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Regret region hypothesis testing using unbiased statistics---  
Application to the left-retracted exponential distribution.

by

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Summary.

Hypothesis testings using unbiased statistics (or estimates) are neglected for a long time. Actually, hypothesis testings for mean and variance of the normal distribution are of this type. The tests for the regression coefficients of the regression analysis and F-tests in the experimental design are also of this type. In this paper we introduce an optimal hypothesis testing with the regret region derived from inverting the minimum random interval based on the unbiased statistic for the parameter obtained by using Lagrange's multiplier. As an example the tests for two parameters of the left-retracted exponential distribution are considered.

*Key Words and Phrases:* unbiased statistics, unbiased estimates, hypothesis testing, minimum random interval, the logistic distribution, regret region, regret-probability function.

1. Introduction.

Let  $X_1, \dots, X_n$  be a random sample of size  $n$  taken from the density  $f(x|\theta)$  with a real parameter  $\theta$ . Let  $T(X_1, \dots, X_n)$  be a statistic for  $\theta$ . Let  $=:$  be the defining property. We call  $T(X_1, \dots, X_n)(=:T)$  an unbiased statistic (or estimate) for  $\theta$  when  $E(T)=\theta$ . In this paper we first, in Sections 2 and 3, consider to test the null hypothesis  $H_0:\theta=\theta_0$  versus the alternative hypothesis  $H_1:\theta\neq\theta_0$  (with known real  $\theta_0$ ). Let  $\alpha$  be the real number such that  $0<\alpha<1$ . We call  $(U_1, U_2)$  a  $(1-\alpha)$  random interval for the parameter  $\gamma$  if  $P_\gamma[U_1 < \gamma < U_2]=1-\alpha$ . To get the optimal two-edged tests we find the minimum  $(1-\alpha)$  random intervals for  $\theta$  based on the unbiased statistic  $T$  and construct the regret regions derived from inverting this random intervals for  $\theta_0$ . The name of the regret region  $D$  comes from the fact that we regret not rejecting  $H_0$  if  $T \in D$  and satisfy to reject  $H_0$  if  $T \notin D$ . We define the regret-probability function by  $\phi(\theta):=P_{\theta_0}(D)$ ,  $\forall \theta$ .

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Our optimal test is the one which makes  $\phi(\theta)$  a concave function from below and maximizes  $\phi(\theta)$  at  $\theta_0$  with  $\phi(\theta_0)=1-\alpha$ . The method introduced here also applies for testing the hypotheses  $H_0': \theta \geq \theta_0$  versus  $H_1': \theta < \theta_0$  and for testing the hypotheses  $H_0'': \theta \leq \theta_0$  versus  $H_1'': \theta > \theta_0$  as well. (See Sections 6 and 7.)

As the example we consider the left-retracted exponential distribution with the density

$$f(x|\theta, b) = b^{-1} e^{-(x-\theta)/b}, \quad \text{for } \theta < x \quad (1)$$

where  $-\infty < \theta < \infty$  and  $b > 0$ . In Sections 2 and 6 we assume that  $b$  is known and in Sections 3 and 7 we deal with the tests for  $\theta$  with unknown  $b$ . In Section 5 we introduce the two-edged test for  $(\theta, b)$ . In Section 4 we test unknown  $b$ . This family (1) of distributions has been used in many areas of application; for example, survival analysis, life testing and reliability theory et. al..

Based on i.i.d. observations  $X_1, \dots, X_n$  from (1) we consider, in Sections 2 and 3 to test the null hypothesis  $H_0: \theta = \theta_0$  versus the alternative hypothesis  $H_1: \theta \neq \theta_0$ . In Section 2, to obtain the optimal two-edged test we find the minimum random interval for  $\theta$ , using the unbiased statistic  $Y := \bar{X} - b$  ( $= n^{-1} \sum_{i=1}^n X_i - b$ ) for  $\theta$  and construct the regret region derived from inverting this random interval for  $\theta_0$ . In Section 3 we deal with the two-edged-test with unknown  $b$ . In Section 4 we consider the two-edged test for  $b$ . Let  $b_0$  be a positive constant. In Section 5 we introduce the two-edged test for testing the hypotheses  $H_0: \theta = \theta_0, b = b_0$  versus  $H_1: \text{At least one equality in } H_0 \text{ fails}$ . In Section 6 we derive the optimal one-edged test for testing the hypothesis  $H_0': \theta \geq \theta_0$  versus the alternative hypothesis  $H_1': \theta < \theta_0$ , with known  $b$ . In Section 7 we deal with the one-edged tests for  $\theta$  with unknown  $b$ .

