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OPTIMAL HOSTAGE RESCUE PROBLEM WHERE ACTION
CAN ONLY BE TAKEN ONCE
—CASE WHERE EFFECT VANISHES THEREAFTER—

by

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OPTIMAL HOSTAGE RESCUE PROBLEM WHERE ACTION CAN ONLY BE TAKEN ONCE
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Abstract

We propose the following model for optimal rescue problems concerning hostages. Suppose that a person is taken as a hostage and that a decision has to be made from among three alternatives: rescuing, no rescuing, or taking one action which will save the situation. It is assumed that the action can only be taken once and it will be effective only at that time, i.e., the effect vanishes thereafter. The objective here is to find the optimal decision rule so as to maximize the probability of the hostage not being killed. Several properties of the optimal rescuing rule are revealed.

1 Introduction

Acts involving hostage taking occur for different reasons, e.g., social inequality, poverty, religious problems, racial problems, political problems, are part of life. The problem has become an urgent issue to be tackled worldwide. Typical examples in recent years include:

1 A 17-year-old youth wielding a knife, hijacked a bus on the Sanyo Expressway and killed a 68-year-old hostage. After 15 hours, the police stormed the bus, the other hostages were rescued, and the hijacker was arrested (May 4, 2000).

2 An armed man took a Finance Ministry official hostage in the Tokyo Stock Exchange building and demanded a meeting with the Finance Minister. He surrendered to the police after a tense, five and half hour standoff (January 12, 1998).

3 Fourteen guerrillas stormed the home of the Japanese ambassador to Peru and took about three hundred people hostage, including diplomats and government officials attending a birthday party for the emperor. All but one of the hostages were rescued though all the rebels were killed when special forces stormed the building (December 17, 1996).

4 A man with a knife broke into a house and took a 2-year-old boy hostage. The police finally rushed into the house, set the uninjured boy free, and arrested the criminal (December 1, 1995).

Although the information is not available for accurate statistics, it could be said that different scenarios of the above continue to occur all over the world. The most important decision for the person in charge of crisis settlement is the timing to enact rescue of the hostages. Wrestling with the problem, needless to say, involves many factors, political, economical, sociological, psychological, and so on, and all must be taken into acount, together with the safety of hostages, the demands of criminals, the repercussions of success or failure in a rescue attempt, and so on. The purpose of this paper is to propose a mathematical model of an optimal hostage rescue problem by using the concept of a sequential stochastic decision processes and examine the properties of an optimal rescuing rule. The author has proposed and examined a model concerning the problem in [1] where only two alternatives, rescuing or no rescuing the hostage, were available. However, as is seen in many hostages cases, negotiators often take certain actions to coax the kidnapper(s) to their way of thinking, for example, presuading the criminal to surrender by subjecting him/her to his/her mother's voice, submitting to his demands to be airlifted to another country, providing a means of escape, paying the ransom, releasing his comrades in prison, and so on. In the paper we propose a model where such an action can only be taken once, which is effective only at that time, i.e., its effect vanishes thereafter. Unfortunately, for this problem, with the exception of the author's paper[1], we are unable to find any reference material based on any mathematical approach. Accordingly, we cannot list references to be directly cited.

2 Model

Consider the following sequential stochastic decision process with a finite planning horizon. Here, for convenience, let points in time be numbered backward from the final point in time of the planning horizon, time 0, as 0, 1, $\cdots$, and so on. Let the time interval between two successive points, say times t and $t-1$, be called the period t. Here, assume that time 0 is the deadline at which a rescue attempt is considered as the only course of action for some reason, say, the hostage's health condition, the degree of criminal desperation, and so on.

Suppose one person is taken as a hostage at any given point in time t, and a decision has to be make from among three alternatives: Rescuing, no rescuing, or taking an action $\mathcal{A}$. Let x denote a decision variable of a certain point in time t where $x = 0$ if there is no rescue attempt, $x = 1$ if the action is taken, and $x = 2$ if rescue is attempted, and X_t to denote the set of possible decisions of time t, i.e., $X_t = \{0, 1, 2\}$ for $t \geq 1$ and $X_0 = \{2\}$.

Let p $(0 < p < 1)$ be the probability of the hostage being killed if $x = 2$, let q and r $(0 < q < 1,\ 0 \leq r < 1$, and $0 < q + r < 1)$ be the probabilities of the hostage being, respectively, killed and set free up to the next point in time if $x = 0$; accordingly, $1 - q - r$ is the probability of the hostage being neither killed nor set free if $x = 0$. Now, taking an action $\mathcal{A}$ will influence the probabilities p, q and r to a greater or lesser degree. Therefore, if $x = 1$, let p' $(0 < p' < 1)$ be the probability of the hostage being killed, let q' and r' $(0 < q' < 1,\ 0 \leq r' < 1$, and $0 < q' + r' < 1)$ be the probabilities of the hostage being, respectively, killed and set free up to the next point in time. Here, let us assume that the action $\mathcal{A}$ can only be taken once and that if this action $\mathcal{A}$ is taken at a certain point in time, then the p, q and r thus far change into p', q' and r', which are effective only at that time, and then back again to p, q and r following thereafter.

The objective here is to maximize the probability of the hostage not being killed. Now, the cases of $p = p' = q = q' = 0$, $p = p' = q = q' = r = r' = 1$, and $q + r = q' + r' = 1$ make the problem trivial. Accordingly, all of these are excluded in the definition of the model.

