{\, \boldsymbol{d}\boldsymbol{x} \mid \boldsymbol{x} \in X \,\} = 0$ by Assumption 1.2 (iii).

In the next section we first introduce a gap function. We then extend the domain of the gap function to $\mathbb{R}^n$ and reformulate (mmF). Section 3 is devoted to a review of the OA method for D.C. optimization problems. Based on this method, we propose an algorithm for (mmF) in Section 4, in which we introduce an ε-optimal solution and investigate the proper range of the parameter ε for the optimality condition. To make the algorithm more efficient, we incorporate a local search technique. Finally, we show that the algorithm with the local search technique terminates after finitely many iterations. Further works will be described in the last section.

Throughout this paper we use the following notations: $\mathbb{R}^n$ denotes the set of n-dimensional column vectors. Let $\mathbb{R}^n_+ = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} \geqq \mathbf{0} \,\}$ and $\mathbb{R}^n_{++} = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} > \mathbf{0} \,\}$. Let $\mathbb{R}_n$ denote the set of n-dimensional row vectors, $\mathbb{R}_{n+}$ and $\mathbb{R}_{n++}$ are defined in the similar way. We use $\boldsymbol{e}$ to denote the row vector of ones, $\mathbf{1}$ to denote the column vector of ones, and $\boldsymbol{e}^i$ to denote the ith unit row or column vector of an appropriate dimension. Let I denote the identity matrix of an appropriate size. We use $\boldsymbol{a}^\top$ and $A^\top$ to denote the transposed vector of $\boldsymbol{a}$ and the transposed matrix of A, respectively. For a set S, we denote the interior of S by $\operatorname{int} S$, the closure of S by $\operatorname{cl} S$, and the relative boundary of S by ∂S. We use P_V to denote the set of vertices of a polyhedron P. For two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, let $[\boldsymbol{v}, \boldsymbol{w}]$ denote the line segment with endpoints $\boldsymbol{v}$ and $\boldsymbol{w}$, and let $(\boldsymbol{v}, \boldsymbol{w}] = [\boldsymbol{v}, \boldsymbol{w}] \backslash \{\boldsymbol{v}\}$. Also $[\boldsymbol{v}, \boldsymbol{w})$ and $(\boldsymbol{v}, \boldsymbol{w})$ are defined in the similar way.

2. Reformulation of (mmF) by the Extended Gap Function

It is known that the *gap function* $g : \mathbb{R}^n \to [-\infty, +\infty]$ given by

$$g(\boldsymbol{x}) = \max\{\, \boldsymbol{e}\boldsymbol{y} \mid \boldsymbol{y} \in X,\ \boldsymbol{y} \geqq \boldsymbol{x} \,\} - \boldsymbol{e}\boldsymbol{x}, \tag{2.1}$$

defines the set of maximal flows X_M as

$$X_M = \{\, \boldsymbol{x} \in X \mid g(\boldsymbol{x}) \leqq 0 \,\}.$$

Then we can rewrite (mmF) as

$$(mmF) \quad \left| \begin{array}{ll} \min\limits_{\boldsymbol{x}} & \boldsymbol{dx} \\ \text{s.t.} & \boldsymbol{x} \in X,\ g(\boldsymbol{x}) \leqq 0. \end{array} \right.$$

The function g has some nice properties such piecewise linearity and concavity; for more information, see e.g., Benson [4] and White [32].

The *domain of* g, denoted by $\operatorname{dom} g$, is the set $\{\, \boldsymbol{x} \in \mathbb{R}^n \mid g(\boldsymbol{x}) > -\infty \,\}$. When we apply the OA method to (mmF), we need to evaluate g at points outside of X. Unless there is a point $\boldsymbol{y} \in X$ satisfying $\boldsymbol{y} \geqq \boldsymbol{v}$, $g(\boldsymbol{v})$ takes $-\infty$, and hence no information is available about how away the point $\boldsymbol{v}$ is from the domain of g. Then we extend the gap function g to $\mathbb{R}^n$ in this section. The *extended gap function* $\bar{g} : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\bar{g}(\boldsymbol{x}) = \max\{\, \boldsymbol{ey} - \bar{\boldsymbol{\beta}}\boldsymbol{t} \mid \boldsymbol{y} \in X,\ \boldsymbol{y} + \boldsymbol{t} \geqq \boldsymbol{x},\ \boldsymbol{t} \geqq \boldsymbol{0} \,\} - \boldsymbol{ex}, \tag{2.2}$$

where the n-dimensional row vector $\bar{\boldsymbol{\beta}}$ will be specified later. Clearly $\bar{g}$ is also a piecewise linear concave function. The following theorem in Yamamoto-Zenke [34] shows that $\bar{g}$ is an extension of g.

Theorem 2.1 *(i) The domain of $\bar{g}$ is $\mathbb{R}^n$ for any $\bar{\boldsymbol{\beta}} \geqq \boldsymbol{0}$.*

(ii) If $\bar{\boldsymbol{\beta}} \geqq n\boldsymbol{e}$ then $\bar{g} = g$ on the domain of g.

Proof: See Appendix for the proof. □

By Theorem 2.1, fixing $\bar{\boldsymbol{\beta}} = n\boldsymbol{e}$, we can replace the constraint $g(\boldsymbol{x}) \leqq 0$ in (mmF) with $\bar{g}(\boldsymbol{x}) \leqq 0$ to obtain an equivalent formulation of (mmF):

$$(mmF) \quad \left| \begin{array}{ll} \min\limits_{\boldsymbol{x}} & \boldsymbol{dx} \\ \text{s.t.} & \boldsymbol{x} \in X,\ \bar{g}(\boldsymbol{x}) \leqq 0, \end{array} \right.$$

which is equivalent to

$$(mmF) \quad \left| \begin{array}{ll} \min\limits_{\boldsymbol{x}} & \boldsymbol{dx} \\ \text{s.t.} & \boldsymbol{x} \in X \backslash \operatorname{int} \bar{G}, \end{array} \right.$$

where

$$\bar{G} = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid \bar{g}(\boldsymbol{x}) \geqq 0 \,\}. \tag{2.3}$$

Note that $\bar{G}$ is a convex set since $\bar{g}$ is a concave function. By the definition of $\bar{g}$, it is clear that $\bar{g}(\boldsymbol{x}) \geqq 0$ for all $\boldsymbol{x} \in X$, i.e., $X \subseteqq \bar{G}$. Since Assumption 1.2 (ii) implies $\boldsymbol{0} \in X \backslash X_M$, we see that $\bar{g}(\boldsymbol{0}) = g(\boldsymbol{0}) > 0$, i.e., $\boldsymbol{0} \in \operatorname{int} \bar{G}$. Additionally we have the following lemma.

Lemma 2.2 *$\bar{g}(\boldsymbol{x}) > 0$ for every point $\boldsymbol{x}$ in the relative interior of X.*

Proof: Let $\boldsymbol{x}$ be a point in the relative interior of X, i.e., $A\boldsymbol{x} = \boldsymbol{0}$ and $\boldsymbol{0} < \boldsymbol{x} < \boldsymbol{c}$. Letting $\boldsymbol{x}' = (1+\varepsilon)\boldsymbol{x}$ for a sufficiently small $\varepsilon > 0$, we see that $A\boldsymbol{x}' = \boldsymbol{0}$ and $\boldsymbol{0} \leqq \boldsymbol{x}' \leqq \boldsymbol{c}$, i.e., $\boldsymbol{x}' \in X$ and $\boldsymbol{x}' \geqq \boldsymbol{x}$. Therefore $\bar{g}(\boldsymbol{x}) = g(\boldsymbol{x}) \geqq \boldsymbol{e}(\boldsymbol{x}' - \boldsymbol{x}) = \varepsilon \boldsymbol{ex} > 0$. □

3. Outer Approximation Method for D.C. Optimization Problems

A set S is said to be a *D.C. set* if there are two convex sets Q and R such that $S = Q \backslash R$. The optimization problem on a D.C. set is called the *D.C. optimization problem*, which is

studied in e.g., Tuy [29, 30] and Horst-Tuy [16]. In this section we explain the OA method for the *canonical form D.C. optimization problem*, abbreviated to (CDC), which is defined as

$$(CDC) \qquad \left| \begin{array}{ll} \min\limits_{\boldsymbol{x}} & \boldsymbol{p}\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in D,\ h(\boldsymbol{x}) \geqq 0, \end{array} \right.$$

where $\boldsymbol{p} \in \mathbb{R}_n$ is the cost vector, $D \subsetneqq \mathbb{R}^n$ is a nonempty compact convex set and $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a convex function. We assume that

$$\operatorname{int}\{\, \boldsymbol{x} \in \mathbb{R}^n \mid h(\boldsymbol{x}) \leqq 0 \,\} = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid h(\boldsymbol{x}) < 0 \,\}.$$

Defining the convex set $H = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid h(\boldsymbol{x}) \leqq 0 \,\}$, (CDC) can be written as

$$(CDC) \qquad \left| \begin{array}{ll} \min\limits_{\boldsymbol{x}} & \boldsymbol{p}\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in D \backslash \operatorname{int} H, \end{array} \right.$$

and hence (CDC) is a D.C. optimization problem. To make this problem simple we further assume that

$$\mathbf{0} \in D \cap \operatorname{int} H, \ \text{and} \ \min\{\, \boldsymbol{p}\boldsymbol{x} \mid \boldsymbol{x} \in D \,\} = 0. \tag{3.1}$$

Note that (CDC) reduces to (mmF) when $D = X, H = \bar{G}$ and $\boldsymbol{p} = \boldsymbol{d}$.

