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by

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Stochastic Analysis of Number of Corporations in a Market Derived from Strategic Policies of Individual Corporations for Market Entry and Retreat

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Abstract

A stochastic model is developed for describing a market life cycle expressed in terms of the number of corporations in the market. Each corporation independently determines the probability of market entry if it is not in the market yet or the probability of market retreat if it is already in the market. These probabilities may depend on time t, the number of corporations in the market at time t and the number of corporations which have retreated from the market by time t. Consequently individual corporations are expressed as temporally inhomogeneous Markov chains, and the whole market state is given by the sum of such Markov chains. An algorithmic procedure is developed for computing the probability distribution of the number of corporations in the market based on spectral analysis of the temporally inhomogeneous Markov chain combined with a bivariate generating function approach. Extensive numerical experiments reveal somewhat surprising results concerning how the market would be affected by interactions among individual corporations with different strategic policies.

0 Introduction

For understanding the growth and decline of a market, a traditional approach has been to model a product life cycle based on analysis of consumer behavior. Bass[1969], for example, developed a diffusion model by assuming that the conditional probability of a consumer purchasing a product under consideration at time t given that he/she has not purchased the product by time t would depend only on the number of consumers who have purchased the product by time t. Horsky and Simon[1983] extended this model by incorporating the level of the advertisement expenditure in addition to the number of consumers who have purchased the product by time t in the dependency structure of the conditional probabilities. Horsky[1990] further strengthened the analysis by introducing the utility structure and incomes of consumers as well as the price of the product into the model, which enabled one

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to combine a decision mechanism of consumers for purchasing the product with the product life cycle analysis for the first time.

The diffusion process approach for modeling a product life cycle through analysis of consumer behavior can be justified simply because there exist sufficiently many consumers despite their discrete nature. In order to analyze the growth and decline of a market from corporate side, however, the diffusion process approach is inappropriate due to the limited number of corporations which are potentially interested in entering into the market. The principal tool employed for this type of the market analysis is an econometric approach where the number of corporations in the market is expressed as a time series governed by the total product sales in the market, technological progress, etc. Many extended models have been developed and the reader is referred to Geroski and Mazzucato[2001] for an extensive summary of the literature.

A major pitfall of the econometric approach above can be found in that it cannot directly connect strategic policies of individual corporations with the market state. The purpose of this paper is to fill this gap by modelling individual corporations as temporally inhomogeneous Markov chains and then expressing the market as a sum of such Markov chains. Despite this rather simple model structure, the temporal inhomogeneity present makes analysis fairly complicated. We conquer this difficulty via spectral analysis of the underlying Markov chains combined with a bivariate generating function approach. By capturing sophisticated interactions among individual corporations with different strategic policies, our model will provide an insight into processes of how the market as a whole would be constructed through separate decisions by individual corporations, revealing somewhat surprising results.

In this paper, the market state is defined in terms of the number of corporations in the market. In parallel with a product life cycle, we introduce a market life cycle consisting of the following four stages: the introduction stage; the growth stage; the maturity stage, and the decline stage. Actual data on the automobile industry and the tire industry in the United States are extracted from Simons[1995] and are depicted in Figures 0.1 and 0.2 respectively.

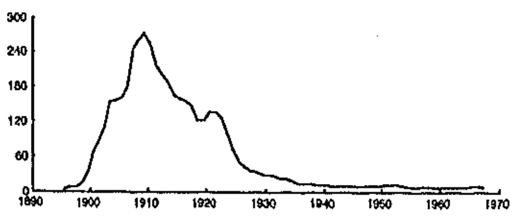

Figure 0.1: Number of Corporations in the US Automobile Industry

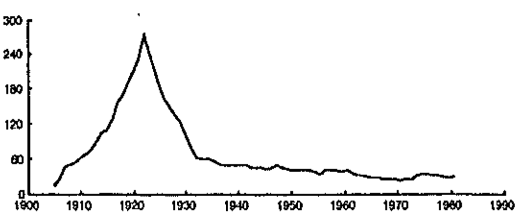

Figure 0.2: Number of Corporations in the US Tire Industry

It should be noted that the decline of the market is only in terms of the number of corporations in the market and does not necessarily mean the decline of the sales volume. The main purpose of this paper is to develop an analytical model and a computational algorithm to explore the relationship between the structure of strategic policies of individual corporations and presence or non-presence of the market life cycle with four stages.

In Section 1, an analytical Model is formally introduced, where strategic policies of individual corporations are expressed in terms of conditional probabilities of entry into and retreat from the market. These conditional probabilities may depend on time t, the number of corporations in the market at time t, $X(t)$, and the number of corporations which have retreated from the market by time t, $Y(t)$. This interdependence is the key to the potential usefulness of our model. The state of a corporation is described as a temporally inhomogeneous Markov chain involving the conditional probabilities above, and the state of the whole market is constructed through a sum of such Markov chains. Section 2 is devoted to spectral analysis of the underlying temporally inhomogeneous Markov chain and the transition probability matrix at time t is derived in a closed form. Based on a bivariate generating function approach, the joint probabilities of $(X(t), Y(t))$ at time t are evaluated. A computational algorithm is summarized in Section 3, and finally numerical results are presented in Section 4. By decomposing the set of corporations potentially interested in entering the market into three categories: RT(Risk-Taking) corporations; RN(Risk-Neutral) corporations, and RA(Risk-Aversive) corporations, numerical experiments reveal rather astonishing behavior of the market size which cannot be explained by the behavioral characteristics of any single category.

1 Model Description

We consider a situation that N corporations are potentially interested in entering into a new product market. Of interest is to develop a stochastic model which captures the market life cycle consisting of the four stages discussed in Section 0 through analysis of strategic actions of individual corporations. More specifically, at time t $(t = 1, 2, \cdots)$ any corporation is in one of the following three states:

$$\begin{cases} 0 & \text{The corporation has not entered the market yet.} \\ 1 & \text{The corporation is in the market.} \\ 2 & \text{The corporation has retreated from the market.} \end{cases} \tag{1.1}$$

It is assumed that if any corporation retreats from the market, it never enters the market again. At time t, each corporation makes an independent decision so as to determine its state at time $t+1$. However, the decision parameters may be time-dependent or may depend on the market state at time t involving all other corporations. Consequently each corporation is modelled to follow a discrete time Markov chain on $\mathcal{S}_C = \{0, 1, 2\}$ which is temporally inhomogeneous having state 2 as the absorbing state. The market as a whole can then be described as an independent sum of such Markov chains. Despite this structural simplicity, the temporal inhomogeneity presents considerable analytical complexity as we will see.

