No. 992

Multiserver Queue with Semi-Markovian Batch Plus Poisson Arrivals with Application to the MPEG Frame Sequence and Interfering Traffic

by

De-An Wu and Hideaki Takagi

June 2002

Multiserver Queue with Semi-Markovian Batch Plus Poisson Arrivals with Application to the MPEG Frame Sequence and Interfering Traffic

De-An Wu
Doctoral Program in Policy and Planning Sciences, University of Tsukuba
1-1-1 Tennoudai, Tsukuba-shi, Ibaraki 305-8573, Japan
phone: +81-298-53-5167; fax: +81-298-53-5167
e-mail: tbu@sk.tsukuba.ac.jp

and

Hideaki Takagi
Vice President, University of Tsukuba
1-1-1 Tennoudai, Tsukuba-shi, Ibaraki 305-8577, Japan
phone: +81-298-53-2005; fax: +81-298-53-6310
e-mail: takagi@sk.tsukuba.ac.jp

Abstract

We consider a multiserver queueing system having a mixture of a semi-Markov process (SMP) and a Poisson process as the arrival process, where each SMP arrival contains a batch of customers. The service times are exponentially distributed. We derive the distributions of the queue size and the waiting times of both SMP and Poisson customers. Based on the result of the analysis, we propose a model to evaluate the waiting time of MPEG video traffic on an ATM network with multiple channels. Here, SMP arrivals correspond to the exact sequence of Motion Picture Experts Group (MPEG) frames and Poisson arrivals are regarded as interfering traffic. In the numerical examples, the mean and variance of the waiting time of the ATM cells generated from the MPEG frames of real video data are evaluated. It is observed that the number of channels does not influence much on the waiting time if the total transmission rate is kept constant.

Key words: Semi-Markov process; batch arrival; multiserver queue; waiting time; MPEG; Rouché's theorem

1 Introduction

We consider a queueing system consisting of multiple identical servers and a common queue. The service time follows an exponential distribution. The queueing system is fed by a mixture of a semi-Markov batch and a Poisson arrival process and the capacity of a waiting room is infinite. This system is denoted by $\mathrm{SMP}^{[\mathrm{X}]}+\mathrm{M}/\mathrm{M}/c$ throughout the paper. The motivation to study the queueing system with semi-Markov process (SMP) arrivals lies in the assumption that it can model the auto-correlated traffic on the high speed network generated by a real time communication, for example, Motion Picture Experts Group (MPEG) -encoded variable bit rate (VBR) video.

The semi-Markov arrival process can be described as follows: There are L types of customers numbered 1 through L. Customers arrive at time epochs $0 = T_0 < T_1 < T_2 < \dots$. Then $A_n := T_n - T_{n-1}$, $n > 1$, is the interarrival time, and we set $A_0 := 0$. Let $S^{(n)}$ denote the type of a customer arriving at epoch T_n. For a given sequence of arrival epochs, all interarrival times are mutually independent. It is assumed that A_{n+1} and $S^{(n+1)}$ depend only on $S^{(n)}$, i.e.,

$$P\{S^{(n+1)} = l, A_{n+1} \le t | S^{(0)}, \dots , S^{(n)}, A_1, \dots , A_n\} = P\{S^{(n+1)} = l, A_{n+1} \le t | S^{(n)}\}; \quad l = 1, \dots , L \, ; \; t \ge 0. \quad (1)$$

Let

$$Q_{lm}(t) := P\{S^{(n+1)} = m, A_{n+1} \le t | S^{(n)} = l\}$$

be the probability that the arrival process moves from state l to state m in time t. Let p_{lm} denote the probability that the arrival of type l is followed by the arrival of type m, and let $A_{lm}(t)$ be the distribution function of the time interval A_{lm} between those successive arrivals. A_{lm} is also referred to as the state sojourn time. Thus, we have

$$Q_{lm}(t) = p_{lm} A_{lm}(t); \quad t \ge 0,$$

where

$$Q_{lm}(\infty) = p_{lm}. \quad (2)$$

Hereafter we use a matrix $\mathbf{P}= (p_{lm})$, which is a stochastic matrix. Let $\boldsymbol{\pi} := [\pi_1, \dots , \pi_L]$ be the stationary distribution of stochastic matrix $\mathbf{P}$. Then, we have the relations

$$\mathbf{P1} = \mathbf{1},$$

and

$$\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}; \quad \boldsymbol{\pi}\mathbf{1} = 1, \quad (3)$$

where $\mathbf{1} := [1, \dots , 1]$.

Since Çinlar [2] first analyzed an SMP/M/1 queue, several queueing systems with arrivals governed by a semi-Markov process (SMP) have been studied, for example, an SMP/M/c queue [10, 18], an SMP/G/1 queue [1, 5, 15], and an SMP/PH/1 queue [14]. On the other hand, Kuczura [8] studies a piecewise Markov process. Based on this theory, Kuczura [7] analyzes a GI+M/M/1 queue in which the arrival process is a mixture of a renewal process and a Poisson process. Yagyu and Takagi [19] consider an SSMP$^{[X]}$+M/M/1 queue, where the SSMP stands for a special semi-Markov process such that the sojourn time in the state of SMP is determined only by the current state. Wu and Takagi [17] extend this model to a more general case, namely, an SMP$^{[X]}$+M/M/1 queue. The method for dealing with an SMP$^{[X]}$+M/M/c queue in this paper is also the theory of piecewise Markov process.

The rest of this paper is organized as follows. In Section 2 we derive the formulas for a birth-and-death process as preliminaries. In Section 3, a Markov chain that describes a queue size process in an $\text{SMP}^{[X]}$+M/M/c system is introduced and the generating function for the steady-state queue size is derived. The stability condition of this system is discussed in Section 4. In Appendix, we prove that the unknown constants contained in the generating function for the queue size can be determined by the zeros of the denominator for this generating function when the sojourn time in the state of SMP follows an exponential distribution. The waiting time distributions for SMP and Poisson customers are studied in Section 5. In Section 6, we use an $\text{SMP}^{[X]}$+M/M/c queue to model the transmission of MPEG frames in multiple channels on an ATM network interfering with other traffic. A numerical example using real data taken from the Jurassic Park video is also given.

2 Transient Behavior of a Birth-and-Death Process

We first consider a birth-and-death process $\{X(t); t \geq 0\}$ for a population whose size changes by the births and deaths of its individuals. $X(t)$ denotes the population size at time t. The process $\{X(t); t \geq 0\}$ is a continuous-time Markov process with state space $\{0, 1, 2, \dots\}$ with $X(0) = i$ $(i \geq 0)$. In a short interval $[t, t + \Delta t)$, $X(t)$ increases by one with probability $\lambda \Delta t$, or decreases by one with probability $i\mu\Delta t$ if $X(t) = i$ is less than c (a constant positive integer), otherwise the death probability is $c\mu\Delta t$. We define the transition probability for the population size

$$P_{i,j}(t) := P\{X(t) = j | X(0) = i\}; \quad i, j = 0, 1, \dots; \quad t \geq 0 \tag{4}$$

and the generating function of its Laplace transform

$$P_i^*(s, z) := \sum_{j=0}^{\infty} P_{i,j}^*(s) z^j, \tag{5}$$

where

$$P_{i,j}^*(s) := \int_0^{\infty} e^{-st} P_{i,j}(t) dt. \tag{6}$$

By the argument that the transition can occur only between adjacent states in a short interval $[t, t + \Delta t)$ for the birth-and-death process, we have the following Kolmogorov forward equations:

$$\begin{aligned}
P'_{i,0}(t) &= -\lambda P_{i,0}(t) + \mu P_{i,1}(t), \\
P'_{i,j}(t) &= \lambda P_{i,j-1}(t) - (\lambda + j\mu) P_{i,j}(t) + (j+1)\mu P_{i,j+1}(t); \quad 1 \leq j \leq c-1, \\
P'_{i,j}(t) &= \lambda P_{i,j-1}(t) - (\lambda + c\mu) P_{i,j}(t) + c\mu P_{i,j+1}(t); \quad j \geq c
\end{aligned}$$

with the initial condition

$$P_{i,j}(0) = \delta_{ij}; \quad i, j = 0, 1, \dots,$$

where δ_{ij} is the Kronecker delta.

Taking the Laplace transform, we obtain

$$
\begin{aligned}
sP^*_{i,0}(s) &= \delta_{i0} - \lambda P^*_{i,0}(s) + \mu P^*_{i,1}(s), && \text{(7a)}\\
sP^*_{i,j}(s) &= \delta_{ij} + \lambda P^*_{i,j-1}(s) - (\lambda + j\mu)P^*_{i,j}(s) + (j+1)\mu P^*_{i,j+1}(s); 1 \le j \le c-1, && \text{(7b)}\\
sP^*_{i,j}(s) &= \delta_{ij} + \lambda P^*_{i,j-1}(s) - (\lambda + c\mu)P^*_{i,j}(s) + c\mu P^*_{i,j+1}(s); \quad j \ge c. && \text{(7c)}
\end{aligned}
$$

Multiplying (7a)–(7c) by z^j and summing over j from 0 to ∞, then changing the order of summation yields

$$
P^*_i(s,z) = \frac{z^{i+1} - (1-z)\mu \sum_{j=0}^{c-1}(c-j)P^*_{i,j}(s)z^j}{sz - (1-z)(c\mu - \lambda z)}. \tag{8}
$$

We note that the denominator on the right-side hand of (8) has a unique zero in the unit disk, which is

$$
z_1 = \frac{\lambda + c\mu + s - \sqrt{(\lambda + c\mu + s)^2 - 4c\lambda\mu}}{2\lambda}.
$$

In order to determine c unknown functions $\{P^*_{i,j}(s);\, 0 \le j \le c-1\}$, the same number of equations are required. One of them is given by the condition that the numerator and denominator on the right-side hand of (8) must vanish at $z = z_1$ since the generation function $P^*_i(s,z)$ is analytic for $|z| < 1$ and $\Re(s) > 0$. Other equations are given by (7a) and (7b). In the analysis of an SMP$^{[X]}$+M/M/c queue, however, it is not necessary to find $\{P^*_{i,j}(s);\, 0 \le j \le c-1\}$ explicitly.

3 Queue Size in an SMP$^{[X]}$+M/M/c System

We now consider a queueing system SMP$^{[X]}$+M/M/c, which consists of c identical severs and a common queue with an infinite buffer served on a first-come first-served (FCFS) basis. The queue is fed by a mixture of a semi-Markov batch arrival process and a Poisson arrival process. The semi-Markov batch arrival process is an extension of the semi-Markov arrival process described in Section 1 such that each arrival consists of a batch of a random number of customers. Let $g_l(k)$ denote the probability of the batch size being k for type l, $l = 1, \dots, L$. We say that the system enters state l when a batch of type l arrives. The rate of the Poisson arrival process is denoted by λ. The service time distribution for each of SMP and Poisson customers is exponential with the same mean $1/\mu$.

We analyze the queue size in an SMP$^{[X]}$+M/M/c system. Let $X(t)$ denote the number of both SMP and Poisson customers present in the system (queue size), including the customers both in service and waiting, at time t. Since the queue size process $X(t)$ behaves exactly like a birth-and-death process, described in Section 2, between the successive arrival epochs of SMP customers, we call the time segment between those arrivals a Markovian segment. The start and termination of each Markovian segment are caused by the arrivals of SMP customers. Thus the queue size immediately after an

SMP arrival determines the queue size immediately before the next SMP batch arrival in accordance with a Markov chain. This fact suggests that the queue size process of an SMP$^{[X]}$+M/M/c system can be analyzed by the theory of a piecewise Markov process proposed by Kuczura [7]. We do so by means of a discrete-time Markov chain of two random variables $\{(X^{(n)}, S^{(n)}); n = 0, 1, 2, \dots\}$, where $X^{(n)}$ denotes the queue size seen by nth SMP arrival, and $S^{(n)}$ denotes the state of the underlying SMP immediately after that arrival.

