No. 947

Optimal Hostage Rescue Problem Where an Action Can Only Be Taken Once
-Case Where Its Effectiveness Lasts up to the Deadline-

by

Fengbo Shi

September 2001

OPTIMAL HOSTAGE RESCUE PROBLEM WHERE AN ACTION CAN ONLY BE TAKEN ONCE

— CASE WHERE ITS EFFECTIVENESS LASTS UP TO THE DEADLINE —

Fengbo Shi
University of Tsukuba, Japan
September 3, 2001

Abstract

We propose the following mathematical model for optimal rescue problems concerning hostages. Suppose that a person is taken as a hostage and that a decision has to be made from among three alternatives: storm for rescue, or wait up to the next point in time for an opportunity to present itself, or take one action of negotiation which might save the situation. It is assumed that the action of negotiation can only be taken once and it will be effective up to the deadline. The objective is to find an optimal decision rule so as to maximize the probability of a hostage not being killed. Several properties of the optimal decision rule are revealed.

1 Introduction

Acts involving hostage taking occur for different reasons, e.g., social inequality, poverty, religious problems, racial problems, political problems, and so on. The problem has become an urgent issue to be tackled worldwide. Typical examples in recent years include:

1 A 17-year-old youth wielding a knife, hijacked a bus on the Sanyo Expressway in Japan and killed a 68-year-old hostage. After 15 hours, the police stormed the bus, the other hostages were rescued, and the hijacker was arrested (May 4, 2000).

2 An armed man took a Finance Ministry official hostage in the Tokyo Stock Exchange building and demanded a meeting with the Finance Minister. He surrendered to the police after a tense, five and half hour standoff (January 12, 1998).

3 Fourteen guerrillas stormed the home of the Japanese ambassador to Peru and took about three hundred people hostage, including diplomats and government officials attending a birthday party for the emperor. All but one of the hostages were rescued while all the rebels were killed when special forces stormed the building (December 17, 1996).

4 A man with a knife broke into a house and took a 2-year-old boy hostage in Japan. The police finally rushed into the house, set the uninjured boy free, and arrested the criminal (December 1, 1995).

Although the information is not available for accurate statistics, it could be said that different scenarios of the above continue to occur all over the world. The most important decision for the person in charge of crisis settlement is the timing to enact rescue of the hostages. Wrestling with the problem, needless to say, involves many factors, political, economical, sociological, psychological, and so on, and all must be taken into acount, together with the safety of hostages, the demands of criminals, the repercussions of success or failure in a rescue attempt, and so on. The purpose of this paper is to propose a mathematical model of an optimal hostage rescue problem by using the concept of a sequential stochastic decision processes and examine properties of an optimal decision rule. The author has proposed and examined a model based on the problem in [1] where only two alternatives, storming for rescue or waiting up to the next point in time for an opportunity to present itsilf, were available. However, as is seen in many hostage cases, negotiators take varied actions to condescend to the kidnapper(s), for example, presuading him/her to surrender by subjecting him/her to his/her mother's voice, or submitting to his/her demands to be airlifted to another country, or providing a means of escape, paying the ransom, releasing his/her comrades in prison, and so on. Therefore, it is necessary to put such an action of negotiation in our rescue decision, that is, we should make a rescue decision from among three alternatives: storm for rescue, or wait up to the next point in time for an opportunity to present itsilf, or take one action of negotiation which hopefully will save the situation. The author has already proposed and examined a basic model in [2] where such an action of negotiation can only be taken once and will be effective only at that time, i.e., the effectiveness vanishes thereafter. In this new paper we propose another basic model where such an action of negotiation can only be taken once and will be effective up to the deadline. Unfortunately, concerning this problem, with the exception of the author's two papers [1] [2], we have been unable to find any reference material based on any mathematical approach. Accordingly, we cannot list references to be directly cited except for the two above.

2 Model

Consider the following sequential stochastic decision process with a finite planning horizon. Here, for convenience, let points in time be numbered backward from the final point in time of the planning horizon, time 0, as 0, 1, $\cdots$, and so on. Let the time interval between two successive points, say times t and $t-1$, be called the period t. Here, assume that time 0 is the deadline at which storming for rescue is considered to be the only course of action, prompted by some reason, say, the hostage's health condition, the degree of criminal desperation, and so on.

Suppose one person is taken hostage at any given point in time t, and a decision has to be make from among three alternatives: storm for rescue, or wait up to the next point in time for an opportunity to present itsilf, or take one action of negotiation which might save the situation. Here, let us assume that the action of negotiation can only be taken once and that if the action of negotiation is taken at a certain point in time, then it is effective up to the deadline.

For simplicity, by S, W, and A let us denote the decisions of, respectively, "storm for rescue", "wait up to the next point in time for an opportunity to present itself", and "take one action of negotiation which might save the situation".

Provided that the action of negotiation has not yet been taken, let p $(0 < p < 1)$ be the probability of the hostage being killed if the decision S is made, let q and r $(0 < q < 1$ and $0 \leq r < 1)$ be the probabilities of the hostage being, respectively, killed and released up to the next point in time if the decision W is made; accordingly $1 - q - r$ $(0 < q + r < 1)$ is the probability of the hostage being neither killed nor set free. Now, noting the fact that taking the action of negotiation will influence the probabilities p, q and r to a greater or lesser degree, in this model let us suppose that the p, q and r thus far change into p', q' and r', respectively, while the action of negotiation is taken at a certain point in time, and that p', q' and r' are effective following thereafter. Consequently, provided that the action of negotiation has already been taken, p' $(0 < p' < 1)$ is the probability of the hostage being killed if the decision S is made, q', r', and $1 - q' - r'$ $(0 < q' < 1$, $0 \leq r' < 1$, and $0 < q' + r' < 1)$ are the probabilities of the hostage being, respectively, killed, released, and neither killed nor released up to the next point in time if the decision W is made. Now, the cases of $p = p' = q = q' = 0$, $p = p' = q = q' = r = r' = 1$, and $q + r = q' + r' = 1$ make the problem trivial. Accordingly, all are excluded in the definition of the model.

The objective here is to maximize the probability of the hostage not being killed.

3 Optimal Equation

Provided that the action of negotiation has not yet (has already) been taken up to time t, let v_t (v'_t) be the maxmium probability of the hostage not being killed, and let $S = 1 - p$ ($S' = 1 - p'$), be the probability of the hostage not being killed if the decision S is made at any time. Then, we have

$$v_t = \max\{S, W_t, A_t\}, \quad t \geq 1, \quad v_0 = S, \tag{3.1}$$

$$v'_t = \max\{S', W'_t\}, \quad t \geq 1, \quad v'_0 = S' \tag{3.2}$$

where, provided that the action of negotiation has not yet been taken up to time t, W_t and A_t are the probabilities of the hostage not being killed from times t to 0, respectively, if the decision W is made and if the decision A is made, and provided that the action negotiation has already been taken up to time t, W'_t is the probability of the hostage not being killed from times t to 0 if the decision W is made. Accordingly, we can express the W_t, A_t, and W'_t for $t \geq 1$ as follows.

$$W_t = r + (1 - q - r)v_{t-1}, \tag{3.3}$$

$$A_t = r' + (1 - q' - r')v'_{t-1}, \tag{3.4}$$

$$W'_t = r' + (1 - q' - r')v'_{t-1} \tag{3.5}$$

where the implications of the A_t and W'_t are different although both have the same expressions. The above three expressions imply the following:

Note that the expression Eq. (3.3) should be rewritten $W_t = q \times 0 + r \times 1 + (1 - q - r)v_{t-1}$. This can be interpreted as follows. Suppose the action of negotiation has not yet been taken up to time t and the decision W is made at time t. Then, if the hostage is killed with the probability q, the probability of the hostage not being killed is equal to $q \times 0$, if the hostage is released with the probability r, the probability of the hostage not being killed is equal to $r \times 1$, and if the hostage is neither killed nor released with the probability $1 - q - r$, the probability of the hostage not being killed over the period from times $t - 1$ to 0 is equal to $(1 - q - r)v_{t-1}$. Further, Eq. (3.4) and Eq. (3.5) can be also similarly interpreted.