Let $\mathcal{N} = \{1, \cdots, N\}$ be a set of corporations under consideration and let $\{N_i(t) : t = 0, 1, 2, \cdots\}$ be a stochastic process describing the state of corporation i at time t. We define two stochastic processes $\{X(t) : t = 0, 1, 2, \cdots\}$ and $\{Y(t) : t = 0, 1, 2, \cdots\}$ where

$$X(t) = \sum_{i \in \mathcal{N}} \delta_{\{N_i(t)=1\}}; \ Y(t) = \sum_{i \in \mathcal{N}} \delta_{\{N_i(t)=2\}}. \tag{1.2}$$

Here $\delta_{\{P\}} = 1$ if the statement P holds and $\delta_{\{P\}} = 0$ otherwise. We note that $X(t)$ is the number of corporations in the market at time t, while $Y(t)$ is the number of corporations which have retreated from the market by time t. Consequently the bivariate stochastic process $\{(X(t), Y(t)) : t = 0, 1, 2, \cdots\}$ represents the state of the whole market at time t. The corresponding state space $\mathcal{S}_M$ is then defined as

$$\mathcal{S}_M = \{(x, y) : 0 \le x + y \le N,\ x, y \in \{0\} \cup \mathcal{N}\}. \tag{1.3}$$

The corresponding state probabilities and the bivariate generating functions are defined respectively by

$$\underline{\underline{m}}(t) = [m(x, y, t)]_{(x,y) \in \mathcal{S}_M};\ m(x, y, t) = P[X(t) = x, Y(t) = y] \tag{1.4}$$

and

$$\psi(u, v, t) = \sum_{(x,y) \in \mathcal{S}_M} m(x, y, t) u^x v^y. \tag{1.5}$$

In order to analyze the market excluding corporation i, we introduce the followings in parallel with (1.2) through (1.5):

$$X_i(t) = \sum_{j \in \mathcal{N} \setminus \{i\}} \delta_{\{N_j(t)=1\}};\ Y_i(t) = \sum_{j \in \mathcal{N} \setminus \{i\}} \delta_{\{N_j(t)=2\}}; \tag{1.6}$$

$$\mathcal{S}_{M \setminus \{i\}} = \{(x, y) : 0 \le x + y \le N - 1,\ x, y \in \{0\} \cup \mathcal{N} \setminus \{i\}\}; \tag{1.7}$$

$$\underline{\underline{m}}_i(t) = [m_i(x, y, t)]_{(x,y) \in \mathcal{S}_{M \setminus \{i\}}};\ m_i(x, y, t) = P[X_i(t) = x, Y_i(t) = y]; \tag{1.8}$$

and

$$\psi_i(u, v, t) = \sum_{(x,y) \in \mathcal{S}_{M \setminus \{i\}}} m_i(x, y, t) u^x v^y. \tag{1.9}$$

Let $\underline{p}_i^T(t)$ be the state probability vector of $\{N_i(t) : t = 0, 1, \cdots\}$, that is,

$$\underline{p}_i^T(t) = [p_{i0}(t), p_{i1}(t), p_{i2}(t)];\ p_{ij}(t) = P[N_i(t) = j], 0 \le j \le 2. \tag{1.10}$$

The corresponding bivariate generating function is defined by

$$\varphi_i(u, v, t) = p_{i0}(t) + p_{i1}(t)u + p_{i2}(t)v. \tag{1.11}$$

We assume that $\{N_i(t) : t = 0, 1, 2, \cdots\}$ is a temporally inhomogeneous Markov chain governed by one step transition probability matrix $\underline{\underline{a}}_i(t)$ at time t specified in the following manner. At time $t = 0$, no corporation is assumed to be in the market so that one has for all $j \in \mathcal{N}$

$$\underline{p}_j^T(0) = [1, 0, 0];\ m_j(x, y, 0) = \delta_{\{x=y=0\}}\ for\ (x, y) \in \mathcal{S}_{M \setminus \{j\}}. \tag{1.12}$$

Suppose that $\underline{p}_j^T(t)$ and $\underline{\underline{m}}_j(t)$ are known for all $j \in \mathcal{N}$. Then $\underline{\underline{a}}_i(t)$ is determined by

$$\underline{\underline{a}}_i(t) = \begin{bmatrix} 1 - \alpha_i(t) & \alpha_i(t) & 0 \\ 0 & \beta_i(t) & 1 - \beta_i(t) \\ 0 & 0 & 1 \end{bmatrix} \tag{1.13}$$

where

$$\alpha_i(t) = \sum_{(x,y) \in \mathcal{S}_{M \setminus \{i\}}} m_i(x, y, t) p_i(t|x, y) \tag{1.14}$$

and

$$\beta_i(t) = \sum_{(x,y)\in\mathcal{S}_{M\setminus\{i\}}} m_i(x-1,y,t)q_i(t|x,y). \tag{1.15}$$

Here $p_i(t|x,y)$ is the probability that corporation i enters the market at time $t+1$ given that it is not in the market at time t, $X(t) = x$ and $Y(t) = y$. Similarly $q_i(t|x,y)$ is the probability that corporation i remains in the market at time $t+1$ given that it is in the market at time t, $X(t) = x$ and $Y(t) = y$. More formally, we define;

$$p_i(t|x,y) = P[N_i(t+1) = 1|N_i(t) = 0, X(t) = x, Y(t) = y] \tag{1.16}$$

and

$$q_i(t|x,y) = P[N_i(t+1) = 1|N_i(t) = 1, X(t) = x, Y(t) = y]. \tag{1.17}$$

When corporation i is not in the market, both $X(t)$ and $Y(t)$ are contributed by other corporations. Accordingly $\alpha_i(t)$ in (1.14) is expressed as a probability mixture of $p_i(t|x,y)$ with corresponding weights $m_i(t|x,y)$ over $(x,y)\in\mathcal{S}_{M\setminus\{i\}}$. For evaluation of $\beta_i(t)$ in (1.15), the mixing weights become $m_i(t|x-1,y)$ over $(x,y)\in\mathcal{S}_{M\setminus\{i\}}$ since corporation i is already in the market.

In summary, the state transition diagram is depicted in Figure 1.1. Because of dependence of individual entry and retreat probabilities on time t, the number of corporations in the market and the number of corporations which have retreated from the market, the model enables one to understand how strategic policies of individual corporations affect the market state, as we will see in Section 4.

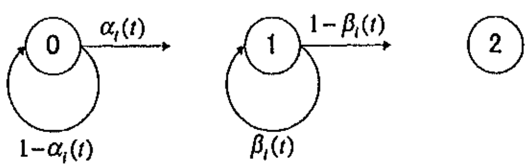

Figure 1.1: State Transition Diagram

It can be seen that

$$\underline{p}_i^T(t+1) = \underline{p}_i^T(t)\underline{\underline{a}}_i(t). \tag{1.18}$$

Equation (1.18) enables one to specify $\varphi_i(u,v,t+1)$ through (1.11) for all $i\in\mathcal{N}$. Once $\underline{\underline{a}}_i(t)$ of (1.13) is given, individual corporations make their decisions independently and one has for each $i\in\mathcal{N}$

$$\psi_i(u,v,t+1) = \prod_{j\in\mathcal{N}\setminus\{i\}} \varphi_j(u,v,t+1). \tag{1.19}$$

By specifying the coefficients of $u^x v^y$ of (1.19), one can see that $\underline{p}_i^T(t)$ and $\underline{\underline{m}}_i(t)$ generate $\underline{p}_i^T(t+1)$ and $\underline{\underline{m}}_i(t+1)$ for all $i\in\mathcal{N}$ via (1.13) through (1.19).

We note that if we define

$$\underline{\underline{P}}_i(t) = \prod_{k=0}^{t} \underline{\underline{a}}_i(k), \tag{1.20}$$

then

$$\underline{p}_i^T(t+1) = \underline{p}_i^T(0)\underline{\underline{P}}_i(t). \tag{1.21}$$

Since $\underline{p}_i^T(0) = [1,0,0]$, $\underline{p}_i^T(t+1)$ is actually the first row of $\underline{\underline{P}}_i(t)$.

2 Spectral Analysis of Market Entry/Retreat Decisions by Individual Corporations

In this section, we analyze the spectral representation of the stochastic matrices $\underline{\underline{a}}_i(t)$ of (1.13) and $\underline{\underline{P}}_i(t)$ of (1.20), which in turn enables one to capture the stochastic structure of market entry/retreat decisions by individual corporations. A few preliminary lemmas are needed.

