Totally Convex Preferences for Gambles

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Abstract

Numerical representations that adimit total convexity of preferences include Chew's weighted linear utility, Dekel's implicit representation, and Fishburn's SSB utility. This paper shows that existence of a universally indifferent gamble and acyclicity of strict preferences have significant implications for general axiomatizations of those representations.

1 Introduction

Last two decades observed that numerous numerical representations and their axiomatic developments have appeared to generalize the traditional expected utility theories by von Neumann and Morgenstern (1944, 1947, 1953), Savage (1954), and others. Many survey papers and books have already been published to consult those research activities (for example, see Fishburn (1988), Karni and Schmeidler (1991), Quiggin (1993), and Schmidt (1998)).

One of the pioneering work for those generalizations is due to Chew's (1983) weighted linear utility, Dekel's (1986) implicit representation, and Fishburn's (1982) SSB utility. Those representations have a common preference structure, to which we refer hereafter as a *totally convex structure*. A typical characteristic of the structure is that every indifference surface in the barycentric coordinate of the probability simplex (or Machina-Degroot triangle), whose elements are indifferent to a gamble, consists of a hyperplane, as opposed to curved surfaces supported by other generalizations such as Quiggin's (1982) anticipated utility, Machina's (1982) generalized expected utility, and other refined and generalized models.

Fishburn (1992, 1983, 1988) provided general representational implications of the totally convex structure. He elaborated on effects of transitivity of the indifference relation on the structure, and showed that transitive indifference forces strict preference to be transitive, so that preferences must

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be weakly ordered. This finding has two representational implications. The first is that a nontransitive convex representation, defined in the next section, is reduced to a transitive convex representation, which is an alternative to Dekel's implicit representation. The second is that SSB utility is reduced to Chew's weighted linear utility.

It has been argued, however, that transitivity of the indifference relation fails to hold due to imperfect powers of discrimination of the human judgments (see for early discussions in economics Armstrong (1939), Georgescu-Roegen (1936), Luce (1956), and many others). Therefore, it may be desirable to investigate axiomatic relationships among numerical representations for the totally convex preferences without assuming transitivity of indifferences.

The aim of the paper is to explore such relationships. It is shown that existence of a universally indifferent gamble and acyclicity of strict preferences have significant implications for general axiomatizations of those numerical representaitons, where a gamble is univerally indifferent if it is indifferent to every gamble. When there is no universally indifferent gamble, acyclicity of strict preferences suffices to discriminate between nontransitive and transitive convex representations, and between SSB utility and weighted linear utility. In this case, acyclicity forces indifference to be transitive.

The paper is organized as follows. Section 2 introduces the totally convex structure and describes two numerical representations, which accommodate preference cycles and intransitive idifference. Section 3 elaborate acyclicity on the nontransitive convex representation. Section 4 presents necessary and sufficient axiom for the separable SSB representation, and then shows that acyclicity reduces SSB utility to weighted linear utility.

2 Totally Convex Preferences

A gamble on a set X of decision outcomes is a nonnegative function p on X for which $\{x \in X : p(x) \neq 0\}$ is finite and $\sum_{x \in X} p(x) = 1$. Every gamble p on X is interpreted as giving an outcome x with probability $p(x)$. Let $\mathscr{G}$ denote the set of all gambles. A probability mixture of two gambles $p, q \in \mathscr{G}$ with respect to a probability number $0 \leq \lambda \leq 1$, denoted $p \bigcirc_\lambda q$, is a nonnegative function on X that takes value $\lambda p(x) + (1 - \lambda) q(x)$ at $x \in X$. Since p and q are gambles, $p \bigcirc_\lambda q$ is also a gamble, so by definition, $p \bigcirc_\lambda q \in \mathscr{G}$.

