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Abstract

In the setting of a discrete-time delayed renewal process, we study counting the number of renewals during a random interval. An example is the number of appearances of a specific pattern in a random number of repeated trials. We obtain closed-form mathematical expressions for the probability mass function and the binomial moments of this number for various distributions of the interrenewal times and the length of the random interval.

keywords: Delayed renewal process; counting process; discrete-time.

1 Introduction

This paper is concerned with counting the number of renewals in a discete-time random interval. Let $N(n)$ be the number of renewals in a fixed discrete-time interval $(0,n]$, where n is a positive integer. The interrenewal times occur according to a sequence of discrete random variables $\{X_1, X_2, \ldots, X_i, \ldots\}$, where X_1 is started at time 0. Let T be a random variable representing a discrete-time interval, which is independent of $\{X_1, X_2, \ldots, X_i, \ldots\}$. Hence $N(T)$ is a random variable which represents the number of renewals occurring in the random interval $(0,T]$. For the sake of convenience, we call T a session time in this paper.

The problem of finding the probability distribution of $N(T)$ in the continuous-time setting has been treated for several specific cases by Cox in his monograph [2, sec. 3.4] under the title "the number of renewals in a random time." Most of the results presented by Cox are based on the *ordinary* renewal process, i.e., all the random variables $X_i, i = 1,2,\ldots$ come from the same distribution [2, p. 25]. Thus it is more general to consider the case in which $X_i, i = 2,3,\ldots$ come from the same distribution as X_2 while only X_1 may come from a different distribution. Such a case is called the *delayed* renewal process [2, p. 28]. As a special case of the delayed renewal process, if X_1 is a residual life of X_2, we have the *equilibrium* renewal process [2, p. 28]. These are the three types of a renewal process introduced by Cox, and often considered by others subsequently.

However, as a further generalization of the delayed renewal process, we may assume that each interrenewal time $X_1, X_2, \ldots$ can have different distribution.

The discrete-time renewal process is dealt with in several textbooks, including Feller [3] and Hunter [4], where it is called the *recurrent event process.* They present a theory for counting the number of renewals in a fixed time interval. As an application, Koutras [5] discusses the number of appearances of a specific pattern in a fixed number of repeated trials. More applications of this type can be found in [1]. However, none of them consider the number of renewals in a random time interval. Thus the theory in this paper extends the previous treatment.

The rest of this paper is organized as follows. In Section 2, we derive the generating function for $N(T)$ when both session times and interrenewal times have general distributions. We also obtain a useful expression when the session time is negative binomially distributed. Based on the general results of Section 2, we obtain in Section 3 the probability mass function and the binomial moments for $N(T)$ when the session time is geometrically distributed while the interrenewal times are general. In Section 4 similar results are obtained when the session times are generally distributed while the interrenewal times are geometrically distributed. Our results in Sections 2 through 4 are valid for the delayed renewal process in the sense defined by Cox [2]. In Section 5, we extend the results to the case of a generalized delayed renewal process such that some or all of the interrenewal times can have different distributions.

2 General Session Time and General Interrenewal Times in a Delayed Renewal Process

In Sections 2 through 4, we assume a delayed renewal process, i.e., a sequence of interrenewal times $\{X_i; i = 1, 2, \ldots\}$ such that $X_i, i = 2, 3, \ldots$ come from the same distribution as X_2. In Section 2, we present a framework for handling the case in which the session time T and the interrenewal times X_1 and X_2 have general distributions respectively. In particular, we derive the generating function for $N(T)$ when the session time has negative binomial distribution.

2.1 Generating function

Let us define the sum of m interrenewal times $\{X_1, X_2, \ldots, X_m\}$ as

$$S_m := \sum_{i=1}^{m} X_i; \quad m = 1, 2, \ldots \tag{1}$$

and let $S_0 := 0$. Let $N(n)$ be a random variable representing the number of renewals in a *fixed* interval $(0, n]$, where n is a positive integer. Thus, the event $\{N(n) \geq m\}$ is equivalent to the event $\{S_m \leq n\}$, i.e., the number of renewals by time n inclusive is not fewer than m if and only if the mth renewal occurs before or at time n. Thus we have

$$\begin{aligned} P[N(n) = m] &= P[N(n) \geq m] - P[N(n) \geq m + 1] \\ &= P[S_m \leq n] - P[S_{m+1} \leq n]. \end{aligned} \tag{2}$$

