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Newsboy problem with pricing policy

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Abstract

In the conventional newsboy problem the buyer is implicitly assumed to be certain to buy an item for the price offered by a seller. In this paper, however, we assume that a buyer will decide whether to buy the item or not based on the price offered by the seller. This assumption leads us to a variation in the newsboy problem in which the optimal decision rules, consisting of the optimal pricing policy and the optimal ordering quantity, are considered. Our main concern here is to reveal the relationship between the optimal decision rules and the model's parameters which include the purchasing cost, salvage value, holding cost, discount factor and buyer arriving probability.

Keywords: Dynamic programming; Newsboy problem; Optimal pricing policy

1 Introduction

In this paper, we consider a product that must be sold within a specified period. Typical examples include foodstuffs such as fish, meat and so on which physically decay as time elapses, garments in fashion which will become outdated due to shifts in consumer preferences, or personal computers which become obsolete due to rapid techonological changes. With such products, sellers will normally lower their prices gradually as time passes rather than adopting a fixed price throughout the period. For instance, the selling price of packages of sashimi in the fish shop may sometimes be marked down by more than fifty percent of the initial price immediately prior to the shop's closing time because once the sashimi is defrosted, it losses its freshness gradually and becomes unfit for sale the next day.

Since the actual demand for such products is unpredictable, the seller faces the risk of having leftover items at the end of the selling period if the quantity ordered has exceeded the actual demand over the period. Items remaining unsold at the end of the period, the deadline, may be disposed of at a giveaway price. Similarly, some of the products which become valueless by the deadline will have to be discarded as industrial waste by paying some cost. This situation gives rise to a problem in determining the optimal ordering quantity at the start of the selling period and the optimal prices to be charged at each point in time over the entire selling period so as to maximize the total expected present discounted net profit gained over the entire period. If reorder of the product is assumed to be prohibited throughout the selling period, this problem can be regarded as a newsboy problem into which a pricing policy is integrated.

There is substantial literature on pricing strategies and inventory strategies for selling perishable products and they can be generally classified into three categories. The literature in the first category assumes that a fixed price is offered by the seller at the beginning of the selling period and all arriving buyers will purchase the product at this price. In this case, only the optimal ordering quantity need be determined. The conventional newsboy problem is included in this category [5]. The literature in the second category assumes that each arriving buyer has his own reservation price† and will decide whether

†By reservation price we mean the maximum permissible buying price of an arriving buyer. The buyer is willing to pay for a product if and only if the price of the product offered by the seller is lower than his maximum permissible buying price.

or not to purchase the product by comparing his reservation price to the price offered by the seller. The literature in this category focuses on determining the optimal pricing policy to be implemented by the seller at each point in time over the selling period. Papers belonging to the second category include [2], [7], and [8]. The optimal ordering quantity has been studied intensively in terms of the conventional newsboy problem, and also a great deal of research has been conducted on pricing policy. However, literature on the optimal ordering quantity into which the optimal pricing policy is integrated, as classified in the third category, is scant and incidental. Brief discussions on the optimal ordering quantity with the optimal pricing policy can be found in [1], [3], and [4].

The pricing problem presented in this paper belongs to the third category. Although we also clarify the properties of the optimal ordering quantity in accordance with the articles belonging to this category, the distinguishing feature of this paper is to investigate the relationship between the optimal decision rules, consisting of the optimal pricing policy and the optimal ordering quantity, and the model's parameters such as purchasing cost, salvage value, holding cost, discount factor and buyer arriving probability. In a sense, the treatment of pricing policy in this paper is basically the same as that in [7] with several differences in our model definitions. However, in order to investigate the properties related to the optimal ordering quantity, we need many properties, in relation to the optimal pricing policy, that are not dealt with in [7]. For this reason and in order to maintain the consistency of our analysis, we have developed new properties that are essential to our analysis as well as reexamine a few properties corresponding to the ones in [7] by borrowing some ideas underlying the proofs of the properties in [7].

A key contribution of this paper is to elucidate the conditions under which the optimal ordering quantity assumes a zero value, finite value greater than zero, or infinite value, called respectively, a *nonviable*, *viable*, or *unpractical* business deal in this paper. These conditions provide two managerial implications as stated below. Suppose that all the parameters are known. The conditions then offer a method that assists the seller in determining whether ordering a product is profitable. When ordering a product is known to be unprofitable according to the conditions, in other words, the business deal is nonviable, the seller, if he is an acute retailer in profit terms, will take steps to change the parameters so as to make the business deal viable. In this case, the conditions also serve as a guideline in helping the seller determine how much effort would be required to make a business deal viable. For example, a seller could contemplate volume buying by cooperating with other sellers to take advantage of quantity discount, thereby reducing the purchasing cost per unit and increasing any profit that may be obtained.

In the process of investigating the properties of optimal ordering quantity, we inevitably examined the monotonicity of the optimal selling price in the remaining time periods up to the deadline. Intuitively, it is conjectured that the shorter the remaining time periods are, the lower the optimal selling price will be. However, in [1] and [8] it is demonstrated that under certain conditions this conjecture may fail to hold. In [7] a numerical example showing that the optimal selling price is not time-monotone is also given. In this paper, we provide conditions under which the optimal selling price is nonincreasing or may be either monotone or unimodal in the remaining time periods. Figure 5.2 in Section 5 also demonstrates examples where the optimal price is nonincreasing, nondecreasing, and unimodal.

Section 2 provides a strict definition of the model examined in the paper. Section 3 defines several functions and examines their properties, and these are used in the subsequent analyses. In Section 4 we derive the optimal equation for the original model and in Section 5 we analyze and identify the properties of the optimal ordering quantity and the optimal selling price. In Section 6 we summarize the conclusions on the optimal decision rules in our models, consisting of the optimal pricing rule and optimal ordering quantity and present their practical implications. Section 7 presents an overall consideration of our research and suggestions for further work.

2 Model

The model discussed in this paper is defined on the assumptions stated below:

A1. Consider the following discrete-time sequential stochastic decision problem of purchasing a certain quantity of items at a certain point in time and then selling them at certain points in time that follows. The points in time are numbered backward from the final point in time of the planning horizon, time 0 (the deadline) as $0, 1, \cdots$ and so on. Accordingly, if time t is the present point in time, the two adjacent times $t+1$ and $t-1$ are the previous and next points in time, respectively. Further, let the time interval between times t and $t-1$ be called the *period* t.

A2. A buyer who requests an item arrives with a probability λ $(0 < \lambda < 1)$ and then the seller offers a selling price z to the buyer.

A3. By w let us denote the reservation price of a buyer, implying that the buyer is willing to buy an item if and only if the selling price z offered for the item by the seller is lower than or equal to w, i.e., $z \leq w$. Here, assume that subsequent buyers' reservation prices w, w', $\cdots$ are independent identically distributed random variables having a known continuous distribution function $F(w)$ with a finite expectation μ. Also, let $f(w)$ denote its probability density function, which is truncated on both sides (see Figure 2.1), and assume that

$$f(w) = 0, \ \ w < a, \qquad f(w) > 0, \ \ a \leq w \leq b, \qquad f(w) = 0, \ \ b < w, \tag{2.1}$$

for certain given numbers a and b such that $0 < a < b < \infty$; clearly $a < \mu < b$. In addition, let us define

$$\underline{f} \ = \ \inf\{f(w) \mid a \leq w \leq b\},$$

assumed to be positive, i.e. $\underline{f} > 0$. Thus, the probability of an arriving buyer buying the item, provided that a price z is offered by the seller, is given by

$$p(z) = \Pr\{z \leq w\},$$

where $0 \leq p(z) \leq 1$. From Eq. (2.1) it can be seen that

$$p(z) \begin{cases} = 1, & z \leq a \quad \cdots (1), \\ < 1, & a < z \quad \cdots (2), \end{cases} \qquad p(z) \begin{cases} > 0, & z < b \quad \cdots (3), \\ = 0, & b \leq z \quad \cdots (4). \end{cases}$$

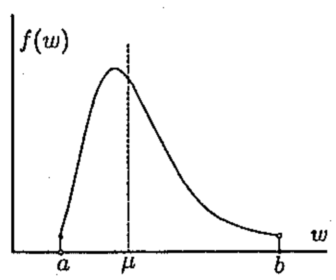


Figure 2.1: Probability density function $f(w)$.

A4. Let $c > 0$ be the purchasing price per item and set $c < b$ as a natural assumption.

A5. Let $h \geq 0$ be the inventory holding cost per item remaining unsold for a period.

A6. An item remaining unsold at time 0 can be sold at the salvage price ρ per item, assumed $\rho < M$ for a sufficiently large $M > b$. Here, $\rho < 0$ implies the disposal cost per item to discard an unsold item.

A7. By β $(0 < \beta \leq 1)$ let us denote the discount factor, implying that the monetary value of one unit a period hence is equivalent to that of β units at the present point in time.

The decision rules of the model consist of:

1. *Ordering rule* prescribing how many items to order at the time when the process starts.
2. *Pricing rule* prescribing what price to offer to an arriving buyer at each point in time.

The objective here is to find the optimal decision rule to maximize the total expected present discounted net profit over the planning horizon, i.e., the total expected present discounted revenue *minus* the total expected present discounted costs paid to purchase items at the start of the process *minus* the total expected present discounted holding cost *plus* the total expected present discounted salvage value.

3 Preliminaries

This section defines the functions that will be used to describe the optimal equations of the model in Section 4. The properties of the functions verified in this section will be applied to the analyses of the model developed in the sections that follow.

3.1 Definitions

For any x define

$$\nu(x) = (1-\beta)x + h, \tag{3.1}$$

$$T(x) = \max_z p(z)(z-x), \tag{3.2}$$

and by $z(x)$ let us designate the z attaining the maximum of the right hand side of Eq. (3.2) if it exists, i.e.,

$$T(x) = p(z(x))(z(x)-x). \tag{3.3}$$

For explanatory simplicity in the subsequent discussions, let

$$L(x) = \lambda\beta T(x). \tag{3.4}$$

Using the function $L(x)$, we shall define the following two functions:

$$K(x) = L(x) - (1-\beta)x \tag{3.5}$$

$$= L(x) - \nu(x) + h, \tag{3.6}$$

$$N(x) = L(x) + \beta x - c - h \tag{3.7}$$

$$= K(x) + x - c - h. \tag{3.8}$$

For convenience in the later discussions, we introduce the following three functions:

$$g(x,y) = L(x) - L(y) + \beta(x-y), \tag{3.9}$$

$$g(x,y,y') = L(x) - 2L(y) + L(y') + \beta(x-y), \tag{3.10}$$

$$g(x,x',y,y') = L(x) - L(x') + L(y') - L(y) + \beta(x-x'). \tag{3.11}$$

Here, by $x^\bullet$, x_0^*, and $x^\diamond$ let us denote the solutions of, respectively, $\nu(x) = 0$, $K(x) = 0$, and $N(x) = 0$ if they exist, i.e.,

$$\nu(x^\bullet) = 0, \qquad K(x_0^*) = 0, \qquad N(x^\diamond) = 0, \tag{3.12}$$

where clearly

$$x^\bullet = -h/(1-\beta), \quad \beta < 1, \tag{3.13}$$

and if $K(x) = 0$ and $N(x) = 0$ have multiple solutions, then let us define the *smallest* solution of $K(x) = 0$ by x_0^* and the *largest* solution of $N(x) = 0$ by $x^\diamond$, respectively. Further, by x_i^*, $i \geq 1$, let us denote the *smallest* solutions of the system of equations below if they exist.

$$K(x_1) = h, \tag{3.14}$$

$$K(x_i) = K(x_{i-1}) + (1-\beta)x_{i-1} + h, \quad i \geq 2. \tag{3.15}$$

Starting with x_1^*, we can successfully solve the x_2^*, x_3^*, $\cdots$, and x_i^*. Noting Eq. (3.14), we can express Eq. (3.15) as follows.

$$K(x_i) = (1-\beta)\sum_{j=1}^{i-1} x_j + ih, \quad i \geq 2. \tag{3.16}$$

Accordingly,

$$K(x_1^*) = h, \tag{3.17}$$

$$K(x_i^*) = K(x_{i-1}^*) + (1-\beta)x_{i-1}^* + h \tag{3.18}$$

$$= (1-\beta)\sum_{j=1}^{i-1} x_j^* + ih, \quad i \geq 2. \tag{3.19}$$

In general, by $x(\gamma)$ let us denote the smallest solution of $K(x) = \gamma$ for any given γ if it exists, i.e.,

$$K(x(\gamma)) = \gamma. \tag{3.20}$$

Furthermore, let

$$G_t(\rho) = \beta^t \rho - h\sum_{k=0}^{t-1} \beta^k - c, \quad t \geq 1, \tag{3.21}$$

and by ρ_t^* let us denote the solution of $G_t(\rho) = 0$, which is given by

$$\rho_t^* = (h\sum_{k=0}^{t-1} \beta^k + c)/\beta^t, \quad t \geq 1. \tag{3.22}$$

Finally, let us define $x^\star$ as follows (see [6]).

$$x^\star = \inf\{x \mid z(x) > a\}.$$

3.2 Properties

Lemma 3.1

(a) *$L(x) > 0$ on $(-\infty, b)$ and $L(x) = 0$ on $[b, \infty)$.*

(b) *$L(x)$ is continuous and nonincreasing on $(-\infty, \infty)$.*

(c) *$L(x) + \beta x$ is strictly increasing on $(-\infty, \infty)$.*

(d) *$\lim_{x\to-\infty} L(x) = \infty$ and $\lim_{x\to-\infty} L(x) + \beta x = -\infty$.*

(e) *If $x \leq (\geq)\, y$, then $0 \leq (\geq)\ L(x) - L(y) \leq (\geq)\, \lambda\beta(y - x)$.*

(f) $|L(x) - L(y)| \leq \lambda\beta|y - x|$ *for any* x *and* y.

Proof. See Appendix A ∎

Lemma 3.2

(a) $z(x) \geq a$ *for any* x.

(b) $z(x)$ *is nondecreasing in* $x \in (-\infty, \infty)$.

(c) *If* $x > (<)\ x^*$, *then* $z(x) > (=)\ a$[†].

Proof. See Appendix B ∎

Lemma 3.3

(a) $K(x)$ *is continuous and nonincreasing on* $(-\infty, \infty)$.

(b) $K(x)$ *is strictly decreasing on* $(-\infty, b]$.

(c) $K(x)$ *is strictly decreasing on* $(-\infty, \infty)$ *if* $\beta < 1$.

(d) $\lim_{x \to -\infty} K(x) = \infty$.

(e) $\lim_{x \to \infty} K(x) = -\infty$ *if* $\beta < 1$, *or else* $K(x) > 0$ *for* $x < b$ *and* $K(x) = 0$ *for* $x \geq b$.

(f) $K(x) + x$ *is strictly increasing in* $x \in (-\infty, \infty)$.

