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Minimizing Conditional Value-at-Risk**

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# The Downside Risk-Averse News-Vendor Minimizing Conditional Value-at-Risk

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## Abstract

In this paper, we consider the minimization of the conditional value-at-risk (CVaR), a most preferable risk measure in financial risk management, in the context of the well-known single-period news-vendor problem which is originally formulated as the maximization of the expected profit, or the minimization of the expected cost. We show that downside risk measures including the CVaR are tractable in the problem due to their convexity, and consequently, under mild assumptions on the probability distribution of product's demand, we provide analytical solutions or linear programming formulation of the minimization of the CVaR measures defined with two different loss functions. Numerical examples are also shown to clarify the difference among the models analyzed in this paper.

**Keywords:** news-vendor problem, conditional value-at-risk (CVaR), downside risk, mean-risk model, convex optimization

## 1 Introduction

Suppose that everyday, facing uncertain demand on a certain product whose value will decrease by the next day, a manager has to decide how many quantities of the product should be ordered. The classical news-vendor model offers a solution to this situation by maximizing the daily expected profit or, equivalently, minimizing the daily expected cost of the product.

In the literature, however, it has been pointed out that maximizing the expected profit is not satisfactory from practical point of view, and managers in the real world are more concerned with the other objectives. For example, some try to attain a predetermined target profit as much as possible. However, such a criterion is still insufficient because it may result in an unacceptably large loss. To reduce such a risk arising from the variation of the profit, some researchers propose to minimize the standard deviation of the profit (e.g., [9]), which originates from Markowitz [10]. On the other hand, it is natural that profit above some target level is not regarded as a risk to be hedged, but more pleasant gain. From this viewpoint, minimizing a downside risk measure which captures a risk of the profit going down to some target level, is more appealing than the other risk measures such as the standard deviation. In the literature of the news-vendor framework, many researchers consider to minimize such downside risk measures as alternatives

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to the expected profit maximization. For example, several researches including Lau [9] and Lau-Lau [7] examine a model which maximizes the probability of exceeding a predetermined fixed target profit, whereas Parlar-Weng [13] considers the expected profit in place of the fixed target. These objectives are very intuitive, but the related optimization problem has no convex structure, and accordingly, they are very tough to handle for general distribution functions. Besides, these models seek higher profit, whereas a possibility of suffering great loss is not considered.

In this paper, we adopt another type of downside risk measure which is called the *conditional value-at-risk* (CVaR) in financial risk management, to the single-period news-vendor situation. The CVaR is known as a risk measure which is coherent ([2]), and consistent with the second (or higher) order stochastic dominance ([14, 12]). These preferable properties are induced from some axiomatization of rational investor's behavior under uncertainty, and thus, these are also valid to a manager who faces uncertain profit/loss situation as in the news-vendor problem. In particular, the consistency with the stochastic dominance implies that minimizing the CVaR never conflicts with maximizing the expectation of any risk-averse utility function ([12]). On the other hand, some researchers directly treat the risk-aversion through the news-vendor's utility function (see Eeckhoudt et al.[3], for example). In practice, utility function is, however, too conceptual to identify and thus, the use of risk measures has advantage over that of utility functions.

Moreover, the lower partiality of the CVaR plays an important role in preserving the concavity of the profit or, equivalently, the convexity of the cost. In financial portfolio management as in [17], the return from an asset portfolio is often represented as a linear function of the portfolio which is to be determined. This is why the standard deviation results in a convex quadratic function. On the contrary, the profit in the news-vendor problem is a nonlinearly concave function of the order quantities. Consequently, minimizing the standard deviation of the profit may turn into a nonconvex optimization, though many researchers introduce it in order to capture the profit variation (e.g., [9]) and develop a CAPM by following the modern portfolio theory (e.g., [1]). In this paper, we show that downside risk measures preserve the concavity of the profit function by virtue of their lower partiality, and the resulting risk minimization becomes a convex program.

The structure of this paper is as follows. In the next section, we review the well-known results of the classical single-period news-vendor problem in which only single product is considered, and define the total cost of a news-vendor as well as the profit. In Section 3, the CVaR is introduced in a general form by following [17], and two types of CVaRs are defined by introducing different loss functions which are called the net-loss and the total cost. By exploiting the formulation developed in [17], we can achieve closed form solutions of the unconstrained minimization of the two different CVaRs. In Section 4, we extend the analysis into the mean-risk framework in which the trade-off between maximizing the profit and minimizing the risk is considered. When only single product is considered without any constraint, closed form solution or simple numerical solution method is derived. On the other hand, when multiple products with many constraints should be dealt with, the problems can be reformulated into equivalent

linear programs. This fact helps managers analyze the mean-risk trade-off structure in a more practical setting with numerous linear constraints. Also, some numerical examples are presented to clarify the difference among the models analyzed in this paper. Finally, we close this paper with some concluding remarks.

## 2 News-Vendor Problem in Single Period

In this section, we briefly summarize the classical single-period news-vendor problem for later comparisons with our results.

### 2.1 Notation

First of all, let us introduce notation used in this paper as follows:

$N$  : index set for products,  $N := \{1, 2, \dots, n\}$ , where  $n := |N|$

$\xi_i$  : daily demand for product  $i$  (random variable),  $\xi_i \in \mathbb{R}_+$

$q_i$  : selling price per unit for product  $i$  (given)

$c_i$  : cost per unit for product  $i$  (given)

$r_i$  : salvage value per unit for product  $i$  (given)

$s_i$  : shortage penalty per unit for product  $i$  (given)

$x_i$  : daily order quantity for product  $i$  (decision variable).

We assume the following condition through the paper:

**Assumption 2.1**  $r_i < c_i < q_i$ ,  $s_i \geq 0$  for all  $i \in N$ .

In the following, we omit the subscript for simplicity when only single product is considered.

### 2.2 Profit Maximization and Cost Minimization

With fixed  $x$ , the daily profit gained from each product is a random variable defined by

$$\mathcal{P}(x, \xi) := q \min \{ \xi, x \} + r \max \{ x - \xi, 0 \} - s \max \{ \xi - x, 0 \} - cx, \quad (1)$$

where the third term in the right-hand side represents an artificial penalty for opportunity cost, and  $s$  is often set to be 0.

Let  $F(\eta)$  denote the distribution function of demand of the product, i.e.,  $F(\eta) := \mathbb{P} \{ \xi \leq \eta \}$ . We note that  $F(0) = 0$ . The classical news-vendor model then maximizes the expected profit:

$$\underset{x}{\text{maximize}} \mu(x) := \mathbb{E} [\mathcal{P}(x, \xi)] = \int_0^\infty \mathcal{P}(x, \xi) dF(\xi), \quad (2)$$

where  $\mathbb{E}[\cdot]$  is the mathematical expectation under the distribution  $F$ . When the inverse of the distribution function exists, an optimal solution of Problem (2) is obtained by solving  $\frac{\partial \mu}{\partial x} = 0$ , as

$$x^* = F^{-1} \left( \frac{U}{E + U} \right), \quad (3)$$

where  $E := c - r$ , and  $U := q + s - c$ . Even when  $F$  does not have the inverse, one can obtain a solution via a simple numerical calculation ([18]).

On the other hand, the daily total cost is defined by

$$\mathcal{Q}(x, \xi) := E[x - \xi]^+ + U[\xi - x]^+, \quad (4)$$

where  $[Y]^+ := \max\{Y, 0\}$ . Here, the first term in the right-hand side of (4) represents the cost for excess order, while the second does the opportunity cost. By noting the relation

$$\mathcal{P}(x, \xi) = V\xi - \mathcal{Q}(x, \xi),$$

where  $V := q - c = U - s$ , the minimization of the expected cost is proved to be equivalent to the maximization of the expected profit:

$$\min_x \mathbb{E}[\mathcal{Q}(x, \xi)] = V \mathbb{E}[\xi] - \max_x \mathbb{E}[\mathcal{P}(x, \xi)].$$

Since  $E, U > 0$  from Assumption 2.1, the expected cost is a convex function of  $x$ , whereas the expected profit is concave one, and therefore, both problems are so-called convex program.

