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by

Kazutoshi Ando

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UNIVERSITY OF TSUKUBA
Tsukuba, Ibaraki 305
JAPAN

Möbius Functions on Rooted Forests and Their Applications to Faigle-Kern's Dual Greedy Polyhedra

Kazutoshi Ando*
Institute of Policy and Planning Sciences
University of Tsukuba
Tsukuba, 305-8573 Ibaraki, Japan

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Abstract

A poset is called a rooted forest if every element has at most one cover. We will show that the Möbius function on a rooted forest maps the (order) ideals to the antichains. Equipped with this linear mapping, the linear programming problem over Faigle-Kern's dual greedy polyhedra can be reduced to those over submodular polyhedra in the ordinary sense. Here, Möbius functions connects these problems in the space of linear programming dual. Using this framework, Intersection Theorem on dual greedy polyhedra can also be reduced to that on submodular polyhedra. Furthermore, we show a new min-max theorem concerning intersection of two such dual greedy polyhedra.

1. Introduction

Let $P = (V, \preceq)$ be a finite poset. A subset $I \subseteq V$ is called an *ideal* of P if $v \preceq w \in I$ implies $v \in I$. We denote by $\mathcal{I}(P)$ the set of ideals of P. It is well-known that $\mathcal{I}(P)$ ordered by set-inclusion forms a distributive lattice with lattice operations being set-union $\cup$ and intersection $\cap$.

Let us denote by $\mathbb{R}$ the set of reals. A function $f : \mathcal{I}(P) \to \mathbb{R}$ on $\mathcal{I}(P)$ is called *submodular* if

$$f(I) + f(J) \geq f(I \cup J) + f(I \cap J) \tag{1.1}$$

holds for each $I, J \in \mathcal{I}(P)$. The pair $(\mathcal{I}(P), f)$ of the set $\mathcal{I}(P)$ of ideals and a submodular function f on $\mathcal{I}(P)$ is called a *submodular system* (cf. [8]) on V. The *submodular polyhedron* $\mathrm{P}(f)$ associated with $(\mathcal{I}(P), f)$ is defined as

$$\mathrm{P}(f) = \{x \mid x \in \mathbb{R}^V, x(I) \leq f(I) \ (I \in \mathcal{I}(P))\}, \tag{1.2}$$

where $x(X) = \sum_{v \in X} x(v)$ for $x \in \mathbb{R}^V$ and $X \subseteq V$. Linear programming problem

$$\left|\begin{array}{lll} \max & \sum\{d(v)x(v) \mid v \in V\} & \\ \text{s.t.} & x(I) \leq f(I) & (I \in \mathcal{I}(P)), \end{array}\right. \tag{1.3}$$

*Email: ando@sk.tsukuba.ac.jp

over submodular polyhedra can be solved by a greedy algorithm (see, e.g., Fujishige [8]). Indeed, the submodularity may be characterized by the validity of the greedy algorithm.

A subset $A \subseteq V$ is an *antichain* of P if $v, w \in A, v \preceq w$ implies $v = w$. We denote by $\mathcal{A}(P)$ the set of antichains of P. For any $A \in \mathcal{A}(P)$ define

$$\mathrm{id}(A) = \{v \mid v \in V, \exists w \in A : v \preceq w\}. \tag{1.4}$$

We call $\mathrm{id}(A)$ the *ideal generated by* A. Conversely, given an ideal I of P define $\max(I)$ by

$$\max(I) = \{v \mid v \in I, \nexists w \in I : v \prec w\}, \tag{1.5}$$

i.e., $\max(I)$ is the set of the maximal elements of the poset $P[I]$ induced by I. It is straightforward to see the mappings $\mathrm{id}: \mathcal{A}(P) \to \mathcal{I}(P)$ and $\max: \mathcal{I}(P) \to \mathcal{A}(P)$ are one-to-one correspondences, and indeed, mapping max is the inverse of mapping id. Since $\mathcal{I}(P)$ is a distributive lattice with lattice operations $\cup$ and $\cap$, the isomorphism $\max: \mathcal{I}(P) \to \mathcal{A}(P)$ induces lattice operations $\vee$ and $\wedge$ given by

$$A \vee B = \max(\mathrm{id}(A) \cup \mathrm{id}(B)), \tag{1.6}$$
$$A \wedge B = \max(\mathrm{id}(A) \cap \mathrm{id}(B)). \tag{1.7}$$

Also, this isomorphism induces the associated partial order $\preceq$ on $\mathcal{A}(P)$: we have for $A, B \in \mathcal{A}(P)$ $A \preceq B$ if and only if $\mathrm{id}(A) \subseteq \mathrm{id}(B)$.

