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A Complexity Bound of a Predictor-Corrector Smoothing Method Using CHKS-Functions for Monotone LCP

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A Complexity Bound of a Predictor-Corrector Smoothing Method Using CHKS-Functions for Monotone LCP

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Abstract

We propose a new smoothing method using CHKS-functions for solving linear complementarity problems. While the algorithm in [6] uses a quite large neighborhood, our algorithm generates a sequence in a relatively narrow neighborhood and employs predictor and corrector steps at each iteration. A complexity bound for the method is also provided under the assumption that the problem is monotone and has a feasible interior point. As a result, the bound can be improved compared to the one in [6].

1 Introduction

This paper deals with the standard linear complementarity problem (LCP):

$$\begin{aligned} \text{LCP:} \quad \text{Find} \quad & (\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{2n} \\ \text{s.t.} \quad & \boldsymbol{y} = M\boldsymbol{x} + \boldsymbol{q}, && (1) \\ & (\boldsymbol{x}, \boldsymbol{y}) \geq \boldsymbol{0}, && (2) \\ & x_i y_i = 0 \quad (i = 1, \ldots, n), && (3) \end{aligned}$$

where M is an $n \times n$ matrix and $\boldsymbol{q}$ is an n-dimensional vector.

We impose the following assumption on the LCP.

Assumption 1.1 *(1) The LCP is monotone, i.e., the matrix M is positive semidefinite.*

(2) The LCP has a feasible interior point, i.e., there exists a point $(\mathring{\boldsymbol{x}}, \mathring{\boldsymbol{y}}) \in \mathbb{R}^{2n}$ satisfying $\mathring{\boldsymbol{y}} = M \mathring{\boldsymbol{x}} + \boldsymbol{q}$ and $(\mathring{\boldsymbol{x}}, \mathring{\boldsymbol{y}}) > 0$.

The basic idea of the smoothing method for the LCP is to rearrange or approximate the system (1) – (3) using some smooth functions so that Newton-type methods can be adopted. Mangasarian[12] first showed a class of such functions, and since then various types of functions and algorithms have been provided (See e.g., [2, 1, 3, 4, 5, 7, 6, 8, 9, 10,

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13, 14, 15, 16]). While it is still an open problem whether we can construct a polynomial-time smoothing algorithm, a complexity bound was shown in [6] for an algorithm using the so-called Chen-Harker-Kanzow-Smale function (CHKS function), which is given by

$$\phi(\mu, a, b) := a + b - \sqrt{(a-b)^2 + 4\mu^2} \tag{4}$$

where $a, b \in \mathbb{R}$ and $\mu > 0$. Based on this result, we provide a new type of algorithm which employs a relatively narrow neighborhood compared to the one in [6] and whose iteration consists of two steps, predictor step and corrector step. As a result, we obtain a better complexity bound than the one in [6].

It should be noted that another analysis has been done for the case where the matrix M is positive definite by Burke and Xu [3]. Their complexity bound deeply depends on the condition number of M, while the size of the generated sequence plays an important role in our analysis.

This paper is organized as follows. The new algorithm is described in Section 2. Some basic results are collected in Section 3. Using them, a complexity bound is derived in Section 4. Concluding remarks are given in Section 5.

We define some symbols used throughout this paper. N means the index set $\{1, \ldots, n\}$. Symbols $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ denote the n-dimensional nonnegative orthant and the n-dimensional positive orthant, respectively. $\boldsymbol{e}$ denotes the vector with all components equal to one. For a given vector $\boldsymbol{x} \in \mathbb{R}^n$, $\mathrm{vec}\{x_i\}$ and $\mathrm{diag}\{x_i\}$ represent the n-dimentional vector whose i-th element is i-th component of $\boldsymbol{x}$ and $n \times n$-diagonal matrix whose i-th diagonal elements is i-th component of $\boldsymbol{x}$, respectively. For example,

$$\mathrm{vec}\{x_i y_i\} = \begin{pmatrix} x_1 y_1 \\ \vdots \\ x_n y_n \end{pmatrix}, \quad \mathrm{diag}\{x_i\} = \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix}.$$

2 A predictor-corrector smoothing method

Let us define the function $\Phi : \mathbb{R}_{++} \times \mathbb{R}^{2n} \to \mathbb{R}^n$ as $\Phi(\mu, \boldsymbol{x}, \boldsymbol{y}) := \mathrm{vec}\{\phi(\mu, x_i, y_i)\}$. Let $\boldsymbol{e}$ denote the vector whose components are 1s. For a given nonnegative vector $\boldsymbol{h} \geq \boldsymbol{e}$, let us consider the system

$$\boldsymbol{y} = M\boldsymbol{x} + \boldsymbol{q}, \quad \Phi(\mu, \boldsymbol{x}, \boldsymbol{y}) = -\mu \boldsymbol{h}.$$

Suppose that the LCP satisfies Assumption 1.1. In the paper [7], the authors show that the above system has a unique solution $(\boldsymbol{x}(\mu), \boldsymbol{y}(\mu))$ for every $\mu > 0$ and $\{(\boldsymbol{x}(\mu), \boldsymbol{y}(\mu)) : \mu > 0\}$ forms a 1-dimensional trajectory whose accumulation point as $\mu \to 0$ is always a solution of the LCP. Let us define the set P as follows.

$$P := \{(\mu, \boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}_+ \times \mathbb{R}^{2n} \mid \boldsymbol{y} = M\boldsymbol{x} + \boldsymbol{q}, \Phi(\mu, \boldsymbol{x}, \boldsymbol{y}) \leq \boldsymbol{0}\}$$

Our algorithm traces the trajectory $\{(\boldsymbol{x}(\mu), \boldsymbol{y}(\mu)) : \mu > 0\}$ using the following two neighborhoods: for given α and β satisfying $0 < \beta < \alpha < 1$ and $\alpha + \beta < 1$, let us define the inner neighborhood,

$$\mathcal{N}(\alpha) := P \cap \{(\mu, \boldsymbol{x}, \boldsymbol{y}) \mid ||\Phi(\mu, \boldsymbol{x}, \boldsymbol{y}) + \mu \boldsymbol{h}|| \ \leq \ \alpha\mu\},$$

and the outer neighborhood

$$\mathcal{N}(\alpha + \beta) := P \cap \{(\mu, \boldsymbol{x}, \boldsymbol{y}) \mid ||\Phi(\mu, \boldsymbol{x}, \boldsymbol{y}) + \mu \boldsymbol{h}|| \ \leq \ (\alpha + \beta)\mu\}.$$

Here the condition $\Phi(\mu, \boldsymbol{x}, \boldsymbol{y}) \le \boldsymbol{0}$ plays a crucial role when we derive the boundedness of the generated sequence, and by this reason, the assumption $\boldsymbol{h} \ge \boldsymbol{e}$ is imposed. In fact, if $(\mu, x, y) \in \mathcal{N}(\alpha + \beta)$ then

$$-(\alpha+\beta)\mu \le \phi(\mu, x_i, y_i) + \mu h_i \le (\alpha+\beta)\mu$$

for every i, which implies that $\phi(\mu, x_i, y_i) \le 0 (i \in N)$. The relationship $\beta < \alpha$ is required only for ease of description in the further discussions.

For given $\alpha \in (0,1)$ and $\boldsymbol{h} \ge \boldsymbol{0}$, suppose that we obtain a point $(\bar{\mu}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \mathcal{N}(\alpha)$. This assumption is not strict: For any $\boldsymbol{x}^0 \in \mathbb{R}^n$,

- set $\boldsymbol{y}^0 := M\boldsymbol{x}^0 + \boldsymbol{q}$,
- choose a μ^0 so that $\mu^0 > \max\{0, x_i^0 y_i^0 \ (i \in N)\}$.

Then we can easily find that $(\mu^0, \boldsymbol{x}^0, \boldsymbol{y}^0) \in P$. Moreover,

- set $\boldsymbol{h} := -\Phi(\mu^0, \boldsymbol{x}^0, \boldsymbol{y}^0)/\mu^0$,

Then the point $(\bar{\mu}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ lies in the inner neighborhood $\mathcal{N}(\alpha)$.