By $\succ$, we denote a binary strict preference relation on $\mathscr{G}$, read as "is preferred to." Let the indifference relation $\sim$ and the weak preference relation $\succeq$ be defined in the usual way: for all $p, q \in \mathscr{G}$, $p \sim q$ if $\neg(p \succ q)$ and $\neg(q \succ p)$, and $p \succeq q$ if $\neg(q \succ p)$. Let $\mathscr{G}^*$ denote the preference interior of $\mathscr{G}$, and $\mathscr{G}_{\max}$ and $\mathscr{G}_{\min}$ respectively denote the preference-maximal and preference-minimal subsets of $\mathscr{G}$, which are defined by

$$\begin{aligned}\mathscr{G}^* &= \{p \in \mathscr{G} : q \succ p \text{ and } p \succ r \text{ for some } q, r \in \mathscr{G}\},\\ \mathscr{G}_{\max} &= \{p \in \mathscr{G} : p \succeq q \text{ for all } q \in \mathscr{G}\},\\ \mathscr{G}_{\min} &= \{p \in \mathscr{G} : q \succeq p \text{ for all } q \in \mathscr{G}\}.\end{aligned}$$

We say that $(\mathscr{G}, \succ)$ is *closed* if $\mathscr{G}_{\max}$ and $\mathscr{G}_{\min}$ are nonempty, *open* if $\mathscr{G}_{\max} = \mathscr{G}_{\min} = \emptyset$, and *half-open* otherwise.

A totally convex structure of preferences is stated as follows.

Definition 2.1 *$(\mathscr{G}, \succ)$ is said to be a* totally convex structure *if the following two axioms hold: for all $p, q, r \in \mathscr{G}$ and all $0 \leq \lambda \leq 1$,*

Axiom C (continuity) *If $p \succ q$ and $q \succ r$, then there is an $0 < \alpha < 1$ such that $q \sim p \bigcirc_\alpha r$.*

Axiom TC (total convexity) *If $p \succ q$ and $p \succeq r$, then $p \succ q \bigcirc_\lambda r$; if $p \sim q$ and $p \sim r$, then $p \sim q \bigcirc_\lambda r$; $q \succ p$ and $r \succeq p$, then $q \bigcirc_\lambda r \succ p$.*

Axioms C and TC were first introduced by Fishburn (1982). Axiom TC implies that a probability number α in a probability mixture $p \bigcirc_\alpha r$ in axiom C is "uniquely" determined.

It is shown in Fishburn (1988, Theorem 4.2) that if $(\mathscr{G}, \succ)$ is a open and totally convex structure, then $(\mathscr{G}, \succ)$ has the following numerical representation, referred to as a *nontransitive convex representation*: there is a functional ϕ on $\mathscr{G} \times \mathscr{G}$ such that, for all $p, q, r \in \mathscr{G}$ and all $0 < \lambda < 1$,

$$\begin{aligned} p \succ q &\Leftrightarrow \phi(p, q) > 0,\\ \phi(p, q) > 0 &\Leftrightarrow \phi(q, p) < 0,\\ \phi(p \bigcirc_\lambda q, r) &= \lambda\phi(p, r) + (1 - \lambda)\phi(q, r).\end{aligned}$$

The second property of ϕ indicates asymmetry of $\succ$, and the third requires linearity in the first argument. The functional ϕ on $\mathscr{G} \times \mathscr{G}$ is said to be *skew-symmeric* if, for all $p, q \in \mathscr{G}$,

$$\phi(p, q) = -\phi(q, p).$$

Then ϕ in the nontransitive convex representation is called an *SSB functional* if it is skew-symmetric and bilinear, i.e., linear separately in each argument of ϕ. Note that skew-symmetry and linearity in the first argument implies linearity in the second argument.

Fishburn (1982) showed that $(\mathscr{G}, \succ)$ has an SSB representation, i.e., there is an SSB functional ϕ on $\mathscr{G} \times \mathscr{G}$ such that, for all $p, q \in \mathscr{G}$,

$$p \succ q \Leftrightarrow \phi(p, q) > 0,$$

if and only if $(\mathscr{G}, \succ)$ is a totally convex structure and the following symmetry axiom holds: for all $p, q, r \in \mathscr{G}$ and all $0 < \lambda < 1$,

Axiom S (symmetry) *If* $p \succ q$, $q \succ r$, $p \succ r$, *and* $q \sim p \bigcirc_{\frac{1}{2}} r$, *then* $p \bigcirc_\lambda r \sim p \bigcirc_{\frac{1}{2}} q \Longleftrightarrow r \bigcirc_\lambda p \sim r \bigcirc_{\frac{1}{2}} q$.