In the case where multiple products are considered, we additionally assume that the total profit (or cost) is just the sum of the ones from each product, i.e., letting  $\mathcal{P}(\mathbf{x}, \boldsymbol{\xi})$  and  $\mathcal{Q}(\mathbf{x}, \boldsymbol{\xi})$  denote the total profit and cost, respectively, we assume

$$\textbf{Assumption 2.2} \quad \mathcal{P}(\mathbf{x}, \boldsymbol{\xi}) = \sum_{i \in N} \mathcal{P}(x_i, \xi_i); \quad \mathcal{Q}(\mathbf{x}, \boldsymbol{\xi}) = \sum_{i \in N} \mathcal{Q}(x_i, \xi_i).$$

### 3 Minimization of CVaR in the News-Vendor Problem

In this section, we introduce the conditional value-at-risk (CVaR) for general distribution functions by following Rockafellar-Uryasev [17], and show that the CVaR minimization leads to a convex problem when the associated loss is represented as a convex function. Also, we examine the parameter sensitivity of its solution.

#### 3.1 Conditional Value-at-Risk and Its Convexity

Let  $\mathcal{L}(\mathbf{x}, \boldsymbol{\xi})$  denote the magnitude of the loss which is a random variable for fixed  $\mathbf{x}$ , and let us denote the distribution function of  $\mathcal{L}$  by  $\Phi(\eta | \mathbf{x}) := \mathbb{P}\{\mathcal{L}(\mathbf{x}, \boldsymbol{\xi}) \leq \eta\}$ . Here, any variable to be minimized can be adopted as the loss  $\mathcal{L}$ , and we will apply two different functions as the loss in the succeeding sections.

For  $\beta \in [0, 1)$ , we define the  $\beta$ -VaR of the distribution by  $\alpha_\beta(\mathbf{x}) := \min\{\alpha \mid \Phi(\alpha | \mathbf{x}) \geq \beta\}$ . By definition, we can expect that the loss  $\mathcal{L}$  exceeds  $\alpha_\beta$  only in  $(1 - \beta) \times 100\%$ .

Rockafellar-Uryasev [17] introduces the  $\beta$ -tail distribution function to focus on the upper tail part of the loss distribution as

$$\Phi_\beta(\eta | \mathbf{x}) := \begin{cases} 0 & \text{for } \eta < \alpha_\beta(\mathbf{x}), \\ \frac{\Phi(\eta | \mathbf{x}) - \beta}{1 - \beta} & \text{for } \eta \geq \alpha_\beta(\mathbf{x}). \end{cases}$$

Using the expectation operator  $E_\beta[\cdot]$  under the  $\beta$ -tail distribution  $\Phi_\beta$ , we define the  $\beta$ -conditional value-at-risk of the loss  $\mathcal{L}$  by  $\phi_\beta(\mathbf{x}) := E_\beta[\mathcal{L}(\mathbf{x}, \boldsymbol{\xi})]$ . Denoting the expectation under the original distribution  $\Phi$  by  $E[\cdot]$ , the following relation shown in [17]:

$$E[\mathcal{L}(\mathbf{x}) | \mathcal{L}(\mathbf{x}) \geq \alpha_\beta(\mathbf{x})] \leq \phi_\beta(\mathbf{x}) \leq E[\mathcal{L}(\mathbf{x}) | \mathcal{L}(\mathbf{x}) > \alpha_\beta(\mathbf{x})] \quad (5)$$

implies that  $\phi_\beta$  is approximately equal to the conditional expectation of  $\mathcal{L}$  which exceeds the threshold  $\alpha_\beta$  with fixed  $\mathbf{x}$ .

In order to minimize  $\phi_\beta(\mathbf{x})$ , Rockafellar-Uryasev [17] introduces a simpler auxiliary function  $F_\beta : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ , defined by

$$F_\beta(\mathbf{x}, \alpha) := \alpha + \frac{1}{1-\beta} E [ [\mathcal{L}(\mathbf{x}, \boldsymbol{\xi}) - \alpha]^+ ], \quad (6)$$

and shows that  $F_\beta$  is convex with respect to  $\alpha$ . Also, they provide a shortcut to minimizing  $\phi_\beta(\mathbf{x})$  as

$$\underset{\mathbf{x} \in X}{\text{minimize}} \phi_\beta(\mathbf{x}) = \underset{(\mathbf{x}, \alpha) \in X \times \mathbb{R}}{\text{minimize}} F_\beta(\mathbf{x}, \alpha), \quad (7)$$

where  $X \subset \mathbb{R}^n$  a feasible region. This relation shows that the minimal value  $\phi_\beta(\mathbf{x}^*)$  can be achieved by minimizing the function  $F_\beta(\mathbf{x}, \alpha)$  with respect to  $\mathbf{x} \in X$  and  $\alpha \in \mathbb{R}$  simultaneously. Furthermore, it is shown in [17] that, with an optimal solution  $(\mathbf{x}^*, \alpha^*)$  of the right-hand side optimization problem,  $\mathbf{x}^*$  is an optimal solution of the left-hand side one, and  $\alpha^*$  is almost (or sometimes exactly) equal to  $\alpha_\beta(\mathbf{x}^*)$ .

In the following, we consider two different loss functions as  $\mathcal{L}$ : one is defined by  $-\mathcal{P}(\mathbf{x}, \boldsymbol{\xi})$  and called the *net loss* of the profit, while the other one is the *total cost*  $\mathcal{Q}(\mathbf{x}, \boldsymbol{\xi})$ . For the two loss functions, we can show that the corresponding CVaR becomes a convex function.

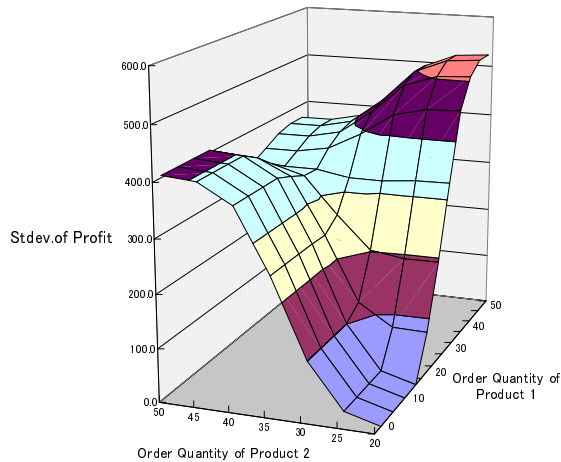
**Proposition 3.1** ([17]) *The function (6) is convex if the loss function  $\mathcal{L}(\cdot, \boldsymbol{\xi})$  from  $\mathbb{R}^n$  to  $(-\infty, \infty]$  is convex.*

**Proof.** See [17]. □

Since both the net loss,  $-\mathcal{P}(\cdot, \boldsymbol{\xi})$ , and the total cost,  $\mathcal{Q}(\cdot, \boldsymbol{\xi})$ , are convex functions with fixed  $\boldsymbol{\xi}$  under Assumptions 2.1 and 2.2, the CVaR minimization problems using these functions are convex. In the following, we call them the net-loss CVaR minimization and the total cost CVaR minimization, respectively.

It is worth noting that this proposition is also valid for a class of downside risk measures including the below-target return defined by  $E[[t - \mathcal{P}(\mathbf{x}, \boldsymbol{\xi})]^+]$  for fixed target  $t \in \mathbb{R}$  ([4]), and maximal loss  $\max_{\xi_i} -\mathcal{R}(\mathbf{x}, \boldsymbol{\xi})$  when  $\boldsymbol{\xi}$  has finite supports ([19]).

**Proposition 3.2** ([15]) *Let  $g$  be a convex function from  $\mathbb{R}^n$  to  $(-\infty, \infty]$ , and let  $\gamma$  be a convex function from  $(-\infty, \infty]$  to  $(-\infty, \infty]$  which is non-decreasing with  $\gamma(\infty) = \infty$ . Then,  $h(x) = \gamma(g(x))$  is convex on  $\mathbb{R}^n$ .*

Figure 1: Nonconvexity of Standard Deviation of  $\mathcal{P}$ 

**Proof.** See [15], for example.  $\square$

From this proposition, we see that minimization of any non-decreasing convex risk measure is formulated as a convex problem, and at the same time, the lower partiality of the risk measures seems crucial for the convexity in the risk minimization for the news-vendor problem. In fact, the variance (or equivalently, the standard deviation) of the net loss or the cost function can have a non-convex structure. Figure 1 shows an example of the non-convexity with respect to  $\mathbf{x}$  of the standard deviation of profit  $\mathcal{P}$  in the two-product case where the underlying distribution has finite supports.