The state transition probability of the time-homogeneous Markov chain $\{(X^{(n)}, S^{(n)});$ $n = 0, 1, 2 \dots\}$ is given by

$$P\{X^{(n+1)} = j, S^{(n+1)} = m | X^{(n)} = i, S^{(n)} = l\} = p_{lm} \sum_{k=1}^{\infty} g_l(k) \int_0^{\infty} P_{i+k,j}(t) dA_{lm}(t)$$
$$i, j = 0, 1, 2, \dots \ ; \ l, m = 1, \dots, L, \quad (9)$$

where $P_{i,j}(t)$ is transition probability of the birth-and-death process defined in (4). Assuming that this Markov chain is ergodic, the limiting distribution

$$P(i, l) := \lim_{n \to \infty} P\{X^{(n)} = i, S^{(n)} = l\}; \quad i = 0, 1, 2, \dots \ ; \ l = 1, \dots, L \qquad (10)$$

satisfies the balance equations

$$P(j, m) = \sum_{i=0}^{\infty} \sum_{l=1}^{L} \sum_{k=1}^{\infty} p_{lm} g_l(k) P(i, l) \int_0^{\infty} P_{i+k,j}(t) dA_{lm}(t); \quad j = 0, 1, 2, \dots ; m = 1, \dots, L, \qquad (11)$$

and the normalization condition

$$\sum_{i=0}^{\infty} \sum_{l=1}^{L} P(i, l) = 1. \qquad (12)$$

We transform (11) to a complex integral, since we want to utilize (8) to convert (11) in terms of the generating function for the queue size. Since $P^*_{i,j}(s)$ is the Laplace transform of $P_{i,j}(t)$, we have the inversion

$$P_{i,j}(t) = \frac{1}{2\pi\mathbf{i}} \int_{b-\mathbf{i}\infty}^{b+\mathbf{i}\infty} e^{st} P^*_{i,j}(s) ds, \qquad (13)$$

where $b > 0$, $\mathbf{i} := \sqrt{-1}$, and the integration $\int_{b-\mathbf{i}\infty}^{b+\mathbf{i}\infty}$ denotes the *Bromwich integral*, being written as $\int_{Br}$ hereafter. Substituting (13) into (11) yields

$$P(j, m) = \sum_{i=0}^{\infty} \sum_{l=1}^{L} \sum_{k=1}^{\infty} p_{lm} g_l(k) P(i, l) \frac{1}{2\pi\mathbf{i}} \int_{Br} P^*_{i+k,j}(s) \alpha_{lm}(-s) ds;$$
$$j = 0, 1, 2, \dots ; m = 1, \dots, L, \qquad (14)$$

where

$$\alpha_{lm}(s) := \int_0^{\infty} e^{-st} dA_{lm}(t)$$

is the Laplace-Stieltjes transform (LST) of $A_{lm}(t)$. Let us introduce the generating function for $\{P(i,l); i = 0,1,2,\dots\}$ by

$$\Phi_l(z) := \sum_{i=0}^{\infty} P(i,l) z^i; \quad l = 1,\dots,L.$$

By definition, we must have

$$\Phi_l(1) = \pi_l; \quad l = 1,\dots,L. \tag{15}$$

Multiplying (14) by z^j and summing over $j = 0,1,2,\dots$, we obtain

$$\Phi_m(z) = \sum_{l=1}^{L} p_{lm} \sum_{i=0}^{\infty} P(i,l) \sum_{k=1}^{\infty} g_l(k) \frac{1}{2\pi\mathbf{i}} \int_{Br} \sum_{j=0}^{\infty} P^*_{i+k,j}(s) z^j \alpha_{lm}(-s) ds. \tag{16}$$

Using (8) in (16) and changing the order of summation and integration, we get the following set of simultaneous equations for $\{\Phi_l(z); l = 1,\dots,L\}$:

$$\Phi_m(z) = \sum_{l=1}^{L} p_{lm} \frac{1}{2\pi\mathbf{i}} \int_{Br} \left[\frac{zG_l(z)\Phi_l(z) - (1-z)\mu \sum_{j=0}^{c-1}(c-j)H_l(j,s)z^j}{zs - (1-z)(c\mu - \lambda z)} \right] \alpha_{lm}(-s) ds;$$
$$m = 1,\dots,L, \quad (17)$$

where

$$H_l(j,s) := \sum_{i=0}^{\infty} \sum_{k=1}^{\infty} P(i,l) g_l(k) P^*_{i+k,j}(s); \quad j = 0,\dots,c-1; \quad l = 1,\dots,L. \tag{18}$$

We note that $H_l(j,s)$ can be written as

$$H_l(j,s) = \sum_{k=1}^{\infty} d_l(k) P^*_{k,j}(s); \quad j = 0,\dots,c-1; \quad l = 1,\dots,L, \tag{19}$$

where

$$d_l(k) := \sum_{i=0}^{k-1} P(i,l) g_l(k-i); \quad k \geq 1 \tag{20}$$

is the convolution of $P(i,l)$ and $g_l(i)$. It is clear that $\sum_{k=1}^{\infty} d_l(k) P^*_{k,j}(s)$ is a convergent series for $\Re(s) > 0$. From (7a) and (7b), we have

$$\begin{aligned} (s+\lambda)H_l(0,s) &= \mu H_l(1,s); \quad l = 1,\dots,L, \\ (s+\lambda+j\mu)H_l(j,s) &= \lambda H_l(j-1,s) + (j+1)\mu H_l(j+1,s) + d_l(j); \\ & \qquad j = 1,\dots,c-2; \quad l = 1,\dots,L. \quad (21) \end{aligned}$$

Following Kuczura [7], we may comment on the Bromwich integral in (17) as follows. Since $P_{i+k,j}(t)$ is the probability, the generating function $P^*_{i+k}(s,z)$ of $P^*_{i+k,j}(s)$ is analytic for $|z| \leq 1$ and $\Re(s) > 0$. Hence the bracketed part of the integrand in (17) is analytic for $|z| \leq 1$ and $\Re(s) > 0$, since it is the convergent series of $\sum_{i=0}^{\infty} \sum_{k=1}^{\infty} P(i,l) g_l(k) P_{i+k}^*(s,z)$. On the other hand, since $A_{lm}(t)$ is the distribution function, $\alpha_{lm}(s)$ is analytic for $\Re(s) > 0$. For $\Re(s) < 0$, $\alpha_{lm}(s)$ may or may not be analytic. However, $\alpha_{lm}(s)$ is meromorphic for $\Re(s) < 0$ in many cases, including the cases in which the distribution of A_{lm} is exponential, Erlang, and a linear combination thereof.

If we assume that $\alpha_{lm}(s)$ is meromorphic for the left-half plane $\Re(s) < 0$, all the poles of $\alpha_{lm}(-s)$ are in the right-half plane $\Re(s) > 0$. Hence the integrand in (17) is meromorphic in the right-half plane. Thus we can use the residue theorem to evaluate the integrand over the contour consisting of the line $(b + \mathbf{i}R, b - \mathbf{i}R)$ and a semicircle of radius R in the right-half plane which connects $b - \mathbf{i}R$ with $b + \mathbf{i}R$ counterclockwise. We can choose b and R such that all the poles of $\alpha_{lm}(-s)$ are interior to this contour for all $l = 1, \ldots, L$. Then the Bromwich integrals in (17) are evaluated only at the poles of $\alpha_{lm}(-s)$'s. Therefore, (17) is not a set of integral equations but simply a set of linear equations for $\{\Phi_l(z);\ l = 1, \ldots, L\}$ containing unknown constants as coefficients. These unknown constants are determined from the condition that the generating function $\Phi_l(z)$ is analytic for $|z| \leq 1$, the equations in (21) and other relations for $\{P(i,l);\ i = 0, \ldots, c-3;\ \ l = 1, \ldots, L\}$. In Appendix, we show explicitly how to determine $\Phi_l(z)$ when the state sojourn time follows an exponential distribution.

4 Stability Condition for an SMP$^{[X]}$M/M/c System

Let us discuss the stability condition for the SMP$^{[X]}$+M/M/c system. To do so, we rewrite (17) as

$$\Phi_m(z) = \sum_{l=1}^{L} p_{lm} \Big[z G_l(z) \Phi_l(z) \Psi_{lm}(z) - (1-z) \hat{B}_{lm}(z) \Big], \tag{22}$$

where

$$\Psi_{lm}(z) := \frac{1}{2\pi \mathbf{i}} \int_{Br} \frac{\alpha_{lm}(-s)}{zs - (1-z)(c\mu - \lambda z)} ds, \tag{23}$$

and

$$\hat{B}_{lm}(z) := \frac{1}{2\pi \mathbf{i}} \int_{Br} \frac{\mu \sum_{j=0}^{c-1} (c-j) H_l(j,s) z^j}{zs - (1-z)(c\mu - \lambda z)} \alpha_{lm}(-s) ds. \tag{24}$$

Evaluating $\Psi_{lm}(z)$ at $z = 1$ yields

$$\Psi_{lm}(1) = \frac{1}{2\pi \mathbf{i}} \int_{Br} \frac{\alpha_{lm}(-s)}{s} ds = 1. \tag{25}$$

Differentiating $\Psi_{lm}(z)$ and evaluating at $z=1$, we obtain

$$\left.\frac{d\Psi_{lm}(z)}{dz}\right|_{z=1} = \frac{1}{2\pi\mathbf{i}}\int_{Br}\frac{\lambda-s-c\mu}{s^2}ds\int_0^\infty e^{st}dA_{lm}(t)$$
$$= \int_0^\infty dA_{lm}(t)\frac{1}{2\pi\mathbf{i}}\int_{Br}\frac{\lambda-s-c\mu}{s^2}e^{st}ds$$
$$= (\lambda-c\mu)\hat{a}_{lm}-1, \tag{26}$$

where $\hat{a}_{lm}$ is mean of the distribution function $A_{lm}(t)$. Evaluating $\hat{B}_{lm}(z)$ at $z=1$, we obtain

$$\hat{B}_{lm}(1) = \frac{1}{2\pi\mathbf{i}}\int_{Br}\frac{\mu\sum_{j=0}^{c-1}(c-j)H_l(j,s)}{s}\alpha_{lm}(-s)ds$$
$$= \mu\int_0^\infty dA_{lm}(t)\frac{1}{2\pi\mathbf{i}}\int_{Br}\frac{\sum_{j=0}^{c-1}(c-j)H_l(j,s)}{s}e^{st}ds$$
$$= \mu\int_0^\infty dA_{lm}(t)\sum_{j=0}^{c-1}(c-j)\sum_{k=1}^{\infty}d_l(k)\int_0^t P_{k,j}(t)dt, \tag{27}$$

where we have used the definition of $H_l(j,s)$ in (19). Thus, $\hat{B}_{lm}(1)$ is positive. These results are used later.

Now, equation (22) can be written in matrix form as

$$\mathbf{\Phi}(z)\hat{\mathbf{F}}(z) = (z-1)\mathbf{1}\mathrm{diag}[\hat{\mathbf{B}}^{\mathrm{t}}(z)\mathbf{P}], \tag{28}$$

where $\mathbf{\Phi}(z) := [\Phi_1(z),\ldots,\Phi_L(z)]$,

$$\hat{\mathbf{F}}(z) := \mathbf{I}_L - z\mathbf{G}(z)\hat{\mathbf{Q}}(z), \tag{29}$$

$$\mathbf{G}(z) := \begin{bmatrix} G_1(z) & 0 & \ldots & 0 \\ 0 & G_2(z) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & G_L(z) \end{bmatrix}, \tag{30}$$

$$\hat{\mathbf{Q}}(z) := \begin{bmatrix} p_{11}\Psi_{11}(z) & p_{12}\Psi_{12}(z) & \ldots & p_{1L}\Psi_{1L}(z) \\ p_{21}\Psi_{21}(z) & p_{22}\Psi_{22}(z) & \ldots & p_{2L}\Psi_{2L}(z) \\ \vdots & \vdots & \ddots & \vdots \\ p_{L1}\Psi_{L1}(z) & p_{L2}\Psi_{L2}(z) & \ldots & p_{LL}\Psi_{LL}(z) \end{bmatrix}, \tag{31}$$

$$\hat{\mathbf{B}}(z) := \begin{bmatrix} \hat{B}_{11}(z) & \hat{B}_{12}(z) & \ldots & \hat{B}_{12}(z) \\ \hat{B}_{21}(z) & \hat{B}_{22}(z) & \ldots & \hat{B}_{2L}(z) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{B}_{L1}(z) & \hat{B}_{L2}(z) & \ldots & \hat{B}_{LL}(z) \end{bmatrix}, \tag{32}$$

and $\mathbf{I}_L$ denotes an $L \times L$ identity matrix. In equation (28), diag$\mathbf{X}$ is a diagonal matrix whose elements are taken from the corresponding elements of $\mathbf{X}$, and $\hat{\mathbf{B}}^{\mathrm{t}}(z)$ is the transpose of $\hat{\mathbf{B}}(z)$.

