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Hypothesis Testing Based on Lagrange's Method
---An Application to The Location Parameter of The Exponential Distribution---

by

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Hypothesis testing based on Lagrange's method
---An application to the location parameter of the exponential distribution---

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Abstract.

In this paper the author considers the exponential distribution with density

$$f(x|\theta)=\begin{cases}(1/b)e^{-(x-\theta)/b}, & \text{for } \theta<x<\infty \\ 0, & \text{otherwise}\end{cases}$$

where $-\infty<\theta<\infty$ and $b>0$.

Based on i.i.d. observations $X_1, \ldots, X_n$ from above density with b=1 she constructs the unbiased two-sided test based on Lagrange's method for testing the hypothesis $H_0:\theta=\theta_0$ versus the alternative hypothesis $H_1:\theta\neq\theta_0$ with some constant θ_0, and derives the one-sided test for testing $H_0:\theta\geqq\theta_0$ versus $H_1:\theta<\theta_0$. For $\theta<\theta_0$ the powers of our tests are equal to $\beta^*(a_1)$ with a_1, a_0 and b there replaced by θ, θ_0 and 1, respectively shown in Problem 3(ii) of page 112 of E. L. Lehmann[1].

The author also comments on another two-sided test when b is unknown.

§1. Introduction.

The author has been considering goodness of the two-sided test with the acceptance region derived from inverting the shortest interval estimate for the parameter of the underlined distribution. The shortest interval estimates are constructed by using the method of Lagrange's multiplier. (See e.g. [2], [3], [4], [5], [6].) This paper is on the same line as such researches. The author would like to call these tests as the tests based on Lagrange's method.

Let $I_A(x)$ be an indicator function so that for a set A $I_A(x)=1$ if $x \in A$; $=0$ if $x \notin A$. In this paper we consider as the underlined distribution the exponential distribution with the density

(1) $$f(x|\theta)=(1/b)e^{-(x-\theta)/b}I_{[\theta,\infty)}(x)$$

where $-\infty<\theta<\infty$ and $b>0$. In Section 2 we assume $b=1$ and introduce an unbiased two-sided test based on Lagrange's method for the problem of testing the hypothesis $H_0:\theta=\theta_0$ versus the alternative hypothesis $H_1:\theta\neq\theta_0$ for some constant θ_0. In Section 3 we consider the uniformly most powerful(UMP) two-sided test shown in Problem 3(i) of page 112 of E. L. Lehmann[1] and compare its power with that of our test. We see that when $\theta<\theta_0$ our test has the same power as his and also his test is not useful for constructing one-sided test for testing $H_0:\theta\geqq\theta_0$ versus $H_1:\theta<\theta_0$. We also see that when b is unknown deviding our test by the same divisor as the UMP test in his Problem 3(iii) leads to the same UMP test. In Section 4 we propose the one-sided test for testing $H_0:\theta\geqq\theta_0$ versus $H_1:\theta<\theta_0$ and see that this test is UMP unbiased.

§2. An unbiased two-sided test.

In this section we let $X_1, \ldots, X_n$ be a random sample of size n taken from $f(x|\theta)$ given by (1) with b=1. We consider the problem of testing $H_0:\theta=\theta_0$ versus $H_1:\theta\neq\theta_0$. To construct the test we first find the shortest interval estimate for θ using the Lagrange's method and then construct the acceptance region derived from inverting this interval estimate for θ_0.

To get an estimate for θ we take $Y=\bar{X}-1$ where $\bar{X}=\Sigma_{i=1}^{n}X_i/n$. We can easily check $E(Y)=\theta$. Let $X_{(i)}$ be the i-th smallest observation. Then, $X_{(1)}\leqq X_{(2)}\leqq\ldots\leqq X_{(n)}$. We first find the joint density of variables $W=X_{(1)}+X_{(2)}+\ldots+X_{(n)}$ $(=X_1+\ldots+X_n)$, $V=X_{(1)}$, $Z_2=X_{(2)}$, ... ,and $Z_{n-1}=X_{(n-1)}$ as follows:

$$g(w,v,z_2,\ \ldots\ ,z_{n-1})=\begin{cases} n!e^{-(w-n\theta)}, & \text{for } \theta\leqq v\leqq z_2\leqq \ \ldots\ \leqq z_{n-1}\leqq w-\Sigma_{i=1}^{n-1}z_i \\ 0, & \text{otherwise.}\end{cases}$$