For $0 \leq \alpha, \beta \leq 1$, we define

$$f(\alpha,\beta) = \frac{\alpha}{\alpha+\beta-1}; \; g(\alpha,\beta) = \frac{1-\beta}{\alpha+\beta-1}. \tag{2.1}$$

We also introduce $\underline{\underline{J}}_1(\alpha,\beta)$, $\underline{\underline{J}}_2(\alpha,\beta)$ and $\underline{\underline{J}}_3(\alpha,\beta)$ as follows:

$$\underline{\underline{J}}_1(\alpha,\beta) = \underline{u}_1\underline{v}_1^T \text{ where } \underline{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } \underline{v}_1^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}; \tag{2.2}$$

$$\begin{aligned} &\underline{\underline{J}}_2(\alpha,\beta) = \underline{u}_2(\alpha,\beta)\underline{v}_2^T \\ &\text{where } \underline{u}_2(\alpha,\beta) = \begin{bmatrix} f(\alpha,\beta) \\ 1 \\ 0 \end{bmatrix} \text{ and } \underline{v}_2^T = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}; \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &\underline{\underline{J}}_3(\alpha,\beta) = \underline{u}_3\underline{v}_3^T(\alpha,\beta) \\ &\text{where } \underline{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \underline{v}_3^T(\alpha,\beta) = \begin{bmatrix} 1 & -f(\alpha,\beta) & g(\alpha,\beta) \end{bmatrix}. \end{aligned} \tag{2.4}$$

When no ambiguity is present, we omit (α,β) and write $\underline{u}_2 = \underline{u}_2(\alpha,\beta)$, $\underline{\underline{J}}_i = \underline{\underline{J}}_i(\alpha,\beta)$, etc. The following lemma then holds true.

Lemma 2.1

a) $\underline{\underline{J}}_i(\alpha,\beta)$, $1 \leq i \leq 3$, *are dyadic and idempotent, i.e.* $\underline{\underline{J}}_i^2(\alpha,\beta) = \underline{\underline{J}}_i(\alpha,\beta)$, $1 \leq i \leq 3$.

b) $\underline{\underline{J}}_i(\alpha,\beta)$, $1 \leq i \leq 3$, *are matrix orthogonal to each other, i.e.* $\underline{\underline{J}}_i(\alpha,\beta)\underline{\underline{J}}_j(\alpha,\beta) = \underline{\underline{0}}$ *if* $i \neq j$, $1 \leq i,j \leq 3$.

c) $\underline{\underline{J}}_2(\alpha_1,\beta_1)\underline{\underline{J}}_2(\alpha_2,\beta_2) = \underline{\underline{J}}_2(\alpha_1,\beta_1)$.

d) $\underline{\underline{J}}_3(\alpha_1,\beta_1)\underline{\underline{J}}_3(\alpha_2,\beta_2) = \underline{\underline{J}}_3(\alpha_2,\beta_2)$.

e) $\underline{\underline{J}}_2(\alpha_1,\beta_1)\underline{\underline{J}}_3(\alpha_2,\beta_2) = \underline{\underline{0}}$.

f) $\underline{\underline{J}}_3(\alpha_1,\beta_1)\underline{\underline{J}}_2(\alpha_2,\beta_2) = \{f(\alpha_2,\beta_2) - f(\alpha_1,\beta_1)\}\underline{u}_3\underline{v}_2^T$.

g) $\underline{u}_3\underline{v}_2^T\underline{\underline{J}}_1 = \underline{\underline{0}}$.

h) $\underline{u}_3\underline{v}_2^T\underline{\underline{J}}_2(\alpha_2,\beta_2) = \underline{u}_3\underline{v}_2^T$.

i) $\underline{u}_3\underline{v}_2^T\underline{\underline{J}}_3(\alpha_2,\beta_2) = \underline{\underline{0}}$.

Proof *We first note that* $\underline{v}_i^T\underline{u}_i = 1$, $1 \le i \le 3$, *while* $\underline{v}_i^T(\alpha,\beta)\underline{u}_j(\alpha,\beta) = 0$, *for* $i \ne j$ $1 \le i,j \le 3$. *Hence parts a), b), e), g), h) and i) follow immediately. For part c), one sees that*

$$\begin{aligned}\underline{\underline{J}}_2(\alpha_1,\beta_1)\underline{\underline{J}}_2(\alpha_2,\beta_2) &= \underline{u}_2(\alpha_1,\beta_1)\underline{v}_2^T\underline{u}_2(\alpha_2,\beta_2)\underline{v}_2^T \\ &= \underline{u}_2(\alpha_1,\beta_1)\underline{v}_2^T \\ &= \underline{\underline{J}}_2(\alpha_1,\beta_1)\end{aligned}$$

since $\underline{v}_2^T\underline{u}_2(\alpha_2,\beta_2) = 1$. *Part d) follows similarly since* $\underline{v}_3^T(\alpha_1,\beta_1)\underline{u}_3 = 1$. *For part f), one has*

$$\begin{aligned}\underline{\underline{J}}_3(\alpha_1,\beta_1)\underline{\underline{J}}_2(\alpha_2,\beta_2) &= \underline{u}_3\underline{v}_3^T(\alpha_1,\beta_1)\underline{u}_2(\alpha_2,\beta_2)\underline{v}_2^T \\ &= \{f(\alpha_2,\beta_2) - f(\alpha_1,\beta_1)\}\underline{u}_3\underline{v}_2^T,\end{aligned}$$

completing the proof. □

Lemma 2.2 *Let* $\underline{\underline{b}}(\alpha,\beta)$ *be a* 3×3 *stochastic matrix given by*

$$\underline{\underline{b}}(\alpha,\beta) = \begin{bmatrix} 1-\alpha & \alpha & 0 \\ 0 & \beta & 1-\beta \\ 0 & 0 & 1 \end{bmatrix}, \ 0 \le \alpha,\beta \le 1. \tag{2.5}$$

Let $f(\alpha,\beta)$ *be as in (2.1). Then the following statements hold true.*

a) $\underline{\underline{b}}(\alpha,\beta) = \underline{\underline{J}}_1 + \beta\underline{\underline{J}}_2(\alpha,\beta) + (1-\alpha)\underline{\underline{J}}_3(\alpha,\beta)$.

b) $\underline{\underline{b}}(\alpha_1,\beta_1)\underline{\underline{b}}(\alpha_2,\beta_2) = \underline{\underline{J}}_1 + \beta_1\beta_2\underline{\underline{J}}_2(\alpha_1,\beta_1) + (1-\alpha_1)(1-\alpha_2)\underline{\underline{J}}_3(\alpha_2,\beta_2)$

$$+ (1-\alpha_1)\beta_2\{f(\alpha_2,\beta_2) - f(\alpha_1,\beta_1)\}\underline{u}_3\underline{v}_2^T.$$

Proof *It can be readily seen that* $\underline{u}_i$ *and* $\underline{v}_i^T$, $1 \le i \le 3$, *are right and left eigenvectors of* $\underline{\underline{b}}(\alpha,\beta)$ *associated with eigenvalues* 1, β, *and* $(1-\alpha)$ *respectively and part a) follows immediately. Part b) can be proven from a) and Lemma 2.1.* □

From (1.13) and (2.5), one sees that

$$\underline{\underline{a}}_i(t) = \underline{\underline{b}}(\alpha_i(t),\beta_i(t)). \tag{2.6}$$

Hence if temporal homogeneity is present, i.e. $\alpha_i(t) = \alpha_i(0)$ and $\beta_i(t) = \beta_i(0)$ for $t = 1, 2, \cdots$, one sees from (1.20) and Lemma 2.2 a) that

$$\begin{aligned}\underline{\underline{P}}(t) &= \underline{\underline{a}}_i^{t+1}(0) \\ &= \underline{\underline{J}}_1 + \beta_i^{t+1}(0)\underline{\underline{J}}_2(\alpha_i(0),\beta_i(0)) + (1-\alpha_i(0))^{t+1}\underline{\underline{J}}_3(\alpha_i(0),\beta_i(0)).\end{aligned}$$

Because of temporal inhomogeneity, however, this simple structure disappears. We overcome this difficulty by using Lemma 2.2 b), as shown in the main theorem of this section below.