For convenience in later discussions, let us define

$$\lambda = 1 - q - r, \quad \lambda' = 1 - q' - r', \quad 0 < \lambda, \lambda' < 1, \tag{3.6}$$

$$\delta = -(r' - r)/(\lambda' - \lambda), \quad \lambda \neq \lambda', \tag{3.7}$$

$$U = r + \lambda S, \quad U' = r' + \lambda' S', \tag{3.8}$$

$$\alpha = U/U', \tag{3.9}$$

$$Z_t = A_t - W_t, \quad t \geq 1. \tag{3.10}$$

Then, from Eqs. (3.3) to (3.5) we clearly have

$$W_t = r + \lambda v_{t-1}, \quad t \geq 2, \quad W_1 = U, \tag{3.11}$$

$$A_t = W'_t = r' + \lambda' v'_{t-1}, \quad t \geq 2, \quad A_1 = W'_1 = U'. \tag{3.12}$$

4 Preliminaries

The two lemmas below will be used in the subsequent sections.

Lemma 4.1 *v_t, v'_t, W_t, A_t, and W'_t are all nondecreasing in t, hence converage to finite numbers v, v', W, A, and W', respectively, as $t \to \infty$.*

Proof. From Eqs. (3.1) and (3.2) we have $v_1 \geq S = v_0$ and $v'_1 \geq S' = v'_0$. Suppose $v_{t-1} \geq v_{t-2}$ and $v'_{t-1} \geq v'_{t-2}$. Then $W_t \geq W_{t-1}$, $A_t \geq A_{t-1}$, and $W'_t \geq W'_{t-1}$ from Eqs. (3.11) and (3.12), hence $v_t = \max\{S, W_t, A_t\} \geq \max\{S, W_{t-1}, A_{t-1}\} = v_{t-1}$ and $v'_t = \max\{S', W'_t\} \geq \max\{S', W'_{t-1}\} = v'_{t-1}$. Accordingly, the monotonicities of v_t and v'_t hold by induction. Further, the monotonicities of W_t, A_t, and W'_t also hold from Eqs. (3.11) and (3.12). Now, noting the fact that v_t, v'_t, W_t, A_t, and W'_t are all bounded because they are all probabilities, it follows that their limits as $t \to \infty$ exist. ∎

Lemma 4.2

(a) *If* $A_t = r' + \lambda' A_{t-1}$ *for* $t \geq 2$, *then* $A_t = r'(1 - \lambda'^{t-1})/(1 - \lambda') + \lambda'^{t-1}U'$ *for* $t \geq 2$.

(b) *If* $W_t = r + \lambda W_{t-1}$ *for* $t \geq 2$, *then* $W_t = r(1 - \lambda^{t-1})/(1 - \lambda) + \alpha\lambda^{t-1}U'$ *for* $t \geq 2$.

(c) *If* $W_t = r + \lambda W_{t-1}$ *for* $t \geq 3$ *and* $v_1 = A_1$, *then* $W_t = r(1 - \lambda^{t-1})/(1 - \lambda) + \lambda^{t-1}U'$ *for* $t \geq 2$.

Proof.

(a) Let $A_t = r' + \lambda' A_{t-1}$ for $t \geq 2$. Then, noting $A_1 = U'$, we have $A_t = r'(1 + \lambda' + \cdots + \lambda'^{t-2}) + \lambda'^{t-1}U' = r'(1 - \lambda'^{t-1})/(1 - \lambda') + \lambda'^{t-1}U'$.

(b) It is from $W_1 = U$ and $\alpha = U/U'$ that the assertion holds in the same way as (a).

(c) Let $v_1 = A_1$. Then $W_2 = r + \lambda U'$ from Eqs. (3.11) and Eq. (3.12). Therefore, in the same way as (a) we can prove that the assertion is true. ∎

5 Analysis

In this section, we examine the properties of the optimal decision rule for the problem, classifying all the possible combinations of the parameters, p, q, r, p', q', and r' into the two cases below:

$$\text{Case A: } S' \geq U', \qquad \text{Case B: } S' < U'.$$

Further, the each of the above two cases is classified into the following three cases.

Case 1: $S > U$ and $S > U'$, Case 2: $U' \geq S$ and $U' \geq U$, Case 3: $U \geq S$ and $U > U'$.

5.1 Case A: $S' \geq U'$

Lemma 5.1 *Assume* $S' \geq U'$. *Then* $v'_t = S'$ *and* $A_t = U'$ *for* $t \geq 1$.

Proof. Assume $S' \geq U'$. From Eqs. (3.2) and (3.12) we have $v'_1 = \max\{S', U'\} = S'$. Suppose $v'_{t-1} = S'$. Then $W'_t = r' + \lambda' S' = U'$ from Eqs. (3.12) and (3.8), hence $v'_t = \max\{S', W'_t\} = \max\{S', U'\} = S'$. Accordingly $v_t = S'$ for $t \geq 1$ by induction. Further, from Eqs. (3.12) and (3.8) we get $A_t = r' + \lambda' S' = U'$ for $t \geq 1$. ∎

5.1.1 Case 1: $S > U$ and $S > U'$

Lemma 5.2 *Assume* $S' \geq U'$, $S > U$, *and* $S > U'$. *Then* $v_t = S$ *for* $t \geq 1$.

Proof. Assume $S' \geq U'$, $S > U$, and $S > U'$. From Eqs. (3.1), (3.11), and (3.12) we have $v_1 = \max\{S, U, U'\} = S$. Suppose $v_{t-1} = S$. Then $W_t = r + \lambda S = U$ from Eqs. (3.11) and (3.8), hence, noting $A_t = U'$ in Lemma 5.1, we get $v_t = \max\{S, W_t, A_t\} = \max\{S, U, U'\} = S$. Accordingly $v_t = S$ for $t \geq 1$ by induction. ∎

5.1.2 Case 2: $U' \geq S$ and $U' \geq U$

Lemma 5.3 *Assume $S' \geq U'$, $U' \geq S$, and $U' \geq U$.*

(a) $v_1 = A_1$ *and* $v_t = \max\{W_t, A_t\}$ *for* $t \geq 1$.

(b) *If* $U' > r/(1-\lambda)$, *then* $v_t = A_t$ *for* $t \geq 2$.

(c) *If* $U' = r/(1-\lambda)$, *then* $v_t = A_t = W_t$ *for* $t \geq 2$.

(d) *If* $U' < r/(1-\lambda)$, *then* $v_t = W_t$ *for* $t \geq 2$.

Proof. Assume $S' \geq U'$, $U' \geq S$, and $U' \geq U$.

(a) From Eqs. (3.1), (3.11), and (3.12) we have $v_1 = \max\{S, U, U'\} = U' = A_1$. In addition, noting $A_t = U'$ in Lemma 5.1, we get $A_t = U' \geq S$ for $t \geq 1$, hence $v_t = \max\{S, W_t, A_t\} = \max\{W_t, A_t\}$ for $t \geq 1$.

(b) Let $U' > r/(1-\lambda)$. Then $A_2 - W_2 = A_2 - r - \lambda A_1 = U' - r - \lambda U' = (1-\lambda)U' - r > 0$ form Eq. (3.11), $v_1 = A_1$, and Lemma 5.1, hence $v_2 = \max\{W_2, A_2\} = A_2$. Suppose $v_{t-1} = A_{t-1}$. Then $W_t = r + \lambda A_{t-1} = r + \lambda U'$ due to Eq. (3.11) and Lemma 5.1. From this we get $A_t - W_t = U' - r - \lambda U' = (1-\lambda)U' - r > 0$, hence $v_t = \max\{W_t, A_t\} = A_t$. Accordingly $v_t = A_t$ for $t \geq 2$ by induction.