We call a function $f: \mathcal{A}(P) \to \mathbb{R}$ *submodular* if for each $A, B \in \mathcal{A}(P)$ we have

$$f(A) + f(B) \geq f(A \vee B) + f(A \wedge B). \tag{1.8}$$

It follows from the above isomorphism between lattices $(\mathcal{I}(P); \cup, \cap)$ and $(\mathcal{A}(P); \vee, \wedge)$ that a submodular function $f: \mathcal{I}(P) \to \mathbb{R}$ on $\mathcal{I}(P)$ can be made correspond to a unique submodular function $f': \mathcal{A}(P) \to \mathbb{R}$ on $\mathcal{A}(P)$ in the following manner:

$$f'(A) = f(\mathrm{id}(A)) \quad (A \in \mathcal{A}(P)), \tag{1.9}$$

and vice versa. Therefore, there will be no confusion to use the same symbol f to denote two functions on $\mathcal{I}(P)$ and on $\mathcal{A}(P)$.

For a poset $P = (V, \preceq)$ and $v \in V$ an element $w \in V$ is called a *cover* of v if $v \preceq w$ and there is no $u \in V$ such that $v \prec u \prec w$. In this case, we also say *w covers v*. A poset is called a *rooted forest* if for each $v \in V$ there is at most one cover of v. Faigle and Kern [5] considered a polyhedron

$$\mathrm{Q}(f) = \{x \mid x \in \mathbb{R}^V, x(A) \leq f(A) \ (A \in \mathcal{A}(P))\} \tag{1.10}$$

and a linear program over $\mathrm{Q}(f)$:

$$\left| \begin{array}{lll} \max & \sum\{c(v)x(v) \mid v \in V\} & \\ \text{s.t.} & x(A) \leq f(A) & (A \in \mathcal{A}(P)). \end{array} \right. \tag{1.11}$$

They showed that a dual greedy algorithm works for Problem (1.11) if $P = (V, \preceq)$ is a rooted forest and $f: \mathcal{A}(P) \to \mathbb{R}$ is submodular. In case that $P = (V, \preceq)$ is the trivial order, i.e., V itself is an antichain, submodularity is equivalent to that in the ordinary sense and the polyhedron $\mathrm{Q}(f)$ is a polymatroid.

The transportation problems with Monge cost functions (cf. [3]) can be formulated as the linear programming dual problem for Problem (1.11) and it was shown in [5] that the famous Northwest Corner Rule is nothing but the dual greedy algorithm for the problem. Also, applications of submodular functions on $\mathcal{A}(P)$ and associated polyhedra in a game theoretic framework can be found in [7].

This paper is addressed to the question: How are $\mathrm{P}(f)$ and $\mathrm{Q}(f)$ related? We give a partial answer for it: When the underlying poset is a rooted forest, they are equivalent via the Möbius transformation in the spaces of linear programming dual. Although the equivalence was already stated in in a recent article of Faigle and Kern [6], our treatment using Moöbius transformation is more explicit and make this equivalence look more transparent. Also, the Möbius transformation gives a simple proof for another result of Faigle and Kern [7]: given two submodular functions $f_1, f_2 : \mathcal{A}(P) \to \mathbb{R}$ with P being a rooted forest the system

$$x(A) \le \min\{f_1(A), f_2(A)\} \quad (A \in \mathcal{A}(P)) \tag{1.12}$$

is totally dual integral. This can be easily shown to follow from the Intersection Theorem for submodular system [4] (see also [8]) through the Möbius transformation. Furthermore, we will show a min-max theorem concerning intersection $\mathrm{Q}(f_1) \cap \mathrm{Q}(f_2)$ of polyhedra associated with submodular functions $f_1, f_2 : \mathcal{A}(P) \to \mathbb{R}$.

