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Hypothesis Testing based on Lagrange's Method:
Application to the Uniform Distribution (II)

by

Yoshiko Nogami

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UNIVERSITY OF TSUKUBA
Tsukuba, Ibaraki 305-8573
JAPAN

Hypothesis testing based on Lagrange's method: Application to the uniform distribution(II).

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Abstract. In this paper the author treats the uniform distribution as follows:

$$f(x|\theta)=\begin{cases} c^{-1}, & \theta+\delta_1\leq x<\theta+\delta_2 \\ 0, & \text{otherwise} \end{cases}$$

where $-\infty<\theta<\infty$, $c=\delta_2-\delta_1$ and δ_1 and δ_2 are real numbers such that $\delta_1<\delta_2$. We refine the test appeared in Nogami(2001a) and compare this refined test with the tests in Chatterjee and Chattopadhyay(1994). We also present two-sided test when c is unknown.

§1. Introduction. For the hypothesis testing based on Lagrange's method we refer to Nogami(2002). In this paper we deal with the uniform distribution over the interval $[\theta+\delta_1, \theta+\delta_2)$ with the density

$$(1) \qquad f(x|\theta)=\begin{cases} c^{-1}, & \text{for } \delta_1 \leqq x-\theta < \delta_2 \\ 0, & \text{otherwise} \end{cases}$$

where $-\infty<\theta<\infty$, $c=\delta_2-\delta_1$ and $\delta_i(i=1,2)$ are real numbers such that $\delta_1<\delta_2$. Based on a random sample $X_1, \ldots, X_n$ from $f(x|\theta)$ we consider the test of the hypothesis $H_0:\theta=\theta_0$ versus the alternative hypothesis $H_1:\theta\neq\theta_0$ for a constant θ_0. Let $X_{(i)}$ be the i-th smallest observation of $X_1, \ldots, X_n$. Let $V=X_{(1)}$ and $W=X_{(n)}$. We estimate θ by an unbiased estimate $Y=(V+W-\delta_0)/2$ with $\delta_0=\delta_1+\delta_2$. Let α be a real number such that $0<\alpha<1$ and let $r=c(1-\alpha^{1/n})/2$. According to Nogami(2001a) and Nogami(2001b) we refine the rejection region of (2.6) in Nogami(2001a) as follows:

$$(2) \qquad D=\{Y\leqq\theta_0-r,\ Y\geqq\theta_0+r,\ \theta_0+\delta_1<V<W<\theta_0+\delta_2\}\cup\{V<\theta_0+\delta_1,\ W-V<c\}\cup\{W\geqq\theta_0+\delta_2,\ W-V<c\}.$$

Our refined test is to reject H_0 if $(V,W)\in D$ and accept H_0 if $(V,W)\notin D$. We can see that this refinement makes the power higher than (3.1) of Nogami(2001a).

As a bibliography of the uniform hypothesis testing we refer to Chatterjee & Chattopadhyay(1994). There they considered the uniform distribution (1) with $\delta_1=0$ and $\delta_2=1$. The tests $\phi^0{}_1$, $\phi^0{}_2$, ϕ_L and $\phi^0{}_*$(by Pratt(1961)) in their paper consist of the same approach and the same direction of thinking. Especially, ϕ_L is nonsence. Even if ϕ_L is the locally best test, no statistician wants to use it in practical fields because the test ϕ_L has the rejection region in the center of the true hypothesis H_0. In commom sense if the sample point falls in such a region we will accept H_0. This kind of curiosity probably comes from making the test uniformly most powerful(UMP) mathematically and not thinking the practical use and the peculiar characteristic of the uniformity of the underlined distribution also makes this kind of curiosity possible. Although

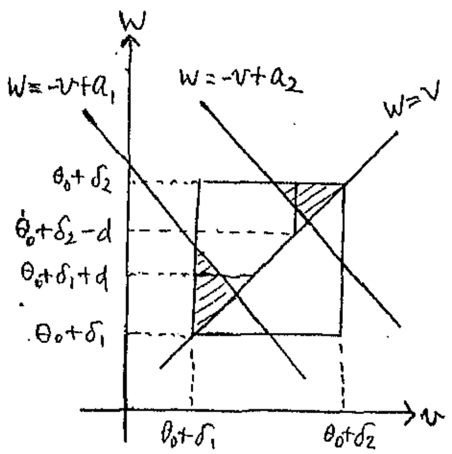
$\phi^0{}_1$ (see (5)) is UMP for the test of $H_0:\theta=\theta_0$ versus $H'_1:\theta>\theta_0$, the rejection region of $\phi^0{}_1$ covers all sample space for H_0 when n gets large. This means that $\phi^0{}_1$ rejects all the time for large n. This is curios. Even if $\phi^0{}_1$ is UMP of size α statisticians would not want to use this test. As far as our problem concerns the author feels that some other point of view rather than merely UMP is necessary.