(c) Let $U' = r/(1-\lambda)$. Then $A_2 - W_2 = (1-\lambda)U' - r = 0$, hence $v_2 = A_2 = W_2$. Suppose $v_{t-1} = A_{t-1} = W_{t-1}$. Then $W_t = r + \lambda U'$, hence $A_t - W_t = (1-\lambda)U' - r = 0$, hence $v_t = A_t = W_t$. Accordingly $v_t = A_t = W_t$ for $t \geq 2$ by induction.

(d) Let $U' < r/(1-\lambda)$. Then $A_2 - W_2 = (1-\lambda)U' - r < 0$, hence $v_2 = W_2$. Suppose $v_{t-1} = W_{t-1}$. Then $A_{t-1} \leq W_{t-1}$ due to (a) and $W_t = r + \lambda W_{t-1}$ from Eq. (3.11). From these we get $A_t - W_t = A_t - r - \lambda W_{t-1} \leq A_t - r - \lambda A_{t-1} = (1-\lambda)U' - r < 0$, hence $v_t = W_t$. Accordingly $v_t = W_t$ for $t \geq 2$ by induction. ∎

5.1.3 Case 3: $U \geq S$ and $U > U'$

Lemma 5.4 *Assume $S' \geq U'$, $U \geq S$, and $U > U'$. Then $v_t = W_t$ for $t \geq 1$.*

Proof. Assume $S' \geq U'$, $U \geq S$, and $U > U'$. Since W_t is nondecreasing in t due to Lemma 4.1, we have $W_t \geq W_1 = U \geq S$ from Eq. (3.11) and $W_t \geq U > U' = A_t$ due to Lemma 5.1. Accordingly $v_t = \max\{S, W_t, A_t\} = W_t$ for $t \geq 1$. ∎

5.2 Case B: $S' < U'$

Lemma 5.5 *Assume $S' < U'$.*

(a) $v'_t = W'_t$ *for* $t \geq 1$.

(b) $A_t = r' + \lambda' A_{t-1}$ *for* $t \geq 2$.

(c) $A = r'/(1-\lambda')$.

(d) A_t *is strictly increasing in* t.

Proof. Assume $S' < U'$.

(a) Since W'_t is nondecreasing in t due to Lemma 4.1, we have $W'_t \geq W'_1 = U' > S'$ for $t \geq 1$ from Eq. (3.12). Accordingly $v'_t = \max\{S', W'_t\} = W'_t$ for $t \geq 1$.

(b) From (a) and Eq. (3.12) we have $v'_t = W'_t = A_t$ for $t \geq 1$, hence $A_t = r' + \lambda' A_{t-1}$ for $t \geq 2$.

(c) Due to (b) and Lemma 4.1 we have $A = r' + \lambda' A$, hence $A = r'/(1-\lambda')$.

(d) From Lemma 4.1 and Eq. (3.12) we have $U' = A_1 \le A_t \le A$, i.e., $A_1 \le A$. Suppose $A_1 = A$. Then $A_t = U'$ for $t \ge 1$, hence $r' + \lambda' v'_{t-1} = U'$ from Eq. (3.12), and further we get $v'_{t-1} = S'$ from Eq. (3.8). Accordingly $S' \ge W'_{t-1} \ge U'$ from Eq. (3.2), which contradicts the assumption $S' < U'$. Consequently, it must be $A_1 < A$. Hence $U' < r'/(1-\lambda')$ due to $U' = A_1$ and (c). Now, from this and (b) we have $A_2 - A_1 = r' - (1-\lambda')A_1 = r' - (1-\lambda')U' > 0$. Suppose $A_t - A_{t-1} = r' - (1-\lambda')A_{t-1} > 0$. Then $A_{t+1} - A_t = r' - (1-\lambda')A_t = r' - (1-\lambda')(r' + \lambda' A_{t-1}) = \lambda'(r' - (1-\lambda')A_{t-1}) > 0$. Accordingly, the assertion holds by induction. ∎

For convenience in the later discussions, let t_S and t_δ be such that, respectively,

$$t_S = \{t \mid A_{t-1} < S \le A_t\}, \quad t_S \ge 2, \tag{5.1}$$

$$t_\delta = \{t \mid A_{t-1} < \delta \le A_t\}, \quad t_\delta \ge 2, \tag{5.2}$$

if they exist. It is clear from Lemma 5.5(d) that each of t_S and t_δ is unique if they exist. Further, let

$$t_z = \min\{t \mid Z_{t-1} < 0 \le Z_t\}, \quad t_z \ge 2, \tag{5.3}$$

which may be infinite.

Now, the lemma below will be used in the subsequent subsections.

Lemma 5.6 *Let $S' < U'$, and for any given $1 \le t' < t'' < \infty$ let $v_t = \max\{W_t, A_t\}$ for $t' \le t \le t''$ ($t' \le t$). Then*

(a) *If $v_{t'} = A_{t'}$ and $(\lambda' - \lambda)A_t + r' - r \ge 0$ for $t' \le t < t''$ ($t' \le t$), then $v_t = A_t$ for $t' \le t \le t''$ ($t' \le t$).*

(b) *If $v_{t'} = W_{t'}$ and $(\lambda' - \lambda)A_t + r' - r \le 0$ for $t' \le t < t''$ ($t' \le t$), then $v_t = W_t$ for $t' \le t \le t''$ ($t' \le t$).*

(c) *If $Z_{t'} < 0$ and $(\lambda' - \lambda)A_t + r' - r < 0$ for $t' \le t < t''$ ($t' \le t$), then $Z_t < 0$ and $v_t = W_t$ for $t' \le t \le t''$ ($t' \le t$).*

(d) *Suppose $Z_{t'} < 0$ and $t_z \notin [1, t')$.*

1 *If $t_z \in [t', t'')$ ($[t', \infty)$) and $(\lambda' - \lambda)A_t + r' - r \ge 0$ for $t_z \le t < t''$ ($t_z \le t$), then $v_t = W_t$ for $t' \le t < t_z$ and $v_t = A_t$ for $t_z \le t \le t''$ ($t_z \le t$).*

2 *If $t_z \notin [t', t'')$ ($[t', \infty)$). Then $v_t = W_t$ for $t' \le t \le t''$ ($t' \le t$).*

Proof. Let $S' < U'$. Then $A_t = r' + \lambda' A_{t-1}$ for $t \ge 2$ due to Lemma 5.5(b). Further, consider t' and t'' such that $1 \le t' < t'' < \infty$, let $v_t = \max\{W_t, A_t\}$ for $t' \le t \le t''$ ($t' \le t$).

(a) Let $v_{t'} = A_{t'}$. Suppose $v_{t-1} = A_{t-1}$ for $t' < t$. Then $W_t = r + \lambda A_{t-1}$ for $t' < t$ from Eq. (3.11). Therefore $A_t - W_t = (\lambda' - \lambda)A_{t-1} + r' - r$ for $t' < t$. Now, since $(\lambda' - \lambda)A_t + r' - r \ge 0$ for $t' \le t < t''$ ($t' \le t$) by the assumption, we immediately get $(\lambda' - \lambda)A_{t-1} + r' - r \ge 0$ for $t' < t \le t''$ ($t' < t$). From this we have $A_t \ge W_t$ for $t' < t \le t''$ ($t' < t$), hence, noting $v_{t'} = A_{t'} = \max\{W_{t'}, A_{t'}\} \ge W_{t'}$, we have $A_t \ge W_t$ for $t' \le t \le t''$ ($t' \le t$). Accordingly $v_t = \max\{W_t, A_t\} = A_t$ for $t' \le t \le t''$ ($t' \le t$).