Hence

$$P[N(n) = m] = F_{S_m}(n) - F_{S_{m+1}}(n); \quad m = 0, 1, 2, \ldots, \tag{3}$$

where $F_{S_m}(n) := P[S_m \leq n]$ is the cumulative distribution function (*cdf*) of the random variable S_m. Note that $F_{S_0}(n) \equiv 1$. We define the probability generating function (*pgf*) for $N(n)$ as

$$G_{N(T)}(n, z) := \sum_{m=0}^{\infty} P[N(n) = m] z^m. \tag{4}$$

Substituting (3) into (4), we get

$$G_{N(T)}(n, z) = 1 + (z - 1) \sum_{m=1}^{\infty} F_{S_m}(n) z^{m-1}. \tag{5}$$

Now, let us define the following generating function

$$G^*_{N(T)}(y, z) := \sum_{n=1}^{\infty} G_{N(T)}(n, z) y^n; \quad |y| < 1. \tag{6}$$

From (5) and (6) we have

$$G^*_{N(T)}(y, z) = \frac{y}{1-y} + (z-1) \sum_{m=1}^{\infty} z^{m-1} \sum_{n=1}^{\infty} F_{S_m}(n) y^n. \tag{7}$$

By substituting $F_{S_m}(n) = \sum_{j=1}^{n} P[S_m = j]$ into (7), we obtain

$$G^*_{N(T)}(y, z) = \frac{y}{1-y} + \frac{z-1}{1-y} \sum_{m=1}^{\infty} z^{m-1} \sum_{j=1}^{\infty} P[S_m = j] y^j. \tag{8}$$

This holds for a general renewal process.

For a delayed renewal process, the relationship between the *pgf* of S_m and the *pgf*'s $G_{X_1}(y)$ and $G_{X_2}(y)$ for the interrenewal times X_1 and X_2, respectively, is given by

$$\sum_{j=1}^{\infty} P[S_m = j] y^j = G_{X_1}(y) \{G_{X_2}(y)\}^{m-1}. \tag{9}$$

Then, by substituting (9) into (8), we obtain

$$G^*_{N(T)}(y, z) = \frac{y}{1-y} + \frac{(z-1) G_{X_1}(y)}{(1-y)[1 - z G_{X_2}(y)]}. \tag{10}$$

2.2 Probability mass function and binomial moments

Let $G_{N(T)}(z)$ be the *pgf* for $N(T)$, the number of renewals in a *random* interval $(0, T]$. It is given by

$$G_{N(T)}(z) := \sum_{n=1}^{\infty} G_{N(T)}(n, z) \; P[T = n], \tag{11}$$

where

$$G_{N(T)}(n,z) := \mathrm{E}\left[z^{N(T)}|T=n\right] = \sum_{j=0}^{\infty} P[N(n)=j|T=n]z^j. \tag{12}$$

Note that this is equivalent to (4).

Once $G_{N(T)}(z)$ is obtained, the *pmf* of $N(T)$ is given by

$$P[N(T)=j] = \frac{1}{j!}\frac{d^j}{dz^j}G_{N(T)}(z)\bigg|_{z=0}; \quad j=0,1,2,\ldots. \tag{13}$$

The ℓth binomial moment of $N(T)$ is given by

$$\mathrm{E}\left[\binom{N(T)}{\ell}\right] = \frac{1}{\ell!}\frac{d^\ell}{dz^\ell}G_{N(T)}(z)\bigg|_{z=1}; \quad \ell=0,1,2,\ldots. \tag{14}$$

2.3 Negative binomially distributed session time

Let us consider a special case in which the random interval T can be fitted by a negative binomial *pmf*, say

$$P[T=n] = \binom{n-1}{r-1}p^r q^{n-r}; \quad n \geq r \tag{15}$$

where $p+q=1$, $0<p,q<1$, and r is a positive integer. Hence

$$G_{N(T)}(z) = \sum_{n=r}^{\infty} G_{N(T)}(n,z)\binom{n-1}{r-1}p^r q^{n-r}. \tag{16}$$

