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Hypothesis Testing Based on Lagrange's Method: Application to The Uniform Distribution.

by

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HYPOTHESIS TESTING BASED ON LAGRANGE'S METHOD: APPLICATION TO THE UNIFORM DISTRIBUTION.

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Abstract. In this paper we deal with the uniform distribution as follows:

$$f(x|\theta)=\theta^{-1} \qquad (0<x<\theta;\ 0<\theta).$$

The author proposes the tests which essentially have the acceptance regions derived from inverting the conditional minimum-length(CML) interval estimates for the function $\ln\theta$ of the parameter θ based on the Lagrange's method. She proposes the two-sided test for testing the hypothesis $H_0:\theta=\theta_0$ versus the alternative hypothesis $H_1:\theta\neq\theta_0$ with a positive constant θ_0. Her test is unbiased. She also propose the uniformly most powerful one-sided test for testing $H'_0:\theta\leqq\theta_0$ versus $H'_1:\theta>\theta_0$.

§1. Introduction.

The idea for the relation between the tests and interval estimates is seen in Neyman(1937) and recently, for example, Matusita(1951) and Matubara & Nogam (1982). For the hypothesis testing based on Lagrange's method we refer to Nogami(2002). Let $I_A(x)$ be the indicator function of the interval A such that $I_A(x)=1$ if $x\in A$; $=0$ if $x\notin A$. Here, we consider to test the parameter θ of the uniform distribution

$$(1) \qquad f(x|\theta)=\theta^{-1}I_{(0,\theta)}(x) \qquad (\theta>0).$$

Let $X_1, \dots, X_n$ be a random sample of size n taken from (1). We apply the similar analysis appeared in Sections 2.3 and 2.4 and Section 3 of Nogami(2002).

In Section 2 we consider the problem for testing the hypothesis $H_0:\theta=\theta_0$ versus the alternative hypothesis $H_1:\theta\neq\theta_0$ with a positive constant θ_0. Let $\doteq$ be the defining property. Let $\theta^*\doteq\ln\theta$ and $Y\doteq\ln X$. We estimate θ^* by the unbiased estimate $U=\bar{Y}+1=n^{-1}\Sigma_{i=1}^{n}Y_i+1$ and construct the conditional minimum-length (CML) interval estimate for θ^* based on the Lagrange's method. Then, we apply this interval estimate to get the two-sided test and show that our test is unbiased. Let α be a real number such that $0<\alpha<1$. For a reference on this problem there is a uniformly most powerful(UMP) test of size α in Ferguson(1967, p.213) (as well as Lehmann(1986, p. 111)). However, this test cannot be applied to the test of $H'_0:\theta\leqq\theta_0$ versus $H'_1:\theta>\theta_0$ because it takes probability of size α from the lower tail only.

In Section 3 we propose the one-sided unbiased test of $H'_0:\theta\leqq\theta_0$ versus $H'_1:\theta>\theta_0$. As references for this problem we refer to Mood, Graybill and Boes(1988, p. 424) (as well as Ferguson(1967, p. 213) for a randomized test and Lehmann (1986, p. 111)). Since our test has the same power as that in Mood, Graybill & Boes(1988, p. 424) for $\theta>\theta_0$, our test is also UMP and of size α.

We call (S_1,S_2) a $(1-\alpha)$ interval estimate for the parameter η if $P_\eta[S_1<\eta<S_2]=1-\alpha$.

§2. The unbiased two-sided test.

Let $U=\bar{Y}+1$. Let $Y_{(1)}$ be the smallest observation of $Y_1, \dots, Y_n$. Let $W\doteq\Sigma_{i=1}^{n}Y_{(i)}(=\Sigma_{i=1}^{n}Y_i)$ and $V=Y_{(n)}$. We first find the density $h_{W,V}(w,v)$ of (W,V).

Then, we find the density $g_W(w)$ of W. Furthermore, letting $T=2n(\theta^*+1-U)$ we obtain the density $h_T(t)$ of T to get the CML $(1-\alpha)$ interval estimate for θ^* based on U.