Differentiating (28) and evaluating the result at $z = 1$, we obtain

$$\mathbf{\Phi}'(1)(\mathbf{I}_L - \mathbf{P}) + \boldsymbol{\pi}\hat{\mathbf{F}}'(1) = \mathbf{1}\mathrm{diag}[\hat{\mathbf{B}}^{\mathrm{t}}(1)\mathbf{P}], \tag{33}$$

Here we have used $\hat{\mathbf{F}}(1) = \mathbf{I}_L - \mathbf{P}$ since $\mathbf{G}(1) = \mathbf{I}_L$ and $\hat{\mathbf{Q}}(1) = \mathbf{P}$. Note also that $\mathbf{\Phi}(1) = \boldsymbol{\pi}$. Multiplying (33) on the right by $\mathbf{1}^{\mathrm{t}} := [1, \dots, 1]^{\mathrm{t}}$ and noting that $(\mathbf{I}_L - \mathbf{P})\mathbf{1}^{\mathrm{t}} = 0$, we get

$$\boldsymbol{\pi}\hat{\mathbf{F}}'(1)\mathbf{1}^{\mathrm{t}} = \mathbf{1}\mathrm{diag}[\hat{\mathbf{B}}^{\mathrm{t}}(1)\mathbf{P}]\mathbf{1}^{\mathrm{t}}. \tag{34}$$

To determine the left-hand side of (34), we differentiate (29) and evaluate the result at $z = 1$, where we use (25) and (26). Then we have

$$\hat{\mathbf{F}}'(1) = -\hat{\mathbf{Q}}'(1) - \mathbf{G}'(1)\mathbf{P} - \mathbf{P}, \tag{35}$$

where

$$\mathbf{G}'(1) = \begin{bmatrix} g_1 & 0 & \dots & 0 \\ 0 & g_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & g_L \end{bmatrix}, \tag{36}$$

and

$$\hat{\mathbf{Q}}'(1) = \begin{bmatrix} p_{11}[(\lambda - c\mu)\hat{a}_{11} - 1] & p_{12}[(\lambda - c\mu)\hat{a}_{12} - 1] & \dots & p_{1L}[(\lambda - c\mu)\hat{a}_{1L} - 1] \\ p_{21}[(\lambda - c\mu)\hat{a}_{21} - 1] & p_{22}[(\lambda - c\mu)\hat{a}_{22} - 1] & \dots & p_{2L}[(\lambda - c\mu)\hat{a}_{2L} - 1] \\ \vdots & \vdots & \ddots & \vdots \\ p_{L1}[(\lambda - c\mu)\hat{a}_{L1} - 1] & p_{L2}[(\lambda - c\mu)\hat{a}_{L1} - 1] & \dots & p_{LL}[(\lambda - c\mu)\hat{a}_{LL} - 1] \end{bmatrix}. \tag{37}$$

Here g_l is the mean batch size of type l customers. Multiplying (34) on the right by $\mathbf{1}^{\mathrm{t}}$ and substituting (36) and (37) yields

$$\begin{aligned} \hat{\mathbf{F}}'(1)\mathbf{1}^{\mathrm{t}} &= -\hat{\mathbf{Q}}'(1)\mathbf{1}^{\mathrm{t}} - \mathbf{G}'(1)\mathbf{1}^{\mathrm{t}} - \mathbf{1}^{\mathrm{t}} \\ &= \begin{bmatrix} \sum_{m=1}^{L} p_{1m}[(c\mu - \lambda)\hat{a}_{1m} + 1] \\ \sum_{m=1}^{L} p_{2m}[(c\mu - \lambda)\hat{a}_{2m} + 1] \\ \vdots \\ \sum_{m=1}^{L} p_{Lm}[(c\mu - \lambda)\hat{a}_{Lm} + 1] \end{bmatrix} - \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_L \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}. \end{aligned} \tag{38}$$

Finally, multiplying (38) on the left by $\boldsymbol{\pi}$, we obtain

$$\boldsymbol{\pi}\hat{\mathbf{F}}'(1)\mathbf{1}^{\mathrm{t}} = \sum_{l=1}^{L} \pi_l \sum_{m=1}^{L} p_{lm}[(c\mu - \lambda)\hat{a}_{lm} + 1] - g - 1 = (c\mu - \lambda)\hat{a} - g, \tag{39}$$

where $\hat{a}$ is the mean of interarrival time for SMP arrivals defined by

$$\hat{a} := \sum_{l=1}^{L} \pi_l \sum_{m=1}^{L} p_{lm}\hat{a}_{lm}, \tag{40}$$

and g is the mean batch size given by

$$g := \sum_{l=1}^{L} \pi_l g_l. \tag{41}$$

The expression in (39) is the left-hand side of (34). Thus we have

$$(c\mu - \lambda)\hat{a} - g = \sum_{l=1}^{L}\sum_{m=1}^{L} \hat{B}_{lm}(1)p_{lm}. \tag{42}$$

From (27) we see that the right-hand side of this equation is positive. Hence we have

$$\lambda + \frac{g}{\hat{a}} < c\mu. \tag{43}$$

The condition in (43) means that the sum of the arrival rates of the SMP and Poisson customers is less than the total service rate. Thus it assures the stability of our system.

5 Waiting Times in an SMP$^{[X]}$+M/M/c System

We proceed to investigate the waiting time for an arbitrary customer in an SMP$^{[X]}$+M/M/c system. In section 5.1, the waiting time distribution for an arbitrary SMP customer in a batch is derived. In section 5.2, based on the theory of Markov renewal processes, the waiting time distribution for an arbitrary Poisson customer is given.

5.1 Waiting time of an SMP Customer

We first consider the waiting time W of an arbitrary SMP customer. Let us focus on a randomly chosen *tagged* customer included in a batch that arrives to bring state l. Let $\hat{G}_l$ denote the number of customers placed before the tagged customer in this batch, and $W_l(t)$ be the waiting time distribution of this tagged customer. The distribution of $\hat{G}_l$ is given by [16, p.46]

$$\hat{g}_l(k) = \frac{\sum_{j=k+1}^{\infty} g_l(k)}{g_l}; \quad k = 0, 1, 2, \ldots, \tag{44}$$

and its generating function is

$$\hat{G}_l(z) = \frac{1 - G_l(z)}{g_l(1-z)}. \tag{45}$$

The probability that the tagged customer need not wait is

$$W_l(0) = \sum_{k=0}^{c-1} \sum_{i=0}^{c-1-k} \hat{g}_l(k) P(i,l).$$

On the other hand, if the tagged customer in a batch of size k arrives and finds that the number $i+k$ of customers in the system is great than c, he must wait until $i+k+1-c$ customers depart before he enters service. Therefore, his waiting time has $(i+k+1-c)$-stage Erlang distribution. Thus, $W_l(t)$ is given by

$$\begin{aligned} W_l(t) = &\sum_{k=0}^{c-1} \sum_{i=0}^{c-1-k} \hat{g}_l(k) P(i,l) + \sum_{k=1}^{c-1} \sum_{i=c-k}^{c-1} \hat{g}_l(k) P(i,l) \int_0^t \frac{c\mu(c\mu y)^{i+k-c} e^{-c\mu y}}{(i+k-c)!} dy \\ &+ \sum_{k=0}^{c-1} \sum_{i=c}^{\infty} \hat{g}_l(k) P(i,l) \int_0^t \frac{c\mu(c\mu y)^{i+k-c} e^{-c\mu y}}{(i+k-c)!} dy \\ &+ \sum_{k=c}^{\infty} \sum_{i=0}^{\infty} \hat{g}_l(k) P(i,l) \int_0^t \frac{c\mu(c\mu y)^{i+k-c} e^{-c\mu y}}{(i+k-c)!} dy. \end{aligned} \tag{46}$$

Taking the LST of $W_l(t)$, we obtain

$$\begin{aligned} \Omega_l(s) = &\frac{[B(s)]^{1-c}}{g_l[1-B(s)]} \Big(1 - G_l[B(s)]\Big) \Phi_l[B(s)] \\ &+ \sum_{k=0}^{c-1} \sum_{i=0}^{c-1-k} \hat{g}_l(k) P(i,l) \left(1 - [B(s)]^{i+k+1-c}\right), \end{aligned} \tag{47}$$

where $B(s) := c\mu/(s+c\mu)$.

Finally we get the LST $\Omega(s)$ of the distribution function for the waiting time W of an arbitrary customer as

$$\begin{aligned} \Omega(s) &= \frac{1}{g} \sum_{l=1}^{L} g_l \Omega_l(s) \\ &= \frac{1}{g} \Bigg\{ \frac{[B(s)]^{1-c}}{1-B(s)} \sum_{l=1}^{L} \Big(1 - G_l[B(s)]\Big) \Phi_l[B(s)] \\ &\quad + \sum_{l=1}^{L} \sum_{k=0}^{c-1} \sum_{i=0}^{c-1-k} g_l \hat{g}_l(k) P(i,l) \Big(1 - [B(s)]^{i+k+1-c}\Big) \Bigg\}. \end{aligned} \tag{48}$$

The mean $E[W]$ and the second moment $E[W^2]$ of the waiting time are then given by

$$E[W] = \frac{1}{gc\mu} \left[\sum_{l=1}^{L} g_l E_l[X] + \frac{g^{(2)}}{2} - \sum_{l=1}^{L} \sum_{k=0}^{c-1} \sum_{i=0}^{c-1-k} g_l \hat{g}_l(k) P(i,l)(i+k+1-c) \right] - \frac{c-1}{c\mu}, \tag{49}$$

$$E[W^2] = \frac{1}{g(c\mu)^2}\Bigg\{ \sum_{l=1}^{L} \Big[g_l \Big(E_l[X^2] + (3-2c)E_l[X] \Big) + g_l^{(2)} E_l[X] \Big] + (2-c)g^{(2)} + \frac{g^{(3)}}{3}$$
$$- \sum_{l=1}^{L} \sum_{k=0}^{c-1} \sum_{i=0}^{c-1-k} g_l \hat{g}_l(k) P(i,l)(i+k+1-c)(i+k+2-c) \Bigg\} + \frac{(c-1)(c-2)}{(c\mu)^2}, \tag{50}$$

where

$$g_l^{(i)} = G_l^{(i)}(1), \quad g^{(i)} = \sum_{l=1}^{L} \pi_l g_l^{(i)}; \quad i = 2, 3,$$

$$E_l[X] = \Phi_l^{(1)}(1), \quad E_l[X^2] = \Phi_l^{(2)}(1) + E_l[X]; \quad l = 1, \ldots, L.$$

5.2 Waiting Time of a Poisson Customer

We next consider the waiting time W^* of an arbitrary Poisson customer. According to the PASTA (Poisson arrivals see time averages) property, the number of customers that an arriving Poisson customer finds in the system has the same distribution as the number X^* of customers present in the system at an arbitrary time in steady state. Thus we will find the generating function $\Phi^*(z)$ for the probability distribution of X^*.

To do so, note that the interval between an arbitrary time and the preceding SMP arrival time corresponds to the backward recurrence time in the Markov renewal process that counts the number of state transitions in the SMP. The joint distribution for the backward recurrence time in state l and the probability that the next state is m is given by

$$\hat{A}_{lm}(t) = \frac{p_{lm}}{E[A_l]} \int_0^t [1 - A_{lm}(x)]dx; \quad t \geq 0, \tag{51}$$

where

$$E[A_l] := \sum_{m=1}^{L} p_{lm} E[A_{lm}]$$

is the mean sojourn time in state l.

