$$u = 1/(1+\delta),$$
$$w = 1/(1+\gamma).$$

Hence, for any sufficiently large i we have

$$(1+\delta)^i = 1 + i\delta + \cdots > i\delta,$$
$$(1+\gamma)^i = 1 + i\gamma + \cdots + \binom{i}{k+\ell+2}\gamma^{k+\ell+2} + \cdots$$
$$> \binom{i}{k+\ell+2}\gamma^{k+\ell+2} = \frac{i!}{(k+\ell+2)!(i-k-\ell-2)!}\gamma^{k+\ell+2},$$

from which we get

$$u^i = 1/(1+\delta)^i < \frac{1}{i\delta},$$
$$w^i = 1/(1+\gamma)^i < \frac{(k+\ell+2)!(i-k-\ell-2)!}{i!\gamma^{k+\ell+2}} = \frac{(k+\ell+2)!}{i(i-1)(i-2)\cdots(i-k-\ell)(i-k-\ell-1)\gamma^{k+\ell+2}}.$$

Consequently, it follows that

$$(1-p)^i = u^i < \frac{1}{i\delta},$$
$$i^{k+\ell+1}(1-q-r)^i = i^{k+\ell+1}w^i < \frac{i^{k+\ell+1}(k+\ell+2)!}{i(i-1)(i-2)\cdots(i-k-\ell)(i-k-\ell-1)\gamma^{k+\ell+2}}$$
$$= \frac{(k+\ell+2)!}{\gamma^{k+\ell+2}}\frac{1}{i(1-\frac{1}{i})(1-\frac{2}{i})\cdots(1-\frac{k+\ell}{i})(1-\frac{k+\ell+1}{i})},$$

both of which converge to 0 as $i \to \infty$. Noting,

$$\lim_{i\to\infty}\left(\frac{i}{i-k-\ell}\right)^{0.5} = \lim_{i\to\infty}\left(\frac{1}{1-\frac{k+\ell}{i}}\right)^{0.5} = 1,$$
$$\lim_{i\to\infty}\left(\frac{i}{i-k-\ell}\right)^{i-k-\ell} = \lim_{i\to\infty}\left(1+\frac{k+\ell}{i-k-\ell}\right)^{i-k-\ell} = e^{k+\ell}.$$

Accordingly, $(1-p)^i$ and $if_{gr}(k,\ell\ i)$ converge to 0 as $i \to \infty$. In quite a similar way, we can show $\lim_{i\to\infty}(1-z)^i = 0$. ∎

5 Analysis

5.1 General Properties of $V_t(i)$

Lemma 5.1 *For all $i \geq 1$ we have*

$$V_1(i) = (1-s)(1-z)^i - (1-p)^i + s, \tag{5.1}$$

$$V_1(1) = p - (1-s)z. \tag{5.2}$$

PROOF. Noting Eqs. (3.2) and (4.1), we can write Eq. (3.12) as follows.

$$\begin{aligned}
V_1(i) &= (1-s)\left(\sum_{\ell=0}^{i-1}\binom{i}{\ell}r^\ell(1-q-r)^{i-\ell}(1-p)^{i-\ell} + r^i\right) + s - (1-p)^i \\
&= (1-s)\left(\sum_{\ell=0}^{i}\binom{i}{\ell}r^\ell\Big((1-q-r)(1-p)\Big)^{i-\ell} - r^i + r^i\right) + s - (1-p)^i \\
&= (1-s)\Big(r + (1-q-r)(1-p)\Big)^i - (1-p)^i + s \\
&= (1-s)\Big(1 - (q + (1-q-r)p)\Big)^i - (1-p)^i + s \\
&= (1-s)(1-z)^i - (1-p)^i + s,
\end{aligned}$$

from which we immediately have Eq. (5.2). ∎

Lemma 5.2 *For any given $i_2 > i_1 \geq 1$ we have*

(a) *If $p \geq z$, the following two inequalities can not coincide.*

$$V_1(i_1) > V_1(i_1+1), \tag{5.3}$$

$$V_1(i_2) < V_1(i_2+1). \tag{5.4}$$

(b) *If $p \leq z$, the following two inequalities can not coincide.*

$$V_1(i_1) < V_1(i_1+1), \tag{5.5}$$

$$V_1(i_2) > V_1(i_2+1). \tag{5.6}$$

PROOF.