At each iteration, we reduce the value of μ by a constant ratio $1 - \bar{\xi}$. As a result, the point $(\mu^{k+1}, \boldsymbol{x}^k, \boldsymbol{y}^k)$ may not lie in the inner neighborhood $\mathcal{N}(\alpha)$, but we will see that it still lies in $\mathcal{N}(\alpha+\beta)$(see Lemma 4.1). To confine the sequence in the inner neighborhood $\mathcal{N}(\alpha)$, we consider the following system to reduce the value of $\|\Phi(\mu^{k+1}, \boldsymbol{x}, \boldsymbol{y}) + \mu^{k+1}\boldsymbol{h}\|$:

$$\left\{ \begin{array}{rcl} \mu &=& \sigma_\mu \bar{\mu}, \\ \boldsymbol{y} - M\boldsymbol{x} - \boldsymbol{q} &=& \boldsymbol{0}, \\ \phi(\mu, x_i, y_i) + \mu h_i &=& \sigma_\phi(\bar{\phi}_i + \bar{\mu} h_i) \quad (i \in N). \end{array} \right. \tag{5}$$

where $\sigma_\mu \in [0,1]$ and $\sigma_\phi \in [0,1]$ are parameters which control the target points of μ and ϕ_i $(i \in N)$ for approximation. Let $(\bar{\mu}.\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) := (\mu^k, \boldsymbol{x}^k, \boldsymbol{y}^k)$. We employ the Newton direction $(\Delta x, \Delta y)$ for approximating the solution of the above system, which is given by

$$\begin{pmatrix} 1 & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & -M & I \\ \boldsymbol{d}_\mu & D_x & D_y \end{pmatrix} \begin{pmatrix} \Delta\mu \\ \Delta\boldsymbol{x} \\ \Delta\boldsymbol{y} \end{pmatrix} = \begin{pmatrix} -(1-\sigma_\mu)\bar{\mu} \\ \boldsymbol{0} \\ -(1-\sigma_\phi)(\bar{\Phi} + \bar{\mu}\boldsymbol{h}) \end{pmatrix} \tag{6}$$

where

$$\begin{array}{rcl} \boldsymbol{d}_\mu &:=& \mathrm{vec}\{h_i - \dfrac{4\bar{\mu}}{\sqrt{(\bar{x}_i - \bar{y}_i)^2 + 4\bar{\mu}^2}}\}, \\ D_x &:=& \mathrm{diag}\{1 - \dfrac{\bar{x}_i - \bar{y}_i}{\sqrt{(\bar{x}_i - \bar{y}_i)^2 + 4\bar{\mu}^2}}\}, \\ D_y &:=& \mathrm{diag}\{1 + \dfrac{\bar{x}_i - \bar{y}_i}{\sqrt{(\bar{x}_i - \bar{y}_i)^2 + 4\bar{\mu}^2}}\}, \\ \bar{\Phi} &:=& \Phi(\bar{\mu}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \ := \ \mathrm{vec}\{\phi(\bar{\mu}, \bar{x}_i, \bar{y}_i)\}. \end{array}$$

Note that the system (6) can be reduced to

$$\Delta\mu = -(1-\sigma_\mu)\bar{\mu}, \tag{7}$$

$$\begin{pmatrix} -M & I \\ D_x & D_y \end{pmatrix} \begin{pmatrix} \Delta\boldsymbol{x} \\ \Delta\boldsymbol{y} \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} \\ (1-\sigma_\mu)\bar{\mu}\boldsymbol{d}_\mu - (1-\sigma_\phi)(\bar{\Phi} + \bar{\mu}\boldsymbol{h}) \end{pmatrix}. \tag{8}$$

By setting $\sigma_\mu = 1$ and $\sigma_\phi = 0$, after solving a finite series of the above systems, we can find a new point $(\mu^{k+1}, \boldsymbol{x}^{k+1}, \boldsymbol{y}^{k+1}) \in \mathcal{N}(\alpha)$ (see Lemma 4.2).

Here, we describe our algorithm in detail.

Algorithm.

Step 0 : **Initialization**

Set $\epsilon > 0$, $k := 0$,
Set $1 > \alpha > \beta > 0$ s.t. $\alpha + \beta < 1$ and $\boldsymbol{h} \in \mathbb{R}^n_+$,
Choose $\boldsymbol{x}^0 \in \mathbb{R}^n$ and calculate $\boldsymbol{y}^0 := M\boldsymbol{x}^0 + \boldsymbol{q}$,
Choose $\mu^0 > 0$ s.t. $(\mu^0, \boldsymbol{x}^0, \boldsymbol{y}^0) \in \mathcal{N}(\alpha)$.
Let us define

$$\zeta := \left(\|\boldsymbol{h}\| + 2\sqrt{n}\right)^2 + 2\sqrt{n}, \tag{9}$$

$$\kappa := \sqrt{n}\|\boldsymbol{h}\| + (\alpha+\beta)^2, \tag{10}$$

$$\bar{\xi} := n\left\{\frac{-\kappa + \sqrt{\kappa^2 + \{\zeta - (\alpha+\beta)^2\}\{(\alpha+\beta)^2 - \alpha^2\}}}{\zeta - (\alpha+\beta)^2}, \frac{1}{2}\right\} < 1. \tag{11}$$

Step 1 : **Stopping Criteria**

If $\mu^k < \epsilon$, then stop.

Step 2 : **Predictor Step**

Let $\mu^{k+1} = (1 - \bar{\xi})\mu^k$.

Step 3 : **Corrector Step**

Set $p := 0$, $(\hat{\mu}, \hat{\boldsymbol{x}}^0, \hat{\boldsymbol{y}}^0) := (\mu^{k+1}, \boldsymbol{x}^k, \boldsymbol{y}^k)$, and let $\hat{\Phi}^0 := \Phi(\hat{\mu}, \hat{\boldsymbol{x}}^0, \hat{\boldsymbol{y}}^0)$.

Step 3.1 : If $(\hat{\mu}, \hat{\boldsymbol{x}}^p, \hat{\boldsymbol{y}}^p) \in \mathcal{N}(\alpha)$, then go to **Step 4**.
Calculate the Newton direction $(\Delta\hat{\mu}^p, \Delta\hat{\boldsymbol{x}}^p, \Delta\hat{\boldsymbol{y}}^p)$ by solving the system (6) with $\sigma_\mu = 1$, $\sigma_\phi = 0$ and $(\bar{\mu}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) = (\hat{\mu}, \hat{\boldsymbol{x}}^0, \hat{y}^0)$.
Set the step size

$$\theta^p := \min\left\{1, \frac{\hat{\mu}||\hat{\Phi}^p + \hat{\mu}\boldsymbol{h}||}{2\{||\Delta\hat{\boldsymbol{x}}^p||^2 + ||\Delta\hat{\boldsymbol{y}}^p||^2\}}\right\}, \tag{12}$$

Calculate $(\hat{\boldsymbol{x}}^{p+1}, \hat{\boldsymbol{y}}^{p+1}) := (\hat{\boldsymbol{x}}^p, \hat{\boldsymbol{y}}^p) + \theta^p(\Delta\hat{\boldsymbol{x}}^p, \Delta\hat{\boldsymbol{y}}^p)$,
Let $\hat{\Phi}^{p+1} := \Phi(\hat{\mu}, \hat{\boldsymbol{x}}^{p+1}, \hat{\boldsymbol{y}}^{p+1})$.
Set $p := p + 1$ and go to **Step 3.1**.

Step 4 :

$(\boldsymbol{x}^{k+1}, \boldsymbol{y}^{k+1}) := (\hat{\boldsymbol{x}}^p, \hat{\boldsymbol{y}}^p)$. Calculate Φ^{k+1} as $\Phi(\mu^{k+1}, \boldsymbol{x}^{k+1}, \boldsymbol{y}^{k+1})$.
Set $k := k + 1$ and go to **Step 1**.

Remark.

1. We can start from any initial point $\boldsymbol{x}^0 \in \mathbb{R}^n$.

2. Since $\zeta \geq 6$ and $\alpha + \beta < 1$, $\bar{\xi}$ is given by a positive real number.

3. The following proposition ensures that Step 3.1 is well-defined.

   Proposition 2.1 *(Lemma 4.1 of [11], (i) of Lemma 8.3.1 of [17]) The system (6) has a unique solution* $(\Delta\mu, \Delta\boldsymbol{x}, \Delta\boldsymbol{y})$ *whenever Assumption 1.1 holds.*

4. In practical use, we may use an inexact line search method to decide the step size θ^p in Step 3.1. The way is to minimize $||\Phi(\hat{\mu}, \hat{\boldsymbol{x}}^p + \theta^p \Delta\hat{\boldsymbol{x}}^p, \hat{\boldsymbol{y}}^p + \theta^p \Delta\hat{\boldsymbol{y}}^p) + \hat{\mu}\boldsymbol{h}||$ subject to $\Phi(\hat{\mu}, \hat{\boldsymbol{x}}^p + \theta^p \Delta\hat{\boldsymbol{x}}^p, \hat{\boldsymbol{y}}^p + \theta^p \Delta\hat{\boldsymbol{y}}^p) \leq \boldsymbol{0}$. As we will see in Lemma 4.2, the obtained step size is larger than the value in (12).