Theorem 2.3 *Let* $f(\alpha,\beta)$, $\underline{\underline{J}}_1$, $\underline{\underline{J}}_2(\alpha,\beta)$ *and* $\underline{\underline{J}}_3(\alpha,\beta)$ *be as in (2.1) through (2.4). Then* $\underline{\underline{P}}_i(t)$ *in (1.20) is given by*

$$\begin{aligned}\underline{\underline{P}}_i(t) &= \underline{\underline{J}}_1 + \prod_{k=0}^{t}\beta_i(k)\underline{\underline{J}}_2(\alpha_i(0),\beta_i(0)) \\ &\quad + \prod_{k=0}^{t}\{1-\alpha_i(k)\}\underline{\underline{J}}_3(\alpha_i(t),\beta_i(t)) + C(t)\underline{u}_3\underline{v}_2^T\end{aligned} \tag{2.7}$$

where $\alpha_i(k)$ *and* $\beta_i(k)$ *are as in (1.14) and (1.15) respectively, and*

$$\begin{aligned}C(t) &= \beta_i(t)C(t-1) + \prod_{k=0}^{t-1}\{1-\alpha_i(k)\}\beta_i(t) \\ &\quad \times\{f(\alpha_i(t),\beta_i(t)) - f(\alpha_i(t-1),\beta_i(t-1))\}, t=1,2,\cdots\end{aligned} \tag{2.8}$$

starting with $C(0)=0$.

Proof *The theorem can be proven by induction as follows. For* $t=0$, *one sees from (2.6) that* $\underline{\underline{P}}_i(0) = \underline{\underline{a}}_i(0) = \underline{\underline{b}}(\alpha_i(0),\beta_i(0))$ *and (2.7) holds true by Lemma 2.2 a). Suppose it is true for* t *and consider* $t+1$. *One sees that*

$$\underline{\underline{P}}_i(t+1) = \underline{\underline{P}}_i(t)\underline{\underline{a}}_i(t+1) = \underline{\underline{P}}_i(t)\underline{\underline{b}}(\alpha_i(t+1),\beta_i(t+1)).$$

Using the induction hypothesis and Lemmas 2.1 and 2.2, the above equation leads to

$$\begin{aligned}\underline{\underline{P}}_i(t+1) &= \Big[\underline{\underline{J}}_1 + \prod_{k=0}^{t}\beta_i(k)\underline{\underline{J}}_2(\alpha_i(0),\beta_i(0)) \\ &\quad + \prod_{k=0}^{t}\{1-\alpha_i(k)\}\underline{\underline{J}}_3(\alpha_i(t),\beta_i(t)) + C(t)\underline{u}_3\underline{v}_2^T\Big] \\ &\quad \times\Big[\underline{\underline{J}}_1 + \beta_i(t+1)\underline{\underline{J}}_2(\alpha_i(t+1),\beta_i(t+1)) \\ &\quad + \{1-\alpha_i(t+1)\}\underline{\underline{J}}_3(\alpha_i(t+1),\beta_i(t+1))\Big]\end{aligned}$$

and the theorem follows from Lemma 2.1. □

3 Development of Algorithm

In this section, an algorithmic procedure is summarized for computing $\underline{p}_i^T(t)$, $\underline{\underline{m}}_i(t)$ and $\underline{\underline{m}}(t)$ of (1.10), (1.8) and (1.4) respectively.

[Input]

N: the number of corporations

T: the time periods for consideration

Strategies of Individual Corporations : $[p_i(t|x,y)]_{(x,y)\in S_{M\setminus\{i\}}}$, $0 \le t \le T-1$, $i \in \mathcal{N}$

$$[q_i(t|x,y)]_{(x,y)\in\mathcal{S}_{M\setminus\{i\}}},\ 0 \le t \le T-1,\ i \in \mathcal{N}$$

[Output]

$\underline{p}_i^T(t)$, $\underline{\underline{m}}_i(t)$, $\underline{\underline{m}}(t)$, $i \in \mathcal{N}$, $0 \le t \le T$

[Algorithm]

[0] $\underline{p}_i^T(0) = [1,0,0]$ for all i; $t \leftarrow 0$.

[1] LOOP: Find $\varphi_i(u,v,t)$ using (1.11) for all $i \in \mathcal{N}$.

[2] Generate $\underline{\underline{m}}_i(t)$ by identifying the coefficients of $\psi_i(u,v,t) = \prod_{j\in\mathcal{N}\setminus\{i\}} \varphi_j(u,v,t)$ for all $i \in \mathcal{N}$.

[3] Generate $\underline{\underline{m}}(t)$ by identifying the coefficients of $\psi(u,v,t) = \prod_{j\in\mathcal{N}} \varphi_j(u,v,t)$.

[4] Compute $(\alpha_i(t), \beta_i(t))$ based on (1.14) and (1.15) for all $i \in \mathcal{N}$.

[5] Compute $\underline{p}_i^T(t+1)$ as the first row of $\underline{\underline{P}}_i(t)$ based on Theorem 2.3.

[6] $\rightarrow (T > t \leftarrow t+1)$ / LOOP

4 Numerical Results

The purpose of this section is to demonstrate the usefulness of the market life cycle model developed in the previous sections through numerical examples. In particular, we will see that the model enables one to capture how the market growth and decline would be affected by strategic policies of individual corporations.

For numerical experiments presented in this section, N corporations are decomposed into three categories, i.e. $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$, $\mathcal{N}_i \cap \mathcal{N}_j = \emptyset$ for $i \neq j$ where

$$\mathcal{N}_1 : \text{the set of } N_1 = |\mathcal{N}_1| \text{ RT(Risk-Taking) corporations;} \tag{4.1}$$

$$\mathcal{N}_2 : \text{the set of } N_2 = |\mathcal{N}_2| \text{ RN(Risk-Neutral) corporations;} \tag{4.2}$$

and

$$\mathcal{N}_3 : \text{the set of } N_3 = |\mathcal{N}_3| \text{ RA(Risk-Aversive) corporations,} \tag{4.3}$$

where $|\mathcal{N}_i|$ denotes the cardinality of $\mathcal{N}_i$, $1 \le i \le 3$. For computational simplicity, we assume that all corporations within one category have a common strategic policy. RT corporations tend to enter the market when the market size $X(t)$ is small, but retreat from the market rather quickly when $X(t)$ becomes large. RN corporations incline to enter the market when $X(t)$ exceeds a certain level x_0, continues to stay in the market as x increases from x_0 to another level, say x_1, and then retreats from the market beyond x_1. RA corporations decide to enter the market only when $X(t)$ is sufficiently large beyond x_0, but retreat from the market at the market size below x_1. In other words, RA corporations tend to enter the market after, but to retreat from the market before RN corporations.