3.1. Regularity and optimality condition

Problem (CDC) is said to be *regular* when

$$D \backslash \operatorname{int} H = \operatorname{cl}(D \backslash H). \tag{3.2}$$

Figure 2 shows an example of (CDC) that is not regular, where $\boldsymbol{x}^* \in D \backslash \operatorname{int} H$, while $\boldsymbol{x}^* \notin \operatorname{cl}(D \backslash H)$.

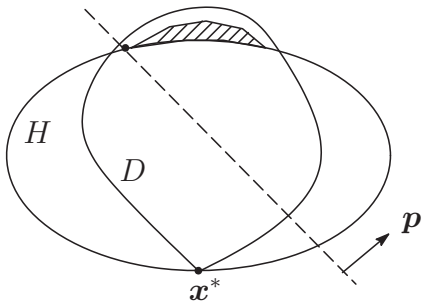


Figure 2: The case where (CDC) is not regular

The regularity assumption yields the optimality condition Theorem 3.1, which was given by Horst-Tuy [16]. To make this paper self-contained, we give the proof in Appendix. In the followings we denote

$$D(\eta) = \{\, \boldsymbol{x} \in D \mid \boldsymbol{p}\boldsymbol{x} \leqq \eta \,\}, \tag{3.3}$$

for $\eta \in \mathbb{R}$.

Theorem 3.1 *Let $\bar{\boldsymbol{x}}$ be a feasible solution of (CDC). Then $\bar{\boldsymbol{x}}$ is an optimal solution if $D(\boldsymbol{p}\bar{\boldsymbol{x}}) \subsetneqq H$.*

Proof: See Appendix for the proof. □

3.2. OA method for (CDC)

Let $\boldsymbol{x}^*$ be an optimal solution of (CDC) and $\bar{\boldsymbol{x}}^k \in D \backslash \operatorname{int} H$ be the incumbent at iteration k. In the OA method, we construct polytopes $P^0, P^1, \cdots, P^k, \cdots$ such that $P^0 \supseteqq P^1 \supseteqq \cdots \supseteqq P^k \supseteqq \cdots \supseteqq D(\boldsymbol{p}\boldsymbol{x}^*)$. If $\boldsymbol{p}\bar{\boldsymbol{x}}^k = 0$, we have done by (3.1). In the case where $\boldsymbol{p}\bar{\boldsymbol{x}}^k > 0$, we check the optimality condition $D(\boldsymbol{p}\bar{\boldsymbol{x}}^k) \subseteqq H$ by evaluating $h(\boldsymbol{v})$ at each vertex $\boldsymbol{v}$ of P^k. Namely, if $h(\boldsymbol{v}) \leqq 0$ for each vertex $\boldsymbol{v}$ of P^k, meaning $P^k \subseteqq H$, then $\bar{\boldsymbol{x}}^k$ solves (CDC). Otherwise we construct P^{k+1} by adding some linear inequality to P^k.

Here we describe the OA method for (CDC).

/** OA method for (CDC) **/

$\langle 0 \rangle$ (initialization) Find an initial feasible solution $\bar{\boldsymbol{x}}^0$ of (CDC) and construct an initial polytope P^0 such that $P^0 \supseteqq D(\boldsymbol{p}\bar{\boldsymbol{x}}^0)$. Compute the vertex set P_V^0 of P^0. Set $k := 0$.

$\langle k \rangle$ (iteration k) Solve $\max\{\, h(\boldsymbol{v}) \mid \boldsymbol{v} \in P_V^k \,\}$ to obtain $\boldsymbol{v}^k$.

- $\langle k1 \rangle$ (termination) If either $\boldsymbol{p}\bar{\boldsymbol{x}}^k = 0$ or $h(\boldsymbol{v}^k) \leqq 0$, meaning $P^k \subseteqq H$, then stop. (The current incumbent $\bar{\boldsymbol{x}}^k$ is an optimal solution of (CDC)). Otherwise, obtain the point $\boldsymbol{x}^k \in [\boldsymbol{0}, \boldsymbol{v}^k) \cap \partial H$.
- $\langle k2 \rangle$ (cutting the polytope) If $\boldsymbol{x}^k \notin D$, set $\bar{\boldsymbol{x}}^{k+1} := \bar{\boldsymbol{x}}^k$ and $P^{k+1} := P^k \cap \{\, \boldsymbol{x} \in \mathbb{R}^n \mid l(\boldsymbol{x}) \leqq 0 \,\}$ for some function $l : \mathbb{R}^n \to \mathbb{R}$ such that $l(\boldsymbol{v}^k) > 0$ and $l(\boldsymbol{x}) \leqq 0$ for all $\boldsymbol{x} \in D(\boldsymbol{p}\bar{\boldsymbol{x}})$. If $\boldsymbol{x}^k \in D$, set $\bar{\boldsymbol{x}}^{k+1} := \boldsymbol{x}^k$ and $P^{k+1} := P^k \cap \{\, \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{p}\boldsymbol{x} \leqq \boldsymbol{p}\bar{\boldsymbol{x}}^{k+1} \,\}$.
- $\langle k3 \rangle$ Compute the vertex set P_V^{k+1} of P^{k+1}. Set $k := k+1$ and go to $\langle k \rangle$.

Remark 3.2 *Note that adding a linear inequality to P^k makes P^{k+1} and the vertex set P_V^k of P^k is at hand. Subroutines for computing the vertex set P_V^{k+1} from the knowledge of P_V^k are provided in e.g., Chen-Hansen-Jaumard [8], Subsection* 7.4 *of Padberg [19] and Chapter* 18 *of Chvátal [9]. Due to the possible degeneracy of P^k, a sophisticated implementation should be needed e.g., Fukuda-Prodon [12].*

4. Outer Approximation Method for (mmF)

By Assumption 1.2 (ii)-(iii), we have

$$\boldsymbol{0} \in X \cap \operatorname{int} \bar{G}, \text{ and } \min\{\, \boldsymbol{d}\boldsymbol{x} \mid \boldsymbol{x} \in X \,\} = 0, \tag{4.1}$$

which correspond to (3.1). Then we can apply the OA method to (mmF) if the regularity condition is met.

4.1. Regularity and optimality condition

Unfortunately, the problem (mmF) is not regular, hence we introduce a positive tolerance ε and consider, instead of (mmF),

$$(mmF_\varepsilon) \qquad \left| \begin{array}{ll} \min\limits_{\boldsymbol{x}} & \boldsymbol{d}\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x} \in X \backslash \operatorname{int} \bar{G}_\varepsilon, \end{array} \right.$$

where

$$\bar{G}_\varepsilon = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid \bar{g}(\boldsymbol{x}) \geqq \varepsilon \,\}. \tag{4.2}$$

We call an optimal solution of (mmF_ε) an *ε-optimal solution of* (mmF).

First we show that any positive ε ensures the regularity of (mmF_ε).