3 Some basic results

In this section, we collect some basic results concerning the CHKS-function. All of which are required in the next section for deriving a complexity bound of the algorithm.

Proposition 3.1 *Let $\phi(\mu, a, b) := a + b - \sqrt{(a-b)^2 + 4\mu^2}$ for any $\mu \geq 0$. The following results hold for every $a, b, c \in \mathbb{R}$.*

(i) *(Lemma 1.1 of [7])*

$$\phi(\mu, a, b) = c \text{ if and only if } (a - c/2, b - c/2) \geq 0 \text{ and } (a - c/2)(b - c/2) = \mu^2.$$

(ii) *(Lemma 2 of [13])*

$$\nabla^2 \phi(\mu, a, b) \;=\; -\frac{4}{(\sqrt{(a-b)^2 + 4\mu^2})^3} \begin{pmatrix} a-b \\ -\mu \\ \mu \end{pmatrix} (a-b, -\mu, \mu)\,, \tag{13}$$

i.e., ϕ is a concave function and

$$||\nabla^2 \phi(\mu, a, b)|| \;\leq\; \frac{4}{(\sqrt{(a-b)^2 + 4\mu^2})^3} \left\| \begin{pmatrix} a-b \\ -\mu \\ \mu \end{pmatrix} \right\|^2 \;\leq\; \frac{4}{\sqrt{(a-b)^2 + 4\mu^2}} \;\leq\; \frac{2}{\mu}. \tag{14}$$

(iii)

$$0 \;<\; 1 \pm \frac{a-b}{\sqrt{(a-b)^2 + 4\mu^2}} \;<\; 2 \quad \text{for } \mu > 0. \tag{15}$$

For every $(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^{2n}$ and $\mu > 0$, define $\bar{\Phi} := \text{vec}\{\phi(\bar{\mu}, \bar{x}_i, \bar{y}_i)\}$ and let

$$(\boldsymbol{x}', \boldsymbol{y}') := (\bar{\boldsymbol{x}} - \bar{\Phi}/2, \bar{\boldsymbol{y}} - \bar{\Phi}/2).$$

(i) of Proposition 3.1 implies that if $\boldsymbol{y} = M\boldsymbol{x} + \boldsymbol{q}$ then the point $(\boldsymbol{x}', \boldsymbol{y}') \in \mathbb{R}^{2n}$ is an analytical center of a perturbed LCP($\boldsymbol{v}$):

$$\begin{array}{ll} \text{Find} & (\boldsymbol{x}', \boldsymbol{y}') \in \mathbb{R}^{2n} \\ \text{s.t.} & \boldsymbol{y}' = M\boldsymbol{x}' + \boldsymbol{q}', (\boldsymbol{x}', \boldsymbol{y}') > \boldsymbol{0} \text{ and } x_i' y_i' = 0 \ (i \in N), \end{array}$$

where $\boldsymbol{q}' = \boldsymbol{q} + (M - I)\boldsymbol{v}/2$. The next proposition gives us more detailed properties of the perturbed problem LCP($\boldsymbol{v}$).

Proposition 3.2 *(Proposition 4 of [6]) Let $(\bar{\mu}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \mathbb{R}_{++} \times \mathbb{R}^{2n}$ be a point satisfying $\bar{\boldsymbol{y}} = M\bar{\boldsymbol{x}} + \boldsymbol{q}$ and let $\bar{\phi}_i = \phi(\bar{\mu}, \bar{x}_i, \bar{y}_i)$ for $i \in N$. Then the following results are true.*

(i) $\sqrt{(\bar{x}_i - \bar{y}_i)^2 + 4\bar{\mu}^2} \;=\; x'_i + y'_i \;\geq\; 2\bar{\mu} \;>\; 0$ *for* $i \in N$.

(ii) *The solution of (8) is the unique solution of the system*

$$\begin{pmatrix} -M & I \\ Y' & X' \end{pmatrix} \begin{pmatrix} \Delta \boldsymbol{x} \\ \Delta \boldsymbol{y} \end{pmatrix} \;=\; \begin{pmatrix} \boldsymbol{0} \\ \bar{\boldsymbol{d}} \end{pmatrix} \tag{16}$$

where

$$\bar{\boldsymbol{d}} \;=\; \frac{1-\sigma_\mu}{2}\{(X'+Y')\bar{\mu}\boldsymbol{h} - 4\bar{\mu}^2\boldsymbol{e}\} - \frac{1-\sigma_\phi}{2}(X'+Y')(\bar{\Phi} + \bar{\mu}\boldsymbol{h}). \tag{17}$$

(iii) *Suppose that Assumption 1.1 is satisfied. Let* $L(\lambda, \mu^0)$ *be the level set of the function* $\|\Phi(\mu, \boldsymbol{x}, \boldsymbol{y})\|$ *i.e.,*

$$L(\lambda, \mu^0) := \{(\mu, \boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}_+ \times \mathbb{R}^{2n} \mid ||\Phi(\mu, \boldsymbol{x}, \boldsymbol{y})|| \leq \lambda, \mu \in (0, \mu^0]\}. \tag{18}$$

If $(\bar{\mu}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \mathbb{R}_{++} \times \mathbb{R}^{2n}$ *lies in the set* $P \cap L(\lambda, \mu^0)$ *for some* $\lambda > 0$ *and* $\mu^0 > 0$ *then*

$$0 \;<\; \frac{\bar{\mu}^2}{3\bar{\gamma}(\lambda, \mu^0)} \;\leq\; x'_i \;\leq\; 2\bar{\gamma}(\lambda, \mu^0), \tag{19}$$

$$0 \;<\; \frac{\bar{\mu}^2}{3\bar{\gamma}(\lambda, \mu^0)} \;\leq\; y'_i \;\leq\; 2\bar{\gamma}(\lambda, \mu^0) \tag{20}$$

for $i \in N$, *where* $\bar{\gamma}(\lambda, \mu^0) \;=\; \max\{\gamma(\lambda, \mu^0), \mu^0\}$,

$$\gamma(\lambda, \mu^0) \;:=\; \frac{n(\mu^0)^2 + (\mathring{\boldsymbol{x}} + (\lambda/2)\boldsymbol{e})^T(\mathring{\boldsymbol{y}} + (\lambda/2)\boldsymbol{e})}{\min_i\{\mathring{x}_i, \mathring{y}_i\}} \;>\; \frac{\lambda}{2} \tag{21}$$

and $(\mathring{\boldsymbol{x}}, \mathring{\boldsymbol{y}})$ *is a feasible interior point whose existence is ensured by Assumption 1.1.*

The proposition below often used in the field of interior point algorithms.

Proposition 3.3 *(Proposition 5 of [6]) Suppose that* M *is an* $n \times n$ *positive semidefinite matrix. For every* $(\boldsymbol{x}', \boldsymbol{y}') > \boldsymbol{0}$ *and* $\bar{\boldsymbol{d}} \in \mathbb{R}^n$, *the system (16) has the unique solution* $(\Delta\boldsymbol{x}, \Delta\boldsymbol{y})$ *which satisfies the following inequalities:*

$$0 \;\leq\; \Delta\boldsymbol{x}^T\Delta\boldsymbol{y} \;\leq\; ||(X'Y')^{-\frac{1}{2}}\bar{\boldsymbol{d}}||, \tag{22}$$

$$||(X'Y')^{-\frac{1}{2}}Y'\Delta\boldsymbol{x}||^2 + ||(X'Y')^{-\frac{1}{2}}X'\Delta\boldsymbol{y}||^2 \;\leq\; ||(X'Y')^{-\frac{1}{2}}\bar{\boldsymbol{d}}||^2. \tag{23}$$

Let $(\bar{\mu}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ be a fixed point. To derive a complexity bound, we need to estimate the value of the function ϕ along the line segment $(\bar{\mu} + \theta\Delta\mu, \bar{\boldsymbol{x}} + \theta\Delta\boldsymbol{x}, \bar{\boldsymbol{y}} + \theta\Delta\boldsymbol{y})$. The following results can be obtained by a similar discussion to the one of Proposition 6 in [6].

