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Hypothesis Testing Based on Lagrange's Method:
Applications to Cauchy, Exponential and Logistic
Distributions.

by

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HYPOTHESIS TESTING BASED ON LAGRANGE'S METHOD: APPLICATIONS TO CAUCHY, EXPONENTIAL AND LOGISTIC DISTRIBUTIONS.

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Summary. This paper proposes the tests which essentially (except for the scale parameter of the Cauchy distribution) have the acceptance regions derived from inverting the shortest interval estimates for the parameters based on Lagrange's method. As the examples we deal with the problems of hypothesis testing for the parameters of the Cauchy distribution, the location parameter of the exponential distribution and the location parameter of the Logistic distribution. We show that these tests are unbiased. In case of Exponential distribution we propose the uniformly most powerful unbiased one-sided test for certain one-sided hypotheses.

1. Introduction. This paper consists of three parts; in Section 2 we deal with the Cauchy distribution, in Section 3 we treat the Exponential distribution, and in Section 4 we deal with the Logistic distribution.

For inferences of the parameters of Cauchy distribution we refer to Haas, Bain and Antle(1970). There, they used Monte Carlo method to obtain the distributions of the maximum likelihood estimates in order to get interval estimates and test the hypotheses. However, their methods are too complicated. For inferences of the location parameter of Exponential distribution we refer to two uniformly most powerful (UMP) tests in Problem 3 of page 112 of Lehmann(1986). Our method will give the two-sided test as good as his UMP test in Problem 3(iii). Furthermore, our test can provide UMP unbiased one-sided test for certain one-sided hypotheses, but his test in Problem 3(i) cannot. For inferences of the location parameter of Logistic distribution we refer to Antle, Klimko and Harkness(1970). There, they presented the interval estimates based on maximum likelihood estimates derived from simulation.

In all three parts of this paper we directly find the tests using the unbiased estimates for the location parameters (or the log-transformed scale parameter in Section 2) in the underlined density. Based on i. i. d. observations $X_1, \ldots, X_n$ from the underlined density the author first obtains the interval estimates of the parameters with the minimum lengths by using the method of the Lagrange's multiplier and then proposes the two-sided tests with the acceptance regions derived from inverting the (shortest) interval estimates for the parameters. We show that these tests are unbiased. Hypothesis testing with the tests so obtained is called the hypothesis testing based on the Lagrange's method. The author has been working on inferences based on the Lagrange's method since Nogami(1992, 1995; See also 2001).

Let α be a real number such that $0<\alpha<1$. We call (U_1, U_2) a $(1-\alpha)$ interval estimate for the parameter γ if $P_\gamma[U_1<\gamma<U_2]=1-\alpha$.

Let $\doteq$ be the defining property. ∎ shows the end of the proof.

2. Cauchy distribution. Throughout Section 2 we deal with the Cauchy distribution with the density

$$(2.1) \qquad f(x|\theta,\xi)=\xi\pi^{-1}\{\xi^2+(x-\theta)^2\}^{-1}, \quad -\infty<x<\infty$$

where $-\infty<\theta<\infty$ and $\xi>0$. In Sections 2.1 and 2.2 we assume ξ is known and in Sections 2.3 and 2.4 we assume θ is known.

In Section 2 the author uses the sample median for the unbiased estimate of θ and median of log-transformation of |observation $-\theta$| for the unbiased estimate of $\log_e \xi$ to get the two-sided tests based on Lagrange's method. Letting $X_1, \ldots, X_n$ be a random sample taken from (2.1) with known ξ we construct, in Section 2.1, the shortest interval estimate for θ and in Section 2.2 propose the unbiased two-sided test inverting the shortest interval estimate for θ_0, for testing the null hypothesis $H_0:\theta=\theta_0$ versus the alternative hypothesis $H_1:\theta\neq\theta_0$ with some constant θ_0. In Sections 2.3 and 2.4 the author uses i.i.d. observations $X_1, \ldots, X_n$ from (2.1) with known θ to construct an unbiased two-sided test for testing the hypotheses $H_0:\xi=\xi_0$ versus $H_1:\xi\neq\xi_0$ with some positive constant ξ_0.

Throughout Section 2 it is enough to assume n is odd (i.e. n=2m+1 with m a nonnegative integer) because if n is even, then we discard one observation.

2.1. The shortest interval estimate for θ. Let $f(x|\theta)\doteq f(x|\theta,\xi)$ in (2.1) with known ξ. Let $X_1, \ldots, X_n$ be the random sample of size n(=2m+1) taken from the density (2.1). Let $X_{(i)}$ be the i-th smallest observation of $X_1, \ldots, X_n$. We estimate θ by $Y\doteq X_{(m+1)}$. Then, we can easily check $E(Y)=\theta$. We first find the density of Y and then get the shortest interval estimate for θ based on Y.