Theorem 4.1 *The problem* (mmF_ε) *is regular for any* $\varepsilon > 0$.
Proof: We show that

$$X\backslash \operatorname{int} \bar{G}_\varepsilon = \operatorname{cl}(X\backslash \bar{G}_\varepsilon), \tag{4.3}$$

holds for any $\varepsilon > 0$.
($\supseteqq$) Since $X\backslash \operatorname{int} \bar{G}_\varepsilon$ is closed and $X\backslash \operatorname{int} \bar{G}_\varepsilon \supseteqq X\backslash \bar{G}_\varepsilon$, we have

$$X\backslash \operatorname{int} \bar{G}_\varepsilon = \operatorname{cl}(X\backslash \operatorname{int} \bar{G}_\varepsilon) \supseteqq \operatorname{cl}(X\backslash \bar{G}_\varepsilon).$$

($\subseteqq$) Let $\boldsymbol{x}$ be an arbitrary point of $X\backslash \operatorname{int} \bar{G}_\varepsilon$ and let $N_\delta(\boldsymbol{x})$ denote its δ-neighborhood, i.e., $N_\delta(\boldsymbol{x}) = \{\, \boldsymbol{x}' \in \mathbb{R}^n \mid \|\boldsymbol{x}' - \boldsymbol{x}\| < \delta \,\}$. We show that there is always a point, say x' in $N_\delta(\boldsymbol{x}) \cap (X\backslash \bar{G}_\varepsilon)$. If $\bar{g}(\boldsymbol{x}) > \varepsilon$ then there exists $\gamma > 0$ such that $\bar{g}(\boldsymbol{x}') > \varepsilon$ for any point $\boldsymbol{x}' \in N_\gamma(\boldsymbol{x})$ by the continuity of $\bar{g}$. This implies $N_\gamma(\boldsymbol{x}) \subseteqq \bar{G}_\varepsilon$, and hence $\boldsymbol{x} \in \operatorname{int} \bar{G}_\varepsilon$. Therefore the assumption $\boldsymbol{x} \in X\backslash \operatorname{int} \bar{G}_\varepsilon$ implies that $\boldsymbol{x} \in X$ and $\bar{g}(\boldsymbol{x}) \leqq \varepsilon$. By Theorem 2.1, we have $\bar{g}(\boldsymbol{x}) = g(\boldsymbol{x})$. When $\bar{g}(\boldsymbol{x}) < \varepsilon$, take $\boldsymbol{x}$ as $\boldsymbol{x}'$. Clearly $\boldsymbol{x}' = \boldsymbol{x} \notin \bar{G}_\varepsilon$ and $\boldsymbol{x}' = \boldsymbol{x} \in N_\delta(\boldsymbol{x})$, and we have done. When $g(\boldsymbol{x}) = \bar{g}(\boldsymbol{x}) = \varepsilon$, there is an optimal solution $\boldsymbol{y}^*$ of $\max\{\, \boldsymbol{ey} \mid \boldsymbol{y} \in X,\ \boldsymbol{y} \geqq \boldsymbol{x} \,\}$ such that $\boldsymbol{e}(\boldsymbol{y}^* - \boldsymbol{x}) = \varepsilon$, and hence $\boldsymbol{y}^* \neq \boldsymbol{x}$. Take λ such that $0 < \lambda < \min\{\, 1,\ \delta/\|\boldsymbol{y}^* - \boldsymbol{x}\| \,\}$ and let $\boldsymbol{x}' = \lambda \boldsymbol{y}^* + (1-\lambda)\boldsymbol{x}$. Since $\|\boldsymbol{x}' - \boldsymbol{x}\| = \lambda \|\boldsymbol{y}^* - \boldsymbol{x}\| < \delta$, we see $\boldsymbol{x}' \in N_\delta(\boldsymbol{x})$. Also we see that $\boldsymbol{x}' \in X$ by the convexity of X, and hence $g(\boldsymbol{x}') = \bar{g}(\boldsymbol{x}')$ by applying Theorem 2.1 again. Since $\boldsymbol{x}' \geqq \boldsymbol{x}$ and $\boldsymbol{x}' \neq \boldsymbol{x}$, we have

$$\begin{aligned}
\bar{g}(\boldsymbol{x}') &= g(\boldsymbol{x}') \\
&= \max\{\, \boldsymbol{ey} \mid \boldsymbol{y} \in X,\ \boldsymbol{y} \geqq \boldsymbol{x}' \,\} - \boldsymbol{ex}' \\
&< \max\{\, \boldsymbol{ey} \mid \boldsymbol{y} \in X,\ \boldsymbol{y} \geqq \boldsymbol{x} \,\} - \boldsymbol{ex} \\
&= \boldsymbol{e}(\boldsymbol{y}^* - \boldsymbol{x}) = \varepsilon.
\end{aligned}$$

Therefore we see that $\boldsymbol{x}' \notin \bar{G}_\varepsilon$. This completes the proof. □

Next we discuss the upper bound of ε, which will be crucial for the convergence of the algorithm.

Lemma 4.2 *If* $\varepsilon \in (0,1)$ *then* $\mathbf{0} \in \mathit{int}\bar{G}_\varepsilon$, *and* $(\mathbf{0}, \boldsymbol{v}) \cap \partial \bar{G}_\varepsilon \neq \emptyset$ *for any point* $\boldsymbol{v}$ *such that* $\bar{g}(\boldsymbol{v}) \leqq 0$.

Proof: We have $\bar{g}(\mathbf{0}) > 0$ since $\mathbf{0} \in \operatorname{int} \bar{G}$. Note that $\bar{g}(\mathbf{0})$, which coincides with $g(\mathbf{0})$, takes an integer value by the integrality property of X, and hence $\bar{g}(\mathbf{0}) \geqq 1$. Then we have $\bar{g}(\mathbf{0}) > \varepsilon$, i.e., $\mathbf{0} \in \operatorname{int} \bar{G}_\varepsilon$ for any $\varepsilon \in (0,1)$. The continuity of $\bar{g}$ ensures the last assertion. □

For the following lemma, we use δ_s to denote the number of arcs leaving node s, i.e.,

$$\delta_s = |\{\, i \mid d_i = +1 \,\}|. \tag{4.4}$$

Lemma 4.3 *Let* $\boldsymbol{x}^*$ *and* $\boldsymbol{x}^*_\varepsilon$ *be an optimal solution and an* ε*-optimal solution of* (mmF), *respectively. Then* $0 \leqq \boldsymbol{dx}^* - \boldsymbol{dx}^*_\varepsilon \leqq \varepsilon\delta_s$.

Proof: Since $\boldsymbol{x}^* \in X$ and $\bar{g}(\boldsymbol{x}^*) \leqq 0$, $\boldsymbol{x}^*$ is a feasible solution of (mmF_ε), and hence $\boldsymbol{dx}^*_\varepsilon \leqq \boldsymbol{dx}^*$. Let $\boldsymbol{y}^*_\varepsilon$ be an optimal solution of $\max\{\, \boldsymbol{ey} \mid \boldsymbol{y} \in X,\ \boldsymbol{y} \geqq \boldsymbol{x}^*_\varepsilon \,\}$. Clearly $\boldsymbol{y}^*_\varepsilon \in X_M$, i.e., $\boldsymbol{y}^*_\varepsilon$ is a feasible solution of (mmF), and hence $\boldsymbol{dx}^* \leqq \boldsymbol{dy}^*_\varepsilon$. We see that $y^*_{\varepsilon i} - x^*_{\varepsilon i} \leqq \varepsilon$ for each $i = 1, \ldots, n$, since $\boldsymbol{y}^*_\varepsilon - \boldsymbol{x}^*_\varepsilon \geqq \mathbf{0}$ and $\boldsymbol{e}(\boldsymbol{y}^*_\varepsilon - \boldsymbol{x}^*_\varepsilon) \leqq \varepsilon$. That implies $\boldsymbol{d}(\boldsymbol{y}^*_\varepsilon - \boldsymbol{x}^*_\varepsilon) \leqq \varepsilon|\{\, i \mid d_i = +1 \,\}| = \varepsilon\delta_s$, hence $\boldsymbol{dx}^*_\varepsilon \leqq \boldsymbol{dx}^* \leqq \boldsymbol{dy}^*_\varepsilon \leqq \boldsymbol{dx}^*_\varepsilon + \varepsilon\delta_s$. □

Theorem 4.4 *Let $\boldsymbol{x}^*_\varepsilon$ be an ε-optimal solution for some $\varepsilon \in (0,\, 1/\delta_s)$. Then $\lceil \boldsymbol{dx}^*_\varepsilon \rceil$ coincides with the optimal value of (mmF).*

Proof: From Lemma 4.3 we see that $0 \leqq \boldsymbol{dx}^* - \boldsymbol{dx}^*_\varepsilon < 1$. This inequality and the integrality of $\boldsymbol{dx}^*$ gives the assertion. □

In the sequel we choose ε from the open interval $(0,\, 1/\delta_s)$.

Note that $\bar{g}(\boldsymbol{x}^*_\varepsilon) \leqq \varepsilon$ holds for an ε-optimal solution $\boldsymbol{x}^*_\varepsilon$ of (mmF). Therefore $\bar{g}(\boldsymbol{x}) \leqq 0$ for any accumulation point $\boldsymbol{x}$ of $\{\boldsymbol{x}^*_\varepsilon\}_{\varepsilon\to 0+}$. This observation leads to the following corollary.