First of all we find the density of Y as follows:

(2) $$g_Y(y)=\exp\{y-\theta^*\}I_{(-\infty,\theta^*)}(y).$$

Since, from (2), $g_{Y_1,\ldots,Y_n}(y_1,\ldots,y_n)=\exp\{\Sigma_{i=1}^{n}y_i-n\theta^*\}I_{(-\infty,\theta^*)}(v)$, we can find the joint density of W, V, $Z_2=Y_{(2)}$, $Z_3=Y_{(3)}$, ... , and $Z_{n-1}=Y_{(n-1)}$ as follows:

(3) $$h(w,v,z_2,\ldots,z_{n-1})=n!\exp\{w-n\theta^*\},$$

for $-\infty<w-v-\Sigma_{i=2}^{n}z_i\leqq z_2\leqq\ldots\leqq z_{n-1}\leqq v\leqq\theta^*$. To get $h_{W,V}(w,v)$ we integrate out (3) with respect to $z_2,\ldots,z_{n-1}$. Then, we obtain

(4) $$h_{W,V}(w,v)=\{n/\Gamma(n-1)\}\exp\{-(n\theta^*-w)\}(nv-w)^{n-2}, \quad \text{for } w\leqq nv\leqq n\theta^*.$$

Taking the marginal density $g_W(w)$ of W we have

(5) $$g_W(w)=(\Gamma(n))^{-1}\exp\{-(n\theta^*-w)\}(n\theta^*-w)^{n-1}, \quad \text{for } w\leqq n\theta^*.$$

Using a variable transformation $T\doteq 2n(\theta^*+1-U)=2\{n(\theta^*+1)-W\}$ we get, from (5), the density of T as follows:

(6) $$h_T(t)=(\Gamma(n))^{-1}t^{n-1}e^{-t/2}2^{-n}I_{[0,\infty)}(t).$$

Let r_1 and r_2 be real numbers such that $r_1<r_2$. To find the CML $(1-\alpha)$ interval estimate for θ^* we want to minimize r_2-r_1 subject to

(7) $$P_\theta[r_1<U-\theta^*<r_2]=1-\alpha.$$

But, by a variable transformation $t=2n(\theta^*+1-u)$ (7) is equal to

(8) $P[t_1<T<t_2]=1-\alpha$

with $t_1=2n(1-r_2)$ and $t_2=2n(1-r_1)$. Hence, we want to minimize t_2-t_1 subject to the condition (8). Let λ be a Lagrange's multiplier and define

$$L=t_2-t_1-\lambda\{\int_{t_1}^{t_2} h_T(t)\,dt-1+\alpha\}.$$

Then, $\partial L/\partial t_1=0=\partial L/\partial t_2$ leads to

(9) $h_T(t_1)=h_T(t_2)\ (=\lambda^{-1})$.

Taking t_1 and t_2 which satisfy (9) and $\partial L/\partial\lambda=0$, noticing that $r_1=1-t_2/(2n)$ and $r_2=1-t_1/(2n)$ we obtain the CML $(1-\alpha)$ interval estimate for θ^* as follows:

(10) $(U-1+t_1/(2n),U-1+t_2/(2n))$.

Hence, by letting $u_1{}^0=\theta_0{}^*+1-t_2/(2n)$ and $u_2{}^0=\theta_0{}^*+1-t_1/(2n)$ and inverting (10) for $\theta_0{}^*$ our test is to reject H_0 if $U\leqq u_1{}^0$ or $u_2{}^0\leqq U$ or $\theta_0{}^*<V$ and to accept H_0 if $u_1{}^0<U<u_2{}^0$ and $V\leqq\theta_0{}^*$.

To check unbiasedness of this test we use (4) and obtain the power of the test as follows:

$$\pi(\theta)=P_\theta[U\leqq u_1{}^0 \text{ or } u_2{}^0\leqq U \text{ or } \theta_0{}^*<V]$$

$$=P_\theta[\theta_0{}^*<V]+P_\theta[W\leqq n\theta_0{}^*-2^{-1}t_2 \text{ and } V\leqq\theta_0{}^*]+P_\theta[n\theta_0{}^*-2^{-1}t_1\leqq W \text{ and } V\leqq\theta_0{}^*]$$

(11)
$$=\begin{cases} 1-(1-\alpha)(\theta_0/\theta)^n, & \text{for } \theta_0<\theta,\\ \int_0^{2n(\theta^*-\theta_0{}^*)+t_1} h_T(t)\,dt+\int_{2n(\theta^*-\theta_0{}^*)-t_2}^{\infty} h_T(t)\,dt, & \\ \qquad\text{for } \theta_0\exp\{-t_1/(2n)\}<\theta\leqq\theta_0, & \end{cases}$$

$$\begin{cases} \int_{2n(\theta^*-\theta_0^*)+t_2}^{\infty} h_T(t)\,dt, & \text{for } \theta_0\exp\{-t_2/(2n)\}<\theta\leqq\theta_0\exp\{-t_1/(2n)\}, \\ 1, & \text{for } 0<\theta\leqq\theta_0\exp\{-t_2/(2n)\}. \end{cases}$$