Proposition 3.4 *Let* $(\bar{\mu}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \mathbb{R}_{++} \times \mathbb{R}^{2n}$ *such that* $\Phi(\bar{\mu}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \;=\; \bar{\Phi} \;\leq\; \boldsymbol{0}$ *and let* $(\Delta\mu, \Delta\boldsymbol{x}, \Delta\boldsymbol{y})$ *be the solution of the system (6).*

(i) *For every* $i \in N$ *and* $\theta \in [0, 1]$,

$$\begin{aligned} &\{1 - \theta(1-\sigma_\phi)\}(\bar{\phi}_i + \bar{\mu}h_i) \\ &\geq\; \phi(\bar{\mu} + \theta\Delta\mu, \bar{x}_i + \theta\Delta x_i, \bar{y}_i + \theta\Delta y_i) + \bar{\mu}h_i \\ &\geq\; \{1 - \theta(1-\sigma_\phi)\}(\bar{\phi}_i + \bar{\mu}h_i) - \frac{\theta^2}{\sigma_\mu\bar{\mu}}((1-\sigma_\mu)^2\bar{\mu}^2 + \Delta x_i^2 + \Delta y_i^2). \end{aligned}$$

(ii) *For every* $\theta \in [0,1]$,

$$\begin{aligned}&\|\Phi(\bar{\mu}+\theta\Delta\mu, \bar{\boldsymbol{x}}+\theta\Delta\boldsymbol{x}, \bar{\boldsymbol{y}}+\theta\Delta\boldsymbol{y})+\bar{\mu}\boldsymbol{h}\| \\ \leq\ & \{1-\theta(1-\sigma_\phi)\}\|\bar{\Phi}+\bar{\mu}\boldsymbol{h}\| + \frac{\theta^2}{\sigma_\mu\bar{\mu}}\{(1-\sigma_\mu)^2\sqrt{n}\bar{\mu}^2 + \|\Delta\boldsymbol{x}\|^2 + \|\Delta\boldsymbol{y}\|^2\}.\end{aligned}$$

The following corollary is a special case of the above proposition.

Corollary 3.5 *Let* $(\bar{\mu}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \mathbb{R}_{++} \times \mathbb{R}^{2n}$ *such that* $\bar{\Phi} \leq \mathbf{0}$ *and let* $(\Delta\mu, \Delta\boldsymbol{x}, \Delta\boldsymbol{y})$ *be the solution of the system (6) with the parameters* $\sigma_\mu = 1$ *(i.e.,* $\Delta\mu = 0$*) and* $\sigma_\phi = 0$.

(i) *For every* $i \in N$ *and* $\theta \in [0,1]$,

$$\begin{aligned}&(1-\theta)(\bar{\phi}_i+\bar{\mu}h_i) \\ \geq\ & \phi(\bar{\mu}, \bar{x}_i+\theta\Delta x_i, \bar{y}_i+\theta\Delta y_i)+\bar{\mu}h_i \\ \geq\ & (1-\theta)(\bar{\phi}_i+\bar{\mu}h_i) - \frac{\theta^2}{\bar{\mu}}(\Delta x_i^2+\Delta y_i^2).\end{aligned}$$

(ii)

$$\begin{aligned}&||\Phi(\bar{\mu}, \bar{\boldsymbol{x}}+\theta\Delta\boldsymbol{x}, \bar{\boldsymbol{y}}+\theta\Delta\boldsymbol{y})+\bar{\mu}\boldsymbol{h}|| \\ \leq\ & (1-\theta)||\bar{\Phi}+\bar{\mu}\boldsymbol{h}|| + \frac{\theta^2}{\bar{\mu}}\{||\Delta\boldsymbol{x}||^2 + ||\Delta\boldsymbol{y}||^2\}.\end{aligned}$$

For ease of notation, we define δ_i for $i \in N$, g, D_μ and $(D_\mu)_{\max}$ as follows:

$$\delta_i(\mu) := \sqrt{(x_i-y_i)^2+4\mu^2}, \tag{24}$$

$$g(\mu) := ||\Phi(\mu, \boldsymbol{x}, \boldsymbol{y})+\mu\boldsymbol{h}||^2, \tag{25}$$

$$D_\mu := \mathrm{diag}\{(d_\mu)_i\} = \mathrm{diag}\{h_i - 4\mu/\delta_i(\mu)\}, \tag{26}$$

$$(D_\mu)_{\max} := \max\{(D_\mu)_{ii}\ (i \in N)\}. \tag{27}$$

Here the vector $\boldsymbol{d}_\mu$ is given in (6). The next results are used to show that $(\mu^{k+1}, \boldsymbol{x}^k, \boldsymbol{y}^k) = ((1-\bar{\xi})\mu^k, \boldsymbol{x}^k, \boldsymbol{y}^k) \in \mathcal{N}(\alpha+\beta)$ in Step 2.

Proposition 3.6 *Let* $(\bar{\mu}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \mathcal{N}(\alpha)$. *Then there exists a* $\Delta\mu$ *for which* $\bar{\mu}+\Delta\mu > 0$ *and*

$$g(\bar{\mu}+\theta\Delta\mu) \leq (\alpha+\beta)^2(\bar{\mu}+\theta\Delta\mu)^2 \tag{28}$$

for any $\theta \in [0,1]$. *For such* $\Delta\mu$, *the following relations hold.*

(i)

$$|g'(\bar{\mu})| \leq 2\alpha\sqrt{n}|(D_{\bar{\mu}})_{max}|\bar{\mu}.$$

(ii)

$$\frac{1}{2}|g''(\bar{\mu}+\theta\Delta\mu)| \leq \left(||\boldsymbol{h}||+2\sqrt{n}\right)^2 + 2(\alpha+\beta)\sqrt{n}$$

for every $\theta \in [0,1]$.

(iii)

$$\begin{aligned} |g(\bar{\mu}+\Delta\mu)| &\le \alpha^2\bar{\mu}^2 + 2\alpha\sqrt{n}|(D_{\bar{\mu}})_{max}|\cdot|\Delta\mu|\bar{\mu} \\ &\quad + \left\{ (||\boldsymbol{h}|| + 2\sqrt{n})^2 + 2(\alpha+\beta)\sqrt{n} \right\} |\Delta\mu|^2. \end{aligned}$$

Proof: Let us consider the function

$$\bar{g}(\mu) := (\alpha+\beta)\mu^2 - g(\mu).$$

Since the function $\bar{g}(\mu)$ is continuous w.r.t. μ and $\bar{g}(\bar{\mu}) > 0$, there exists a $\Delta\mu$ such that $\bar{g}(\mu+\theta\Delta\mu) \ge 0$, i.e.,

$$g(\bar{\mu}+\theta\Delta\mu) \le (\alpha+\beta)(\bar{\mu}+\theta\Delta\mu)^2 \tag{29}$$

for every $\theta \in [0,1]$.

By the definitions (24) of $\delta_i(\mu)$ and (25) of $g(\mu)$, we see that

$$\begin{aligned} g(\mu) &= \sum_{i=1}^n \{\phi(\mu, x_i, y_i) + \mu h_i\}^2 \\ &= \sum_{i=1}^n \{x_i + y_i - \delta_i(\mu) + \mu h_i\}^2, \\ \delta_i'(\mu) &= \frac{8\mu}{2\sqrt{(x_i-y_i)^2 + 4\mu^2}} = \frac{4\mu}{\delta_i(\mu)}. \end{aligned}$$

By a direct calculation, we have

$$\begin{aligned} g'(\mu) &= 2\sum_{i=1}^n (\phi_i + \mu h_i)\left(h_i - \frac{4\mu}{\delta_i(\mu)}\right), \\ g''(\mu) &= 2\sum_{i=1}^n \left\{ \left(h_i - \frac{4\mu}{\delta_i(\mu)}\right)^2 + (\phi_i + \mu h_i)\left(-\frac{4}{\delta_i(\mu)}\right)\left(1 - \frac{4\mu^2}{\delta_i(\mu)^2}\right) \right\} \end{aligned}$$

where the second term of the twice derivative follows from

$$\left\{ h_i - \frac{4\mu}{\delta_i(\mu)} \right\}' = -\frac{4\delta_i(\mu) - 4\mu\cdot\frac{4\mu}{\delta_i(\mu)}}{\delta_i(\mu)^2} = -\frac{4}{\delta_i(\mu)} + \frac{16\mu^2}{\delta_i(\mu)^3} = -\frac{4}{\delta_i(\mu)}\left(1 - \frac{4\mu^2}{\delta_i(\mu)}\right).$$