Corollary 4.5 *Let $\{\boldsymbol{x}^*_\varepsilon\}_{\varepsilon\to 0+}$ be a sequence of ε-optimal solutions of (mmF) for ε converging to 0 from above. Then any accumulation point of $\{\boldsymbol{x}^*_\varepsilon\}_{\varepsilon\to 0+}$ is an optimal solution of (mmF).*

As seen in Theorem 3.1, letting

$$X(\eta) = \{\, \boldsymbol{x} \in X \mid \boldsymbol{dx} \leqq \eta \,\}, \tag{4.5}$$

for $\eta \in \mathbb{R}$, the optimality condition of the OA method is $X(\boldsymbol{d}\bar{\boldsymbol{x}}_\varepsilon) \subseteq \bar{G}_\varepsilon$ for some $\bar{\boldsymbol{x}}_\varepsilon \in X \backslash \operatorname{int} \bar{G}_\varepsilon$. We can further relax this condition.

Theorem 4.6 *Let $\bar{\boldsymbol{x}}_\varepsilon \in X \backslash int \bar{G}_\varepsilon$ for some $\varepsilon \in (0,\, 1/\delta_s)$. If $X(\lceil \boldsymbol{d}\bar{\boldsymbol{x}}_\varepsilon - 1 \rceil) \subseteq \bar{G}_{\varepsilon'}$ for some $\varepsilon' > 0$ then $\lceil \boldsymbol{d}\bar{\boldsymbol{x}}_\varepsilon \rceil$ coincides with the optimal value of (mmF).*

Proof: Let $\boldsymbol{x}^*$ and $\boldsymbol{x}^*_\varepsilon$ be an optimal solution and an ε-optimal solution of (mmF), respectively. Since $\bar{\boldsymbol{x}}_\varepsilon$ is a feasible solution of (mmF_ε), we have $\boldsymbol{dx}^*_\varepsilon \leqq \boldsymbol{d}\bar{\boldsymbol{x}}_\varepsilon$. It is also clear that $\boldsymbol{dx}^*_\varepsilon \leqq \boldsymbol{dx}^*$. If $\boldsymbol{dx}^* < \boldsymbol{d}\bar{\boldsymbol{x}}_\varepsilon$ then we have $\boldsymbol{x}^* \in X(\lceil \boldsymbol{d}\bar{\boldsymbol{x}}_\varepsilon - 1 \rceil) \subseteq \bar{G}_{\varepsilon'}$ since $\boldsymbol{dx}^*$ is integer, and hence $\bar{g}(\boldsymbol{x}^*) \geqq \varepsilon' > 0$, which contradicts that $\bar{g}(\boldsymbol{x}^*) = 0$. Then we have $\boldsymbol{d}\bar{\boldsymbol{x}}_\varepsilon \leqq \boldsymbol{dx}^*$. Hence by Lemma 4.3 we obtain $\boldsymbol{dx}^*_\varepsilon \leqq \boldsymbol{d}\bar{\boldsymbol{x}}_\varepsilon \leqq \boldsymbol{dx}^* \leqq \boldsymbol{dx}^*_\varepsilon + \varepsilon\delta_s < \boldsymbol{dx}^*_\varepsilon + 1$. This completes the proof. □

We construct a polytope P satisfying $X(\lceil \boldsymbol{d}\bar{\boldsymbol{x}}_\varepsilon - 1 \rceil) \subseteq P$ for some $\bar{\boldsymbol{x}}_\varepsilon \in X \backslash \operatorname{int} \bar{G}_\varepsilon$. Let $\boldsymbol{v}^*$ be a vertex minimizing $\bar{g}(\boldsymbol{v})$ over P_V and $\varepsilon' = \bar{g}(\boldsymbol{v}^*)$. For any $\boldsymbol{x} \in P$ we have $\bar{g}(\boldsymbol{x}) \geqq \bar{g}(\boldsymbol{v}^*)$, i.e., $0 \leqq \bar{g}(\boldsymbol{x}) - \bar{g}(\boldsymbol{v}^*) = \bar{g}(\boldsymbol{x}) - \varepsilon'$, and hence $P \subseteq \bar{G}_{\varepsilon'}$. This implies that $X(\lceil \boldsymbol{d}\bar{\boldsymbol{x}}_\varepsilon - 1 \rceil) \subseteq \bar{G}_{\varepsilon'}$. Therefore if $\varepsilon' > 0$ then the optimal value of (mmF) is obtained by Theorem 4.6.

4.2. Local search

For $\boldsymbol{v} \in X_M \cap X_V$, we define the set of efficient vertices linked to $\boldsymbol{v}$ by an edge as

$$\begin{aligned} N_M(\boldsymbol{v}) &= \{\, \boldsymbol{v}' \in X_M \cap X_V \mid [\boldsymbol{v}, \boldsymbol{v}'] \text{ is an edge of } X \,\} \\ &= \{\, \boldsymbol{v}' \in X_V \mid [\boldsymbol{v}, \boldsymbol{v}'] \text{ is an edge of } X \text{ and } g(\boldsymbol{v}') \leqq 0 \,\}. \end{aligned} \tag{4.6}$$

Whenever we find a feasible solution $\boldsymbol{w} \in X_M$, we apply the Local Search procedure starting with $\boldsymbol{w}$ (LS($\boldsymbol{w}$) for short) for further improvement.

The procedure is described as follows.

/** LS($\boldsymbol{w}$) procedure **/

⟨0⟩ (initialization) If $\boldsymbol{w} \notin X_V$ then find the face F of X containing $\boldsymbol{w}$ in its relative interior and solve $\min\{\, \boldsymbol{dx} \mid \boldsymbol{x} \in F \,\}$ to obtain a vertex $\boldsymbol{v}^0 \in X_M \cap X_V$, otherwise set $\boldsymbol{v}^0 := \boldsymbol{w}$. Set $k := 0$.

$\langle k \rangle$ (iteration k) Solve $\min\{\, \boldsymbol{dv} \mid \boldsymbol{v} \in N_M(\boldsymbol{v}^k) \,\}$ to obtain a solution $\boldsymbol{v}^*$. If $\boldsymbol{dv}^* \geqq \boldsymbol{dv}^k$ then stop, $\boldsymbol{v}^k$ is the local optimal vertex of (mmF). Otherwise set $\boldsymbol{v}^{k+1} := \boldsymbol{v}^*$, $k := k+1$ and go to $\langle k \rangle$.

Remark 4.7 *If $\boldsymbol{w} \in X_M$, the face F of X containing $\boldsymbol{w}$ in its relative interior is contained in X_M since X_M is a connected union of several faces of X.*

4.3. OA method for (mmF)

We describe the OA method for (mmF) as follows.

/** OA method for (mmF) **/

$\langle 0 \rangle$ (initialization) Find an initial feasible vertex $\boldsymbol{w}^0 \in X_M \cap X_V$ of (mmF). If $N_M(\boldsymbol{w}^0) = \emptyset$ then stop. ($\boldsymbol{w}^0$ is a unique feasible solution of (mmF)). Otherwise, apply the LS($\boldsymbol{w}^0$) procedure to obtain a local optimal vertex $\bar{\boldsymbol{x}}^0 \in X_M \cap X_V$. Solve $\zeta := \max\{\, \boldsymbol{ex} \mid \boldsymbol{x} \in X,\ \boldsymbol{dx} \leqq \boldsymbol{d}\bar{\boldsymbol{x}}^0 - 1 \,\}$ and construct an initial polytope $P^0 \supseteqq X(\boldsymbol{d}\bar{\boldsymbol{x}}^0 - 1)$ by setting $P^0 := \{\, \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{ex} \leqq \zeta,\ \boldsymbol{dx} \leqq \boldsymbol{d}\bar{\boldsymbol{x}}^0 - 1,\ \boldsymbol{x} \geqq \boldsymbol{0} \,\}$. Compute the vertex set P_V^0 of P^0. Set $k := 0$.

$\langle k \rangle$ (iteration k) Solve $\min\{\, \bar{g}(\boldsymbol{v}) \mid \boldsymbol{v} \in P_V^k \,\}$ to obtain a vertex $\boldsymbol{v}^k$.

$\langle k1 \rangle$ (termination) If either $\boldsymbol{d}\bar{\boldsymbol{x}}^k = 0$ or $\bar{g}(\boldsymbol{v}^k) > 0$ then stop. (The optimal value of (mmF) is $\lceil \boldsymbol{d}\bar{\boldsymbol{x}}^k \rceil$). Otherwise, obtain the point $\boldsymbol{x}_\varepsilon^k \in (\boldsymbol{0}, \boldsymbol{v}^k) \cap \partial \bar{G}_\varepsilon$. (Note that Lemma 4.2 ensures that $(\boldsymbol{0}, \boldsymbol{v}^k) \cap \partial \bar{G}_\varepsilon \neq \emptyset$).

$\langle k2 \rangle$ (update) If $\boldsymbol{x}_\varepsilon^k \in X$, obtain the point $\boldsymbol{x}^k \in (\boldsymbol{0}, \boldsymbol{v}^k] \cap \partial \bar{G}$.