Conditioning on the number of customers and the states of the SMP at the preceding and the next arrival points, and integrating with the backward recurrence time distribution in (51), the steady-state distribution of X^* is given by

$$P(X^* = j) = \sum_{i=0}^{\infty} \sum_{l=1}^{L} P(i,l) \sum_{k=1}^{\infty} g_l(k) \int_0^{\infty} P_{i+k,j}(t) d\hat{A}_l(t); \quad j = 0, 1, 2, \ldots, \tag{52}$$

where

$$\hat{A}_l(t) := \frac{E[A_l]}{E[A]} \sum_{m=1}^{L} \hat{A}_{lm}(t) = \frac{1}{E[A]} \sum_{m=1}^{L} p_{lm} \int_0^t [1 - A_{lm}(x)]dx; \quad t \geq 0$$

is the conditional distribution function for the backward recurrence time in state l. The mean interarrival time $E[A]$ between the batches of SMP customers is given by

$$E[A] := \sum_{l=1}^{L} \pi_l E[A_l].$$

From (52), the generating function $\Phi^*(z)$ for X^* is given by

$$\Phi^*(z) := \sum_{j=0}^{\infty} P(X^* = j) z^j = \sum_{i=0}^{\infty} \sum_{l=1}^{L} P(i,l) \sum_{k=1}^{\infty} g_l(k) \int_0^{\infty} \sum_{j=0}^{\infty} P_{i,j}^*(t) z^j d\hat{A}_l(t). \tag{53}$$

In the same way as deriving (17), we obtain

$$\Phi^*(z) = \sum_{l=1}^{L} \frac{1}{2\pi \mathbf{i}} \int_{Br} \left[\frac{zG_l(z)\Phi_l(z) - (1-z)\mu \sum_{j=0}^{c-1} (c-j) H_l(j,s) z^j}{zs - (1-z)(c\mu - \lambda z)} \right] \hat{\alpha}_l(-s) ds, \tag{54}$$

where $H_l(j,s)$; $j = 1, \ldots, c-1$, are given in (18), and $\hat{\alpha}_l(s)$ is the LST of $\hat{A}_l(t)$. Again, the Bromwich integrals are evaluated only at the poles of $\hat{\alpha}_l(-s)$'s in the right-half plane $\Re(s) > 0$ in most cases.

The LST $\Omega^*(s)$ of the distribution function for the waiting time W^* of an arbitrary Poisson customer is expressed as

$$\begin{aligned} \Omega^*(s) &= \sum_{j=0}^{c-1} P(X^* = j) + \sum_{j=c}^{\infty} P(X^* = j)[B(s)]^{j+1-c} \\ &= [B(s)]^{1-c} \Phi^*[B(s)] + \sum_{j=0}^{c-1} P(X^* = j) \left(1 - [B(s)]^{j+1-c}\right). \end{aligned} \tag{55}$$

The mean $E[W^*]$ and the second moment $E[(W^*)^2]$ are then given by

$$E[W^*] = \frac{1}{c\mu} \left(E[X^*] - (c-1) + \sum_{j=0}^{c-1} P(X^* = j)(c-j-1) \right), \tag{56}$$

$$\begin{aligned} E[(W^*)^2] = \frac{1}{(c\mu)^2} \Bigg(& (c-1)(c-2) + (3-2c)E[X^*] + E[(X^*)^2] \\ & - \sum_{j=0}^{c-1} P(X^* = j)(c-j-1)(c-j-2) \Bigg), \end{aligned} \tag{57}$$

respectively, where $E[X^*]$ and $E[(X^*)^2]$ are obtained from $\Phi^*(z)$.

6 Application to the MPEG Frame Sequence

Let us use the $\mathrm{SMP}^{[X]} + \mathrm{M/M}/c$ system to model the traffic in multiple channels on an ATM network in which the transmission of MPEG frames interferes with other traffic.

The waiting time of an arbitrary ATM cell generated from MPEG frames is studied. In Section 6.1, a brief description of MPEG coding scheme is given. In section 6.2, the transmission of MPEG frame sequence with interfering traffic is modeled by an $\mathrm{SMP}^{[X]}$ + M/M/c system. Assuming that the MPEG frame arrival process is also Poisson, we obtain the formula for evaluating the waiting time of an arbitrary ATM cell. In Section 6.3, some numerical results using the statistics of a real video film are presented.

6.1 MPEG Video Coding Scheme

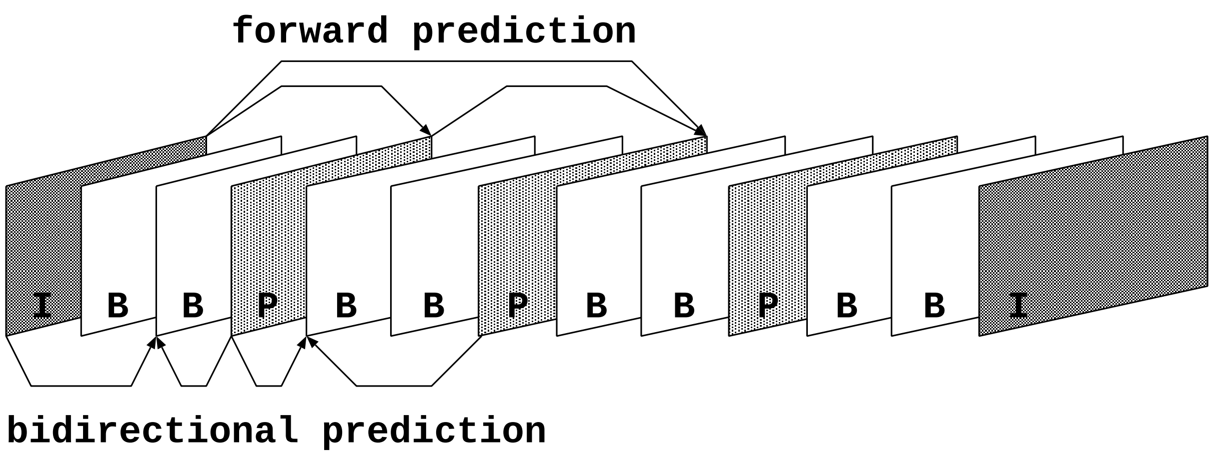

Figure 1: Group of pictures (GOP) of an MPEG stream [9].

In the MPEG coding [9], a video traffic is compressed using the following three types of frames.

- I-frames are generated independently of B- or P-frames and inserted periodically.
- P-frames are encoded for the motion compensation with respect to the previous I- or P-frame.
- B-frames are similar to P-frames, except that the motion compensation can be done with respect to the previous I- or P-frame, the next I- or P-frame, or the interpolation between them.

These frames are arranged in a deterministic sequence "IBBPBBPBBPBB," which is called a Group of Pictures (GOP) as shown in Figure 1. The length of the GOP is 12 frames. The traffic stream generated by the MPEG coding is characterized by two features, namely (i) the deterministic frame pattern in the GOP, and (ii) the distinguishable frame size distributions for the three types of frames (I, B and P).

6.2 Traffic Model for MPEG Frame Sequence

We are now in a position to apply the analysis results of the $\mathrm{SMP}^{[\mathrm{X}]}$+M/M/$c$ system to modeling the transmission of MPEG frame sequence on an ATM network with multiple channels interfering with other traffic. In this model, the Markov chain underlying the SMP has twelve states corresponding to the frame pattern "IBBPBBPBBPBB" with cyclic transitions. We index this sequence which represents the states in the Markov chain as 0 through 11. As shown in Figure 2, for any given state, the transition probability to the next state is unity, since the frame pattern is deterministic.

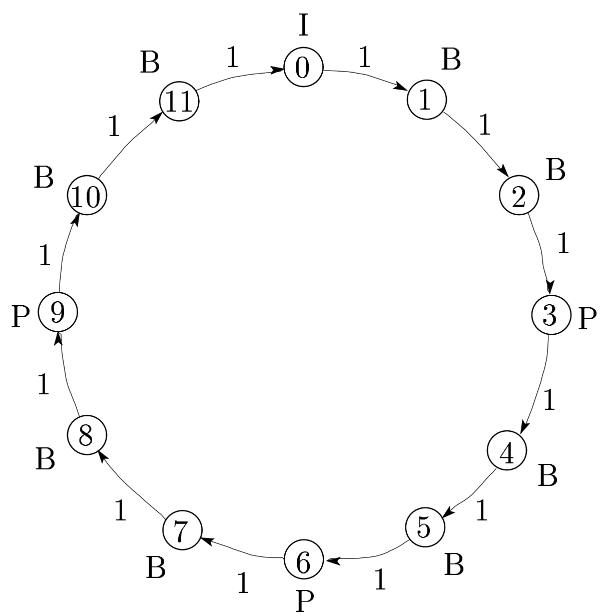

Figure 2: State transition diagram of the MPEG frame pattern.

The stationary distribution of this Markov chain is given by

$$\pi_l = \tfrac{1}{12} \quad ; \quad l = 0, \dots, 11.$$

For the sake of simplicity in the expressions, we assume that the arrival process of the frames is Poisson with rate α as a (very) special case of the SMP. Let $G_l(z)$ denote the probability generating function for the number of ATM cells generated from the lth frame, $l = 0, \dots, 11$, and let c be the number of channels for transmitting the MPEG frames. Equations in (17) then become

$$\Phi_m(z) = \frac{1}{q(z)} \left[zG_{m-1}(z)\Phi_{m-1}(z) - (1-z)\mu \sum_{j=0}^{c-1} (c-j) H_{m-1}(j) z^j \right]; \quad m = 0, \dots, 11, \tag{58}$$

where

$$q(z) := z - \frac{1}{\alpha}(1-z)(c\mu - \lambda z).$$

We note that $\{H_m(j);\ j = 0, \dots, c-1;\ m = 0, \dots, 11\}$ are constants to be determined. Hereafter state "$-m$ " should read state "$12 - m$". Solving the set of equations in (58), we get

$$\Phi_m(z) = \frac{(z-1)\mu \sum_{k=0}^{11} z^k [q(z)]^{11-k} \Big[\sum_{j=0}^{c-1} (c-j) H_{m+k-1}(j) z^j \Big] \prod_{l=m-k}^{m-1} G_l(z)}{T(z)}; \qquad m = 0, \dots, 11, \tag{59}$$

where

$$T(z) := [q(z)]^{12} - z^{12} \prod_{l=0}^{11} G_l(z). \tag{60}$$

It is shown in Appendix that there are twelve zeros of $T(z)$ in $|z| \le 1$ under the condition

$$\alpha g + \lambda \ < c\mu. \tag{61}$$

Here

$$g := \frac{1}{12} \sum_{l=0}^{11} g_l$$

is the mean size of an MPEG frame. We also have the relations

$$\begin{aligned} (\alpha+\lambda) H_m(0) &= \mu H_m(1); \quad m = 0, \dots, 11, \\ (\alpha + \lambda + j\mu) H_m(j) &= \lambda H_m(j-1) + (j+1)\mu H_m(j+1) + d_m(j); \\ & \qquad j = 1, \dots, c-2; \quad m = 0, \dots, 11, \end{aligned} \tag{62}$$

and

$$P(i,m) = \frac{1}{i!} \frac{d^i \Phi_m(z)}{dz^i}\bigg|_{z=0}; \quad i = 0, \dots, c-3; \quad m = 0, \dots, 11. \tag{63}$$

In order to determine the unknown constants $\{H_m(j);\ m = 0, \dots, 11;\ j = 0, \dots, c-1\}$ in (59) and $\{P(i,m);\ i = 0, \dots, c-3;\ m = 0, \dots, 11\}$ included in $d_m(j)$ in (62), $12(2c-2)$ equations are required. Out of them, 12 equations are obtained from the zeros of $T(z)$, and the remaining $12(2c-3)$ equations are given by (62) and (63). This completes the determination of parameters in our formulation.

6.3 Numerical Examples

Let us evaluate the waiting time of an arbitrary ATM cell in the model. The real video film data for the Jurassic Park is prepared by Rose [11], and it can be downloaded from the web site *http://nero.informatik.uni-wuerzburg.de/MPEG/*. In addition, we need to assume some distribution for the number of cells in each frame (frame size) so that we can calculate the distribution of the waiting time numerically.

Table 1: Statistics for the frame size in ATM cells calculated from the MPEG frame trace of the Jurassic Park video.

I-frame			B-frame			P-frame		
mean	var	c.v.	mean	var	c.v.	mean	var	c.v.
143.427	918.704	0.211	19.033	135.021	0.612	37.659	632.568	0.667

Table 2: Parameters of the negative binomial distributions for the frame size of the Jurassic Park video.