(b) Let $v_{t'} = W_{t'}$. Suppose $v_{t-1} = W_{t-1}$ for $t' < t$. Then $A_{t-1} \le W_{t-1}$ for $t' < t$ and $W_t = r + \lambda W_{t-1}$ for $t' < t$ from Eq. (3.11). Therefore

$$\begin{aligned} A_t - W_t &= r' + \lambda' A_{t-1} - r - \lambda W_{t-1} \le r' + \lambda' A_{t-1} - r - \lambda A_{t-1} \\ &= (\lambda' - \lambda)A_{t-1} + r' - r \le 0, \quad t' < t \le t'' \ (t' < t) \end{aligned}$$

due to the assumption; that is, $A_t \le W_t$ for $t' < t \le t''$ ($t' < t$), hence, noting $v_{t'} = W_{t'} \ge A_{t'}$, we have $A_t \le W_t$ for $t' \le t \le t''$ ($t' \le t$). Accordingly $v_t = W_t$ for $t' \le t \le t''$ ($t' \le t$).

(c) Let $Z_{t'} < 0$. Then $A_{t'} < W_{t'}$ due to Eq. (3.10). Suppose $Z_{t-1} < 0$ for $t' < t$. Then $A_{t-1} < W_{t-1}$, hence $v_{t-1} = W_{t-1}$ for $t' < t$. Further, from Eq. (3.11) we get $W_{t-1} = r + \lambda W_{t-1}$, therefore $Z_t = A_t - W_t < (\lambda' - \lambda)A_{t-1} + r' - r < 0$ for $t' < t \leq t''$ $(t' < t)$ due to the assuption. Accordingly, noting $Z_{t'} < 0$, we have $Z_t < 0$ for $t' \leq t \leq t''$ $(t' \leq t)$, i.e., $A_t < W_t$, hence $v_t = W_t$ for $t' \leq t \leq t''$ $(t' \leq t)$.

(d) Let $Z_{t'} < 0$ and $t_z \notin [1, t')$.

(d1) Assume that $t_z \in [t', t'')$ $([t', \infty))$. Then, from the definition of t_z and the assumption $Z_{t'} < 0$ we have $Z_t < 0$ for $t' \leq t < t_z$, i.e., $A_t < W_t$ for $t' \leq t < t_z$, hence $v_t = W_t$ for $t' \leq t < t_z$. Now, $Z_{t_z} \geq 0$ by the assumption, i.e., $A_{t_z} \geq W_{t_z}$, hence $v_{t_z} = A_{t_z}$, noting the assumption $(\lambda' - \lambda)A_t + r' - r \geq 0$ for $t_z \leq t < t''$ $(t_z \leq t)$, we immediately get $v_t = A_t$ for $t_z \leq t \leq t''$ $(t_z \leq t)$ due to (a).

(d2) Assume that $t_z \notin [t', t'')$ $([t', \infty))$. Then, it follows from the definition of t_z and $Z_{t'} < 0$ that $Z_t < 0$ for $t' \leq t \leq t''$ $(t' \leq t)$. Hence $A_t < W_t$ for $t' \leq t \leq t''$ $(t' \leq t)$, i.e., $v_t = W_t$ for $t' \leq t \leq t''$ $(t' \leq t)$. ∎

5.2.1 Case 1: $S > U$ and $S > U'$

Lemma 5.7 *Assume $S' < U'$, $S > U$, and $S > U'$.*

(a) $v_1 = S$.

(b) *Let $S \geq A$, then $v_t = S$ for $t \geq 1$.*

(c) *Let $U' < S < A$.*

1 *There must exist a unique $t_S \geq 2$.*

2 $v_t = S$ *for* $1 \leq t < t_S$.

3 $v_{t_S} = A_{t_S}$ *and* $v_t = \max\{W_t, A_t\}$ *for* $t \geq t_S$.

4 *Suppose $\lambda' - \lambda \geq 0$. Then $v_t = A_t$ for $t \geq t_S$.*

5 *Suppose $\lambda' - \lambda < 0$.*

i *If $U' < S < A \leq \delta$, then $v_t = A_t$ for $t \geq t_S$.*

ii *If $U' < S < \delta < A$, there must exist a unique $t_s > t_S$, hence $v_t = A_t$ for $t_S \leq t \leq t_s$ and $v_t = W_t$ for $t > t_s$.*

iii *If $\delta \leq S < A$, then $v_t = W_t$ for $t > t_S$ whether $\delta > U'$ or $\delta \leq U'$.*

Proof. Assume $S' < U'$, $S > U$, and $S > U'$.

(a) From Eqs. (3.1), (3.11), and (3.12) we have $v_1 = \max\{S, U, U'\} = S$.

(b) Let $S \geq A$. Then $A_t \leq S$ for $t \geq 1$ due to Lemma 4.1, hence $v_t = \max\{S, W_t, A_t\} = \max\{S, W_t\}$. Now, the assertion is ture for $t = 1$ due to (a). Suppose $v_{t-1} = S$. Then $W_t = r + \lambda S = U < S$ from Eqs. (3.11) and (3.8), hence $v_t = S$. Accordingly $v_t = S$ for $t \geq 1$ by induction.

(c) Let $U' < S < A$. Then $A_1 < S < A$ from Eq. (3.12).

(c1) Since A_t is strictly increasing in t due to Lemma 5.5(d), there must exist a unique $t_S \geq 2$ by the definition of t_S, hence $A_t < S$ for $1 \leq t < t_S$ and $A_t \geq S$ for $t \geq t_S$.

(c2) In almost the same way as (b), we obtain $v_t = S$ for $1 \leq t < t_S$ due to $A_t < S$ for $1 \leq t < t_S$.

(c3) From (c2) we have $v_{t_S-1} = S$, hence $W_{t_S} = r + \lambda S = U < S$ from Eqs. (3.11) and (3.8). Noting the fact that $A_t \geq S$ for $t \geq t_S$, we get $v_{t_S} = \max\{S, W_{t_S}, A_{t_S}\} = A_{t_S}$ and $v_t = \max\{W_t, A_t\}$ for $t \geq t_S$ from Eq. (3.1).

(c4) Let $\lambda' - \lambda \geq 0$. Since $A_{t_S-1} < S \leq A_{t_S}$, we have $A_{t_S-1} < S \leq A_{t_S} \leq A_t$ for $t \geq t_S$ due to Lemma 4.1, hence noting Eq. (3.8) and Lemma 5.5(b), we get

$$(\lambda'-\lambda)A_t + r' - r \geq (\lambda'-\lambda)S + r' - r = r' + \lambda' S - U$$
$$> r' + \lambda' A_{t_S-1} - U = A_{t_S} - U \geq S - U > 0, \quad t \geq t_S.$$

Accordingly, noting $v_{t_S} = A_{t_S}$, from Lemma 5.6(a) we have $v_t = A_t$ for $t \geq t_S$ where $t' = t_S$.

(c5) Let $\lambda' - \lambda < 0$.

(c5i) Let $U' < S < A \leq \delta$. Then $A_t \leq \delta$ for $t \geq 1$ due to Lemma 4.1. Thus $(\lambda'-\lambda)A_t + r' - r \geq 0$ for $t \geq 1$ from Eq. (3.7). Accordingly, noting $v_{t_S} = A_{t_S}$, from Lemma 5.6(a) we have $v_t = A_t$ for $t \geq t_S$ where $t' = t_S$.