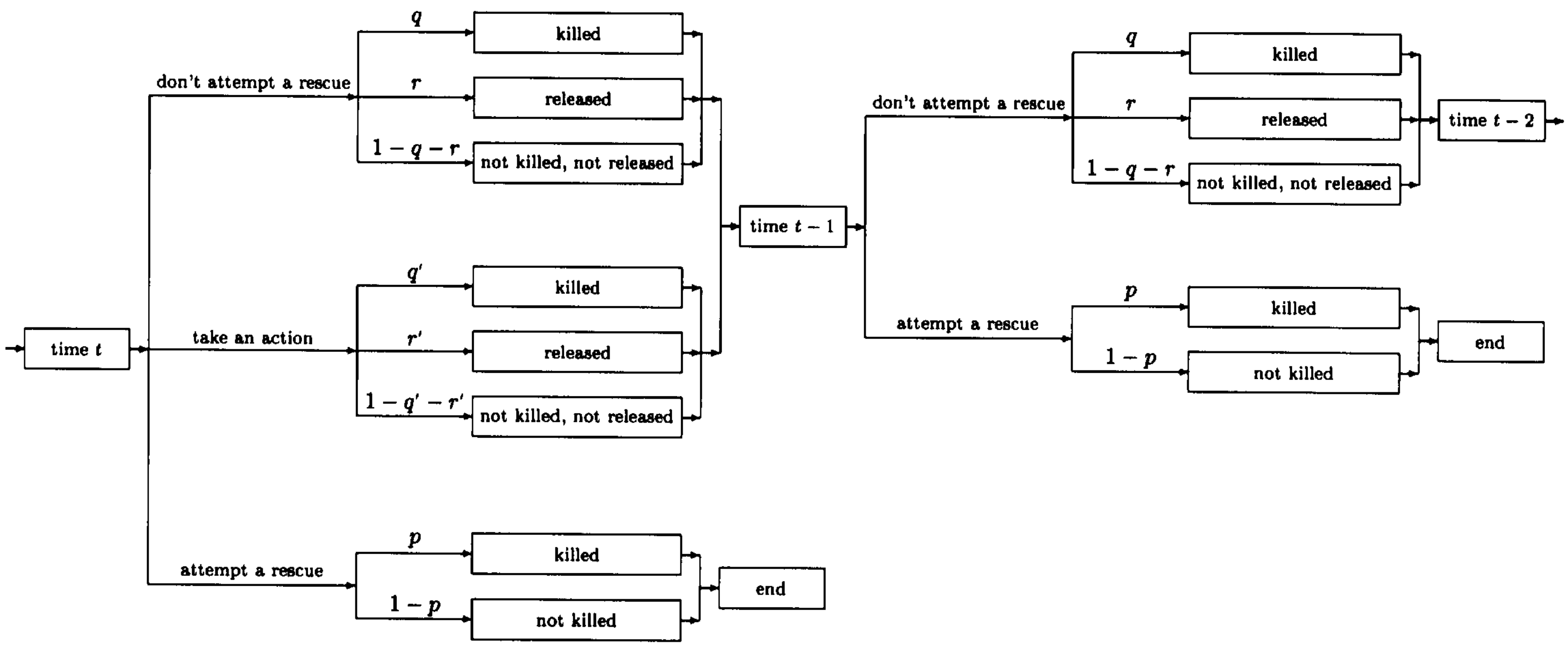

Figure 1: Decision Tree

3 Optimal Equation

Let v_t be the maxmium probability of the hostage not being killed, provided that the action $\mathcal{A}$ has already been taken up to time t, and let $\bar{v}_t$ be the maxmium probability of the hostage not being killed, provided that the action $\mathcal{A}$ has not yet been taken up to time t. For convenience, let $P = 1 - p$, the probability of the hostage not being killed if a rescue attempt is made ($x = 2$) at any time. Then, we have

$$v_0 = P, \tag{3.1}$$

$$\bar{v}_0 = P, \tag{3.2}$$

$$v_t = \max\{P, V_t\}, \quad t \geq 1, \tag{3.3}$$

$$\bar{v}_t = \max\{P, \bar{V}_t, V_t'\}, \quad t \geq 1, \tag{3.4}$$

Now, for convenience let

$$V_0 = \bar{V}_0 = V_0' = P. \tag{3.5}$$

Therefore, Eqs. (3.3) and (3.4) hold also for $t = 0$. Accordingly, Eq. (3.1) to Eq. (3.4) can be rewritten as follows.

$$v_t = \max\{P, V_t\}, \quad t \geq 0, \tag{3.6}$$

$$\bar{v}_t = \max\{P, \bar{V}_t, V_t'\}, \quad t \geq 0. \tag{3.7}$$

where V_t is the probability of the hostage not being killed over the period from time t to 0 (the deadline) if no rescue attempt is made ($x = 0$), provided that the action $\mathcal{A}$ has already been taken up to time t, $\bar{V}_t$ and V_t' are the probabilities of the hostage not being killed over the period from time t to 0 (the deadline), respectively, if no rescue attempt is made ($x = 0$) and if the action $\mathcal{A}$ is taken ($x = 1$), provided that the action $\mathcal{A}$ has not yet been taken up to time t. Accordingly, we can express V_t, $\bar{V}_t$ and V_t', respectively, for $t \geq 1$, as follows.