Integrating out the above density with respect to z_2 through z_{n-1} we get the marginal density of (w,v) as follows:

(2) $$g(w,v|\theta)=\begin{cases}(n/\Gamma(n-1))e^{-(w-n\theta)}(w-nv)^{n-2}, & \text{for } \theta\leqq v\leqq w/n<\infty \\ 0, & \text{otherwise.}\end{cases}$$

Taking the marginal density $g_W(w|\theta)$ of W we obtain

(3) $$g_W(w|\theta)=(1/\Gamma(n))e^{-(w-n\theta)}(w-n\theta)^{n-1}I_{[n\theta,\infty)}(w).$$

Furthermore, letting $t=2n(n^{-1}w-\theta)$ we have the density of T so that

(4) $$h_T(t)=(1/\Gamma(n))e^{-t/2}t^{n-1}2^{-n}I_{[0,\infty)}(t)$$

which is the Chi-square distribution with 2n degrees of freedom.

Let α be a real number such that $0<\alpha<1$. We call a random interval (U_1,U_2) as a $(1-\alpha)$ interval estimate for θ if $P_\theta(U_1<\theta<U_2)=1-\alpha$. Let r_1 and r_2 be real numbers such that $r_1<r_2$. To find the shortest $(1-\alpha)$ interval estimate for θ we want to minimize r_2-r_1 subject to

(5) $$P_\theta[r_1<Y-\theta<r_2]=1-\alpha.$$

But, it follows by a variable transformation $t=2n(y+1-\theta)$ that

(6) the left hand side of $(5)=P_\theta[2n(r_1+1)<T<2n(r_2+1)]=1-\alpha$.

Hence, we want to minimize t_2-t_1 with $t_i=2n(r_i+1)$ $(i=1,2)$ subject to the condition (6). Let γ be a real number and define

$$L\doteq L(t_1,t_2;\gamma)\doteq t_2-t_1-\gamma\{\int_{t_1}^{t_2} h_T(t)\,dt\ -1+\alpha\}$$

where $h_T(t)$ is defined by (4). Then, by the Lagrange's method we have that

$$(7)\quad \begin{cases} \partial L/\partial t_1 = -1+\gamma h_T(t_1)=0 \\ \partial L/\partial t_2 = 1-\gamma h_T(t_2)=0. \end{cases}$$

By (7) we get

$$(8)\qquad h_T(t_1)=h_T(t_2)\ \ (=\gamma^{-1}).$$

Taking t_1 and t_2 which satisfy (8) and (6) and noticing that $t_1<T=2n(Y+1-\theta)<t_2$ we obtain the $(1-\alpha)$ interval estimate $(Y+1-t_4, Y+1-t_3)$ for θ with $t_3=t_1/(2n)$ and $t_4=t_2/(2n)$.

Hence, our test is to reject H_0 if $Y\leqq\theta_0+t_3-1$ or $Y\geqq\theta_0+t_4-1$ or $V<\theta_0$ and to accept H_0 if $\theta_0+t_3-1<Y<\theta_0+t_4-1$ and $V\geqq\theta_0$. Here, we emphasize on the necessity of having the set $\{V<\theta_0\}$ in the rejection region. Using a test function we can write this test by

$$\phi(y,v)=\begin{cases} 1, & \text{if } y\leqq\theta_0+t_3-1 \text{ or } \theta_0+t_4-1\leqq y \text{ or } v<\theta_0 \\ 0, & \text{if } \theta_0+t_3-1<y<\theta_0+t_4-1 \text{ and } v\geqq\theta_0. \end{cases}$$