(g) *For any* x *and* y *we have*

$$|K(x) - K(y)| \leq |x - y|, \tag{3.23}$$

$$|K(x) + x - K(y) - y| \leq \beta|x - y|. \tag{3.24}$$

Proof. See Appendix C. ∎

Lemma 3.4

(a) *If* $\beta < 1$, *then* $x(\gamma)$ *uniquely exists for any given* γ.

(b) *Let* $\beta = 1$. *Then* $x(0) = b$, *and* $x(\gamma)$ *uniquely exists if* $\gamma > 0$.

(c) *The solutions of* Eqs. (3.14) *and* (3.15) *exist.*

(d) *Let* $\beta < 1$.

1. x_0^* *and* x_1^* *uniquely exist.*
2. $0 < x_0^* < b$.
3. $x < (= (>))\ x_0^* \Leftrightarrow K(x) > (= (<))\ 0$.
4. *If* $h = 0$, *then* $x_0^* = x_1^*$.

(e) *Let* $\beta = 1$.

1. $x_0^* = b$.
2. $x < (\geq)\ x_0^* \Leftrightarrow K(x) > (=)\ 0$.
3. *If* $h = 0$, *then* $x_1^* = b$, *and if* $h > 0$, *then* x_1^* *is given by the unique solution of* $K(x) = h$.

(f) *If* $h > 0$, *then* $x_1^* < x_0^* \leq b$, *or else* $x_1^* = x_0^* \leq b$.

(g) *Let* $(1 - \beta)^2 + h^2 \neq 0$. *Then* $x < (= (>))\ x_1^* \Leftrightarrow K(x) > (= (<))\ h$.

[†]Any one of $z(x^*) = a$ and $z(x^*) > a$ may occur due to the fact that $z(x)$ might be a discontinuous function of x as stated in Remark 6.1 [6]. Fortunately, this fact does not directly relate to the discussions in this paper.

(h) *If* $\beta < 1$, *then* x_i^* *uniquely exists for* $i \geq 0$.

(i) *Let* $(1-\beta)^2 + h^2 = 0$. *Then* $x_i^* = b$ *for* $i \geq 0$.

Proof. See Appendix D. ∎

Lemma 3.5

(a) $N(x)$ *is strictly increasing on* $(-\infty, \infty)$.

(b) x° *uniquely exists.*

(c) $x < (= (>))\ x^\circ \Leftrightarrow N(x) < (= (>))\ 0$.

(d) $x_1^* > (= (<))\ c \Leftrightarrow x^\circ < (= (>))\ x_1^*$ *and* $x^\circ < (= (>))\ c$.

(e) *Let* $(1-\beta)^2 + h^2 = 0$. *Then* $x_1^* > c > x^\circ$.

(f) *Let* $(1-\beta)^2 + h^2 \neq 0$. *Then* $x^\circ < \rho_t^*$ *for* $t \geq 1$.

Proof. See Appendix E. ∎

Lemma 3.6

(a) *If* $x \leq (\geq)\, y$, *then* $g(x, y) \leq (\geq)\, 0$.

(b) *If* $x \leq y \leq y'$ $(x \geq y \geq y')$, *then* $g(x, y, y') \leq (\geq)\, 0$.

(c) *If* $x \leq (\geq)\, x'$ *and* $y \leq (\geq)\, y'$, *then* $g(x, x', y, y') \leq (\geq)\, 0$ *where*

$$g(x, x', y, y') \leq (\geq)\, \beta(1-\lambda)(x - x'). \tag{3.25}$$

Proof. See Appendix F. ∎

4 Optimal Equation

4.1 Basic Equation

Suppose that a certain amount of a product had been purchased at a certain past point in time and that i items remain unsold at a time t after that. Let $u_t(i, \phi)$ and $u_t(i, 1)$ be the maximum total expected present discounted profits, respectively, with no buyer and with a buyer. Then, clearly

$$u_0(i, \phi) = \rho i, \qquad u_t(0, \phi) = u_t(0, 1) = 0, \quad t \geq 0, \quad i \geq 0, \tag{4.1}$$

and

$$u_t(i, \phi) = \beta\big(\lambda u_{t-1}(i, 1) + (1-\lambda) u_{t-1}(i, \phi)\big) - hi, \qquad t \geq 1, \quad i \geq 0. \tag{4.2}$$

$$u_t(i, 1) = \max_z \{p(z)\big(z + u_t(i-1, \phi)\big) + (1 - p(z)) u_t(i, \phi)\}, \qquad t \geq 0, \quad i \geq 1. \tag{4.3}$$

Optimal Selling Price I *The optimal selling price of time* $t \geq 0$ *with* $i \geq 1$ *items remaining unsold is given by the* z *attaining the maximum of the right hand side of* Eq. (4.3) *if it exists, denoted by* $z_t(i)$.

By $v_t(i)$ let us define the maximum of the total expected present discounted net profit, provided that i items are ordered at time t with no buyer. Here, assume that there is no lead time between ordering and receiving a product, i.e., the delivery is instantaneous and that the items are sold at the optimal selling price at each point in time that follows up to time 0. Then we have

$$v_t(i) = u_t(i, \phi) - ci \quad \text{or equivalently} \quad u_t(i, \phi) = v_t(i) + ci, \qquad t \geq 0, \quad i \geq 0, \tag{4.4}$$

where

$$v_t(0) = 0, \qquad v_0(i) = (\rho - c)i, \qquad t \geq 0, \quad i \geq 0. \tag{4.5}$$

Optimal Ordering Quantity *The optimal ordering quantity when the process starts from time t is given by the smallest i maximizing $v_t(i)$ on $i \geq 0$ if it exists, denoted by i_t^*, that is,*

$$v_t(i_t^*) = \max_{i \geq 0} v_t(i).$$

If i_t^* does not exist, then let $i_t^* = \infty$ for the explanatory convenience.

4.2 Transformation of $u_t(i, \phi)$ and $u_t(i, 1)$

We shall define

$$U_t(i) = u_t(i, \phi) - u_t(i-1, \phi), \qquad t \geq 0, \quad i \geq 1. \tag{4.6}$$

Noting Eq. (4.1), we have

$$U_0(i) = \rho, \qquad i \geq 1. \tag{4.7}$$

Using the L-function defined by Eq. (3.4), we can rewrite Eq. (4.3) multiplied by $\lambda\beta$ as follows.

$$\lambda\beta u_t(i, 1) = L(U_t(i)) + \lambda\beta u_t(i, \phi), \qquad t \geq 0, \quad i \geq 1. \tag{4.8}$$

Then Optimal Selling Price I can be restated as follows.

Optimal Selling Price II *The optimal selling price $z_t(i)$ with $t \geq 0$ and $i \geq 1$ is given by the z attaining the maximum of $p(z)(z - U_t(i))$ in $L(U_t(i))$ over $(-\infty, \infty)$[‡], which is given by $z_t(i) = z(U_t(i))$ from the definition of $z(x)$ (see* Eq. (3.3)*).*

Remark 4.1 For convenience in the later discussions, let

$$U_t(0) = M, \qquad t \geq 0, \tag{4.9}$$

for a sufficiently large $M > b$. Noting $L(U_t(0)) + \lambda\beta u_t(0, \phi) = L(M) = 0 = \lambda\beta u_t(0, 1)$ due to Eq. (4.1) and Lemma 3.1(a), we see that Eq. (4.8) also holds for $i \geq 0$ instead of $i \geq 1$. □

Since $\lambda\beta u_{t-1}(i, 1) - \lambda\beta u_{t-1}(i, \phi) = L(U_{t-1}(i))$ for $t \geq 1$ and $i \geq 0$ from Eq. (4.8) and Remark 4.1, we can rewrite Eq. (4.2) as follows.

$$u_t(i, \phi) = L(U_{t-1}(i)) + \beta u_{t-1}(i, \phi) - hi, \qquad t \geq 1, \quad i \geq 0. \tag{4.10}$$

Accordingly, using Eq. (3.5) we can express $U_t(i)$ as follows.

$$U_t(i) = L(U_{t-1}(i)) - L(U_{t-1}(i-1)) + \beta U_{t-1}(i) - h \tag{4.11}$$

$$= K(U_{t-1}(i)) + U_{t-1}(i) - K(U_{t-1}(i-1)) - (1-\beta)U_{t-1}(i-1) - h, \qquad t \geq 1, \quad i \geq 1. \tag{4.12}$$

Since $L(U_{t-1}(0)) = L(M) = 0$ and $K(U_{t-1}(0)) + (1-\beta)U_{t-1}(0) = L(U_{t-1}(0)) = 0$ for $t \geq 1$ due to Lemma 3.1(a), from Eqs. (4.11) and (4.12) we immediately obtain

$$U_t(1) = L(U_{t-1}(1)) + \beta U_{t-1}(1) - h \tag{4.13}$$

$$= K(U_{t-1}(1)) + U_{t-1}(1) - h, \qquad t \geq 1, \tag{4.14}$$

where

$$U_1(1) = L(U_0(1)) + \beta U_0(1) - h = L(\rho) + \beta\rho - h, \tag{4.15}$$

due to Eq. (4.7). In addition, noting Eq. (4.7), from Eq. (4.11) we have

[‡]Note that $z_t(i) \geq a$ from Lemma 3.2(a).

$$U_1(i) = L(\rho) - L(\rho) + \beta\rho - h = \beta\rho - h, \qquad i \geq 2. \tag{4.16}$$

For convenience in the discussions throughout the remainder of this paper, let the notations Δ and ∇ represent the differences with respect to t and i, respectively, and define

$$\begin{aligned}
&\Delta U_t(i) = U_t(i) - U_{t-1}(i), && t \geq 1, \quad i \geq 0,\\
&\nabla U_t(i) = U_t(i) - U_t(i-1), && t \geq 0, \quad i \geq 1,\\
&\Delta u_t(i,\phi) = u_t(i,\phi) - u_{t-1}(i,\phi), && t \geq 1, \quad i \geq 0,\\
&\Delta u_t(i,1) = u_t(i,1) - u_{t-1}(i,1), && t \geq 1, \quad i \geq 0,\\
&\nabla u_t(i,\phi) = u_t(i,\phi) - u_t(i-1,\phi), && t \geq 0, \quad i \geq 1.
\end{aligned}$$

Then, $U_t(i)$ can be rewritten

$$U_t(i) = \nabla u_t(i,\phi), \qquad t \geq 0, \quad i \geq 1. \tag{4.17}$$

Noting Eq. (3.1), from Eqs. (4.7), (4.15), and (4.16) we immediately obtain

$$\begin{aligned}
\Delta U_1(i) &= U_1(i) - \rho, \qquad i \geq 1\\
&= \begin{cases} L(\rho) - \nu(\rho), & \text{if } i = 1 \cdots (1),\\ -\nu(\rho), & \text{if } i \geq 2 \cdots (2). \end{cases}
\end{aligned} \tag{4.18}$$

Further, from Eqs. (4.11) we have

$$\begin{aligned}
\Delta U_t(i) = {} & L(U_{t-1}(i)) - L(U_{t-2}(i)) + L(U_{t-2}(i-1)) - L(U_{t-1}(i-1))\\
& + \beta(U_{t-1}(i) - U_{t-2}(i)), \qquad t \geq 2, \quad i \geq 1.
\end{aligned} \tag{4.19}$$

Then noting Eqs. (3.9) and (3.11), since $L(U_{t-1}(0)) = L(U_{t-2}(0)) = L(M) = 0$ due to Eq. (4.9) and Lemma 3.1(a), we can rewrite Eq. (4.19) as follows.

$$\Delta U_t(i) = \left\{ \begin{array}{ll} g(U_{t-1}(1), U_{t-2}(1)), & \text{if } i = 1 \cdots (1),\\ g(U_{t-1}(i), U_{t-2}(i), U_{t-1}(i-1), U_{t-2}(i-1)), & \text{if } i \geq 2 \cdots (2) \end{array} \right\}, \qquad t \geq 2. \tag{4.20}$$

Noting Eq. (3.10), from Eq. (4.11) we get

$$\begin{aligned}
\nabla U_t(i) &= L(U_{t-1}(i)) - 2L(U_{t-1}(i-1)) + L(U_{t-1}(i-2)) + \beta(U_{t-1}(i) - U_{t-1}(i-1))\\
&= g(U_{t-1}(i), U_{t-1}(i-1), U_{t-1}(i-2)), \qquad t \geq 1, \quad i \geq 2.
\end{aligned} \tag{4.21}$$

Finally, define $U(i) = \lim_{t\to\infty} U_t(i)$ for any given $i \geq 1$ if it exists.

4.3 Transformation of $v_t(i)$

Let us define

$$\nabla v_t(i) = v_t(i) - v_t(i-1), \qquad t \geq 0, \quad i \geq 1, \tag{4.22}$$

$$\nabla^2 v_t(i) = \nabla v_t(i) - \nabla v_t(i-1), \qquad t \geq 0, \quad i \geq 2. \tag{4.23}$$

Then from Eqs. (4.17) and (4.4) we get

$$U_t(i) = \nabla v_t(i) + c, \qquad t \geq 0, \quad i \geq 1, \tag{4.24}$$

from which we obtain

$$U_t(1) = v_t(1) + c \quad \text{or equivalently} \quad v_t(1) = U_t(1) - c, \qquad t \geq 0 \tag{4.25}$$

due to Eq. (4.5). Noting Eq. (3.1), from Eqs. (4.4), (4.10), and (3.5) we have

$$
\begin{aligned}
v_t(i) &= L(U_{t-1}(i)) + \beta u_{t-1}(i,\phi) - hi - ci \\
&= L(U_{t-1}(i)) + \beta v_{t-1}(i) - \nu(c)i && (4.26)\\
&= K(U_{t-1}(i)) + (1-\beta)U_{t-1}(i) + \beta v_{t-1}(i) - \nu(c)i, \qquad t \geq 1, \quad i \geq 0, && (4.27)
\end{aligned}
$$

from which we get

$$
\begin{aligned}
v_t(1) &= K(U_{t-1}(1)) + (1-\beta)U_{t-1}(1) + \beta v_{t-1}(1) - \nu(c) \\
&= K(U_{t-1}(1)) + U_{t-1}(1) - c - h \quad \text{(See Eq. (4.25))} \\
&= N(U_{t-1}(1)), \qquad t \geq 1 \qquad \text{(See Eq. (3.8))}. && (4.28)
\end{aligned}
$$

Finally, define $v(i) = \lim_{t\to\infty} v_t(i)$ for any given $i \geq 1$ if it exists.

5 Analysis

This section is devoted to examining the properties of the optimal decision rules, consisting of the optimal pricing rule and the optimal ordering rule.