### 3.2 Unconstrained Minimization of CVaR for Single Product

In this subsection, we consider the case dealing with only single product without constraint, i.e.,  $X = \mathbb{R}$ , and present analytical results of the CVaR minimization problems with two loss functions. In addition, let us assume for simplicity that there exists the inverse of the distribution function  $F$ , and let us denote its density by  $f$ .

**a) The Net-Loss CVaR Minimization** First, we define the net-loss by  $-\mathcal{P}$ , and adopt it as the loss function  $\mathcal{L}$ , so that a manager can consider the profit lower than  $\alpha_\beta$ .

The minimization of (6) with  $\mathcal{L} = -\mathcal{P}$  is represented as the following convex program:

$$\left| \begin{array}{l} \text{minimize} \\ x \in \mathbb{R}, \alpha \in \mathbb{R} \end{array} \right. p(x, \alpha) := \alpha + \frac{1}{1-\beta} \int_0^\infty [-\mathcal{P}(x, \xi) - \alpha]^+ f(\xi) d\xi. \quad (8)$$

Note that the integral part of the objective in (8) can be expanded as

$$\int_0^x [-\{V\xi - E(x - \xi)\} - \alpha]^+ f(\xi) d\xi + \int_x^\infty [-\{V\xi - U(\xi - x)\} - \alpha]^+ f(\xi) d\xi. \quad (9)$$

Then, consider the following three cases (see Figure 2):

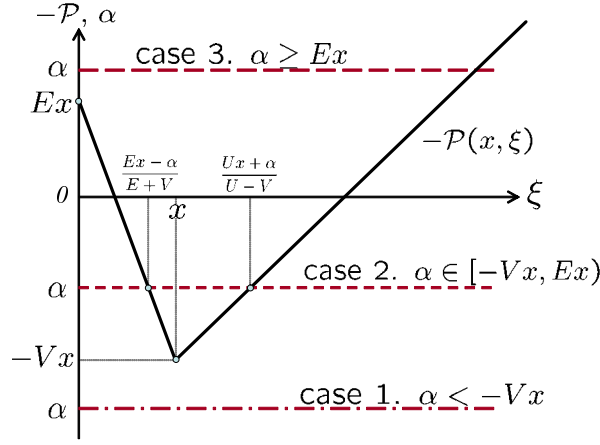


Figure 2: Three Cases in Minimization of Net-Loss CVaR

⟨⟨ **case 1.**  $\alpha < -Vx$  ⟩⟩ The integral part (9) becomes

$$\int_0^x [-\{V\xi - E(x - \xi)\} - \alpha] f(\xi) d\xi + \int_x^\infty [-\{V\xi - U(\xi - x)\} - \alpha] f(\xi) d\xi.$$

From the first-order condition of Problem (8), one has a solution  $(x^*, \alpha^*)$  satisfying  $x^* = F^{-1}\left(\frac{U}{E+U}\right)$  and  $\alpha^* < -Vx^*$ , only when one sets  $\beta = 0$ .

⟨⟨ **case 2.**  $\alpha \in [-Vx, Ex)$  ⟩⟩ When  $s > 0$ , the integral part (9) becomes

$$\int_0^{\frac{Ex-\alpha}{E+V}} [-\{V\xi - E(x - \xi)\} - \alpha] f(\xi) d\xi + \int_{\frac{Ux+\alpha}{U-V}}^\infty [-\{V\xi - U(\xi - x)\} - \alpha] f(\xi) d\xi,$$

while the second integral vanishes when  $s = 0$ . From the first-order condition, one has a solution  $(x^*, \alpha^*)$  defined by

$$\begin{cases} x^* = \frac{E+V}{E+U} F^{-1}\left(\frac{U(1-\beta)}{E+U}\right) + \frac{U-V}{E+U} F^{-1}\left(\frac{E\beta+U}{E+U}\right), \\ \alpha^* = \frac{E(U-V)}{E+U} F^{-1}\left(\frac{E\beta+U}{E+U}\right) - \frac{U(E+V)}{E+U} F^{-1}\left(\frac{U(1-\beta)}{E+U}\right). \end{cases}$$

It is easy to see that this solution  $(x^*, \alpha^*)$  satisfies  $\alpha^* \in [-Vx^*, Ex^*)$  under Assumption 2.1. Also, we note that this  $x^*$  includes the solution in the previous case when  $\beta = 0$ .

⟨⟨ **case 3.**  $\alpha \geq Ex$  ⟩⟩ When  $s > 0$ , the integral part (9) becomes

$$\int_{\frac{Ux+\alpha}{U-V}}^\infty [-\{V\xi - U(\xi - x)\} - \alpha] f(\xi) d\xi,$$

while the integral part is 0 and, thus the problem has no bounded solution when  $s = 0$ . By differentiating this equation, we observe that this case has no optimal solution.

Here, we summarize the discussion above.



**Proposition 3.3** *Assume that there exists the inverse of the distribution function of the product demand. Then, the problem (8) with  $\beta \in [0, 1)$  has an optimal solution  $(x^*, \alpha^*)$  defined by*

$$\begin{cases} x^* = \frac{E+V}{E+U} F^{-1}\left(\frac{U(1-\beta)}{E+U}\right) + \frac{U-V}{E+U} F^{-1}\left(\frac{E\beta+U}{E+U}\right), \\ \alpha^* = \frac{E(U-V)}{E+U} F^{-1}\left(\frac{E\beta+U}{E+U}\right) - \frac{U(E+V)}{E+U} F^{-1}\left(\frac{U(1-\beta)}{E+U}\right). \end{cases} \quad (10)$$

*In particular, when  $\beta = 0$ , any  $\alpha^*$  with  $\alpha^* \leq -Vx^*$  also satisfies optimality.*

In particular, when the artificial penalty  $s$  is set to be 0, i.e.,  $V = U$ , one has the following result:

**Corollary 3.4** *Under the same assumption as in Proposition 3.3 with  $s = 0$ , one has an optimal solution  $(x^*, \alpha^*)$  defined by*

$$x^* = F^{-1}\left(\frac{U}{E+U}(1-\beta)\right); \quad \alpha^* = -Ux^*. \quad (11)$$

*In particular, when  $\beta = 0$ , any  $\alpha^*$  with  $\alpha^* \leq -Ux^*$  also satisfies optimality.*

From these proposition and corollary, we see that the difference between the solution  $x^*$  given by (10) or (11) and the classical one (3) depends on two parameters  $s$  and  $\beta$ . In particular, from Corollary 3.4, we see that it is only the coefficient in the argument of the inverse  $F^{-1}$  when  $s = 0$ , whereas when  $s > 0$ , it may be much more complex. However, this CVaR minimization gives a simple generalization of the classical problem since the solution with  $\beta = 0$  is equal to that for the classical one with any  $s \geq 0$ . This consequence is consistent with the definition of the  $\beta$ -CVaR.

**b) The Total Cost CVaR Minimization** Next, we consider the total cost  $\mathcal{Q}$  as the loss  $\mathcal{L}$ . By minimizing the  $\beta$ -CVaR defined on the total cost, a manager may avoid an unduly large cost which consists of the excess order cost and the opportunity cost.

The corresponding problem is then represented as

$$\left| \begin{array}{l} \text{minimize} \\ x \in \mathbb{R}, \alpha \in \mathbb{R} \end{array} \right. \quad q(x, \alpha) := \alpha + \frac{1}{1-\beta} \int_0^\infty [\mathcal{Q}(x, \xi) - \alpha]^+ f(\xi) d\xi. \quad (12)$$

Similarly to the previous discussion, we consider three cases by taking into account that the integral part of the objective in (12) can be transformed into

$$\int_0^x [E(x-\xi) - \alpha]^+ f(\xi) d\xi + \int_x^\infty [U(\xi-x) - \alpha]^+ f(\xi) d\xi. \quad (13)$$

We can calculate this integral by considering the following three cases (see Figure 3).

$\langle\langle$  **case 1.**  $\alpha < 0$   $\rangle\rangle$  The integral part (13) becomes

$$\int_0^x \{E(x-\xi) - \alpha\} f(\xi) d\xi + \int_x^\infty \{U(\xi-x) - \alpha\} f(\xi) d\xi.$$

From the first-order conditions of Problem (12), one has a solution  $(x^*, \alpha^*)$  satisfying  $x^* = F^{-1}\left(\frac{U}{E+U}\right)$  and  $\alpha^* < 0$ , only when one sets  $\beta = 0$ .