$\langle k2.1 \rangle$ If $\boldsymbol{x}^k \in X$, meaning $\boldsymbol{x}^k \in X_M$, then obtain the local optimal vertex $\boldsymbol{z}^k \in X_M \cap X_V$ by applying the LS($\boldsymbol{x}^k$) procedure, and further obtain the point $\boldsymbol{z}_\varepsilon^k \in (\boldsymbol{0}, \boldsymbol{z}^k) \cap \partial \bar{G}_\varepsilon$. Set $\bar{\boldsymbol{x}}^{k+1} := \operatorname{argmin}\{\, \boldsymbol{dz}_\varepsilon^k,\ \boldsymbol{dx}_\varepsilon^k \,\}$ and $P^{k+1} := P^k \cap \{\, \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{dx} \leqq \lceil \boldsymbol{d}\bar{\boldsymbol{x}}^{k+1} - 1 \rceil \,\}$.

$\langle k2.2 \rangle$ If $\boldsymbol{x}^k \notin X$, meaning $\boldsymbol{x}^k \notin X_M$, then set $\bar{\boldsymbol{x}}^{k+1} := \boldsymbol{x}_\varepsilon^k$ and $P^{k+1} := P^k \cap \{\, \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{dx} \leqq \lceil \boldsymbol{d}\bar{\boldsymbol{x}}^{k+1} - 1 \rceil,\ l(\boldsymbol{x}) \leqq 0 \,\}$.

$\langle k3 \rangle$ If $\boldsymbol{x}_\varepsilon^k \notin X$ then set $\bar{\boldsymbol{x}}^{k+1} := \bar{\boldsymbol{x}}^k$ and $P^{k+1} := P^k \cap \{\, \boldsymbol{x} \in \mathbb{R}^n \mid l(\boldsymbol{x}) \leqq 0 \,\}$.

$\langle k4 \rangle$ Compute the vertex set P_V^{k+1} of P^{k+1}. Set $k := k+1$ and go to $\langle k \rangle$.

Remark 4.8 *The function $l : \mathbb{R}^n \to \mathbb{R}$ in Step k2.2 and Step k3 is given by one of the inequalities $\pm A\boldsymbol{x} \leqq 0$ and $\boldsymbol{x} \leqq \boldsymbol{c}$ not satisfied by the point $\boldsymbol{v}^k$, i.e.,*

(i) $l(\boldsymbol{x}) = \boldsymbol{e}^j\boldsymbol{x} - c_j$ for some $j \in \{1, \ldots, n\}$ such that $v_j^k > c_j$, or

(ii) $l(\boldsymbol{x}) = sgn(\boldsymbol{a}^i\boldsymbol{v}^k)\boldsymbol{a}^i\boldsymbol{x}$ for some $i \in \{1, \ldots, m\}$ such that $\boldsymbol{a}^i\boldsymbol{v}^k \neq 0$, where $\boldsymbol{a}^i$ is the ith row of A, and

$$sgn(\alpha) = \begin{cases} +1 & when\ \alpha > 0 \\ -1 & when\ \alpha < 0. \end{cases}$$

Lemma 4.9 *Let $\boldsymbol{z}^k$ be the local optimal vertex obtained by applying the LS($\boldsymbol{x}^k$) procedure starting with $\boldsymbol{x}^k$ in Step k2.1 at iteration k, and suppose $\boldsymbol{dz}^k > 0$. Then $\boldsymbol{dz}^{k'} < \boldsymbol{dz}^k$ for iteration k' such that $k' > k$.*

Proof: It suffices to show that $\boldsymbol{dz}^{k+1} < \boldsymbol{dz}^k$. By the construction of P^{k+1} we have $P^{k+1} \subseteqq \{\, \boldsymbol{x} \mid \boldsymbol{dx} \leqq \lceil \boldsymbol{d}\bar{\boldsymbol{x}}^{k+1} - 1 \rceil \,\}$. Since $\boldsymbol{x}^{k+1} \in (0, \boldsymbol{v}^{k+1}] \subseteqq P^{k+1}$ and $\boldsymbol{z}^{k+1}$ is obtained by LS($\boldsymbol{x}^{k+1}$), we have

$$\boldsymbol{dz}^{k+1} \leqq \boldsymbol{dx}^{k+1} \leqq \lceil \boldsymbol{d}\bar{\boldsymbol{x}}^{k+1} - 1 \rceil.$$

Since we assume that $\boldsymbol{dz}^k > 0$, we have $0 < \boldsymbol{dz}^k_\varepsilon < \boldsymbol{dz}^k$ by the choice of $\boldsymbol{z}^k_\varepsilon$. Therefore in Step k2.1

$$\lceil \boldsymbol{d}\bar{\boldsymbol{x}}^{k+1} - 1 \rceil < \boldsymbol{d}\bar{\boldsymbol{x}}^{k+1} = \min\{\boldsymbol{dx}^k_\varepsilon, \boldsymbol{dz}^k_\varepsilon\} < \boldsymbol{dz}^k.$$

Combining the two inequalities yields the desired result. □

Theorem 4.10 *The OA method for* (mmF) *works correctly and terminates after finitely many iterations.*

Proof: (correctness) If $N_M(\boldsymbol{w}^0) = \emptyset$ at the initialization step, we can conclude from the connectedness of X_M that $\boldsymbol{w}^0$ is a unique feasible solution of (mmF) and hence solves the problem. When the algorithm terminates in Step k1, the optimal value of (mmF) is equal either to zero by Assumption 1.2 (iii), or to $\lceil \boldsymbol{d}\bar{\boldsymbol{x}}^k \rceil$ by Theorem 4.6. So the optimal value is obtained whenever the algorithm terminates.

We suppose that the algorithm has not yet terminated at iteration k, i.e., $\boldsymbol{d}\bar{\boldsymbol{x}}^k > 0$ and $\bar{g}(\boldsymbol{v}^k) \leqq 0$, and show that each step of the algorithm can be done. Lemma 4.2 ensures that there are points $\boldsymbol{x}^k_\varepsilon \in (\boldsymbol{0}, \boldsymbol{v}^k) \cap \partial \bar{G}_\varepsilon$ and $\boldsymbol{z}^k_\varepsilon \in (\boldsymbol{0}, \boldsymbol{z}^k) \cap \partial \bar{G}_\varepsilon$, in Step k1 and Step k2.1, respectively. Since $\boldsymbol{0} \in \operatorname{int} \bar{G}$ and $\boldsymbol{v}^k \notin \operatorname{int} \bar{G}$, there also exists a point $\boldsymbol{x}^k \in (\boldsymbol{0}, \boldsymbol{v}^k] \cap \partial \bar{G}$. When $\boldsymbol{x}^k_\varepsilon \notin X$, clearly $\boldsymbol{v}^k \notin X$, and hence the function $l : \mathbb{R}^n \to \mathbb{R}$ of Remark 4.8 can be found in Step k3. To show that the function $l : \mathbb{R}^n \to \mathbb{R}$ is found in Step k2.2 we have only to show that $\boldsymbol{v}^k \notin X$. Suppose the contrary, i.e., $\boldsymbol{v}^k \in X$. By the assumption that $\bar{g}(\boldsymbol{v}^k) \leqq 0$ and the fact that $\bar{g}(\boldsymbol{x}) \geqq 0$ for all $\boldsymbol{x} \in X$, we have $\bar{g}(\boldsymbol{v}^k) = 0$, i.e., $\boldsymbol{v}^k \in \partial \bar{G}$, and hence $\boldsymbol{v}^k \in X \backslash \operatorname{int} \bar{G} = X_M$. This implies $\boldsymbol{x}^k = \boldsymbol{v}^k \in X_M$ by the choice of $\boldsymbol{x}^k$, which contradicts that we are currently at iteration k2.2. Therefore we have seen that $\boldsymbol{v}^k \notin X$ in Step k2.2.