On the other hand, we have

$$\frac{G^*_{N(T)}(y,z)}{y} = \sum_{n=1}^{\infty} G_{N(T)}(n,z)y^{n-1}. \tag{17}$$

Hence the $(r-1)$th derivative is given by

$$\frac{1}{(r-1)!}\frac{\partial^{r-1}}{\partial y^{r-1}}\left[\frac{G^*_{N(T)}(y,z)}{y}\right] = \sum_{n=r}^{\infty}\binom{n-1}{r-1}G_{N(T)}(n,z)y^{n-r}. \tag{18}$$

Thus we can obtain

$$G_{N(T)}(z) = \frac{p^r}{(r-1)!}\frac{\partial^{r-1}}{\partial y^{r-1}}\left[\frac{G^*_{N(T)}(y,z)}{y}\right]\bigg|_{y=q}. \tag{19}$$

This is the discrete-time version of Equation 4 in Section 3.4 of [2].

3 Geometrically Distributed Session Time and General Interrenewal Times

As a special case of the general setting in Section 2, let us assume that the session time T is modeled by a geometric *pmf*, say

$$P[T = n] = pq^{n-1}; \quad n = 1, 2, \ldots \tag{20}$$

with mean $\mathbf{E}[T] = 1/p$, where $p + q = 1$, $0 < p, q < 1$. This is a special case of $r = 1$ in (15). Then the *pgf* of $N(T)$ is given by

$$G_{N(T)}(z) = \frac{p}{q} G^*_{N(T)}(q, z) = 1 + \frac{(z-1)G_{X_1}(q)}{q[1 - zG_{X_2}(q)]}. \tag{21}$$

The jth derivative of this *pgf* is given by

$$\frac{d^j}{dz^j} G_{N(T)}(z) = \frac{j! G_{X_1}(q)[1 - G_{X_2}(q)][G_{X_2}(q)]^{j-1}}{q[1 - zG_{X_2}(q)]^{j+1}}; \quad j = 1, 2, \ldots . \tag{22}$$

Substituting (22) into (13), we obtain the *pmf* of $N(T)$ as

$$P[N(T) = j] = \begin{cases} 1 - \dfrac{G_{X_1}(q)}{q} & ; \ j = 0 \\ \dfrac{1}{q} G_{X_1}(q)[1 - G_{X_2}(q)][G_{X_2}(q)]^{j-1} & ; \ j = 1, 2, \ldots \end{cases} . \tag{23}$$

Substituting (22) into (14), we obtain the ℓth binomial moment of $N(T)$ as

$$\mathbf{E}\left[\binom{N(T)}{\ell}\right] = \frac{G_{X_1}(q)[G_{X_2}(q)]^{\ell-1}}{q[1 - G_{X_2}(q)]^{\ell}} \quad \ell = 1, 2, \ldots . \tag{24}$$

In particular, we have the mean

$$\mathbf{E}[N(T)] = \frac{G_{X_1}(q)}{q[1 - G_{X_2}(q)]} \tag{25}$$

and the variance

$$\mathbf{Var}[N(T)] = \frac{2G_{X_1}(q)G_{X_2}(q)}{q[1 - G_{X_2}(q)]^2} + \mathbf{E}[N(T)] - \mathbf{E}^2[N(T)]. \tag{26}$$

4 General Session Time and Geometrically Distributed Interrenewal Times

In this section, we consider a general *pmf* for the session time T and geometrically distributed interrenewal times as follows:

$$P[X_i = k] = p_i q_i^{k-1}; \quad k = 1, 2, \ldots, \tag{27}$$

where $p_i + q_i = 1,\ 0 < p_i, q_i < 1;\ i = 1, 2$. The *pgf* of X_i is then given by

$$G_{X_i}(y) = \frac{p_i y}{1 - q_i y}; \quad i = 1, 2. \tag{28}$$

Substituting (28) into (10), we obtain the *pgf* for $N(n)$ as follows:

$$G^*_{N(T)}(y, z) = \frac{y}{1-y} + \frac{(z-1)(1-q_2 y)p_1 y}{(1-y)(1-q_1 y)[1-(q_2+p_2 z)y]}. \tag{29}$$