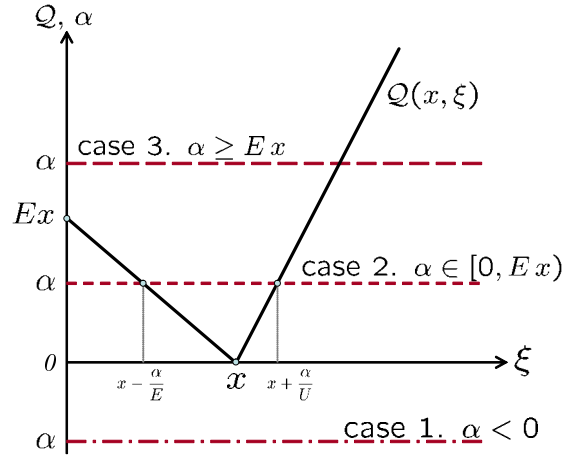


Figure 3: Three Cases in Minimization of Total Cost CVaR

⟨⟨ **case 2.**  $\alpha \in [0, Ex)$  ⟩⟩ The integral part (13) becomes

$$\int_0^{x - \frac{\alpha}{E}} \{E(x - \xi) - \alpha\} f(\xi) d\xi + \int_{x + \frac{\alpha}{U}}^{\infty} \{U(\xi - x) - \alpha\} f(\xi) d\xi.$$

From the first-order condition, one has a solution  $(x^*, \alpha^*)$  defined by

$$\begin{cases} x^* = \frac{E}{E+U} F^{-1}\left(\frac{U(1-\beta)}{E+U}\right) + \frac{U}{E+U} F^{-1}\left(\frac{E\beta+U}{E+U}\right), \\ \alpha^* = \frac{EU}{E+U} \left( F^{-1}\left(\frac{E\beta+U}{E+U}\right) - F^{-1}\left(\frac{U(1-\beta)}{E+U}\right) \right). \end{cases}$$

We note that this  $x^*$  includes the solution in the previous case when  $\beta = 0$ .

⟨⟨ **case 3.**  $\alpha \geq Ex$  ⟩⟩ The integral part (13) becomes

$$\int_{x + \frac{\alpha}{U}}^{\infty} \{U(\xi - x) - \alpha\} f(\xi) d\xi.$$

By differentiating this equation, we observe that this case has no optimal solution.

Therefore, we obtain the following proposition.

**Proposition 3.5** *Assume that there exists the inverse of the distribution function  $F$  of the product demand  $\xi$ . Then, the problem (12) with  $\beta \in [0, 1)$  has an optimal solution  $(x^*, \alpha^*)$  defined by*

$$\begin{cases} x^* = \frac{E}{E+U} F^{-1}\left(\frac{U(1-\beta)}{E+U}\right) + \frac{U}{E+U} F^{-1}\left(\frac{E\beta+U}{E+U}\right), \\ \alpha^* = \frac{EU}{E+U} \left( F^{-1}\left(\frac{E\beta+U}{E+U}\right) - F^{-1}\left(\frac{U(1-\beta)}{E+U}\right) \right). \end{cases} \quad (14)$$

*In particular, when  $\beta = 0$ , any  $\alpha^*$  with  $\alpha^* \leq 0$  also satisfies optimality.*

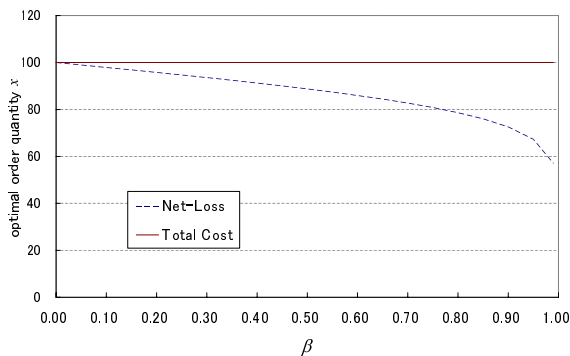
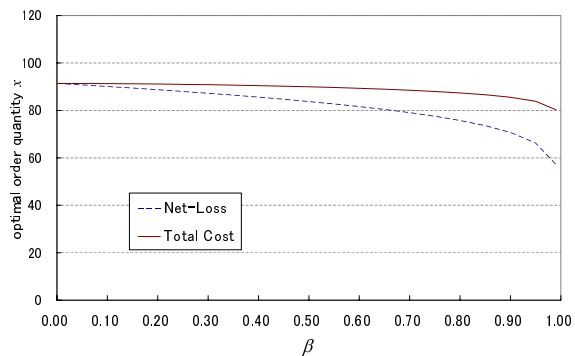
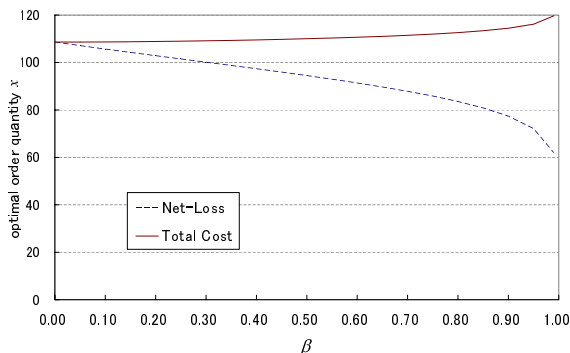
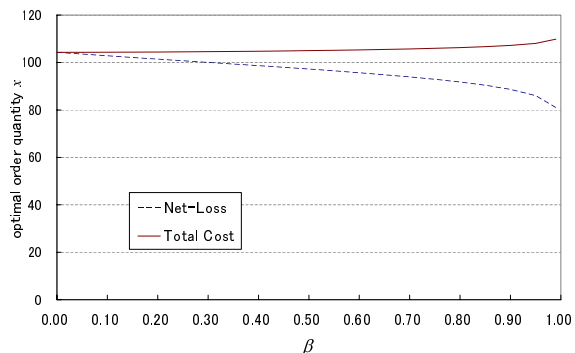
(a)  $(E, U, V) = (60, 60, 50); N(100, 20^2)$ (b)  $(E, U, V) = (60, 30, 20); N(100, 20^2)$ (c)  $(E, U, V) = (30, 60, 50); N(100, 20^2)$ (d)  $(E, U, V) = (30, 60, 50); N(100, 10^2)$ 

Figure 4: Optimal Solutions of CVaR Minimization with Two Loss Functions

By comparing solutions (10) and (14) of the two CVaR minimization, we observe that the solution  $(x^*, \alpha^*)$  of the total cost CVaR minimization (12) can be far different from that of the net-loss CVaR minimization (8), which does not hold in the classical problem in which maximizing the profit and minimizing the cost are equivalent (see the results in Section 2). However, this CVaR minimization (12) also provides a generalization of the classical maximizing profit model because this solution with  $\beta = 0$  is the same as the solution (3), which is consistent with the definition of the  $\beta$ -CVaR.

Figures 4 (a) to (d) illustrate the differences among the three optima (3), (10) and (14) when  $s$  is set to be 10, i.e.,  $V = U - 10$ , and  $\xi$  follows a normal distribution under a couple of parameter settings. Noting that the solution with  $\beta = 0$  is equal to the classical one (3), we see from Figures 4 (a) to (d), that the net-loss CVaR minimization implies less order quantity than the classical one and the difference becomes larger as  $\beta$  gets higher, while the optimal solution of the total cost CVaR depends on parameters  $E$  and  $U$ . In particular, when  $E = U$  holds as in (a), the solution (14) is independent of  $\beta$ , and accordingly, equal to the classical solution, and when  $E < U$  holds as in (c) and (d), two solutions with different loss show reverse trends with  $\beta$ . From Figures 4 (c) and (d), we see that the difference between two solutions becomes smaller as the variance of normal distribution decreases.

### 3.3 Sensitivity Analysis

Here, we analyze the parameter sensitivity of the solutions obtained so far. Table 1 summarizes the sign of the partial derivative of the optimal solution  $x^*$  with respect to parameters,  $q$ ,  $c$ ,  $r$ ,  $s$  and  $\beta$ . From this table, we see that all the signs of the sensitivity to  $c$ ,  $r$  and  $s$  remain the same as that of the classical model, whereas those to  $\beta$  and  $q$  can differ from model to model.