(finiteness) Suppose that the polytope P^ν at iteration ν meets the condition

$$P^\nu \subsetneqq X \text{ and } P^\nu \cap X_M = \emptyset, \tag{4.7}$$

after updated either in Step k2 or in Step k3, and consider the next iteration. Since $\boldsymbol{v}^\nu$ is chosen from P^ν, we have $\boldsymbol{v}^\nu \in X \backslash X_M$ and consequently $\bar{g}(\boldsymbol{v}^\nu) > 0$. Then the algorithm stops at Step k1. Therefore we have only to prove that (4.7) holds within a finite number of iterations. Note first that both Step k2.2 and Step k3 are done at most a finite number of times. Indeed, the polytope, say $P^{k'}$, when $2m + n$ cuts $l(\boldsymbol{x}) \leqq 0$ have been added to the initial polytope P^0, is contained in X. Therefore $\boldsymbol{v}^{k'}$ as well as $\boldsymbol{x}^{k'}_\varepsilon$ lies in X, and hence we obtain that $\boldsymbol{x}^{k'} = \boldsymbol{v}^{k'} \in X_M$ in the same way as in the former part of this proof. Therefore we come to neither Step k2.2 nor Step k3 after iteration k'. Namely, Step k2.1 followed by Step k4 repeats itself after iteration k'. For iteration k with $k \geqq k' + 1$, we have $\boldsymbol{x}^k \in X_M$. We then locate $\boldsymbol{z}^k \in X_M \cap X_V$ by applying the LS($\boldsymbol{x}^k$) procedure and obtain a point $\boldsymbol{z}^k_\varepsilon \in (\boldsymbol{0}, \boldsymbol{z}^k) \cap \partial \bar{G}_\varepsilon$. If $\boldsymbol{dz}^k = 0$ for some $k \geqq k' + 1$ then we set $\bar{\boldsymbol{x}}^{k+1} := \boldsymbol{z}^k_\varepsilon$ since $\boldsymbol{dz}^k_\varepsilon = \boldsymbol{dz}^k = 0 \leqq \boldsymbol{dx}^k_\varepsilon$. Then the incumbent value $\boldsymbol{d}\bar{\boldsymbol{x}}^{k+1}$ becomes zero, and hence the algorithm stops in Step k1 at the next iteration. If $\boldsymbol{dz}^k > 0$ for all k with $k \geqq k' + 1$, we see that $\boldsymbol{dz}^{k+1} < \boldsymbol{dz}^k$ for all $k \geqq k' + 1$ by Lemma 4.9. Since $|X_M \cap X_V|$ is finite, we eventually obtain a point $\boldsymbol{z}^{\nu-1} \in X_M \cap X_V$ such that $\boldsymbol{dz}^{\nu-1} \leqq \boldsymbol{dz}$ for all $\boldsymbol{z} \in X_M \cap X_V$. Also we have $\boldsymbol{dz}^{\nu-1}_\varepsilon < \boldsymbol{dz}^{\nu-1}$ by the choice of $\boldsymbol{z}^{\nu-1}_\varepsilon$. The polytope P^ν is then defined as $P^\nu := P^{\nu-1} \cap \{\, \boldsymbol{x} \mid \boldsymbol{dx} \leqq \lceil \boldsymbol{d}\bar{\boldsymbol{x}}^\nu - 1 \rceil \,\}$, where $\bar{\boldsymbol{x}}^\nu$ satisfies that $\boldsymbol{d}\bar{\boldsymbol{x}}^\nu = \min\{\, \boldsymbol{dx}^{\nu-1}_\varepsilon, \boldsymbol{dz}^{\nu-1}_\varepsilon \,\} < \boldsymbol{dz}^{\nu-1}$. This means that $P^\nu \cap (X_M \cap X_V) = \emptyset$. Since X_M is a connected union of several faces of X, we see that $\boldsymbol{dz}^{\nu-1} \leqq \boldsymbol{dx}$ for all $\boldsymbol{x} \in X_M$. Therefore we conclude that $P^\nu \cap X_M = \emptyset$. □

We illustrate the OA method for (mmF) in Figure 3, in which we use a two-dimensional general polyhedron $X = \{\, \boldsymbol{x} \in \mathbb{R}^2 \mid B\boldsymbol{x} \leqq \boldsymbol{b},\ \boldsymbol{x} \geqq \boldsymbol{0} \,\}$ for $B \in \mathbb{R}^{m\times 2}$ and $\boldsymbol{b} \in \mathbb{R}^m$ because the set of feasible flows $X = \{\, \boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} = \boldsymbol{0},\ \boldsymbol{0} \leqq \boldsymbol{x} \leqq \boldsymbol{c} \,\}$ is unsuitable for illustration. We obtain a local optimal vertex $\bar{\boldsymbol{x}}^0 \in X_M \cap X_V$ and set up an initial polytope P^0 (See (a)). It is easy to enumerate all vertices of P^0 because this polytope is simply given by $P^0 := \{\, \boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{e}\boldsymbol{x} \leqq \zeta,\ \boldsymbol{d}\boldsymbol{x} \leqq \boldsymbol{d}\bar{\boldsymbol{x}}^0 - 1,\ \boldsymbol{x} \geqq \boldsymbol{0} \,\}$. We obtain a point $\boldsymbol{v}^0$ minimizing $\bar{g}(\boldsymbol{v})$ over P^0_V, and a point $\boldsymbol{x}^0_\varepsilon \in (\boldsymbol{0}, \boldsymbol{v}^0) \cap \partial\bar{G}_\varepsilon$ (See (b)). We see that $\boldsymbol{x}^0_\varepsilon \notin X$, and hence set $\bar{\boldsymbol{x}}^1 := \bar{\boldsymbol{x}}^0$ and cut off $\boldsymbol{v}^0$ from P^0 (See (c)). Using P^0_V, we compute P^1_V. In the next iteration, we obtain points $\boldsymbol{v}^1, \boldsymbol{x}^1_\varepsilon$ and $\boldsymbol{x}^1$. Since $\boldsymbol{x}^1 \in X_M$, we apply the LS($\boldsymbol{x}^1$) procedure to obtain a point $\boldsymbol{z}^1$, and obtain a point $\boldsymbol{z}^1_\varepsilon \in (\boldsymbol{0}, \boldsymbol{z}^1) \cap \partial\bar{G}_\varepsilon$ (See (d)). We find a point $\boldsymbol{z}^1_\varepsilon \in X \backslash \operatorname{int}\bar{G}_\varepsilon$ such that $\boldsymbol{d}\boldsymbol{z}^1_\varepsilon < \boldsymbol{d}\boldsymbol{x}^1_\varepsilon$. We then set $\bar{\boldsymbol{x}}^2 := \boldsymbol{z}^1_\varepsilon$ and construct P^2 by adding the cut $\boldsymbol{d}\boldsymbol{x} \leqq \lceil \boldsymbol{d}\bar{\boldsymbol{x}}^2 - 1 \rceil$ to P^1 (See (e)). Because $\bar{g}(\boldsymbol{v}) > 0$ for all vertices $\boldsymbol{v}$ of P^2 (See (f)), we terminate at the next iteration with the optimal value $\lceil \boldsymbol{d}\bar{\boldsymbol{x}}^2 \rceil$.

5. Further Works

The OA method provides the optimal value but may fail to provide an optimal solution of (mmF). Finding an optimal solution is still a hard task even when its value is known, however, the following lemma affords a clue to the way of finding an optimal solution.

Lemma 5.1 *Let $\varepsilon \in (0,1)$, $\boldsymbol{x}^*_\varepsilon$ be an ε-optimal solution of (mmF) and*

$$\Delta_\varepsilon = \{\, \boldsymbol{\xi} \in \mathbb{R}^n \mid A\boldsymbol{\xi} = \boldsymbol{0},\ \boldsymbol{\xi} \geqq \boldsymbol{0},\ \boldsymbol{e}\boldsymbol{\xi} \leqq \varepsilon \,\}. \tag{5.1}$$

If $\boldsymbol{x}^*_\varepsilon + \bar{\boldsymbol{\xi}}$ is an integer vector for some $\bar{\boldsymbol{\xi}} \in \Delta_\varepsilon$ then $\boldsymbol{x}^*_\varepsilon + \bar{\boldsymbol{\xi}}$ is an optimal solution of (mmF).

Proof: (feasibility) Let $\boldsymbol{x}^* = \boldsymbol{x}^*_\varepsilon + \bar{\boldsymbol{\xi}}$ and $\boldsymbol{y}^*$ be an optimal solution of $\max\{\, \boldsymbol{e}\boldsymbol{y} \mid \boldsymbol{y} \in X,\ \boldsymbol{y} \geqq \boldsymbol{x}^* \,\}$. Note that

$$\boldsymbol{e}\boldsymbol{x}^* \text{ is integer,} \tag{5.2}$$

$$\boldsymbol{e}\boldsymbol{x}^*_\varepsilon \leqq \boldsymbol{e}\boldsymbol{x}^* \leqq \boldsymbol{e}\boldsymbol{y}^*, \tag{5.3}$$

and also

$$\boldsymbol{e}\boldsymbol{y}^* \text{ is integer,} \tag{5.4}$$

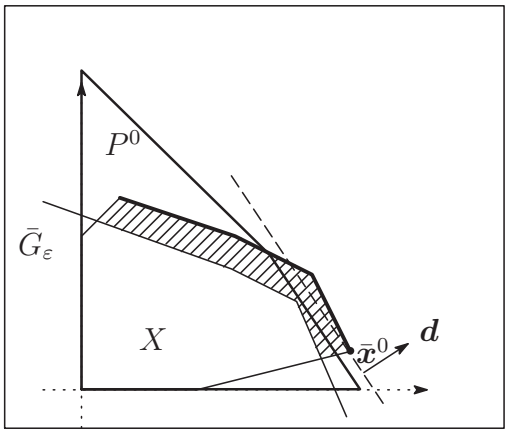
since $X \cap \{\, \boldsymbol{y} \mid \boldsymbol{y} \geqq \boldsymbol{x}^* \,\}$ inherits the integrality property of X.