I-frame		B-frame		P-frame	
n_{I}	p_{I}	n_{B}	p_{B}	n_{P}	p_{P}
19	0.132	2	0.105	2	0.053

Frey and Nguen-Quang [3] and Sarkar et al. [13] propose the gamma distribution for the frame size. As a discrete version of the gamma distribution, let us assume that the distribution of the frame size is *negative binomial.* Thus the probability generating functions for the frame size are given by

$$G_l(z) = \left(\frac{p_l z}{1 - q_l z}\right)^{n_l}; \quad q_l := 1 - p_l; \quad l = 0, \dots, 11.$$

where, by referring to Figure 2, we set

$$\begin{aligned} p_l &= p_{\mathrm{I}}, \quad n_l = n_{\mathrm{I}}; \quad l = 0, \\ p_l &= p_{\mathrm{B}}, \quad n_l = n_{\mathrm{B}}; \quad l = 1, 2, 4, 5, 7, 8, 10, 11, \\ p_l &= p_{\mathrm{P}}, \quad n_l = n_{\mathrm{P}}; \quad l = 3, 6, 9. \end{aligned}$$

Table 1 shows the mean, variance, and coefficient of variation (c.v.) for the number of ATM cells in each frame type for the Jurassic Park video, which have been calculated by assuming that every frame is divided into a group of cells each with a payload of 48 bytes. The fitting parameters for the negative binominal distributions determined from the statistics in Table 1 are given in Table 2.

To compare the influence of the number of channels, we keep the total transmission rate at 10 Mbps, which corresponds to 2350 cells/sec, and vary c from 2 to 3. For each set of parameters we have exactly twelve zeros in the unit disk. The zeros of $T(z)$ for $c = 2$ and $\mu = (2350/2)$ cells/sec are plotted in the complex z-plane in Figure 3.

Figures 4 and 5 show the mean and the variance, respectively, of the waiting time of an arbitrary ATM cell for $c = 2$, $\mu = (2350/2)$ cells/sec. Figures 6 and 7 show the mean and the variance, respectively, of an arbitrary ATM cell for $c = 3$, $\mu = (2350/3)$ cells/sec. For completeness, the mean and variance for the waiting time of interfering Poisson cells

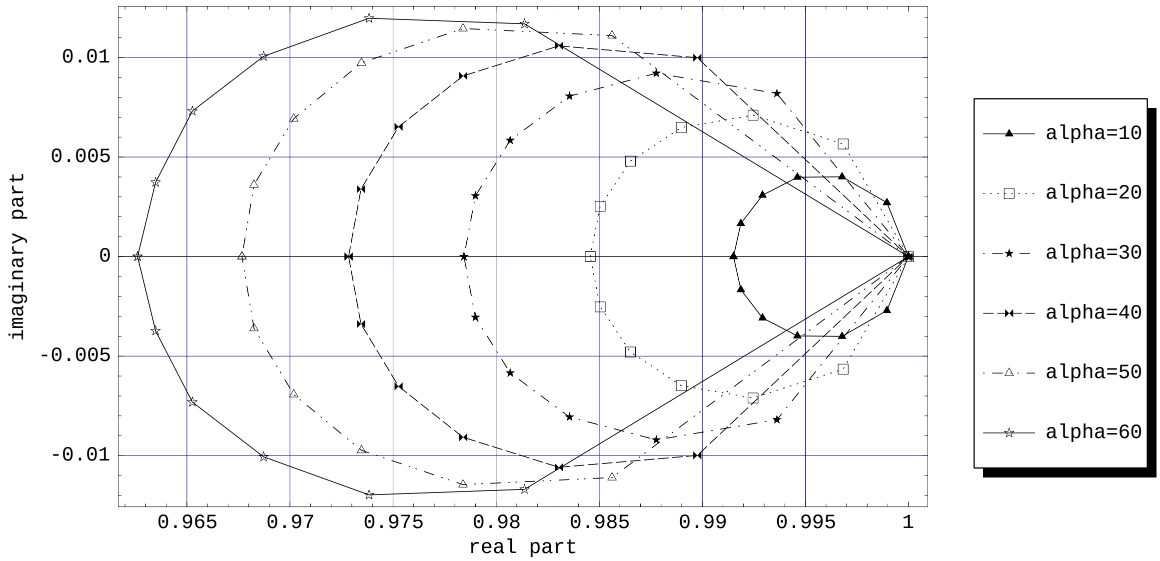

Figure 3: Zeros of $T(z)$ in the unit disk for $c = 2$, $\mu = (2350/2)$ cells/sec.

are also plotted in corresponding figures. It is observed that the mean and variance for the waiting time of MPEG cells increase as the arrival rate of the interfering Poisson cells. It is also observed that the difference between the curves for $c = 2$ and $c = 3$ is very small for both the mean and variance if we keep the total transmission rate at a constant $c\mu = 2350$ cells/sec. In other words the number of channels does not influence much on the mean and variance of the waiting time when the total transmission rate is kept constant. This is just like the situation in an M/M/c queueing system.

7 Summary

In this paper, we have first analyzed the queue size in an SMP$^{[X]}$+M/M/c system, where the underlying SMP has L states. The formulas of the mean and variance of the waiting time for both an arbitrary SMP customer and an arbitrary Poisson customer have been derived. When the state sojourn times are exponentially distributed, we have proved that there exist L^2 zeros in the unit disk in the denominator of the generating function for the queue size if the arrival rate is less than the total service rate. Then we have modeled the arrival of ATM cells in the MPEG frame sequence as an SMP batch arrival process, whereas the interfering traffic is represented by Poisson arrivals. This model captures two major features of the MPEG coding scheme: (i) the deterministic frame pattern and (ii) the distinct distributions for the size of the three types of frames. The waiting time of each ATM cell has been evaluated. It is observed that the mean and variance of MPEG cells increase as the arrival rate of the interfering Poisson cells. It is also observed that, just like the situation in an M/M/c queueing system, the number of channels does not influence much on the mean and variance of the waiting time as far

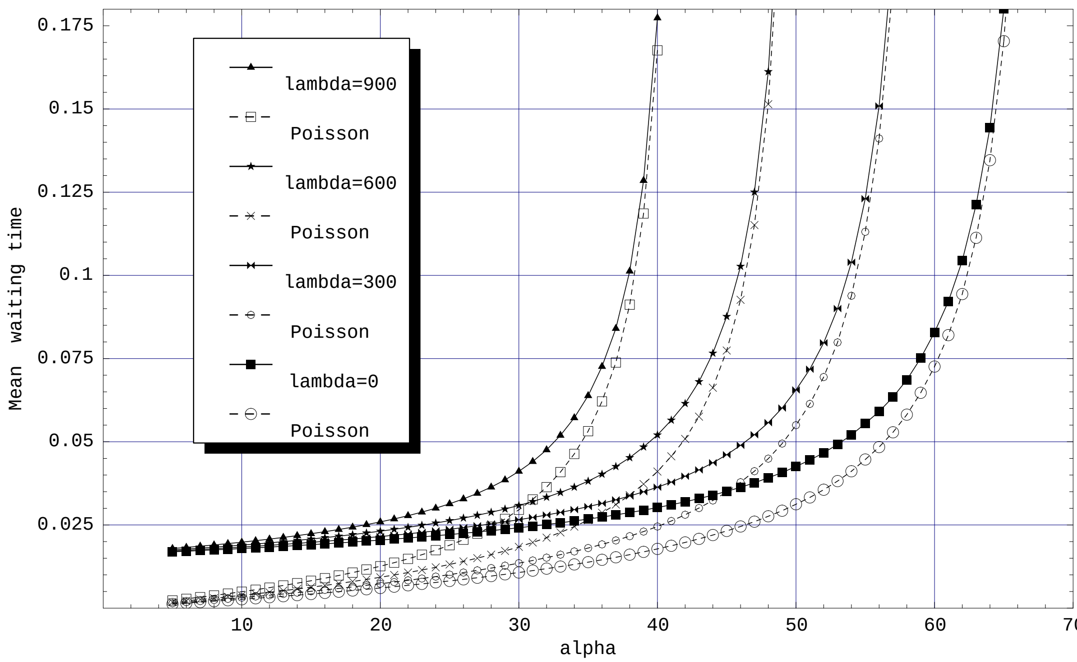

Figure 4: Mean waiting time for an arbitrary cell for $c = 2$, $\mu = (2350/2)$ cells/sec.

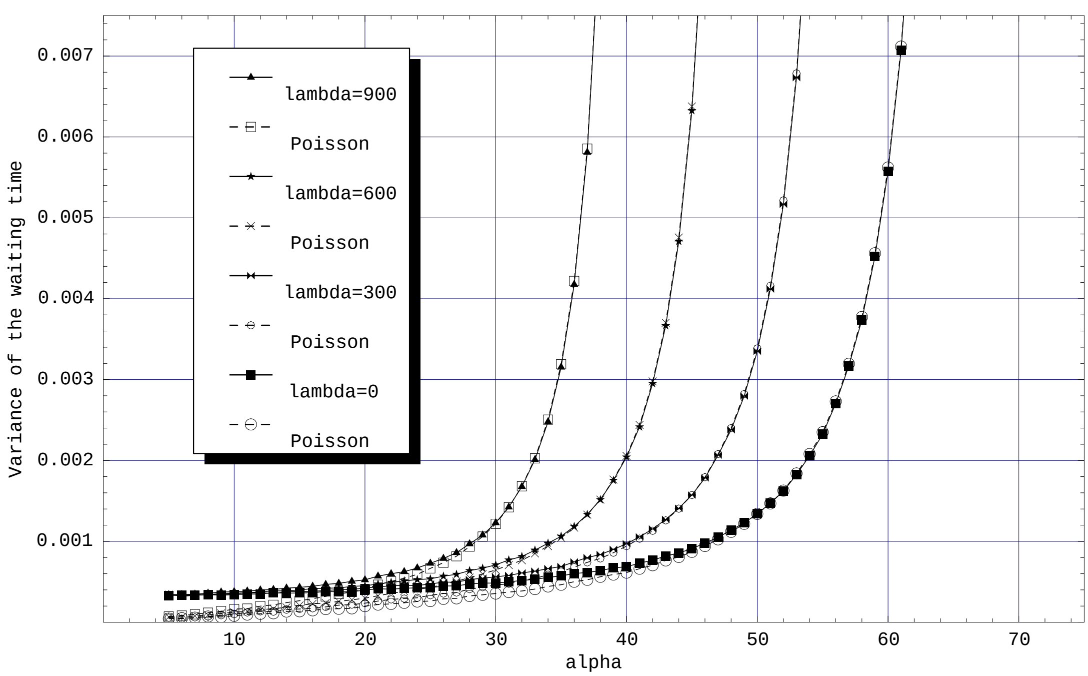

Figure 5: Variance of the waiting time for an arbitrary cell for $c = 2$, $\mu = (2350/2)$ cells/sec.

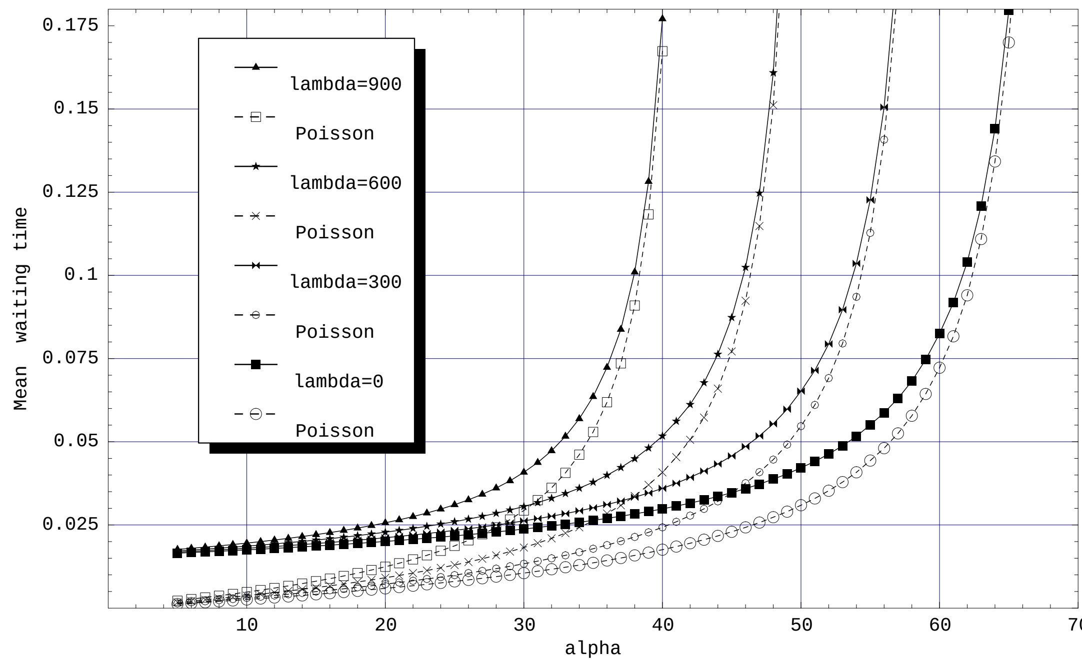

Figure 6: Mean waiting time for an arbitrary cell for $c = 3$, $\mu = (2350/3)$ cells/sec.