(i) By the definition of $D_{\bar{\mu}}$, we have

$$\begin{aligned} |g'(\bar{\mu})| &= 2\left|\sum_{i=1}^n (\bar{\phi}_i + \bar{\mu}h_i)\left(h_i - \frac{4\bar{\mu}}{\delta_i(\bar{\mu})}\right)\right| = 2\left|\boldsymbol{e}^T D_{\bar{\mu}}(\bar{\Phi} + \bar{\mu}\boldsymbol{h})\right| \\ &\le 2||\boldsymbol{e}||\cdot||D_{\bar{\mu}}(\bar{\Phi}+\bar{\mu}\boldsymbol{h})|| = 2\sqrt{n}\sqrt{\sum_{i=1}^n (D_{\bar{\mu}})_{ii}^2(\bar{\phi}_i + \bar{\mu}h_i)^2} \\ &\le 2\sqrt{n}\sqrt{\max_i\{(D_{\bar{\mu}})_{ii}\}^2}\sqrt{\sum_{i=1}^n (\bar{\phi}_i + \bar{\mu}h_i)^2} \\ &= 2\sqrt{n}\left|\max_i\{(D_{\bar{\mu}})_{ii}\}\right| ||\bar{\Phi} + \bar{\mu}\boldsymbol{h}||. \end{aligned}$$

Since $(\bar{\mu}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in \mathcal{N}(\alpha)$, i.e., $||\Phi(\bar{\mu}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) + \bar{\mu}\boldsymbol{h}|| = ||\bar{\Phi} + \bar{\mu}\boldsymbol{h}|| \le \alpha\bar{\mu}$ by the assumption, we obtain (i).
(ii) For every $\theta \in (0,1)$ and $i \in N$, define

$$\omega_i(\theta) := \frac{4(\bar{\mu} + \theta\Delta\mu)}{\delta_i(\bar{\mu} + \theta\Delta\mu)} \tag{30}$$

(iii) of Proposition 3.1 implies that $\omega_i(\theta) \in [0,2]$,

$$\|\boldsymbol{\omega}(\theta)\| \le 2\sqrt{n},\ 0 \le 1 - \frac{\omega_i(\theta)^2}{4} \le 1 \tag{31}$$

for every $i \in N$ and $\theta \in [0,1]$. Recall that, by the definition (4),

$$\begin{aligned}\delta_i(\mu + \theta\Delta\mu) &= \bar{x}_i + \bar{y}_i - \phi(\bar{\mu} + \theta\Delta\mu, \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \\ &= (\bar{x}_i - \phi(\bar{\mu} + \theta\Delta\mu, \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})/2) + (\bar{y}_i - \phi(\bar{\mu} + \theta\Delta\mu, \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})/2).\end{aligned} \tag{32}$$

Let us define

$$x_i'(\theta) := (\bar{x}_i - \phi(\bar{\mu} + \theta\Delta\mu, \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})/2), \quad y_i'(\theta) := (\bar{y}_i - \phi(\bar{\mu} + \theta\Delta\mu, \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})/2). \tag{33}$$

Then

$$x_i'(\theta) + y_i'(\theta) \ge 2(\bar{\mu} + \theta\Delta\mu)$$

holds by (i) of Proposition 3.2. Thus, we obtain

$$||(X'(\theta) + Y'(\theta))^{-1}|| \ \le \ \frac{1}{2(\bar{\mu} + \theta\Delta\mu)} \tag{34}$$

and

$$\begin{aligned}
&\frac{1}{2}|g''(\bar{\mu} + \theta\Delta\mu)| \\
&= \left| \sum_{i=1}^{n} \left\{ (h_i - \omega_i(\theta))^2 \right. \right. \\
&\quad \left. \left. - (\phi(\bar{\mu} + \theta\Delta\mu, \bar{x}_i, \bar{y}_i) + (\bar{\mu} + \theta\Delta\mu)h_i) \left(-\frac{4}{\delta_i(\bar{\mu} + \theta\Delta\mu)} \right) \left(1 - \frac{\omega_i(\theta)^2}{4} \right) \right\} \right| \\
&\quad \text{(by the definition (30) of } \omega_i) \\
&\le \sum_{i=1}^{n} (h_i - \omega_i(\theta))^2 + \sum_{i=1}^{n} \left| \frac{4}{x_i'(\theta) + y_i'(\theta)} (\phi(\bar{\mu} + \theta\Delta\mu, \bar{x}_i, \bar{y}_i) + (\bar{\mu} + \theta\Delta\mu)h_i) \right| \\
&\quad \text{(by (31), (32) and (33))} \\
&= ||\boldsymbol{h} - \boldsymbol{\omega}(\theta)||^2 + 4||(X' + Y')^{-1} \left(\Phi(\bar{\mu} + \theta\Delta\mu, \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) + (\bar{\mu} + \theta\Delta\mu)\boldsymbol{h} \right)||_1 \\
&\le (||\boldsymbol{h}|| + ||\boldsymbol{\omega}(\theta)||)^2 + 4\sqrt{n}||(X' + Y')^{-1}||||\Phi(\mu + \theta\Delta\mu, \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) + (\mu + \theta\Delta\mu)\boldsymbol{h}|| \\
&\le (||\boldsymbol{h}|| + 2\sqrt{n})^2 + 2\sqrt{n} \cdot \frac{\sqrt{g(\bar{\mu} + \theta\Delta\mu)}}{\bar{\mu} + \theta\Delta\mu} \\
&\quad \text{(by (31), (34) and the definition of } g) \\
&\le (||\boldsymbol{h}|| + 2\sqrt{n})^2 + 2(\alpha + \beta)\sqrt{n}. \\
&\quad \text{(by (29))}
\end{aligned}$$

(iii) By Taylor's expansion, the value of $g(\mu + \Delta\mu)$ is given by

$$g(\bar{\mu} + \Delta\bar{\mu}) \;=\; g(\bar{\mu}) + g'(\bar{\mu})\Delta\bar{\mu} + \frac{1}{2}g''(\bar{\mu} + \theta\Delta\bar{\mu})\Delta\bar{\mu}^2$$

for some $\theta \in (0,1)$. Combining (i) and (ii) with this equation, we conclude that

$$\begin{aligned}
&|g(\bar{\mu} + \Delta\mu)| \\
&\leq\; |g(\bar{\mu})| + |g'(\bar{\mu})||\Delta\mu| + \frac{1}{2}|g''(\bar{\mu} + \theta\Delta\mu)||\Delta\mu|^2 \\
&\leq\; (\alpha\bar{\mu})^2 + 2\alpha\sqrt{n}|(D_{\bar{\mu}})_{max}|\bar{\mu}\cdot|\Delta\mu| + \left\{(||\boldsymbol{h}|| + 2\sqrt{n})^2 + 2(\alpha+\beta)\sqrt{n}\right\}|\Delta\mu|^2.
\end{aligned}$$

∎

4 A complexity analysis

In this section, we derive a complexity bound of the algorithm described in Section 2. The lemma below shows that a reduction rate of μ in Step 2 can be evaluated explicitly. In particular, the rate is better than $O(1 - \frac{1}{n})$ if $\|\boldsymbol{h}\| = 0$ and is better than $O(1 - \frac{1}{\sqrt{n}})$ if $\|\boldsymbol{h}\| = 1$.

Lemma 4.1 *Let* $(\mu^k, \boldsymbol{x}^k, \boldsymbol{y}^k) \in \mathcal{N}(\alpha)$.

(i) *Then* $((1-\xi)\mu^k, \boldsymbol{x}^k, \boldsymbol{y}^k) \in \mathcal{N}(\alpha+\beta)$ *for every* $\xi \in [0, \bar{\xi}]$. *Here* $\bar{\xi}$ *is defined by (11).*

(ii) *If* $\boldsymbol{h} = \boldsymbol{e}$, *then there exists a value* $\hat{\xi} = O(1/n)$ *for which* $\hat{\xi} \leq \bar{\xi}$.