3 Transitive Convex Representations

We say that $(\mathscr{G}, \succ)$ has a *transitive convex representation* if there is a functional u on $\mathscr{G}$ such that, for all $p, q \in \mathscr{G}$,

$$\begin{aligned} p \succ q &\quad\Leftrightarrow\quad u(p) > u(q), \\ p \succ q &\quad\Rightarrow\quad u(p \bigcirc_\lambda q) \text{ is continuous and increasing in } \lambda. \end{aligned}$$

It is shown by Fishburn (1983, Theorem 1) that $(\mathscr{G}, \succ)$ has a transitive convex representation if and only if $(\mathscr{G}, \succ)$ is a countably bounded, totally convex structure with transitive indifference $\sim$, where we say that $(\mathscr{G}, \succ)$ is *countablly bounded* if there is a countable subset Q of $\mathscr{G}$ such that for every $p \in \mathscr{G}$, there are $q, r \in Q$ for which $q \succeq p$ and $p \succeq r$. If $(\mathscr{G}, \succ)$ is closed, it is clearly countably bounded; otherwise, it need not be. Fishburn (1983b) demonstrated that the countable boundedness cannot be dropped to ensure the existence of the transitive convex representation.

In place of transitivity of the indifference relation, we shall consider acyclic strict preferences for the totally convex structure. For a given positive integer $n > 2$, $\succ$ is said to be *n-acyclic* if, for all $p_1, \ldots, p_n \in \mathscr{G}$, $\neg(p_n \succ p_1)$ whenever $p_i \succ p_{i+1}$ for $i = 1, \ldots, n-1$. We say that $\succ$ is *acyclic* if it is n-acyclic for all integers $n > 0$.

Lemma 3.1 *If $(\mathscr{G}, \succ)$ is a totally convex structure, then* 3*-acyclicity implies acyclicity.*

Proof. Suppose that $(\mathscr{G}, \succ)$ is a totally convex structure, and that $\succ$ is 3-acyclic. We show acyclicity of $\succ$ by induction. Assume that $\succ$ is k-acyclic for $k = 3, \ldots, n-1$. We are to show that it is n-acyclic. Let $p_1 \succ \cdots \succ p_n$ and $p_n \succ p_1$. By the induction hypothesis, $p_1 \succeq p_3$ and $p_2 \succ p_4$. Thus by axiom TC, $p_1 \succ p_2 \bigcirc_\lambda p_3$ and $p_2 \bigcirc_\lambda p_3 \succ p_4$ for all $0 < \lambda < 1$. Since $p_1 \succ p_2 \bigcirc_\lambda p_3 \succ p_4 \succ \cdots \succ p_n \succ p_1$, $(n-1)$-acyclicity is violated. □

For a subset Q of $\mathscr{G}$ and $p \in \mathscr{G}$, we shall write $p \sim$ (respectively, $\succ$)Q when $p \sim$ (respectively, $\succ$)q for all $q \in Q$, and $Q \succ p$ when $q \succ p$ for all $q \in Q$. Let $\mathscr{G}^0 = \{p \in \mathscr{G} : p \sim \mathscr{G}\}$. A gamble in $\mathscr{G}^0$ is said to be *universally indifferent*. Given a finite subset $\{p_1, \ldots, p_{n+1}\}$ of $\mathscr{G}$, let the convex hull be defined by

$$\begin{aligned} &H(\{p_1, \ldots, p_{n+1}\}) \\ &\quad = \{(\cdots(p_1 \bigcirc_{\lambda_1} p_2) \bigcirc_{\lambda_2} p_3) \cdots) \bigcirc_{\lambda_n} p_n : 0 \leq \lambda_i \leq 1 \text{ for } i = 1, \ldots, n\}. \end{aligned}$$

We say that a convex hull $H(\{p_1, \ldots, p_n\})$ contains a cyclic preference (or simply a cycle) if there are p, q, r in the hull such that $p \succ q$, $q \succ r$, and $r \succ p$.

Lemma 3.2 *Suppose that $(\mathscr{G}, \succ)$ is a 3-acyclic, totally convex structure. For all $p, q, r \in \mathscr{G}$, if $p \sim \{q, r\}$ and $q \succ r$, then $p \in \mathscr{G}^0$.*

Proof. Suppose that the hypotheses of the lemma hold. Assume that p is not a universally indifferent gamble. Then there is an $s \in \mathscr{G}$ such that either $p \succ s$ or $s \succ p$. We are to show that $H(p, q, r, s)$ has a cycle, contradicting 3-acyclicity.