$$V_t = r + (1 - q - r)v_{t-1}, \tag{3.8}$$

$$\bar{V}_t = r + (1 - q - r)\bar{v}_{t-1}, \tag{3.9}$$

$$V_t' = r' + (1 - q' - r')v_{t-1}, \tag{3.10}$$

The right hand side of the above three expressions imply the following:

1 V_t: Suppose the action $\mathcal{A}$ has already been taken up to time t and no rescue attempt is made at time t. Then, if the hostage is released with the probability r, the probability of the hostage not being killed is equal to $r \times 1$, if the hostage is killed with the probability q, the probability of the hostage not being killed is equal to $q \times 0$, and if the hostage is neither released nor killed with the probability $1 - q - r$, the probability of the hostage not being killed over the period from time $t - 1$ to 0 is equal to $(1 - q - r)v_{t-1}$.

2 $\bar{V}_t$: Suppose the action $\mathcal{A}$ has not yet been taken up to time t and no rescue attempt is made at time t. Then, if the hostage is released with the probability r, the probability of the hostage not being

killed is equal to $r \times 1$, if the hostage is killed with the probability q, the probability of the hostage not being killed is equal to $q \times 0$, and if the hostage is neither released nor killed with the probability $1-q-r$, the probability of the hostage not being killed over the period from time $t-1$ to 0 is equal to $(1-q-r)\bar{v}_{t-1}$.

3 V'_t: Suppose the action $\mathcal{A}$ has not yet been taken up to time t and the action is taken at time t. Then, if the hostage is released with the probability r', the probability of the hostage not being killed is equal to $r' \times 1$, if the hostage is killed with the probability q', the probability of the hostage not being killed is equal to $q' \times 0$, and if the hostage is neither released nor killed with the probability $1-q'-r'$, the probability of the hostage not being killed over the period from time $t-1$ to 0 is equal to $(1-q'-r')v_{t-1}$.

For convenience, let

$$U = r + (1-q-r)P, \tag{3.11}$$

$$U' = r' + (1-q'-r')P, \tag{3.12}$$

$$\delta = \frac{r'-r}{q'-q+r'-r}. \tag{3.13}$$

Then, clearly

$$V_1 = \bar{V}_1 = U \tag{3.14}$$

$$V'_1 = U'. \tag{3.15}$$

4 Preliminaries

This section provides the two lemmas which are used in the subsequent sections.

Lemma 4.1 *All of v_t, $\bar{v}_t$, V_t, V'_t and $\bar{V}_t$ are nondecreasing in t, hence converage to finite numbers v, $\bar{v}$, V, V' and $\bar{V}$, respectively, as $t \to \infty$.*

Proof. From Eq. (3.6) we have $v_1 \geq P = v_0$. Suppose $v_{t-1} \geq v_{t-2}$. Then $V_t \geq V_{t-1}$ due to Eq. (3.8), hence $v_t = \max\{P, V_t\} \geq \max\{P, V_{t-1}\} = v_{t-1}$. Accordingly, by induction the assertion for v_t holds. In a similar way we can prove the assertions for $\bar{v}_t$, V_t, V'_t and $\bar{V}_t$ hold. Hence, from the fact that v_t, $\bar{v}_t$, V_t, V'_t and $\bar{V}_t$ are all bounded since they are probabilities, their limits as $t \to \infty$ exist. ∎

For convenience we define

$$A(\tau) = -(r'q - rq')\sum_{k=0}^{\tau-1}(1-q-r)^k, \quad \tau \geq 1. \tag{4.1}$$

Lemma 4.2 *Let $U \geq P$. Then,*

(a) *$v_t = V_t$ for all $t \geq 1$.*

(b) *$V = r/(q+r)$.*

(c) *If $q'-q+r'-r < 0$, then $r'/q' > (\leq)\, r/q \Longleftrightarrow V > (\leq)\, \delta$.*

(d) *Assume that a certain $t^\circ \geq 1$ exists such that $\bar{v}_{t^\circ} = V'_{t^\circ}$.*

1 *If $r'/q' > r/q$, then $\bar{V}_{t^\circ+\tau} < V'_{t^\circ+\tau}$ for $\tau \geq 1$.*

2 *If* $r'/q' \leq r/q$, *then* $\bar{V}_{t^\circ+\tau} - V'_{t^\circ+\tau} = A(\tau) \geq 0$, *hence,* $\bar{V}_{t^\circ+\tau} \geq V'_{t^\circ+\tau}$ *for* $\tau \geq 1$.

Proof. Let $U \geq P$. From Lemma 4.1 and Eq. (3.14), for all $t \geq 1$ we have

$$V_t \geq V_1 = U \geq P, \tag{4.2}$$

$$\bar{V}_t \geq \bar{V}_1 = U \geq P. \tag{4.3}$$

(a) From Eqs. (3.3) and (4.2) we immediately get $v_t = V_t$ for all $t \geq 1$.

(b) From (a), Eqs. (3.8), and (3.10) we get, for $t \geq 1$,

$$V_t = r + (1-q-r)V_{t-1}, \tag{4.4}$$

$$V'_t = r' + (1-q'-r')V_{t-1}. \tag{4.5}$$

Hence $V = r + (1-q-r)V$, from which $V = r/(q+r)$.