The signs of  $\frac{\partial x^*}{\partial \beta}$  for the net-loss and the total cost CVaR minimization depend on the underlying distribution because for the net-loss CVaR minimizer (10) with  $s = U - V > 0$ , one has

$$\frac{\partial x^*}{\partial \beta} = \frac{1}{(E+U)^2} \left( \frac{E(U-V)}{f(F^{-1}(G_2))} - \frac{U(E+V)}{f(F^{-1}(G_1))} \right),$$

where  $G_1 = \frac{U(1-\beta)}{E+U}$  and  $G_2 = \frac{E\beta+U}{E+U}$ , and for the total cost CVaR minimizer (14), one has

$$\frac{\partial x^*}{\partial \beta} = \frac{EU}{(E+U)^2} \left( \frac{1}{f(F^{-1}(G_2))} - \frac{1}{f(F^{-1}(G_1))} \right).$$

To illustrate how the shape of distribution affects the derivatives, let us consider the S-D distribution ([6]) whose distribution and density functions are defined, respectively, by

$$F(\eta) = \begin{cases} d - \{(a - \eta)/b\}^{\frac{1}{l}}, & \text{for } \eta \in [H_1, a), \\ d + \{(\eta - a)/b\}^{\frac{1}{l}}, & \text{for } \eta \in [a, H_2], \end{cases}$$

and

$$f(\eta) = \frac{1}{bl} \left| \frac{a - \eta}{b} \right|^{\frac{1-l}{l}}, \quad \text{for } H_1 \leq \eta \leq H_2,$$

where  $a, b, d, l, H_1$  and  $H_2$  are constant with  $b > 0, l > 0$  and  $d \in \left[ \{(a - H_1)/b\}^{1/l}, 1 - \{(H_2 - a)/b\}^{1/l} \right]$ .

For the net-loss CVaR minimizer, one then has  $\frac{\partial x^*}{\partial \beta} < 0$  for  $l = 1$ ,

$$\begin{cases} \frac{\partial x^*}{\partial \beta} \geq 0 & \text{if } d \in [(1-\nu)G_1 + \nu G_2, (1-\theta)G_1 + \theta G_2] \\ \frac{\partial x^*}{\partial \beta} \leq 0 & \text{if } d < (1-\nu)G_1 + \nu G_2 \text{ or } d \geq (1-\theta)G_1 + \theta G_2 \end{cases} \quad (\text{for } l > 1),$$

and

$$\begin{cases} \frac{\partial x^*}{\partial \beta} \geq 0 & \text{if } d \in [(1-\theta)G_1 + \theta G_2, (1-\nu)G_1 + \nu G_2] \\ \frac{\partial x^*}{\partial \beta} \leq 0 & \text{if } d < (1-\theta)G_1 + \theta G_2 \text{ or } d \geq (1-\nu)G_1 + \nu G_2 \end{cases} \quad (\text{for } 0 < l < 1),$$

where  $\nu := B^{1/(1-l)}/(B^{1/(1-l)} - 1)$ ,  $\theta := B^{1/(1-l)}/(B^{1/(1-l)} + 1)$  and  $B := \frac{U(E+V)}{E(U-V)}$ . On the other

hand, for the total cost CVaR solution, one has  $\frac{\partial x^*}{\partial \beta} = 0$  for  $l = 1$ ,

$$\frac{\partial x^*}{\partial \beta} > 0 \quad \text{if } d < \frac{G_1 + G_2}{2}; \quad \frac{\partial x^*}{\partial \beta} \leq 0 \quad \text{if } d \geq \frac{G_1 + G_2}{2} \quad (\text{for } l > 1),$$

Table 1: Sign of Partial Derivative of Each Minimizer

	$\frac{\partial x^*}{\partial q}$	$\frac{\partial x^*}{\partial c}$	$\frac{\partial x^*}{\partial r}$	$\frac{\partial x^*}{\partial s}$	$\frac{\partial x^*}{\partial \beta}$
net-loss CVaR ( $s = 0$ ) (11)	+	-	+	0	-
net-loss CVaR ( $s > 0$ ) (10)	case-by-case	-	+	+	case-by-case
total cost CVaR (14)	+	-	+	+	case-by-case
Classical (3)	+	-	+	+	0

and

$$\frac{\partial x^*}{\partial \beta} \geq 0 \quad \text{if } d \geq \frac{G_1 + G_2}{2}; \quad \frac{\partial x^*}{\partial \beta} < 0 \quad \text{if } d < \frac{G_1 + G_2}{2} \quad (\text{for } 0 < l < 1).$$

From the above results, we see that the sensitivity with respect to  $\beta$  depends on the parameters determining the shape of the S-D distribution. In particular, the skewness parameter  $d$  is crucial for the solutions above. Also, we can see that the sign of the derivative with respect to  $q$  of the net-loss CVaR minimizer with  $s > 0$  depends only on parameters  $d$  and  $l$ .

## 4 Mean-CVaR Models and LP Formulation

Since the Markowitz's seminal work, the trade-off between risk and return has been considered in various situations by using mathematical optimization techniques. This trade-off model is known as the *mean-risk model* (see [11], for example), which is formulated as the optimization of a composite objective consisting of the expected return and a certain risk measure  $\rho(\mathbf{x})$ :

$$\left| \begin{array}{ll} \underset{\mathbf{x}}{\text{maximize}} & \text{E}[\mathcal{P}(\mathbf{x}, \boldsymbol{\xi})] - \lambda \rho(\mathbf{x}) \\ \text{subject to} & \mathbf{x} \in X, \end{array} \right. \quad (15)$$

where  $X$  a convex set representing some constraints on the portfolio  $\mathbf{x}$ , and  $\lambda \geq 0$  a trade-off parameter, or the minimization of the risk while the return is kept at least as large as a predetermined target:

$$\left| \begin{array}{ll} \underset{\mathbf{x}}{\text{minimize}} & \rho(\mathbf{x}) \\ \text{subject to} & \text{E}[\mathcal{P}(\mathbf{x}, \boldsymbol{\xi})] \geq \bar{\mu}, \\ & \mathbf{x} \in X, \end{array} \right. \quad (16)$$

where  $\bar{\mu}$  is the minimum level of the expected profit. It is known that the both formulations give the same efficient frontier, which is a graph of Pareto efficient pairs of expected return and some risk measure  $\rho$ , when the expected return is a concave function of  $\mathbf{x}$  while the risk is a convex one. Exploiting the results in the previous section and applying this framework to the news-vendor problem by using the CVaR measures  $\phi_\beta$  as the risk  $\rho$ , the corresponding mean-risk models (15) and (16) with  $\rho(\mathbf{x}) = \phi_\beta(\mathbf{x})$  are convex programs, and consequently, result in the same efficient frontier.

#### 4.1 Unconstrained Mean-CVaR Models for Single Product

**a) Mean-Net-Loss CVaR Model** First, we consider the unconstrained mean-risk model using the net-loss CVaR for single-product case as in the previous section. The problem is then represented by

$$\left| \begin{array}{l} \text{maximize} \\ x, \alpha \end{array} \int_0^\infty \mathcal{P}(x, \xi) f(\xi) d\xi - \lambda \left( \alpha + \frac{1}{1-\beta} \int_0^\infty [-\mathcal{P}(x, \xi) - \alpha]^+ f(\xi) d\xi \right). \quad (17) \right.$$

By the same reasoning in the net-loss CVaR minimization, we consider the following three cases. Throughout the below analysis (except for propositions), we omit the case of  $\lambda = 0$ , since the mean-risk model is equal to the profit maximization (2).

$\langle\langle$  **case 1.**  $\alpha \leq -Vx$   $\rangle\rangle$  Let  $h(x) := \mu(x) - \lambda p(x, \alpha)$ . Since one has the following first-order condition:

$$\begin{cases} \frac{\partial h}{\partial x} = - \left( 1 + \frac{\lambda}{1-\beta} \right) \{ (E+U)F(x) - U \} = 0 \\ \frac{\partial h}{\partial \alpha} = -\lambda \left( 1 - \frac{1}{1-\beta} \right) = 0, \end{cases} \quad (18)$$

only when  $\beta = 0$  holds, one has a solution  $(x^*, \alpha^*)$  satisfying (3) and  $\alpha^* \leq -Vx^*$ .