This paper is organized as follows. In Section 2, we investigate the Möbius functions and their inverses Zeta-functions on finite posets viewed as linear mapping on $\mathbb{R}^V$ and show that if P is a rooted forest, the Möbius function for P maps ideals to antichains. Equipped with this observation, in Section 3, we will see a simple reduction of Problem (1.11) to Problem (1.3) and a simple derivation of the total dual integrality of the system (1.12). The min-max theorem stated above will be proved there. Finally, in Section 4, the Möbius transformations associated with general posets will be discussed.

2. Möbius function on rooted forests

Let $P = (V, \preceq)$ be a poset. Define the *Zeta-function* $\zeta : V \times V \to \mathbb{Z}$ for P by

$$\zeta(v, w) = \begin{cases} 1 & \text{if } v \le w, \\ 0 & \text{otherwise} \end{cases} \quad ((v, w) \in V \times V). \tag{2.1}$$

The *Möbius function* $\mu : V \times V \to \mathbb{Z}$ for P is defined recursively as

$$\mu(v, w) = \begin{cases} 1 & \text{if } v = w, \\ -\sum_{v \preceq u \prec w} \mu(v, u) & \text{if } v \prec w, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

We consider a mapping $\alpha : V \times V \to \mathbb{R}$ as a linear mapping $\alpha : \mathbb{R}^V \to \mathbb{R}^V$ in the following sense: for $x \in \mathbb{R}^V$ $\alpha x \in \mathbb{R}^V$ is defined as

$$(\alpha x)(v) = \sum_{w \in V} \alpha(v, w) x(w) \quad (v \in V). \tag{2.3}$$

Then, it is well-known that

Proposition 2.1 (Möbius Inversion (cf. [1])): *The product $\zeta\mu$ is the identity mapping, that is, for each $(v, w) \in V \times V$ it holds that*

$$\sum_{u \in V} \zeta(v, u) \mu(u, w) = \begin{cases} 1 & \text{if } v = w, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

□

We call an ideal I of P *principal* if I is generated by a singleton set, i.e., $I = \mathrm{id}(v)$ for some $v \in V$. The Möbius function for P maps a principal ideal to a singleton antichain, and conversely, the Zeta-function maps a singleton antichain to a principal ideal as the following proposition shows. For $X \subseteq V$ we define its *characteristic vector* $\chi_X\colon V \to \mathbb{R}$ by $\chi_X(v) = 1$ if $v \in X$ and $\chi_X(v) = 0$ otherwise.

Proposition 2.2: *Let $P = (V, \preceq)$ be a poset. For a principal ideal I of P, we have*

$$\mu\chi_I = \chi_{\max(I)}. \tag{2.5}$$

Conversely, for each $v \in V$ we have

$$\zeta\chi_v = \chi_{\mathrm{id}(v)}. \tag{2.6}$$

(Proof) Let $I = \mathrm{id}(v)$ for some $v \in V$. Then, we have $\max(I) = \{v\}$ and for each $w \in V$

$$\begin{aligned} (\mu\chi_I)(w) &= \sum_{u \in V} \mu(w,u)\chi_I(u) && (2.7)\\ &= \sum_{w \preceq u} \mu(w,u)\chi_I(u) && (2.8)\\ &= \sum_{w \preceq u \in I} \mu(w,u) && (2.9)\\ &= \sum_{w \preceq u \leq v} \mu(w,u) && (2.10)\\ &= \begin{cases} 1 & \text{if } w = v, \\ 0 & \text{otherwise} \end{cases} && (2.11)\\ &= \chi_v(w). && (2.12) \end{aligned}$$

The latter assertion follows from the Möbius Inversion (Proposition 2.1) and (2.5). □

Lemma 2.3: *For a poset $P = (V, \preceq)$ P is a rooted forest if and only if each ideal of P is a disjoint union of principal ideals.*