(c5ii) Let $U' < S < \delta < A$. Then $A_1 < S < \delta < A$ from Eq. (3.12). Since A_t is strictly increasing in t due to Lemma 5.5(d), there must exist a unique $t_\delta \geq 2$ by the definition of t_δ, further, since $\delta > S$, we have $t_\delta > t_S$, hence $A_t < \delta$ for $t_S \leq t < t_\delta$ and $A_t \geq \delta$ for $t \geq t_\delta$. Now, from Eq. (3.7) we have

$$(\lambda'-\lambda)A_t + r' - r > 0, \quad t_S \leq t < t_\delta, \tag{5.4}$$

$$(\lambda'-\lambda)A_t + r' - r \leq 0, \quad t \geq t_\delta. \tag{5.5}$$

Accordingly, noting $v_{t_S} = A_{t_S}$ and Eq. (5.4), from Lemma 5.6(a) we get $v_t = A_t$ for $t_S \leq t \leq t_\delta$ where $t' = t_S$ and $t'' = t_\delta$. Thus $v_{t_\delta} = A_{t_\delta}$. From this we have $W_{t_\delta+1} = r + \lambda A_{t_\delta}$ from Eq. (3.11), further noting Lemma 5.5(b) and Eq. (5.5) we get $A_{t_\delta+1} - W_{t_\delta+1} = (\lambda'-\lambda)A_{t_\delta} + r' - r \leq 0$, i.e., $A_{t_\delta+1} \leq W_{t_\delta+1}$, hence $v_{t_\delta+1} = \max\{W_{t_\delta+1}, A_{t_\delta+1}\} = W_{t_\delta+1}$. Accordingly, from Lemma 5.6(b) and Eq. (5.5) we have $v_t = W_t$ for $t \geq t_{\delta+1}$ where $t' = t_\delta + 1$, i.e., $v_t = W_t$ for $t > t_\delta$.

(c5iii) Let $\delta \leq S < A$. Then, noting the fact that $A_t \geq S$ for $t \geq t_S$ according to the proof of (c1), we have $A_t \geq \delta$ for $t \geq t_S$ whether $\delta > U'$ or $\delta \leq U'$, hence $(\lambda'-\lambda)A_t + r' - r \leq 0$ for $t \geq t_S$ from Eq. (3.7). Since $v_{t_S} = A_{t_S}$, from Lemma 5.5(b) and Eq. (3.11) we get $A_{t_S+1} - W_{t_S+1} = (\lambda'-\lambda)A_{t_S} + r' - r \leq 0$, i.e., $A_{t_S+1} \leq W_{t_S+1}$, hence $v_{t_S+1} = W_{t_S+1}$. Accordingly, it follows from Lemma 5.6(b) that $v_t = W_t$ for $t \geq t_S + 1$ where $t' = t_S + 1$, i.e., $v_t = W_t$ for $t > t_S$. ∎

5.2.2 Case 2: $U' \geq S$ and $U' \geq U$

Lemma 5.8 *Assume $S' < U'$, $U' \geq S$, and $U' \geq U$.*

(a) $v_1 = A_1$, $Z_1 \geq 0$, *and* $v_t = \max\{W_t, A_t\}$ *for* $t \geq 1$.

(b) *Let* $(\lambda'-\lambda)U' + r' - r \geq 0$.

  1 *Suppose* $\lambda' - \lambda \geq 0$. *Then* $v_t = A_t$ *for* $t \geq 1$.

  2 *Suppose* $\lambda' - \lambda < 0$.

    i $\delta \geq U'$.

    ii *If* $\delta \geq A$, *then* $v_t = A_t$ *for* $t \geq 1$.

    iii *If* $U' < \delta < A$, *there must exist a unique* $t_\delta \geq 2$, *hence* $v_t = A_t$ *for* $1 \leq t \leq t_\delta$ *and* $v_t = W_t$ *for* $t > t_\delta$.

    iv *If* $\delta = U'$, *then* $v_2 = A_2 = W_2$ *and* $v_t = W_t$ *for* $t \geq 3$.

(c) *Let* $(\lambda'-\lambda)U' + r' - r < 0$.

  1 $v_2 = W_2$ *and* $Z_2 < 0$.

  2 *Suppose* $\lambda' - \lambda \leq 0$. *Then* $v_t = W_t$ *for* $t \geq 2$.

  3 *Suppose* $\lambda' - \lambda > 0$.

    i $\delta > U'$.

ii *If $\delta \geq A$, then $v_t = W_t$ for $t \geq 2$.*

iii *If $U' < \delta < A$, then $v_t = W_t$ for $2 \leq t < t_z$ and $v_t = A_t$ for $t \geq t_z$ where t_z may be infinite.*

Proof. Assume $S' < U'$, $U' \geq S$, and $U' \geq U$.

(a) Noting Eqs. (3.11) and (3.12), we have $v_1 = \max\{S, U, U'\} = U' = A_1$ due to Eq. (3.1), and $Z_1 = U' - U \geq 0$ due to Eq. (3.10). Further, from Lemma 4.1 we get $A_t \geq A_1 = U' \geq S$ for $t \geq 1$, hence $v_t = \max\{W_t, A_t\}$ for $t \geq 1$ from Eq. (3.1).

(b) Let $(\lambda' - \lambda)U' + r' - r \geq 0$.

(b1) Let $\lambda' - \lambda \geq 0$. From Lemma 4.1 and Eq. (3.12) we have $(\lambda' - \lambda)A_t + r' - r \geq (\lambda' - \lambda)A_1 + r' - r = (\lambda' - \lambda)U' + r' - r \geq 0$ for $t \geq 1$. Accordingly, noting $v_1 = A_1$, it is from Lemma 5.6(a) that $v_t = A_t$ for $t \geq 1$ where $t' = 1$.

(b2) Let $\lambda' - \lambda < 0$.

(b2i) From $(\lambda' - \lambda)U' + r' - r \geq 0$ and Eq. (3.7) we get $U' \leq -(r' - r)/(\lambda' - \lambda) = \delta$.

(b2ii) Let $\delta \geq A$. Then $A_t \leq \delta$ for $t \geq 1$ due to Lemma 4.1. From Eq. (3.7) we obtain $(\lambda' - \lambda)A_t + r' - r \geq 0$ for $t \geq 1$. Accordingly, noting $v_1 = A_1$, from Lemma 5.6(a) we have $v_t = A_t$ for $t \geq 1$ where $t' = 1$.

(b2iii) Let $U' < \delta < A$. Then $A_1 < \delta < A$ from Eq. (3.12). Since A_t is strictly increasing in t due to Lemma 5.5(d), there must exist a unique $t_s \geq 2$ by the definition of t_s, hence, $A_t < \delta$ for $1 \leq t < t_s$ and $A_t \geq \delta$ for $t \geq t_s$. From these and Eq. (3.7) we have

$$(\lambda' - \lambda)A_t + r' - r > 0, \quad 1 \leq t < t_s, \tag{5.6}$$

$$(\lambda' - \lambda)A_t + r' - r \leq 0, \quad t \geq t_s. \tag{5.7}$$

Accordingly, noting $v_1 = A_1$ and Eq. (5.6), it is from Lemma 5.6(a) that $v_t = A_t$ for $1 \leq t \leq t_s$ where $t' = 1$ and $t'' = t_s$. Thus $v_{t_s} = A_{t_s}$. From this, Lemma 5.5(b), Eqs. (3.11), and (5.7) we have $A_{t_s+1} - W_{t_s+1} = (\lambda' - \lambda)A_{t_s} + r' - r \leq 0$, hence $v_{t_s+1} = \max\{W_{t_s+1}, A_{t_s+1}\} = W_{t_s+1}$. Accordingly, from Lemma 5.6(b) and Eq. (5.7) we obtain $v_t = W_t$ for $t \geq t_s + 1$ where $t' = t_s + 1$, i.e., $v_t = W_t$ for $t > t_s$.

(b2iv) Let $\delta = U'$. Then $\delta = A_1$, hence $(\lambda' - \lambda)A_1 + r' - r = 0$ from Eq. (3.7). Since $v_1 = A_1$, we get $A_2 - W_2 = (\lambda' - \lambda)A_1 + r' - r = 0$ due to Lemma 5.5(b) and Eq. (3.11), hence $v_2 = A_2 = W_2$. Further, from Lemma 5.5(d) we have $A_t > A_1 = U' = \delta$ for $t \geq 2$, hence $(\lambda' - \lambda)A_t + r' - r < 0$ for $t \geq 2$ from Eq. (3.7). Accordingly, it follows that $v_t = W_t$ for $t \geq 2$ due to Lemma 5.6(b) where $t' = 2$. Thus $v_t = W_t$ for $t \geq 3$.

