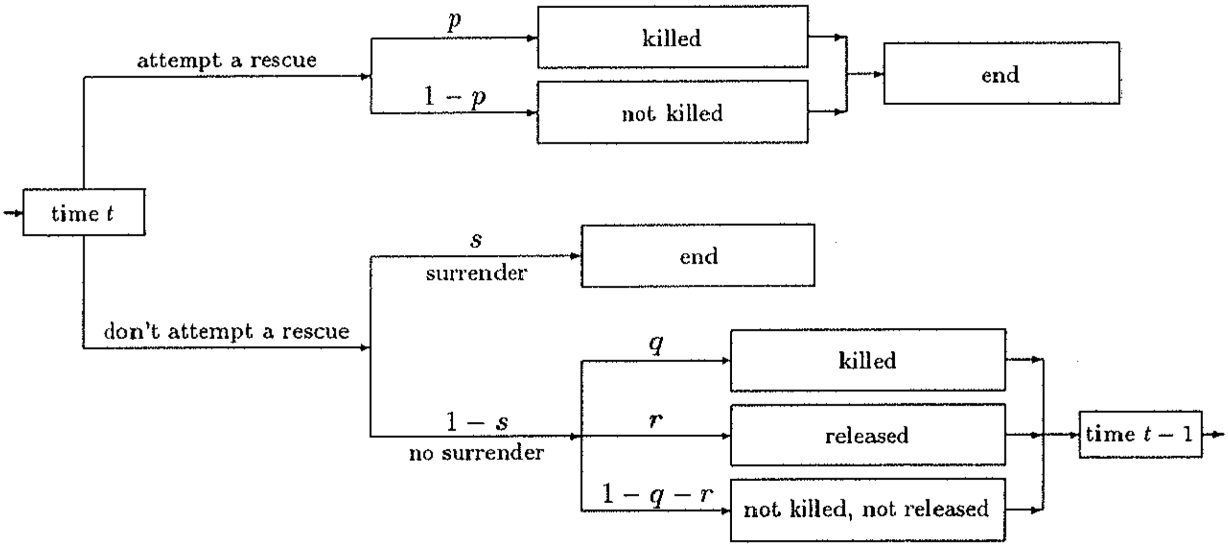

Figure 2.1: Decision Tree

3 Optimal Equation

Let $f_p(m|i)$ be the probability of m hostages being killed among i hostages if rescue attempt is made ($x = 1$), which is given by

$$f_p(m|i) = \binom{i}{m} p^m (1-p)^{i-m}, \qquad i \geq 1, \quad 0 \leq m \leq i. \tag{3.1}$$

Let the probability of no hostage being killed if a rescue attempt is made ($x = 1$) at any time t be denoted by $P(i)$. Then

$$P(i) = f_p(0|i) = (1-p)^i, \qquad i \geq 1. \tag{3.2}$$

Further, let $f_{qr}(k, \ell|i)$ be the probability of k hostages being killed and ℓ hostages being set free among i hostages if no rescue attempt ($x = 0$) and criminal(s) not surrendering up to the next point in time, which is given by

$$f_{qr}(k, \ell|i) = \frac{i!}{k!\ell!(i-k-\ell)!} q^k r^\ell (1-q-r)^{i-k-\ell}, \qquad i \geq 1, \quad 0 \leq k + \ell \leq i. \tag{3.3}$$

Now, let $v_t(i)$ be the maximum probability of no hostage being killed, starting from time $t \geq 0$ with i hostages, expressed as

$$v_0(i) = P(i), \qquad i \geq 1, \tag{3.4}$$

$$v_t(i) = \max\{P(i), W_t(i)\}, \qquad i \geq 1, \quad t \geq 1, \tag{3.5}$$

where $W_t(i)$ is the probability of no hostage being killed over the period from time t to 0 (deadline) if no rescue attempt is made ($x = 0$). Noting $k + \ell \leq i$ and $k = 0$, we can express $W_t(i)$ as

$$W_t(i) = (1-s)\left(\sum_{\ell \leq i-1} f_{qr}(0,\ell|i)v_{t-1}(i-\ell) + f_{qr}(0,i|i) \times 1\right) + s \times 1, \quad i \geq 1, \quad t \geq 1, \tag{3.6}$$

the right hand side of which implies the following:

At any given point in time t,

(a) Suppose the criminal(s) surrender with probability s. All the hostages are released, in other words, no hostage is killed; accordingly, the probability of no hostage being killed is equal to 1.