(a) Assume $p \geq z$, and let $b(i) = V_1(i+1) - V_1(i)$. Then, from Eq. (5.1) we have

$$\begin{aligned}
b(i) &= (1-s)(1-z)^{i+1} - (1-p)^{i+1} + s - (1-s)(1-z)^i + (1-p)^i - s \\
&= (1-s)(1-z)^i(1-z-1) - (1-p)^i(1-p-1) \\
&= p(1-p)^i - (1-s)z(1-z)^i. \qquad (5.7)
\end{aligned}$$

Since $V_1(i+1) = V_1(i) + b(i)$, if Eqs. (5.3) and (5.4) are both satisfied, then

$$V_1(i_1) > V_1(i_1) + b(i_1),$$
$$V_1(i_2) < V_1(i_2) + b(i_2),$$

from which $b(i_1) < 0$ and $b(i_2) > 0$; equivalently,

$$p(1-p)^{i_1} < (1-s)z(1-z)^{i_1},$$
$$p(1-p)^{i_2} > (1-s)z(1-z)^{i_2}.$$

Consquently, we obtain

$$(1-s)z\Big(\frac{1-z}{1-p}\Big)^{i_2} < p < (1-s)z\Big(\frac{1-z}{1-p}\Big)^{i_1}.$$

Thus, we get

$$\Big(\frac{1-z}{1-p}\Big)^{i_2} < \Big(\frac{1-z}{1-p}\Big)^{i_1}.$$

Accordingly, since $i_2 > i_1 \geq 1$ by assumption, it must be that $(1-z)/(1-p) < 1$, i.e., $p < z$, which is a contradiction, hence, it follows that Eqs. (5.3) and (5.4) can not coincide.

(b) Almost the same as the proof of (a). ∎

From Lemma 5.2 we immediately get the following corollary.

Corollary 5.1 *For any given $i_2 > i_1 \geq 1$ we have*

(a) *If $p \geq z$, the following two inequalities can not coincide.*

$$V_1(i_1) > 0 \geq V_1(i_1+1), \tag{5.8}$$
$$V_1(i_2) \leq 0 < V_1(i_2+1). \tag{5.9}$$

(b) *If $p \leq z$, the following two inequalities can not coincide.*

$$V_1(i_1) < 0 \leq V_1(i_1+1), \tag{5.10}$$
$$V_1(i_2) \geq 0 > V_1(i_2+1). \tag{5.11}$$

Lemma 5.3 *If $V_1(i) < (=) 0$ for all $i \geq 1$, then $V_t(i) = V_1(i) < (=) 0$ for all $i \geq 1$ and $t \geq 1$.*

Proof. It is evident for $t = 1$. Suppose $V_{t-1}(i) = V_1(i) < (=) 0$ for all $i \geq 1$. Then, since $\max\{0, V_{t-1}(i-\ell)\} = 0$ for all i and ℓ, we immediately get $V_t(i) = V_1(i) < (=) 0$ from Eq. (3.13). This completes the induction. ∎

Lemma 5.4 *$V_t(i)$ is nondecreasing in t for all $i \geq 1$.*

PROOF. Since $\max\{0, V_{t-1}(i-\ell, 1)\} \geq 0$, from Eq. (3.13) we have $V_t(i) \geq V_1(i)$ for all $t \geq 1$ and $i \geq 1$, hence $V_2(i) \geq V_1(i)$ for all $i \geq 1$. Suppose $V_{t-1}(i) \geq V_{t-2}(i)$ for all $i \geq 1$. Then, from Eq. (3.13) we have

$$\begin{aligned} V_t(i) &= V_1(i) + (1-s)\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)\max\{0, V_{t-1}(i-\ell)\} \\ &\geq V_1(i) + (1-s)\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)\max\{0, V_{t-2}(i-\ell)\} = V_{t-1}(i). \end{aligned}$$

This completes the induction. ∎

Lemma 5.5 $\lim_{i\to\infty} V_t(i) = s$ *for all* $i \geq 1$ *and* $t \geq 1$.

PROOF. Let

$$A_t(i) = \sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)\max\{0, V_{t-1}(i-\ell)\}.$$

Then, Eq. (3.13) can be rewritten as follows.

$$V_t(i) = V_1(i) + (1-s)A_t(i).$$

Now, since clearly $|\max\{0, V_t(i)\}| \leq 1$ for all $i \geq 1$ and $t \geq 1$, we obtain

$$0 \leq |A_t(i)| \leq \sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)|\max\{0, V_{t-1}(i-\ell)\}| \leq \sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i).$$