Let $F(x|\theta)$ be the cumulative distribution function (c.d.f.) of X. Then, by (2.1) we get

(2.2) $$F(x)\doteq F(x|\theta)=\pi^{-1}\tan^{-1}((x-\theta)/\xi) + 2^{-1}, \quad \text{for } -\infty<x<\infty.$$

Hence, the density of Y is of form

(2.3) $$g_Y(y|\theta)=k(F(y))^m(1-F(y))^m f(y|\theta), \quad \text{for } -\infty<y<\infty$$

where

$$k=\Gamma(2m+2)/(\Gamma(m+1))^2.$$

Let r_1 and r_2 be real numbers such that $r_1 < r_2$. To find the shortest $(1-\alpha)$ interval estimate for θ we want to minimize $r_2 - r_1$ subject to

(2.4) $P_\theta[r_1 < Y - \theta < r_2] = 1-\alpha$.

Let λ be a Lagrange's multiplier. Using a variable transformation $W=F(Y)$ we define

$$L \doteq r_2 - r_1 - \lambda \{ \int_{F(r_1+\theta)}^{F(r_2+\theta)} h_W(w)\, dw - 1 + \alpha \}$$

where $h_W(w)$ is the density of W given by

(2.5) $h_W(w) = k w^m (1-w)^m$, for $0 < w < 1$.

Then, $\partial L / \partial r_1 = 0 = \partial L / \partial r_2$ leads to

(2.6) $h_W(F(r_1+\theta)) f(r_1+\theta \mid \theta) = h_W(F(r_2+\theta)) f(r_2+\theta \mid \theta)$ $(= \lambda^{-1})$.

Let $\beta(\alpha/2)$ be the positive number such that

(2.7) $\int_0^{\beta(\alpha/2)} h_W(w)\, dw = \alpha/2$.

Taking

(2.8) $F(r_1+\theta) = \beta(\alpha/2)$ and $F(r_2+\theta) = 1-\beta(\alpha/2)$

we obtain by (2.2) that $r_2 = -r_1 (\doteq r)$ where

(2.9) $r = \xi \tan[(2^{-1} - \beta(\alpha/2))\pi]$.

From (2.8) with $r_2=-r_1=r$, (2.5) and (2.1), we also have that $h_W(F(-r+\theta))=h_W(F(r+\theta))$ and $f(-r+\theta|\theta)=f(r+\theta|\theta)$. Thus, when $r_2=-r_1=r$, (2.6) and $\partial L/\partial\lambda=0$ are satisfied. Therefore, in view of (2.4), the shortest $(1-\alpha)$ interval estimate for θ is given by

(2.10) $(Y-r,\ Y+r)$.

In the next section we introduce the two-sided test for θ.

2.2. The unbiased two-sided test for θ. We consider the problem of testing the hypotheses $H_0:\theta=\theta_0$ versus $H_1:\theta\neq\theta_0$. Throughout this section we let $y_1=\theta_0-r$ and $y_2=\theta_0+r$ with r given by (2.9). Inverting the interval estimate (2.10) for θ_0 we obtain the two-sided test which rejects H_0 if $Y\in(-\infty,y_1]\cup[y_2,\infty)$ and accepts H_0 if $Y\in(y_1,y_2)$. Now, we show that this test is unbiased and of size α.

Define the test acceptance function $\phi(\theta)$ of θ by

(2.11) $$\phi(\theta)=\int_{y_1}^{y_2} g_Y(y|\theta)\,dy$$

where $g_Y(y|\theta)$ is defined by (2.3). From the construction $\phi(\theta_0)=1-\alpha$. Hence, the test with acceptance region (y_1,y_2) is of size α.

To show unbiasedness of our test we want to see that $\phi(\theta)$ is maximized at $\theta=\theta_0$; namely,

(2.12) $$[d\phi(\theta)/d\theta]_{\theta=\theta_0}=g_Y(y_1|\theta_0)-g_Y(y_2|\theta_0)=0$$

and

$$[d^2\phi(\theta)/d\theta^2]_{\theta=\theta_0}<0.$$

Since from the construction the equality (2.6) with $r_2=-r_1=r$ and $\theta=\theta_0$ is satisfied, it follows from (2.3) and (2.5) that $g_Y(y_1|\theta_0)=g_Y(y_2|\theta_0)$. Hence, the second equality of (2.12) is satisfied. Thus, it is sufficient to prove the following theorem:

THEOREM 1. *When $n=2m+1$ and $0<\beta(\alpha/2)<2^{-1}$,*

(2.13) $[d^2\phi(\theta)/d\theta^2]_{\theta=\theta_0}<0.$

PROOF. By (2.11) we have that

(2.14) $[d^2\phi(\theta)/d\theta^2]_{\theta=\theta_0}=[dg_Y(y_1|\theta)/d\theta]_{\theta=\theta_0}-[dg_Y(y_2|\theta)/d\theta]_{\theta=\theta_0}.$