## 2. The Two-edged test for $\theta$ with known $b$ .

Let  $X_1, \dots, X_n$  be a random sample of size  $n$  taken from (1) with known  $b$ . We consider the problem of testing the hypotheses  $H_0: \theta = \theta_0$  versus  $H_1: \theta \neq \theta_0$ . We first derive necessary distributions to find the minimum random interval for  $\theta$ .

As the statistic for  $\theta$  we take  $Y := \bar{X} - b$ . We can easily check  $E(Y) = \theta$ . Let  $X_{(i)}$  be the  $i$ -th smallest observation such that  $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$ . We find the



But, by the transformation  $t=2n(y+b-\theta)/b$  (4) is equivalent to

$$P[t_1 < T < t_2] = 1 - \alpha \quad (5)$$

where  $t_i := 2n(r_i + b)/b$  for  $i=1, 2$ . Hence, we want to minimize  $t_2 - t_1$  subject to the condition (5). Let  $\lambda$  be a Lagrange's multiplier and define

$$L := t_2 - t_1 - \lambda \left\{ \int_{t_1}^{t_2} h_n(t) dt - 1 + \alpha \right\}. \quad (6)$$

Then,  $\partial L / \partial t_1 = 0 = \partial L / \partial t_2$  leads to

$$h_n(t_1) = h_n(t_2) \quad (= \lambda^{-1}). \quad (7)$$

Taking  $t_1$  and  $t_2$  which satisfy (7) and  $\partial L / \partial \lambda = 0$ , noticing that  $t_1 < T = 2n(Y + b - \theta) / b < t_2$  and letting  $t_3 = bt_1 / (2n)$  and  $t_4 = bt_2 / (2n)$  we obtain the minimum  $(1 - \alpha)$  random interval for  $\theta$  as follows:

$$(Y + b - t_4, Y + b - t_3). \quad (8)$$

Hence, by inverting (8) for  $\theta_0$  our regret region based on  $(W, V)$  becomes  $\theta_0 + t_3 - b < Y < \theta_0 + t_4 - b$  and  $V \geq \theta_0$  where  $Y = n^{-1}W - b$ . Here, we emphasize the necessity of having the set  $\{V \geq \theta_0\}$  in the regret region. To check the optimality of this test we obtain the regret-probability function of the test as follows:

$$\phi(\theta) = P_0[\theta_0 + t_3 - b < Y < \theta_0 + t_4 - b \text{ and } V \geq \theta_0]$$

$$\phi(\theta) = \begin{cases} (1 - \alpha) \exp\{-n(\theta_0 - \theta) / b\}, & \text{for } \theta < \theta_0 \\ P[t_1 - 2n(\theta - \theta_0) / b < T < t_2 - 2n(\theta - \theta_0) / b], & \text{for } \theta_0 \leq \theta < \theta_0 + t_3 \\ P[0 < T \leq t_2 - 2n(\theta - \theta_0) / b], & \text{for } \theta_0 + t_3 \leq \theta < \theta_0 + t_4 \\ 0, & \text{for } \theta_0 + t_4 \leq \theta. \end{cases} \quad (9)$$

Hence,  $d\phi(\theta) / d\theta > 0$  for  $\theta < \theta_0$ ;  $d\phi(\theta) / d\theta = 2nb^{-1} \{ h_T(t_1 - 2n(\theta - \theta_0) / b) - h_T(t_2 - 2n(\theta - \theta_0) / b) \}$

$< 0$  for  $\theta_0 \leq \theta < \theta_0 + t_3$  because of (7) and (3), and  $d\phi(\theta)/d\theta < 0$  for  $\theta_0 + t_3 \leq \theta < \theta_0 + t_4$ . Since  $\phi'(\theta_0) = 0$  by (7) and  $\phi(\theta_0) = 1 - \alpha$ , we have that  $\phi(\theta) \leq 1 - \alpha$  for real  $\theta$ . Thus, the optimality of the test is proved.