5.1 Optimal Pricing Rule

Lemma 5.1

(a) *For a given $t' \geq 0$, if $U_{t'+1}(i) \leq (\geq)\, U_{t'}(i)$ for $i \geq 0$, then $U_t(i)$ is nonincreasing (nondecreasing) in $t \geq t'$ for $i \geq 0$.*

(b) *$U_t(i) \leq M$ for $t \geq 0$ and $i \geq 0$.*

(c) *$U_t(i)$ is nonincreasing in $i \geq 0$ for $t \geq 0$.*

(d) *$U_t(i)$ is bounded in i for $t \geq 0$.*

(e) *$U_t(i)$ is bounded in t for $i \geq 0$.*

Proof. (a) The assertion is clearly true for $t = t'$ by the assumption. For any given $t > t'$ assume $U_{t+1}(i) \leq (\geq)\, U_t(i)$ for $i \geq 0$ as the induction hypothesis, hence $U_{t+1}(i-1) \leq (\geq)\, U_t(i-1)$ for $i \geq 1$. Then, from Eq. (4.20) and Lemma 3.6(a,c) we have

$$
\begin{aligned}
\Delta U_{t+2}(1) &= g(U_{t+1}(1), U_t(1)) \leq (\geq)\, 0, \\
\Delta U_{t+2}(i) &= g(U_{t+1}(i), U_t(i), U_{t+1}(i-1), U_t(i-1)) \leq (\geq)\, 0, \qquad i \geq 2,
\end{aligned}
$$

thus $U_{t+2}(i) \leq (\geq)\, U_{t+1}(i)$ for $i \geq 1$. Therefore, by induction it follows that $U_t(i)$ is nonincreasing (nondecreasing) in $t \geq t'$ for $i \geq 1$. In addition, since $U_t(0) = M$ is nonincreasing and nondecreasing in $t \geq 0$, eventually the assertion holds.

(b) $U_t(0) = M$ for $t \geq 0$ from Eq. (4.9), hence the assertion holds for $i = 0$. From this result, Eq. (4.7), and the assumption of $\rho < M$ we have $U_0(i) \leq M$ for $i \geq 0$. Suppose $U_{t-1}(i) \leq M$ for $i \geq 0$, hence $U_{t-1}(i-1) \leq M$ for $i \geq 1$. Then from Lemma 3.1(c) and the fact that $L(M) = 0$ due to Lemma 3.1(a) we have $L(U_{t-1}(i)) + \beta U_{t-1}(i) \leq L(M) + \beta M = \beta M \leq M$ for $i \geq 0$. In addition, we also obtain $L(U_{t-1}(i-1)) \geq L(M) = 0$ for $i \geq 1$ from Lemma 3.1(b). Therefore, from Eq. (4.11) we have $U_t(i) \leq M - h \leq M$ for $t \geq 1$ and $i \geq 1$, thus for $i \geq 0$.

(c) Since $U_0(0) = M > \rho = U_0(i)$ for $i \geq 1$ by assumption, Eqs. (4.9), and (4.7), the assertion is clearly true for $t = 0$. Let $t \geq 1$. Suppose $U_{t-1}(i)$ are nonincreasing in $i \geq 0$ as the induction hypothesis. Then $U_{t-1}(i) \leq U_{t-1}(i-1) \leq U_{t-1}(i-2)$ for $i \geq 2$. Thus from Eq. (4.21) and Lemma 3.6(b), we have

$\nabla U_t(i) \leq 0$ for $i \geq 2$, i.e., $U_t(i) \leq U_t(i-1)$ for $i \geq 2$. Since $U_t(1) \leq M = U_t(0)$ for $t \geq 0$ from (b), it follows that $U_t(i) \leq U_t(i-1)$ for $i \geq 1$, implying that $U_t(i)$ is nonincreasing in $i \geq 0$. Accordingly, the assertion holds by induction.

(d) First, $U_t(i)$ is clearly upper bounded in i for any given $t \geq 0$ from (b). Since $U_t(0) = M$ for $t \geq 0$ due to Eq. (4.9) and since $U_0(i) = \rho$ for $i \geq 1$ from Eq. (4.7), it follows that $U_0(i)$ is lower bounded in i. Furthermore, since $U_{t-1}(i) \leq U_{t-1}(i-1)$ for $i \geq 1$ and $t \geq 1$ from (c), we have $L(U_{t-1}(i)) \geq L(U_{t-1}(i-1))$ due to Lemma 3.1(b), thus from Eq. (4.11) we get $U_t(i) \geq \beta U_{t-1}(i) - h$ for $t \geq 1$ and $i \geq 1$. Accordingly, since $U_1(i) \geq \beta U_0(i) - h = \beta\rho - h$ for $i \geq 1$, we immediately obtain $U_t(i) \geq \beta^t\rho - (1+\beta+\cdots+\beta^{t-1})h \cdots (1^*)$ for $t \geq 1$ and $i \geq 1$. Hence, $U_t(i)$ is lower bounded in i for any given $t \geq 0$.

(e) Since $U_t(i)$ is upper bounded in t for any given $i \geq 0$ from (b), it suffices to prove that $U_t(i)$ is lower bounded in t for any given $i \geq 0$. First, it is clear from Eq. (4.9) that the assertion is true for $i = 0$. Below let us prove that it holds for $i \geq 1$. Note that $U_0(i) = \rho$ for $i \geq 1$ from Eq. (4.7).

1. Let $\beta < 1$. Then $U_t(i) \geq \beta^t\rho - h/(1-\beta)$ for $t \geq 1$ and $i \geq 1$ from (1^*). If $\rho \geq 0$, then $U_t(i) \geq -h/(1-\beta)$, and if $\rho < 0$, then $U_t(i) \geq \rho - h/(1-\beta)$. Hence, $U_t(i)$ is lower bounded in t for any $i \geq 1$ whether $\rho \geq 0$ or $\rho < 0$.
2. Let $\beta = 1$ and $h = 0$. Then $U_t(i) \geq \rho$ from (1^*), hence $U_t(i)$ is lower bounded in t for any $i \geq 1$.
3. Let $\beta = 1$ and $h > 0$.
   i. First, let us prove the lower boundedness of $\Delta u_t(i, \phi)$ in $t \geq 1$ for $i \geq 0$. From Eq. (4.3) we obtain

$$\begin{aligned}|\Delta u_t(i,1)| &= |u_t(i,1) - u_{t-1}(i,1)| \\ &\leq \max_z\{p(z)|\Delta u_t(i-1,\phi)| + (1-p(z))|\Delta u_t(i,\phi)|\}, \qquad t \geq 1, \quad i \geq 1.\end{aligned}$$

   Here, define $A_t(i) = \max_{i \geq j \geq 0} |\Delta u_t(j,\phi)| \geq 0$ for $t \geq 1$ and $i \geq 0$, which is nondecreasing in i. Then since $A_t(i) \geq |\Delta u_t(i,\phi)| \cdots (2^*)$ for $t \geq 1$ and $i \geq 0$ and since $A_t(i) \geq |\Delta u_t(i-1,\phi)|$ for $t \geq 1$ and $i \geq 1$, we obtain

$$|\Delta u_t(i,1)| \leq \max_z\{p(z)A_t(i) + (1-p(z))A_t(i)\} = A_t(i), \qquad t \geq 1, \quad i \geq 1. \tag{5.1}$$

   Further, from Eq. (4.2) we have

$$|\Delta u_t(j,\phi)| \leq \lambda|\Delta u_{t-1}(j,1)| + (1-\lambda)|\Delta u_{t-1}(j,\phi)|, \qquad t \geq 2, \quad j \geq 0.$$

   Now, for $t \geq 2$ and $j \geq 1$ we have $|\Delta u_{t-1}(j,1)| \leq A_{t-1}(j)$ from Eq. (5.1) and $|\Delta u_{t-1}(j,\phi)| \leq A_{t-1}(j)$ from (2^*). Hence we obtain

$$|\Delta u_t(j,\phi)| \leq \lambda A_{t-1}(j) + (1-\lambda)A_{t-1}(j) = A_{t-1}(j), \qquad t \geq 2, \quad j \geq 1. \tag{5.2}$$

   In addition, clearly $|\Delta u_t(0,\phi)| = 0 \leq A_{t-1}(0) \cdots (3^*)$ for $t \geq 2$. Accordingly, it follows that $|\Delta u_t(j,\phi)| \leq A_{t-1}(j)$ for $t \geq 2$ and $j \geq 0$. Further, due to the monotonicity of $A_{t-1}(j)$ in $j \geq 0$ for $t \geq 2$, we get $|\Delta u_t(j,\phi)| \leq A_{t-1}(j) \leq A_{t-1}(i)$ for $i \geq j \geq 0$ and $t \geq 2$, thus $A_t(i) \leq A_{t-1}(i)$ for any $t \geq 2$ and $i \geq 0$, i.e., $A_t(i)$ is nonincreasing in $t \geq 1$ for any $i \geq 0$. Therefore, from Eq. (5.2) we have $|\Delta u_t(j,\phi)| \leq A_1(j)$ for $t \geq 2$ and $j \geq 1$. Accordingly, from (3^*) we have $|\Delta u_t(j,\phi)| \leq A_1(j)$ for $t \geq 2$ and $j \geq 0$. Without loss of generality, since the notation j can be replaced with i, we get $|\Delta u_t(i,\phi)| \leq A_1(i)$ for $t \geq 2$ and $i \geq 0$, hence $t \geq 1$ for $i \geq 0$ due to (2^*) with $t = 1$, implying that $\Delta u_t(i,\phi)$ is bounded in $t \geq 1$ for $i \geq 0$.

   ii. Next, using the above result, let us prove the boundedness of $U_t(i)$ in t. Since $\beta = 1$ by assumption, Eq. (4.12) can be rewritten

$$U_t(i) = K(U_{t-1}(i)) + U_{t-1}(i) - K(U_{t-1}(i-1)) - h, \qquad t \geq 1, \quad i \geq 1, \tag{5.3}$$

which can be rearranged as follows.

$$K(U_{t-1}(i)) = K(U_{t-1}(i-1)) + U_t(i) - U_{t-1}(i) + h, \qquad t \geq 1, \quad i \geq 1.$$

Therefore, from Eq. (4.6) we have

$$\begin{aligned} K(U_{t-1}(i)) &= K(U_{t-1}(i-1)) + u_t(i,\phi) - u_t(i-1,\phi) - u_{t-1}(i,\phi) + u_{t-1}(i-1,\phi) + h \\ &= K(U_{t-1}(i-1)) + \Delta u_t(i,\phi) - \Delta u_t(i-1,\phi) + h, \qquad t \geq 1, \quad i \geq 1. \qquad (5.4) \end{aligned}$$

Here, note that $U_t(i)$ is upper bounded in t for $i \geq 0$ from (b) and that $U_t(0)$ is lower bounded in t from Eq. (4.9). Suppose $U_t(i-1)$ is lower bounded in $t \geq 0$ for any given $i \geq 1$, hence upper and lower bounded in t for any given $i \geq 1$. Then $K(U_t(i-1))$ is also bounded in t for the i due to the fact that $K(x)$ is continuous on $(-\infty, \infty)$ from Lemma 3.3(a). Since $\Delta u_t(i,\phi)$ and $\Delta u_t(i-1,\phi)$ are already proven to be bounded in t for the i, Eq. (5.4), i.e., $K(U_{t-1}(i))$ is also bounded in t. In addition, since $K(x) = L(x)$ for $\beta = 1$ from Eq. (3.5), it follows that $L(U_{t-1}(i))$ is bounded in t, implying that there exists a number B such that $L(U_{t-1}(i)) \leq B$ for all t. Assume that $U_{t-1}(i)$ is not lower bounded in t. Then there exists at least one t, denoted by $t(B')$, such that $U_{t-1}(i) < B'$ for any given $B' < 0$; in other words, $U_{t(B')-1}(i) < B'$. Therefore, $L(B') \leq L(U_{t(B')-1}(i)) \cdots (4^*)$ due to Lemma 3.1(b). Since $\lim_{x \to -\infty} L(x) = \infty$ due to Lemma 3.1(d), we can consider a B' satisfying $B < L(B')$. Further, we have $L(U_{t(B')-1}(i)) \leq B$ since $L(U_{t-1}(i)) \leq B$ for all t. Accordingly, we obtain $L(U_{t(B')-1}(i)) \leq B < L(B')$, which contradicts (4^*). Thus $U_{t-1}(i)$ must be lower bounded in t for the $i \geq 1$. Since all the terms on the right-hand of Eq. (5.3) are lower bounded in t for the $i \geq 1$, it eventually follows that $U_t(i)$ is also bounded in t. This complete the induction ∎

Lemma 5.2

(a) *Let $\rho > x_0^*$. Then $U_t(i)$ is nonincreasing in $t \geq 0$ for $i \geq 0$.*

(b) *Let $\rho \leq x_0^*$. If $\rho \leq (\geq)\ x_1^*$, then $U_t(1)$ is nondecreasing (nonincreasing) in $t \geq 0$.*

(c) *Let $(1-\beta)^2 + h^2 \neq 0$ and $\rho > x_1^*$. Then $U_t(1)$ is strictly decreasing in $t \geq 0$.*

(d) *Let $(1-\beta)^2 + h^2 = 0$. Then $U_t(i) = \rho$ for any given $t \geq 0$ and $i > t$.*

Proof. (a) Let $\rho > x_0^*$. Then $\rho > 0$ due to $x_0^* > 0$ from Lemma 3.4(d2,e1).

1. The assertion clearly holds for $i = 0$ from Eq. (4.9).
2. From Eq. (4.7) and Lemma 3.4(d3,e2) we have $K(U_0(1)) = K(\rho) \leq 0$. Thus, from Eq. (4.14) we have $U_1(1) = K(U_0(1)) + U_0(1) - h \leq U_0(1)$. Assume $U_{t-1}(1) \leq U_{t-2}(1)$. Then from Eq. (4.14) and Lemma 3.3(f) we have $U_t(1) \leq K(U_{t-2}(1)) + U_{t-2}(1) - h = U_{t-1}(1)$. Hence, by induction the assertion holds for $i = 1$.
3. From Eqs. (4.18) and (3.1) we have $\Delta U_1(2) = -\nu(\rho) = -(1-\beta)\rho - h \leq 0$, i.e., $U_1(2) \leq U_0(2)$. Assume $U_{t-1}(2) \leq U_{t-2}(2)$. Since $U_{t-1}(1) \leq U_{t-2}(1)$ has already been proven above, it follows from Eq. (4.20 (2)) and Lemma 3.6(c) that $\Delta U_t(2) \leq 0$, i.e., $U_t(2) \leq U_{t-1}(2)$. Hence, by induction we have $U_t(2) \leq U_{t-1}(2)$ for all $t \geq 1$.
4. Assume $U_t(i-1)$ is nonincreasing in $t \geq 0$ for any given $i \geq 3$.
   i. From Eqs. (4.18) and (3.1) we have $\Delta U_1(i) = -\nu(\rho) = -(1-\beta)\rho - h \leq 0$ for $i \geq 3$, i.e., $U_1(i) \leq U_0(i)$ for $i \geq 3$.
   ii. Assume $U_{t-1}(i) \leq U_{t-2}(i)$ for $i \geq 3$. Since $U_{t-1}(2) \leq U_{t-2}(2)$ has already been proven above, the assumption can be restated as $U_{t-1}(i) \leq U_{t-2}(i)$ for $i \geq 2$, hence $U_{t-1}(i-1) \leq U_{t-2}(i-1)$

for $i \geq 3$. Therefore, from Eq. (4.20 (2)) and Lemma 3.6(c) we have $\Delta U_t(i) \leq 0$ for $i \geq 3$, i.e., $U_t(i) \leq U_{t-1}(i)$ for $i \geq 3$. Hence, by induction it follows that $U_t(i) \leq U_{t-1}(i)$ for all $t \geq 1$ and $i \geq 3$.