We have the following five cases to examine:

Case 1. either $s \succ \{p, q, r\}$ or $\{p, q, r\} \succ s$;
Case 2. either $q \succ s \succ p$ or $p \succ s \succ r$;
Case 3. either $\{q, r\} \sim s \succ p$ or $p \succ s \sim \{q, r\}$;
Case 4. either $q \sim s \succ \{p, r\}$ or $\{p, q\} \succ s \sim r$;
Case 5. either $r \succ s \succ p$ and $s \sim q$, or $p \succ s \succ q$ and $s \sim r$.

Case 1. Assume $s \succ \{p, q, r\}$. The other case can be similarly proved. By C and TC, $q \sim s \bigcirc_\alpha r$ for a unique $0 < \alpha < 1$. Take any $0 < \beta < \alpha$, so that $q \succ s \bigcirc_\beta r$. Since $s \succ s \bigcirc_\beta r \succ p$, axioms C and TC give $s \bigcirc_\beta r \succ s \bigcirc_\gamma p$ for small $\gamma > 0$. By TC, $s \bigcirc_\gamma p \succ q$. Hence $H(\{p, q, r, s\})$ contains a cycle.

Case 2. Assume $q \succ s \succ p$. A similar proof applies to the other possibility. By C and TC, $s \succ q \bigcirc_\alpha p$ and $r \bigcirc_\alpha q \succ s$ for small $\alpha > 0$. Since, by TC, $q \sim q \bigcirc_\alpha p \succ r$, axiom TC gives $q \bigcirc_\alpha p \succ q \bigcirc_\lambda r$ for all $0 < \lambda < 1$. Hence $H(\{p, q, r, s\})$ contains a cycle.

Case 3. Assume that $\{q, r\} \sim s \succ p$. The other case is similar. By TC, $s \bigcirc_\alpha q \succ r$ for any $0 < \alpha < 1$, so $s \bigcirc_\alpha q \succ q \bigcirc_\beta r$ for any $0 < \beta < 1$. It follows from Case 1 that $H(\{p, r, s \bigcirc_\alpha q, q \bigcirc_\beta r\})$ has a cycle.

Case 4. Assume that $q \sim s \succ \{p, r\}$. The other case is similar. By TC, $s \succ q \bigcirc_\alpha r$ for any $0 < \alpha < 1$. Thus by Case 1, $H(\{p, r, s, q \bigcirc_\alpha r\})$ has a cycle.

Case 5. Assume that $r \succ s \succ p$ and $s \sim q$. The other case is similar. By TC, $q \bigcirc_\alpha r \succ s$ for any $0 < \alpha < 1$. By Case 2, $H(\{p, r, s, q \bigcirc_\alpha r\})$ has a cycle. □

The main result of this section is given by the following theorem, which says that if there is no universally indifferent gamble, then the transitive indifference can be replaced by 3-acyclicity in Fishburn's transitive convex representation theorem.

Theorem 3.1 *$(\mathscr{G}, \succ)$ is a 3-acyclic, countably bounded, totally convex* *structure, and* $\mathscr{G}^0 = \emptyset$ *if and only if* $(\mathscr{G}, \succ)$ *has a transitive convex representation.*

Proof. Suppose that $\mathscr{G}^0 = \emptyset$. Then by Lemma 3.2, 3-acyclicity implies that $\sim$ is transitive. Therefore the conclusion of the theorem follows from Theorem 1 in Fishburn (1983). A transitive convex representation requires that $\succ$ is a weak order. Since $\mathscr{G}^0 = \mathscr{G}_{\max} \cap \mathscr{G}_{\min}$, $\mathscr{G}^0 = \emptyset$ is necessary for the representation. $\square$

4 Separable SSB Representations

A linear functional on $\mathscr{G}$ is a real valued function u for which, for all $p, q \in \mathscr{G}$ and all $0 < \lambda < 1$,

$$u(p \bigcirc_\lambda q) = \lambda u(p) + (1 - \lambda) u(q).$$

We say that $(\mathscr{G}, \succ)$ has a *separable SSB representation* (u, w) if u and w are are linear functionals on $\mathscr{G}$ such that, for all $p, q \in \mathscr{G}$,

$$p \succ q \Longleftrightarrow u(p)w(q) > u(q)w(p).$$

If we let $\phi(p, q) = u(p)w(q) - u(q)w(p)$ for all $p, q \in \mathscr{G}$, then ϕ is an SSB functional on $\mathscr{G} \times \mathscr{G}$. Therefore, if $(\mathscr{G}, \succ)$ has a separable SSB representation, then it is a symmetric totally convex structure.