In the next section we consider the two-edged test with unknown  $b$ .

### 3. The two-edged tests for $\theta$ with unknown $b$ .

In this section we let  $b$  be unknown. We test the hypotheses  $H_0: \theta = \theta_0$  versus  $H_1: \theta \neq \theta_0$ . Let  $W$  and  $V$  be as defined in Section 2. Since  $(W - nV)/(n-1)$  is an unbiased statistic for  $b$ , we use the statistic

$$Z := (n-1)(n^{-1}W - \theta)/(W - nV)$$

and furthermore the test statistic

$$S := \{\log_e(n/(n-1)) + \log_e Z\}^{1/2} \quad (9)$$

whose density has one peak. Since from (2) and a little calculation, the p. d. f. of  $Z$  is obtained by

$$g_Z(z) = (n-1)^n / \{n^{n-1} z^n\}, \quad \text{for } (n-1)/n \leq z; = 0, \text{ otherwise,} \quad (10)$$

from the transformation (9) applied to (10) we obtain the density of  $S$  as follows:

$$g_S(s) = 2(n-1) \exp\{-(n-1)s^2\} s, \quad \text{for } 0 \leq s; = 0, \text{ for } s < 0. \quad (11)$$

Let  $r_1$  and  $r_2$  be positive numbers such that  $r_1 < r_2$ ,

$$\int_{r_1}^{r_2} g_S(s) ds = 1 - \alpha$$

and

$$g_S(r_1) = g_S(r_2).$$

For such  $r_1$  and  $r_2$  we have that  $r_1 < S < r_2$  or  $\exp\{r_1^2\} < (W - n\theta)/(W - nV) < \exp\{r_2^2\}$ .

Hence, the minimum  $(1-\alpha)$  random interval for  $\theta$  is

$$(n^{-1}(W-\exp\{r_2^2\}(W-nV)), n^{-1}(W-\exp\{r_1^2\}(W-nV))). \quad (12)$$

Thus, our regret region based on  $(W, V)$  for testing  $H_0$  versus  $H_1$  is as follows:

$$\{(W, V): n^{-1}(W-\exp\{r_2^2\}(W-nV)) < \theta_0 < n^{-1}(W-\exp\{r_1^2\}(W-nV)) \text{ and } \theta_0 \leq V\}.$$

In the next section we show two-edged tests for  $b$ .

#### 4. The two-edged tests for $b$ .

In this section we consider to test the hypotheses  $H_0: b=b_0$  versus  $H_1: b \neq b_0$ .

We first assume that  $\theta$  is known. Let  $W$  and  $V$  be as defined in Section 2.

We use the test statistic  $T:=2(W-n\theta)/b$ . From Section 2 the density of  $T$  is given by (3). Taking positive numbers  $t_1$  and  $t_2$  which satisfy (5), (6) and (7) we obtain the minimum  $(1-\alpha)$  random interval for  $b$  as follows:

$$((W-n\theta)/t_2, (W-n\theta)/t_1). \quad (13)$$

By inverting (13) for  $b_0$  our regret region based on  $W$  becomes

$$n\theta + b_0 t_1 < W < n\theta + b_0 t_2.$$

Secondly, we assume that  $\theta$  is unknown. We consider another test based on the statistic  $S:=2(W-nV)/b$ . From (2) and a little computation we get the density  $h_{n-1}(s)$  of  $S$  where  $h_n(s)$  is given by (3). Hereafter, we proceed the same way as above. Taking positive numbers  $t_1$  and  $t_2$  which satisfy (5), (6) and (7) with  $T$  and  $n$  there replaced by  $S$  and  $n-1$ , respectively we obtain the minimum random interval for  $b$  as follows:

$$((W-nV)/t_2, (W-nV)/t_1). \quad (14)$$

Thus, by inverting (14) for  $b_0$  the regret region of our test is as follows:

$$\{(W, V) : (W-nV)/t_2 < b_0 < (W-nV)/t_1\}.$$

In the next section we introduce the two-edged test for  $(\theta, b)$ .