(c) Using Eq. (3.13), we immediately have

$$V - \delta = \frac{r(q'+r') - r'(q+r)}{(q+r)(q'-q+r'-r)} = \frac{-(r'q - rq')}{(q+r)(q'-q+r'-r)}. \tag{4.6}$$

Hence, the assertion is true due to the assumption $q' - q + r' - r < 0$.

(d) Suppose $\bar{v}_{t^\circ} = V'_{t^\circ}$ with $t^\circ \geq 1$. Then $\bar{V}_{t^\circ+1} = r + (1-r-q)V'_{t^\circ}$ from Eq. (3.9). Accordingly, from Eqs. (4.4) and (4.5) we get

$$\begin{aligned}
\bar{V}_{t^\circ+1} - V'_{t^\circ+1} &= r + (1-q-r)V'_{t^\circ} - r' - (1-q'-r')V_{t^\circ} \\
&= r + (1-q-r)\big(r' + (1-q'-r')V_{t^\circ-1}\big) - r' - (1-q'-r')\big(r + (1-q-r)V_{t^\circ-1}\big) \\
&= r + r'(1-q-r) - r' - r(1-q'-r') = -(r'q - rq').
\end{aligned} \tag{4.7}$$

(d1) Let $r'/q' > r/q$. Then, it follows from Eq. (4.7) that the assertion holds for $\tau = 1$. Suppose it holds for $\tau - 1$, i.e., $\bar{V}_{t^\circ+\tau-1} < V'_{t^\circ+\tau-1}$. Now, from Eq. (4.3) we have $P \leq \bar{V}_{t^\circ+\tau-1}$. Accordingly, we have $P \leq \bar{V}_{t^\circ+\tau-1} < V'_{t^\circ+\tau-1}$. Then, we get $\bar{v}_{t^\circ+\tau-1} = V'_{t^\circ+\tau-1}$ due to Eq. (3.4). Therefore, from Eq. (3.9) we have $\bar{V}_{t^\circ+\tau} = r + (1-q-r)V'_{t^\circ+\tau-1}$. Noting Eqs. (4.4) and (4.5), we obtain

$$\begin{aligned}
\bar{V}_{t^\circ+\tau} - V'_{t^\circ+\tau} &= r + (1-q-r)V'_{t^\circ+\tau-1} - r' - (1-q'-r')V_{t^\circ+\tau-1} \\
&= r + (1-q-r)\big(r' + (1-q'-r')V_{t^\circ+\tau-2}\big) \\
&\quad - r' - (1-q'-r')\big(r + (1-q-r)V_{t^\circ+\tau-2}\big) \\
&= r + r'(1-q-r) - r' - r(1-q'-r') = -(r'q - rq') < 0.
\end{aligned}$$

Then $\bar{V}_{t^\circ+\tau} < V'_{t^\circ+\tau}$ for $\tau \geq 1$. Accordingly, by induction the assertion holds.

(d2) Let $r'/q' \leq r/q$. From Eq. (4.7) the assertion clearly holds for $\tau = 1$. Suppose it holds for $\tau - 1$, i.e., $\bar{V}_{t^\circ+\tau-1} - V'_{t^\circ+\tau-1} = A(\tau-1) \geq 0$. Then $\bar{V}_{t^\circ+\tau-1} = V'_{t^\circ+\tau-1} + A(\tau-1)$, hence $\bar{V}_{t^\circ+\tau-1} \geq V'_{t^\circ+\tau-1}$. Accordingly, since $\bar{V}_{t^\circ+\tau-1} \geq P$ due to Eq. (4.3), we have $\bar{v}_{t^\circ+\tau-1} = \bar{V}_{t^\circ+\tau-1}$ due to Eq. (3.4), hence

$\bar{V}_{t^\circ+\tau} = r + (1-q-r)\bar{V}_{t^\circ+\tau-1}$ from Eq. (3.9). Consequently, noting Eqs. (4.4) and (4.5), we get

$$\begin{aligned}
\bar{V}_{t^\circ+\tau} - V'_{t^\circ+\tau} &= r + (1-q-r)(V'_{t^\circ+\tau-1} + A(\tau-1)) - r' - (1-q'-r')V_{t^\circ+\tau-1} \\
&= r + (1-q-r)V'_{t^\circ+\tau-1} - r' - (1-q'-r')V_{t^\circ+\tau-1} + (1-q-r)A(\tau-1) \\
&= r + (1-q-r)\big(r' + (1-q'-r')V_{t^\circ+\tau-2}\big) \\
&\quad - r' - (1-q'-r')\big(r + (1-q-r)V_{t^\circ+\tau-2}\big) + (1-q-r)A(\tau-1) \\
&= r + r'(1-q-r) - r' - r(1-q'-r') + (1-q-r)A(\tau-1) \\
&= -(r'q - rq') - (1-q-r)(r'q-rq')\sum_{k=0}^{\tau-2}(1-q-r)^k \\
&= -(r'q-rq')\Big(1 + \sum_{k=0}^{\tau-2}(1-q-r)^{k+1}\Big) \\
&= -(r'q-rq')\sum_{k=0}^{\tau-1}(1-q-r)^k = A(\tau) \geq 0.
\end{aligned}$$