5. From all the above, it follows that $U_t(i)$ is nonincreasing in $t \geq 0$ for $i \geq 0$.

(b) Let $\rho \leq x_0^*$. If $\rho \leq (\geq)\ x_1^*$, then $K(\rho) \geq (\leq)\ K(x_1^*)$ due to Lemma 3.3(a). Hence from Eqs. (4.14), (4.7), and (3.17) we have $U_1(1) - U_0(1) = K(\rho) - h = K(\rho) - K(x_1^*) \geq (\leq)\ 0$, i.e., $U_1(1) \geq (\leq)\ U_0(1)$. The monotonicity can be easily proven by induction starting with the result by use of Lemma 3.3(f) and Eq. (4.14).

(c) Let $(1-\beta)^2 + h^2 \neq 0$ and $\rho > x_1^*$. Then from Lemma 3.4(g) we get $K(\rho) < h$. Hence from Eqs. (4.14) and (4.7) we have $U_1(1) - U_0(1) = K(\rho) - h < 0$, i.e., $U_1(1) < U_0(1)$. The monotonicity can be easily proven by induction starting with the result by use of Lemma 3.3(f) and Eq. (4.14).

(d) Let $(1-\beta)^2 + h^2 = 0$. Then, since $U_0(i) = \rho$ for $i > 0$ from Eq. (4.7), the assertion holds for $t = 0$. Suppose $U_{t-1}(i) = \rho$ for $i > t-1$, thus $U_{t-1}(i-1) = \rho$ for $i > t$. Now, from Eq. (4.11) we get $U_t(i) = L(U_{t-1}(i)) - L(U_{t-1}(i-1)) + U_{t-1}(i)$ for $t \geq 1$ and $i \geq 1$. Hence $U_t(i) = L(\rho) - L(\rho) + \rho = \rho$ for $i > t$. ▮

Lemma 5.3

(a) *Let $\nu(\rho) \leq 0$. Then $U_0(i) \leq U_1(i)$ for $i \geq 1$.*

(b) *Let $\nu(\rho) > 0$.*

1 *If $\rho < (= (>))\ x_1^*$, then $U_0(1) < (= (>))\ U_1(1)$.*

2 *$U_0(i) > U_1(i)$ for $i \geq 2$.*

Proof. (a) Let $\nu(\rho) \leq 0$. Since $L(\rho) \geq 0$ due to Lemma 3.1(a), we get $\Delta U_1(1) \geq 0$ from Eq. (4.18 (1)), hence $U_1(1) \geq U_0(1)$. If $\nu(\rho) \leq 0$, then $\Delta U_1(i) \geq 0$ for $i \geq 2$ from Eq. (4.18 (2)), hence $U_1(i) \geq U_0(i)$ for $i \geq 2$.

(b) Let $\nu(\rho) > 0$, implying that $(1-\beta)^2 + h^2 \neq 0$ because $(1-\beta)^2 + h^2 = 0$ leads to the contradiction of $\nu(\rho) = 0$.

(b1) Note that $\Delta U_1(1) = L(\rho) - \nu(\rho) = K(\rho) - h$ from Eq. (4.18 (1)) and (3.6). If $\rho < (= (>))\ x_1^*$, then $K(\rho) > (= (<))\ h$ due to Lemma 3.4(g), thus we get $\Delta U_1(1) > (= (<))\ 0$, hence $U_1(1) > (= (<))\ U_0(1)$.

(b2) From Eq. (4.18 (2)) we have $\Delta U_1(i) < 0$, hence $U_1(i) < U_0(i)$ for $i \geq 2$. ▮

For a given $i \geq 2$ let $t(i)$ be the largest t such that $U_t(i)$ is strictly decreasing in t if it exists; clearly $t(i) \geq 1$. The $t(i)$ is used only in Lemma 5.4(b2ii) stated below in which $U_t(i)$ is proven not to oscillate. For convenience let $t(i) = \infty$ if $t(i)$ does not exist. Here, for any given $t \geq 2$, by S_t let us define the statement below.

$$S_t = \left\{ \begin{array}{lll} U_0(i) > U_1(i), & i \geq 2 & \cdots \text{Eq}'.(2), \\ U_1(i) > U_2(i), & i \geq 3 & \cdots \text{Eq}'.(3), \\ \vdots & & \\ U_{\tau-2}(i) > U_{\tau-1}(i), & i \geq \tau & \cdots \text{Eq}'.(\tau), \\ \vdots & & \\ U_{t-3}(i) > U_{t-2}(i), & i \geq t-1 & \cdots \text{Eq}'.(t-1), \\ U_{t-2}(i) > U_{t-1}(i), & i \geq t & \cdots \text{Eq}'.(t) \end{array} \right\}, \qquad t \geq 2.$$

Then S_{t+1} can be expressed as

$$S_{t+1} = \{S_t, U_{t-1}(i) > U_t(i),\ i \geq t+1\}, \qquad t \geq 2. \tag{5.5}$$

Lemma 5.4

(a) *Let $\nu(\rho) \leq 0$. Then $U_t(i)$ is nondecreasing in $t \geq 0$ for $i \geq 0$.*

(b) *Let $\nu(\rho) > 0$.*

1. *Let $\rho \geq x_1^*$. Then $U_t(i)$ is nonincreasing in $t \geq 0$ for $i \geq 0$.*
2. *Let $\rho < x_1^*$. Then:*
   i. *$U_t(1)$ is nondecreasing in $t \geq 0$.*
   ii. *Let $i \geq 2$.*
      1. *Let $t \geq 2$. If $i \geq t$, then $U_\tau(i)$ is strictly decreasing in $\tau < t$ with $\tau \geq 0$.*
      2. *Let $t(i) = \infty$ for a given $i \geq 2$. Then $t(i') = \infty$ for any $i' \geq i \geq 2$, hence $U_t(i')$ is strictly decreasing in $t \geq 0$ for any $i' \geq i$.*
      3. *If $t(i') < \infty$ for a given $i' \geq 2$, then $t(i) < \infty$ for $2 \leq i \leq i'$.*
      4. *Let $t(i) < \infty$ for a given $i \geq 2$.*
         i′. *$U_{t(i)+1}(i+1) < U_{t(i)}(i+1)$.*
         ii′. *$U_t(i)$ is nondecreasing in $t \geq t(i)$.*

Proof. Note that $U_t(0) = M$ for $t \geq 0$ from Eq. (4.9), which is nondecreasing and nonincreasing in $t \geq 0$.

(a) Immediate from Lemmas 5.3(a) and 5.1(a) with $t' = 0$.

(b) Let $\nu(\rho) > 0$.

(b1) Immediate from Lemmas 5.3(b) and 5.1(a).

(b2) Let $\rho < x_1^*$. Below, note that $x_1^* \leq x_0^*$ due to Lemmas 3.4(f).

(b2i) Since $\rho < x_1^* \leq x_0^*$, it follows from Lemma 5.2(b) that $U_t(1)$ is nondecreasing in $t \geq 0$.

(b2ii) Let $i \geq 2$.

(b2ii1) First, $S_2 = \{U_0(i) > U_1(i), i \geq 2\}$, which is clearly true due to Lemma 5.3(b2). Assume that S_t holds for any given $t \geq 2$. Then $U_{t-2}(i) > U_{t-1}(i)$ for all $i \geq t \geq 2$, hence $U_{t-2}(i+1) > U_{t-1}(i+1)$ for all $i \geq t \geq 2$. Now, from Eq. (4.20 (2)) we have, for any $t \geq 2$, $\Delta U_t(i+1) = g(U_{t-1}(i+1), U_{t-2}(i+1), U_{t-1}(i), U_{t-2}(i))$ for $i \geq 1$, hence $i \geq 2$. Therefore, from Eq. (3.25) we have $\Delta U_t(i+1) \leq \beta(1-\lambda)(U_{t-1}(i+1) - U_{t-2}(i+1)) < 0$ for $i \geq t \geq 2$, i.e., $U_{t-1}(i+1) > U_t(i+1)$ for $i \geq t \geq 2$ or equivalently $U_{t-1}(i) > U_t(i)$ for $i \geq t+1$ and $t \geq 2$. Thus, from Eq. (5.5) it follows that S_{t+1} holds for $t \geq 2$. Accordingly, S_t holds for all $t \geq 2$ by induction, implying that $U_0(i) > U_1(i) > \cdots > U_{t-3}(i) > U_{t-2}(i) > U_{t-1}(i)$ for $i \geq t$. Hence the assertion holds.

(b2ii2) Let $t(i) = \infty$ for a given $i \geq 2$.

1. The assertion is clearly true for $i' = i$ by the assumption.
2. Assume $t(i') = \infty$ for any given $i' > i \geq 2$. Then from the definition of $t(i')$ it follows that $U_t(i')$ is strictly decreasing in $t \geq 0$. Hence $U_{t-1}(i') < U_{t-2}(i')$ for $t \geq 2$.
   i. Note that $U_1(i'+1) < U_0(i'+1)$ due to Lemma 5.3(b2).
   ii. Suppose $U_{t-1}(i'+1) < U_{t-2}(i'+1)$ for $t \geq 2$. Then, from Eq. (4.20 (2)) and Eq. (3.25) we have

$$\begin{aligned}\Delta U_t(i'+1) &= g(U_{t-1}(i'+1), U_{t-2}(i'+1), U_{t-1}(i'), U_{t-2}(i')) \\ &\leq \beta(1-\lambda)(U_{t-1}(i'+1) - U_{t-2}(i'+1)) < 0,\end{aligned}$$

   i.e., $U_t(i'+1) < U_{t-1}(i'+1)$ for $t \geq 2$. Thus by induction we have $U_t(i'+1) < U_{t-1}(i'+1)$ for all $t \geq 1$, implying that $U_t(i'+1)$ is strictly decreasing in $t \geq 0$. Hence $t(i'+1) = \infty$.

This completes the induction.

(b2ii3) If $t(i) = \infty$ for a certain $i < i'$, we have the contradiction of $t(i') = \infty$ from (b2ii2), hence the assertion must hold.

(b2ii4) Let $t(i) < \infty$ for a given $i \geq 2$.

(b2ii4i′) Since $U_0(i) > U_1(i)$ for $i \geq 2$ from Lemma 5.3(b2), we have $U_0(i+1) > U_1(i+1)$ for $i \geq 1$. Then from Eqs. (4.20 (2)) and (3.25) we have

$$\begin{aligned} \Delta U_2(i+1) &= g(U_1(i+1), U_0(i+1), U_1(i), U_0(i)) \\ &\leq \beta(1-\lambda)(U_1(i+1) - U_0(i+1)) < 0, \qquad i \geq 2, \end{aligned}$$

hence $U_1(i+1) > U_2(i+1)$ for $i \geq 2$. Further, by the definition of $t(i)$ we have $U_0(i) > U_1(i) > U_2(i) > \cdots > U_{t(i)-1}(i) > U_{t(i)}(i)$ for the $i \geq 2$. Then from $U_1(i) > U_2(i)$ and $U_1(i+1) > U_2(i+1)$ we can get $U_2(i+1) > U_3(i+1)$ in the same way as the above. Repeating the same procedure up to $t(i)+1$, we obtain $U_{t(i)}(i+1) > U_{t(i)+1}(i+1)$ for $i \geq 2$.

(b2ii4ii′) From Lemma 5.3(b2) and the definition of $t(i)$, we have $t(i) \geq 1$ for $i \geq 2$.

S1. Let $i = 2$. Then note that $U_t(1) \geq U_{t-1}(1)$ for $t \geq 1$ from (b2i). Clearly, $U_{t(2)+1}(2) \geq U_{t(2)}(2)$ by the definition of $t(2)$. Assume $U_t(2) \geq U_{t-1}(2)$ for any given $t > t(2)$. Then $\Delta U_{t+1}(2) = g(U_t(2), U_{t-1}(2), U_t(1), U_{t-1}(1)) \geq 0$ from Eq. (4.20 (2)) and Lemma 3.6(c), hence $U_{t+1}(2) \geq U_t(2)$. Thus, by induction $U_t(2)$ is nondecreasing in $t \geq t(2)$, implying that the assertion holds for $i = 2$.

S2. Let $i \geq 3$. Then $t(i-1) < \infty$, $t(i-2) < \infty$, $\cdots$, $t(2) < \infty$ for the $i \geq 3$ from (b2ii3). Further, the inequality of $t(i-1) < \infty$ leads to $U_{t(i-1)+1}(i) < U_{t(i-1)}(i)$ due to (b2ii4i′), implying that $t(i) \geq t(i-1)+1 > t(i-1) \cdots (1^*)$.

   i. Since $t(2) < \infty$, we have $U_{t(2)+1}(2) \geq U_{t(2)}(2)$ by the definition of $t(2)$. Thus, in the same way as in S1 we can show that $U_t(2)$ is nondecreasing in $t \geq t(2)$. Similarly we can show that $U_t(3)$ is nondecreasing in $t \geq t(3)$ due to $t(3) < \infty$, $t(3) > t(2)$, and the fact that $U_t(2)$ is nondecreasing in $t \geq t(2)$. Repeating the same discussion, we have that $U_t(i-1)$ is nondecreasing in $t \geq t(i-1)$; in other words, $U_t(i-1) \geq U_{t-1}(i-1)$ for $t > t(i-1)$, hence $t > t(i)$ due to (1^*).

   ii. Clearly $U_{t(i)+1}(i) \geq U_{t(i)}(i)$ by the definition of $t(i)$.

   iii. Assume $U_t(i) \geq U_{t-1}(i)$ for any given $t > t(i)$. Then from Eq. (4.20 (2)) and Lemma 3.6(c) we have $\Delta U_{t+1}(i) = g(U_t(i), U_{t-1}(i), U_t(i-1), U_{t-1}(i-1)) \geq 0$, hence, $U_{t+1}(i) \geq U_t(i)$. Thus, by induction it follows that $U_t(i)$ is nondecreasing in $t \geq t(i)$.