Proof: (i) Let us consider $(1-\eta)\mu^k$ for some $\eta \in [0,1]$. Since $(\mu^k, \boldsymbol{x}^k, \boldsymbol{y}^k) \in \mathcal{N}(\alpha)$ and $\beta > 0$, we have

$$g(\mu^k) \;=\; ||\Phi(\mu^k, \boldsymbol{x}^k, \boldsymbol{y}^k) + \mu\boldsymbol{h}||^2 \;\leq\; \alpha^2(\mu^k)^2 \;<\; (\alpha+\beta)^2(\mu^k)^2.$$

Thus, by the continuity of $g(\mu)$ with respect to μ, there exists an $\bar{\eta} \in (0,1)$ such that

$$g((1-\eta)\mu^k) \;\leq\; (\alpha+\beta)^2((1-\eta)(\mu^k))^2.$$

for every $\eta \in [0, \bar{\eta}]$. Let us find such $\bar{\eta}$. For every $\eta \in (0,1)$, (iii) of Proposition 3.6 holds and hence,

$$\begin{aligned}
&g((1-\eta)\mu^k) \\
&\leq\; \alpha^2(\mu^k)^2 + 2\alpha\sqrt{n}|(D_{\mu^k})_{\max}|\eta(\mu^k)^2 + \left((||\boldsymbol{h}|| + 2\sqrt{n})^2 + 2(\alpha+\beta)\sqrt{n}\right)\eta^2(\mu^k)^2 \\
&=\; \left\{\alpha^2 + 2\alpha\sqrt{n}|(D_{\mu^k})_{\max}|\eta + \left((||\boldsymbol{h}|| + 2\sqrt{n})^2 + 2(\alpha+\beta)\sqrt{n}\right)\eta^2\right\}(\mu^k)^2 \\
&\leq\; \left\{\alpha^2 + 2\sqrt{n}|(D_{\mu^k})_{\max}|\eta + \left((||\boldsymbol{h}|| + 2\sqrt{n})^2 + 2\sqrt{n}\right)\eta^2\right\}(\mu^k)^2.
\end{aligned}$$

Thus if $\bar{\eta}$ satisfies

$$\alpha^2 + 2\sqrt{n}|(D_{\mu^k})_{\max}|\bar{\eta} + \left((||\boldsymbol{h}|| + 2\sqrt{n})^2 + 2\sqrt{n}\right)\bar{\eta}^2 \leq (\alpha+\beta)^2(1-\bar{\eta})^2 \tag{35}$$

then $g((1-\eta)\mu^k) \leq (\alpha+\beta)^2((1-\eta)(\mu^k))^2$ for every $\eta \in [0, \bar{\eta}]$. By the definition (27) of $(D_\mu)_{\max}$, we have $|(D_{\mu^k})_{\max}| \leq \kappa$. Hence the above inequality (35) can be replaced by

$$\alpha^2 + 2\sqrt{n}\kappa\bar{\eta} + \left((||\boldsymbol{h}|| + 2\sqrt{n})^2 + 2\sqrt{n}\right)\bar{\eta}^2 \leq (\alpha+\beta)^2(1-\bar{\eta})^2$$

By rearranging the inequality (35) with the definitions (9) and (10) of ζ and κ, we obtain

$$(\zeta - (\alpha+\beta)^2)\bar{\eta}^2 + 2\kappa\bar{\eta} - \{(\alpha+\beta)^2 - \alpha^2\} \ \le\ 0.$$

Since $\zeta \ge 6$ by its definition (9) and since we assume $\alpha + \beta < 1$, there exists a positive solution of the above inequality, which gives us a bound

$$\bar{\eta} = \frac{-\kappa + \sqrt{\kappa^2 + \{\zeta - (\alpha+\beta)^2\}\{(\alpha+\beta)^2 - \alpha^2\}}}{\zeta - (\alpha+\beta)^2}.$$

Thus we obtain the assertion (i).

(ii): Since $\alpha + \beta \in (0,1)$ and $\zeta \ge 4n$ by the definition (9), it is easy to see that

$$\begin{aligned}
&\frac{-\kappa + \sqrt{\kappa^2 + \{\zeta - (\alpha+\beta)^2\}\{(\alpha+\beta)^2 - \alpha^2\}}}{\zeta - (\alpha+\beta)^2} \\
\ge\ & \frac{-\kappa + \sqrt{\kappa^2 + (\zeta-1)(2\alpha+\beta)\beta}}{\zeta} \\
=\ & \frac{1}{\zeta} \cdot \frac{-\kappa^2 + \{\kappa^2 + (\zeta-1)(2\alpha+\beta)\beta\}}{\kappa + \sqrt{\kappa^2 + (\zeta-1)(2\alpha+\beta)\beta}} \\
\ge\ & \frac{1}{\zeta} \cdot \frac{(\zeta-1)(2\alpha+\beta)\beta}{2\kappa + \sqrt{(\zeta-1)(2\alpha+\beta)\beta}} \\
=\ & \frac{\zeta-1}{\zeta} \cdot \frac{(2\alpha+\beta)\beta}{2\kappa + \sqrt{(\zeta-1)(2\alpha+\beta)\beta}} \\
\ge\ & \frac{3}{4} \cdot \frac{(2\alpha+\beta)\beta}{2\kappa + \sqrt{(\zeta-1)(2\alpha+\beta)\beta}} \\
=\ & O\left(\frac{1}{\kappa + \sqrt{\zeta}}\right).
\end{aligned}$$

Thus, by the definitions (9) and (10), we obtain the assertion (ii). ∎

The following lemma shows a reduction rate of the value $||\hat{\Phi}^p + \hat{\mu}\boldsymbol{h}||$ at each iteration p in Step 3.1.

Lemma 4.2 *At each iteration p in Step 3, the following inequality holds.*

$$||\hat{\Phi}^{p+1} + \hat{\mu}\boldsymbol{h}|| \ \le\ \max\left\{1 - \frac{\hat{\mu}||\hat{\Phi}^p + \hat{\mu}\boldsymbol{h}||}{4\{||\Delta\hat{\boldsymbol{x}}^p||^2 + ||\Delta\hat{\boldsymbol{y}}^p||^2\}}, \frac{1}{2}\right\} ||\hat{\Phi}^p + \hat{\mu}\boldsymbol{h}||.$$

Proof: At the beginning of Step 3, we have $(\hat{\mu}, \hat{\boldsymbol{x}}^0, \hat{\boldsymbol{y}}^0) := (\mu^{k+1}, \boldsymbol{x}^k, \boldsymbol{y}^k)$ which satisfies $(\hat{\mu}, \hat{\boldsymbol{x}}^0, \hat{\boldsymbol{y}}^0) \in \mathcal{N}(\alpha+\beta)$, i.e.,

$$||\Phi(\hat{\mu}, \hat{\boldsymbol{x}}^0, \hat{\boldsymbol{y}}^0) + \hat{\mu}\boldsymbol{h}|| \le (\alpha+\beta)\hat{\mu},\ \hat{\boldsymbol{y}}^0 = M\hat{\boldsymbol{x}}^0 + \boldsymbol{q} \text{ and } \Phi(\hat{\mu}, \hat{\boldsymbol{x}}^0, \hat{\boldsymbol{y}}^0) < 0.$$

Throughout Step 3, we set the parameters $\sigma_\mu := 1$ and $\sigma_\phi := 0$, respectively. Therefore, by (ii) of Corollary 3.5, we have

$$||\Phi(\hat{\mu}, \hat{\boldsymbol{x}}^p + \theta\Delta\hat{\boldsymbol{x}}^p, \hat{\boldsymbol{y}}^p + \theta\Delta\hat{\boldsymbol{y}}^p) + \hat{\mu}\boldsymbol{h}|| \ \le\ (1-\theta)||\hat{\Phi}^p + \hat{\mu}\boldsymbol{h}|| + \frac{\theta^2}{\hat{\mu}}\{||\Delta\hat{\boldsymbol{x}}^p||^2 + ||\Delta\hat{\boldsymbol{y}}^p||^2\}$$

at each iteration p. Let us define $s^p(\theta) = (1-\theta)||\hat{\Phi}^p + \hat{\mu}\boldsymbol{h}|| + \frac{\theta^2}{\hat{\mu}}\{||\Delta\hat{\boldsymbol{x}}^p||^2 + ||\Delta\hat{\boldsymbol{y}}^p||^2\}$. It attains the minimum at

$$\bar{\theta}^p := \frac{\hat{\mu}||\hat{\Phi}^p + \hat{\mu}\boldsymbol{h}||}{2\{||\Delta\hat{\boldsymbol{x}}^p||^2 + ||\Delta\hat{\boldsymbol{y}}^p||^2\}}$$

and the value is given by

$$s^p(\bar{\theta}^p) \;=\; \left\{1 - \frac{\hat{\mu}||\hat{\Phi}^p + \hat{\mu}\boldsymbol{h}||}{4\{||\Delta\hat{\boldsymbol{x}}^p||^2 + ||\Delta\hat{\boldsymbol{y}}^p||^2\}}\right\} ||\hat{\Phi}^p + \hat{\mu}\boldsymbol{h}||. \tag{36}$$

If $\bar{\theta}^p < 1$ then we set $\theta^p := \bar{\theta}^p$, and hence,

$$||\Phi(\hat{\mu}, \hat{\boldsymbol{x}}^p + \theta\Delta\hat{\boldsymbol{x}}^p, \hat{\boldsymbol{y}}^p + \theta\Delta\hat{\boldsymbol{y}}^p) + \hat{\mu}\boldsymbol{h}|| \;\le\; s^p(\bar{\theta}^p).$$