Then $\bar{V}_{t^\circ+\tau} \geq V'_{t^\circ+\tau}$ for $\tau \geq 1$. This completes the induction. ∎

5 Analysis

In this section, we examine the properties of the optimal decision rule for the problem, classifying all the possible combinations of the parameters, p, q, r, q' and r' into the following three cases:

$$\text{Case 1}: \quad \left\{\begin{array}{l} P > U \\ P > U' \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{ll} (q+r)P > r & (1) \\ (q'+r')P > r' & (2) \end{array}\right\} \tag{5.1}$$

$$\text{Case 2}: \quad \left\{\begin{array}{l} U' \geq P \\ U' > U \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{ll} (q'+r')P \leq r' & (1) \\ (q'-q+r'-r)P < r'-r & (2) \end{array}\right\} \tag{5.2}$$

$$\text{Case 3}: \quad \left\{\begin{array}{l} U \geq P \\ U \geq U' \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{ll} (q+r)P \leq r & (1) \\ (q'-q+r'-r)P \geq r'-r & (2) \end{array}\right\} \tag{5.3}$$

5.1 Case of $P > U$ and $P > U'$

Theorem 5.1 *Let $P > U$ and $P > U'$. Then, $v_t = \bar{v}_t = P$ for all $t \geq 0$.*

Proof. Assume $P > U$ and $P > U'$. Then, $v_1 = \max\{P, U\} = P$ and $\bar{v}_1 = \max\{P, U, U'\} = P$ due to Eqs. (3.3) and (3.4). Hence, the assertion holds for $t = 1$. Suppose $v_{t-1} = \bar{v}_{t-1} = P$. Then, from Eq. (3.8) to (3.12) we get $V_t = \bar{V}_t = r + (1-q-r)P = U$ and $V'_t = r' + (1-q'-r')P = U'$; accordingly $v_t = P$ and $\bar{v}_t = P$ for $t \geq 0$ due to Eqs. (3.6) and Eq. (3.7). ∎

5.2 Case of $U' \geq P$ and $U' > U$

Theorem 5.2 *Let $U' \geq P$ and $U' > U$.*

(a) *Suppose $U < P$, then $v_t = P$ and $\bar{v}_t = V'_t = U'$ for all $t \geq 1$.*

(b) *Suppose $U \geq P$, then*

1 *If* $r'/q' > r/q$, *then* $\bar{v}_t = V'_t$ *for all* $t \geq 1$.

2 *If* $r'/q' \leq r/q$, *then* $\bar{v}_1 = V'_1$ *and* $\bar{v}_t = \bar{V}_t$ *for* $t \geq 2$.

Proof. Let $U' \geq P$ and $U' > U$. Then $\bar{v}_1 = \max\{P, U, U'\} = U' = V'_1$ due to Eqs. (3.4) and (3.15).

(a) Let $U < P$, i.e., $r-(q+r)P < 0$ due to Eq. (5.1 (1)). Then $v_1 = P$ from Eq. (3.3). Noting $\bar{v}_1 = V'_1 = U'$, the assertion is true for $t = 1$. Suppose it is true for $t-1$, i.e., $v_{t-1} = P$ and $\bar{v}_{t-1} = V'_{t-1} = U'$. Then, from Eqs. (3.8) and (3.11) we have $V_t = r + (1-q-r)P = U$, hence $v_t = \max\{P, U\} = P$ for all $t \geq 1$. From Eqs. (3.9) and (3.10), and (3.12) we get $\bar{V}_t = r+(1-q-r)U'$ and $V'_t = r'+(1-q'-r')P = U' \geq P$. Hence $\bar{V}_t - V'_t = r + (1-q-r)U' - U' = r - (q+r)U' \leq r - (q+r)P < 0$, i.e., $\bar{V}_t < V'_t$. Accordingly $\bar{v}_t = \max\{P, \bar{V}_t, V'_t\} = V'_t = U'$ for all $t \geq 1$.

(b) Let $U \geq P$. Since $\bar{v}_1 = V'_1$, from Lemma 4.2(d) we have $t^\circ = 1$.

(b1) Let $r'/q' > r/q$. Then, from Lemma 4.2(d1) and Eq. (4.3) we have $P \leq \bar{V}_t < V'_t$ for all $t \geq 2$. Accordingly $\bar{v}_t = V'_t$ for $t \geq 2$ due to Eq. (3.4), hence $\bar{v}_t = V'_t$ for $t \geq 1$.

(b2) Let $r'/q' \leq r/q$. Then, from Lemma 4.2(d2) and Eq. (4.3) we have $\bar{V}_t \geq V'_t$ and $\bar{V}_t \geq P$ for $t \geq 2$, hence $\bar{v}_t = \bar{V}_t$ for $t \geq 2$ due to Eq. (3.4). ∎

5.3 Case of $U \geq P$ and $U \geq U'$

Lemma 5.1 *Let* $U \geq P$ *and* $U \geq U'$. *If* $q' - q + r' - r < 0$, *then* $\delta > 0$.

Proof. From Eq. (5.3 (2)), the assumption $q' - q + r' - r < 0$ yields $r' - r < 0$ due to $P > 0$, hence, $\delta > 0$. ∎

Theorem 5.3 *Let* $U \geq P$ *and* $U \geq U'$. *Then,*

(a) *Suppose* $q' - q + r' - r \geq 0$, *then* $\bar{v}_t = \bar{V}_t = V_t$ *for* $t \geq 1$.