Concerning the dependency structure of $p_i(t|x,y)$ and $q_i(t|x,y)$ on t, x and y, two models are constructed.

[Model I] Both $p_i(t|x,y)$ and $q_i(t|x,y)$ depend only on t.

[Model II] Both $p_i(t|x,y)$ and $q_i(t|x,y)$ depend only on x and y.

4.1 Numerical Exploration for Model I

For Model I, $[p_i(t|x,y)]$ and $[q_i(t|x,y)]$ are assumed to be independent of (x,y) and depend only on time t. Let $H(A,B,t)$ be defined by

$$H(A,B,t) = e^{-\{A(t-B)\}^2}. \tag{4.4}$$

We note that the function $e^{-(ax)^2}$ is symmetric about the y-axis with the maximum point $(0,1)$ and the points of inflection $\left(\pm\frac{1}{a\sqrt{2}}, \frac{1}{\sqrt{e}}\right)$. Hence it may be appropriate to characterize the three categories RT, RN and RA by making $[p_i(t|x,y)]$ and $[q_i(t|x,y)]$ of the form $H(A,B,t)$ with different parameter values A and B. For Model I, these parameter values are summarized in Table 4.1.1 below, and the corresponding $p_i(t|x,y)$ and $q_i(t|x,y)$ are depicted in Figures 4.1.2 and 4.1.3.

		A	B
RT	$p_i(t\|x,y)$	$\frac{\sqrt{2}}{7}$	10
	$q_i(t\|x,y)$	$\frac{\sqrt{2}}{22}$	10
RN	$p_i(t\|x,y)$	$\frac{\sqrt{2}}{13}$	30
	$q_i(t\|x,y)$	$\frac{\sqrt{2}}{52}$	30
RA	$p_i(t\|x,y)$	$\frac{\sqrt{2}}{13}$	35
	$q_i(t\|x,y)$	$\frac{\sqrt{2}}{52}$	35

Table 4.1.1: A and B for Model I

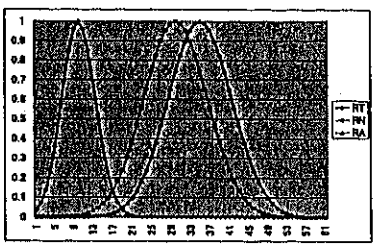

Figure 4.1.2: $p_i(t|x,y)$ for Model I

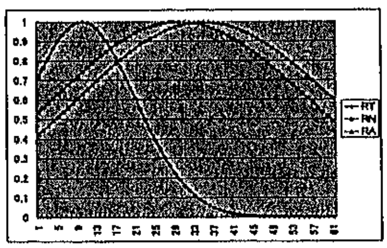

Figure 4.1.3: $q_i(t|x,y)$ for Model I

We note that RT corporations enter into the market rather quickly. Probability to stay in the market remains high in an early stage and then decreases beyond this early period. For RN and RA corporations, these probabilities are shifted to the right with widening spread.

Figures 4.1.4 through 4.1.9 depict $E[X(t)]$ and $Var[X(t)]$ for $(N_1, N_2, N_3) = (35,0,0)$, $(0,35,0)$ and $(0,0,35)$ respectively, capturing the characteristics of the three categories.

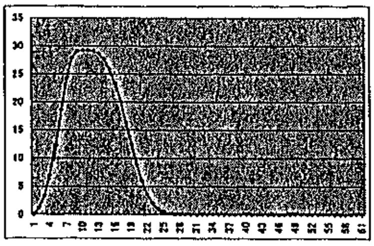

Figure 4.1.4: $E[X(t)]$ for $(N_1, N_2, N_3) = (35, 0, 0)$

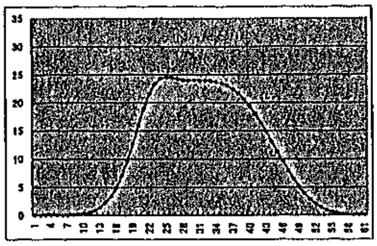

Figure 4.1.5: $E[X(t)]$ for $(N_1, N_2, N_3) = (0, 35, 0)$

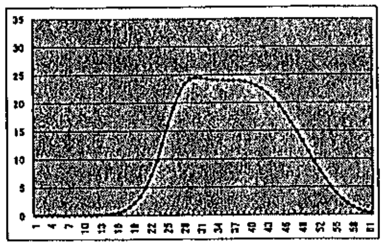

Figure 4.1.6: $E[X(t)]$ for $(N_1, N_2, N_3) = (0, 0, 35)$

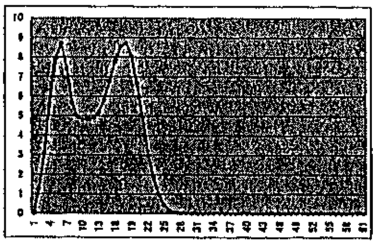

Figure 4.1.7: $Var[X(t)]$ for $(N_1, N_2, N_3) = (35, 0, 0)$

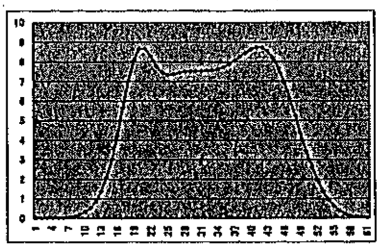

Figure 4.1.8: $Var[X(t)]$ for $(N_1, N_2, N_3) = (0, 35, 0)$

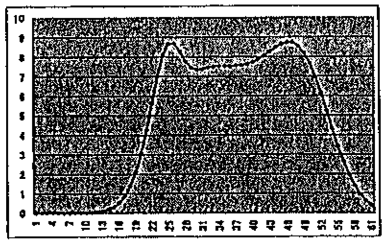

Figure 4.1.9: $Var[X(t)]$ for $(N_1, N_2, N_3) = (0, 0, 35)$

We observe that the peak of $E[X(t)]$ shifts to the right in the order of RT, RN and RA. While the maturity stage is hardly present and the market grows and shrinks rapidly for RT, the market life cycle with four stages can be clearly observed for RN and RA. Concerning $Var[X(t)]$ for RT, RN and RA, it increases rapidly in the growth stage, drops a little bit during the initial period of the maturity stage, becomes stable, grows again gradually toward the end of the maturity stage, and drops in the decline stage.

Figures 4.1.10 and 4.1.11 exhibit $E[X(t)]$ and $Var[X(t)]$ for $(N_1, N_2, N_3) = (5, 15, 15)$ and individual contributions by RT, RN and RA.

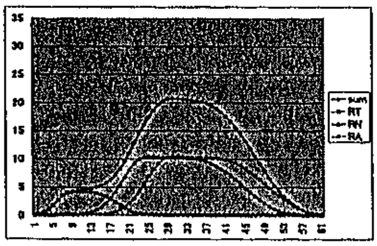

Figure 4.1.10: $E[X(t)]$ for $(N_1, N_2, N_3) = (5, 15, 15)$

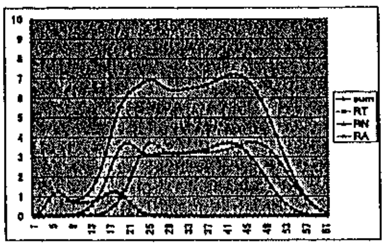

Figure 4.1.11: $Var[X(t)]$ for $(N_1, N_2, N_3) = (5, 15, 15)$

Since market entry and retreat probabilities of corporation i depend only on time t and are independent of other corporations, contribution patterns of the three categories are similar to the previous figures for separate individual categories. However the market life cycle is captured much better due to the shift of the peak to the right in the order of RT, RN and RA. By the same reason, $E[X(t)]$ during the maturity stage for $(N_1, N_2, N_3) = (5, 15, 15)$ is much smaller than that for $(N_1, N_2, N_3) = (0, 35, 0)$ and $(N_1, N_2, N_3) = (0, 0, 35)$.