(Proof) ("only if" part:) Let I be an ideal of P and let $\mathcal{J} = \{\mathrm{id}(v) \,|\, v \in I\}$. It is clear that $I = \cup_{v \in I}\mathrm{id}(v)$. Hence, it suffices to show hat $\mathcal{J}$ is *laminar*, i.e., for each $I, J \in \mathcal{J}$ we have $I \subseteq J$, $I \supseteq J$, or $I \cap J = \emptyset$. Suppose that for some $v, v' \in I$ we have $\mathrm{id}(v) \cap \mathrm{id}(v') \neq \emptyset$. Let $w \in \mathrm{id}(v) \cap \mathrm{id}(v')$. Then, we must have $v \preceq v'$ or $v \succeq v'$ since P is a rooted forest, and hence, $\mathrm{id}(v) \subseteq \mathrm{id}(v')$ or $\mathrm{id}(v) \supseteq \mathrm{id}(v')$.

("if" part:) Conversely, suppose that P is not a rooted forest. Then, there exists an $v, w, u \in V$ such that both w and u are covers of v. Let us consider ideal $I = \mathrm{id}(\{w, u\})$. Then, for each representation of $I = \cup_{v \in J}\mathrm{id}(v)$ by a union of principal ideals, we must have $\{w, u\} \subseteq J$. However, $\mathrm{id}(w)$ and $\mathrm{id}(u)$ cannot be disjoint since $v \in \mathrm{id}(w) \cap \mathrm{id}(u)$. □

Theorem 2.4: *Suppose that $P = (V, \preceq)$ be a rooted forest. Let $I \subseteq V$ be an ideal of P. Then, we have*

$$\mu\chi_I = \chi_{\max(I)}. \tag{2.13}$$

Conversely, for an antichain A of P we have

$$\zeta\chi_A = \chi_{\mathrm{id}(A)}. \tag{2.14}$$

(Proof) If $P = (V, \preceq)$ is a rooted forest, each ideal I is a disjoint union of principal ideals $\mathrm{id}(v)$ $(v \in W)$ by Lemma 2.3. Then, it follows from Proposition 2.2 that

$$\begin{aligned} \mu\chi_I &= \mu\left(\sum_{v\in W}\chi_{\mathrm{id}(v)}\right) && (2.15)\\ &= \sum_{v\in W}\chi_v && (2.16)\\ &= \chi_W && (2.17)\\ &= \chi_{\max(I)}. && (2.18)\end{aligned}$$

The latter follows from Möbius Inversion (Proposition 2.1) and (2.13). □

Remark 2.5: If $P = (V, \preceq)$ is not a rooted forest, there exists distinct $v, w, u \in V$ such that v is covered by w and u as in the proof of Lemma 2.3. For an ideal I generated by antichain $\{w, u\}$, we have

$$(\mu\chi_I)(v) = \sum_{v\preceq w}\mu(v,w)\chi_I(w) = \mu(v,v) + \mu(v,w) + \mu(v,u) = -1, \qquad (2.19)$$

and hence, Theorem 2.4 fails to hold. □

We observe that the Möbius function on a rooted forest is quite simple.

Proposition 2.6: *Suppose that $P = (V, \preceq)$ is a rooted forest. Then, for each $(v,w) \in V \times V$ we have*

$$\mu(v,w) = \begin{cases} 1 & \text{if } v = w,\\ -1 & \text{if } w \text{ covers } v,\\ 0 & \text{otherwise.}\end{cases} \qquad (2.20)$$

□

Then, it is obvious to see the following.

Proposition 2.7: *Let $P = (V, \preceq)$ be a rooted forest. A vector $c: V \to \mathbb{R}$ is nonnegative if and only if ζc is nonnegative and monotone nonincreasing. Equivalently, a vector $d: V \to \mathbb{R}$ is nonnegative and monotone nonincreasing if and only if μd is nonnegative.* □

3. Applications

Throughout this section, we assume $P = (V, \preceq)$ is a rooted forest. We denote by μ and ζ, respectively, the Möbius function and the Zeta-function for P.