Given a separable SSB representation (u, w), let $\mathscr{G}_w = \{p \in \mathscr{G} : w(p) > 0\}$. An interesting feature of the representation (u, w) is that, for all $p, q \in \mathscr{G}_w$ and all $0 < \lambda < 1$,

$$\begin{aligned} p \succ q \quad &\Leftrightarrow \quad v(p) > v(q), \\ v(p \bigcirc_\lambda q) \quad &= \quad \frac{\lambda w(p) v(p) + (1 - \lambda) w(q) v(q)}{\lambda w(p) + (1 - \lambda) w(q)}, \end{aligned}$$

where $v(p) = u(p)/w(p)$ for all $p \in \mathscr{G}_w$.

A separable SSB representation (u, w) is said to be a *weighted linear representation* if $w \geq 0$, and $w(p) > 0$ whenever $(\mathscr{P}, \succ)$ is open or closed. If $(\mathscr{G}, \succ)$ is half-open, the representation requires $w(p) > 0$ for every $p \in \mathscr{G}^*$. However, some half-open situations force w to vanish on the one of $\mathscr{G}_{\max}$ and $\mathscr{G}_{\min}$ that is not empty. In all cases, w must be positive throughout $\mathscr{G}^*$. Fishburn (1983) showed that $(\mathscr{G}, \succ)$ is an indifference-transitive, symmetric, and totally convex structure if and only if $(\mathscr{G}, \succ)$ has a weighted linear representation (u, w).

The aim of this section is to show a necessary and sufficient axiom for the separable SSB representation and to explore the effects of acyclicity on the representation. The following axiom, which applies to all $p, q, r \in \mathscr{G}$, is necessary and sufficient for separable SSB representability of the symmetric, totally convex preferences.

Axiom UI (universal indifference) *If* $p \sim q$*,* $q \sim r$*, and* $p \succ r$*, then* $q \sim \mathscr{G}$*.*

The axiom idnetifies a condition that the set of universal indifferent gambles satisfies. It says that if a gamble q is indifferent to both of gambles p and r, and p and r are not indifferent, then q must be universally indifferent.

A characterization of the separable SSB representation is given by the following theorem.

Theorem 4.1 *Suppose that* $(\mathscr{G}, \succ)$ *is a nonempty, symmetric totally convex structure. Then the universal indifference axiom* UI *holds if and only if* $(\mathscr{G}, \succ)$ *has a separable SSB representation.*

Proof. Let (u, w) be a separable SSB representation for $\succ$ on $\mathscr{P}$. The necessities of the axioms C, TC, and S easily obtain. To see necessity of axiom UI, assume that $p \sim q$, $q \sim r$, and $p \succ r$. Then we have

$$\begin{aligned} u(p)w(q) &= u(q)w(p), \\ u(q)w(r) &= u(r)w(q), \\ u(p)w(r) &> u(r)w(p). \end{aligned}$$

If $w(r) = 0$ or $w(p) = 0$, then $w(q) = u(q) = 0$, so that $q \sim \mathscr{G}$. Thus we sssume that $w(r) \neq 0$ and $w(p) \neq 0$. If $w(q) \neq 0$, then the above three equations give

$$\frac{u(q)}{w(q)} = \frac{u(p)}{w(p)} > \frac{u(r)}{w(r)} = \frac{u(q)}{w(q)},$$

a contradiction. Therefore, $w(q) = 0$. The second equation gives $u(q) = 0$, so that $q \sim \mathscr{G}$.

Next we show sufficiencies of axioms C, TC, S, and UI. Suppose that axioms C, TC, S, and UI hold. Then there is an SSB functional ϕ on $\mathscr{G} \times \mathscr{G}$ such that, for all $p, q \in \mathscr{G}$,

$$p \succ q \Longleftrightarrow \phi(p, q) > 0.$$

For all $p, q, r, s \in \mathscr{G}$, let

$$\Phi(p, q, r, s) = \phi(p, q)\phi(r, s) + \phi(p, r)\phi(s, q) + \phi(p, s)\phi(q, r).$$

It suffices to show that $\Phi(p, q, r, s) = 0$ for all $p, q, r, s \in \mathscr{G}$.