4.2 Numerical Exploration for Model II

For Model II, $[p_i(t|x,y)]$ and $[q_i(t|x,y)]$ are assumed to be independent of time t and depend only on $(x,y) \in S_{M\setminus\{i\}}$. Let $H(A,B,x)$ and $H(C,D,y)$ be as in (4.4). Then it may be appropriate to characterize the three categories RT, RN and RA by making $[p_i(t|x,y)]$ and $[q_i(t|x,y)]$ of the form $H(A,B,x) \times H(C,D,y)$ with different parameter values A, B, C and D where numbers for x and y are replaced by percentages against the whole population $N = 35$. For Model II, these parameter values are summarized in Table 4.2.1 below, and the corresponding $p_i(t|x,y)$ and $q_i(t|x,y)$ are depicted in Figures 4.2.2 through 4.2.7.

		A	B	C	D
RT	$p_i(t\|x,y)$	$\frac{20\sqrt{2}}{5}$	$\frac{1}{5}$	0	-
	$q_i(t\|x,y)$	$5\sqrt{2}$	$\frac{1}{5}$	0	-
RN	$p_i(t\|x,y)$	$\frac{20\sqrt{2}}{3}$	$\frac{3}{10}$	0	-
	$q_i(t\|x,y)$	$\frac{\sqrt{2}}{5}$	$\frac{3}{10}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{5}$
RA	$p_i(t\|x,y)$	$\frac{20\sqrt{2}}{3}$	$\frac{1}{2}$	0	-
	$q_i(t\|x,y)$	$\frac{\sqrt{2}}{5}$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	0

Table 4.2.1: A, B, C and D for Model II

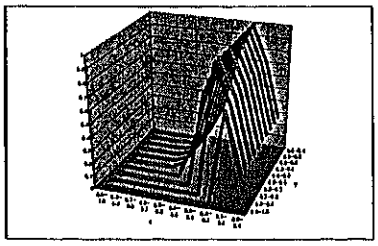
Figure 4.2.2: $[p_i(t|x,y)]$ of RT for Model II

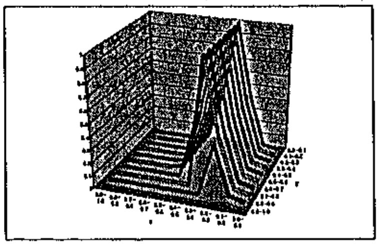
Figure 4.2.3: $[p_i(t|x,y)]$ of RN for Model II

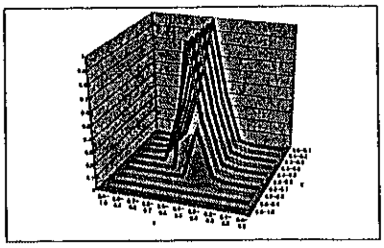
Figure 4.2.4: $[p_i(t|x,y)]$ of RA for Model II

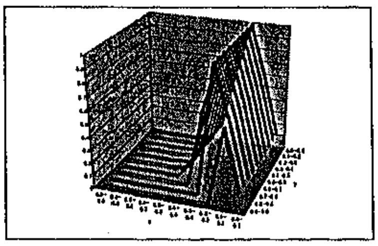
Figure 4.2.5: $[q_i(t|x,y)]$ of RT for Model II

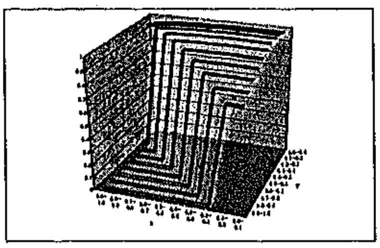
Figure 4.2.6: $[q_i(t|x,y)]$ of RN for Model II

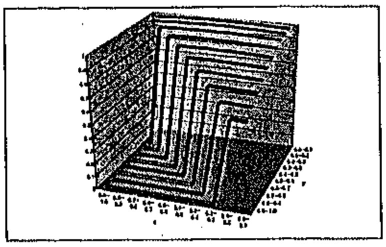
Figure 4.2.7: $[q_i(t|x,y)]$ of RA for Model II

In order to observe the characteristics of RT, RN and RA separately, we first consider three cases where corporations from only one category constitute the entire market. Figures 4.2.8 through 4.2.13 exhibit $E[X(t)]$ and $Var[X(t)]$ for the three cases $(N_1, N_2, N_3) = (35,0,0)$, $(0,35,0)$ and $(0,0,35)$.

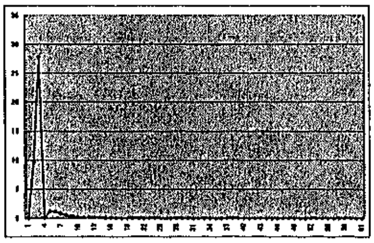

Figure 4.2.8: $E[X(t)]$ for $(N_1, N_2, N_3) = (35, 0, 0)$

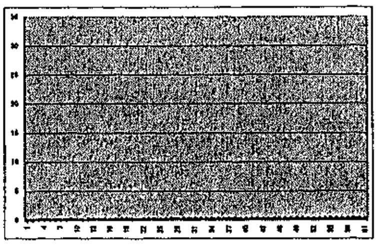

Figure 4.2.9: $E[X(t)]$ for $(N_1, N_2, N_3) = (0, 35, 0)$

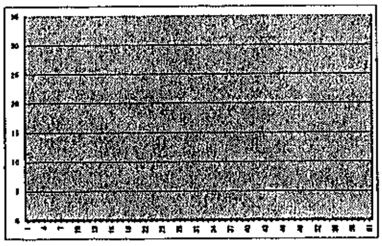

Figure 4.2.10: $E[X(t)]$ for $(N_1, N_2, N_3) = (0, 0, 35)$

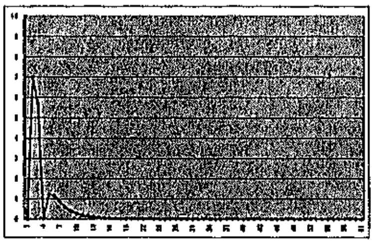

Figure 4.2.11: $Var[X(t)]$ for $(N_1, N_2, N_3) = (35, 0, 0)$

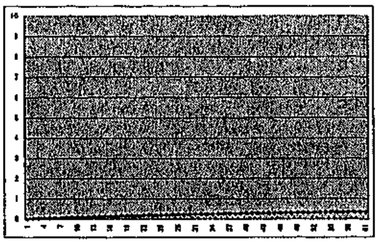

Figure 4.2.12: $Var[X(t)]$ for $(N_1, N_2, N_3) = (0, 35, 0)$

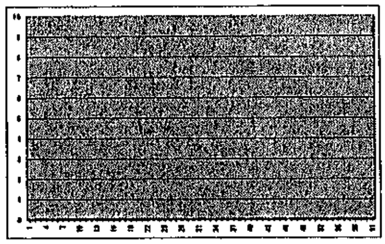

Figure 4.2.13: $Var[X(t)]$ for $(N_1, N_2, N_3) = (0, 0, 35)$

We observe that when the market involves only RT corporations, it grows and declines very rapidly without having the maturity stage at all. On the other hand, when only RN or RA corporations are present, the market can hardly be formed.