3.1. Linear optimization

Let $f: \mathcal{A}(P) \to \mathbb{R}$ be a submodular function and let $c: V \to \mathbb{R}$. We consider the following linear optimization problem over $\mathrm{Q}(f)$.

$$(\mathrm{P})\left|\begin{array}{ll} \max & \sum_{v\in V} c(v)x(v)\\ \text{s.t.} & x(A) \le f(A) \qquad (A \in \mathcal{A}(P))\end{array}\right. \qquad (3.1)$$

It is easy to see that Problem (P) is always feasible and that (P) has an optimum solution if and only if c is nonnegative (cf. [2]). The dual problem for (P) is:

$$
\text{(D)}\left|\begin{array}{lll} \min & \sum\{f(A)y(A) \mid A \in \mathcal{A}(P)\} & \\ \text{s.t.} & \sum\{y(A) \mid v \in A \in \mathcal{A}(P)\} = c(v) & (v \in V), \\ & y(A) \geq 0 & (A \in \mathcal{A}(P)). \end{array}\right. \tag{3.2}
$$

It follows from Theorem 2.4 and Proposition 2.7 that the Möbius transformation make the dual problem (D) equivalent to

$$
\text{(D}'\text{)}\left|\begin{array}{lll} \min & \sum\{f(A)y(A) \mid A \in \mathcal{A}(P)\} & \\ \text{s.t.} & \sum\{y(A) \mid v \in \mathrm{id}(A), A \in \mathcal{A}(P)\} = (\zeta c)(v) & (v \in V), \\ & y(A) \geq 0 & (A \in \mathcal{A}(P)). \end{array}\right. \tag{3.3}
$$

This is equivalent to

$$
\text{(D}''\text{)}\left|\begin{array}{lll} \min & \sum\{f(I)y(I) \mid I \in \mathcal{I}(P)\} & \\ \text{s.t.} & \sum\{y(I) \mid v \in I \in \mathcal{I}(P)\} = (\zeta c)(v) & (v \in V), \\ & y(I) \geq 0 & (I \in \mathcal{I}(P)) \end{array}\right. \tag{3.4}
$$

and is the dual of

$$
\text{(P}''\text{)}\left|\begin{array}{lll} \max & \sum_{v\in V}(\zeta c)(v)x(v) & \\ \text{s.t.} & x(I) \leq f(I) & (I \in \mathcal{I}(P)). \end{array}\right. \tag{3.5}
$$

This is a result of Faigle and Kern [6].

3.2. Intersection

Theorem 3.1 (Faigle and Kern [7]): *Let $P = (V, \preceq)$ be a rooted forest and let $f_1, f_2\colon \mathcal{A}(P) \to \mathbb{R}$ be submodular. Then, the system*

$$
\begin{array}{ll} x(A) \leq f_1(A) & (A \in \mathcal{A}(P)), \\ x(A) \leq f_2(A) & (A \in \mathcal{A}(P)) \end{array} \tag{3.6}
$$

is totally dual integral.

(Proof) Let $c\colon V \to \mathbb{Z}$ and let us consider the following linear program.

$$
\text{(P)}\left|\begin{array}{lll} \max & \sum_{v\in V} c(v)x(v) & \\ \text{s.t.} & x(A) \leq f_1(A) & (A \in \mathcal{A}(P)), \\ & x(A) \leq f_2(A) & (A \in \mathcal{A}(P)) \end{array}\right. \tag{3.7}
$$

The dual problem for (P) is

$$
\text{(D)}\left|\begin{array}{lll} \min & \sum\{f_1(A)y_1(A) + f_2(A)y_2(A) \mid A \in \mathcal{A}(P)\} & \\ \text{s.t.} & \sum\{y_1(A) \mid v \in A \in \mathcal{A}(P)\} & \\ & +\sum\{y_2(A) \mid v \in A \in \mathcal{A}(P)\} = c(v) & (v \in V), \\ & y_1(A), y_2(A) \geq 0 & (A \in \mathcal{A}(P)). \end{array}\right. \tag{3.8}
$$

It follows form Theorem 2.4 and Proposition 2.7 that Problem (D) is equivalent to the following (D$'$).