This completes the proof. ∎

Corollary 5.1 *Let $\nu(\rho) > 0$ and $\rho < x_1^*$. For a given $i \geq 2$ we have:*

(a) *If $t(i) = \infty$, then $U_t(i)$ is strictly decreasing in $t \geq 0$.*

(b) *If $t(i) < \infty$, then $U_t(i)$ is strictly decreasing in $t \leq t(i)$ and nondecreasing in $t \geq t(i)$.*

Proof. (a,b) Evident from the definition of $t(i)$ and Lemma 5.4(b2ii1,b2ii4ii′). ∎

Lemma 5.5 *$U_t(i)$, $L(U_t(i))$, $K(U_t(i))$, and $N(U_t(i))$ converge to, respectively, $U(i)$, $L(U(i))$, $K(U(i))$, and $N(U(i))$ as $t \to \infty$ for $i \geq 1$.*

Proof. Since $U_t(i)$ is bounded in t for $i \geq 0$ due to Lemma 5.1(e) and monotone in $t \geq \tau'$ for a sufficiently large $\tau' \geq 0$ due to Lemma 5.4(a,b1,b2i) and Corollary 5.1, it follows that $U_t(i)$ converges to a finite $U(i)$ as $t \to \infty$. Further, note that $|L(U_t(i)) - L(U(i))| \leq \lambda\beta|U_t(i) - U(i)| \cdots (1^*)$ from

Lemma 3.1(f), $|K(U_t(i)) - K(U(i))| \leq |U_t(i) - U(i)| \cdots (2^*)$ from Eq. (3.23), and $|N(U_t(i)) - N(U(i))| = |K(U_t(i)) + U_t(i) - K(U(i)) - U(i)| \leq \beta |U_t(i) - U(i)| \cdots (3^*)$ from Eqs. (3.8) and (3.24). Therefore, eventually all of the right hand sides of Eqs.(1*), (2*), and (3*) converges to 0, implying that the assertion holds. ▮

Theorem 5.1

(a) *$z_t(i)$ is nonincreasing in $i \geq 0$ for $t \geq 0$.*

(b) *If $U_t(i) > (<)\ x^\star$, then $z_t(i) > (=)\ a$.*

(c) *Let $\nu(\rho) \leq 0$. Then $z_t(i)$ is nondecreasing in $t \geq 0$ for $i \geq 0$.*

(d) *Let $\nu(\rho) > 0$.*

1. *Let $\rho \geq x_1^*$. Then $z_t(i)$ is nonincreasing in $t \geq 0$ for $i \geq 0$.*
2. *Let $\rho < x_1^*$.*
   i. *$z_t(1)$ is nondecreasing in $t \geq 0$.*
   ii. *$z_t(i)$ does not oscillate in $t \geq 0$ for $i \geq 2$.*

Proof. Note Lemma 3.2(b) and the definition of $z_t(i)$ (Optimal Selling Price II).

(a) Immediate from Lemma 5.1(c).

(b) Evident from Lemma 3.2(c).

(c) Obvious from Lemma 5.4(a).

(d1) Clear from Lemma 5.4(b1).

(d2i) Evident from Lemma 5.4(b2i).

(d2ii) Immediate from Corollary 5.1 (see the numerical examples below). ▮

Numerical Examples Here, using two numerical examples, we shall demonstrate the monotonicity of $U_t(i)$ and $z_t(i)$ in $t \geq 0$. Let $\beta = 0.99$, $\lambda = 0.5$, $c = 0.4$, $h = 0.01$, and let $F(w)$ be the uniform distribution on $[1.5, 2.5]$, i.e., $a = 1.5$ and $b = 2.5$. Then from [6] we have

$$p(z) = \begin{cases} 1, & z \leq 1.5, \\ 2.5 - z, & 1.5 \leq z \leq 2.5, \\ 0, & 2.5 \leq z, \end{cases}$$

from which we obtain

$$T(x) = \begin{cases} 1.5 - x, & x \leq 0.5 & \Rightarrow & z(x) = 1.5, \\ 0.25(2.5 - x)^2, & 0.5 \leq x \leq 2.5 & \Rightarrow & z(x) = (x + 2.5)/2, \\ 0, & 2.5 \leq x & \Rightarrow & z(x) = 2.5. \end{cases}$$

Since $x^\star = 2a - b$ for uniform distribution [6], we obtain $x^\star = 2 \times 1.5 - 2.5 = 0.5$. Furthermore, x_1^*, the solution of $K(x) = h$, can be easily obtained by numerical calculation using Eq. (3.14), which is approximately equal to 2.0071, i.e., $x_1^* \simeq 2.0071$. Below, let us present three graphs of $U_t(i)$ and $z_t(i)$ with $(\rho, i) = (0.8, 10)$, $(0.8, 20)$, and $(0.2, 10)$, respectively.

1. Let $\rho = 0.8$. Then, we have $x_1^* \simeq 2.0071 > 0.80 = \rho$, and $\nu(\rho) = (1 - 0.99) \times 0.80 + 0.01 = 0.018 > 0$, i.e., the two conditions in Lemma 5.4(b,b2) are satisfied. Figure 5.2 (Graph I) depicts the case where both $U_t(10)$ and $z_t(10)$ are strictly decreasing in $t \leq t(10)$ and strictly increasing in $t \geq t(10)$.

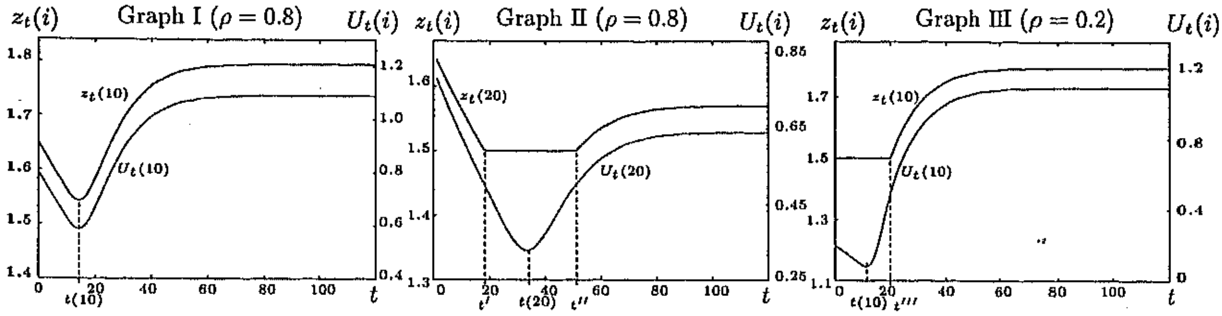


Figure 5.2: Relationship between $U_t(i)$ and $z_t(i)$

Figure 5.2 (Graph II) shows that $U_t(20)$ is strictly decreasing in $t \le t(20)$ and strictly increasing in $t \ge t(20)$, while $z_t(20)$ is strictly decreasing in $t \le t'$, constant in $t \in (t', t'')$ and strictly increasing in $t \ge t''$ with $t' = 19$ and $t'' = 50$. From Graph II we observe that $z_t(20)$ is truncated at $a = 1.5$ if $U_t(20) < x^\star = 0.5$. This result confirms the statement proven in Theorem 5.1(b).

2. Let $\rho = 0.2$. Then, we have $x_1^* \simeq 2.0071 > 0.20 = \rho$ and $\nu(\rho) = (1 - 0.99) \times 0.20 + 0.01 = 0.012 > 0$, hence the two conditions in Lemma 5.4(b,b2) are satisfied. Figure 5.2 (Graph III) demonstrates an example where $U_t(10)$ is strictly decreasing in $t \le t(10)$ and strictly increasing in $t \ge t(10)$, while $z_t(10)$ is constant in $t \le t''' = 20$ and strictly increasing in $t \ge t'''$.

5.2 Optimal Ordering Rule

Noting that $U_t(i) \le U_t(i-1)$ for $i \ge 1$ and $t \ge 0$ due to Lemma 5.1(c), for $t \ge 1$ we get:

$$
\begin{aligned}
\nabla v_t(i) &= U_t(i) - c && \text{(see Eq. (4.24))}, && i \ge 1, \\
&= L(U_{t-1}(i)) - L(U_{t-1}(i-1)) + \beta U_{t-1}(i) - h - c && \text{(see Eq. (4.11))}, && i \ge 1, \\
&= L(U_{t-1}(i)) - L(U_{t-1}(i-1)) + \beta \nabla v_{t-1}(i) - \nu(c) && \text{(see Eq. (4.24),Eq. (3.1))}, && i \ge 1, \quad (5.6) \\
&\le \lambda\beta(U_{t-1}(i-1) - U_{t-1}(i)) + \beta \nabla v_{t-1}(i) - \nu(c) && \text{(see Lemma 3.1(e))}, && i \ge 1, \\
&= \lambda\beta(\nabla v_{t-1}(i-1) - \nabla v_{t-1}(i)) + \beta \nabla v_{t-1}(i) - \nu(c) && \text{(see Eq. (4.24))}, && i \ge 2, \\
&= \beta(1-\lambda)\nabla v_{t-1}(i) + \lambda\beta \nabla v_{t-1}(i-1) - \nu(c), && && i \ge 2, \quad (5.7) \\
&= \beta(1-\lambda)(v_{t-1}(i) - v_{t-1}(i-1)) + \lambda\beta(v_{t-1}(i-1) - v_{t-1}(i-2)) - \nu(c), && && i \ge 2. \quad (5.8)
\end{aligned}
$$

Further, from Eqs. (4.26), (4.7), and (4.5) we obtain

$$
v_1(i) = L(\rho) + \beta(\rho - c)i - \nu(c)i = L(\rho) + (\beta\rho - h - c)i, \quad i \ge 1. \tag{5.9}
$$

Lemma 5.6

(a) *$v_t(i)$ is concave in $i \ge 0$ for $t \ge 0$.*

(b) *If $v_t(1) \le 0$ for a certain $t \ge 0$, then $v_t(i)$ is nonincreasing in $i \ge 0$ with $v_t(i) \le 0$ for $i \ge 0$.*

(c) *If $\rho < c$, then $v_t(i)$ is strictly decreasing in $i \ge t \ge 0$.*

(d) *If $\rho = c$ and $(1-\beta)^2 + h^2 \ne 0$, then $v_t(i)$ is strictly decreasing in $i \ge t \ge 1$.*

(e) *If $\rho \le (\ge)\ x_1^*$, then $v_t(1)$ is nondecreasing (nonincreasing) in $t \ge 0$.*

(f) *If $(1-\beta)^2 + h^2 \ne 0$ and $\rho > x_1^*$, then $v_t(1)$ is strictly decreasing in $t \ge 0$.*

(g) *If $(1-\beta)^2 + h^2 = 0$, then $v_t(i)$ is nondecreasing in $t \ge 0$ for $i \ge 0$.*

(h) *$\nabla v_t(i) \ge (=)\ G_t(\rho)$ for any given $t \ge 1$ and all $i \ge 1$ $(i > t)$.*

Proof. (a) In general, let a series a_x, $x = 0, 1, \cdots$, be said to be concave (convex) in x if the difference $a_x - a_{x-1}$ is nonincreasing (nondecreasing) in x. Note that for any given $t \geq 1$ we have $U_{t-1}(i) \leq U_{t-1}(i-1) \leq U_{t-1}(i-2)$ for $i \geq 2$ from Lemma 5.1(c). Since $v_0(i) = (\rho - c)i$ from Eq. (4.5), the assertion holds for $t = 0$. Suppose it holds for $t-1$, hence $\nabla^2 v_{t-1}(i) \leq 0$ for $i \geq 2$. Accordingly, from Eqs. (4.23), (5.6), and (3.10) for $t \geq 1$, we have

$$\begin{aligned}\nabla^2 v_t(i) &= L(U_{t-1}(i)) - 2L(U_{t-1}(i-1)) + L(U_{t-1}(i-2)) + \beta \nabla^2 v_{t-1}(i) \\ &= g(U_{t-1}(i), U_{t-1}(i-1), U_{t-1}(i-2)) - \beta(U_{t-1}(i) - U_{t-1}(i-1)) + \beta \nabla^2 v_{t-1}(i), \quad i \geq 2,\end{aligned}$$

in which $g(U_{t-1}(i), U_{t-1}(i-1), U_{t-1}(i-2)) \leq 0$ due to Lemma 3.6(b). Thus

$$\nabla^2 v_t(i) \leq -\beta(U_{t-1}(i) - U_{t-1}(i-1)) + \beta \nabla^2 v_{t-1}(i) = -\beta \nabla^2 v_{t-1}(i) + \beta \nabla^2 v_{t-1}(i) = 0, \qquad i \geq 2,$$

due to Eq. (4.24). Accordingly, by induction it follows that $\nabla^2 v_t(i) \leq 0$ for all $t \geq 0$ and $i \geq 2$, i.e., $v_t(i)$ is concave in $i \geq 0$ for all $t \geq 0$.

(b) Let $v_t(1) \leq 0$ for a certain $t \geq 0$. Then $\nabla v_t(1) = v_t(1) - v_t(0) = v_t(1) \leq 0 = v_t(0)$ due to Eq. (4.5). Suppose $v_t(i-1) \leq 0$ and $\nabla v_t(i-1) \leq 0$. Then from (a) we have $\nabla v_t(i) \leq \nabla v_t(i-1) \leq 0$, hence $v_t(i) \leq v_t(i-1) \leq 0$. Accordingly, by induction we have $v_t(i) \leq v_t(i-1) \leq 0$ for $t \geq 0$, hence the assertion holds.

(c) Let $\rho < c$. Clearly $\nabla v_0(i) = v_0(i) - v_0(i-1) = \rho - c < 0$ for $i \geq 1$ from Eq. (4.5), i.e., $v_0(i) < v_0(i-1)$ for $i \geq 1$, hence the assertion holds for $t = 0$. Let $t \geq 1$, and suppose $\nabla v_{t-1}(i) < 0$ for $i \geq t$, i.e., $v_{t-1}(i) < v_{t-1}(i-1)$ for $i \geq t$ or equivalently $v_{t-1}(i-1) < v_{t-1}(i-2)$ for $i \geq t+1$. Then from Eq. (5.8) and the fact that $\nu(c) \geq 0$ we get $\nabla v_t(i) < 0$ for $i \geq t+1$, i.e., $v_t(i) < v_t(i-1)$ for $i \geq t+1$. Thus by induction it follows that $v_t(i)$ is strictly decreasing in $i \geq t$ for $t \geq 0$.

(d) Let $\rho = c$ and $(1-\beta)^2 + h^2 \neq 0$. Then from Eqs. (5.9) and (3.1) we obtain $\nabla v_1(i) = v_1(i) - v_1(i-1) = \beta\rho - h - c = -(1-\beta)c - h = -\nu(c) < 0$ for $i \geq 2$. Let $t \geq 2$, and suppose $\nabla v_{t-1}(i) < 0$ for $i \geq t$, i.e., $v_{t-1}(i) < v_{t-1}(i-1)$ for $i \geq t$, hence $v_{t-1}(i-1) < v_{t-1}(i-2)$ for $i \geq t+1$. Then from Eq. (5.8) we get $\nabla v_t(i) < 0$, i.e., $v_t(i) < v_t(i-1)$ for $i \geq t+1$. Thus by induction it follows that $v_t(i)$ is strictly decreasing in $i \geq t$ for $t \geq 1$.