In Figures 4.2.14 and 4.2.15, $E[X(t)]$ and $Var[X(t)]$ are depicted for the case $(N_1, N_2, N_3) = (10, 15, 10)$ with separate three curves showing contributions of individual categories.

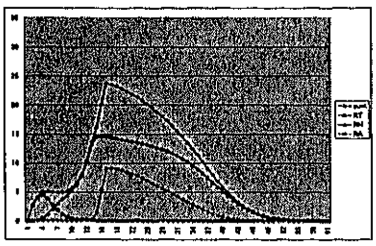

Figure 4.2.14: $E[X(t)]$ for $(N_1, N_2, N_3) = (10, 15, 10)$

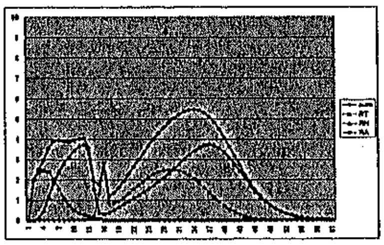

Figure 4.2.15: $Var[X(t)]$ for $(N_1, N_2, N_3) = (10, 15, 10)$

It should be noted that the market life cycle with four stages is clearly present. One can see that RT corporations trigger the first market growth, and then retreat from the market rather quickly, as the market growth is picked up next by RN corporations. RA corporations then start to join the market creating the third market growth. Both RN and RA corporations sustain the maturity stage. While RA corporations retreat from the market gradually, RN corporations tend to stay on and then begin to retreat rapidly. Consequently the decline stage is present largely due to RN corporations. As we saw in Figures 4.2.8 through 4.2.13, any category of corporations alone is incapable of creating the market life cycle of this sort. It is remarkable to observe that interactions among the three categories change the market behavior so drastically.

As for the variance, a peculiar move similar to the one found for Model I in Section 4.1 can be observed again. It increases rapidly during the growth stage, drops a little in the initial period of the maturity stage due to quick retreat of RT corporations, begins to climb slowly toward the end of the maturity period because of RN and RA corporations gradually entering into and retreating from the market, and finally drops to zero during the decline stage.

We next conduct numerical experiments to understand the effect of interactions among the three categories in further detail. The total population $N = 35$ is fixed. In Figures 4.2.16 through 4.2.21, we assume no presence of RA corporations, i.e. $N_3 = 0$, and show $E[X(t)]$ and $Var[X(t)]$ for $(N_1, N_2, N_3) = (35,0,0)$, $(30,5,0)$, $(25,10,0)$, $(20,15,0)$, $(15,20,0)$, $(10,25,0)$, $(9,26,0)$, $(8,27,0)$, $(7,28,0)$, $(6,29,0)$ and $(5,30,0)$.

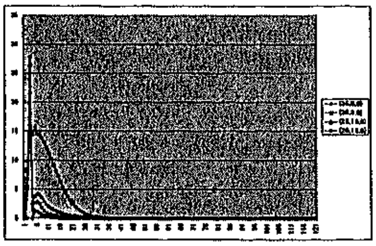

Figure 4.2.16: $E[X(t)]$ when $N_3 = 0$ part 1

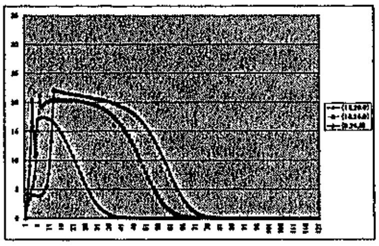

Figure 4.2.17: $E[X(t)]$ when $N_3 = 0$ part 2

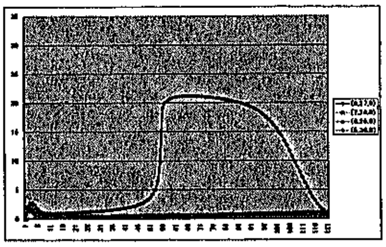

Figure 4.2.18: $E[X(t)]$ when $N_3 = 0$ part 3

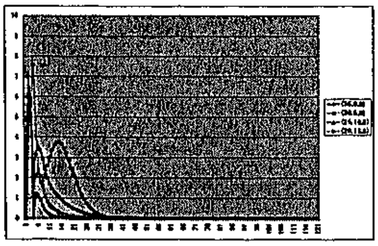

Figure 4.2.19: $Var[X(t)]$ when $N_3 = 0$ part 1

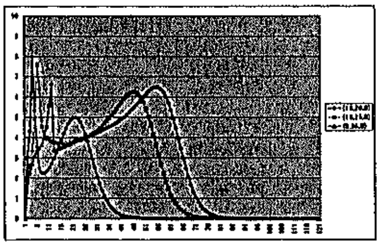

Figure 4.2.20: $Var[X(t)]$ when $N_3 = 0$ part 2

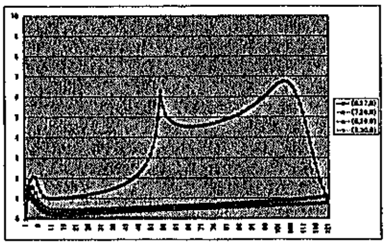

Figure 4.2.21: $Var[X(t)]$ when $N_3 = 0$ part 3

It can be seen that the maturity stage starts to appear and becomes longer as N_1 decreases and N_2 increases up to $N_1 = 8$ and $N_2 = 27$. However, beyond $N_1 = 7$ or less, the market loses its growth momentum and almost disappears as N_1 decreases further. It is rather astonishing to observe that the market life cycle with four stages is clearly present for $N_1 = 8$, while it suddenly disappears for $N_1 = 7$ or less.

We next observe the market evolution and decline when only RT and RA corporations interact each other without presence of RN corporations. We assume that $N = 35$ and $N_2 = 0$. In Figures 4.2.22 through 4.2.27, $E[X(t)]$ and $Var[X(t)]$ are exhibited for $(N_1, N_2, N_3) = (35,0,0)$, $(30,0,5)$, $(25,0,10)$, $(24,0,11)$, $(23,0,12)$, $(22,0,13)$, $(21,0,14)$, $(20,0,15)$, $(19,0,16)$, $(15,0,20)$, $(10,0,25)$, $(5,0,30)$ and $(0,0,35)$.