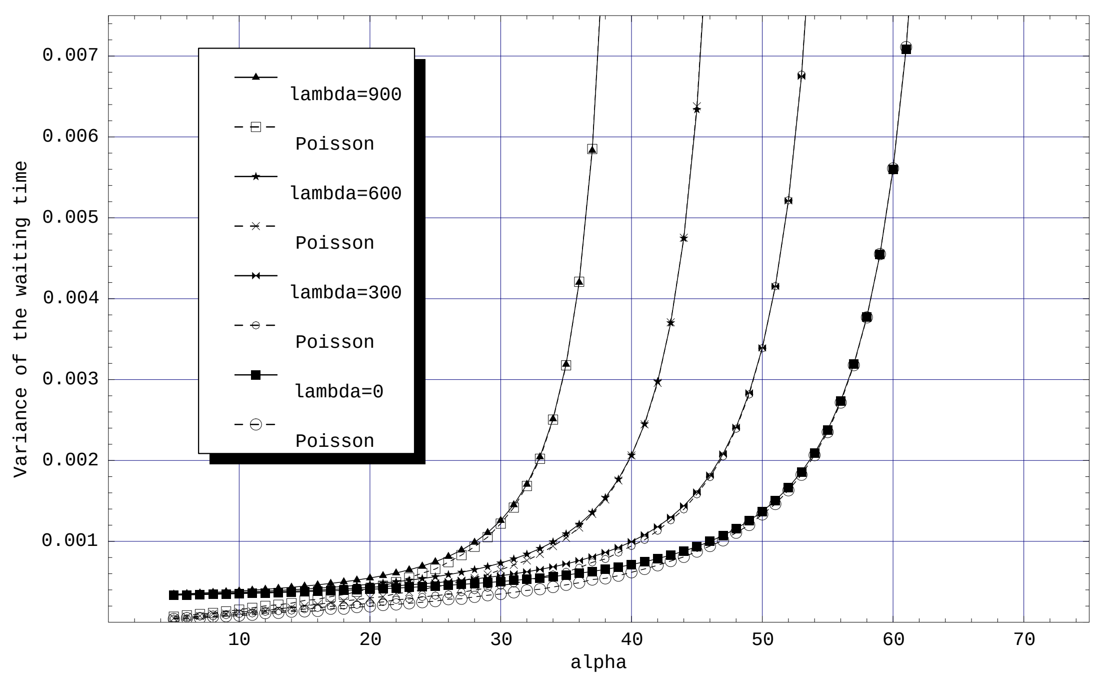

Figure 7: Variance of the waiting time for an arbitrary cell for $c = 3$, $\mu = (2350/3)$ cells/sec.

as the total transmission rate is kept constant.

References

[1] E. Arjas, "On a fundamental identity in the theory of semi-Markov processes," *Advances in Applied Probability*, Vol.4, No.2, pp.258–270, August 1972.

[2] E. Çinlar, "Queues with semi-Markovian arrivals," *Journal of Applied Probability*, Vol.4, No.2, pp.365-379, August 1967.

[3] M. Frey and S. Nguyen-Quang, "A gamma-based framework for modeling variable-rate MPEG video source: The GOP GBAR model," *IEEE/ACM Transactions on Networking*, Vol.8, No.6, pp.710–719, December 2000.

[4] H. R. Gail, S. L. Hantler and B. A. Taylor, " Use of characteristic roots for solving infinite state Markov chains," *Computational Probability*, W. K. Grassmann (editor), pp.205–255, Kluwer Academic Publishers, 2000.

[5] G. Haβlinger, "Waiting time, busy periods and output models of a server analyzed via Wiener-Hopf factorization," *Performance Evaluation*, Vol.40, No.1–3, pp.3–26, March 2000.

[6] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.

[7] A. Kuczura, "Queues with mixed renewal and Poisson inputs," *The Bell System Technical Journal*, Vol.51, No.6, pp.1305–1326, July–August 1972.

[8] A. Kuczura, "Piecewise Markov processes," *SIAM Journal on Applied Mathematics*, Vol.24, No.2, pp.169–181, March 1973

[9] D. Le Gall, "MPEG: A video compression standard for multimedia applications," *Communications of the ACM*, Vol.34, No.4, pp.46–58, April 1991.

[10] M. F. Neuts, *Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach*, The Johns Hopkins University Press, Baltimore, 1981.

[11] O. Rose, "Simple and efficient models for variable bit rate MPEG video traffic," *Performance Evaluation*, Vol.30, No.1–2, pp.69–85, July 1997.

[12] T. L. Saaty, *Elements of Queueing Theory with Applications*, McGraw-Hill Book Company, New York, 1961. Republished by Dover Publications, New York, 1983.

[13] U. K. Sarkar, S. Ramakrishnan, and D. Sarkar, "Segmenting full-length VBR video into slots for modeling with Markov-modulated gamma-based framework," *Proceedings of the SPIE*, Vol.4519, 2001.

[14] B. Sengupta, "The semi-Markov queue: theory and applications," *Communications in Statistics–Stochastic Models*, Vol.6, No.3, pp.383–413, September 1990.

[15] J. H. de Smit, "The single server semi-Markov queue," *Stochastic Processes and their Applications*, Vol.22, No.1, pp.37–50, May 1986.

[16] H. Takagi, *Queueing Analysis: A Foundation of Performance Evaluation, Volume 1: Vacation and Priority Systems, Part 1*, Elsevier, 1991.

[17] D. Wu and H. Takagi, "A queue with semi-Markovian batch plus Poisson arrivals with application to the MPEG frame sequence," Discussion paper No.958, Institute of Policy and Planning Sciences, University of Tsukuba, November 2001.

[18] D. Wu and H. Takagi, "Analysis of a multiserver queue with semi-Markovian batch arrival with application to the MPEG frame sequence," Discussion paper No.985, Institute of Policy and Planning Sciences, University of Tsukuba, May 2002.

[19] S. Yagyu and H. Takagi, "A queueing model with input of MPEG fame sequence and interfering traffic (a revised version)," Discussion paper No.951, Institute of Policy and Planning Sciences, University of Tsukuba, September 2001. To appear in the *Journal of the Operations Research Society of Japan.*

Appendix: Queue Size in an SMP$^{[X]}$+M/M/c System When the State Sojourn Time Follows an Exponential Distribution

We show that the unknown constants contained in the generating function (17) can be determined through the zeros of the denominator for this generating function in the unit disk when the sojourn time in the state of SMP follows an exponential distribution.

If the state sojourn time A_{lm} follows an exponential distribution with mean $1/\alpha_{lm}$, equation (17) is free from the Bromwich integral, and it is reduced to

$$\Phi_m(z) = \sum_{l=1}^{L} \frac{p_{lm}}{q_{lm}(z)} \left[zG_l(z)\Phi_l(z) - (1-z)B_{lm}(z) \right]; \quad m = 1, \dots, L, \tag{A.1}$$

where

$$q_{lm}(z) := z - \frac{1}{\alpha_{lm}}(1-z)(c\mu - \lambda z); \quad l, m = 1, \dots, L, \tag{A.2}$$

$$B_{lm}(z) := \mu \sum_{j=0}^{c-1} (c-j) H_{lm}(j) z^j, \tag{A.3}$$

$$H_{lm}(j) := \sum_{k=1}^{\infty} d_l(k) P^*_{k,j}(\alpha_{lm}); \quad j = 0, \dots, c-1,$$

and

$$d_l(k) := \sum_{i=0}^{k-1} P(i,l) g_l(k-i); \quad k \geq 1. \tag{A.4}$$

We also have the relations

$$\begin{aligned}(s+\lambda)H_{lm}(0) &= \mu H_{lm}(1); \quad l,m=1,\ldots,L,\\ (s+\lambda+j\mu)H_{lm}(j) &= \lambda H_{lm}(j-1)+(j+1)\mu H_{lm}(j+1)+d_l(j);\\ & \qquad j=1,\ldots,c-2; \quad l,m=1,\ldots,L. \end{aligned} \tag{A.5}$$

and

$$P(i,l)=\frac{1}{i!}\frac{d^i\Phi_l(z)}{dz^i}\bigg|_{z=0}; \quad l=1,\ldots,L, \tag{A.6}$$

for $\{P(i,l);\ i=0,\ldots,c-3\}$ that appear in $d_l(j)$ in (A.5). Note that each $B_{lm}(z)$ is a polynomial in z. The set $\{B_{lm}(z);\ l,m=1,\ldots,L\}$ contains cL^2 unknown constants $\{H_{lm}(j);\ j=0,\ldots,c-1;\ l,m=1,\ldots,L\}$. In addition, the relations in (A.5) include $(c-2)L$ unknown constants $\{P(i,l);\ i=0,\ldots,c-3;\ l=1,\ldots,L\}$. Thus the total number of unknown constants is $cL^2+(c-2)L$.

Now, equation (A.1) can be written in matrix form as

$$\mathbf{\Phi}(z)\mathbf{V}(z)=z\mathbf{\Phi}(z)\mathbf{G}(z)\mathbf{Q}(z)-(1-z)\mathbf{1}\mathrm{diag}[\mathbf{B}^{\mathrm{t}}(z)\mathbf{Q}(z)], \tag{A.7}$$

where

$$\mathbf{V}(z):=\begin{bmatrix}\prod_{j=1}^{L}q_{j1}(z) & 0 & \ldots & 0\\ 0 & \prod_{j=1}^{L}q_{j2}(z) & \ldots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \ldots & \prod_{j=1}^{L}q_{jL}(z)\end{bmatrix}, \tag{A.8}$$

$$\mathbf{Q}(z):=\begin{bmatrix}p_{11}\prod_{j\neq1}q_{j1}(z) & p_{12}\prod_{j\neq1}q_{j2}(z) & \ldots & p_{1L}\prod_{j\neq1}q_{jL}(z)\\ p_{21}\prod_{j\neq2}q_{j1}(z) & p_{22}\prod_{j\neq2}q_{j2}(z) & \ldots & p_{2L}\prod_{j\neq2}q_{jL}(z)\\ \vdots & \vdots & \ddots & \vdots\\ p_{L1}\prod_{j\neq L}q_{j1}(z) & p_{L2}\prod_{j\neq L}q_{j2}(z) & \ldots & p_{LL}\prod_{j\neq L}q_{jL}(z)\end{bmatrix}, \tag{A.9}$$

$$\mathbf{B}(z):=\begin{bmatrix}B_{11}(z) & B_{12}(z) & \ldots & B_{1L}(z)\\ B_{21}(z) & B_{21}(z) & \ldots & B_{2L}(z)\\ \vdots & \vdots & \ddots & \vdots\\ B_{L1}(z) & B_{L2}(z) & \ldots & B_{LL}(z)\end{bmatrix}, \tag{A.10}$$

and $\mathbf{G}(z)$ is given by (45). We may rewrite (A.7) as

$$\mathbf{\Phi}(z)\mathbf{F}(z) = (z-1)\mathbf{1}\text{diag}[\mathbf{B}^{\text{t}}(z)\mathbf{Q}(z)], \tag{A.11}$$

where

$$\mathbf{F}(z) := \mathbf{V}(z) - z\mathbf{G}(z)\mathbf{Q}(z). \tag{A.12}$$

Let adj$\mathbf{F}(z)$ denote the adjoint matrix of $\mathbf{F}(z)$. Multiplying (A.11) on the right by adj$\mathbf{F}(z)$, we have

$$\mathbf{\Phi}(z) = \frac{(z-1)\mathbf{1}\text{diag}[\mathbf{B}^{\text{t}}(z)\mathbf{Q}(z)]\text{adj}\mathbf{F}(z)}{\det \mathbf{F}(z)}. \tag{A.13}$$

Note that (A.13) contains the set of cL^2 unknown constants $\{H_{lm}(j);\ j=0,\dots,c-1;\ l,m=1,\dots,L\}$. We will show that there are L^2 zeros for $\det \mathbf{F}(z)$ in the unit disk $|z| \le 1$ if the condition

$$\alpha g + \lambda < c\mu \tag{A.14}$$

is satisfied, which is the special case of the stability condition in (43). Here

$$\alpha := \frac{1}{\displaystyle\sum_{l=1}^{L} \pi_l \sum_{m=1}^{L} \frac{p_{lm}}{\alpha_{lm}}} \tag{A.15}$$

is the arrival rate of the batches of SMP customers, and g is the average batch size given in (41). In order to determine the unknown constants $\{H_{lm}(j),\ P(i,l);\ j=0,\dots,c-1;\ i=0,\dots,c-3;\ l,m=1,\dots,L\}$, $cL^2+(c-2)L$ equations are required. Out of them, L^2 equations are obtained from the zeros of det$\mathbf{F}$(z), and the remaining $(c-1)L^2+(c-2)L$ equations are given by (A.5) and (A.6).