Expanding (29) in partial fractions in y and then inverting, we obtain

$$G_{N(T)}(n, z) = (1-z)A(z)q_1^n + B(z)(q_2 + p_2 z)^n, \tag{30}$$

where

$$A(z) := \frac{q_2 - q_1}{q_2 - q_1 + p_2 z}, \quad B(z) := \frac{p_1 z}{q_2 - q_1 + p_2 z}. \tag{31}$$

Substituting (30) into (11), we get

$$G_{N(T)}(z) = (1-z)A(z)G_T(q_1) + B(z)G_T(q_2 + p_2 z), \tag{32}$$

where $G_T(z)$ is the *pgf* for T.

We need the jth derivative of $G_{N(T)}(z)$ in order to find the *pmf* and the moments of $N(T)$. It is given by

$$\begin{aligned}
\frac{1}{j!}\frac{d^j G_{N(T)}(z)}{dz^j} &= \frac{(-1)^j p_1 (q_2 - q_1) {p_2}^{j-1}}{(q_2 - q_1 + p_2 z)^{j+1}} G_T(q_1) \\
&+ p_1(q_2 - q_1){p_2}^{j-1} \sum_{i=0}^{j-1} \frac{(-1)^{j-i-1} G_T^{(i)}(q_2 + p_2 z)}{i!(q_2 - q_1 + p_2 z)^{j-i+1}} \\
&+ \frac{p_1 p_2^j z}{j!(q_2 - q_1 + p_2 z)} G_T^{(j)}(q_2 + p_2 z); \qquad j = 1, 2, \ldots
\end{aligned} \tag{33}$$

where $G_T^{(j)}(y) := d^j G_T(y)/dy^j$.

From (33), we find the *pmf* for $N(T)$ as

$$P[N(T) = j] \\
= \begin{cases} G_T(q_1) & j = 0 \\ \dfrac{p_1}{p_2}\left(\dfrac{p_2}{q_1 - q_2}\right)^j \left[G_T(q_1) - \displaystyle\sum_{i=0}^{j-1} \frac{(q_1 - q_2)^i}{i!} G_T^{(i)}(q_2)\right] & j = 1, 2, \ldots \end{cases} \tag{34}$$

The ℓth binomial moment of $N(T)$ is given by

$$\begin{aligned}
\mathbf{E}\left[\binom{N(T)}{\ell}\right] &= \frac{q_1 - q_2}{p_2}\left(-\frac{p_2}{p_1}\right)^\ell \left\{G_T(q_1) - \sum_{i=0}^{\ell-1} \frac{(-p_1)^i}{i!} G_T^{(i)}(1)\right\} \\
&+ \frac{{p_2}^\ell}{\ell!} G_T^{(\ell)}(1); \quad \ell = 1, 2, \ldots.
\end{aligned} \tag{35}$$

For example, the mean is given by

$$\mathrm{E}[N(T)] = \frac{q_2 - q_1}{p_1}[1 - G_T(q_1)] + p_2\mathrm{E}[T]. \tag{36}$$

We remark that the corresponding *pgf* for the ordinary and equilibrium renewal processes can be obtained by making $q_1 = q_2 = q$ and $p_1 = p_2 = p$. In such a case, we get $G_{N(T)}(n, z) = (q + pz)^n$, which is the *pgf* of a binomial random variable. Thus we simply have

$$G_{N(T)}(z) = G_T(q + pz), \tag{37}$$

which leads to

$$P[N(T) = j] = \frac{p^j}{j!}\, G_T^{(j)}(q); \quad j = 0, 1, 2, \ldots \tag{38}$$

and

$$\mathrm{E}\left[\binom{N(T)}{\ell}\right] = \frac{p^\ell}{\ell!}\, G_T^{(\ell)}(1); \quad \ell = 0, 1, 2, \ldots. \tag{39}$$

5 Generalized Delayed Renewal Process

Let us generalize the delayed renewal process so that each interrenewal time $X_1, X_2, \ldots$ may have different distribution.