Now, it follows from Eq. (4.3) of Lemma 4.1 that for any infinitesimal number $\varepsilon > 0$ there exists a certain $I(\varepsilon|\ell)$ such that $if_{qr}(0,\ell|i) < \varepsilon$, or $f_{qr}(0,\ell|i) < \varepsilon/i$ for all $i > I(\varepsilon|\ell)$. Hence,

$$\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i) < \sum_{\ell=0}^{i-1} \varepsilon/i = \varepsilon, \qquad i > I(\varepsilon|\ell).$$

Thus, $\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)$ converges to 0 as $i \to \infty$. Consequently, it follows that $\lim_{i\to\infty}|A_t(i)| = 0$, so $\lim_{i\to\infty} A_t(i) = 0$. From Eq. (5.1) and Eq. (4.2) of Lemma 4.1 we immediately obtain $\lim_{i\to\infty} V_1(i) = s$. Eventually it follows that

$$\lim_{i\to\infty} V_t(i) = \lim_{i\to\infty} V_1(i) + (1-s)\lim_{i\to\infty} A_t(i) = s. \quad ∎$$

5.2 Case of $s = 0$

Since $p - z = (q+r)p - q$ due to Eq. (4.1), noting Eq. (5.2), we have

$$p > (= (<))\, z \iff p > (= (<))\, q/(q+r) \iff V_1(1) > (= (<))\, 0. \tag{5.12}$$

Lemma 5.6 *Let $s = 0$. If $p \geq (<)\, q/(q+r)$, then $V_t(i) \geq (<)\, 0$ for all $i \geq 1$ and $t \geq 1$.*

PROOF. Since $s = 0$, Eq. (5.1) becomes

$$\begin{aligned} V_1(i) &= (1-z)^i - (1-p)^i \\ &= \Big((1-z) - (1-p)\Big)\Big((1-z)^{i-1} + (1-z)^{i-2}(1-p) + \cdots + (1-z)(1-p)^{i-2}(1-p)^{i-1}\Big) \\ &= (p-z)\Big((1-z)^{i-1} + (1-z)^{i-2}(1-p) + \cdots + (1-z)(1-p)^{i-2}(1-p)^{i-1}\Big). \end{aligned} \tag{5.13}$$

From this and Eq. (5.12), the assertion for $t = 1$ becomes true. Accordingly, if $p \geq q/(q+r)$, i.e., $V_1(i) \geq 0$ for all i from Eq. (5.12), hence $V_t(i) \geq 0$ for all $i \geq 1$ and $t \geq 1$ from Lemma 5.4, and if $p < q/(q+r)$, i.e., $V_1(i) < 0$ for all i, hence $V_t(i) < 0$ for all $i \geq 1$ and $t \geq 1$ from Lemma 5.3. Thus the assertion holds. ∎

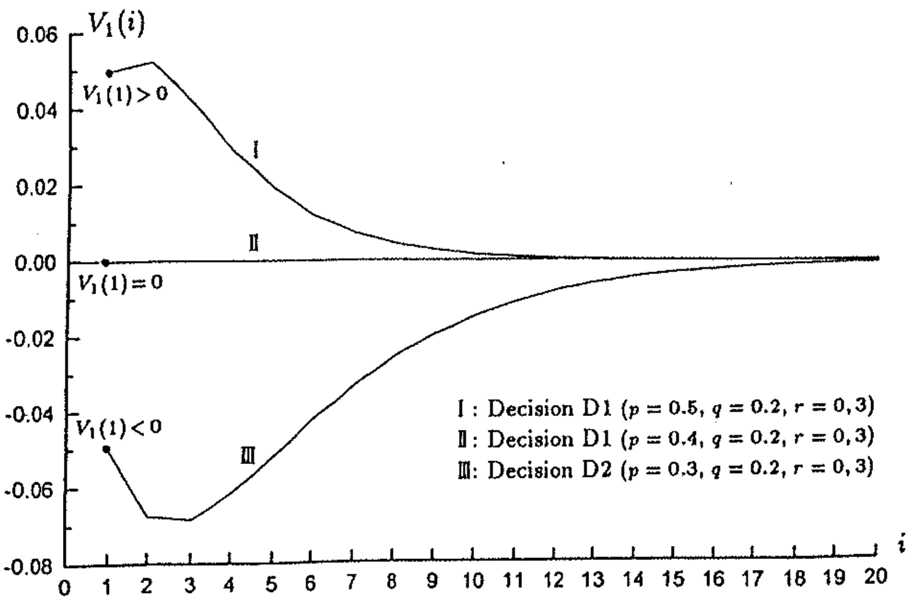

Figure 5.1: The graph of $V_1(i)$ with i ($s = 0$)

5.3 Case of $s > 0$ and $p \geq (1-s)z$

Lemma 5.7 *Let $s > 0$ and $p \geq (1-s)z$. Then, $V_t(i) > 0$ for all $i > 1$ and $t \geq 1$.*

PROOF. Assume $s > 0$.