Suppose we have the inequality

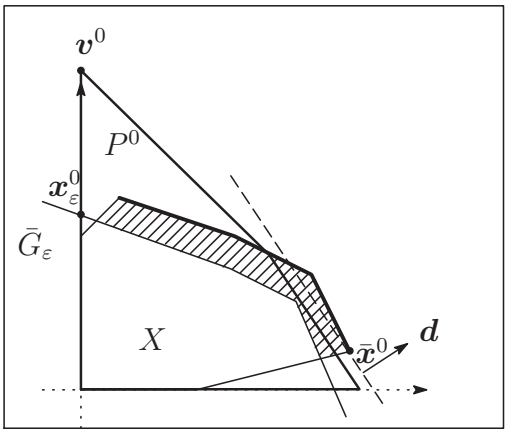
$$\boldsymbol{e}\boldsymbol{y}^* < \boldsymbol{e}\boldsymbol{x}^*_\varepsilon + 1. \tag{5.5}$$

Then by (5.3) and (5.5) together with the integrality of $\boldsymbol{e}\boldsymbol{x}^*$ and $\boldsymbol{e}\boldsymbol{y}^*$ we see that $\boldsymbol{e}\boldsymbol{x}^* = \boldsymbol{e}\boldsymbol{y}^*$. Hence $\bar{g}(\boldsymbol{x}^*) = \boldsymbol{e}\boldsymbol{y}^* - \boldsymbol{e}\boldsymbol{x}^* = 0$, meaning that $\boldsymbol{x}^* \in X_M$.

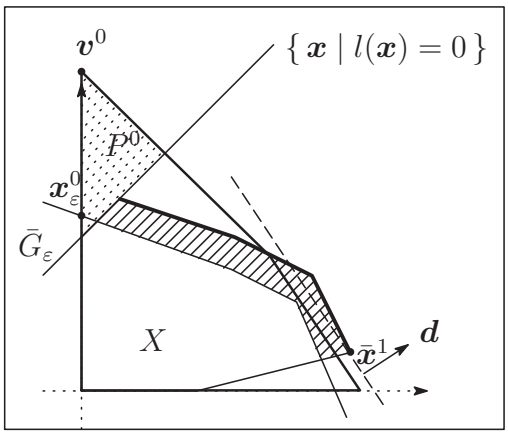
The inequality (5.5) is seen as follows. Let $\boldsymbol{y}^*_\varepsilon$ be an optimal solution of $\max\{\, \boldsymbol{e}\boldsymbol{y} \mid \boldsymbol{y} \in X,\ \boldsymbol{y} \geqq \boldsymbol{x}^*_\varepsilon \,\}$, and let $\boldsymbol{\xi}^* = \boldsymbol{y}^*_\varepsilon - \boldsymbol{x}^*_\varepsilon$. We see that $A\boldsymbol{\xi}^* = A\boldsymbol{y}^*_\varepsilon - A\boldsymbol{x}^*_\varepsilon = \boldsymbol{0}$, $\boldsymbol{\xi}^* \geqq \boldsymbol{0}$ and $\boldsymbol{e}\boldsymbol{\xi}^* = \boldsymbol{e}(\boldsymbol{y}^*_\varepsilon - \boldsymbol{x}^*_\varepsilon) = g(\boldsymbol{x}^*_\varepsilon) \leqq \varepsilon$, and hence $\boldsymbol{\xi}^* \in \Delta_\varepsilon$. Then $\boldsymbol{e}\boldsymbol{y}^*_\varepsilon = \boldsymbol{e}(\boldsymbol{x}^*_\varepsilon + \boldsymbol{\xi}^*) \leqq \boldsymbol{e}\boldsymbol{x}^*_\varepsilon + \varepsilon < \boldsymbol{e}\boldsymbol{x}^*_\varepsilon + 1$. The point $\boldsymbol{y}^*$ is a feasible solution of $\max\{\, \boldsymbol{e}\boldsymbol{y} \mid \boldsymbol{y} \in X,\ \boldsymbol{y} \geqq \boldsymbol{x}^*_\varepsilon \,\}$, since $\boldsymbol{y}^* \in X$ and $\boldsymbol{y}^* \geqq \boldsymbol{x}^* = \boldsymbol{x}^*_\varepsilon + \bar{\boldsymbol{\xi}} \geqq \boldsymbol{x}^*_\varepsilon$. Then we see that $\boldsymbol{e}(\boldsymbol{y}^*_\varepsilon - \boldsymbol{y}^*) \geqq 0$, and hence $\boldsymbol{e}\boldsymbol{y}^* \leqq \boldsymbol{e}\boldsymbol{y}^*_\varepsilon < \boldsymbol{e}\boldsymbol{x}^*_\varepsilon + 1$. (optimality) We show that $\boldsymbol{x}^*$ solves (mmF). Clearly, $\boldsymbol{d}\bar{\boldsymbol{\xi}} \leqq \boldsymbol{e}\bar{\boldsymbol{\xi}}$ since $\boldsymbol{d} \leqq \boldsymbol{e}$ and $\bar{\boldsymbol{\xi}} \geqq \boldsymbol{0}$. For any $\boldsymbol{v} \in X_M \cap X_V$, we see that $g(\boldsymbol{v}) \leqq \varepsilon$, and $\boldsymbol{v}$ is an integer vector by the integrality property of X. Since $\boldsymbol{x}^*_\varepsilon = \boldsymbol{x}^* - \bar{\boldsymbol{\xi}}$ is an optimal solution of (mmF_ε), we have $\boldsymbol{d}\boldsymbol{x}^*_\varepsilon \leqq \boldsymbol{d}\boldsymbol{x}$ for



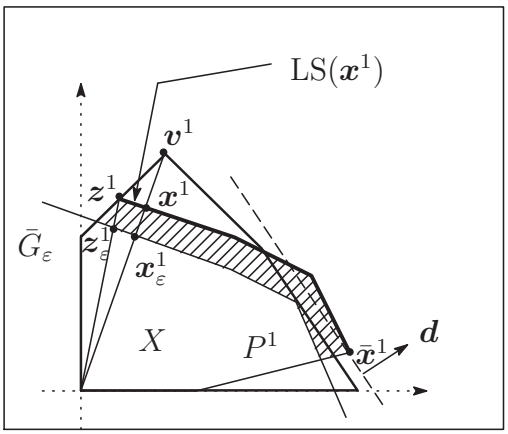
(a) polytope P^0



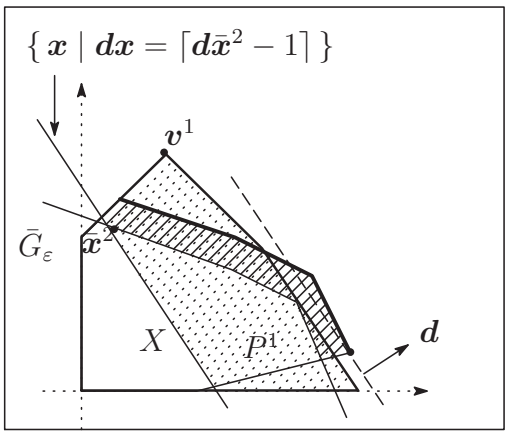
(b) obtaining $\boldsymbol{v}^0$ and $\boldsymbol{x}^0_\varepsilon$



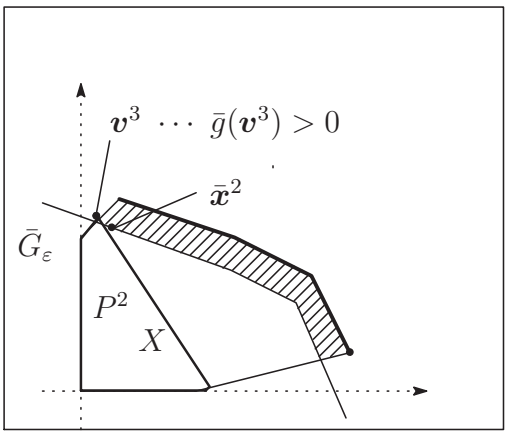
(c) cutting off $\boldsymbol{v}^0$ from P^0



(d) obtaining $\boldsymbol{v}^1$, $\boldsymbol{x}^1_\varepsilon$ and $\boldsymbol{z}^1_\varepsilon$



(e) cutting off $\boldsymbol{v}^1$ from P^1



(f) termination

Figure 3: An example of the OA method for (mmF)

all $\boldsymbol{x} \in X$ such that $g(\boldsymbol{x}) \leqq \varepsilon$, and hence $\boldsymbol{dx}^*_\varepsilon \leqq \boldsymbol{dv}$ for all $\boldsymbol{v} \in X_M \cap X_V$. Then we see that $\boldsymbol{dx}^* = \boldsymbol{dx}^*_\varepsilon + \boldsymbol{d\bar{\xi}} \leqq \boldsymbol{dv} + \boldsymbol{e\bar{\xi}} < \boldsymbol{dv} + 1$. Since both $\boldsymbol{x}^*$ and $\boldsymbol{v}$ are integer vectors, we have $\boldsymbol{dx}^* \leqq \boldsymbol{dv}$ for all $\boldsymbol{v} \in X_M \cap X_V$. □

Since the dimension of X is $n-m$, it would be desirable to reduce the number of variables that we have to handle in the algorithm. Yamamoto-Zenke explains an idea in [34], however, with the proviso that it does not work generally. Computational experiment should be carried out to improve the efficiency of the algorithm in this paper.