(c) Let $(\lambda' - \lambda)U' + r' - r < 0$.

(c1) Since $v_1 = A_1 = U'$, we have $A_2 - W_2 = (\lambda' - \lambda)U' + r' - r < 0$ due to Lemma 5.5(b) and Eq. (3.11), i.e., $A_2 < W_2$, hence $v_2 = \max\{W_2, A_2\} = W_2$ and $Z_2 = A_2 - W_2 < 0$.

(c2) Let $\lambda' - \lambda \leq 0$. From Lemma 4.1 and Eq. (3.12) we get $(\lambda' - \lambda)A_t + r' - r \leq (\lambda' - \lambda)A_1 + r' - r = (\lambda' - \lambda)U' + r' - r < 0$ for $t \geq 1$. Accordingly, noting $v_2 = W_2$, from Lemma 5.6(b) we have $v_t = W_t$ for $t \geq 2$ where $t' = 2$.

(c3) Suppose $\lambda' - \lambda > 0$.

(c3i) From $(\lambda' - \lambda)U' + r' - r < 0$ and Eq. (3.7) we get $U' < -(r' - r)/(\lambda' - \lambda) = \delta$.

(c3ii) Let $\delta \geq A$. Then $A_t \leq \delta$ for $t \geq 1$ due to Lemma 4.1. Then $(\lambda' - \lambda)A_t + r' - r \leq 0$ for $t \geq 1$ form Eq. (3.7). Accordingly, noting $v_2 = W_2$, from Lemma 5.6(b) we have $v_t = W_t$ for $t \geq 2$ where $t' = 2$.

(c3iii) Let $U' < \delta < A$. Then $A_1 < \delta < A$ due to Eq. (3.12). Since A_t is strictly increasing in t due to

Lemma 5.5(d), there must exist a unique $t_s \geq 2$ by the definition of t_s, hence $A_t < \delta$ for $1 \leq t < t_s$ and $A_t \geq \delta$ for $t \geq t_s$.

1 Let $t \in [2, t_s)$. From the fact that $A_t < \delta$ for $2 \leq t < t_s$ we have $(\lambda' - \lambda)A_t + r' - r < 0$ for $2 \leq t < t_s$ due to Eq. (3.7). Accordingly, notint $Z_2 < 0$, from Lemma 5.6(c) we have $Z_t < 0$ and $v_t = W_t$ for $2 \leq t \leq t_s$ where $t' = 2$ and $t'' = t_s$, implying that $t_z > t_s$ if it exists.

2 Let $t \in [t_s, \infty)$. Here, note $Z_{t_s} < 0$. Let $t_z \in [t_s, \infty)$. From the fact that $A_t \geq \delta$ for $t \geq t_s$ we get $A_t \geq \delta$ for $t \geq t_z$, hence $(\lambda' - \lambda)A_t + r' - r \geq 0$ for $t \geq t_z$ from Eq. (3.7). Accordingly, from Lemma 5.6(d1) we have $v_t = W_t$ for $t_s \leq t < t_z$ and $v_t = A_t$ for $t \geq t_z$ where $t' = t_s$. Let $t_z \notin [t_s, \infty)$. Then, from Lemma 5.6(d2) we have $v_t = W_t$ for $t \geq t_s$ where $t' = t_s$.

According, if $t_z \in [2, \infty)$, then $v_t = W_t$ for $2 \leq t < t_z$ and $v_t = A_t$ for $t \geq t_z$, and if $t_z \notin [2, \infty)$, then $v_t = W_t$ for $t \geq 2$. Consequently, $v_t = W_t$ for $2 \leq t < t_z$ and $v_t = A_t$ for $t \geq t_z$ where t_z may be infinite. ∎

5.2.3 Case 3: $U \geq S$ and $U > U'$

Lemma 5.9 *Assume $S' < U'$, $U \geq S$, and $U > U'$.*

(a) $v_1 = W_1$, $Z_1 < 0$, $v_t = \max\{W_t, A_t\}$ *for* $t \geq 1$ *and* $\alpha > 1$.

(b) *Let* $(\lambda' - \alpha\lambda)U' + r' - r \geq 0$.

  1 $v_2 = A_2$.

  2 *Suppose* $\lambda' - \lambda \geq 0$. *Then* $v_t = A_t$ *for* $t \geq 2$.

  3 *Suppose* $\lambda' - \lambda < 0$.

    i $\delta > U'$.

    ii *If* $\delta \geq A$, *then* $v_t = A_t$ *for* $t \geq 2$.

    iii *If* $U' < \delta < A$, *there must exist a unique* $t_s \geq 2$, *hence* $v_t = A_t$ *for* $2 \leq t \leq t_s$ *and* $v_t = W_t$ *for* $t > t_s$.

(c) *Let* $(\lambda' - \alpha\lambda)U' + r' - r < 0$.

  1 *Suppose* $\lambda' - \lambda \geq 0$.

    i *If* $\delta \geq A$, *then* $v_t = W_t$ *for* $t \geq 1$.

    ii *If* $\delta < A$, *then* $v_t = W_t$ *for* $1 \leq t < t_z$ *and* $v_t = A_t$ *for* $t \geq t_z$ *where* t_z *may be infinite, whether* $\delta > U'$ *or* $\delta \leq U'$.

  2 *Suppose* $\lambda' - \lambda < 0$.

    i *If* $\delta \geq A$, *then* $v_t = W_t$ *for* $1 \leq t < t_z$ *and* $v_t = A_t$ *for* $t \geq t_z$ *where* t_z *may be infinite.*

    ii *If* $U' < \delta < A$, *there must exist a unique* $t_s \geq 2$, *hence* $v_t = W_t$ *for* $1 \leq t < t_z$, $v_t = A_t$ *for* $t_z \leq t \leq t_s$, *and* $v_t = W_t$ *for* $t > t_s$ *where* t_z *may be infinite.*

    iii *If* $\delta \leq U'$, *then* $v_t = W_t$ *for* $t \geq 1$.

Proof. Assume $S' < U'$, $U \geq S$, and $U > U'$.

(a) Noting Eqs. (3.11) and (3.12), we get $v_1 = \max\{S, U, U'\} = U = W_1$ due to Eq. (3.1), and $Z_1 = U' - U < 0$ due to Eq. (3.10). From Lemma 4.1 we get $W_t \geq W_1 = U \geq S$ for $t \geq 1$, hence $v_t = \max\{W_t, A_t\}$ for $t \geq 1$ from Eq. (3.1). Further, using the assumption $U > U'$ and Eq. (3.9), we have $\alpha > 1$.

(b) Let $(\lambda' - \alpha\lambda)U' + r' - r \geq 0$.

(b1) Since $v_1 = W_1 = U$, we have $A_2 - W_2 = r' + \lambda' U' - r - \lambda U = (\lambda' - \alpha\lambda)U' + r' - r \geq 0$ from Eqs. (3.11), (3.12), and (3.9), i.e., $A_2 \geq W_2$, hence $v_2 = \max\{W_2, A_2\} = A_2$.

(b2) Let $\lambda' - \lambda \geq 0$. Since $\alpha > 1$, we have $\lambda' - \lambda > \lambda' - \alpha\lambda$. Thus $(\lambda' - \lambda)A_t + r' - r \geq (\lambda' - \lambda)A_1 + r' - r = (\lambda' - \lambda)U' + r' - r > (\lambda' - \alpha\lambda)U' + r' - r \geq 0$ for $t \geq 1$ due to Lemma 4.1. Accordingly, noting $v_2 = A_2$, from Lemma 5.6(a) we obtain $v_t = A_t$ for $t \geq 2$ where $t' = 2$.