$\langle\langle$  **case 2.**  $\alpha \in (-Vx, Ex)$   $\rangle\rangle$  When  $s > 0$ , we have the first-order condition:

$$\begin{cases} E \left\{ F(x) + \frac{\lambda}{1-\beta} F\left(\frac{Ex-\alpha}{E+V}\right) \right\} + U \left\{ F(x) + \frac{\lambda}{1-\beta} F\left(\frac{Ux+\alpha}{U-V}\right) \right\} = U \left( 1 + \frac{\lambda}{1-\beta} \right), \\ F\left(\frac{Ux+\alpha}{U-V}\right) - F\left(\frac{Ex-\alpha}{E+V}\right) = \beta. \end{cases}$$

Then, an optimal  $\alpha$  is given by

$$\alpha^* = Ex^* - (E+V)F^{-1}(A(x^*)), \quad (19)$$

where  $x^*$  solves the equation:

$$(E+V)F^{-1}(A(x)) + (U-V)F^{-1}(A(x)+\beta) - (E+U)x = 0, \quad (20)$$

where  $A(x) := \frac{1-\beta}{\lambda} \left\{ \frac{U}{E+U}(1+\lambda) - F(x) \right\}$ .

$\langle\langle$  **case 3.**  $\alpha \geq Ex$   $\rangle\rangle$  We have a solution  $(x^*, \alpha^*)$  defined by

$$x^* = F^{-1}\left(\frac{U}{E+U}(1+\lambda)\right); \quad \alpha^* = (U-V)F^{-1}(\beta) - Ux^*. \quad (21)$$

Considering the condition  $\alpha \geq Ex$ ,  $\beta$  and  $\lambda$  should satisfy the relation

$$F^{-1}\left(\frac{U}{E+U}(1+\lambda)\right) \leq \frac{U-V}{E+U}F^{-1}(\beta). \quad (22)$$

Therefore, for  $\lambda > 0$  and  $s > 0$ , we can find an optimal solution through the following steps:

1. If  $\beta = 0$ , then  $(x^*, \alpha^*)$  satisfying (3) and  $\alpha^* \leq -Vx^*$  is a solution.

2. If  $\beta$  and  $\lambda$  satisfy the relation (22), then  $(x^*, \alpha^*)$  satisfying (21) is an optimal solution.
3. Otherwise, search  $x$  satisfying Equation (20), and  $\alpha$  defined by (19).

In particular when we assume  $s = 0$ , we can obtain a closed form solution of Problem (17), by similar discussion to the first one in the previous section.

**Proposition 4.1** *For  $s = 0$ ,  $\beta \in [0, 1)$  and  $\lambda \geq 0$ , the mean-CVaR model (17) has an optimal solution  $(x^*, \alpha^*)$  defined by*

$$x^* = F^{-1}\left(\frac{U}{E+U} \cdot \frac{1+\lambda}{1+\lambda(1-\beta)^{-1}}\right); \quad \alpha^* = -Ux^*.$$

*In particular, when  $\lambda = 0$  or  $\beta = 0$  holds, any  $\alpha^*$  with  $\alpha^* \leq -Ux^*$  also satisfies optimality. Moreover, for  $\lambda \in [0, \frac{E+U}{E}\beta - 1)$ , a solution  $(x^{**}, \alpha^{**})$  defined by*

$$x^{**} = F^{-1}\left(\frac{U-\lambda E}{E+U}\right); \quad \alpha^{**} = Ex^{**} - (E+U)F^{-1}(1-\beta)$$

*also achieves the optimal value. In this case, so does a solution  $(\hat{x}, \hat{\alpha})$  satisfying  $\hat{x} = (1-t)x^* + tx^{**}$  and  $\hat{\alpha} = (1-t)\alpha^* + t\alpha^{**}$  for any  $t \in (0, 1)$ .*

**Proof.** When  $s = 0$  holds, we consider the following three cases.

⟨⟨ **case 1.**  $\alpha < -Ux$  ⟩⟩ Noting that  $s = 0$  is equivalent to  $U = V$ , we have the same result with the general one.

⟨⟨ **case 2.**  $\alpha = -Ux$  ⟩⟩ From the first-order condition, we have a solution  $(x^*, \alpha^*)$  defined by

$$x^* = F^{-1}\left(\frac{U}{E+U} \cdot \frac{1+\lambda}{1+\frac{\lambda}{1-\beta}}\right); \quad \alpha^* = -Ux^*.$$

⟨⟨ **case 3.**  $\alpha > -Ux$  ⟩⟩ By exploiting Proposition 3.3, we have a solution defined by

$$\begin{cases} x^* = F^{-1}\left(\frac{U-\lambda E}{E+U}\right), \\ \alpha^* = Ex^* - (E+U)F^{-1}(1-\beta). \end{cases}$$

Combining with the condition  $\alpha^* > -Ux^*$ , this is valid only for  $\lambda < \frac{E+U}{U}\beta - 1$ . Note that the optimal solution set is convex since the problem is convex, then the result follows.  $\square$

**b) Mean-Total Cost CVaR Model** The analysis of the mean-risk model using the total cost CVaR can be conducted in a similar manner. For single product case, the model is formulated as follows:

$$\left| \begin{array}{l} \text{maximize} \\ x, \alpha \end{array} \int_0^\infty \mathcal{P}(x, \xi) f(\xi) d\xi - \lambda \left( \alpha + \frac{1}{1-\beta} \int_0^\infty [\mathcal{Q}(x, \xi) - \alpha]^+ f(\xi) d\xi \right). \quad (23) \right.$$

In this analysis, we omit the trivial case of  $\lambda = 0$ .

⟨⟨ **case 1.**  $\alpha \leq 0$  ⟩⟩ By similar discussion to the case 1 in the proof for Proposition 3.5, only when  $\beta = 0$  holds, one has a solution  $(x^*, \alpha^*)$  satisfying (3) and  $\alpha^* \leq 0$ , since one has the first-order condition (18).

⟨⟨ **case 2.**  $\alpha \in (0, Ex)$  ⟩⟩ We have the first-order condition:

$$\begin{cases} E \left\{ F(x) + \frac{\lambda}{1-\beta} F(x - \frac{1}{E}\alpha) \right\} + U \left\{ F(x) + \frac{\lambda}{1-\beta} F(x + \frac{1}{U}\alpha) \right\} = U \left( 1 + \frac{\lambda}{1-\beta} \right), \\ F(x + \frac{1}{U}\alpha) - F(x - \frac{1}{E}\alpha) = \beta. \end{cases}$$

Then, an optimal  $\alpha$  is given by

$$\alpha^* = E \left\{ x^* - F^{-1}(A(x^*)) \right\}, \quad (24)$$

where  $x^*$  solves the equation:

$$E F^{-1}(A(x)) + U F^{-1}(A(x) + \beta) - (E + U)x = 0. \quad (25)$$

⟨⟨ **case 3.**  $\alpha \geq Ex$  ⟩⟩ We have a solution  $(x^*, \alpha^*)$  satisfying

$$x^* = F^{-1} \left( \frac{U}{E+U} (1 + \lambda) \right); \quad \alpha^* = U F^{-1}(\beta) - U x^*. \quad (26)$$

Considering the condition  $\alpha \geq Ex$ ,  $\beta$  and  $\lambda$  should satisfy the relation

$$F^{-1} \left( \frac{U}{E+U} (1 + \lambda) \right) \leq \frac{U}{E+U} F^{-1}(\beta). \quad (27)$$

Therefore, for  $\lambda > 0$ , we can find an optimal solution through the following steps:

1. If  $\beta = 0$ , then  $(x^*, \alpha^*)$  satisfying (3) and  $\alpha^* \leq 0$  is a solution.
2. If  $\beta$  and  $\lambda$  satisfy the relation (27), then  $(x^*, \alpha^*)$  satisfying (26) is an optimal solution.
3. Otherwise, search  $x$  satisfying Equation (25), and  $\alpha$  defined by (24).

## 4.2 Constrained Mean-CVaR Models for Multiple Products

In this subsection, we address how to compute an optimal solution when multiple products are considered with multiple constraints which are given as a system of linear inequalities.