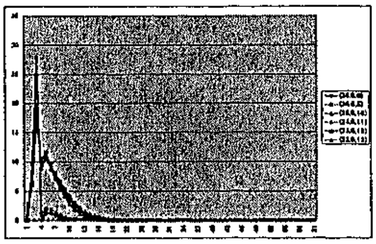

Figure 4.2.22: $E[X(t)]$ when $N_2 = 0$ part 1

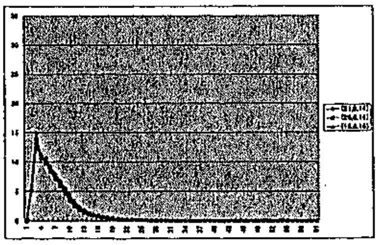

Figure 4.2.23: $E[X(t)]$ when $N_2 = 0$ part 2

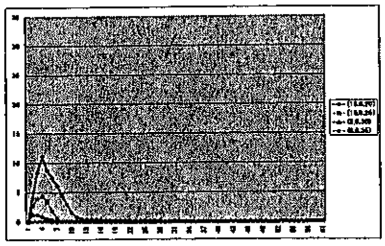

Figure 4.2.24: $E[X(t)]$ when $N_2 = 0$ part 3

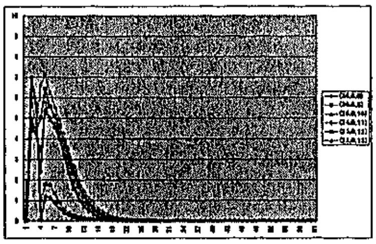

Figure 4.2.25: $Var[X(t)]$ when $N_2 = 0$ part 1

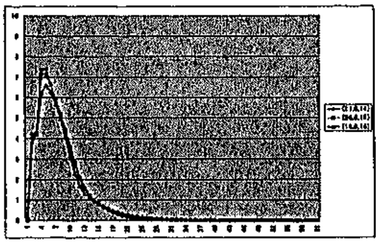

Figure 4.2.26: $Var[X(t)]$ when $N_2 = 0$ part 2

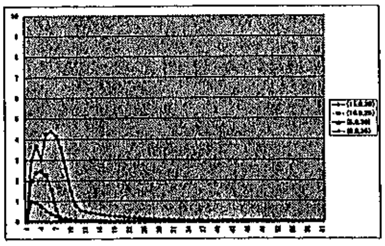

Figure 4.2.27: $Var[X(t)]$ when $N_2 = 0$ part 3

As N_1 decreases from 35 to 21 while N_3 increases from 0 to 14, the market seems to start forming the maturity stage by gradual entries of RA corporations triggered by rapid entries of RT corporations. However, this momentum is much weaker than the case of the RT-RN combination observed previously, and disappears when N_1 becomes 20 or less. We may conclude that the ability of RA corporations to sustain the maturity stage is much weaker than that of RN corporations.

The fact that RA corporations lack the ability to sustain the maturity stage alone can also be observed from a different angle. By comparing Figure 4.2.14 for $(N_1, N_2, N_3) = (10, 15, 10)$ with Figure 4.2.17 for $(N_1, N_2, N_3) = (10, 25, 0)$, one realizes that the maturity stage for the former is shorter than that for the latter. One may then suspect that RA corporations indeed contribute to shorten the maturity stage. In order to examine this point, we next fix $N = 35$ and $N_1 = 10$, and change (N_2, N_3). $E[X(t)]$ and $Var[X(t)]$ for $(N_1, N_2, N_3) = (10, 25, 0)$, $(10, 20, 5)$, $(10, 15, 10)$, $(10, 14, 11)$, $(10, 13, 12)$, $(10, 12, 13)$, $(10, 11, 14)$, $(10, 10, 15)$, $(10, 5, 20)$ and $(10, 0, 25)$ are depicted in Figures 4.2.28 through 4.2.31.

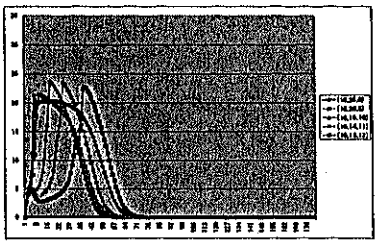

Figure 4.2.28: $E[X(t)]$ when $N_1 = 10$ part 1

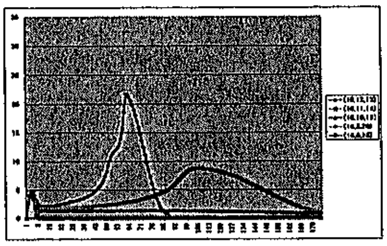

Figure 4.2.29: $E[X(t)]$ when $N_1 = 10$ part 2

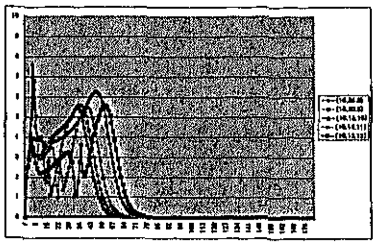

Figure 4.2.30: $Var[X(t)]$ when $N_1 = 10$ part 1

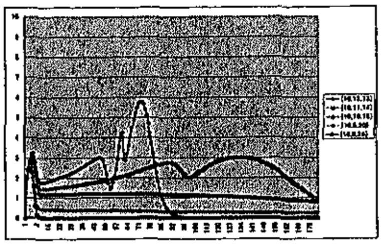

Figure 4.2.31: $Var[X(t)]$ when $N_1 = 10$ part 2

It can be clearly seen that the maturity stage becomes shorter as N_2 decreases and N_3 increases. Furthermore the shape of the market life cycle experiences a sudden distortion at $(N_1, N_2, N_3) = (10, 11, 14)$ and the market loses its growth momentum and almost disappears beyond $N_2 = 10$ or less.

5 Concluding Remarks

In this paper, an analytical model is developed for understanding the market life cycle through strategic policies of individual corporations potentially interested in entering into the market. Strategic policies of individual corporations are expressed in terms of probabilities of entry into and retreat from the market, which may depend on time t, the number of corporations in the market at time t, $X(t)$, and the number of corporations which have retreated from the market, $Y(t)$. Accordingly, each corporation is modelled as a temporally inhomogeneous Markov chain, and $\{(X(t), Y(t)) : t \geq 0\}$ is expressed as a sum of such Markov chains. Through spectral analysis of the underlying temporally inhomogeneous Markov chains combined with bivariate generating function approach, a numerical algorithm is developed for computing the joint probability distribution of $(X(t), Y(t))$ for $t = 1, 2, \cdots$, capturing the characteristics of the market life cycle in terms of $E[X(t)]$ and $Var[X(t)]$.

Concerning entry and retreat probabilities of individual corporations, two models are considered. In Model I, these probabilities depend only on time t, and do not depend on $(X(t), Y(t))$, while in Model II, they depend only on $(X(t), Y(t))$ and do not depend on time t. For Model I, there is no interaction effect among the three categories, and a direct sum of strategic policies of individual corporations constitutes the market life cycle as a whole. Nevertheless, we could still observe the market life cycle with four stages largely due to RN and RA corporations. For Model II, the market life cycle cannot be expressed as a straightforward sum of individual corporations because of sophisticated interactions among the three categories, RT, RN and RA. Each category plays a distinguishable role in forming the market life cycle. The characteristics of each category can be summarized as follows.

A) No category alone can constitute a typical market life cycle with distinguishable four stages.

B) RT corporations trigger the creation of the market, motivating RN and RA corporations to join the market.

C) RN corporations play a major role in the growth stage and the maturity stage, stabilizing the market state.

D) RA corporations also contribute to form the maturity stage but take a leading role in initializing the decline stage.

In summary, Model II enables one to understand how strategic policies of individual corporations collectively form the market life cycle with four stages. While individual corporations make their own decisions separately, the market as a whole may emerge in a way that cannot be explained in terms of the characteristics of individual categories. Constructing this mechanism through an analytical model is the major contribution of this paper.

References

[1] Bass, F. M. (1969), "A New Product Growth Model for Consumer Durables," *Management Science*, Vol. 15 (January), pp. 215-227.

[2] Geroski, P. A. and Mazzucato, M. (2001), "Modelling the Dynamics of Industry Populations," *International Journal of Industrial Organization*, Vol. 19 (July), pp. 1003-1022.

[3] Horsky, D. (1990), "A Diffusion Model Incorporating Product Benefits, Price, Income and Information," *Marketing Science*, Vol. 9 (Fall), pp.342-365.

[4] Horsky, D. and L. S. Simon (1983), "Advertising and the Diffusion of New Products," *Marketing Science*, Vol. 2 (Winter), pp. 1-17.

[5] Simons, K. (1995), "Shakeouts: Firm Survival and Technological Change in New Manufacturing Industries," PhD dissertation, Department of Social and Decision Sciences, Carnegie Mellon University.