To check unbiasedness of this test we obtain the power function as follows:

$$\pi(\theta)=E_\theta(\phi(Y,V))=P_\theta[Y\leqq\theta_0+t_3-1 \text{ or } \theta_0+t_4-1\leqq Y \text{ or } V<\theta_0]$$

$$(9)\qquad =\begin{cases} 1-(1-\alpha)e^{-n(\theta_0-\theta)}, & \text{for } \theta<\theta_0 \\ \int_0^{t_1-2n(\theta-\theta_0)} h_T(t)\,dt + \int_{t_2-2n(\theta-\theta_0)}^{\infty} h_T(t)\,dt, & \text{for } \theta_0\leqq\theta<\theta_0+t_3 \\ \int_{t_2-2n(\theta-\theta_0)}^{\infty} h_T(t)\,dt, & \text{for } \theta_0+t_3\leqq\theta<\theta_0+t_4 \\ 1, & \text{for } \theta_0+t_4\leqq\theta. \end{cases}$$

Hence, $d\pi(\theta)/d\theta<0$ for $\theta<\theta_0$; $d\pi(\theta)/d\theta=2n\{h_T(t_2-2n(\theta-\theta_0))-h_T(t_1-2n(\theta-\theta_0))\}>0$ for $\theta_0<\theta<\theta_0+t_3$ because of (8) and the form of $h_T(t)$; $d\pi(\theta)/d\theta=h_T(t_2-2n(\theta-\theta_0))>0$ for $\theta_0+t_3\leqq\theta<\theta_0+t_4$. Since $\pi'(\theta_0)=0$ by (8) and $\pi(\theta_0)=\alpha$, we have that $\pi(\theta)\geqq\alpha$ for real θ. Thus, unbiasedness of the test is proved.

Furthermore, we note that (9) for $\theta<\theta_0$ is equal to $\beta^*(a_1)$ with a_1, a_0 and b replaced by θ, θ_0 and 1, respectively in Problem 3(ii) of page 112 of E. L. Lehmann[1].

In the next section we compare our test with the UMP test proposed in Problem 3(i) of page 112 of E. L. Lehmann[1] when b=1 and see that when b is unknown deviding our test by the same divisor as the UMP test in his Problem 3(iii) leads to the same UMP test.

§3. Comparison with the UMP tests in E. L. Lehmann[1].

In this section we consider the UMP tests in Problem 3(i) and (iii) of page 112 of E. L. Lehmann[1]. We assume a_1 and a_0 there to be θ and θ_0, respectively. We also assume the density determined by (1). Based on a random sample $X_1, \ldots, X_n$ from the density (1) we test the hypothesis $H_0:\theta=\theta_0$ versus the alternative hypothesis $H_1:\theta\neq\theta_0$. Let $V=X_{(1)}$ as in Section 2.

We first consider the solution of Problem 3(i). Assume b=1 until the 4-th line from the top of page 6. Let α be a real number such that $0<\alpha<1$. Since e^{-x} has the uniform distribution on $[0, e^{-\theta}]$ we obtain the UMP two-sided test as follows:

$$\phi(v)=\begin{cases}1, & \text{when } V<\theta_0 \text{ or } \theta^*\leqq V\\ 0, & \text{otherwise}\end{cases}$$

where $\theta^*=\theta_0-n^{-1}\ln\alpha$. Hence, the power function of this test is given by

$$\pi_*(\theta)=E_\theta(\phi(V))=P_\theta[V<\theta_0 \text{ or } \theta^*\leqq V]$$

$$=\begin{cases}1-(1-\alpha)e^{-n(\theta_0-\theta)}, & \text{for } \theta<\theta_0\\ \alpha e^{-n(\theta_0-\theta)}, & \text{for } \theta_0\leqq\theta<\theta^*\\ 1, & \text{for } \theta^*\leqq\theta.\end{cases}$$

For $\theta<\theta_0$, $\pi_*(\theta)$ is equal to (9), but, for $\theta>\theta_0$, $\pi_*(\theta)$ is higher than (9). However, we cannot construct the one-sided test for testing $H_0:\theta\geqq\theta_0$ versus $H_1:\theta<\theta_0$ from this approach, which is because the test takes the probability of size α from upper tail only.