On the other hand, by (2.3) we have that

(2.15) $dg_Y(y|\theta)/d\theta=kmf(y|\theta)(dF(y)/d\theta)(F(y))^{m-1}(1-F(y))^{m-1}(1-2F(y))$

$$+k(F(y))^m(1-F(y))^m(df(y|\theta)/d\theta).$$

Since from (2.8) and (2.2) $[F(y_1)]_{\theta=\theta_0}=[1-F(y_2)]_{\theta=\theta_0}=\beta(\alpha/2)$ and $dF(y)/d\theta=-f(y|\theta)$ and since $[df(y_2|\theta)/d\theta]_{\theta=\theta_0}=-[df(y_1|\theta)/d\theta]_{\theta=\theta_0}=2r\pi\xi^{-1}(f(y_2|\theta_0))^2$ and $f(y_1|\theta_0)=f(y_2|\theta_0)$, putting these together leads to

$$[dg_Y(y_2|\theta)/d\theta]_{\theta=\theta_0}=k(f(y_2|\theta_0))^2(1-\beta(\alpha/2))^{m-1}(\beta(\alpha/2))^{m-1}$$

$$\cdot\{m(1-2\beta(\alpha/2))+2r\pi\xi^{-1}\beta(\alpha/2)(1-\beta(\alpha/2))\}\ (>0)$$

and $[g_Y(y_1|\theta)/d\theta]_{\theta=\theta_0}=-[dg_Y(y_2|\theta)/d\theta]_{\theta=\theta_0}$. By (2.14) we obtain (2.13).∎

In the next two sections we deal with the scale parameter ξ.

2.3. The interval estimate for ξ. Let $f(x|\xi)\doteq f(x|\theta,\xi)$ in (2.1) with known θ. Let $X_1,\ldots,X_n$ be a random sample of size $n(=2m+1)$ taken from the density (2.1). In this section we obtain a $(1-\alpha)$ interval estimate for ξ. Putting $\xi^*=\log_e\xi$ we have that

$$f(x|\xi)=\pi^{-1}\exp(-\xi^*)\{1+\exp\{2(\log_e|x-\theta|-\xi^*)\}\}^{-1},\quad \text{for } -\infty<x<\infty.$$

Letting $Z=\log_e|X-\theta|$ and $Z_{(i)}$ be the i-th smallest observation of $Z_1, \ldots, Z_n$ we estimate ξ^* by $U \doteq Z_{(m+1)}$. We can easily check $E(U)=\xi^*$. We first derive the density of U and find the $(1-\alpha)$ interval estimate for ξ based on U. To do so we beforehand find the distribution of Z.

Since $x-\theta=e^z$ for $x>\theta$; $x-\theta=-e^z$ for $x<\theta$; $z=-\infty$ for $x=\theta$, by a variable transformation $Z=\log_e|X-\theta|$ the density of Z is obtained as follows:

$$(2.16) \quad q_Z(z) \doteq q_Z(z|\xi)=2\pi^{-1}e^{z-\xi^*}\{1+\exp\{2(z-\xi^*)\}\}^{-1}, \quad -\infty<z<\infty$$

where $-\infty<\xi^*<\infty$. We can easily see that $q_Z(z)$ is symmetric about $z=\xi^*$ and the unimodal function with the mode ξ^*. Letting $Q_Z(z|\xi)$ be the c.d.f. of Z we obtain by (2.16) that

$$(2.17) \quad Q_Z(z) \doteq Q_Z(z|\xi)=2\pi^{-1}\tan^{-1}\{\exp(z-\xi^*)\}, \quad -\infty<z<\infty.$$

Hence, in view of (2.3), the density of U is derived as follows:

$$(2.18) \quad g_U(u|\xi)=k(Q_Z(u))^m(1-Q_Z(u))^m q_Z(u), \quad -\infty<u<\infty.$$

We now find the $(1-\alpha)$ interval estimate for ξ. Let s_1 and s_2 be real numbers such that $s_1<s_2$. We try to find s_1 and s_2 which minimize s_2-s_1 subject to

$$(2.19) \quad P_\xi[s_1<U-\xi^*<s_2]=1-\alpha.$$

Let λ be a Lagrange's multiplier. Using the variable transformation $W=Q_Z(U)$ we define

$$A \doteq s_2-s_1-\lambda\left\{\int_{Q_Z(\xi^*+s_1)}^{Q_Z(\xi^*+s_2)} h_W(w)\,dw-1+\alpha\right\}$$

where $h_W(w)$ is given by (2.5). Then, $\partial A/\partial s_1=0=\partial A/\partial s_2$ leads to

$$(2.20)\quad h_W(Q_Z(\xi^*+s_1))q_Z(\xi^*+s_1)=h_W(Q_Z(\xi^*+s_2))q_Z(\xi^*+s_2)\ \ (=\lambda^{-1}).$$