### 5. The two-edged test for $(\theta, b)$ .

In this section we consider to test the hypothesis  $H_0: \theta = \theta_0, b = b_0$  versus the alternative hypothesis  $H_1$ : At least one equality in  $H_0$  fails. Let  $W$  and  $V$  be as defined in Section 2. We let  $S := (W-nV)^{1/2}$ . To get the two-edged test for  $(\theta, b)$  we use the statistic

$$Z := 2\sqrt{n-1}(n^{-1}W - \theta) / (\sqrt{b}S).$$

We find the density of  $Z$ , apply Lagrange's multiplier to get the minimum random interval for  $(\theta, b)$  and from this random interval obtain the regret region based on  $(W, V)$  for the two-edged test with respect to  $(\theta, b)$ .

We first find the density of  $Z$  and show that this density of  $Z$  has one peak. Let  $U := n(V - \theta)/S$ . To get the density of  $Z$  we first find the density of  $(S, V)$  from (2) of  $(W, V)$ , then the density of  $(S, U)$  from the density of  $(S, V)$  and finally the density of  $(S, Z)$  with the transformation  $Z = \{2\sqrt{n-1}/(n\sqrt{b})\}(S+U)$  from the density of  $(S, U)$ . Obtained density of  $(S, Z)$  is as follows:

$$h_{S, Z}(s, z) = \begin{cases} \frac{n}{\{b^{n-1/2} \Gamma(n-1) \sqrt{n-1}\}} s^{2n-2} \exp\{-nzs / (2\sqrt{(n-1)b})\}, & (15) \\ 0, & \text{for } 0 \leq \{2\sqrt{n-1}s / (n\sqrt{b})\} \leq z \\ & \text{otherwise.} \end{cases}$$

Let

$$H_{2n-1}(z) := \int_{-\infty}^z h_{2n-1}(t) dt, \quad \forall z.$$

Integrating out  $s$  from (15) we get the density of  $z$  as follows:

$$h_z(z) = cz^{-(2n-1)} H_{2n-1}(n^2 z^2 / (n-1)), \quad \forall z \geq 0; = 0, \quad \forall z < 0,$$

where

$$c := (n-1)^{n-1} 2^{2n-1} \Gamma(2n-1) / \{n^{2n-2} \Gamma(n-1)\}.$$

Since  $h_z(0) = 0 = h_z(-\infty)$ , we show that there exists unique  $z (> 0)$  such that  $h_z'(z) = 0$ . Now,

$$h_z'(z) = c \{ -(2n-1) z^{-2n} H_{2n-1}(n^2 z^2 / (n-1)) + \{2n^2 / (n-1)\} z^{-(2n-2)} h_{2n-1}(n^2 z^2 / (n-1)) \}.$$

Let  $\xi(z) := (2n^2 z^2 / \{(n-1)(2n-1)\}) h_{2n-1}(n^2 z^2 / (n-1)) = c_1 z^2 h_{2n-1}(n^2 z^2 / (n-1))$ . We would like to show that there exists unique  $z$  such that  $\xi(z) = H_{2n-1}(n^2 z^2 / (n-1))$ . Since

$$\xi'(z) = c_1 z \{4n-2-n^2 z^2 / (n-1)\} h_{2n-1}(z)$$

and

$$dH_{2n-1}(n^2 z^2 / (n-1)) / dz = 2n^2 z / (n-1) h_{2n-1}(n^2 z^2 / (n-1)),$$

we have that  $\xi'(z) > dH_{2n-1}(n^2 z^2 / (n-1)) / dz (> 0)$  for  $0 < z < \sqrt{(2n-1)(n-1)} / n$ . Thus, there exists unique  $z (> 0)$  such that  $h_z'(z) = 0$ . Therefore,  $h_z(z)$  is the density with one peak.

To find the minimum  $(1-\alpha)$  random interval for  $(\theta, b)$  we let  $r_1$  and  $r_2$  be positive numbers such that  $r_1 < r_2$  and want to minimize  $r_2 - r_1$  subject to

$$P_\theta[r_1 < Z < r_2] = 1 - \alpha. \tag{16}$$

In the same fashion as (5), (6) and (7) we obtain

$$h_z(r_1) = h_z(r_2). \tag{17}$$

Hence, the minimum  $(1-\alpha)$  random interval for  $(\theta, b)$  is as follows:

$$\{(\theta, b): r_1 < 2\sqrt{n-1}(n^{-1}W-\theta)/\sqrt{b(W-nV)} < r_2\},$$

where  $r_1$  and  $r_2$  satisfy (16) and (17). Thus, our regret region based on  $(W, V)$  becomes as follows:

$$\{(W, V): r_1 < 2\sqrt{n-1}(n^{-1}W-\theta_0)/\sqrt{b_0(W-nV)} < r_2 \text{ and } \theta_0 \leq V\}.$$

In the next two sections we deal with the one-edged tests.