Otherwise,

$$\frac{\hat{\mu}||\hat{\Phi}^p + \hat{\mu}\boldsymbol{h}||^2}{2\{||\Delta\hat{\boldsymbol{x}}^p||^2 + ||\Delta\hat{\boldsymbol{y}}^p||^2\}} \;\ge\; 1 \tag{37}$$

and θ^p turns out to be 1. Thus, we obtain

$$s^p(1) \;=\; \frac{||\Delta\hat{\boldsymbol{x}}^p||^2 + ||\Delta\hat{\boldsymbol{y}}^p||^2}{\hat{\mu}} \;\le\; \frac{1}{2}||\hat{\Phi}^p + \hat{\mu}\boldsymbol{h}||.$$

The assertion follows from the definition of (12) of θ^p in Step 3.1. ∎

Here, $(\mu, \boldsymbol{x}, \boldsymbol{y}) \in \mathcal{N}(\alpha+\beta)$ implies that

$$||\Phi(\mu, \boldsymbol{x}, \boldsymbol{y})|| \;\le\; ||\Phi(\mu, \boldsymbol{x}, \boldsymbol{y}) + \mu\boldsymbol{h}|| + \mu||\boldsymbol{h}|| \;\le\; (\alpha+\beta)\mu + \mu||\boldsymbol{h}|| \;\le\; (\alpha+\beta+||\boldsymbol{h}||)\mu^0.$$

Thus, we have

$$\mathcal{N}(\alpha+\beta) \subset P \cap L(\bar{\lambda}, \mu^0)$$

where $\bar{\lambda} = (\alpha+\beta+\|\boldsymbol{h}\|)\mu^0$, and by (19) and (20), $x'_i \le 2\bar{\gamma}(\bar{\lambda}, \mu^0)$ and $y'_i \le 2\bar{\gamma}(\bar{\lambda}, \mu^0)$ for every $i \in N$. These bounds lead us to the fact that

$$||X' + Y'|| \;\;\le\;\; 4\bar{\gamma}(\bar{\lambda}, \mu^0). \tag{38}$$

Since $X'Y' = (\bar{X} - \bar{\Phi}/2)(\bar{Y} - \bar{\Phi}/2) = \bar{\mu}^2 I$, the inequality (23) implies that

$$\frac{1}{\bar{\mu}}||Y'\Delta\boldsymbol{x}||^2 + \frac{1}{\bar{\mu}}||X'\Delta\boldsymbol{y}||^2 \;\;\le\;\; \frac{1}{\bar{\mu}}||\bar{\boldsymbol{d}}||^2$$

i.e.,

$$||Y'\Delta\boldsymbol{x}||^2 + ||X'\Delta\boldsymbol{y}||^2 \;\;\le\;\; ||\bar{\boldsymbol{d}}||^2.$$

Therefore, both $||Y'\Delta\boldsymbol{x}||$ and $||X'\Delta\boldsymbol{y}||$ are bounded by $||\bar{\boldsymbol{d}}||$. In addition, $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ lies in the bounded simplex $P \cap L(\bar{\lambda}, \mu^0)$. Hence, (19) and (20) give us the bounds

$$||(X')^{-1}|| \le \frac{3\bar{\gamma}(\bar{\lambda}, \mu^0)}{\bar{\mu}^2}, \;\; ||(Y')^{-1}|| \le \frac{3\bar{\gamma}(\bar{\lambda}, \mu^0)}{\bar{\mu}^2}.$$

Thus we obtain

$$\begin{aligned} ||\Delta\boldsymbol{x}||^2 + ||\Delta\boldsymbol{y}||^2 \;&\le\; ||(Y')^{-1}||^2||Y'\Delta\boldsymbol{x}||^2 + ||(X')^{-1}||^2||X'\Delta\boldsymbol{y}||^2 \\ &\le\; \frac{9\bar{\gamma}(\bar{\lambda}, \mu^0)^2}{\bar{\mu}^4}\{||Y'\Delta\boldsymbol{x}||^2 + ||X'\Delta\boldsymbol{y}||^2\} \;\le\; \frac{9\bar{\gamma}(\bar{\lambda}, \mu^0)^2}{\bar{\mu}^4}||\bar{\boldsymbol{d}}||^2. \end{aligned} \tag{39}$$

Using this result, we show the following main theorem.

Theorem 4.3 **(i)** *At each iteration k, Step 3.1 terminates after*

$$P^k \leq \left\lceil 2\max\left\{\frac{2\cdot 36\bar{\gamma}(\bar{\lambda},\mu^0)^4\alpha}{(\mu^{k+1})^4},1\right\}\right\rceil$$

Newton iterations.

(ii) *The total number of Newton iterations in the algorithm is bounded by*

$$\left\lceil\frac{2^{16}\bar{\gamma}(\bar{\lambda},\mu^0)^4}{\bar{\xi}\epsilon^4}\right\rceil + 3\left(\frac{1}{\bar{\xi}}\log\frac{\mu^0}{\epsilon}+1\right)$$

where $\bar{\gamma}(\bar{\lambda},\mu^0)$ and $\bar{\xi}$ are defined by (21) and (11), 4.1 and $\bar{\lambda}=(\alpha+\beta+\|\boldsymbol{h}\|)\mu^0$.

Proof: (i) In Step 3, we set the parameter $\sigma_\mu = 1 (i.e., \Delta\hat{\mu}^p = 0)$ and $\sigma_\phi = 0$. Thus the system (16) is given by

$$\begin{pmatrix} -M & I \\ Y' & X' \end{pmatrix}\begin{pmatrix} \Delta\hat{\boldsymbol{x}} \\ \Delta\hat{\boldsymbol{y}} \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{\boldsymbol{d}}^p \end{pmatrix}$$

where $\hat{\boldsymbol{d}}^p = -\frac{1}{2}(\hat{\Phi}^p + \hat{\mu}\boldsymbol{h})$. By the inequality (38), the norm of $\hat{\boldsymbol{d}}^p$ is bounded by

$$||\hat{\boldsymbol{d}}^p|| \leq \frac{1}{2}||X'+Y'||||\hat{\Phi}^p+\hat{\mu}\boldsymbol{h}|| \leq 2\bar{\gamma}(\bar{\lambda},\mu^0)||\hat{\Phi}^p+\hat{\mu}\boldsymbol{h}||.$$

Combining the inequality (39) with the above, one has

$$||\Delta\hat{\boldsymbol{x}}^p||^2+||\Delta\hat{\boldsymbol{y}}^p||^2 \leq \frac{9\bar{\gamma}(\lambda,\mu^0)^2}{\hat{\mu}^4}||\hat{\boldsymbol{d}}^p||^2 \leq \frac{4\cdot 9\bar{\gamma}(\bar{\lambda},\mu^0)^4}{\hat{\mu}^4}||\hat{\Phi}^p+\hat{\mu}\boldsymbol{h}||^2.$$

and

$$\begin{aligned}\frac{\hat{\mu}||\hat{\Phi}^p+\hat{\mu}\boldsymbol{h}||}{4\{||\Delta\hat{\boldsymbol{x}}^p||^2+||\Delta\hat{\boldsymbol{y}}^p||^2\}} &\geq \frac{\hat{\mu}^5}{4\cdot 36\bar{\gamma}(\bar{\lambda},\mu^0)^4||\hat{\Phi}^p+\hat{\mu}\boldsymbol{h}||} \\ &\geq \frac{\hat{\mu}^5}{4\cdot 36\bar{\gamma}(\bar{\lambda},\mu^0)^4(\alpha+\beta)\hat{\mu}} \\ &\geq \frac{(\mu^{k+1})^4}{4\cdot 36\bar{\gamma}(\bar{\lambda},\mu^0)^4}.\end{aligned}$$

where the last two inequalities are derived from the facts $||\hat{\Phi}^p + \hat{\mu}\boldsymbol{h}|| \leq (\alpha+\beta)\hat{\mu}$ and $\hat{\mu} = \hat{\mu}^0 = \mu^{k+1}$. Therefore, by Lemma 4.2, the value of $||\hat{\Phi}^p+\hat{\mu}\boldsymbol{h}||$ is reduced at least by the factor $(1-\delta^k)$ at each inner iteration p, where $\delta^k := \min\left\{\frac{(\mu^{k+1})^4}{4\cdot 36\bar{\gamma}(\bar{\lambda},\mu^0)^4},\frac{1}{2}\right\}$. Let us consider the number of iteration P^k for which the point $(\hat{\mu},\hat{\boldsymbol{x}}^{P^k},\hat{\boldsymbol{y}}^{P^k})$ satisfies the criterion in Step 3.1, i.e.,

$$\|\Phi(\hat{\mu},\hat{\boldsymbol{x}}^{P^k},\hat{\boldsymbol{y}}^{P^k})+\hat{\mu}\boldsymbol{h}\| \leq \alpha\hat{\mu}.$$