Taking

$$(2.21)\quad Q_Z(\xi^*+s_1)=\beta(\alpha/2) \text{ and } Q_Z(\xi^*+s_2)=1-\beta(\alpha/2)$$

we obtain by (2.17) that

$$(2.22)\quad \begin{cases} s_1{}^0 = \log_e[\tan\{2^{-1}\pi\beta(\alpha/2)\}] \\ s_2{}^0 = \log_e[\tan\{2^{-1}\pi(1-\beta(\alpha/2))\}]. \end{cases}$$

From (2.21) and (2.5), $h_W(Q_Z(\xi^*+s_1{}^0))=h_W(Q_Z(\xi^*+s_2{}^0))$. Furthermore, $q_Z(\xi^*+s_1{}^0)=\pi^{-1}\sin\{\pi\beta(\alpha/2)\}=\pi^{-1}\sin\{\pi(1-\beta(\alpha/2))\}=q_Z(\xi^*+s_2{}^0)$. Hence, for $s_1=s_1{}^0$ and $s_2=s_2{}^0$ (2.20) and $\partial A/\partial\lambda=0$ are satisfied. Therefore, noticing $U=\log_e\{(|X-\theta|)_{(m+1)}\}$ and (2.19) we obtain the $(1-\alpha)$ interval estimate for ξ as follows:

$$(2.23)\quad ((|X-\theta|)_{(m+1)}\exp\{-s_2{}^0\},\ (|X-\theta|)_{(m+1)}\exp\{-s_1{}^0\}).$$

In the next section we introduce the two-sided test for ξ.

2.4. Unbiased two-sided test for ξ. We consider the problem of testing the hypothesis $H_0:\xi=\xi_0$ versus the alternative hypothesis $H_1:\xi\neq\xi_0$. Let U be as defined in Section 2.3. Throughout this section we let $u_1=\xi_0{}^*+s_1{}^0$ and $u_2=\xi_0{}^*+s_2{}^0$ with $s_1{}^0$ and $s_2{}^0$ given by (2.22). Inverting the interval estimate (2.23) for ξ_0 we obtain the two-sided test which rejects H_0 if $U\in(-\infty,u_1]\cup[u_2,\infty)$ and accepts H_0 if $U\in(u_1,u_2)$. We show that this test is unbiased and of size α.

Define the test acceptance function $\phi_1(\xi)$ of ξ by

$$(2.24)\quad \phi_1(\xi)=\int_{u_1}^{u_2} g_U(u|\xi)\,du$$

where $g_U(u|\xi)$ is given by (2.18). From the construction $\phi_1(\xi_0)=1-\alpha$. Hence, the test with acceptance region (u_1, u_2) is of size α.

To show unbiasedness of our test we want to see that $\phi_1(\xi)$ is maximized at $\xi=\xi_0$; namely,

(2.25) $$[d\phi_1(\xi)/d\xi]_{\xi=\xi_0} = \xi_0^{-1}\{g_U(u_1|\xi_0) - g_U(u_2|\xi_0)\}=0$$

and

$$[d^2\phi_1(\xi)/d\xi^2]_{\xi=\xi_0} < 0.$$

From the construction (2.20) with ξ^*+s_1 and ξ^*+s_2 replaced by u_1 and u_2, respectively holds. Hence, we obtain by (2.18) and (2.5) that $g_U(u_1|\xi_0)=g_U(u_2|\xi_0)$. Therefore, the second equality of (2.25) holds. Thus, we need to show the following theorem:

THEOREM 2. *When $n=2m+1$ and $0<\beta(\alpha/2)<2^{-1}$,*

(2.26) $$[d^2\phi_1(\xi)/d\xi^2]_{\xi=\xi_0} < 0.$$

PROOF. By (2.24) and (2.25) we have that

(2.27) $$[d^2\phi_1(\xi)/d\xi^2]_{\xi=\xi_0} = \xi_0^{-1}\{[dg_U(u_1|\xi)/d\xi]_{\xi=\xi_0} - [dg_U(u_2|\xi)/d\xi]_{\xi=\xi_0}\}.$$