Let $p = (1-s)z$. Then, from Eq. (5.2) we have $V_1(1) = 0$. Now, since $z > 0$ and $s > 0$, clearly $z > (1-s)z = p$, so $1-p > 1-z$. Then, from Eq. (5.7) we get for all $i > 1$,

$$V_1(i+1) - V_1(i) = p(1-p)^i - p(1-z)^i = p\Big((1-p)^i - (1-z)^i\Big) > 0,$$

hence it follows that $V_1(i)$ is strictly increasing in i, so $V_1(i) > 0$ for all $i > 1$.

Let $p > (1-s)z$. Then, from Eq. (5.2) we have $V_1(1) > 0$. If $p < z$, so $1-p > 1-z$. Then, from Eq. (5.7) we have, for all $i > 1$,

$$\begin{aligned} V_1(i+1) - V_1(i) &> (1-s)z(1-p)^i - (1-s)z(1-z)^i \\ &= (1-s)z\Big((1-p)^i - (1-z)^i\Big) > 0, \end{aligned}$$

hence it follows that $V_1(i)$ is strictly increasing in i, so $V_1(i) > 0$ for all $i > 1$. If $p \geq z$. Then, from (a) of Corollary 5.1 we obtain, for all $i \geq 1$, that if Eq. (5.8) is satisfied, then Eq. (5.9) must necessarily occur because the limit of $V_1(i)$ in i is positive due to the assumption $s > 0$ and Lemma 5.5. This is a contradiction; therefore, Eq. (5.8) does not occur at all; in other words, it must be that $V_1(i) > 0$ for all $i > 1$.

Thus, it follows that $V_1(i) > 0$ for all $i > 1$ if $p \geq (1-s)z$. Accordingly, from Lemma 5.4 we get $V_t(i) \geq V_1(i) > 0$ for all $i > 1$ and $t \geq 1$. ∎

5.4 Case of $s > 0$ and $p < (1-s)z$

By $i(t)$ let us define the i satisfying

$$V_t(i-1) < 0 \leq V_t(i), \quad i > 1, \quad t \geq 1, \tag{5.14}$$

if it exists.

Lemma 5.8 *Let $s > 0$ and $p < (1-s)z$. Then:*

(a) *If* $V_1(i) < 0$ *and* $V_1(i+1) = 0$ *for a certain* $i \geq 1$, *we have* $V_1(i+2) > 0$ *for that* i.

(b) *There exists a unique* $i(1)$, *hence* $V_1(i) < 0$ *for* $i < i(1)$, $V_1(i(1), 1) \geq 0$ *and* $V_1(i) > 0$ *for* $i > i(1)$.

(c) $V_t(1) < 0$ *for all* t.

(d) *If* $V_1(i') < 0$ *for a certain* $i' \geq 1$, *then* $V_t(i) = V_1(i) < 0$ *for all* $i \leq i'$ *and* $t \geq 1$.

(e) $i(t)$ *is unique for all* $t \geq 1$ *and independent of* t.

PROOF. Let $s > 0$ and $p < (1-s)z$. Then, from the fact of $z > (1-s)z$ due to $z > 0$ and $s > 0$, we get $z > p$, and from Eq. (5.2) we have $V_1(i) < 0$.

(a) If $V_1(i) < 0$ and $V_1(i+1) = 0$ for a certain i, then from $b(i) = V_1(i+1) - V_1(i)$ and $b(i+1) = V_1(i+2) - V_1(i+1)$ by definition (See the proof of Lemma 5.2) we immediately get, for that i,

$$b(i) > 0, \tag{5.15}$$

$$V_1(i+2) = b(i+1). \tag{5.16}$$

From Eqs. (5.15) and (5.7) we have

$$p(1-p)^i > (1-s)z(1-z)^i. \tag{5.17}$$

Noting Eqs. (5.17) and (5.7), Eq. (5.16) can be expressed as follows.

$$\begin{aligned} V_1(i+2) &= p(1-p)^{i+1} - (1-s)z(1-z)^{i+1} \\ &> (1-s)z(1-z)^i(1-p) - (1-s)z(1-z)^{i+1} \\ &= (1-s)z(1-z)^i(1-p-1+z) \\ &= (1-s)z(1-z)^i(z-p) > 0. \end{aligned}$$

(b) Since $V_1(1) < 0$, and the fact that $V_1(i) > 0$ for a sufficiently large i due to Lemma 5.5, clearly, $i(1)$ exists. Now, if Eq. (5.10) is satisfied, then, Eq. (5.11) does not occur at all. Noting this and (a) we immediately obtain that $i(1)$ is unique. Thus $V_1(i) < 0$ for $i < i(1)$, $V_1(i(1), 1) \geq 0$ and $V_1(i) > 0$ for $i > i(1)$.