Differentiating (A.11) and evaluating the result at $z=1$, we obtain

$$\mathbf{\Phi}'(1)(\mathbf{I}_L - \mathbf{P}) + \boldsymbol{\pi}\mathbf{F}'(1) = \mathbf{1}\text{diag}[\mathbf{B}^{\text{t}}(z)\mathbf{P}]. \tag{A.16}$$

Here we have used $\mathbf{F}(1) = \mathbf{I}_L - \mathbf{P}$ since $\mathbf{V}(1) = \mathbf{G}(1) = \mathbf{I}_L$ and $\mathbf{Q}(1) = \mathbf{P}$. Note also that $\mathbf{\Phi}(1) = \boldsymbol{\pi}$. Multiplying (A.16) on the right by $\mathbf{1}^{\text{t}}$ and noting that $(\mathbf{I}_L - \mathbf{P})\mathbf{1}^{\text{t}} = 0$, we get

$$\boldsymbol{\pi}\mathbf{F}'(1)\mathbf{1}^{\text{t}} = \mathbf{1}\text{diag}[\mathbf{B}^{\text{t}}(1)\mathbf{P}]\mathbf{1}^{\text{t}}. \tag{A.17}$$

To determine the left-hand side of (A.17), we differentiate (A.12) and evaluate the result at $z=1$. Then we have

$$\begin{aligned}\mathbf{F}'(1) &= \mathbf{V}'(1) - \mathbf{G}(1)\mathbf{Q}(1) - \mathbf{G}'(1)\mathbf{Q}(1) - \mathbf{G}(1)\mathbf{Q}'(1) \\ &= \mathbf{V}'(1) - \mathbf{P} - \mathbf{G}'(1)\mathbf{P} - \mathbf{Q}'(1),\end{aligned} \tag{A.18}$$

where

$$\mathbf{V}'(1) = \begin{bmatrix} \sum_{j=1}^{L} \frac{\alpha_{j1} + c\mu - \lambda}{\alpha_{j1}} & 0 & \cdots & 0 \\ 0 & \sum_{j=1}^{L} \frac{\alpha_{j2} + c\mu - \lambda}{\alpha_{j2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{j=1}^{L} \frac{\alpha_{jL} + c\mu - \lambda}{\alpha_{jL}} \end{bmatrix}, \tag{A.19}$$

$$\mathbf{Q}'(1) = \begin{bmatrix} p_{11} \sum_{j \neq 1} \frac{\alpha_{j1} + c\mu - \lambda}{\alpha_{j1}} & p_{12} \sum_{j \neq 1} \frac{\alpha_{j2} + c\mu - \lambda}{\alpha_{j2}} & \cdots & p_{1L} \sum_{j \neq 1} \frac{\alpha_{jL} + c\mu - \lambda}{\alpha_{jL}} \\ p_{21} \sum_{j \neq 2} \frac{\alpha_{j1} + c\mu - \lambda}{\alpha_{j1}} & p_{22} \sum_{j \neq 2} \frac{\alpha_{j2} + c\mu - \lambda}{\alpha_{j2}} & \cdots & p_{2L} \sum_{j \neq 2} \frac{\alpha_{jL} + c\mu - \lambda}{\alpha_{jL}} \\ \vdots & \vdots & \ddots & \vdots \\ p_{L1} \sum_{j \neq L} \frac{\alpha_{j1} + c\mu - \lambda}{\alpha_{j1}} & p_{L2} \sum_{j \neq L} \frac{\alpha_{j2} + c\mu - \lambda}{\alpha_{j2}} & \cdots & p_{LL} \sum_{j \neq L} \frac{\alpha_{jL} + c\mu - \lambda}{\alpha_{jL}} \end{bmatrix}, \tag{A.20}$$

and $\mathbf{G}'(1)$ is given by (36). Multiplying (A.18) on the right by $\mathbf{1}^{\mathrm{t}}$ and substituting (A.19), (36), and (A.20) yields

$$\begin{aligned} \mathbf{F}'(1)\mathbf{1}^{\mathrm{t}} &= \mathbf{V}'(1)\mathbf{1}^{\mathrm{t}} - \mathbf{1}^{\mathrm{t}} - \mathbf{G}'(1)\mathbf{1}^{\mathrm{t}} - \mathbf{Q}'(1)\mathbf{1}^{\mathrm{t}} \\ &= \begin{bmatrix} \sum_{j=1}^{L} \frac{\alpha_{j1} + c\mu - \lambda}{\alpha_{j1}} \\ \sum_{j=1}^{L} \frac{\alpha_{j2} + c\mu - \lambda}{\alpha_{j2}} \\ \vdots \\ \sum_{j=1}^{L} \frac{\alpha_{jL} + c\mu - \lambda}{\alpha_{jL}} \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} - \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_L \end{bmatrix} - \begin{bmatrix} \sum_{k=1}^{L} p_{1k} \sum_{j \neq 1} \frac{\alpha_{jk} + c\mu - \lambda}{\alpha_{jk}} \\ \sum_{k=1}^{L} p_{2k} \sum_{j \neq 2} \frac{\alpha_{jk} + c\mu - \lambda}{\alpha_{jk}} \\ \vdots \\ \sum_{k=1}^{L} p_{Lk} \sum_{j \neq L} \frac{\alpha_{jk} + c\mu - \lambda}{\alpha_{jk}} \end{bmatrix}. \end{aligned} \tag{A.21}$$

Finally, multiplying (A.21) on the left by $\boldsymbol{\pi}$, we obtain

$$\begin{aligned} \boldsymbol{\pi}\mathbf{F}'(1)\mathbf{1}^{\mathrm{t}} = &\sum_{l=1}^{L} \pi_l \sum_{k=1}^{L} \frac{\alpha_{kl} + c\mu - \lambda}{\alpha_{kl}} - \sum_{l=1}^{L} \pi_l - \sum_{l=1}^{L} \pi_l g_l \\ &- \sum_{l=1}^{L} \pi_l \sum_{k=1}^{L} p_{lk} \sum_{j \neq l} \frac{\alpha_{jk} + c\mu - \lambda}{\alpha_{jk}} \end{aligned}$$

$$= \sum_{l=1}^{L} \pi_l \sum_{k=1}^{L} \frac{\alpha_{kl} + c\mu - \lambda}{\alpha_{kl}} - 1 - g$$
$$- \sum_{l=1}^{L} \pi_l \sum_{k=1}^{L} p_{lk} \sum_{j=1}^{L} \frac{\alpha_{jk} + c\mu - \lambda}{\alpha_{jk}} + \sum_{l=1}^{L} \pi_l \sum_{k=1}^{L} p_{lk} \frac{\alpha_{lk} + c\mu - \lambda}{\alpha_{lk}}.$$

However, from the relations $\sum_{l=1}^{L} \pi_l p_{lk} = \pi_k$, $k = 1, \ldots, L$, we have

$$\sum_{l=1}^{L} \pi_l \sum_{k=1}^{L} p_{lk} \sum_{j=1}^{L} \frac{\alpha_{jk} + c\mu - \lambda}{\alpha_{jk}} = \sum_{k=1}^{L} \sum_{j=1}^{L} \frac{\alpha_{jk} + c\mu - \lambda}{\alpha_{jk}} \sum_{l=1}^{L} \pi_l p_{lk}$$
$$= \sum_{k=1}^{L} \pi_k \sum_{j=1}^{L} \frac{\alpha_{jk} + c\mu - \lambda}{\alpha_{jk}}.$$

Thus we get

$$\boldsymbol{\pi} \mathbf{F}'(1) \mathbf{1}^{\mathrm{t}} = (c\mu - \lambda) \sum_{l=1}^{L} \pi_l \sum_{k=1}^{L} \frac{p_{lk}}{\alpha_{lk}} - g = \frac{c\mu - \lambda}{\alpha} - g. \tag{A.22}$$

From the stability condition (43), we must have

$$\alpha g + \lambda < c\mu, \tag{A.23}$$

which is the condition in (A.14).

Recall that $\boldsymbol{\Phi}(1) = \boldsymbol{\pi}$. Since $\det \mathbf{F}(1) = \det[\mathbf{I}_L - \mathbf{P}] = 0$, the point $z = 1$ is the common zero of the denominator and the numerator for the right-hand side of (A.13). Thus we investigate the value of the derivative of $\det \mathbf{F}(z)$ at $z = 1$:

$$\gamma = \frac{d}{dz} \det \mathbf{F}(z) \bigg|_{z=1}.$$

Theorem 1 *If $\alpha g + \lambda < c\mu$, then $\gamma > 0$.*

Proof. To determine γ, we use the well-known relations in linear algebra:

$$\mathbf{F}(z) \mathrm{adj} \mathbf{F}(z) = \det \mathbf{F}(z) \mathbf{I}_L = \mathrm{adj} \mathbf{F}(z) \mathbf{F}(z). \tag{A.24}$$

Differentiating the second equality, evaluating the value at $z = 1$, and multiplying on the right by $\mathbf{1}^{\mathrm{t}}$, we obtain

$$\gamma \mathbf{1}^{\mathrm{t}} = \mathrm{adj} \mathbf{F}(1) \mathbf{F}'(1) \mathbf{1}^{\mathrm{t}}. \tag{A.25}$$

An expression for $\mathrm{adj}\mathbf{F}(1)$ may be found as follows. Evaluating (A.24) at $z = 1$ and using $\det \mathbf{F}(1) = 0$, we have

$$\mathbf{P} \mathrm{adj} \mathbf{F}(1) = \mathrm{adj} \mathbf{F}(1) = \mathrm{adj} \mathbf{F}(1) \mathbf{P}.$$

Since $\mathbf{P}$ is an irreducible stochastic matrix, the first equality implies that each column of adj$\mathbf{F}(1)$ is a multiple of $\mathbf{1}^{\mathrm{t}}$ (recall that $\mathbf{P}\mathbf{1}^{\mathrm{t}} = \mathbf{1}^{\mathrm{t}}$). Similarly, the second equality implies that each row of adj$\mathbf{F}(1)$ is a multiple of $\boldsymbol{\pi}$ (recall that $\boldsymbol{\pi}\mathbf{P} = \boldsymbol{\pi}$). It follows that there is a constant h such that

$$\mathrm{adj}\mathbf{F}(1) = h \begin{bmatrix} \boldsymbol{\pi} \\ \vdots \\ \boldsymbol{\pi} \end{bmatrix}. \tag{A.26}$$

We claim that adj$\mathbf{F}(1)$ is a positive matrix [6, p.359]. From the form of (A.26), it is enough to show that the diagonal elements, say, κ_l, $l = 1, \dots, L$, of adj$\mathbf{F}(1)$ are positive. To see this, note that

$$\kappa_l = (-1)^{l+l} \det[\mathbf{F}_{(l,l)}(1)] = \det[\mathbf{I}_{L-1} - \mathbf{P}_{(l,l)}],$$

where $\mathbf{P}_{(l,l)}$ is the matrix $\mathbf{P}$ with its lth row and lth column removed. Since $\mathbf{P}$ is irreducible, the spectral radius of $\mathbf{P}_{(l,l)}$ is strictly less than unity. This implies that $\det[\mathbf{I}_{L-1} - t\mathbf{P}_{(l,l)}] \neq 0$ for real t satisfying $0 \le t \le 1$. Since this determinant function of t is positive for $t = 0$ and never zero, by continuity it is also positive for $t = 1$, i.e., $\kappa_l > 0$. Thus adj$\mathbf{F}(1)$ is positive, and we conclude that $h > 0$ in (A.26).