According to Result 2.1 in Chatterjee and Chattopadhyay(1994) our two tests in Sections 2 and 3 are both unbiased. We see that our test (2) has a natural rejection region and has the same power as that for $\phi^0{}_*$ (see (4)) locally for n=5 and α=.05. Also, when we derive from (2) one-sided test for H_0 versus H'_1, our test has the same power as $\phi^0{}_1$ locally.

Let A be the acceptance region of the test. We call $P_\theta(A)$ the test-acceptance function of θ. In Sections 2 and 3 we assume that c(>0) is known. In Section 4 we assume that c is unknown. In Section 2 we show the test-acceptance function of θ for our refined test (2) and compare this with $\phi^0{}_*$. In Section 3 we show another test derived from (2) and compare it with $\phi^0{}_1$. Test-acceptance functions for $\phi^0{}_*$, $\phi^0{}_1$ and $\phi^0{}_2$ are shown in Appendix. In Section 4 we introduce a two-sided test for testing H_0 versus H_1 when c is unknown.

§2. The test-acceptance function for θ. Let $\bar{B}$ be the complement of the set B. In this section we compute $P_\theta(\bar{D})$ and compare it with that for the test $\phi^0{}_*$ in Pratt(1961). In order to do so we first find the joint density of (V,W) as follows:

(3) $$g(v,w)=n(n-1)c^{-n}(w-v)^{n-2} \quad \text{for } \theta+\delta_1<v<w<\theta+\delta_2.$$



In (2) the area $\{Y\leq\theta_0-r,\ Y\geq\theta_0+r,\ \theta_0+\delta_1<v<w<\theta_0+\delta_2\}$ is the shaded area in FIGURE 1 (here, $d<2^{-1}$ for $\delta_0=0$ and $\delta_1=1$). Here, $a_1=2(\theta_0+\delta_1)+c\alpha^{1/n}$ and $a_2=2(\theta_0+\delta_2)-c\alpha^{1/n}$. It is easy to see $P_{\theta_0}(D)=\alpha$. Since $1-P_\theta(\bar{D})=P_\theta\{V+W\leq a_1,\ V+W\geq a_2,\ \theta_0+\delta_1<V<W<\theta_0+\delta_2\}+P_\theta\{V<\theta_0+\delta_1, W-V<c\}+P_\theta\{W>\theta_0+\delta_2,\ W-V<c\}$, we compute these three probabilities separately and then add them finally. (See FIGURE 2.) Then, we obtain that when $0<\alpha^{1/n}<2/3$

FIGURE 1.

$$P_\theta(\bar{D})=\begin{cases} 0, & \text{for } \theta<\theta_0-2^{-1}c-r \\ 2^{-1}c^{-n}\{2(\theta-\theta_0+c)-c\alpha^{1/n}\}^n, & \text{for } \theta_0-2^{-1}c-r\leqq\theta<\theta_0-2r \\ c^{-n}(\theta-\theta_0+c)^n-2^{-1}\alpha, & \text{for } \theta_0-2r\leqq\theta<\theta_0-2^{-1}c+r \\ c^{-n}(\theta-\theta_0+c)^n-2^{-1}\alpha-2^{-1}c^{-n}\{2(\theta-\theta_0)+c\alpha^{1/n}\}^n, & \text{for } \theta_0-2^{-1}c+r\leqq\theta<\theta_0 \\ c^{-n}(\theta_0-\theta+c)^n-2^{-1}\alpha-2^{-1}c^{-n}\{2(\theta_0-\theta)+c\alpha^{1/n}\}^n, & \text{for } \theta_0\leqq\theta<\theta_0+2^{-1}c-r \\ c^{-n}(\theta_0-\theta+c)^n-2^{-1}\alpha, & \text{for } \theta_0+2^{-1}c-r\leqq\theta<\theta_0+2r \\ 2^{-1}c^{-n}\{2(\theta_0-\theta+c)-c\alpha^{1/n}\}^n, & \text{for } \theta_0+2r\leqq\theta<\theta_0+2^{-1}c+r \\ 0, & \text{for } \theta_0+2^{-1}c+r\leqq\theta \end{cases}$$