But, by (2.18), and in view of (2.15) and $dQ_Z(u)/d\xi=-\xi^{-1}q_Z(u)$ we have that

$$dg_U(u|\xi)/d\xi = -km\xi^{-1}(q_Z(u))^2(Q_Z(u))^{m-1}(1-Q_Z(u))^{m-1}(1-2Q_Z(u))$$

$$+k(Q_Z(u))^m(1-Q_Z(u))^m(dq_Z(u)/d\xi).$$

Since $dq_Z(u)/d\xi=2(\pi\xi)^{-1}e^{u-\xi^*}(e^{2(u-\xi^*)}-1)\{1+\exp\{2(u-\xi^*)\}\}^{-2}$, we have that $[dq_Z(u_2)/d\xi]_{\xi=\xi_0}=(2\pi\xi_0)^{-1}\sin(2\pi\beta(\alpha/2))=-[dq_Z(u_1)/d\xi]_{\xi=\xi_0}$. From (2.21) and (2.16) we also have that $[Q_Z(u_1)]_{\xi=\xi_0}=1-[Q_Z(u_2)]_{\xi=\xi_0}=\beta(\alpha/2)$ and $[q_Z(u_1)]_{\xi=\xi_0}=\pi^{-1}\sin(\pi\beta(\alpha/2))=[q_Z(u_2)]_{\xi=\xi_0}$. Putting these together leads to

(2.28) $[dg_U(u_2|\xi)/d\xi]_{\xi=\xi_0}$

$$=km(\pi^2\xi_0)^{-1}\sin^2(\pi\beta(\alpha/2))(1-\beta(\alpha/2))^{m-1}(\beta(\alpha/2))^{m-1}(1-2\beta(\alpha/2))$$

$$+k(\beta(\alpha/2))^m(1-\beta(\alpha/2))^m(2\pi\xi_0)^{-1}\sin(2\pi\beta(\alpha/2))$$

and $[dg_U(u_1|\xi)/d\xi]_{\xi=\xi_0}=-[dg_U(u_2|\xi)/d\xi]_{\xi=\xi_0}$. Therefore, noticing that $\sin(2\pi\beta(\alpha/2))>0$ for $0<\beta(\alpha/2)<2^{-1}$ we have that (2.28)>0. Hence, in view of (2.27), (2.26) holds.∎

3. The Exponential distribution. Let $I_A(x)$ be an indicator function so that for a set A $I_A(x)=1$ if $x\in A$; $=0$ if $x\notin A$. Throughout Section 3 we consider the problem of hypothesis testing with respect to the location parameter θ of the Exponential distribution with the density

(3.1) $f(x|\theta)=b^{-1}e^{-(x-\theta)/b}\, I_{[\theta,\infty)}(x)$

where $-\infty<\theta<\infty$ and $b>0$. In Sections 3.1 and 3.3 we assume b is known and in Section 3.2 we deal with unknown b.

Based on i.i.d. observations $X_1, \ldots, X_n$ from (3.1) the author considers, in Sections 3.1 and 3.2, to test the hypothesis $H_0:\theta=\theta_0$ versus the alternative hypothesis $H_1:\theta\neq\theta_0$ with some constant θ_0. In Section 3.1 we construct the two-sided test which is unbiased. To do so we find the shortest interval estimate for θ using an unbiased estimate $Y=\bar{X}-1(=n^{-1}\Sigma_{i=1}^{n}X_i-1)$ for θ and construct the acceptance region derived from inverting this interval estimate for θ_0. In Section 3.2 we compare our test in Section 3.1 with two UMP tests in Problem 3 of Lehmann(1986). In Section 3.3 we derive the one-sided test for testing the hypothesis $H'_0:\theta\geq\theta_0$ versus the alternative hypothesis $H'_1:\theta<\theta_0$ and show that this test is UMP unbiased and of size α.

3.1. An unbiased two-sided test for θ. Let $X_1, \ldots, X_n$ be a random sample of size n taken from (3.1) with known b. We consider the problem of testing the hypotheses $H_0:\theta=\theta_0$ versus $H_1:\theta\neq\theta_0$. We first derive necessary distributions to find the shortest interval estimate for θ.

As an estimate for θ we take $Y=\bar{X}-1$. We can easily check $E(Y)=\theta$. Let $X_{(i)}$ be the i-th smallest observation such that $X_{(1)}\leqq X_{(2)}\leqq \ldots \leqq X_{(n)}$. We find the joint density of variables $W=X_{(1)}+\ldots+X_{(n)}$ $(=X_1+\ldots+X_n)$, $V=X_{(1)}$, $Z_2=X_{(2)}$, ... $Z_{n-1}=X_{(n-1)}$ as follows:

$$g(w,v,z_2,z_3,\ldots,z_{n-1}|\theta)=\begin{cases} n!b^{-n}e^{-(w-n\theta)/b}, & \\ \quad \text{for } \theta\leqq v\leqq z_2\leqq z_3\leqq \ldots \leqq z_{n-1}\leqq w-\sum_{i=1}^{n-1} z_i & \\ 0, \quad \text{otherwise.} & \end{cases}$$

Integrating out the above density with respect to z_2 through z_{n-1} we get the joint density of (W,V) as follows:

$$(3.2)\qquad g(w,v|\theta)=\begin{cases} (n/\Gamma(n-1))b^{-n}e^{-(w-n\theta)/b}(w-nv)^{n-2}, & \text{for } \theta\leqq v\leqq w/n<\infty \\ 0, & \text{otherwise.} \end{cases}$$

Taking the marginal density of W and furthermore, letting $t=2(w-n\theta)/b$ we have the density of T so that

$$(3.3)\qquad h_T(t)=(1/\Gamma(n))e^{-t/2}t^{n-1}2^{-n}I_{[0,\infty)}(t)$$

which is the chi-square density for 2n degrees of freedom.