Substituting (A.26) into (A.25) and noting (A.22) yields

$$\gamma = h\Big(\frac{c\mu - \lambda}{\alpha} - g\Big). \tag{A.27}$$

Using $h > 0$ and the condition (A.23), we see that γ is positive. □

We next show that there are L^2 zeros for $\det \mathbf{F}(z)$ in the unit disk. To do so, we use a lemma in [4, p.239]: *Let $f(z,t)$ be a function analytic for z within and on a closed contour $\mathcal{C}$, and continuous for t in some interval $\mathcal{I}$. If $f(z,t) \neq 0$ for $z \in \mathcal{C}$ and $t \in \mathcal{I}$, then the number of zeros of $f(z,t)$ inside $\mathcal{C}$ is the same for all $t \in \mathcal{I}$.*

For our purpose, let

$$f(z,t) := \det \mathbf{F}(z,t),$$

where

$$\mathbf{F}(z,t) := \mathbf{V}(z) - zt\mathbf{G}(z)\mathbf{Q}(z).$$

We choose a closed contour $\mathcal{C} := \{z; |z| = 1\}$ and an interval $\mathcal{I} := \{t; t \in [0,1)\}$. Obviously, $f(z,t)$ is analytic in $\mathcal{C}$ and continuous for $t \in \mathcal{I}$. We first prove that $f(z,t) \neq 0$ for $z \in \mathcal{C}$ and $t \in \mathcal{I}$, and then prove that there are L^2 zeros for $f(z,1) = \det \mathbf{F}(z)$ in $\mathcal{C}$ using the above lemma.

Theorem 2

(a) $\det \mathbf{F}(z,t) \neq 0$ *for* $|z| = 1$ *and* $t \in [0,1)$.

(b) $\det \mathbf{F}(z) \neq 0$ *for* $|z| = 1,\ z \neq 1$.

Proof. We consider $\det \mathbf{F}(z,t)$ for $|z|=1$ and $t \in [0,1]$. Note that $\det \mathbf{F}(z) = \det \mathbf{F}(z,1)$. Then $\mathbf{F}(z,t)$ can be written as

$$\begin{aligned} \mathbf{F}(z,t) &= \mathbf{V}(z) - zt\mathbf{G}(z)\mathbf{Q}(z) \\ &= \mathbf{V}(z) - zt\mathbf{G}(z)\mathbf{L}(z)\mathbf{V}(z) \\ &= [\mathbf{I}_L - zt\mathbf{G}(z)\mathbf{L}(z)]\mathbf{V}(z), \end{aligned} \tag{A.28}$$

where

$$\mathbf{L}(z) := \begin{bmatrix} \frac{p_{11}}{q_{11}(z)} & \frac{p_{12}}{q_{12}(z)} & \cdots & \frac{p_{1L}}{q_{1L}(z)} \\ \frac{p_{21}}{q_{21}(z)} & \frac{p_{22}}{q_{22}(z)} & \cdots & \frac{p_{2L}}{q_{2L}(z)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{p_{L1}}{q_{L1}(z)} & \frac{p_{L2}}{q_{L2}(z)} & \cdots & \frac{p_{LL}}{q_{LL}(z)} \end{bmatrix}. \tag{A.29}$$

Therefore we have

$$\det \mathbf{F}(z,t) = \det[\mathbf{I}_L - zt\mathbf{G}(z)\mathbf{L}(z)] \cdot \det \mathbf{V}(z). \tag{A.30}$$

Since

$$|q_{jk}(z)| = \left| \frac{1}{\alpha_{jk}} [(\alpha_{jk} + \lambda + c\mu)z - (\lambda z^2 + c\mu)] \right| \geq \frac{1}{\alpha_{jk}} [\alpha_{jk} + \lambda + c\mu - (\lambda + c\mu)] = 1$$

for $|z|=1$, we see that

$$|\det \mathbf{V}(z)| = \left| \prod_{k=1}^{L} \prod_{j=1}^{L} q_{jk}(z) \right| \geq 1, \quad \text{for } |z| = 1.$$

It follows that $\det \mathbf{V}(z) \neq 0$ for $|z|=1$.

We next prove that $\mathbf{I}_L - zt\mathbf{G}(z)\mathbf{L}(z)$ is nonsingular for $|z|=1$ and $t \in [0,1)$ and that $\mathbf{I}_L - z\mathbf{G}(z)\mathbf{L}(z)$ is nonsingular for $|z|=1,\ z \neq 1$. These are equivalent to that $\det[\mathbf{I}_L - zt\mathbf{G}(z)\mathbf{L}(z)]$ and $\det[\mathbf{I}_L - z\mathbf{G}(z)\mathbf{L}(z)]$ are invertible, respectively. To do this, we use the notion of maximum row sum matrix norm [6, p.295] and a corollary in [6, p.301]. We state it as follows: *Suppose there is a $L \times L$ matrix $\mathbf{A}_L = \{a_{ij}\}$. The maximum row sum matrix norm of $\mathbf{A}_L$ is defined by*

$$|||\mathbf{A}_L|||_\infty := \max_{1 \leq i \leq L} \sum_{j=1}^{L} |a_{ij}|.$$

A matrix $\mathbf{A}_L$ is invertible if there is a matrix norm (e.g. $|||\cdot|||_\infty$) such that $|||\mathbf{I}_L - \mathbf{A}_L||| < 1$, if this condition is satisfied,

$$\mathbf{A}_L^{-1} = \sum_{k=0}^{\infty} (\mathbf{I}_L - \mathbf{A}_L)^k.$$

From (45) and (A.29) we have

$$\mathbf{I}_L - zt\mathbf{G}(z)\mathbf{L}(z)$$
$$= \begin{bmatrix} 1 - ztG_1(z)\frac{p_{11}}{q_{11}(z)} & -ztG_1(z)\frac{p_{12}}{q_{12}(z)} & \dots & -ztG_1(z)\frac{p_{1L}}{q_{1L}(z)} \\ -ztG_2(z)\frac{p_{21}}{q_{21}(z)} & 1 - ztG_2(z)\frac{p_{22}}{q_{22}(z)} & \dots & -ztG_2(z)\frac{p_{2L}}{q_{2L}(z)} \\ \vdots & \vdots & \ddots & \vdots \\ -ztG_L(z)\frac{p_{L1}}{q_{L1}(z)} & -ztG_L(z)\frac{p_{L2}}{q_{L2}(z)} & \dots & 1 - ztG_L(z)\frac{p_{LL}}{q_{LL}(z)} \end{bmatrix}. \tag{A.31}$$

For our purpose, we define

$$\mathbf{A}(z,t) := \mathbf{I}_L - zt\mathbf{G}(z)\mathbf{L}(z), \tag{A.32}$$

and then

$$\mathbf{I}_L - \mathbf{A}(z,t) = zt\mathbf{G}(z)\mathbf{L}(z). \tag{A.33}$$

The absolute sum of ith row for $zt\mathbf{G}(z)\mathbf{L}(z)$ on $|z| = 1$ and $0 \le t \le 1$ is satisfied the following relations

$$\sum_{j=1}^{L} \Big| ztG_i(z)\frac{p_{ij}}{q_{ij}(z)} \Big| \le t\Big|G_i(z)\Big| \sum_{j=1}^{L} p_{ij} = t\Big|G_i(z)\Big|. \tag{A.34}$$

Thus we have

$$\Big|\Big|\Big|\mathbf{I}_L - \mathbf{A}(z,t)\Big|\Big|\Big|_\infty \le t \max_i \Big|G_i(z)\Big|. \tag{A.35}$$

For case (a) in which $|z| = 1$ and $t \in [0,1)$, we see that

$$\Big|\Big|\Big|\mathbf{I}_L - \mathbf{A}(z,t)\Big|\Big|\Big|_\infty = \Big|\Big|\Big|tz\mathbf{G}(z)\mathbf{L}(z)\Big|\Big|\Big|_\infty < \max_i \Big|G_i(z)\Big| \le 1.$$

For case (b) in which $|z| = 1,\ z \ne 1$ and $t = 1$, since $|G_i(z)| < 1$, we see that

$$\Big|\Big|\Big|\mathbf{I}_L - \mathbf{A}(z,t)\Big|\Big|\Big|_\infty = \Big|\Big|\Big|z\mathbf{G}(z)\mathbf{L}(z)\Big|\Big|\Big|_\infty \le \max_i \Big|G_i(z)\Big| < 1.$$

From the Corollary [6, p.301], it follows that $\mathbf{I}_L - zt\mathbf{G}(z)\mathbf{L}$ is nonsingular for both $|z| = 1, t \in [0,1)$, and $|z| = 1, z \ne 1, t = 1$. From (A.30), we conclude that $\det\mathbf{F}(z,t) \ne 0$ for $|z| = 1, t \in [0,1)$ and $\det\mathbf{F}(z) \ne 0$ for $|z| = 1, z \ne 1$. □

Theorem 3 *If $\gamma > 0$, $\det \mathbf{F}(z)$ has $L^2 - 1$ zeros in $|z| < 1$, and it has a simple zero at $z = 1$.*

Proof. Our proof follows [4, p.241]. We first observe that $\det \mathbf{F}(z,0) = \det \mathbf{V}(z)$ has L^2 zeros in $|z| \leq 1$, because each element $q_{ij}(z)$ in $\mathbf{V}(z)$ has a single zero at

$$z_{ij} = \frac{\lambda + c\mu + \alpha_{ij} - \sqrt{(\lambda + c\mu + \alpha_{ij})^2 - 4c\lambda\mu}}{2\lambda}$$

in $|z| \leq 1$. From Theorem 2, we have $\det \mathbf{F}(z,t) \neq 0$ for $|z| = 1$ and $t \in [0,1)$. Thus, according to the above lemma, there are L^2 zeros of $\det \mathbf{F}(z,t)$ in $|z| < 1$ for all $t \in [0,1)$.

We next investigate $\det \mathbf{F}(z,t)$ at $t = 1$. Note that

$$\det \mathbf{F}(1,1) = \det \mathbf{F}(1) = \det[\mathbf{I}_L - \mathbf{P}] = 0.$$

If $\gamma > 0$, the point $z = 1$ is a simple zero of the function $\det \mathbf{F}(z,1) = \det \mathbf{F}(z)$. Since $\det \mathbf{F}(1,1) = 0$, then $\det \mathbf{F}(1-\varepsilon,1) < 0$ for small $\varepsilon > 0$. By continuity in $t \in [0,1)$, there is small τ so that $\det \mathbf{F}(1-\varepsilon, 1-\tau) < 0$. However, $\det \mathbf{F}(1,0) = \det \mathbf{V}(1) = 1$ and $\det \mathbf{F}(1,t) \neq 0$ for $0 \leq t < 1$ as shown above. By continuity, $\det \mathbf{F}(1,t) > 0$ for $0 \leq t < 1$, so in particular, $\det \mathbf{F}(1, 1-\tau) > 0$. Therefore, $\det \mathbf{F}(1-\varepsilon_1, 1-\tau) = 0$ for some $0 < \varepsilon_1 < \varepsilon$. The same argument holds for $\tau \to 0$, so the simple zero at $z = 1$ is the limit of zeros from inside the unit disk. It follows that $\det \mathbf{F}(z,1) = \det \mathbf{F}(z)$ has L^2 zeros in $|z| \leq 1$. From Theorem 2(b), $\det \mathbf{F}(z)$ has $L^2 - 1$ zeros in $|z| < 1$. □