(b) Suppose criminal(s) do not surrender with probability $1-s$, Then, the probability of no hostage being killed is $f_{qr}(0,\ell|i)$ if ℓ hostages are released.

   1. If i hostages are released, then clearly $\ell = i$, hence the probability of no hostage being killed is equal to 1.
   2. If $i-1$ or less hostages are released ($\ell \leq i-1$), then the number of remaining hostages becomes $i-\ell$, implying that the probability of no hostage being killed over the period from time $t-1$ to 0 is equal to $v_{t-1}(i-\ell)$ by definition.

Eq. (3.6) can be easily rearranged as follows.

$$W_t(i) = (1-s)\left(\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)v_{t-1}(i-\ell) + r^i\right) + s, \quad i \geq 1, \quad t \geq 1, \tag{3.7}$$

where

$$W_1(i) = (1-s)\left(\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)P(i-\ell) + r^i\right) + s, \quad i \geq 1. \tag{3.8}$$

Now, let

$$V_t(i) = W_t(i) - P(i), \quad i \geq 1, \quad t \geq 1, \tag{3.9}$$

where clearly $-1 \leq V_t(i) \leq 1$ for all i and t. Then, Eq. (3.5) can be rewritten as follows.

$$v_t(i) = \max\{0, V_t(i)\} + P(i), \quad i \geq 1, \quad t \geq 1. \tag{3.10}$$

Hence, the optimal decision rule can be stated as follows: If $V_t(i) < 0$, attempt a rescue, or else, do not attempt a rescue. Furthermore, noting Eqs. (3.7) and (3.10), we can rewrite Eq. (3.9) for $t \geq 2$ as follows.

$$V_t(i) = (1-s)\left(\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)\Big(\max\{0, V_{t-1}(i-\ell)\} + P(i-\ell)\Big) + r^i\right) + s - P(i)$$

$$= (1-s)\left(\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)P(i-\ell) + r^i\right) + s - P(i)$$

$$+ (1-s)\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)\max\{0, V_{t-1}(i-\ell)\}, \quad i \geq 1, \quad t \geq 2, \tag{3.11}$$

and noting Eq. (3.8), we can rearrange Eq. (3.9) for $t = 1$ as follows.

$$V_1(i) = (1-s)\left(\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)P(i-\ell) + r^i\right) + s - P(i), \quad i \geq 1. \tag{3.12}$$

Accordingly, Eq. (3.11) becomes as follows.

$$V_t(i) = V_1(i) + (1-s)\sum_{\ell=0}^{i-1} f_{qr}(0,\ell|i)\max\{0, V_{t-1}(i-\ell)\}, \quad i \geq 1, \quad t \geq 2. \tag{3.13}$$

4 Preliminaries

Let

$$z = q + (1-q-r)p, \tag{4.1}$$

where $0 < z < 1$ due to the assumptions of p, q and r.

Lemma 4.1 *For all $i \geq 1$ we have*

$$\lim_{i\to\infty}(1-p)^i = \lim_{i\to\infty}(1-z)^i = 0, \tag{4.2}$$

$$\lim_{i\to\infty} i f_{qr}(k,\ell|i) = 0. \tag{4.3}$$

Proof. Using the Stirling asymptotic formula $i! \sim \sqrt{2\pi}\; i^{i+0.5}e^{-i}$, we obtain

$$i f_{qr}(k,\ell|i) = i\frac{i!}{k!\ell!(i-k-\ell)!}q^k r^\ell (1-q-r)^{i-k-\ell}$$

$$\sim \frac{q^k r^\ell}{k!\ell!(1-q-r)^{k+\ell}}\frac{\sqrt{2\pi}\, i\, i^{i+0.5}e^{-i}}{\sqrt{2\pi}(i-k-\ell)^{i-k-\ell+0.5}e^{-(i-k-\ell)}}(1-q-r)^i$$

$$= \frac{q^k r^\ell e^{-(k+\ell)}}{k!\ell!(1-q-r)^{k+\ell}}\left(\frac{i}{i-k-\ell}\right)^{0.5}\left(\frac{i}{i-k-\ell}\right)^{i-k-\ell} i^{k+\ell+1}(1-q-r)^i.$$

Now, for convenience, let $u = 1-p$ and $w = 1-q-r$ where $0 < u < 1$ and $0 < w < 1$ due to the assumptions of p, q and r. Then, consider $\delta > 0$ and $\gamma > 0$ such that