Noting that, for every permutation σ on $\{1, 2, 3, 4\}$,

$$\Phi(p_1, p_2, p_3, p_4) = (-1)^{\delta(\sigma)}\Phi(p_{\sigma(1)}, p_{\sigma(2)}, p_{\sigma(3)}, p_{\sigma(4)}),$$

where $\delta(\sigma) = 1$ if σ is an odd permutation, and 0 otherwise, we have four cases to examine:

Case 1. $p \sim q$ and $q \sim r$.
Case 2. $p \sim q$, $p \succ r$, and $r \succ q$.
Case 3. $p \sim q$ and either $r \succ \{p, q\}$ or $\{p, q\} \succ r$.
Case 4. neither two of p, q, r, and s are indifferent.

Case 1. Assume that $p \sim q$ and $q \sim r$. Then $\phi(p, q) = \phi(q, r) = 0$. If $p \sim r$, then $\phi(p, r) = 0$, so that $\Phi(p, q, r, s) = 0$. If $\neg(p \sim r)$, then, by UI, $s \sim q$, so $\phi(s, q) = 0$. Thus $\Phi(p, q, r, s) = 0$.

Case 2. Assume that $p \sim q$, $p \succ r$, and $r \succ q$. By C and TC, $r \sim p \bigcirc_\alpha q$ for a unique $0 < \alpha < 1$. Since, by TC, $p \sim p \bigcirc_\alpha q$, it follows from UI that $p \bigcirc_\alpha q \sim s$.

Since $p \bigcirc_\alpha q \sim \{r, s\}$, we have

$$\begin{aligned} \alpha\phi(p, r) + (1 - \alpha)\phi(q, r) &= 0, \\ \alpha\phi(p, s) + (1 - \alpha)\phi(q, s) &= 0, \end{aligned}$$

which give $\phi(p, r)\phi(q, s) = \phi(p, s)\phi(q, r)$. Hence $\Phi(p, q, r, s) = 0$, since $\phi(p, q) = 0$.

Case 3. Assume that $p \sim q$ and $r \succ \{p, q\}$. The proof for the other case is similar. We have two subcases to examine: $\{p, q\} \succ s$; $s \succ \{p, q\}$.

Assume first that $\{p, q\} \succ s$. Then by C, TC, and UI, $\{p, q\} \sim r \bigcirc_\alpha s$ for a unique $0 < \alpha < 1$. A similar analysis to Case 2 gives $\Phi(p, q, r, s) = 0$.

Assume next that $s \succ \{p, q\}$. Suppose that $s \succ r$. When $r \succ s$, the proof is similar. Then by C and TC, $r \sim \{s \bigcirc_\alpha p, s \bigcirc_\beta q\}$ for unique $\alpha, \beta \in (0, 1)$. Thus we have

$$\begin{aligned} \alpha\phi(r, s) + (1 - \alpha)\phi(r, p) &= 0, \\ \beta\phi(r, s) + (1 - \beta)\phi(r, q) &= 0, \end{aligned}$$

which give

$$\alpha^*\phi(r, p) = \beta^*\phi(r, q),$$

where $\lambda^* = (1 - \lambda)/\lambda$ for any $0 < \lambda < 1$.

By UI, $s \bigcirc_\alpha p \sim s \bigcirc_\beta q$. Thus we have

$$\beta^*\phi(s, q) = \alpha^*\phi(s, p),$$

which together with the last equation of the preceding paragraph gives

$$\phi(r, p)\phi(s, q) = \phi(r, q)\phi(s, p).$$

Hence $\Phi(p, q, r, s) = 0$.