(b3) Let $\lambda' - \lambda < 0$.

(b3i) Since $\lambda' - \lambda > \lambda' - \alpha\lambda$, we get $(\lambda' - \lambda)U' + r' - r > (\lambda' - \alpha\lambda)U' + r' - r \geq 0$. Accordingly, $U' < -(r' - r)/(\lambda' - \lambda) = \delta$ from Eq. (3.7).

(b3ii) Let $\delta \geq A$. Then $A_t \leq \delta$ for $t \geq 1$ from Lemma 4.1. Hence $(\lambda' - \lambda)A_t + r' - r \geq 0$ for $t \geq 1$ from Eq. (3.7). Accordingly, noting $v_2 = A_2$, from Lemma 5.6(a) we have $v_t = A_t$ for $t \geq 2$ where $t' = 2$.

(b3iii) Let $U' < \delta < A$. Then $A_1 < \delta < A$ from Eq. (3.12). Since A_t is strictly increasing in t due to Lemma 5.5(d), there must exist a unique $t_s \geq 2$ by the definition of t_s, hence $A_t < \delta$ for $1 \leq t < t_s$ and $A_t \geq \delta$ for $t \geq t_s$. From these and Eq. (3.7) we obtain

$$(\lambda' - \lambda)A_t + r' - r > 0, \quad 1 \leq t < t_s, \tag{5.8}$$

$$(\lambda' - \lambda)A_t + r' - r \leq 0, \quad t \geq t_s. \tag{5.9}$$

Accordingly, noting $v_2 = A_2$ and Eq. (5.8), it is from Lemma 5.6(a) that $v_t = A_t$ for $2 \leq t \leq t_s$ where $t' = 2$ and $t'' = t_s$. Thus $v_{t_s} = A_{t_s}$. From this, Lemma 5.5(b), Eqs. (3.11), (5.9) we have $W_{t_s+1} - A_{t_s+1} = (\lambda' - \lambda)A_{t_s} + r' - r \leq 0$, hence $v_{t_s+1} = W_{t_s+1}$. Accordingly, from Lemma 5.6(b) we have $v_t = W_t$ for $t \geq t_s + 1$ where $t' = t_{s+1}$, i.e., $v_t = W_t$ for $t > t_s$.

(c) Let $(\lambda' - \alpha\lambda)U' + r' - r < 0$.

(c1) Let $\lambda' - \lambda \geq 0$.

(c1i) Let $\delta \geq A$. Then $A_t \leq \delta$ for $t \geq 1$ due to Lemma 4.1. From Eq. (3.7) we have $(\lambda' - \lambda)A_t + r' - r \leq 0$ for $t \geq 1$. Accordingly, noting $v_1 = W_1$, from Lemma 5.6(b) we have $v_t = W_t$ for $t \geq 1$ where $t' = 1$.

(c1ii) Let $\delta < A$.

1 Let $U' < \delta < A$. Then $A_1 < \delta < A$ from Eq. (3.12). Since A_t is strictly increasing in t due to Lemma 5.5(d), there must exist a unique $t_s \geq 2$ by the the definition of t_s, hence $A_t < \delta$ for $1 \leq t < t_s$ and $A_t \geq \delta$ for $t \geq t_s$.

   i. Let $t \in [1, t_s)$. From Eq. (3.7) and the fact that $A_t < \delta$ for $1 \leq t < t_s$ we get $(\lambda' - \lambda)A_t + r' - r < 0$ for $1 \leq t < t_s$. Accordingly, noting $Z_1 < 0$, from Lemma 5.6(c) we have $Z_t < 0$ and $v_t = W_t$ for $1 \leq t \leq t_s$ where $t' = 1$ and $t'' = t_s$, implying that $t_z > t_s$ if it exists.

   ii. Let $t \in [t_s, \infty)$. Here, note $Z_{t_s} < 0$. Let $t_z \in [t_s, \infty)$. From the fact that $A_t \geq \delta$ for $t \geq t_s$ we get $A_t \geq \delta$ for $t \geq t_z$, hence $(\lambda' - \lambda)A_t + r' - r \geq 0$ for $t \geq t_z$ from Eq. (3.7). Accordingly, from Lemma 5.6(d1) we obtain $v_t = W_t$ for $t_s \leq t < t_z$ and $v_t = A_t$ for $t \geq t_z$ where $t' = t_s$. Let $t_z \notin [t_s, \infty)$. Then, from Lemma 5.6(d2) we have $v_t = W_t$ for $t \geq t_s$ where $t' = t_s$.

   Accordingly, if $t_z \in [1, \infty)$, then $v_t = W_t$ for $1 \leq t < t_z$ and $v_t = A_t$ for $t \geq t_z$, and if $t_z \notin [1, \infty)$, then $v_t = W_t$ for $t \geq 1$.

2 Let $\delta \leq U'$. From Lemma 4.1 and Eq. (3.12) we have $A_t \geq A_1 = U' \geq \delta$ for $t \geq 1$. Here, note $Z_1 < 0$. Let $t_z \in [1, \infty)$. Then $A_t \geq \delta$ for $t \geq t_z$, hence $(\lambda' - \lambda)A_t + r' - r \geq 0$ for $t \geq t_z$ from Eq. (3.7). Accordingly, from Lemma 5.6(d1) we get $v_t = W_t$ for $1 \leq t < t_z$ and $v_t = A_t$ for $t \geq t_z$ where $t' = 1$. Let $t_z \notin [1, \infty)$. Then, from Lemma 5.6(d2) we obtain $v_t = W_t$ for $t \geq 1$ where $t' = 1$.

Consequently, whether $\delta > U'$ or $\delta \le U'$, we get $v_t = W_t$ for $1 \le t < t_z$ and $v_t = A_t$ for $t \ge t_z$ where t_z may be infinite.

(c2) Let $\lambda' - \lambda < 0$.

(c2i) Let $\delta \ge A$. Then $A_t \le \delta$ for $t \ge 1$ due to Lemma 4.1. Here note $Z_1 < 0$. Let $t_z \in [1, \infty)$. Then $A_t \le \delta$ for $t \ge t_z$, hence $(\lambda' - \lambda)A_t + r' - r \ge 0$ for $t \ge t_z$ from Eq. (3.7). Accordingly, from Lemma 5.6(d1) we have $v_t = W_t$ for $1 \le t < t_z$ and $v_t = A_t$ for $t \ge t_z$ where $t' = 1$. Let $t_z \notin [1, \infty)$. Then, from Lemma 5.6(d2) we have $v_t = W_t$ for $t \ge 1$ where $t' = 1$. Consequently, $v_t = W_t$ for $1 \le t < t_z$ and $v_t = A_t$ for $t \ge t_z$ where t_z may be infinite.

(c2ii) Let $U' < \delta < A$. Then $A_1 < \delta < A$ from Eq. (3.12). Since A_t is strictly increasing in t due to Lemma 5.5(d), there must exist a unique $t_s \ge 2$ by the definition of t_s, hence $A_t < \delta$ for $1 \le t < t_s$ and $A_t \ge \delta$ for $t \ge t_s$.