$$(\mathrm{D}')\left|\begin{array}{lll}\min & \sum\{f_1(A)y_1(A)+f_2(A)y_2(A)\mid A\in\mathcal{A}(P)\} & \\ \text{s.t.} & \sum\{y_1(A)\mid v\in \mathrm{id}(A), A\in\mathcal{A}(P)\} & \\ & +\sum\{y_2(A)\mid v\in \mathrm{id}(A), A\in\mathcal{A}(P)\}=(\zeta c)(v) & (v\in V), \\ & y_1(A), y_2(A)\geq 0 & (A\in\mathcal{A}(P)).\end{array}\right. \tag{3.9}$$

Rewriting (D$'$), we have

$$(\mathrm{D}'')\left|\begin{array}{lll}\min & \sum\{f_1(I)y_1(I)+f_2(I)y_2(I)\mid I\in\mathcal{I}(P)\} & \\ \text{s.t.} & \sum\{y_1(I)\mid v\in I\in\mathcal{I}(P)\} & \\ & +\sum\{y_2(I)\mid v\in I\in\mathcal{I}(P)\}=(\zeta c)(v) & (v\in V), \\ & y_1(I), y_2(I)\geq 0 & (I\in\mathcal{I}(P)).\end{array}\right. \tag{3.10}$$

Since ζc is integral vector, it follows from Intersection Theorem for submodular system [4] (see also [8]) that (D$''$) has an integral optimal solution. This solution is also an optimal solution for (D). □

For a rooted forest $P=(V,\preceq)$ let $c=\mathbf{1}$, where $\mathbf{1}(v)=1$ $(v\in V)$. Then, for $v\in V$ $(\zeta c)(v)$ is the *depth* of v in P, that is, the maximum path length from v to the unique topmost element:

$$(\zeta c)(v)=\max\{k\mid v=v_1\prec v_2\prec\cdots\prec v_k, v_i\in V\ (i=1,\cdots,k)\}. \tag{3.11}$$

Then, V can be partitioned into $V_i=\{v\mid v\in V, (\zeta c)(v)=i\}$ $(i=1,2,\cdots,l)$.

Theorem 3.2: *Suppose that $P=(V,\preceq)$ is a rooted forest and $f_1,f_2\colon\mathcal{A}(P)\to\mathbb{R}$ be submodular. Then, we have*

$$\max\{x(V)\mid x\in \mathrm{Q}(f_1)\cap \mathrm{Q}(f_2)\}=\min\{\sum_{i=1}^{l} f_1(X\cap V_i)+f_2(V_i-X)\mid X\subseteq V\}. \tag{3.12}$$

(Proof) Let $x\in \mathrm{Q}(f_1)\cap \mathrm{Q}(f_2)$ and $X\subseteq V$. Then, we have

$$\begin{aligned} x(V) &= \sum_{i=1}^{l} x(V_i) && (3.13)\\ &= \sum_{i=1}^{l} x(X\cap V_i)+x(V_i-X) && (3.14)\\ &\leq \sum_{i=1}^{l} f_1(X\cap V_i)+f_2(V_i-X). && (3.15)\end{aligned}$$

Hence, the inequality max $\leq$ min holds.

Let us consider Problem (D$''$) in (3.10) with $c=\mathbf{1}$. It follows from Theorem 3.1 that there exists and integral optimal solution (y_1,y_2) for Problem (D$''$).

Define

$$\mathcal{I}=\{I\mid I\in\mathcal{I}(P), y_1(I)>0\},\ \mathcal{J}=\{J\mid J\in\mathcal{I}(P), y_2(J)>0\}. \tag{3.16}$$