and when $2/3\leqq\alpha^{1/n}<1$,

$$P_\theta(\bar{D})=\begin{cases} 0, & \text{for } \theta<\theta_0-2^{-1}c-r \\ 2^{-1}c^{-n}\{2(\theta-\theta_0+c)-c\alpha^{1/n}\}^n, & \text{for } \theta_0-2^{-1}c-r\leqq\theta<\theta_0-2^{-1}c+r \\ 2^{-1}c^{-n}\{2(\theta-\theta_0+c)-c\alpha^{1/n}\}^n-2^{-1}c^{-n}\{2(\theta-\theta_0)+c\alpha^{1/n}\}^n, & \\ & \text{for } \theta_0-2^{-1}c+r\leqq\theta<\theta_0-2r \\ c^{-n}(\theta-\theta_0+c)^n-2^{-1}\alpha-2^{-1}c^{-n}\{2(\theta-\theta_0)+c\alpha^{1/n}\}^n, & \text{for } \theta_0-2r\leqq\theta<\theta_0 \\ c^{-n}(\theta_0-\theta+c)^n-2^{-1}\alpha-2^{-1}c^{-n}\{2(\theta_0-\theta)+c\alpha^{1/n}\}^n, & \text{for } \theta_0\leqq\theta<\theta_0+2r \\ 2^{-1}c^{-n}\{2(\theta_0-\theta+c)-c\alpha^{1/n}\}^n-2^{-1}c^{-n}\{2(\theta_0-\theta)+c\alpha^{1/n}\}^n, & \\ & \text{for } \theta_0+2r\leqq\theta<\theta_0+2^{-1}c-r \\ 2^{-1}c^{-n}\{2(\theta_0-\theta+c)-c\alpha^{1/n}\}^n, & \text{for } \theta_0+2^{-1}c-r\leqq\theta<\theta_0+2^{-1}c+r \\ 0, & \text{for } \theta_0+2^{-1}c+r\leqq\theta. \end{cases}$$

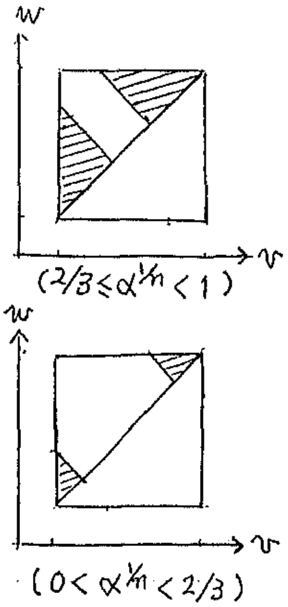


FIGURE 2.

Let $\delta_1=0$ and $\delta_2=1$ in (1) until the eighth line below (4). Let $\underline{x}=(x_1, \ldots, x_n)$. Define

$$(4)\qquad \phi^0_*(\underline{x})=\begin{cases}1, & \text{if } \{\underline{x}\notin[\theta_0,\theta_0+1) \text{ or } V\geq\theta_0-d+1 \text{ and/or } W\leq\theta_0+d\}\\ 0, & \text{Otherwise.}\end{cases}$$

FIGURE 1 shows the difference of two rejection regions, namely D and that for $\phi^0_*(\underline{x})$. We compare our test with $\phi^0_*(\underline{x})$ for $d\leq 1/2$. From (9) in Appendix we can easily see that when $n=5$ and $\alpha=.05$, $P_\theta(\overline{D})=E_\theta(1-\phi^0_*(\underline{X}))$ for $\theta_0-2^{-1}\alpha^{1/n}\leq\theta<\theta_0+2^{-1}\alpha^{1/n}$. However, most of the time d must be larger than 2^{-1} and in this case the test ϕ^0_* has the rejection region covering large area of the sample space for H_0. The author is suspicious to take such rejection region.

On the other hand, our acceptance region is $\theta_0+2^{-1}\delta_0-r<2^{-1}(V+W)<\theta_0+2^{-1}\delta_0+r$ in the sample space for H_0. Since as $n\to\infty$ $r\to 0$, for large n $2^{-1}(V+W)$ accumulates at the center of the interval $[\theta_0+\delta_1, \theta_0+\delta_2)$. To avoid this accumulation we may need to devide the test statistic by some scale factor.

In the next section we show another rejection region for testing $H_0:\theta=\theta_0$ versus $H_1:\theta\neq\theta_0$ and compare it with ϕ^0_1.