Let r_1 and r_2 be real numbers such that $r_1<r_2$. To find the shortest $(1-\alpha)$ interval estimate for θ we want to minimize r_2-r_1 subject to

$$(3.4)\qquad P_\theta[r_1<Y-\theta<r_2]=1-\alpha.$$

But, by a variable transformation $t=2n(y+1-\theta)/b$ (3.4) is equivalent to

(3.5) $\quad P[t_1<T<t_2]=1-\alpha.$

where $t_i=2n(r_i+1)/b$ for $i=1,2$. Hence, we want to minimize t_2-t_1 subject to the condition (3.5). Let γ be a Lagrange's multiplier and define

$$L \doteq t_2-t_1 -\gamma\{\int_{t_1}^{t_2} h_T(t)\,dt -1+\alpha\}.$$

Then, $\partial L/\partial t_1=0=\partial L/\partial t_2$ leads to

(3.6) $\quad h_T(t_1)=h_T(t_2) \quad (=\gamma^{-1}).$

Taking t_1 and t_2 which satisfy (3.6) and $\partial L/\partial\gamma=0$, noticing that $t_1<T=2n(Y+1-\theta)/b <t_2$ and letting $t_3=bt_1/(2n)$ and $t_4=bt_2/(2n)$ we obtain the shortest $(1-\alpha)$ interval estimate for θ as follows:

(3.7) $\quad (Y+1-t_4, Y+1-t_3).$

Hence, by inverting (3.7) for θ_0 our test is to reject H_0 if $Y\leq\theta_0+t_3-1$ or $Y\geq\theta_0+t_4-1$ or $V<\theta_0$ and to accept H_0 if $\theta_0+t_3-1<Y<\theta_0+t_4-1$ and $V\geq\theta_0$. Here, we emphasize the necessity of having the set $\{V<\theta_0\}$ in the rejection region. To check unbiasedness of this test we obtain the power function as follows:

$$\pi(\theta)= P_\theta[Y\leq\theta_0+t_3-1 \text{ or } \theta_0+t_4-1\leq Y \text{ or } V<\theta_0]$$

(3.8)
$$=\begin{cases} 1-(1-\alpha)\exp\{-n(\theta_0-\theta)/b\}, & \text{for } \theta<\theta_0 \\ P[0<T\leq t_1-2n(\theta-\theta_0)/b]+P[t_2-2n(\theta-\theta_0)/b<T], & \text{for } \theta_0\leq\theta<\theta_0+t_3 \\ P[t_2-2n(\theta-\theta_0)/b<T], & \text{for } \theta_0+t_3\leq\theta<\theta_0+t_4 \\ 1, & \text{for } \theta_0 t_4\leq\theta. \end{cases}$$

Hence, $d\pi(\theta)/d\theta<0$ for $\theta<\theta_0$; $d\pi(\theta)/d\theta=2nb^{-1}\{h_T(t_2-2n(\theta-\theta_0)/b)-h_T(t_1-2n(\theta-\theta_0)/b)\}$

>0 for $\theta_0 \leqq \theta < \theta_0 + t_3$ because of (3.6) and (3.3), and $d\pi(\theta)/d\theta > 0$ for $\theta_0 + t_3 \leqq \theta < \theta_0 + t_4$. Since $\pi'(\theta_0)=0$ by (3.6) and $\pi(\theta_0)=\alpha$, we have that $\pi(\theta) \geqq \alpha$ for real θ. Thus, unbiasedness of the test is proved.

Furthermore, we note that (3.8) for $\theta < \theta_0$ is equal to $\beta^*(a_1)$ with a_1 and a_0 replaced by θ and θ_0, respectively in problem 3(ii) of Lehmann(1986).

In the next section we compare our test with two UMP tests; one from Problem 3(i) and the other from Problem 3(ii) of Lehmann(1986).

3.2. Another UMP two-sided test. In this section we assume a_1 and a_0 in Problem 3 of Lehmann(1986) to be θ and θ_0, respectively and consider to test the hypotheses $H_0: \theta = \theta_0$ versus $H_1: \theta \neq \theta_0$. Let $V \doteq X_{(1)}$ as in Section 3.1.