(b) *Suppose* $q' - q + r' - r < 0$. *We have*

1 *If* $V \leq \delta$, *then* $\bar{v}_t = \bar{V}_t = V_t$ *for* $t \geq 1$.

2 *If* $U > \delta$, *then* $\bar{v}_1 = \bar{V}_1 = V_1$ *and* $\bar{v}_t = V'_t$ *for* $t \geq 2$.

3 *If* $U \leq \delta < V$, *there exists a unique* $t^* \geq 2$, *such that* $V_{t^*-1} \leq \delta < V_{t^*}$.

Hence, $\bar{v}_t = \bar{V}_t = V_t$ *for* $1 \leq t \leq t^*$ *and* $\bar{v}_t = V'_t$ *for* $t > t^*$.

Proof. Let $U \geq P$ and $U \geq U'$. Then, $\bar{v}_1 = \max\{P, U, U'\} = U = \bar{V}_1 = V_1$ due to Eqs. (3.4) and (3.14).

(a) Clearly the assertion is true for $t = 1$. Suppose it is true for $t-1$, i.e., $\bar{v}_{t-1} = \bar{V}_{t-1} = V_{t-1}$, hence $\bar{V}_t = r + (1-q-r)V_{t-1} = V_t$ due to Eqs. (3.9) and (4.4). Now, from Eqs. (5.3 (2)), (4.2), and assumption $q' - q + r' - r \geq 0$ we have

$$(q' - q + r' - r)V_{t-1} \geq r' - r, \quad t \geq 1. \tag{5.4}$$

Then, using Eq. (4.5), we obtain

$$\begin{aligned}\bar{V}_t - V'_t = V_t - V'_t &= \big(r + (1-q-r)V_{t-1}\big) - \big(r' + (1-q'-r')V_{t-1}\big) \\ &= (q' - q + r' - r)V_{t-1} - (r' - r) \geq 0, \quad t \geq 1,\end{aligned}$$

due to Eq. (5.4), hence $\bar{V}_t = V_t \geq V_t'$, and noting Eq. (4.3), we get $\bar{v}_t = \bar{V}_t = V_t$ for $t \geq 1$ due to Eq. (3.3).

(b) Let $q' - q + r' - r < 0$. Then $\delta > 0$ due to Lemma 5.1.

(b1) Let $V \leq \delta$. Then $V_{t-1} \leq \delta$ for all $t \geq 1$ due to Lemma 4.1, hence from Eq. (3.13) we get

$$(q' - q + r' - r)V_{t-1} \geq r' - r, \quad t \geq 1. \tag{5.5}$$

Accordingly, in the same way as in the proof of (a), the assertion can be proven, but in this case the Eq. (5.5) is used instead of Eq. (5.4).

(b2) Let $U > \delta$. That is $V_1 > \delta$, then $V_{t-1} > \delta$ for all $t \geq 2$ due to Lemma 4.1, hence from Eq. (3.13) we have

$$(q' - q + r' - r)V_{t-1} < r' - r, \quad t \geq 2. \tag{5.6}$$

Noting $\bar{v}_1 = V_1$, from Eq. (3.9) we have $\bar{V}_2 = r + (1 - q - r)V_1$. Then, from Eq. (4.5) we get $\bar{V}_2 - V_2' = (q' - q + r' - r)V_1 - (r' - r) < 0$ due to Eq. (5.6), hence $\bar{V}_2 < V_2'$. Now, since $\bar{V}_2 \geq P$ due to Eq. (4.3), we get $\bar{v}_2 = V_2'$, hence $t^\circ = 2$ from Lemma 4.2(d). Now, since $V_1 > \delta$, we have $V > \delta$ due to Lemma 4.1, hence $r'/q' > r/q$ due to Lemma 4.2(c). Accordingly, from Lemma 4.2(d1) we immediately get $\bar{V}_t < V_t'$ for $t \geq 3$. Noting Eq. (4.3), we obtain $\bar{v}_t = \max\{P, \bar{V}_t, V_t'\} = V_t'$ for all $t \geq 3$. Accordingly, we have $\bar{v}_1 = \bar{V}_1 = V_1$ and $\bar{v}_t = V_t'$ for $t \geq 2$.

(b3) Let $U \leq \delta < V$. That is $V_1 \leq \delta < V$, then $r'/q' > r/q$ due to Lemma 4.2(c). Now, since V_t is nondecreasing in t with $\delta < V$, there must exist a unique $t^* \geq 2$ such that $V_t \leq \delta$ for $t < t^*$ and $V_t > \delta$ for $t \geq t^*$ due to Lemma 4.1. If $V_{t^*-1} \leq \delta$, then $V_{t-1} \leq \delta$ for $1 \leq t \leq t^*$ due to Lemma 4.1. Hence, from Eq. (3.13) we have

$$(q' - q + r' - r)V_{t-1} \geq r' - r, \quad 1 \leq t \leq t^*. \tag{5.7}$$

Accordingly, in the same way as in the proof of (a), we can prove that $\bar{v}_t = \bar{V}_t = V_t$ for $1 \leq t \leq t^*$, but in this case the Eq. (5.7) is used instead of Eq. (5.4). Now from this we have $\bar{v}_{t^*} = V_{t^*}$. Hence $\bar{V}_{t^*+1} = r + (1 - q - r)V_{t^*}$ due to Eq. (3.9). Since $V_{t^*} > \delta$, then from Eq. (3.13) we get