§3. Another test-acceptance function for θ. Let $s=2^{-1}c(1-(2\alpha)^{1/n})$ and assume $0<2\alpha<1$. In this section we construct the another rejection region for the test of H_0 versus H_1 as follows:

$$D_1=\{Y\geq\theta_0+s,\ \theta_0+\delta_1<V<W<\theta_0+\delta_2\}\cup\{V\leq\theta_0+\delta_1,\ W-V<c\}\cup\{W\geq\theta_0+\delta_2,\ W-V<c\}.$$

It is easy to see that $P_{\theta_0}(D_1)=\alpha$. In the similar way to Section 2 we compute $P_\theta(\overline{D}_1)$ to get

$$P_\theta(\widetilde{D}_1)=\begin{cases} 0, & \text{for } \theta<\theta_0-c \\ c^{-n}(\theta-\theta_0+c)^n, & \text{for } \theta_0-c\leqq\theta<\theta_0-2^{-1}c+s \\ c^{-n}(\theta-\theta_0+c)^n+2^{-1}c^{-n}\{2(\theta-\theta_0)+c(2\alpha)^{1/n}\}^n, & \text{for } \theta_0-2^{-1}c+s\leqq\theta<\theta_0 \\ c^{-n}(\theta_0-\theta+c)^n-\alpha, & \text{for } \theta_0\leqq\theta<\theta_0+2s \\ 2^{-1}c^{-n}\{2(\theta_0-\theta+c)-c(2\alpha)^{1/n}\}^n, & \text{for } \theta_0+2s\leqq\theta<\theta_0+2^{-1}c+s \\ 0, & \text{for } \theta_0+2^{-1}c+s\leqq\theta. \end{cases}$$

Let $\delta_1=0$ and $\delta_2=1$ in (1) until the end of this section. $\phi^0{}_1$ is the test given by

$$\text{(5)}\qquad \phi^0{}_1(\underset{\sim}{x})=\begin{cases} 1, & \text{if } \underset{\sim}{x}\notin[\theta_0,\theta_0+1) \text{ or } \theta_0+1-\alpha^{1/n}\leqq V<W<\theta_0+1, \\ 0, & \text{otherwise.} \end{cases}$$

From (10) in Appendix $P_\theta(\widetilde{D}_1)=E_\theta(1-\phi^0{}_1(\underset{\sim}{X}))$ for $\theta\leqq\theta<\theta_0+1-(2\alpha)^{1/n}$. So, our test behaves as good as $\phi^0{}_1$, locally. Consider the test of $H_0:\theta=\theta_0$ versus $H'_1:\theta>\theta_0$. Since as $n\to\infty$ $\alpha^{1/n}\to1$, the rejection region of $\phi^0{}_1$ covers all sample space for H_0 when n gets large. This means that $\phi^0{}_1$ rejects H_0 all the time for large n. On the other hand, our test rejects H_0 if $2^{-1}(V+W)\geqq\theta_0+2^{-1}\delta_0+2^{-1}(1-(2\alpha)^{1/n})$ in the sample space for H_0. So, the rejection region becomes $2^{-1}(V+W)\geqq\theta_0+2^{-1}\delta_0+\Delta$ (with small $\Delta>0$) for large n. It seems that this is more natural than $\phi^0{}_1$. (Same argument holds between our test and $\phi^0{}_2$ for the test of H_0 versus $H''_1:\theta<\theta_0$. We merely put the test-acceptance function of θ for $\phi^0{}_2$ in Appendix.)

In the next section we consider the two-sided test when c is unknown.

§4. Two-sided test when c is unknown. Let $Z=W-V$. For unknown $c(>0)$ we use unbiased estimate $(n+1)(n-1)^{-1}Z$ for c. To test $H_0:\theta=\theta_0$ versus $H_1:\theta\neq\theta_0$ we use the test statistic

(6) $$T=\{2^{-1}(V+W)-2^{-1}\delta_0-\theta\}/\{(n+1)(n-1)^{-1}Z\}.$$

We try to find the distribution of T.

Let $U=V+W$. By variable transformations we have from (3) that

(7) $$h_{U,Z}(u,z)=n(n-1)2^{-1}c^{-n}z^{n-2},\quad \text{for } \theta+\delta_1<2^{-1}(u-z)<2^{-1}(u+z)<\theta_0+\delta_2.$$

Again, using variable transformations we obtain from (7) that

(8) $$h_{T,Z}(t,z)=n(n+1)c^{-n}z^{n-1},$$

for $0\leq|t|\leq c(n-1)\{2(n+1)z\}^{-1}-(n-1)\{2(n+1)\}^{-1}$ and $0\leq z\leq c$. Since the range of (t,z) in above (8) is also expressed by $0\leq z\leq c\{2(n+1)(n-1)^{-1}|t|+1\}^{-1}$ and $0\leq|t|<\infty$, integrating out z from (8) we obtain

$$h_T(t)=(n+1)\{2(n+1)(n-1)^{-1}|t|+1\}^{-n},\quad \text{for } 0\leq|t|<\infty.$$