Suppose that the probability distribution is given by a finite number of scenarios. Let  $K$  denote a finite set of scenarios, and let  $P\{\boldsymbol{\xi} = \boldsymbol{\xi}_k\} = p_k$  for  $k \in K$  where  $\boldsymbol{\xi}_k := (\xi_{k,1}, \xi_{k,2}, \dots, \xi_{k,n})^\top$ . Moreover,  $X$  is supposed to be a polytope given by  $X := \{\boldsymbol{x} \mid C\boldsymbol{x} \leq \boldsymbol{b}\}$  where  $C \in \mathbb{R}^{m \times n}$  and  $\boldsymbol{b} \in \mathbb{R}^m$ . Then, using the net-loss CVaR, the mean-risk model (15) with  $\rho = \phi_\beta$  is formulated as

$$\begin{cases} \text{maximize}_{\boldsymbol{x}, \alpha} & \sum_{k \in K} p_k \mathcal{P}(\boldsymbol{x}, \boldsymbol{\xi}_k) - \lambda \left( \alpha + \frac{1}{1-\beta} \sum_{k \in K} p_k [-\mathcal{P}(\boldsymbol{x}, \boldsymbol{\xi}_k) - \alpha]^+ \right) \\ \text{subject to} & \boldsymbol{x} \in X, \end{cases} \quad (28)$$



which is equivalent to the linear program (LP):

$$\begin{array}{l}
\left| \begin{array}{l}
\text{maximize}_{\mathbf{x}, \alpha, \mathbf{v}, \mathbf{w}, \mathbf{z}} \quad \sum_{k \in K} p_k \sum_{i \in N} V_i \xi_{k,i} - \sum_{k \in K} p_k \sum_{i \in N} E_i w_{k,i} - \sum_{k \in K} p_k \sum_{i \in N} U_i z_{k,i} - \lambda \alpha - \frac{\lambda}{1-\beta} \sum_{k \in K} p_k v_k \\
\text{subject to} \quad v_k \geq - \sum_{i \in N} V_i \xi_{k,i} + \sum_{i \in N} E_i w_{k,i} + \sum_{i \in N} U_i z_{k,i} - \alpha, \quad v_k \geq 0, \quad k \in K, \\
\quad w_{k,i} \geq x_i - \xi_{k,i}, \quad w_{k,i} \geq 0, \quad k \in K, \quad i \in N, \\
\quad z_{k,i} \geq \xi_{k,i} - x_i, \quad z_{k,i} \geq 0, \quad k \in K, \quad i \in N, \\
\quad \mathbf{x} \in X.
\end{array} \right. \quad (29)
\end{array}$$

**Proposition 4.2** *Let  $(\mathbf{x}^*, \alpha^*, \mathbf{v}^*, \mathbf{w}^*, \mathbf{z}^*)$  be an optimal solution of (29). Then,  $(\mathbf{x}^*, \alpha^*)$  is also optimal to (28), and the optimal values of both problems meet.*

The minimization of the net-loss CVaR with an expected profit constraint, i.e., (16) with  $\rho(\mathbf{x}) = \phi_\beta(\mathbf{x})$ , is also transformed into an LP. The problem is formulated as follows:

$$\left| \begin{array}{l}
\text{minimize}_{\mathbf{x}, \alpha} \quad \alpha + \frac{1}{1-\beta} \sum_{k \in K} p_k [-\mathcal{P}(\mathbf{x}, \xi_k) - \alpha]^+ \\
\text{subject to} \quad \sum_{k \in K} p_k \mathcal{P}(\mathbf{x}, \xi_k) \geq \bar{\mu}, \\
\quad \mathbf{x} \in X,
\end{array} \right. \quad (30)$$

which is equivalent to the LP:

$$\left| \begin{array}{l}
\text{minimize}_{\mathbf{x}, \alpha, \mathbf{v}, \mathbf{w}, \mathbf{z}} \quad \alpha + \frac{1}{1-\beta} \sum_{k \in K} p_k v_k \\
\text{subject to} \quad \sum_{k \in K} p_k \sum_{i \in N} V_i \xi_{k,i} - \sum_{k \in K} p_k \sum_{i \in N} E_i w_{k,i} - \sum_{k \in K} p_k \sum_{i \in N} U_i z_{k,i} \geq \bar{\mu}, \\
\quad v_k \geq - \sum_{i \in N} V_i \xi_{k,i} + \sum_{i \in N} E_i w_{k,i} + \sum_{i \in N} U_i z_{k,i} - \alpha, \quad v_k \geq 0, \quad k \in K, \quad (31) \\
\quad w_{k,i} \geq x_i - \xi_{k,i}, \quad w_{k,i} \geq 0, \quad k \in K, \quad i \in N, \\
\quad z_{k,i} \geq \xi_{k,i} - x_i, \quad z_{k,i} \geq 0, \quad k \in K, \quad i \in N, \\
\quad \mathbf{x} \in X.
\end{array} \right.$$

As easily guessed, the other variants using the total cost CVaR can be also transformed into equivalent linear programs. In fact, only getting rid of the constant term,  $-\sum_{i \in N} V_i \xi_{k,i}$ , from the constraint:

$$v_k \geq - \sum_{i \in N} V_i \xi_{k,i} + \sum_{i \in N} E_i w_{k,i} + \sum_{i \in N} U_i z_{k,i} - \alpha, \quad v_k \geq 0, \quad k \in K,$$

which is the first constraint of Problem (29) and the second constraint of Problem (31), we can obtain the two kinds of mean-risk models with the total cost CVaR.

The advantage of LP formulation is overwhelming since it can deal with a huge number of constraints and variables, and consequently, it is expected to provide a well-approximating optimal solution when any explicit solution cannot be achieved in such a case that many constraints

on multiple products should be dealt with. In addition, even when one cannot achieve any closed form solution, we can figure the distribution as a histogram in an approximate manner.

Figures 5 (a1) to (e2) show histograms of optimal distributions of the profit  $\mathcal{P}$  and total cost  $\mathcal{Q}$  via several models discussed above when single product is considered and one thousand scenarios of its demand are drawn from a normal distribution  $N(150, 45^2)$ . In spite of the normality of the demand distribution, every optimal distribution of profit or total cost is much skewed and, accordingly, different from the normal one because of nonlinearity of the profit or the total cost. Also, we can observe some interesting difference among proposed and classical models. From Figures (a1), (a2), (c1) and (c2), we see that the resulting optimal distributions through the classical expected profit maximization and the cost CVaR minimization show similar shapes. From (b1), we see the net-loss CVaR minimization achieves an optimal profit distribution with small dispersion though it results in smaller expected or maximal profit than the classical solution or the cost CVaR minimizer. We note that the empirical probability that the daily profit becomes negative with net-loss CVaR is only 0.1% which corresponds to one scenario among thousand, while the probability with the classical model is 1.4%. The mean-risk models achieve the medium distribution of those through the mean maximization and the CVaR minimization.

Figure 6 shows the convex efficient frontiers of the mean-net-loss CVaR model with different  $\beta$ s. Another advantage of the use of such LP formulations is efficient computation of the frontier by using the dual simplex algorithm. This fact helps a manager capture the profit-CVaR trade-off relation.

Figure 7 shows the CPU time spent in solving the LP with  $|N| = 3$  and  $|K| = 1,000, 750, 500$  scenarios which are drawn from a multivariate normal distribution. Computation is conducted on a personal computer with Pentium4 processor (1.6GHz) and 256M bite memory, and Xpress-MP (ver.2003G) for Windows is used for solving linear programs. From the figure, we see that the CPU time becomes smaller as  $\beta$  gets closer to 1.0. This fact provides a tailwind in practice because from a viewpoint of risk management, large loss is often more concerned with, and large  $\beta$ , say 0.95 or 0.99, should be thus taken.

## 5 Concluding Remarks

In this paper, we adopt two kinds of the conditional value-at-risk measures to the classical single-period news-vendor problem. This measure captures a risk of the profit going down to a certain level in a predetermined significance, and its minimization or related constraints have a convex structure. It is shown that its convex structure plays an important role in seeking optimal solutions to problems which include the CVaR measure in objective and constraints. In particular, one can achieve a closed form solution in single-product case when no constraint is imposed. Even with multiple constraints represented by a polyhedron, one can compute a solution by solving a linear program, if distribution of demand is given by a finite number of scenarios. By exploiting these computational advantages, we can apply this risk measure into more complex problems.

The excess cost,  $\mathcal{Q}_E := E[x - \xi]^+$ , or the opportunity cost,  $\mathcal{Q}_U := U[\xi - x]^+$ , can be also

applied as the loss  $\mathcal{L}$  and preserve the convexity since these are convex functions with fixed  $\xi$ . Since there have been many extended researches of the news-vendor problem, applications of the CVaR to various settings are future works.