(e) Note that $x_1^* \leq x_0^*$ from Lemmas 3.4(f). Then from Lemma 5.2(b,a) we see that $U_t(1)$ is nondecreasing (nonincreasing) in $t \geq 0$, hence also is $v_t(1)$ due to Eq. (4.25).

(f) From Lemma 5.2(c) we see that $U_t(1)$ is strictly decreasing in $t \geq 0$, hence also is $v_t(1)$ due to Eq. (4.25).

(g) Let $(1-\beta)^2 + h^2 = 0$, hence $\nu(c) = 0$. Then from Eq. (4.26) we have $v_t(i) = L(U_{t-1}(i)) + v_{t-1}(i)$ for $t \geq 1$ and $i \geq 0$. Since $L(U_{t-1}(i)) \geq 0$ due to Lemma 3.1(a), we have $v_t(i) \geq v_{t-1}(i)$ for $t \geq 1$ and $i \geq 0$, hence the assertion holds.

(h) From Eqs. (5.9), (3.21), and the fact that $L(\rho) \geq 0$ due to Lemma 3.1(a) we have $\nabla v_1(i) = \beta\rho - h - c = G_1(\rho)$ for $i \geq 2$ and $\nabla v_1(1) = v_1(1) - v_1(0) = v_1(1) = L(\rho) + \beta\rho - h - c \geq \beta\rho - h - c = G_1(\rho)$, thus the assertion holds for $t = 1$. Below let $t > 1$.

S1. Suppose $\nabla v_{t-1}(i) \geq G_{t-1}(\rho)$ for all $i \geq 1$. Since $U_{t-1}(i) \leq U_{t-1}(i-1)$ for $i \geq 1$ due to Lemma 5.1(c), it follows from Lemma 3.1(b) that $L(U_{t-1}(i)) \geq L(U_{t-1}(i-1))$ for $i \geq 1$. Thus, from Eqs. (5.6), (3.21), and (3.1) we have

$$\nabla v_t(i) \geq \beta \nabla v_{t-1}(i) - \nu(c) \geq \beta G_{t-1}(\rho) - \nu(c) = \beta(\beta^{t-1}\rho - h\sum_{k=0}^{t-2}\beta^k - c) - \nu(c)$$
$$= \beta^t\rho - h\sum_{k=1}^{t-1}\beta^k - \beta c - (1-\beta)c - h = \beta^t\rho - h\sum_{k=0}^{t-1}\beta^k - c = G_t(\rho), \qquad i \geq 1.$$

Hence, by induction $\nabla v_t(i) \geq G_t(\rho)$ for any given $t \geq 1$ and all $i \geq 1$.

S2. Suppose $\nabla v_{t-1}(i) = G_{t-1}(\rho)$ for $i > t-1$ or equivalently $\nabla v_{t-1}(i-1) = G_{t-1}(\rho)$ for $i > t > 1$, so $i \geq 2$. Hence from Eq. (5.7) we have

$$\nabla v_t(i) \leq \beta(1-\lambda)G_{t-1}(\rho) + \lambda\beta G_{t-1}(\rho) - \nu(c) = \beta G_{t-1}(\rho) - \nu(c), \qquad i > t,$$

which can be easily rearranged as $\nabla v_t(i) \leq \beta^t\rho - h\sum_{k=0}^{t-1}\beta^k - c = G_t(\rho)$ for $i > t$. Since $\nabla v_t(i) \geq G_t(\rho)$ from (S1), we have $\nabla v_t(i) = G_t(\rho)$ for $i > t$.

Thus the assertion holds. ∎

Lemma 5.7

(a) *If $\rho < c$, then $\lim_{i\to\infty} v_t(i) = -\infty$ for $t \geq 0$.*

(b) *Let $\rho = c$.*

1. *If $(1-\beta)^2 + h^2 \neq 0$, then $\lim_{i\to\infty} v_t(i) = -\infty$ for $t \geq 1$.*
2. *Let $(1-\beta)^2 + h^2 = 0$.*
   i. *For any given $t \geq 0$ we have $v_t(i) = v_t(t)$ for $i \geq t$.*
   ii. *$v_t(1) > 0$ for any given $t \geq 1$.*

(c) *Let $\rho > c$ and $(1-\beta)^2 + h^2 = 0$. Then $v_t(i)$ diverges to ∞ as $i \to \infty$ for $t \geq 0$.*

(d) *$v_t(1)$ converges to a finite $v(1)$ with $U(1) = x_1^*$ and $v(1) = x_1^* - c$ if $(1-\beta)^2 + h^2 \neq 0$, and $U(1) \geq x_1^*$ and $v(1) \geq x_1^* - c$ if $(1-\beta)^2 + h^2 = 0$.*

Proof. (a) Let $\rho < c$. Then clearly $\lim_{i\to\infty} v_0(i) = -\infty$ from Eq. (4.5). Suppose $\lim_{i\to\infty} v_{t-1}(i) = -\infty$. Since $U_{t-1}(i)$ is bounded in i due to Lemma 5.1(d) and $K(x)$ is continuous on $(-\infty, \infty)$ due to Lemma 3.3(a), the term $K(U_{t-1}(i)) + (1-\beta)U_{t-1}(i)$ is also bounded in i, hence $\lim_{i\to\infty} v_t(i) = -\infty$ from Eq. (4.27) and the fact that $\nu(c) \geq 0$.

(b) Let $\rho = c$. Then $v_0(i) = 0$ from Eq. (4.5).

(b1) Let $(1-\beta)^2 + h^2 \neq 0$, hence $\nu(c) > 0$. Then, from Eqs. (4.26) and (4.7) we get $v_1(i) = L(U_0(i)) + \beta v_0(i) - \nu(c)i = L(\rho) - \nu(c)i$ for $i \geq 1$, hence $\lim_{i\to\infty} v_1(i) = -\infty$. Thus starting with this result, by almost the same induction as in (a) we obtain $\lim_{i\to\infty} v_t(i) = -\infty$ for all $t \geq 1$.

(b2) Let $(1-\beta)^2 + h^2 = 0$. Then $\nu(c) = 0$ and $x_1^* > c > x^\diamond$ from Lemma 3.5(e).

(b2i) Note that from Eq. (4.24) we get $\nabla v_t(i) = U_t(i) - c$ for $t \geq 0$ and $i \geq 1$. Then, for any given $t \geq 0$ we have $\nabla v_t(i) = \rho - c = 0$ for $i > t$ from Lemma 5.2(d), i.e., $v_t(i) = v_t(i-1)$ for $i > t$, hence the assertion holds.

(b2ii) First, from Eq. (4.15) we obtain $U_1(1) = L(\rho) + \rho \geq \rho$ due to $L(\rho) \geq 0$ from Lemma 3.1(a). Suppose $U_{t-1}(1) \geq \rho$. Then $U_t(1) \geq L(\rho) + \rho \geq \rho$ for $t \geq 1$ from Eq. (4.13) and Lemma 3.1(c,a). By induction we get $U_t(1) \geq \rho$ for $t \geq 1$, hence $U_t(1) \geq \rho$ for $t \geq 0$ due to Eq. (4.7). Thus $U_t(1) \geq \rho = c > x^\diamond$ for $t \geq 0$. Accordingly, from Eq. (4.28) and Lemma 3.5(c) we have $v_t(1) = N(U_{t-1}(1)) > 0$ for $t \geq 1$.

(c) Let $\rho > c$ and $(1-\beta)^2 + h^2 = 0$, thus $\nu(c) = 0$. Then $v_t(i) = K(U_{t-1}(i)) + v_{t-1}(i)$ for $t \geq 1$ and $i \geq 0$ from Eq. (4.27). Clearly $\lim_{i\to\infty} v_0(i) = \infty$ from Eq. (4.5). Suppose $\lim_{i\to\infty} v_{t-1}(i) = \infty$. Since

$U_{t-1}(i)$ is bounded in i due to Lemma 5.1(d) and $K(x)$ is continuous on $(-\infty,\infty)$ due to Lemma 3.3(a), the term $K(U_{t-1}(i))$ is also bounded in i, hence $\lim_{i\to\infty} v_t(i) = \infty$.

(d) Since $U_t(1)$ converges to a finite value $U(1)$ as $t \to \infty$ due to Lemma 5.5, we have $v(1) = U(1) - c$ from Eq. (4.25). Further, from Eqs. (4.14) and (3.24) we have

$$|U_t(1) - K(U(1)) - U(1) + h| = |K(U_{t-1}(1)) + U_{t-1}(1) - K(U(1)) - U(1)| \leq \beta|U_{t-1}(1)) - U(1)|,$$

which converges to 0 as $t \to \infty$. Thus $U_t(1)$ converges to $K(U(1)) + U(1) - h$, i.e, $U(1) = K(U(1)) + U(1) - h$, hence $K(U(1)) = h$. Accordingly, if $\beta < 1$, then $U(1) = x_1^*$ from Lemma 3.4(d1) and Eq. (3.14). Let $\beta = 1$. If $h > 0$, then $U(1) = x_1^*$ due to Lemma 3.4(e3). Therefore, $v(1) = x_1^* - c$ for $(1-\beta)^2 + h^2 \neq 0$. Let $(1-\beta)^2 + h^2 = 0$. Since $0 = h = K(U(1)) = L(U(1))$ from Eq. (3.5), it follows that $U(1) \geq b$ from Lemma 3.1(a). Now, since $x_1^* = b$ due to Lemma 3.4(e3), we have $U(1) \geq x_1^*$. Accordingly, we obtain $v(1) \geq x_1^* - c$. ▮

Lemma 5.8

(a) *Let* $(1-\beta)^2 + h^2 \neq 0$.

1. ρ_t^* *is strictly increasing in* $t \geq 1$.
2. $\rho_t^* > c$ *for* $t \geq 1$.

(b) *Let* $v_t(1) > 0$ *for a given* $t \geq 1$.

1. *If* $\rho > \rho_t^*$, *then* $i_t^* = \infty$.
2. *If* $\rho \leq \rho_t^*$, *then* $1 \leq i_t^* \leq t$.

(c) *If* $\rho \leq c$, *then* $i_0^* = 0$, *or else* $i_0^* = \infty$.

Proof. (a) Let $(1-\beta)^2 + h^2 \neq 0$. Then $(1-\beta)c + h > 0$ due to the assumption of $c > 0$ and $h \geq 0$.

(a1) From Eq. (3.22) we have

$$\rho_t^* - \rho_{t-1}^* = (h\sum_{k=0}^{t-1}\beta^k + c)/\beta^t - (h\sum_{k=0}^{t-2}\beta^k + c)/\beta^{t-1}$$

$$= (h(\sum_{k=0}^{t-1}\beta^k - \sum_{k=1}^{t-1}\beta^k) + (1-\beta)c)/\beta^t = ((1-\beta)c + h)/\beta^t > 0, \qquad t \geq 2,$$

i.e., $\rho_t^* > \rho_{t-1}^*$ for $t \geq 2$. Hence ρ_t^* is strictly increasing in $t \geq 1$.

(a2) Since $\rho_1^* = (h+c)/\beta > c$ from Eq. (3.22) and the assumption of $(1-\beta)^2 + h^2 \neq 0$, it follows that $\rho_t^* > \rho_1^* > c$ due to (a1).

(b) Let $v_t(1) > 0$ for a given $t \geq 1$. Since $v_t(0) = 0$ from Eq. (4.5), we have $v_t(1) > v_t(0)$, implying that $i_t^* \geq 1$.

(b1-b2) Note that ρ_t^* is the solution of $G_t(\rho) = 0$ and that $G_t(\rho)$ is strictly increasing in ρ from Eq. (3.21). If $\rho > \rho_t^*$, then $G_t(\rho) > 0$, thus $\nabla v_t(i) \geq G_t(\rho) > 0$ for $i \geq 1$ from Lemma 5.6(h). Hence we obtain $v_t(i) > v_t(i-1)$ for $i \geq 1$, so that $v_t(i)$ is strictly increasing in $i \geq 0$, implying that $i_t^* = \infty$. If $\rho \leq \rho_t^*$, then $G_t(\rho) \leq 0$, from which we get $\nabla v_t(i) = G_t(\rho) \leq 0$ for $i > t$ due to Lemma 5.6(h). Thus $v_t(i) \leq v_t(i-1)$ for $i > t$; in other words, $v_t(i)$ is nonincreasing in $i \geq t$. Accordingly, it follows from the definition of i_t^* that $i_t^* \leq t$, hence $1 \leq i_t^* \leq t$.

(c) Let $\rho \leq c$. Then from Eq. (4.5) we have $v_0(i) \leq 0 = v_0(0)$ for $i \geq 0$, hence $i_0^* = 0$. Let $\rho > c$. Then $v_0(i)$ is strictly increasing in $i \geq 0$ from Eq. (4.5), hence $i_0^* = \infty$. ▮

Theorem 5.2 *Let $t \geq 1$.*

(a) *Let $(1-\beta)^2 + h^2 = 0$.*

1. *Let $\rho < c$.*
   i. *If $\rho \leq x^\circ$, there exists a finite $t^* \geq 1$ such that if $1 \leq t \leq t^*$, then $i_t^* = 0$, or else $1 \leq i_t^* \leq t$.*
   ii. *If $\rho > x^\circ$, then $1 \leq i_t^* \leq t$.*
2. *Let $\rho = c$. Then $1 \leq i_t^* \leq t$.*
3. *Let $\rho > c$. Then $i_t^* = \infty$.*

(b) *Let $(1-\beta)^2 + h^2 \neq 0$.*

1. *Let $\rho \leq c$.*
   i. *Let $x_1^* \leq c$. Then $i_t^* = 0$.*
   ii. *Let $x_1^* > c$.*
      1. *If $\rho \leq x^\circ$, then there exists a finite $t^* \geq 1$ such that if $1 \leq t \leq t^*$, then $i_t^* = 0$, or else $1 \leq i_t^* \leq t$.*
      2. *If $\rho > x^\circ$, then $1 \leq i_t^* \leq t$.*
2. *Let $\rho > c$.*
   i. *Let $x_1^* < c$.*
      1. *If $\rho \leq x^\circ$, then $i_t^* = 0$.*
      2. *Let $\rho > x^\circ$. Then there exists a finite $t^* \geq 1$ such that:*
         i′. *If $t > t^*$, then $i_t^* = 0$.*
         ii′. *If $1 \leq t \leq t^*$:*
            1′. *if $\rho > \rho_t^*$, then $i_t^* = \infty$.*
            2′. *if $\rho \leq \rho_t^*$, then $1 \leq i_t^* \leq t$.*
   ii. *Let $x_1^* \geq c$.*
      1. *If $\rho > \rho_t^*$, then $i_t^* = \infty$.*
      2. *If $\rho \leq \rho_t^*$, then $1 \leq i_t^* \leq t$.*

Proof. Note that $v_t(0) = v(0) = 0$ for all $t \geq 0$ from Eq. (4.5). In addition, from Eq. (4.5) we get $v_0(i) - v_0(i-1) = \rho - c$ for $i \geq 1$. Further, from Eq. (4.28) we obtain

$$v_t(1) = v_t(1) - v_t(0) = N(U_{t-1}(1)), \qquad t \geq 1,$$

hence from Eq. (4.7) we get

$$v_1(1) = N(\rho). \tag{5.10}$$

(a) Let $(1-\beta)^2 + h^2 = 0$. Then $v_t(i)$ is nondecreasing in $t \geq 0$ for $i \geq 0$ due to Lemma 5.6(g). Also, we obtain $x_1^* > c > x^\circ$ from Lemma 3.5(e). Hence we have $v(1) - v(0) \geq x_1^* - c > 0$ from Lemma 5.7(d), i.e., $v(1) > v(0) = 0$.