Suppose next that $s \sim r$. By TC, $s \succ r \bigcirc_\lambda q \succ \{p, q\}$ for all $0 < \lambda < 1$. Therefore, it follows from the preceding two paragraphs that $\Phi(p, q, r \bigcirc_\lambda$

$q, s) = 0$, so that

$$\begin{aligned} & \Phi(p, q, r \bigcirc_\lambda q, s) \\ = \quad & \phi(p, r \bigcirc_\lambda q)\phi(s, q) + \phi(p, s)\phi(q, r \bigcirc_\lambda q) \\ = \quad & \lambda\Phi(p, q, r, s). \end{aligned}$$

Hence $\Phi(p, q, r, s) = 0$.

Case 4. Assume that neither two of p, q, r, and s are indifferent. Let $p \succ q \succ r$ and $p \succ r$ with no loss of generality. By C and TC, $q \sim p \bigcirc_\alpha r$ for a unique $0 < \alpha < 1$. Then $\alpha\phi(p, q) = (1 - \alpha)\phi(q, r)$. By Case 3, $\Phi(q, p \bigcirc_\alpha r, p, s) = \Phi(q, p \bigcirc_\alpha r, p, r) = 0$, each of which gives

$$\begin{aligned} \phi(q, p)\phi(s, p \bigcirc_\alpha r) &= \phi(q, s)\phi(p, p \bigcirc_\alpha r), \\ \phi(q, p)\phi(r, p \bigcirc_\alpha r) &= \phi(q, r)\phi(p, p \bigcirc_\alpha r). \end{aligned}$$

Since the both sides of the second equality are not zero, we divide the first by the second, and use bilinearity of ϕ to get

$$[\alpha\phi(s, p) + (1 - \alpha)\phi(s, r)]\phi(q, r) = \alpha\phi(q, s)\phi(r, p).$$

Substituting $(1 - \alpha)\phi(q, r) = \alpha\phi(p, q)$ for the above, skew-symmetry of ϕ yields $\Phi(p, q, r, s) = 0$. □

We note that a separable SSB representation may allow for a preference cycle as shown by the following example.

Example 4.1. Let (u, w) be a separable SSB representation for $(\mathscr{G}, \succ)$. For a positive number ϵ, let $(u(p), w(p)) = (-2\epsilon, -\epsilon)$, $(u(q), w(q)) = (\epsilon, -\epsilon)$, and $(u(r), w(r)) = (\epsilon, \epsilon)$. Then

$$\begin{aligned} u(p)w(q) - u(q)w(p) &= 3\epsilon^2, \\ u(q)w(r) - u(r)w(q) &= 2\epsilon^2, \\ u(r)w(p) - u(p)w(r) &= \epsilon^2, \end{aligned}$$

so that $p \succ q$, $q \succ r$, and $r \succ p$.

The following theorem that acyclicity of strict preference forces$w$ in a separable SSB representation (u, w) to be nonnegative. Note by Lemma 3.1 that 3-acyclicity implies acyclicity.

Theorem 4.2 *Suppose that $(\mathscr{G}, \succ)$ is a nonempty, symmetric, totally convex structure. Then $\succ$ is 3-acyclic on $\mathscr{G}$ if and only if $(\mathscr{G}, \succ)$ has a separable SSB representation (u, w) with $w \geq 0$.*

Proof. Suppose first that $(\mathscr{G}, \succ)$ has a separable SSB represenation (u, w) with $w \geq 0$. Necessity of C, TC, and S easily follows. To show necessity of

3-acyclicity of $\succ$, assume that $p \succ q$, $q \succ r$, and $r \succ p$. Then we derive a contradiction. By the representation, we have

$$\begin{aligned} u(p)w(q) &> u(q)w(p), \\ u(q)w(r) &> u(r)w(q), \\ u(r)w(p) &> u(p)w(r). \end{aligned}$$

If $w(q) = 0$, then the first inequality gives $w(p) > 0$ and $u(q) < 0$, and the second inequality gives $w(r) > 0$ and $u(q) > 0$, a contradiction. When $w(r) = 0$ or $w(p) = 0$, we obtain similar contradictions. Therefore, we must have $w(p) > 0$, $w(q) > 0$, and $w(r) > 0$. Hence the above three inequalities respectively give

$$\frac{u(p)}{w(p)} > \frac{u(q)}{w(q)}, \quad \frac{u(q)}{w(q)} > \frac{u(r)}{w(r)}, \quad \frac{u(r)}{w(r)} > \frac{u(p)}{w(p)},$$

a contradiction. Hence 3-acyclicity of $\succ$ must hold.