Suppose that the first R interrenewal times $X_1, X_2, \ldots, X_R$ may have different distributions for which the *pgf*s are given by $G_{X_1}(y)$, $G_{X_2}(y)$, $\ldots$, $G_{X_R}(y)$, respectively, and that the subsequent interrenewal times X_{R+1}, X_{R+2}, $\ldots$ have the same distribution as X_R. As special cases of this process, we have an ordinary renewal process for $R = 1$, a delayed renewal process for $R = 2$, and the case in which all interrenewal times can be distinct for $R = \infty$.

For this process, we have

$$\sum_{j=1}^{\infty} P[S_m = j] y^j = \begin{cases} \prod_{r=1}^{m} G_{X_r}(y) & 1 \le m \le R-1 \\ G_R(y)[G_{X_R}(y)]^{m-R} & m \ge R \end{cases}, \tag{40}$$

where we have introduced for notational convenience

$$G_R(y) := \prod_{r=1}^{R} G_{X_r}(y). \tag{41}$$

Then, by substituting (40) into (8), we obtain

$$G^*_{N(T)}(y, z) = \frac{y}{1-y} + \frac{z-1}{1-y}\sum_{m=1}^{R-1} z^{m-1}\prod_{r=1}^{m} G_{X_r}(y) + \frac{(z-1)z^{R-1}G_R(y)}{(1-y)[1 - zG_{X_R}(y)]}. \tag{42}$$

5.1 Geometrically distributed session time

If the session time T is geometrically distributed with mean $\mathbf{E}[T] = 1/p$ as in (20), we have the *pgf* for $N(T)$ as

$$\begin{aligned} G_{N(T)}(z) &= \frac{p}{q} G^*_{N(T)}(q, z) \\ &= 1 + \frac{z-1}{q} \sum_{m=1}^{R-1} z^{m-1} \prod_{r=1}^{m} G_{X_r}(q) + \frac{(z-1)z^{R-1} G_R(q)}{q[1 - zG_{X_R}(q)]}. \end{aligned} \tag{43}$$

It is straightforward as before to obtain the *pmf* and moments of $N(T)$ from (43). As the coefficient of z^j in the expansion of (43) in powers of z, the *pmf* of $N(T)$ is given by

$$P[N(T) = j] = \begin{cases} 1 - \dfrac{G_{X_1}(q)}{q} & ; \ j = 0 \\[2ex] \dfrac{1 - G_{X_{j+1}}(q)}{q} \displaystyle\prod_{r=1}^{j} G_{X_r}(q) & ; \ 1 \le j \le R-1 \\[2ex] \dfrac{G_R(q)}{q} [1 - G_{X_R}(q)] \, [G_{X_R}(q)]^{j-R} & ; \ j \ge R \end{cases} . \tag{44}$$

As the coefficient of $(z-1)^\ell$ in the expansion of (43) in powers of $z - 1$, the ℓth binomial moment of $N(T)$ is given by

$$\mathbf{E}\left[\binom{N(T)}{\ell}\right] = \begin{cases} \dfrac{1}{q} \left[\displaystyle\sum_{j=\ell}^{R-1} \binom{j-1}{\ell-1} \prod_{r=1}^{j} G_{X_r}(q) + G_R(q) \sum_{j=0}^{\ell-1} \binom{R-1}{\ell-j-1} \frac{[G_{X_R}(q)]^j}{[1 - G_{X_R}(q]^{j+1}} \right] ; \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \le \ell \le R-1 \\[2ex] \dfrac{G_R(q)}{q} \displaystyle\sum_{j=\ell-R}^{\ell-1} \binom{R-1}{\ell-j-1} \frac{[G_{X_R}(q)]^j}{[1 - G_{X_R}(q)]^{j+1}} ; \quad \ell \ge R \end{cases} \tag{45}$$

In particular, the mean is given by

$$\mathbf{E}[N(T)] = \frac{1}{q} \left[\sum_{j=1}^{R-1} \prod_{r=1}^{j} G_{X_r}(q) + \frac{G_R(q)}{1 - G_{X_R}(q)} \right] . \tag{46}$$

All the above expressions reduce to those in Section 3 when $R = 2$.