$$(q' - q + r' - r)V_{t^*} < r' - r. \tag{5.8}$$

Noting Eq. (4.5) we get $\bar{V}_{t^*+1} - V_{t^*+1}' = (q' - q + r' - r)V_{t^*} - (r' - r) < 0$ due to Eq. (5.8), hence $\bar{V}_{t^*+1} < V_{t^*+1}'$. Using Eq. (4.3), we get $\bar{v}_{t^*+1} = V_{t^*+1}'$, hence $t^\circ = t^* + 1$ from Lemma 4.2(d). Accordingly, noting $r'/q' > r/q$, from Lemma 4.2(d1) we have $\bar{V}_t < V_t'$ for $t \geq t^* + 2$, i.e., $t > t^* + 1$. Noting Eq. (4.3), we get $\bar{v}_t = \max\{P, \bar{V}_t, V_t'\} = V_t'$ for $t > t^* + 1$. Accordingly, we obtain $\bar{v}_t = V_t'$ for $t \geq t^* + 1$, i.e., $t > t^*$. ∎

6 Conclusion

Our model revealed that any one of the following seven decision rules would be made according to the given parameters p, q, r, q' and r' noting that a rescue is always attempted at time 0 (the deadline) by the assumption.

DR-A When a hostage event occurs at time $t \geq 0$, attempt to rescue immediately.

DR-B When a hostage event occurs at time $t \geq 1$, take the action $\mathcal{A}$ immediately and if the criminal(s) do not surrender, attempt to rescue at the next time $t-1$.

DR-C When a hostage event occurs at time $t \geq 1$, take the action $\mathcal{A}$ immediately and if the criminal(s) do not surrender, wait up to time 0 and attempt to rescue.

DR-D When a hostage event occurs at time $t \geq 2$, take the action $\mathcal{A}$ immediately and if the criminal(s) do not surrender, wait up to time 0 and attempt to rescue. When a hostage event occurs at time 1, wait up to time 0 and attempt to rescue.

DR-E When a hostage event occurs at time $t \geq 1$, wait up to time 0 and attempt to rescue.

DR-F When a hostage event occurs at time $t \geq 2$, wait up to time 1. If the criminal(s) do not surrender, take the action $\mathcal{A}$ at time 1 and further to this, if the criminal(s) do not surrender, attempt to rescue at time 0. When a hostage event occurs at time 1, take the action $\mathcal{A}$ and if the criminal(s) do not surrender, attempt to rescue at time 0.

DR-G There exists a $t^* \geq 2$ such that when a hostage event occurs at time t ($t > t^*$), take the action $\mathcal{A}$ immediately and if the criminal(s) do not surrender, wait up to time 0 and attempt to rescue. When a hostage event occurs at time t ($1 \leq t \leq t^*$), wait up to time 0 and attempt to rescue.

Corollary 6.1 *Suppose a rescue attempt is always made at time* 0 (*the deadline*). *Then*

(a) *Assume $P > U$ and $P > U'$. Then* DR-A *is optimal.*

(b) *Assume $U' \geq P$ and $U' > U$. Then*

  1 *Let $U < P$. Then* DR-B *is optimal.*

  2 *Let $U \geq P$. Then*

    i *If $r'/q' > r/q$, then* DR-C *is optimal.*

    ii *If $r'/q' \leq r/q$, then* DR-F *is optimal.*

(c) *Assume $U \geq P$ and $U \geq U'$. Then*

  1 *Let $q' - q + r' - r \geq 0$. Then* DR-E *is optimal.*

  2 *Let $q' - q + r' - r < 0$. Then*

    i *If $V \leq \delta$, then* DR-E *is optimal.*

    ii *If $U > \delta$, then* DR-D *is optimal.*

    iii *If $U \leq \delta < V$, then* DR-G *is optimal.*

Proof. Note that the action $\mathcal{A}$ can be regarded as having not yet been taken at the time when a hostage event occurs, i.e., it is sufficient to consider only $\bar{v}_t$ when a hostage event occurs.

(a) Assume $P > U$ and $P > U'$. Then, from Theorem 5.1 we have $\bar{v}_t = P$ for $t \geq 0$, implying that a rescue attempt is made when a hostage event occurs at time $t \geq 0$, i.e., DR-A is optimal.

(b) Assume $U' \geq P$ and $U' > U$.

(b1) Let $U < P$ Then $\bar{v}_t = V_t'$ and $v_t = P$ for $t \geq 1$ due to Theorem 5.2(a). Hence, it follows that take the action $\mathcal{A}$ due to $\bar{v}_t = V_t'$ when a hostage event occurs at time $t \geq 1$. Since the action $\mathcal{A}$ can only be taken once by definition, if the criminal(s) do not surrender, it is sufficient to consider only $v_\tau = P$ for time $\tau < t$, hence it means that a rescue attempt is made at the next time $t-1$, i.e., DR-B is optimal.

(b2) Let $U \geq P$. Then $v_t = V_t$ for $t \geq 1$ due to Lemma 4.2(a).