Suppose next that axioms C, TC, and S hold, and that $\succ$ is 3-acyclic. By Lemma 2.2, C, TC, and 3-acyclicity of $\succ$ imply UI. Thus it follows from Theorem 3.1 that $(\mathscr{G}, \succ)$ has a separable SSB representaion (u, w). We are to show that w can be transformed to a linear functional $\sigma \geq 0$ on $\mathscr{G}$ so that (u, σ) is also a separable SSB representation for $(\mathscr{G}, \succ)$.

Let $\mathbb{R}^2$ be two-dimensional Euclidean space. The range of the separable SSB representation (u, w) is defined by

$$\mathscr{R}(u, w) = \{(u(p), w(p)) \in \mathbb{R}^2 : p \in \mathscr{P}\}.$$

Let $\mathscr{R}^0(u, w)$ be the set of all interior points of $\mathscr{R}(u, w)$. We show that $(0, 0) \notin \mathscr{R}^0(u, w)$. Suppose on the contrary that $(0, 0) \in \mathscr{R}^0(u, w)$. Then there exists an open disk D centered at the origin $(0, 0)$ such that $D \subset \mathscr{R}^0(u, w)$. As in Example 4.1, we take $\epsilon > 0$ so small that $(-2\epsilon, -\epsilon)$, $(\epsilon, -\epsilon)$ and (ϵ, ϵ) are contained in the disk D. Since there are gambles p, q, r such that $(u(p), w(p)) = (-2\epsilon, -\epsilon)$, $(u(q), w(q)) = (\epsilon, -\epsilon)$ and $(u(r), w(r)) = (\epsilon, \epsilon)$, those gambels form a preference cycle, $p \succ q \succ r \succ p$, contradicting 3-acyclicity of $\succ$.

Since $(0, 0) \notin \mathscr{R}^0(u, w)$, it is easy to see that there is a straight line L through the origin that divides $\mathbb{R}^2$ into two regions, one of which includes $\mathscr{R}^0(u, w)$. Suppose that L is the y-axis. If $u \geq 0$, then $(-u - w, u)$ is a separable SSB representation. To see this, we note

$$\begin{aligned} p \succ q \iff & u(p)w(q) > u(q)w(p) \\ \iff & -u(q)w(p) > -u(p)w(q) \\ \iff & (-u(p) - w(p))u(q) > (-u(q) - w(q))u(p). \end{aligned}$$

If $u \leq 0$, then so is similarly $(-u - w, -u)$.

Suppose next that the y-axis cannot be L. Then there is a real number a such that either $w(p) \geq au(p)$ for all $p \in \mathscr{G}$ or $w(p) \leq au(p)$ for all $p \in \mathscr{G}$. If $w(p) \geq au(p)$ for all $p \in \mathscr{G}$, then let $\omega(p) = w(p) - au(p)$, so that, for all $p, q \in \mathscr{G}$,

$$\begin{aligned} p \succ q &\iff u(p)w(q) > u(q)w(p) \\ &\iff u(p)(\omega(q) + au(q)) > u(q)(\omega(p) + au(p)) \\ &\iff u(p)\omega(q) > u(q)\omega(p). \end{aligned}$$

Hence (u, ω) with $\omega \geq 0$ is a separable SSB repesentation for $(\mathscr{G}, \succ)$.

If $w(p) \leq au(p)$ for all $p \in \mathscr{G}$, then let $u' = -u$ and $w' = -w$, so that (u', w') is also a separable SSB representation for $(\mathscr{G}, \succ)$ and $w' \geq au'$. Hence it follows from the preceding paragraph that $(u', w' - au')$ with $w' - au' \geq 0$ is a separable SSB representation for $(\mathscr{G}, \succ)$. □

Noting that if (u, w) is a separabble SSB representation for $(\mathscr{G}, \succ)$, and $\mathscr{G}^0 \neq \emptyset$, then $(u(p), w(p)) = (0, 0)$ for no $p \in \mathscr{G}$, an immdediate corollary of Theorem 4.2 is given by

Corollary 4.1 *$(\mathscr{G}, \succ)$ is a nonempty, symmetric,* 3*-acyclic, totally convex structure, and $\mathscr{G}^0 = \emptyset$ if and only if $(\mathscr{G}, \succ)$ has a weighted linear representation.*

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