We first consider the solution of Problem 3(i). Let $\theta_* = \theta_0 - n^{-1} b \log_e \alpha$. We obtain the UMP two-sided test to reject H_0 if $V < \theta_0$ or $\theta_* \leqq V$ and to accept H_0 if $\theta_0 \leqq V < \theta_*$. Hence, the power function of this test is equal to (3.8) for $\theta < \theta_0$ and is larger than (3.8) for $\theta > \theta_0$. However, we cannot construct the one-sided test for testing the hypotheses $H'_0: \theta \geqq \theta_0$ versus $H'_1: \theta < \theta_0$ from this approach, because the test takes the probability of size α from upper tail only. In the next section we show that our one-sided test is UMP unbiased size-α for testing the hypotheses H'_0 versus H'_1 when b is known.

Secondly, we consider Problem 3(iii). Here, we assume unknown b. The two-sided test introduced in Problem 3(iii) is to reject H_0 if $U=(V-\theta_0)/(\Sigma_{i=1}^{n} X_i - nV)$ $\leqq C_1$ or $\geqq C_2$ where C_1 and C_2 are some constants. From Section 3.1 we can propose another two-sided test which rejects H_0 if

(3.9) $\quad S=(Y+1-\theta_0)/(\Sigma_{i=1}^{n} X_i - nV) \leqq C_3$ or $\geqq C_4$

where $Y=\bar{X}-1$ as in Section 3.1 and C_3 and C_4 are some constants. From the definition of Y we have the relation $S=U+n^{-1}$. Therefore, from Problrm 3(iv) of Lehmann(1986), the two-sided test (3.9) is also UMP size-α test of testing $H_0: \theta = \theta_0$ versus $H_1: \theta \neq \theta_0$.

3.3. The UMP one-sided test. In this section we consider to test the hypotheses $H'_0: \theta \geqq \theta_0$ versus $H'_1: \theta < \theta_0$ based on a random sample $X_1, \ldots, X_n$ from (3.1) with known b. Let Y, V and T be as defined in Section 3.1. Let $t_6 = bt_5/(2n)$ where t_5 is given by

$$\int_0^{t_5} h_T(t)\, dt = \alpha .$$

Our test is to reject H_0 if $Y \leqq \theta_0 + t_6 - 1$ or $V < \theta_0$ and to accept H_0 if $\theta_0 + t_6 - 1 < Y$ and $\theta_0 \leqq V$. Let $g_{Y,V}(y, v|\theta)$ be the joint density of (Y, V). Then, from (3.2) and the relation $g_{Y,V}(y, v|\theta) = g(n(y+1), v|\theta)n$ we can easily get the power function of above test as follows:

$$\pi(\theta) = P_\theta[Y \leqq \theta_0 + t_6 - 1 \text{ or } V < \theta_0]$$

$$= \begin{cases} 1-(1-\alpha)\exp\{-n(\theta_0-\theta)/b\}, & \text{for } \theta < \theta_0 \\ P[0 < T < t_5 - 2n(\theta-\theta_0)/b], & \text{for } \theta_0 \leqq \theta < \theta_0 + t_6 \\ 0, & \text{for } \theta_0 + t_6 \leqq \theta \end{cases}$$

Since $d\pi(\theta)/d\theta < 0$ for $\theta < \theta_0 + t_6$ and hence $\pi(\theta) \geqq \alpha = \pi(\theta_0)$ for $\theta < \theta_0$, our test is unbiased, of size α and furthermore UMP because of Problem 3(ii) of Lehmann(1986).

4. The Logistic distribution. In Section 4 we deal with the Logistic distribution whose density is given as follows:

$$(4.1) \qquad f(x|\theta) = b^{-1}e^{-(x-\theta)/b}\{1+\exp\{-(x-\theta)/b\}\}^{-2}, \quad \text{for } -\infty < x < \infty$$

provided that $-\infty < \theta < \infty$ and $b > 0$. Here, we assume b is known.

Let $X_1, \ldots, X_n$ be a random sample of size n taken from (4.1). Throughout Section 4, as in Section 2, it is enough to assume n is odd (i.e. $n = 2m+1$ with m a nonegative integer). Let $X_{(i)}$ be the i-th smallest observation of $X_1, \ldots, X_n$. We estimate θ by the sample median $Y \doteq X_{(m+1)}$. Then, we can easily check $E(Y) = \theta$. We find, in Section 4.1, the shortest $(1-\alpha)$ interval estimate for θ using Y and derive in Section 4.2 the two-sided test for testing the null hypothesis $H_0:\theta=\theta_0$ versus the alternative hypothesis $H_1:\theta\neq\theta_0$ for some constant θ_0 by inverting the shortest $(1-\alpha)$ interval estimate for θ_0. We show that this test is unbiased and of size α.