From the symmetricity of $h_T(t)$ at the origin it is enough to obtain t_0 so that

$$\int_{t_0}^{\infty} h_T(t)\,dt=\alpha/2.$$

Then, we get $t_0=(n-1)\{2(n+1)\}^{-1}(\alpha^{-1/(n-1)}-1)$. Thus, the acceptance region of our two-sided test is given by

$$-t_0<\{2^{-1}(V+W)-2^{-1}\delta_0-\theta_0\}/\{(n+1)(n-1)^{-1}Z\}<t_0$$

or equivalently,

$$2^{-1}(V+W)-2^{-1}\delta_0-t_0(n+1)(n-1)^{-1}Z<\theta_0<2^{-1}(V+W)-2^{-1}\delta_0+t_0(n+1)(n-1)^{-1}Z.$$

Here, since as $n\to\infty$ $t_0\to0$, T by (5) with $\theta=\theta_0$ accumulates at the origin when $n\to\infty$.

§5. Appendix. Here, we show the test-acceptance functions of $\phi^0{}_*$, $\phi^0{}_1$ and $\phi^0{}_2$. We can easily obtain that when $d \leqq 1/2$,

$$(9) \qquad E_\theta\{1-\phi^0{}_*(\underset{\sim}{X})\}=\begin{cases} 0, & \text{for } \theta<\theta_0+d-1 \\ (\theta+1-\theta_0)^n-d^n, & \text{for } \theta_0+d-1\leqq\theta<\theta_0-d \\ (\theta+1-\theta_0)^n-(\theta-\theta_0+d)^n-d^n, & \text{for } \theta_0-d\leqq\theta<\theta_0 \\ (\theta_0+1-\theta)^n-(\theta_0-\theta+d)^n-d^n, & \text{for } \theta_0\leqq\theta<\theta_0+d \\ (\theta_0+1-\theta)^n-d^n, & \text{for } \theta_0+d\leqq\theta<\theta_0-d+1 \\ 0, & \text{for } \theta_0-d+1\leqq\theta \end{cases}$$

and when $d>1/2$,

$$E_\theta(1-\phi^0{}_*(\underset{\sim}{X}))=\begin{cases} 0, & \text{for } \theta<\theta_0+d-1 \\ (\theta+1-\theta_0)^n-(\theta-\theta_0+d)^n-d^n+(2d-1)^n, & \text{for } \theta_0+d-1\leqq\theta<\theta_0 \\ (\theta_0+1-\theta)^n-(\theta_0-\theta+d)^n-d^n+(2d-1)^n, & \text{for } \theta_0\leqq\theta<\theta_0-d+1 \\ 0, & \text{for } \theta_0-d+1\leqq\theta. \end{cases}$$

$$(10) \qquad E_\theta(1-\phi^0{}_1(\underset{\sim}{X}))=\begin{cases} 0, & \text{for } \theta<\theta_0-1 \\ (\theta+1-\theta_0)^n, & \text{for } \theta_0-1\leqq\theta<\theta_0-\alpha^{1/n} \\ (\theta+1-\theta_0)^n-(\theta-\theta_0+\alpha^{1/n})^n, & \text{for } \theta_0-\alpha^{1/n}\leqq\theta<\theta_0 \\ (\theta_0+1-\theta)^n-\alpha, & \text{for } \theta_0\leqq\theta\leqq\theta_0+1-\alpha^{1/n} \\ 0, & \text{for } \theta_0+1-\alpha^{1/n}\leqq\theta. \end{cases}$$

$$
E_\theta(1-\phi^0_2(\underline{X}))=\begin{cases}0, & \text{for } \theta<\theta_0+\alpha^{1/n}-1\\ (\theta+1-\theta_0)^n-\alpha, & \text{for } \theta_0+\alpha^{1/n}-1\leqq\theta<\theta_0\\ (\theta_0+1-\theta)^n-(\theta_0+\alpha^{1/n}-\theta)^n, & \text{for } \theta_0\leqq\theta<\theta_0+\alpha^{1/n}\\ (\theta_0+1-\theta)^n, & \text{for } \theta_0+\alpha^{1/n}\leqq\theta<\theta_0+1\\ 0, & \text{for } \theta_0+1<\theta.\end{cases}
$$

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