Appendix

Proof of Theorem 2.1

Proof: (i) The extended gap function $\bar{g}(\boldsymbol{x})$ of (2.2) is given by the optimal value of

$$(\overline{P_G(\boldsymbol{x})}) \qquad \left| \begin{array}{ll} \max\limits_{\boldsymbol{y},\boldsymbol{t}} & \boldsymbol{ey} - \boldsymbol{ex} - \bar{\boldsymbol{\beta}}\boldsymbol{t} \\ \text{s.t.} & A\boldsymbol{y} = \boldsymbol{0},\ \boldsymbol{0} \leqq \boldsymbol{y} \leqq \boldsymbol{c}, \\ & \boldsymbol{y} + \boldsymbol{t} \geqq \boldsymbol{x},\ \boldsymbol{t} \geqq \boldsymbol{0}, \end{array} \right.$$

whose dual problem is

$$(\overline{D_G(\boldsymbol{x})}) \qquad \left| \begin{array}{ll} \min\limits_{\boldsymbol{\pi},\boldsymbol{\alpha},\boldsymbol{\beta}} & \boldsymbol{\alpha c} - \boldsymbol{\beta x} - \boldsymbol{ex} \\ \text{s.t.} & (\boldsymbol{\pi}, \boldsymbol{\alpha}, \boldsymbol{\beta}) \in \bar{\Omega}, \end{array} \right.$$

where

$$\bar{\Omega} = \{\, (\boldsymbol{\pi}, \boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbb{R}_{m+2n} \mid \boldsymbol{\pi}A + \boldsymbol{\alpha} - \boldsymbol{\beta} \geqq \boldsymbol{e},\ \boldsymbol{\alpha} \geqq \boldsymbol{0},\ \boldsymbol{0} \leqq \boldsymbol{\beta} \leqq \bar{\boldsymbol{\beta}} \,\}.$$

For any $\boldsymbol{x} \in \mathbb{R}^n$, $(\overline{D_G(\boldsymbol{x})})$ is feasible, e.g., take $\boldsymbol{\pi} = \boldsymbol{\beta} = \boldsymbol{0}$ and $\boldsymbol{\alpha} \geqq \boldsymbol{e}$, and has the finite optimal value. By the duality theorem of linear programming, for any $\boldsymbol{x} \in \mathbb{R}^n$, $(\overline{P_G(\boldsymbol{x})})$ also has the finite optimal value, and hence $\bar{g}(\boldsymbol{x})$ is finite for any $\boldsymbol{x} \in \mathbb{R}^n$.
(ii) Let $\boldsymbol{x}$ be a point in the domain of g. By the similar observation in (i), the gap function $g(\boldsymbol{x})$ of (2.1) is given by the optimal value of

$$(D_G(\boldsymbol{x})) \qquad \left| \begin{array}{ll} \min\limits_{\boldsymbol{\pi},\boldsymbol{\alpha},\boldsymbol{\beta}} & \boldsymbol{\alpha c} - \boldsymbol{\beta x} - \boldsymbol{ex} \\ \text{s.t.} & (\boldsymbol{\pi}, \boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Omega, \end{array} \right.$$

where

$$\Omega = \{\, (\boldsymbol{\pi}, \boldsymbol{\alpha}, \boldsymbol{\beta}) \in \mathbb{R}_{m+2n} \mid \boldsymbol{\pi}A + \boldsymbol{\alpha} - \boldsymbol{\beta} \geqq \boldsymbol{e},\ \boldsymbol{\alpha}, \boldsymbol{\beta} \geqq \boldsymbol{0} \,\}.$$

If $\bar{\boldsymbol{\beta}}$ is so large that every vertex $\boldsymbol{v}$ of Ω satisfies $\boldsymbol{v} \leqq \bar{\boldsymbol{\beta}}$ then we have $\bar{g}(\boldsymbol{x}) = g(\boldsymbol{x})$ by the theory of linear programming. Replacing $\boldsymbol{\pi}$ by $\boldsymbol{\pi}^1 - \boldsymbol{\pi}^2$ with $\boldsymbol{\pi}^1, \boldsymbol{\pi}^2 \geqq \boldsymbol{0}$ and introducing a slack variable vector $\boldsymbol{\gamma} \geqq \boldsymbol{0}$, Ω is rewritten as

$$\Omega = \left\{ \begin{pmatrix} (\boldsymbol{\pi}^1)^\top \\ (\boldsymbol{\pi}^2)^\top \\ \boldsymbol{\alpha}^\top \\ \boldsymbol{\beta}^\top \\ \boldsymbol{\gamma}^\top \end{pmatrix} \;\middle|\; \begin{pmatrix} A^\top & -A^\top & I & -I & -I \end{pmatrix} \begin{pmatrix} (\boldsymbol{\pi}^1)^\top \\ (\boldsymbol{\pi}^2)^\top \\ \boldsymbol{\alpha}^\top \\ \boldsymbol{\beta}^\top \\ \boldsymbol{\gamma}^\top \end{pmatrix} = \boldsymbol{1},\ \begin{pmatrix} (\boldsymbol{\pi}^1)^\top \\ (\boldsymbol{\pi}^2)^\top \\ \boldsymbol{\alpha}^\top \\ \boldsymbol{\beta}^\top \\ \boldsymbol{\gamma}^\top \end{pmatrix} \geqq \boldsymbol{0} \right\}.$$

Let $\boldsymbol{v}$ be a vertex of Ω. Then it is a basic solution of the system defining Ω, i.e., $\boldsymbol{v} = (\boldsymbol{w}^B, \boldsymbol{w}^N) = (B^{-1}\mathbf{1}, \mathbf{0})$ for some nonsingular $n \times n$ submatrix B of $(A^\top \ -A^\top \ I \ -I \ -I)$. Since the incidence matrix A is totally unimodular, so is $(A^\top \ -A^\top \ I \ -I \ -I)$. Therefore the matrix B^{-1} is composed of -1, 0 and $+1$, and hence $B^{-1}\mathbf{1} \leqq n\mathbf{1}$. This completes the proof. □

Proof of Theorem 3.1

Proof: Suppose that $\bar{\boldsymbol{x}} \in D \backslash \mathrm{int} H$ is not an optimal solution of (CDC), i.e., there exists $\boldsymbol{y} \in D \backslash \mathrm{int} H$ such that $\boldsymbol{py} < \boldsymbol{p\bar{x}}$. Clearly, $\boldsymbol{y} \in D(\boldsymbol{p\bar{x}})$ and $h(\boldsymbol{y}) \geqq 0$. If $h(\boldsymbol{y}) > 0$ then $\boldsymbol{y}$ is not contained in H, and hence $\boldsymbol{y} \in D(\boldsymbol{p\bar{x}}) \backslash H$. By the regularity assumption, if $h(\boldsymbol{y}) = 0$, i.e., $\boldsymbol{y} \in \partial H$ then we can take $\boldsymbol{y}' \in N_\delta(\boldsymbol{y}) \cap D$ such that $\boldsymbol{py}' < \boldsymbol{p\bar{x}}$ and $h(\boldsymbol{y}') > 0$ for a sufficiently small $\delta > 0$, where $N_\delta(\boldsymbol{y}) = \{\, \boldsymbol{y}' \in \mathbb{R}^n \mid \|\boldsymbol{y}' - \boldsymbol{y}\| < \delta \,\}$, and hence we see that $\boldsymbol{y}' \in D(\boldsymbol{p\bar{x}}) \backslash H$. □

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