(a1) Let $\rho < c$, hence $\rho < x_1^*$. In addition, we also have $i_t^* \leq t$ for $t \geq 1$ from Lemma 5.6(c) and the definition of i_t^*.

(a1i) Let $\rho \leq x^\circ$. Then $N(\rho) \leq 0$ due to Lemma 3.5(c), thus $v_1(1) \leq 0$ from Eq. (5.10). In addition, since $v(1) > 0$, from the monotonicity of $v_t(1)$ it follows that there exists a $t^* \geq 1$ such that $v_t(1) > 0$ for $t > t^*$ and $v_t(1) \leq 0$ for $t \leq t^*$. Accordingly, owing to $v_t(0) = 0$, if $t > t^*$, then $1 \leq i_t^* \leq t$, and if $1 \leq t \leq t^*$, then $i_t^* = 0$ from Lemma 5.6(b).

(a1ii) Let $\rho > x^\circ$. Then $N(\rho) > 0$ due to Lemma 3.5(c), thus $v_1(1) > 0 = v_1(0)$ due to Eq. (5.10). Hence $v_t(1) > 0 = v_t(0)$ for $t \geq 1$ due to the monotonicity of $v_t(1)$, implying that $1 \leq i_t^* \leq t$.

(a2) Let $\rho = c$. Then $v_t(1) > 0 = v_t(0)$ for any given $t \geq 1$ from Lemma 5.7(b2ii), implying that $i_t^* \geq 1$. Further, from Lemma 5.7(b2i) and the definition of i_t^* we have $i_t^* \leq t$, hence $1 \leq i_t^* \leq t$.

(a3) Immediate from Lemma 5.7(c).

(b) Let $(1-\beta)^2 + h^2 \neq 0$. Then from Lemma 5.7(d) we get

$$v(1) - v(0) = v(1) = x_1^* - c. \tag{5.11}$$

Furthermore, if $\rho < c$ or if $\rho = c$ and $(1-\beta)^2 + h^2 \neq 0$, from Lemma 5.6(c,d) and the definition of i_t^* it follows that $i_t^* \leq t$ for $t \geq 1$.

(b1) Let $\rho \leq c$. Then $i_t^* \leq t$ for $t \geq 1$ and $v_0(1) = \rho - c \leq 0$ from Eq. (4.5).

(b1i) Let $x_1^* \leq c$. Then $v(1) \leq 0$ from Eq. (5.11), and $c \leq x^\diamond$ due to Lemma 3.5(d). Noting Lemma 5.6(e), if $\rho < x_1^*$, then since $v(1) \leq 0$, it follows that $v_t(1) \leq 0$ for $t \geq 0$, and if $\rho \geq x_1^*$, then $v_t(1) \leq 0$ for $t \geq 0$. Accordingly, $v_t(1) \leq 0 = v_t(0)$ for $t \geq 0$ whether $\rho < x_1^*$ or $\rho \geq x_1^*$, implying that $i_t^* = 0$ due to Lemma 5.6(b).

(b1ii) Let $x_1^* > c$. Then $x^\diamond < c$ due to Lemma 3.5(d) and $v(1) > 0 = v(0)$ due to Eq. (5.11). Now, from the assumption of $\rho \leq c$ we have $\rho < x_1^*$, hence $v_t(1)$ is nondecreasing in $t \geq 0$ due to Lemma 5.6(e).

(b1ii1) Let $\rho \leq x^\diamond$. Then $N(\rho) \leq 0$ due to Lemma 3.5(c); thus $v_1(1) \leq 0 = v_1(0)$ from Eq. (5.10). In addition, since $v(1) > 0$, from the monotonicity of $v_t(1)$ in $t \geq 0$ it follows that there exists a $t^* \geq 1$ such that $v_t(1) > 0 = v_t(0)$ for $t > t^*$ and $v_t(1) \leq 0 = v_t(0)$ for $1 \leq t \leq t^*$, implying that $1 \leq i_t^* \leq t$ if $t > t^*$, or else $i_t^* = 0$ due to Lemma 5.6(b).

(b1ii2) Let $\rho > x^\diamond$. Then since $N(\rho) > 0$ due to Lemma 3.5(c), we have $v_1(1) > 0$ from Eq. (5.10). Hence $v_t(1) > 0 = v_t(0)$ for $t \geq 1$ due to the monotonicity of $v_t(1)$ in $t \geq 0$, implying that $1 \leq i_t^* \leq t$ for $t \geq 1$.

(b2) Let $\rho > c$.

(b2i) Let $x_1^* < c$. Then $c < x^\diamond$ due to Lemma 3.5(d) and $v(1) < 0$ due to Eq. (5.11). Now, from the assumption of $\rho > c$ we have $\rho > x_1^*$, hence $v_t(1)$ is strictly decreasing in $t \geq 0$ due to Lemma 5.6(f).

(b2i1) Let $\rho \leq x^\diamond$. Then $N(\rho) \leq 0$ due to Lemma 3.5(c), thus $v_1(1) \leq 0 = v_1(0)$ from Eq. (5.10). Since $v_t(1) \leq 0 = v_t(0)$ for $t \geq 1$ due to the monotonicity of $v_t(1)$ in $t \geq 0$, it follows that $i_t^* = 0$ for $t \geq 1$ due to Lemma 5.6(b).

(b2i2′,b2i2ii′1′,b2i2ii′2′) Let $\rho > x^\diamond$. Then $N(\rho) > 0$ due to Lemma 3.5(c), thus $v_1(1) > 0$ from Eq. (5.10). Since $v(1) < 0$, it follows due to the monotonicity of $v_t(1)$ in $t \geq 0$ that there exists a $t^* \geq 1$ such that $v_t(1) \leq 0 = v_t(0)$ for $t > t^*$ and $v_t(1) > 0 = v_t(0)$ for $1 \leq t \leq t^*$. Therefore, if $t > t^*$, then $i_t^* = 0$ due to Lemma 5.6(b), or else since $v_t(1) > 0$ for $1 \leq t \leq t^*$, the assertions hold for $1 \leq t \leq t^*$ due to Lemma 5.8(b).

(b2ii1,b2ii2) Let $x_1^* = c$. Then $x^\diamond = c$ due to Lemma 3.5(d). Now, from the assumption of $\rho > c$ we have $\rho > x^\diamond = x_1^*$. Hence, $N(\rho) > 0$ from Lemma 3.5(c), thus $v_1(1) > 0 = v_1(0)$ from Eq. (5.10). Since $\rho > x_1^*$, it follows that $v_t(1)$ is strictly decreasing in $t \geq 0$ due to Lemma 5.6(f). In addition, since $v(1) = v(0) = 0$ from Eq. (5.11), it follows that $v_t(1) > 0 = v_t(0)$ for $t \geq 2$.

Let $x_1^* > c$. Then $c > x^\diamond$ due to Lemma 3.5(d), hence $\rho > x^\diamond$ due to the assumption of $\rho > c$. Accordingly $N(\rho) > 0$ from Lemma 3.5(c), thus $v_1(1) > 0 = v_1(0)$ from Eq. (5.10). In addition, from Eq. (5.11) we have $v(1) > 0 = v(0)$. If $\rho > x_1^*$, then $v_t(1)$ is strictly decreasing in $t \geq 0$ due to Lemma 5.6(f), hence $v_t(1) > v(1) > 0$ for $t \geq 0$. If $\rho \leq x_1^*$, then $v_t(1)$ is nondecreasing in $t \geq 0$ due to Lemma 5.6(e), hence $v_t(1) \geq v_1(1) > 0$ for $t \geq 1$. Since $v_t(1) > 0$ for $t \geq 1$, the assertions hold due to

Lemma 5.8(b). ∎

Corollary 5.2

(a) *Let* $(1-\beta)^2 + h^2 = 0$. *Then* $x^\circ < c$.

(b) *Let* $(1-\beta)^2 + h^2 \neq 0$. *For all* $t \geq 1$*:*

1. *If* $x_1^* > c$*, then* $x^\circ < c < \rho_t^*$.
2. *If* $x_1^* = c$*, then* $x^\circ = c < \rho_t^*$.
3. *If* $x_1^* < c$*, then* $c < x^\circ < \rho_t^*$.

Proof. (a) Evident from Lemma 3.5(e).

(b) Let $(1-\beta)^2 + h^2 \neq 0$. Then $c < \rho_t^*$ from Lemma 5.8(a2) and $x^\circ < \rho_t^*$ from Lemma 3.5(f) for $t \geq 1$.

(b1-b2) Let $x_1^* > (=)\ c$. Then $x^\circ < (=)\ c < \rho_t^*$ from Lemma 3.5(d).

(b3) Let $x_1^* < c$. Then $c < x^\circ < \rho_t^*$ from Lemma 3.5(d). ∎

6 Conclusions

This section summarizes the properties of the optimal decision rules derived in Section 5 and considers their practical implications.

6.1 Optimal Selling Price $z_t(i)$

Monotonicity in i

In order to avoid leftover items at the deadline, a seller may become more compelled to sell if he has a substantial inventory level at a point in time, implying that he will lower the selling price $z_t(i)$ as the inventory level i increases. This commonly observed phenomenon in the business world is consistent with Theorem 5.1(a).

Monotonicity in t

Intuitively, it is conjectured that the shorter the time left until deadline, the lower the offered selling price will be in order to avoid leftover items at the deadline; in other words, $z_t(i)$ is nondecreasing in $t \geq 0$. However, it was shown through the analysis of our model that this conjecture does not always hold. Table 6.1 lists the conditions for the monotonicity of $z_t(i)$ in t for any given $i \geq 1$, obtained in Theorem 5.1, where $x^\bullet$ and x_1^* are defined by, respectively, Eqs. (3.13) and (3.17). From the table we see that the above conjecture holds in case C1 and in case C2 with $i = 1$. The conjecture may not hold in case C2 with $i \geq 2$ as depicted in Figure 5.2, and does not hold in case C3 at all. Besides, by the numerical examples (Figure 5.2), we demonstrate that $z_t(i)$ may be nonincreasing or unimodal in t. In addition, we also showed that $z_t(i)$ does not oscillate in t.

Furthermore, from Theorem 5.1(b), we see that $z_t(i)$ may be truncated at a, more precisely, if $U_t(i) < x^\star$, then $z_t(i) = a$. This implies that even though $U_t(i)$ is strictly decreasing in t, there may exist a $\dot{t} \geq 0$ such that $z_t(i) = a$ in $t \geq \dot{t}$ as depicted in Figure 5.2 (II,III). For case C2, since $z_t(i)$ may be truncated at a, $z_t(i)$ may not inherit the strict decreasingness of $U_t(i)$ if $t \leq t(i)$ (see Corollary 5.1). Accordingly, $z_t(i)$ is nonincreasing in $t \leq t(i)$ as can be observed in Figure 5.2 (II,III). Unfortunately, the proof of the existence of $t(i) = \infty$ is difficult and this task is left as a future study.

Table 6.1: Conditions for the monotonicity of $z_t(i)$

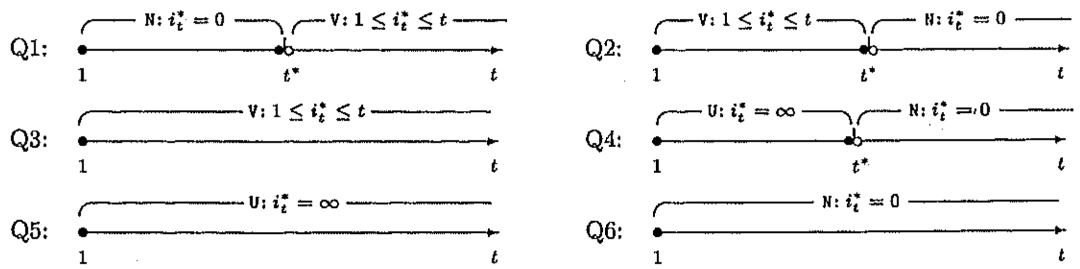
Case	$\beta = 1$	$\beta < 1$	$z_t(i)$
C1	$h = 0$	$\rho \leq x^\bullet$	$z_t(i)$ is nondecreasing in $t \geq 0$ for $i \geq 0$.
C2	$h > 0$, $x_1^* > \rho$	$x^\bullet < \rho < x_1^*$	$z_t(1)$ is nondecreasing in $t \geq 0$. $z_t(i)$ does not oscillate in $t \geq 0$ for $i \geq 2$.
C3	$h > 0$, $x_1^* \leq \rho$	$x_1^* \leq \rho$	$z_t(i)$ is nonincreasing in $t \geq 0$ for $i \geq 0$.

Here, we propose a reasoning for the occurrence of the counterintuitive phenomenon where $z_t(i)$ is nonincreasing in t near the deadline. If the salvage price is sufficiently high, the selling price offered by a seller to a buyer may increase as the deadline draws near since the seller is able to sell the items at the high salvage price even if an arriving buyer refuses to buy the item at the high selling price. This may result in the seller being more willing to offer a high selling price in an attempt to obtain greater profit by selling the leftover items to the salvage dealer. While this reasoning seems plausible near the deadline, the occurrence of the phenomenon where $z_t(i)$ is nonincreasing in t throughout the entire time horizon as stated in case C3 is unexplainable within our present knowledge due to the complicated interdependency of the model's parameters.

6.2 Optimal Ordering Quantity i_t^*

For convenience, if $1 \leq i_t^* \leq t$, let a business deal to order a product be said to be *viable* (V), if $i_t^* = 0$, *nonviable* (N), or if $i_t^* = \infty$, *unpractical* (U). Then, the optimal ordering quantity i_t^* for $t > 0$ can be classified into the following six possible cases according to the conditions in Theorem 5.2. Table 6.3 provides strict explanations of these cases. In the explanation below, by the term "finite amount" we mean an amount greater than or equals to 1 and less than or equals to t.