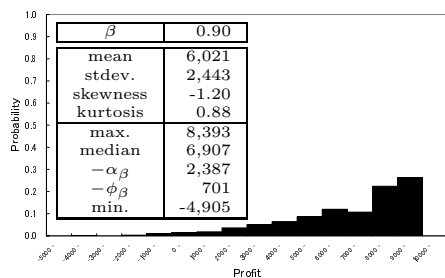
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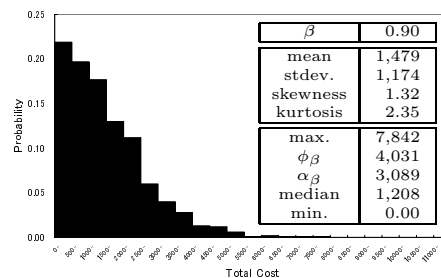
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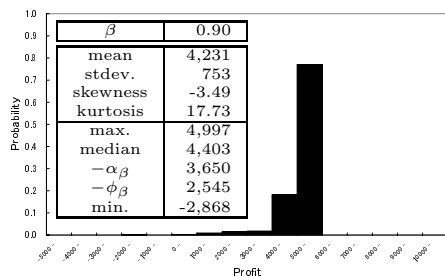
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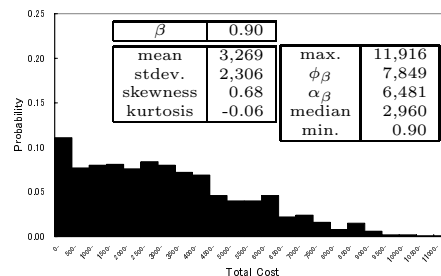
(a1) Profit by Classical Model



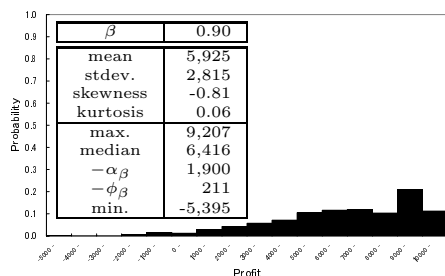
(a2) Total Cost by Classical Model



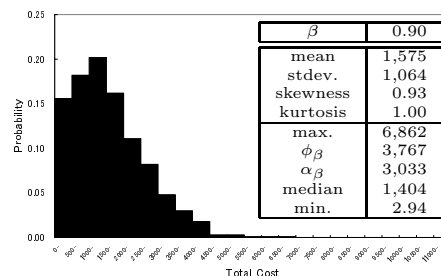
(b1) Profit by Net-Loss CVaR Min.



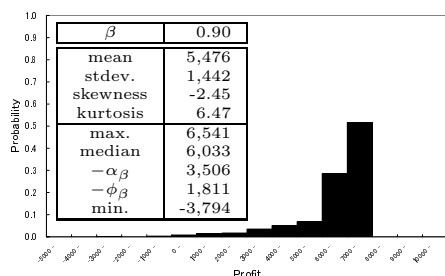
(b2) Total Cost by Net-Loss CVaR Min.



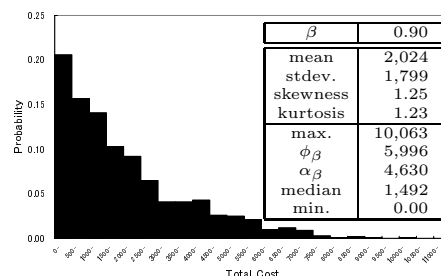
(c1) Profit by Total Cost CVaR Min.



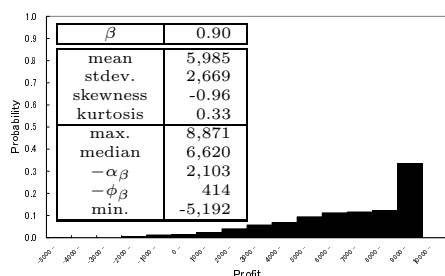
(c2) Total Cost by Total Cost CVaR Min.



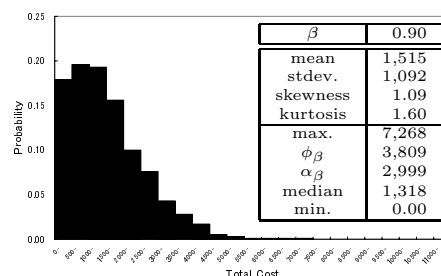
(d1) Profit by Mean-Net-Loss CVaR ( $\lambda = 1$ )



(d2) Total Cost by Mean-Net-Loss CVaR ( $\lambda = 1$ )

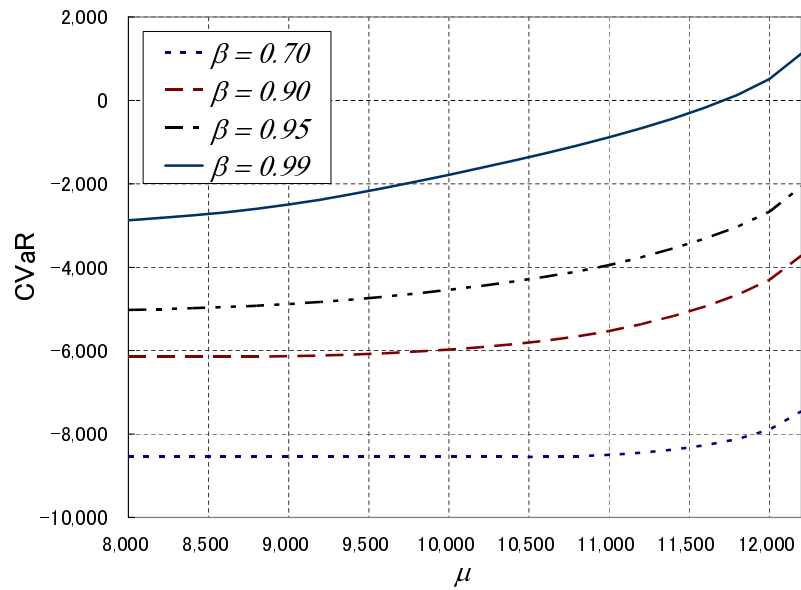
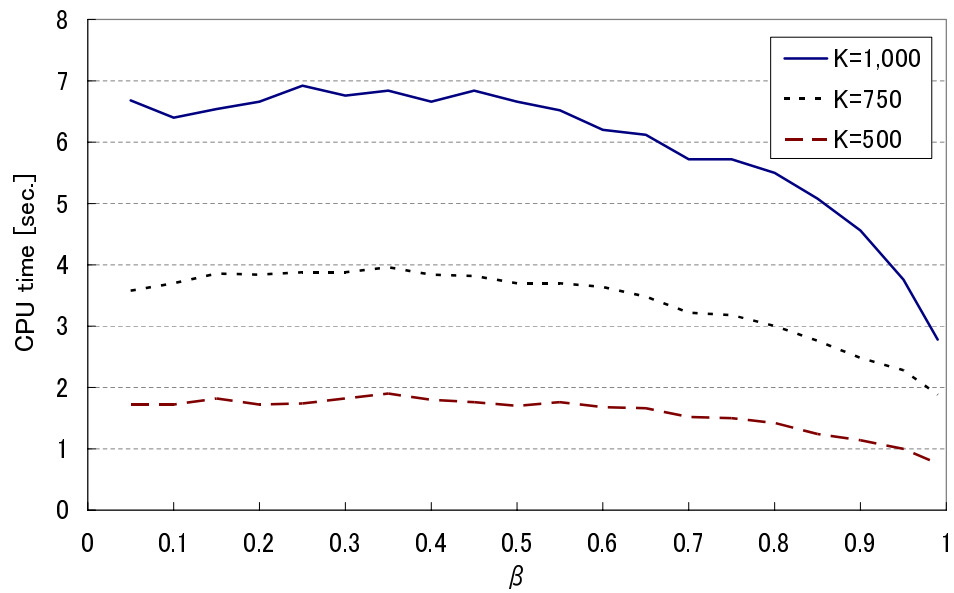


(e1) Profit by Mean-Total Cost CVaR ( $\lambda = 0.5$ )



(e2) Total Cost by Mean-Total Cost CVaR ( $\lambda = 0.5$ )

Figure 5: Histograms of Optimal Distributions of Profit and Total Cost

Figure 6: Efficient Frontier of Mean-Net-Loss CVaR Model with Different  $\beta$ Figure 7: Average CPU Time for Solving LP Formulation and  $\beta$  [sec.]