Hence, $d\pi(\theta)/d\theta>0$ for $\theta_0<\theta$, $d\pi(\theta)/d\theta=2n\theta^{-1}\{h_T(2n(\theta^*-\theta_0^*)+t_1)-h_T(2n(\theta^*-\theta_0^*)+t_2)\}$ <0 for $\theta_0\exp\{-t_1/(2n)\}<\theta<\theta_0$ and $d\pi(\theta)/d\theta<0$ for $\theta_0\exp\{-t_2/(2n)\}<\theta\leqq\theta_0\exp\{-t_1/(2n)\}$. The second inequality above follows because of (9) and (6). Thus, we have $\pi(\theta)\geqq\alpha=\pi(\theta_0)$ for real θ. Therefore, unbiasedness of the test is proved.

From the construction it is easily seen from (7) that our test is of size α.

We note that there is a UMP-size α test in Ferguson(1976, p. 213) (as well as Lehmann(1986, p. 111)) for this problem. The power of this test is the same as (11) for $\theta_0<\theta$. However, since most of the time we have $\exp\{-2^{-1}t_2\}<\alpha^{1/n}$, our power for $\theta_0\exp\{-t_2/(2n)\}<\theta<\theta_0\alpha^{1/n}$ is no better than Ferguson(1976, p. 213). However, As I stated in Section 1, this (his) test is not applicable for the test of $H'_0:\theta\leqq\theta_0$ versus $H'_1:\theta_0<\theta$. In the next section we show that our test of H'_0: $\theta\leqq\theta_0$ versus $H'_1:\theta_0<\theta$ is UMP and of size α.

§3. The UMP one-sided test.

In this section we first consider the test of $H'_0:\theta\leqq\theta_0$ versus $H'_1:\theta_0<\theta$. As in Section 2 we let $\theta^*\doteq\ln\theta$, $U=\bar{Y}+1$ and $V=Y_{(n)}$. We furthermore define $u_2^*=\theta_0^*+1-t_1/(2n)$ where t_1 here is defined by

(12) $\quad P[T<t_1]=\alpha.$

Then, our one-sided test is to reject H'_0 if $u_2^*\leqq U$ or $\theta_0^*<V$ and to accept H'_0 if $U<u_2^*$ and $V\leqq\theta_0^*$. From Section 2 we can easily get the power of the test as follows:

$$\pi(\theta)=\ P_\theta[u_2^*\leqq U \text{ or } \theta_0^*<V]$$

$$
= \begin{cases} 1-(1-\alpha)(\theta_0/\theta)^n, & \text{for } \theta_0<\theta \\ \int_0^{2n(\theta^*-\theta_0^*)+t_1} h_T(t)\,dt, & \text{for } \theta_0\exp\{-t_1/(2n)\}<\theta\leqq\theta_0 \\ 0, & \text{for } 0<\theta\leqq\theta_0\exp\{-t_1/(2n)\}. \end{cases}
$$

Since $d\pi(\theta)/d\theta>0$ for $\theta_0<\theta$ and $d\pi(\theta)/d\theta=2n\theta^{-1}h_T(2n(\theta^*-\theta_0^*)+t_1)>0$ for $\theta<\theta_0$, $\pi(\theta_0)=\alpha\leqq\pi(\theta)$ for real θ such that $\theta_0<\theta$. Hence, this test is unbiased.

It is immediate from (12) that our test is of size α.

Historically, there is a randomized test in Ferguson(1976, p. 213 #7(c)) which is better than our test in the sense of the power. However, it is more natural to compare our test with the test appeared in Mood, Graybill & Boes (1988, p.424) which rejects H_0 if $V>\theta_0(1-\alpha)^{1/n}$ and accepts H_0 if $V\leqq\theta_0(1-\alpha)^{1/n}$. This test has the same power as our power for $\theta_0<\theta$. Hence, our test is also UMP and of size α.

§ 4. Remark.

The term "interval estimate" used in this paper is due to Fabian & Hannan (1985).

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