#### 6. The one-edged test for $\theta$ with known $b$ .

In this section we consider to test the hypotheses  $H_0': \theta \geq \theta_0$  versus  $H_1': \theta < \theta_0$  based on a random sample  $X_1, \dots, X_n$  from (1) with known  $b$ . Let  $Y, W, V$  and  $T$  be as defined in Section 2. Let  $t_\alpha := bt_\alpha/(2n)$  where  $t_\alpha$  is given by

$$\int_{t_\alpha}^{\infty} h_n(t) dt = 1 - \alpha.$$

Here,  $h_n(t)$  is given by (3). From Section 2 we can easily see that our test has the following regret region based on  $(W, V)$ :

$$\{(W, V): \theta_0 + t_\alpha - b < Y \text{ and } \theta_0 \leq V\}$$

where  $Y := n^{-1}W - b$ . Let  $g_{Y, V}(y, v|\theta)$  be the joint density of  $(Y, V)$ . Then, from (2) and the relation  $g_{Y, V}(y, v|\theta) = g(n(y+b), v|\theta)n$  we can easily get the regret-probability function of above test as follows:

$$\phi(\theta) = P_\theta[\theta_0 + t_\alpha - b < Y \text{ and } \theta_0 \leq V]$$

$$\phi(\theta) = \begin{cases} (1-\alpha) \exp\{-n(\theta_0 - \theta)/b\}, & \text{for } \theta < \theta_0 \\ P[t_\alpha - 2n(\theta - \theta_0)/b \leq T], & \text{for } \theta_0 \leq \theta < \theta_0 + t_\alpha \\ 0, & \text{for } \theta_0 + t_\alpha \leq \theta. \end{cases}$$

Since  $d\phi(\theta)/d\theta > 0$  for  $\theta < \theta_0 + t_0$  and hence  $\phi(\theta) \leq 1 - \alpha = \phi(\theta_0)$  for  $\theta < \theta_0$ , our test has optimal property.

In the next section we consider one-edged tests for  $\theta$  with unknown  $b$ .

#### 7. The one-edged tests for $\theta$ with unknown $b$ .

We consider to test the hypotheses  $H_0' : \theta \geq \theta_0$  versus  $H_1' : \theta < \theta_0$  based on a random sample  $X_1, \dots, X_n$  from (1) with unknown  $b$ . Let  $W$  and  $V$  be as in Section 2.

Let  $r_1$  be a positive number such that

$$\int_{r_1}^{\infty} g_S(s) ds = 1 - \alpha$$

where  $g_S(s)$  is given by (11). From (12) in Section 3 we can easily see that the minimum  $(1 - \alpha)$  random interval for  $\theta$  is obtained as follows:

$$(-\infty, n^{-1}(W - \exp\{r_1^2\}(W - nV))).$$

Hence, our regret region based on  $(W, V)$  is as follows:

$$\{(W, V) : \theta_0 < n^{-1}(W - \exp\{r_1^2\}(W - nV)) \text{ and } \theta_0 \leq V\}.$$

We next consider to test the hypotheses  $H_0'' : \theta \leq \theta_0$  versus  $H_1'' : \theta > \theta_0$ . Let  $r_2$  be a positive number such that

$$\int_0^{r_2} g_S(s) ds = 1 - \alpha.$$

From (13) in Section 3 the minimum  $(1 - \alpha)$  random interval for  $\theta$  is as follows:

$$(n^{-1}(W - \exp\{r_2^2\}(W - nV)), \infty).$$

Hence, our regret region based on  $(W, V)$  is as follows:

$$\{(W, V) : n^{-1}(W - \exp\{r_2^2\}(W - nV)) < \theta_0\}.$$