Table 6.2: Six possible cases



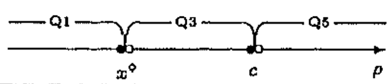
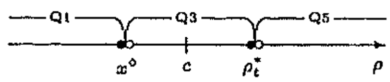
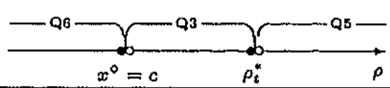
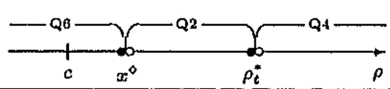
With reference to Theorem 5.2 and Corollary 5.2, the conditions for the occurrence of each case can be summarized in Table 6.4, implying that the optimal ordering quantity can be prescribed as follows:

1. Suppose $(1-\beta)^2 + h^2 = 0$. Let $\rho \leq x^\diamond$. Then if $t < t^*$, do not order any item, or else order a finite amount. Let $x^\diamond < \rho \leq c$, then for all $t \geq 1$ order a finite amount. Let $\rho > c$, implying that salvage value per item is greater than ordering cost per item. Then we obtain an unrealistic result that ordering an infinite amount is optimal for all $t \geq 1$. In fact, the result of ordering an infinite amount for this condition is obvious due to $\rho > c$ since $\beta = 1$ and $h = 0$.

Table 6.3: Explanations of the six possible cases (V: viable, N: nonviable, U: unpractical)

Case		Explanation
Q1	N/V	There exists a time threshold t^* such that if $1 \le t \le t^*$, then do not order any quantity (*nonviable*), and if $t > t^*$, then order a finite amount (*viable*).
Q2	V/N	This is the inverse of case Q1.
Q3	V	Order a finite amount at any point in time (*viable*).
Q4	U/N	There exists a time threshold t^* such that if $1 \le t \le t^*$, then order an infinite amount (*unpractical*), and if $t > t^*$, then do not order any quantity (*nonviable*).
Q5	U	Order an infinite amount at any point in time (*unpractical*).
Q6	N	Do not order any quantity at any point in time (*nonviable*).

Table 6.4: Summary of Theorem 5.2 (Note Corollary 5.2)

$(1-\beta)^2 + h^2 = 0$			Q1 : (a1i) Q3 : (a1ii,a2) Q5 : (a3)
$(1-\beta)^2 + h^2 \neq 0$	$x_1^* > c$		Q1 : (b1ii1) Q3 : (b1ii2,b2ii2) Q5 : (b2ii1)
	$x_1^* = c$		Q3 : (b2ii2) Q5 : (b2ii1) Q6 : (b1i)
	$x_1^* < c$		Q2 : (b2i2ii′2′) Q4 : (b2i2ii′1′) Q6 : (b1i,b2i1)

2. Suppose $(1-\beta)^2 + h^2 \neq 0$.

   i. Assume $x_1^* > c$. Let $\rho \le x^\diamond$. Then if $t < t^*$, do not order any item, or else order a finite amount. Let $x^\diamond < \rho \le \rho_t^*$. Then order a finite amount for all $t \ge 1$. Let $\rho > \rho_t^*$. Then order an infinite amount for all $t \ge 1$.

   ii. Assume $x_1^* = c$. Let $\rho \le x^\diamond$. Then do not order any item for all $t \ge 1$. Let $x^\diamond < \rho \le \rho_t^*$. Then order a finite amount for all $t \ge 1$. Let $\rho > \rho_t^*$. Then order an infinite amount for all $t \ge 1$.

   iii. Assume $x_1^* < c$. Let $\rho \le x^\diamond$. Then do not order any item for all $t \ge 1$. Let $x^\diamond < \rho \le \rho_t^*$. Then if $t > t^*$, do not any item, or else order a finite amount. Let $\rho > \rho_t^*$. Then if $t > t^*$, do not order any item, or else order an infinite amount.

Lastly, we shall draw attention to the following point. If the salvage value per item ρ is assumed to be greater than the ordering cost per item c, we seem, at first glance, to be led to an unrealistic result as stated earlier. Indeed, such is the case when $\beta = 1$ and $h = 0$ as proven in Theorem 5.2(a3). However, if $\beta < 1$ or $h > 0$, we also obtained results that $1 \le i_t^* \le t$ instead of $i_t^* = \infty$.

7 Considerations and Suggested Future Studies

In this paper we have proposed a model of newsboy problem with pricing policy by introducing a concept of reservation price. Although our original model bears some resemblance to the one proposed in [1], [2], [3], [4], [7], and [8], we exhaustively analyzed the relationship between the optimal decision rule and the related model's parameters without excessively constraining the parameters space. This leads us to the conditions that distinguish a deal to order among viable, nonviable, and unpractical deal. Below, we shall reemphasize the three points that are derived from our analysis.

1. The monotonicity in t for the optimal selling price $z_t(i)$ is partially against our intuitive conjecture.
2. All the model's parameters interrelate in determining the viability and practicality of a deal to order a product.
3. The length of the planning horizon also plays an important role in determining whether a deal is viable, unpractical or nonviable.

Finally, in order to make our model more practical, the subjects of study listed below should be considered.

1. Theoretically prove the existence of $t(i) = \infty$ for the case of $\nu(\rho) > 0$ and $\rho < x_1^*$.
2. A seller may have the opportunity to sell a product to a salvage dealer not only on the deadline but also at any point in time before the deadline if he wishes.
3. In our model we deal only with perishable products. For a nonperishable product we can extend our model to the inventory control problem.
4. A seller may be sometimes offered a quantity discount.
5. The buyer arriving probability λ may be dependent on the price offered by the seller, i.e., $\lambda(z)$.

Appendices : Proofs

Here, note that $L(x) = \lambda\beta T(x)$ from Eq. (3.4). Furthermore, the proofs of Lemmas 3.1(a,b) and 3.2(a,b) are basically the same as the proofs presented in [7]. However, owing to the distribution function defined in this paper being slightly different from the one in [7], we shall refer to the proofs of lemmas in [6] for the proofs of Lemma 3.1-Lemma 3.3.

A. Lemma 3.1

For (a) see Lemma 6.15 (a), (b) see Lemma 6.12 (a) and Lemma 6.14 (a), (c) see Lemma 6.14 (e2), (d) see Lemma 6.19 (c,d), and (e,f) see Lemma 6.17. ∎

B. Lemma 3.2

For (a) see Lemma 6.12 (f), (b) see Lemma 6.13, and (c) see Lemma 6.18 (c). ∎

C. Lemma 3.3

Note that in this paper, the notation s (the search cost) defined in [6] is assumed to be equal to 0, i.e., $s = 0$. Then, for (a-c) see Lemma 6.90 (a-d), (d) see Lemma 6.90 (i), (e) see Lemma 6.90 (j,k), (f) see

Lemma 6.90 (f), and (g) see Lemma 6.89. ▮

D. Lemma 3.4

From Lemma 3.1(a) and Eq. (3.5) we have

$$K(x) = -(1-\beta)x \leq 0, \quad x \geq b. \tag{D.1}$$

(a) Evident from Lemma 3.3(c,d,e).

(b) Let $\beta = 1$. Then $K(x) = L(x) \geq 0$ from Eq. (3.5) and Lemma 3.1(a). Let $\gamma_{,,} = 0$. Then $0 = K(x(0)) = L(x(0))$ from Eq. (3.20), hence we have $x(0) = b$ due to Lemma 3.1(a) and the definition of $x(0)$. Let $\gamma > 0$. The assertion is evident from Lemma 3.3(b,d,e).

(c) The existences of the solution of $K(x_1) = h$ is evident from (a,b). Let $i \geq 2$. If $\beta = 1$, then $K(x_i) = ih \geq 0$ for $i \geq 2$ from Eq. (3.16), hence the solutions of the Eq. (3.15) exists due to (b). Let $\beta < 1$. Then noting Lemma 3.3(c,d,e), we can easily see that the solutions of the Eq. (3.15) uniquely exist for $i \geq 2$ by induction starting with the fact the solution x_1^* exists.

(d) Let $\beta < 1$.

(d1)The solutions of $K(x) = 0$ and $K(x) = h$ uniquely exist due to (a), hence the assertion holds.

(d2) Noting Eq. (3.5), we have $K(0) = L(0) > 0$ due to Lemma 3.1(a) and $K(b) < 0$ due to Eq. (D.1), hence $0 < x_0^*$ and $x_0^* < b$ from Lemma 3.3(c) and Eq. (3.12).

(d3) Evident from the definition of x_0^* and Lemma 3.3(c).

(d4) If $h = 0$, then $K(x_1^*) = 0$ from Eq. (3.17), hence $x_0^* = x_1^*$ due to Eq. (3.12) and (d1).

(e) Let $\beta = 1$. Then $K(x) = L(x)$ for any x from Eq. (3.5).

(e1) Since $K(x) = L(x) = 0$ for $x \geq b$ and $K(x) = L(x) > 0$ for $x < b$ due to Lemma 3.1(a), by the definition of x_0^* it follows that $x_0^* = b$.

(e2) Evident from the definition of x_0^* and Lemma 3.3(b,d,e).

(e3) If $h = 0$, then x_1^* is the smallest solution of $K(x) = 0$ due to Eq. (3.14). In addition, x_0^* is also the smallest solution of $K(x) = 0$ from Eq. (3.12). Therefore, since $x_0^* = b$ from (e1), it follows that $x_1^* = x_0^* = b$. The latter half of the assertion is evident from (b).

(f) Let $h > 0$. Since $K(x_1^*) = h > 0 = K(x_0^*)$ due to Eqs. (3.17) and (3.12), it follows from Lemma 3.3(a) that $x_1^* < x_0^*$. The inequality of $x_0^* \leq b$ is from (d2,e1), hence the former half is true. The latter half is from (d2,d4,e1,e3).

(g) Let $(1-\beta)^2 + h^2 \neq 0$. Then x_1^* uniquely exists from (d1,e3). In addition, we have $x_1^* < b$ from (d2,d4,f). Hence, the assertion holds due to these results, Eq. (3.17), and the fact that $K(x)$ is strictly decreasing on the neighborhood of $x = x_1^*$ from Lemma 3.3(b).

(h) Let $\beta < 1$. Then the unique existence of x_i^* for each $i \geq 0$ is immediate from Lemma 3.3(c,d,e).

(i) Let $(1-\beta)^2 + h^2 = 0$. Then $K(x_1^*) = K(x_0^*) = 0$ from Eqs. (3.17) and (3.12), and $K(x_i^*) = 0$ for $i \geq 2$ from Eq. (3.19), hence $K(x_i^*) = 0$ for $i \geq 0$. Since $K(x) = L(x)$ for all x from Eq. (3.5), we have $K(x) > 0$ for $x < b$ and $K(x) = 0$ for $x \geq b$ from Lemma 3.1(a). Accordingly, $x_i^* = b$ for $i \geq 0$ due to the definition of x_i^*. ▮

E. Lemma 3.5

(a) Immediate from Eq. (3.7) and Lemma 3.1(c).

(b) Since $L(x) = 0$ for $x \geq b$ from Lemma 3.1(a), clearly $\lim_{x\to\infty} N(x) = \infty$ from Eq. (3.7). Further, from Lemma 3.1(d) and Eq. (3.7) we have $\lim_{x\to-\infty} N(x) = -\infty$. From the result and (a) it must be that x° uniquely exists.

(c) Evident from (a) and the definition of x°.

(d) We have $N(x_1^*) = K(x_1^*) + x_1^* - c - h = x_1^* - c$ due to Eqs. (3.8) and (3.17). Thus, if $x_1^* > (= (<))\ c$, then $N(x_1^*) > (= (<))\ 0$, hence $x^\circ < (= (>))\ x_1^*$ due to (c). In addition, we have $N(c) = K(c) + c - c - h = K(c) - h = K(c) - K(x_1^*)$ due to Eqs. (3.8) and (3.17). Here, note that $x_1^* \leq b$ from Lemma 3.4(f). Since $c < b$ by the assumption, $K(x)$ is strictly decreasing on the neighborhood of $x = c$ due to Lemma 3.3(b). Accordingly, if $x_1^* > (= (<))\ c$, then $K(x_1^*) < (= (>))\ K(c)$, hence $N(c) > (= (<))\ 0$, implying $x^\circ < (= (>))\ c$ due to (c).

(e) Let $(1-\beta)^2 + h^2 = 0$. Then $x_1^* = b > c$ from Lemma 3.4(e1,e3) and the assumption of $c < b$. Hence $x_1^* > c > x^\circ$ due to (d).

(f) Let $(1-\beta)^2 + h^2 \neq 0$. Suppose $x^\circ \geq \rho_t^*$ for $t \geq 1$. Then from (c) and Eq. (3.7) we have $0 \geq N(\rho_t^*) = L(\rho_t^*) + \beta\rho_t^* - c - h \cdots (1^*)$ for $t \geq 1$. Substituting Eq. (3.22) into (1^*) we obtain

$$\begin{aligned} 0 &\geq L(\rho_t^*) + \beta((h\sum_{k=0}^{t-1}\beta^k + c)/\beta^t) - c - h \\ &= L(\rho_t^*) + (h\sum_{k=0}^{t-2}\beta^k + c + h\beta^{t-1})/\beta^{t-1} - c - h \\ &= L(\rho_t^*) + (h\sum_{k=0}^{t-2}\beta^k + (1-\beta^{t-1})c)/\beta^{t-1} > L(\rho_t^*), \qquad t \geq 1, \end{aligned}$$

due to $(h\sum_{k=0}^{t-2}\beta^k + (1-\beta^{t-1})c) > 0$ by the assumption of $(1-\beta)^2 + h^2 \neq 0$ and $c > 0$. Since $0 > L(\rho_t^*)$ contradicts Lemma 3.1(a), it must be that $x^\circ < \rho_t^*$ for $t \geq 1$. ∎

F. Lemma 3.6

(a) Let $x \leq (\geq)\ y$. Then from Eq. (3.9) and Lemma 3.1(e) we have

$$g(x,y) \leq (\geq)\, \lambda\beta(y-x) + \beta(x-y) = \beta(1-\lambda)(x-y) \leq (\geq)\, 0.$$

(b) Let $x \leq y \leq y'$ $(x \geq y \geq y')$. Now, $g(x,y,y')$ can be expressed as

$$g(x,y,y') = L(x) - L(y) + L(y') - L(y) + \beta(x-y).$$

Since $L(x) - L(y) \leq (\geq)\ \lambda\beta(y-x)$ from Lemma 3.1(e) and $L(y') - L(y) \leq (\geq)\ 0$ from Lemma 3.1(b), we have

$$g(x,y,y') \leq (\geq)\, \lambda\beta(y-x) + \beta(x-y) = \beta(1-\lambda)(x-y) \leq (\geq)\, 0.$$

(c) Let $x \leq (\geq)\, x'$ and $y \leq (\geq)\, y'$. Then $L(x) - L(x') \leq (\geq)\, \lambda\beta(x'-x)$ from Lemma 3.1(e) and $L(y') - L(y) \leq (\geq)\, 0$ from Lemma 3.1(b). Accordingly, from Eq. (3.11) we have

$$g(x,x',y,y') \leq (\geq)\, \lambda\beta(x'-x) + \beta(x-x') = \beta(1-\lambda)(x-x') \leq (\geq)\, 0. \quad ∎$$

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