We can assume, without loss of generality, that $\mathcal{I}$ and $\mathcal{J}$ are chains. Let $\mathcal{I} = \{I_i \mid i = 1, \cdots, p\}$ and $\mathcal{J} = \{J_i \mid i = 1, \cdots, q\}$ so that

$$I_1 \supset I_2 \supset \cdots \supset I_p \tag{3.17}$$

and

$$J_1 \supset J_2 \supset \cdots \supset J_q. \tag{3.18}$$

Since $I_1 \cup J_1 \supseteq V_1$, we must have $I_1 \cup J_1 = V$. Also, since $(I_1 \cap J_1) \cap V_1 = \emptyset$ and P is a rooted forest, we have $I_1 \cap J_1 = \emptyset$, and hence, $I_i \cap J_i = \emptyset$ for each $i = 1, \cdots, l$, where I_i for $i > p$ and J_i for $j > q$ are, respectively, understood as $I_i = \emptyset$ and $J_i = \emptyset$. By induction, we have

$$V_i = (I_i \cup J_i) - (I_{i+1} \cup J_{i+1}) \quad (i = 1, \cdots, l-1). \tag{3.19}$$

Claim: For each $i = 1, 2, \cdots, l$

$$\max(I_i) = I_i \cap V_i, \ \max(J_i) = J_i \cap V_i. \tag{3.20}$$

(Proof of Claim) Let $v \in I_i - V_i$. Since $v \in V_j$ for some $j > i$, there exists $v' \in V_i$ such that $v \prec v'$. Then, we must have $v' \in I_i \cup J_i$. If $v' \in J_i$, we have $v \in I_i \cap J_i$, a contradiction. □ □

Now the optimal value of Problem (D$''$) in (3.10) is

$$\begin{aligned} \sum_{i=1}^{l} f_1(I_i) + f_2(J_i) &= \sum_{i=1}^{l} f_1(\max(I_i)) + f_2(\max(J_i)) && (3.21)\\ &= \sum_{i=1}^{l} f_1(I_i \cap V_i) + f_2(J_i \cap V_i) && (3.22)\\ &= \sum_{i=1}^{l} f_1(I_1 \cap V_i) + f_2(J_1 \cap V_i), && (3.23) \end{aligned}$$

which is equal to the optimal value of (P) in (3.7), and the assertion of the theorem follows. □

4. General posets

Let $P = (V, \preceq)$ be an arbitrary poset. Ando [2] showed that following theorem.

Theorem 4.1: *For each nonnegative vector $c: V \to \mathbb{R}$ there uniquely exist a chain*

$$\mathcal{C}: A_1 \prec \cdots \prec A_k \tag{4.1}$$

of antichains of P and positive reals $\lambda_i > 0$ $(i = 1, \cdots, k)$ such that

$$c = \lambda_1 \chi_{A_1} + \cdots + \lambda_k \chi_{A_k}. \tag{4.2}$$

□

Suppose that $c: V \to \mathbb{R}_+$ is represented as (4.2). Define $\phi(c)$ by

$$\phi(c) = \lambda_1 \chi_{\mathrm{id}(A_1)} + \cdots + \lambda_k \chi_{\mathrm{id}(A_k)}. \tag{4.3}$$

The vector $\phi(c)$ is nonnegative and monotone nonincreasing function on $P = (V, \preceq)$. The representation (4.3) of $\phi(c)$ is unique in the following sense.

Proposition 4.2 (Lovász [9]): *For each nonnegative and monotone nonincreasing function* $d: V \to \mathbb{R}$ *on* $P = (V, \preceq)$ *there uniquely exist a chain*

$$\mathcal{C}: I_1 \subset \cdots \subset I_k \tag{4.4}$$

of ideals of P *and positive reals* $\lambda_i > 0$ $(i = 1, \cdots, k)$ *such that*

$$d = \lambda_1 \chi_{I_1} + \cdots + \lambda_k \chi_{I_k}. \tag{4.5}$$

□

We thus have

Proposition 4.3:

(i) *The function* ϕ *defined in* (4.3) *gives a one-to-one correspondence between* $\mathbb{R}_+^V$ *and the set of nonnegative monotone nonincreasing functions on* $P = (V, \preceq)$.

(ii) *We have* $\phi = \zeta$ *if and only if* $P = (V, \preceq)$ *is a rooted forest.*

(Proof) (i) follows from Theorem 4.1 and Proposition 4.2 and (ii) follows form Theorem 2.4 and Remark 2.5. □

Hence, in case that P is a rooted forest, the mapping $\phi: \mathbb{R}_+^V \to \mathbb{R}^V$ is linear. However, in general ϕ is only piecewise linear mapping.

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