Since $\|\hat{\Phi}^0+\hat{\mu}\boldsymbol{h}\| \leq (\alpha+\beta)\hat{\mu}$, a sufficient condition is

$$(1-\delta^k)^{P^k}(\alpha+\beta) \leq \alpha \tag{40}$$

By taking logarithms in both sides above and using the inequality

$$\log(1-\delta^k) \le -\delta^k < 0$$

we can derive a lower bound of P^k

$$P^k \ge \left\lceil \frac{1}{\delta^k} \log \frac{\alpha+\beta}{\alpha} \right\rceil .$$

By the assumption $0<\beta<\alpha<1$, we obtain $\frac{\alpha+\beta}{\alpha} \le 2$ and the assertion (i).
(ii) An upper bound of the total number of Newton iterations is given by

$$\sum_{k=0}^{K} P^k = \sum_{k=0}^{K} \left\lceil 2\max\{\frac{2\cdot 36\bar{\gamma}(\bar{\lambda},\mu^0)^4}{(\mu^{k+1})^4}, 1\} \right\rceil .$$

Here, we can see that

$$\begin{aligned}\sum_{k=0}^{K} \left\lceil 2\max\{\frac{2\cdot 36\bar{\gamma}(\bar{\lambda},\mu^0)^4}{(\mu^{k+1})^4}, 1\} \right\rceil &\le \sum_{k=0}^{K} \left\lceil 2\left(\frac{2\cdot 36\bar{\gamma}(\bar{\lambda},\mu^0)^4}{(\mu^{k+1})^4}+1\right) \right\rceil \\ &\le \left\lceil 4\cdot 36\bar{\gamma}(\bar{\lambda},\mu^0)^4 \sum_{k=0}^{K} \frac{1}{(\mu^{k+1})^4} \right\rceil + 3(K+1).\end{aligned}$$

Since we set $\mu^{k+1} = (1-\bar{\xi})\mu^k$, we know that $\mu^{k+1} = (1-\bar{\xi})^{k+1}\mu^0$ and

$$\frac{1}{(\mu^{k+1})^4} = \frac{1}{(\mu^0)^4(1-\bar{\xi})^4} \cdot \left(\frac{1}{1-\bar{\xi}}\right)^{4k}.$$

Thus, the sum is given by

$$\begin{aligned}\sum_{k=0}^{K} \frac{1}{(\mu^{k+1})^4} &= \frac{1}{(\mu^0)^4(1-\xi)^4} \sum_{k=0}^{K} \left(\frac{1}{1-\bar{\xi}}\right)^{4k} \\ &= \frac{1}{(\mu^0)^4(1-\xi)^4} \cdot \frac{\left(\frac{1}{1-\bar{\xi}}\right)^{4(K+1)}-1}{\left(\frac{1}{1-\bar{\xi}}\right)^{4}-1}. \end{aligned} \tag{41}$$

The stopping criteria $\mu^k < \epsilon$ and the reduction rate $1-\bar{\xi}$ of μ^k ensure that

$$(1-\bar{\xi})\epsilon \le (1-\bar{\xi})^K \mu^0 = \mu^K < \epsilon.$$

The first inequality above implies that

$$\left(\frac{1}{1-\bar{\xi}}\right)^{4(K+1)} \le \left(\frac{1}{1-\bar{\xi}}\right)^{8} \frac{(\mu^0)^4}{\epsilon^4}$$

Substituting this into (41), we obtain that

$$\frac{1}{(\mu^0)^4(1-\bar{\xi})^4} \cdot \frac{\left(\frac{1}{1-\bar{\xi}}\right)^{4(K+1)}-1}{\left(\frac{1}{1-\bar{\xi}}\right)^{4}-1}$$

$$
\begin{aligned}
&\leq \frac{1}{(\mu^0)^4(1-\bar{\xi})^4} \cdot \frac{\left(\frac{1}{1-\bar{\xi}}\right)^8 \cdot \frac{(\mu^0)^4}{\epsilon^4} - 1}{\left(\frac{1}{1-\bar{\xi}}\right)^4 - 1} \\
&< \frac{1}{(\mu^0)^4(1-\bar{\xi})^4} \frac{\left(\frac{1}{1-\bar{\xi}}\right)^8 \cdot \frac{(\mu^0)^4}{\epsilon^4}}{\left(\frac{1}{1-\bar{\xi}}\right)^4 - 1} \\
&= \frac{1}{(\mu^0)^4(1-\bar{\xi})^4} \cdot \frac{1}{(1-\bar{\xi})^4 - (1-\bar{\xi})^8} \cdot \frac{(\mu^0)^4}{\epsilon^4} \\
&= \frac{1}{(\mu^0)^4(1-\bar{\xi})^8} \cdot \frac{1}{1-(1-\bar{\xi})^4} \cdot \frac{(\mu^0)^4}{\epsilon^4} \\
&= \frac{1}{(1-\bar{\xi})^8} \cdot \frac{1}{1-(1-\bar{\xi})^4} \cdot \frac{1}{\epsilon^4} \\
&= \frac{1}{(1-\bar{\xi})^8} \cdot \frac{1}{\{1-(1-\bar{\xi})\}\{1+(1-\bar{\xi})\}\{1+(1-\bar{\xi})^2\}} \cdot \frac{1}{\epsilon^4} \\
&= \frac{1}{\bar{\xi}(1-\bar{\xi})^8\{1+(1-\bar{\xi})\}\{1+(1-\bar{\xi})^2\}} \cdot \frac{1}{\epsilon^4} \\
&\leq \frac{1}{\bar{\xi}(1/2)^8} \cdot \frac{1}{\epsilon^4} < \frac{2^8}{\bar{\xi}\epsilon^4}
\end{aligned}
$$

where the third inequality follows from the definition (11) i.e., $\bar{\xi} \leq 1/2$.

By a similar discussion in the proof of (i) of the theorem, the inequality (41) gives us a bound $K \leq \frac{1}{\bar{\xi}} \log \frac{\mu^0}{\epsilon}$. Thus, we can conclude that

$$
\sum_{k=0}^{K} P^k \leq \left\lceil \frac{2^{16}\bar{\gamma}(\bar{\lambda}, \mu^0)^4}{\bar{\xi}\epsilon^4} \right\rceil + 3\left(\frac{1}{\bar{\xi}} \log \frac{\mu^0}{\epsilon} + 1\right)
$$

∎

The following corollary follows from (ii) and (iii) of Lemma 4.1.

Corollary 4.4 *Suppose that* $\boldsymbol{h} \in \mathbb{R}^n_+$ *satisfies* $\boldsymbol{h} = \boldsymbol{e}$. *Then the algorithm terminate in*

$$
O\left(\frac{\bar{\gamma}(\bar{\lambda}, \mu^0)^4 n}{\epsilon^4}\right)
$$

number of iterations.

5 Concluding remarks

We propose a new smoothing algorithm for the LCP and derive its complexity bound when the problem satisfies Assumption 1.1. In a previous work, Hotta, Inaba and Yoshise [6] proposed another smoothing algorithm whose complexity bound is $O\left(n\frac{\bar{\gamma}^6}{\epsilon^6} \log \frac{\bar{\gamma}^2 n}{\epsilon^2}\right)$. Combining the predictor-corrector strategy with the idea of using relatively narrow neighborhood, we can improve the bound as $O\left(n\frac{\bar{\gamma}^4}{\epsilon^4}\right)$ (Corollary 4.4). As a by-product, we show that there exists a point $(\mu', \boldsymbol{x}', \boldsymbol{y}') \in \mathcal{N}(\alpha+\beta)$ for each $(\mu, \boldsymbol{x}, \boldsymbol{y}) \in \mathcal{N}(\alpha)$, where

$\mu' = (1-\bar{\xi})\mu$ and $(1-\bar{\xi}) = O(1-1/n)$ ((ii) of Lemma 4.1). For further research, It will be important to find an algorithm which reduces the value of $\|\Phi\|$ polynomially, and/or to evaluate the value of $\bar{\gamma}$ more tightly to construct a polynomial-time smoothing method.

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