4.1. The shortest interval estimate for θ. Let $Y \doteq X_{(m+1)}$. We first find the density of Y and obtain the shortest $(1-\alpha)$ interval estimate for θ.

Let $F(x|\theta)$ be the c.d.f. of X. Then, by (4.1) we get

$$(4.2)\qquad F(x)\doteq F(x|\theta)=\{1+e^{-(x-\theta)/b}\}^{-1}, \qquad \text{for } -\infty<x<\infty.$$

The density of Y is the same form as (2.3) with $F(y)$ and $f(x|\theta)$ given by (4.2) and (4.1), respectively.

Let r_1 and r_2 be real numbers such that $r_1<r_2$. We go through the same process as that from the second line above (2.4) till (2.7). Taking r_1 and r_2 which satisfy (2.8) with $F(Y)$ given by (4.2), we obtain $r_2=-r_1(\doteq r_*)$ where

$$(4.3)\qquad r_*=b\log_e\{(1-\beta(\alpha/2))/\beta(\alpha/2)\}.$$

We define $W=F(Y)$ and $h_W(w)$ by (2.5). Then, we have that $h_W(F(-r_*+\theta))=h_W(F(r_*+\theta))$ and furthermore $f(-r_*+\theta|\theta)=f(r_*+\theta|\theta)$. Thus, when $r_2=-r_1=r_*$, (2.6) and $\partial L/\partial\lambda=0$ are satisfied. Therefore, in view of (2.4) the shortest $(1-\alpha)$ interval estimate for θ is given by

$$(4.4)\qquad (Y-r_*, Y+r_*).$$

In the next section we introduce the two-sided test for θ.

4.2. The unbiased two-sided test for θ. We test the hypotheses $H_0:\theta=\theta_0$ versus $H_1:\theta\neq\theta_0$. Let Y be as defined in Section 4.1. Throughout this section we let $y_1=\theta_0-r_*$ and $y_2=\theta_0+r_*$ with r_* given by (4.3). Inverting the interval estimate (4.4) for θ_0 we obtain the two-sided test which rejects H_0 if $Y\in(-\infty,y_1]\cup[y_2,\infty)$ and accepts H_0 if $Y\in(y_1,y_2)$. Now we show that this test is unbiased and of size α.

Define $\phi(\theta)$ by (2.11) where $g_Y(y|\theta)$ is given by (2.3) with $F(y)$ defined by (4.2). From the construction $\phi(\theta_0)=1-\alpha$. Hence, the test with the acceptance region (y_1, y_2) is of size α.

To show unbiasedness of our test we want to see that (2.12) and (2.13) hold. Since from the construction the equality (2.6) with $r_2=-r_1=r_*$ and $\theta=\theta_0$ is satisfied, it follows from (2.3) and (2.5) that $g_Y(y_1|\theta_0)=g_Y(y_2|\theta_0)$. Hence, the second equality of (2.12) is satisfied. Thus, it is sufficient to prove the following theorem:

THEOREM 3. *When $n=2m+1$ and $0<\beta(\alpha/2)<2^{-1}$,*

(4.5) $\quad [d^2\phi(\theta)/d\theta^2]_{\theta=\theta_0} < 0.$

PROOF. From (2.11) we have (2.14). By (2.3) we also have (2.15). Since $df(y|\theta)/d\theta=b^{-2}e^{-(y-\theta)/b}(1-e^{-(y-\theta)/b})(1+e^{-(y-\theta)/b})^{-3}$, we have that $[df(y_2|\theta)/d\theta]_{\theta=\theta_0}=b^{-1}(1-2\beta(\alpha/2))f(y_2|\theta_0)=-[df(y_1|\theta)/d\theta]_{\theta=\theta_0}$. From (2.8) and (4.2) we also have that $[F(y_1)]_{\theta=\theta_0}=\beta(\alpha/2)=1-[F(y_2)]_{\theta=\theta_0}$ and $f(y_1|\theta_0)=f(y_2|\theta_0)$ $=b^{-1}\beta(\alpha/2)(1-\beta(\alpha/2))$. Applying these and the fact that $dF(y)/d\theta=-f(y|\theta)$ to (2.15) leads to

$$[dg_Y(y_2|\theta)/d\theta]_{\theta=\theta_0}=kb^{-2}(1-\beta(\alpha/2))^{m+1}(\beta(\alpha/2))^{m+1}(1-2\beta(\alpha/2))(m+1) \quad (>0) \ .$$

and $[dg_Y(y_1|\theta)/d\theta]_{\theta=\theta_0}=-[dg_Y(y_2|\theta)/d\theta]_{\theta=\theta_0}$. Therefore, in view of (2.14), (4.5) holds. ∎

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