Secondly, we consider Problem 3(iii). Here, we assume b to be unknown. The two-sided test introduced in Problem 3(iii) is to reject H_0 if

$$U \doteq (V-\theta_0)/(\Sigma_{i=1}^{n} X_i - nV) \leqq C_1 \quad \text{or} \quad \geqq C_2$$

where C_1 and C_2 are some constants. From Section 2 we can propose another two-sided test which rejects H_0 if

$$\text{(10)} \qquad S \doteq (Y+1-\theta_0)/(\Sigma_{i=1}^{n} X_i - nV) \leqq C_1^* \quad \text{or} \quad \geqq C_2^*$$

where $Y \doteq \bar{X}-1$ as in Section 2 and C_1^* and C_2^* are some constants. Since, from the definition of Y, $S=(\bar{X}-\theta_0)/(\Sigma_{i=1}^{n} X_i - nV)$, we have the relation $S=U+n^{-1}$. Therefore, from Problem 3(iv) of page 112 of E. L. Lehmann[1], the two-sided test(10) is also UMP level-α test of $H_0:\theta=\theta_0$ versus $H_1:\theta\neq\theta_0$.

In the next section we introduce the UMP unbiased one-sided test for testing $H_0:\theta\geqq\theta_0$ versus $H_1:\theta<\theta_0$ when b=1.

§4. The UMP unbiased one-sided test.

In this section we consider the problem of testing $H_0:\theta\geqq\theta_0$ versus $H_1:\theta<\theta_0$ based on a random sample $X_1, \ldots, X_n$ from the density (1) with b=1. Let Y and V be as defined in Section 2. Let α be a real number so that $0<\alpha<1$ and $h_T(t)$ be defined by (4). Our test is to reject H_0 when $Y\leqq\theta_0+t_6-1$ or $V<\theta_0$ where $t_6 = t_5/(2n)$ and t_6 is determined by

$$\int_0^{t_6} h_T(t)\,dt=\alpha.$$

Let

$$\text{(11)} \qquad \phi(y,v)=\begin{cases} 1, & \text{if } y\leqq\theta_0+t_6-1 \text{ or } v<\theta_0 \\ 0, & \text{if } \theta_0+t_6-1<y \text{ and } \theta_0\leqq v. \end{cases}$$

From (2), we can easily get the joint density of (Y, V) as follows:

$$g(y,v|\theta)=\begin{cases}(n^2/\Gamma(n-1))e^{-n(y+1-\theta)}\{n(y+1-v)\}^{n-2}, & \text{for } \theta<v<y+1<\infty \\ 0, & \text{otherwise.}\end{cases}$$

Thus, we can easily compute the power function of above test(11) as follows:

$$\pi(\theta)=E_\theta(\phi(Y,V))$$

$$(12)\qquad =\begin{cases}1-(1-\alpha)e^{-n(\theta_0-\theta)}, & \text{for } \theta<\theta_0 \\ \int_0^{t_6-2n(\theta-\theta_0)} h_T(t)\,dt, & \text{for } \theta_0\leqq\theta<\theta_0+t_6 \\ 0, & \text{for } \theta_0+t_6\leqq\theta.\end{cases}$$

Hence, $d\pi(\theta)/d\theta<0$ for $\theta<\theta_0+t_6$ and $\sup_{\theta_0\leq\theta}\pi(\theta)=\alpha$. Since $\pi(\theta)\geqq\alpha=\pi(\theta_0)$ for $\theta<\theta_0$, the test (11) is unbiased and furthermore UMP because from (12) $\pi(\theta)$ for $\theta<\theta_0$ is equal to $\beta^*(a_1)$ with a_1, a_0 and b there replaced by θ, θ_0 and 1, respectively in Problem 3(ii) of page 112 of E. L. Lehmann[1].

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