(c)Since $V_1(1) < 0$. Hence, the assertion is true for $t = 1$. Suppose $V_{t-1}(1) < 0$. Then, from Eq. (3.13) we get $V_t(1) = V_1(1) < 0$. This completes the induction.

(d) From (b) we immediately obtain that the assertion is true if $t = 1$. Suppose the assertion is true for $t - 1$; that is, $V_{t-1}(i) = V_1(i) < 0$ for $i \leq i'$, so $V_{t-1}(i - \ell) < 0$ for $0 \leq \ell \leq i - 1$ and

$i \le i'$. Accordingly, from Eq. (3.13) we immediately get $V_t(i) = V_1(i) < 0$. This completes the induction.

(e) From (c) and Lemma 5.5 we immediately get $i(t)$ exists, and from (b) we have a unique $i(1)$, hence $V_1(i) < 0$ for $i \le i(1) - 1$, $V_1(i(1)) \ge 0$, and $V_1(i) > 0$ for $i > i(1)$. Now, letting $i' = i(1) - 1$, from (d) we have $V_t(i) = V_1(i) < 0$ for $i \le i'$, and from Lemma 5.4 we have $V_t(i(1)) \ge V_1(i(1)) \ge 0$ and $V_t(i) \ge V_1(i) > 0$ for $i > i(1)$. Accordingly, by the definition of $i(t)$ it follows that $i(t) = i(1)$. This implies that $i(t)$ is unique and independent of t for all $t \ge 1$. ∎

Now, noting Eq. (5.2), we immediately have

$$p > (= (<))\, (1 - s)z \iff V_1(1) > (= (<))\, 0. \tag{5.18}$$

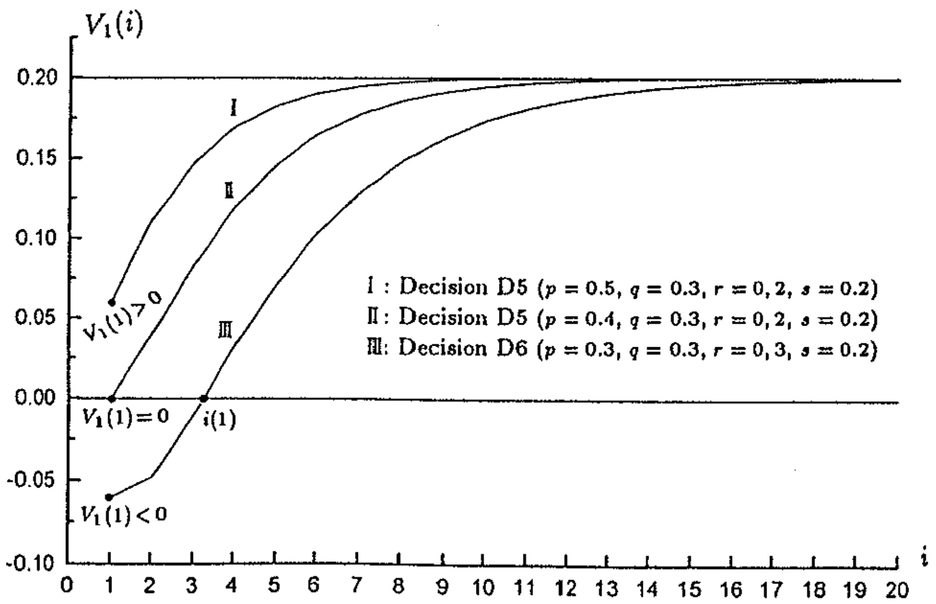

Figure 5.2: The graph of $V_1(i)$ with i ($s > 0$)

6 Concluding Remarks

6.1 Case of $s = 0$

In hostage plots perpetrated by a person who is determined to go through with it no matter what, and not surrender on any terms, he knows that, if arrested, he will be condemned to death or life imprisonment. This can be regarded as the case of $s = 0$. From Lemma 5.6 the optimal