$$
\begin{aligned}
V_t(i) &= (1-s)\left(\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)\Big(\max\{0, V_{t-1}(i-\ell)\} + P(i-\ell)\Big) + r^i\right) + s - P(i) \\
&= (1-s)\left(\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)P(i-\ell) + r^i\right) + s - P(i) \\
&\quad + (1-s)\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)\max\{0, V_{t-1}(i-\ell)\}, \quad i \geq 1, \quad t \geq 2,
\end{aligned} \tag{3.11}
$$

and noting Eq. (3.8), we can rearrange Eq. (3.9) for $t = 1$ as follows.

$$
V_1(i) = (1-s)\left(\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)P(i-\ell) + r^i\right) + s - P(i), \quad i \geq 1. \tag{3.12}
$$

Accordingly, Eq. (3.11) becomes as follows.

$$
V_t(i) = V_1(i) + (1-s)\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)\max\{0, V_{t-1}(i-\ell)\}, \quad i \geq 1, \quad t \geq 2. \tag{3.13}
$$

4 Preliminaries

Let

$$
z = q + (1-q-r)p, \tag{4.1}
$$

where $0 < z < 1$ due to the assumptions of p, q and r.

Lemma 4.1 *For all $i \geq 1$ we have*

$$
\lim_{i\to\infty} (1-p)^i = \lim_{i\to\infty} (1-z)^i = 0, \tag{4.2}
$$

$$
\lim_{i\to\infty} i f_{qr}(k,\ell|i) = 0. \tag{4.3}
$$

PROOF. Using the Stirling asymptotic formula $i! \sim \sqrt{2\pi}\, i^{i+0.5}e^{-i}$, we obtain

$$
\begin{aligned}
i f_{qr}(k,\ell|i) &= i\frac{i!}{k!\ell!(i-k-\ell)!}q^k r^\ell (1-q-r)^{i-k-\ell} \\
&\sim \frac{q^k r^\ell}{k!\ell!(1-q-r)^{k+\ell}} \frac{\sqrt{2\pi} i i^{i+0.5} e^{-i}}{\sqrt{2\pi}(i-k-\ell)^{i-k-\ell+0.5}e^{-(i-k-\ell)}}(1-q-r)^i \\
&= \frac{q^k r^\ell e^{-(k+\ell)}}{k!\ell!(1-q-r)^{k+\ell}}\left(\frac{i}{i-k-\ell}\right)^{0.5}\left(\frac{i}{i-k-\ell}\right)^{i-k-\ell} i^{k+\ell+1}(1-q-r)^i.
\end{aligned}
$$

Now, for convenience, let $u = 1-p$ and $w = 1-q-r$ where $0 < u < 1$ and $0 < w < 1$ due to the assumptions of p, q and r. Then, consider $\delta > 0$ and $\gamma > 0$ such that

$$u = 1/(1+\delta),$$
$$w = 1/(1+\gamma).$$

Hence, for any sufficiently large i we have

$$\begin{aligned}(1+\delta)^i &= 1 + i\delta + \cdots > i\delta,\\ (1+\gamma)^i &= 1 + i\gamma + \cdots + \binom{i}{k+\ell+2}\gamma^{k+\ell+2} + \cdots\\ &> \binom{i}{k+\ell+2}\gamma^{k+\ell+2} = \frac{i!}{(k+\ell+2)!(i-k-\ell-2)!}\gamma^{k+\ell+2},\end{aligned}$$

from which we get

$$u^i = 1/(1+\delta)^i < \frac{1}{i\delta},$$
$$w^i = 1/(1+\gamma)^i < \frac{(k+\ell+2)!(i-k-\ell-2)!}{i!\gamma^{k+\ell+2}} = \frac{(k+\ell+2)!}{i(i-1)(i-2)\cdots(i-k-\ell)(i-k-\ell-1)\gamma^{k+\ell+2}}.$$

Consequently, it follows that

$$(1-p)^i = u^i < \frac{1}{i\delta},$$
$$\begin{aligned}i^{k+\ell+1}(1-q-r)^i = i^{k+\ell+1}w^i &< \frac{i^{k+\ell+1}(k+\ell+2)!}{i(i-1)(i-2)\cdots(i-k-\ell)(i-k-\ell-1)\gamma^{k+\ell+2}}\\ &= \frac{(k+\ell+2)!}{\gamma^{k+\ell+2}}\frac{1}{i(1-\frac{1}{i})(1-\frac{2}{i})\cdots(1-\frac{k+\ell}{i})(1-\frac{k+\ell+1}{i})},\end{aligned}$$

both of which converge to 0 as $i \to \infty$. Noting,

$$\lim_{i\to\infty}\left(\frac{i}{i-k-\ell}\right)^{0.5} = \lim_{i\to\infty}\left(\frac{1}{1-\frac{k+\ell}{i}}\right)^{0.5} = 1,$$
$$\lim_{i\to\infty}\left(\frac{i}{i-k-\ell}\right)^{i-k-\ell} = \lim_{i\to\infty}\left(1+\frac{k+\ell}{i-k-\ell}\right)^{i-k-\ell} = e^{k+\ell}.$$

Accordingly, $(1-p)^i$ and $if_{qr}(k,\ell\ i)$ converge to 0 as $i \to \infty$. In quite a similar way, we can show $\lim_{i\to\infty}(1-z)^i = 0$. ∎

5 Analysis

5.1 General Properties of $V_t(i)$