(b2i) If $r'/q' > r/q$, then $\bar{v}_t = V_t'$ for $t \geq 1$ due to Theorem 5.2(b1); accordingly, when a hostage event occurs at time $t \geq 1$, take the action $\mathcal{A}$ due to $\bar{v}_t = V_t'$, and if the criminal(s) do not surrender, it is sufficient to consider only $v_\tau = V_\tau$ for $\tau < t$, hence wait up to time 0. From the assumption we obtain that a rescue attempt is made at time 0, i.e., DR-C is optimal.

(b2ii) If $r'/q' \leq r/q$, then $\bar{v}_t = \bar{V}_t$ for $t \geq 2$ and $\bar{v}_1 = V_1'$ due to Theorem 5.2(b2); therefore, when a hostage event occurs at time $t \geq 2$, wait up to time 1 due to $\bar{v}_t = \bar{V}_t$, and if the criminal(s) do not surrender, it is necessary to consider continually $\bar{v}_t$ when $t = 1$, i.e., $\bar{v}_1$, hence take the action $\mathcal{A}$ at time 1 due to $\bar{v}_1 = V_1'$, and further to this, if the criminal(s) do not surrender, attempt to rescue at time 0; and when a hostage event occurs at time 1, take the action $\mathcal{A}$ due to $\bar{v}_1 = V_1'$, and if the criminal(s) do not surrender, attempt to rescue at time 0, i.e., DR-F is optimal.

(c) Assume $U \geq P$ and $U \geq U'$, then $v_t = V_t$ for $t \geq 1$ due to Lemma 4.2(a).

(c1) Let $q' - q + r' - r \geq 0$. Then $\bar{v}_t = \bar{V}_t$ for $t \geq 1$ due to Theorem 5.3(a); consequently, when a hostage event occurs at time $t \geq 1$, wait up to time 0 due to $\bar{v}_t = \bar{V}_t$, and attempt to rescue, i.e., DR-E is optimal.

(c2) Let $q' - q + r' - r < 0$. Then $\delta > 0$ due to Lemma 5.1.

(c2i) If $V \leq \delta$, then $\bar{v}_t = \bar{V}_t$ for $t \geq 1$ due to Theorem 5.3(b1); accordingly, when a hostage event occurs at time $t \geq 1$, wait up to time 0 due to $\bar{v}_t = \bar{V}_t$, and attempt to rescue, i.e., DR-E is optimal.

(c2ii) If $U > \delta$, then $\bar{v}_t = V_t'$ for $t \geq 2$ and $\bar{v}_1 = \bar{V}_1$ due to Theorem 5.3(b2); consequently, when a hostage event occurs at time $t \geq 2$, take the action $\mathcal{A}$ immediately due to $\bar{v}_t = V_t'$, and if the criminal(s) do not surrender, it is sufficient to consider only $v_\tau = V_\tau$ for $\tau < t$, hence wait up to time 0, and attempt to rescue; and when a hostage event occurs at time 1, wait up to time 0 due to $\bar{v}_1 = \bar{V}_1$, and attempt to rescue, i.e., DR-D is optimal.

(c2iii) If $U \leq \delta < V$, there exists a unique $t^* \geq 2$ such that $V_{t^*-1} \leq \delta < V_{t^*}$, and " $\bar{v}_t = V_t'$ for $t > t^*$ and $\bar{v}_t = \bar{V}_t$ for $1 \leq t \leq t^*$ " due to Theorem 5.3(b3). This means that when a hostage event occurs at time t ($t > t^*$), take the action $\mathcal{A}$ immediately due to $\bar{v}_t = V_t'$, and if the criminal(s) do not surrender, it is sufficient to consider only $v_\tau = V_\tau$ for $\tau < t$, hence wait up to time 0, and attempt to rescue; and that when a hostage event occurs at time t ($1 \leq t \leq t^*$), wait up to time 0 due to $\bar{v}_t = \bar{V}_t$, and attempt to rescue, i.e., DR-G is optimal. ∎

7 Future Studies

In this paper we propose a basic model of an optimal rescuing problem involving hostages. Taking different real hostage situations into account, we feel a need to modify the model from the following viewpoints:

1 We should consider the case where there is an action which can only be taken once and its effect will last to time 0, the deadline. More practically the effect of an action decreases gradually after it is taken.

2 In real hostage event, several actions are available. The problem arises as to when and what action should be taken.

3 In many real cases, more than one hostage is taken.

4 In the present paper, all the hostages are implicitly assumed to be homogenous. As seen in many hostage crises, however, special considerations are given for females, the aged, the sick, children, and so on. Models in which such nonhomogenous classes of hostages are taken into consideration should also be proposed.

5 In many real cases, criminal(s) operate with confused motives. This causes the probabilities p, q and r to change randomly from one minute to the next. This consideration leads us to the model in which p, q and r are random variables with a known or unknown distribution function $F(p, q, r)$. When it is unknown, we can and must update its unknown parameters by using Bays' theorem.

6 Cases where the deadline cannot always be known.

Finally, in order for the model to be realistically effective, the probabilities p, q and r for each hostage crisis must be measured and known in advance. Although such a measurement would be a very difficult task, it should be tackled through united efforts of researchers in different fields, say, psychologists, sociologists, political scientists, engineers.

8 Reference

[1] F. Shi: An optimal hostage rescue problem. *Discussion Paper Series, No. 872. July, 2000 (Institute of Policy and Planning Sciences)*