For $R = \infty$, by assuming that $\lim_{R\to\infty} G_R(q) = 0$ for $q > 0$, the *pgf* of $N(T)$ is given by

$$G_{N(T)}(z) = 1 + \frac{z-1}{q} \sum_{m=1}^{\infty} z^{m-1} \prod_{r=1}^{m} G_{X_r}(q). \tag{47}$$

Thus we have

$$P[N(T)=j]=\begin{cases} 1-\dfrac{G_{X_1}(q)}{q} & ; \ j=0 \\ \dfrac{1-G_{X_{j+1}}(q)}{q}\displaystyle\prod_{r=1}^{j} G_{X_r}(q) & ; \ j=1,2,\ldots \end{cases} \tag{48}$$

and

$$\mathbf{E}\left[\binom{N(T)}{\ell}\right]=\frac{1}{q}\sum_{j=\ell}^{\infty}\binom{j-1}{\ell-1}\prod_{r=1}^{j} G_{X_r}(q); \quad \ell=1,2,\ldots. \tag{49}$$

5.2 General session time and geometrically distributed inter-renewal times

As an extension to the case of Section 4, we can consider a general session time and geometrically distributed interrenewal times as

$$G_{X_r}(y)=\frac{p_r y}{1-q_r y}; \quad r=1,2,\ldots,R, \tag{50}$$

where $p_r+q_r=1,\ 0<p_r,q_r<1$. Let us assume for simplicity that all p_r's (thus all q_r's) are distinct.

Substituting (50) into (42) and expanding in partial fractions in y yields

$$G^{*}_{N(T)}(y,z)=-1+(1-z)\sum_{r=1}^{R-1}\frac{A_r(z)}{1-q_r y}+\frac{B(z)}{1-(q_R+p_R z)y}, \tag{51}$$

where

$$A_r(z) := \frac{(q_R-q_r+p_R z)\displaystyle\sum_{j=r}^{R-2} z^{j-1}\left[\prod_{i=1}^{j} p_i\right]\left[\prod_{i=j+1}^{R-1}(p_i-p_r)\right]+z^{R-2}(p_r-p_R)\prod_{j=1}^{R-1} p_j}{p_r(q_R-q_r+p_R z)\displaystyle\prod_{j=1(j\neq r)}^{R-1}(p_j-p_r)};$$

$$r=1,2,\ldots,R-1 \tag{52}$$

and

$$B(z):=z^{R-1}\prod_{r=1}^{R-1}\frac{p_r}{q_R-q_r+p_R z}. \tag{53}$$

Substituting the inversion of (51) into (11), we obtain the *pgf* of $N(T)$ as

$$G_{N(T)}(z)=(1-z)\sum_{r=1}^{R-1}A_r(z)G_T(q_r)+B(z)G_T(q_R+p_R z), \tag{54}$$

where $G_T(y)$ is the *pgf* of the session time T.

It is possible to derive the *pmf* and moments of $N(T)$ from (54). For example, the mean is given by

$$\mathrm{E}[N(T)] = -\sum_{r=1}^{R-1} A_r(1) G_T(q_r) + R + p_R \left(\mathrm{E}[T] - \sum_{r=1}^{R} \frac{1}{p_r} \right), \tag{55}$$

where

$$A_r(1) = \frac{p_r \sum_{j=r}^{R-2} \left[\prod_{i=1}^{j} p_i \right] \left[\prod_{i=j+1}^{R-1} (p_i - p_r) \right] + (p_r - p_R) \prod_{j=1}^{R-1} p_j}{{p_r}^2 \prod_{j=1 (j \neq r)}^{R-1} (p_j - p_r)};$$

$$r = 1, 2, \ldots, R-1. \tag{56}$$

These results reduce to those in Section 4 when $R = 2$.

6 Conclusions

We have obtained several closed-form formulas for the *pmf* and the binomial moments of the discrete random variable $N(T)$ which counts the number of renewals in a random discrete time interval (called a session time). These results are valid for both the conventional delayed renewal process and our generalized delayed renewal process.

We have also derived the generating functions of $N(T)$ for the case of a generally distributed session time and generally distributed interrenewal times as well as for its special case of a negative binomially distributed session time. Based on these results we have obtained the *pmf* and the binomial moments when the session time is geometrically distributed while the interrenewal times are general and when the session time is general while the interrenewal times are geometrically distributed.

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