1. Let $t \in [1, t_s)$. Here, note $Z_1 < 0$. Let $t_z \in [1, t_s)$. From the fact that $A_t < \delta$ for $1 \le t < t_s$ we have $A_t < \delta$ for $t_z \le t < t_s$, hence $(\lambda' - \lambda)A_t + r' - r > 0$ for $t_z \le t < t_s$ from Eq. (3.7). Accordingly, from Lemma 5.6(d1) we get $v_t = W_t$ for $1 \le t < t_z$ and $v_t = A_t$ for $t_z \le t \le t_s$ where $t' = 1$ and $t'' = t_s$. Let $t_z \notin [1, t_s)$. Then, from Lemma 5.6(d2) we obtain $v_t = W_t$ for $1 \le t \le t_s$ where $t' = 1$ and $t'' = t_s$.
2. Let $t \in [t_s, \infty)$. Then, from Eq. (3.7) and the fact that $A_t \ge \delta$ for $t \ge t_s$ we have $(\lambda' - \lambda)A_t + r' - r \le 0$ for $t \ge t_s$. If $t_z \in [1, t_s)$, then $v_{t_s} = A_{t_s}$, and further from Lemma 5.5(b) and Eq. (3.11) we have $A_{t_s+1} - W_{t_s+1} = (\lambda' - \lambda)A_{t_s} + r' - r \le 0$, hence $v_{t_s+1} = W_{t_s+1}$. Accordingly, from Lemma 5.6(b) we obtain $v_t = W_t$ for $t \ge t_s + 1$ where $t' = t_s + 1$, i.e., $v_t = W_t$ for $t > t_s$. If $t_z \notin [1, t_s)$, then $v_{t_s} = W_{t_s}$. Then, it is from Lemma 5.6(b) that $v_t = W_t$ for $t \ge t_s$ where $t' = t_s$, implying that $t_z \notin [t_s, \infty)$.

Accordingly, if $t_z \in [1, \infty)$, it follows that the t_z must be on $[1,\, t_s)$, i.e., $t_z \le t_s$. Hence $v_t = W_t$ for $1 \le t < t_z$, $v_t = A_t$ for $t_z \le t \le t_s$ and $v_t = W_t$ for $t > t_s$. If $t_z \notin [1, \infty)$, then $v_t = W_t$ for $t \ge 1$. Consequently, $v_t = W_t$ for $1 \le t < t_z$, $v_t = A_t$ for $t_z \le t \le t_s$ and $v_t = W_t$ for $t > t_s$ where t_z may be infinite.

(c2iii) Let $\delta \le U'$. Then $A_t \ge A_1 = U' \ge \delta$ for $t \ge 1$ due to Lemma 4.1 and Eq. (3.12). From Eq. (3.7) we have $(\lambda' - \lambda)A_t + r' - r \le 0$ for $t \ge 1$. Accordingly, noting $v_1 = W_1$, from Lemma 5.6(b) we get $v_t = W_t$ for $t \ge 1$ where $t' = 1$. ∎

5.2.4 Calculations of t_s, t_s, and t_z

From Eq. (3.12) we have $A_1 = U'$, and from Lemmas 5.5(b) and 4.2(a) we get the expression of A_t for $t \ge 2$. Therefore, t_s and t_s can be immediately obtained by Eqs. (5.1) and (5.2), respectively, if they exist. The conditions on which whether or not they exist have already been stated in Lemmas 5.7, 5.8 and 5.9.

The lemma below is about the calculation of t_z.

Lemma 5.10 *Suppose that $S' < U'$, t_z exist, and $v_t = \max\{W_t, A_t\}$ for $t \ge 1$.*

(a) *The t_z in* Lemma 5.8 *can be found by using of* Eq. (5.3) *where*

$$Z_t = r'(1 - \lambda'^{t-1})/(1 - \lambda') - r(1 - \lambda^{t-1})/(1 - \lambda) + (\lambda'^{t-1} - \lambda^{t-1})U', \quad 2 \le t \le t_z. \tag{5.10}$$

(b) *The t_z in* Lemma 5.9 *can be found by using of* Eq. (5.3) *where*

$$Z_t = r'(1 - \lambda'^{t-1})/(1 - \lambda') - r(1 - \lambda^{t-1})/(1 - \lambda) + (\lambda'^{t-1} - \alpha\lambda^{t-1})U', \quad 2 \le t \le t_z. \tag{5.11}$$

Proof. Let $S' < U'$. Then $A_t = r' + \lambda' A_{t-1}$ for $t \geq 2$ from Lemma 5.5(b). Further, assume that t_z exist and $v_t = \max\{W_t, A_t\}$ for $t \geq 1$.
(a) From Lemma 5.8(a) and (c1) we have $Z_1 \geq 0$ and $Z_2 < 0$. Then $Z_t < 0$ for $2 \leq t < t_z$ and $Z_{t_z} \geq 0$ by the assumption of t_z, i.e., $A_1 \geq W_1$ and $A_t < W_t$ for $2 \leq t < t_z$ from Eq. (3.10), hence $v_1 = \max\{W_1, A_1\} = A_1$ and $v_t = \max\{W_t, A_t\} = W_t$ for $2 \leq t < t_z$. Further, from this and Eq. (3.11) we get $W_t = r + \lambda W_{t-1}$ for $3 \leq t \leq t_z$. Accordingly, we have the assertion is true from Eq. (3.10), Lemma 4.2(a) and (c).
(b) From Lemma 5.9(a) we have $Z_1 < 0$. Then $Z_t < 0$ for $1 \leq t < t_z$ and $Z_{t_z} \geq 0$ by the definition of t_z, i.e., $A_t < W_t$ for $1 \leq t < t_z$, hence $v_t = W_t$ for $1 \leq t < t_z$. Further, from this and Eq. (3.11) we get $W_t = r + \lambda W_{t-1}$ for $2 \leq t \leq t_z$. Accordingly, the assertion holds due to Eq. (3.10), Lemma 4.2(a) and (b). ∎

Although t_z in Lemmas 5.8 and 5.9 can be calculated by using Lemma 5.10(a) and (b), respectively, if it exists, the condition on which whether or not it exists on $[2, \infty)$ can not be found. However, for a given starting point in time t°, we can confirm whether or not t_z exists on $[2, t^\circ]$ by calculating all of Z_2, $Z_3, \cdots$, and Z_{t° using Lemma 5.10; in other words, if Z_2, $Z_3, \cdots$, and Z_{t° are all negative, we can say that t_z does not exist on $[2, t^\circ]$, or else exists, which is given by the first t at which $Z_t \geq 0$, $2 \leq t \leq t^\circ$.

6 Summary of Conclusions

Taking all the results obtained up to the previous sections together, we are led to the conclusion that one of the following eight decision rules becomes optimal, according to parameters p, q, r, p', q' and r', when a hostage event occurs at time t.

DR-A Storm for rescue immediately.

DR-B Take the action of negotiation immediately, and if the criminal(s) do not surrender at the next time, i.e., time $t-1$, storm for rescue.

DR-C Take the action of negotiation immediately, and if the criminal(s) do not surrender at the next time, i.e., time $t-1$, wait up to time 0 and storm for rescue.

DR-D Wait up to time 0 and storm for rescue.

DR-E Wait up to time 1 and take the action of negotiation at time 1; if the criminal(s) do not surrender at time 0 and storm for rescue.

DR-F Wait up to time 2 and take the action of negotiation at time 2; if the criminal(s) do not surrender at time 1, wait up to time 0 and storm for rescue.

DR-G Wait up to time t_s and take the action of negotiation at that time; if the criminal(s) do not surrender at time $t_s - 1$, wait up to time 0 and storm for rescue.

DR-H Wait up to time t_s and take the action of negotiation at that time; if the criminal(s) do not surrender at time $t_s - 1$, wait up to time 0 and storm for rescue.

Note that clearly the action of negotiation has not yet taken at time t when a hostage event occurs by the definition of the model. It is suffcient to consider only v_t at that time, and once the action of negotiation is taken at a certain time $t^\circ \leq t$, it is suffcient to consider only v'_t thereafter so long as the criminal(s) do not surrender at the next time, i.e., time $t^\circ - 1$. Noting the above and the assumption which storming for rescue is always made at the time 0 (the deadline), from Lemmas 5.1 to 5.5 and further from Lemmas 5.7 to 5.10 we can exhaustively prescribe the optimal decision rules of our model as in Optimal Decision Rules 6.1 to 6.4 below.

Optimal Decision Rule 6.1 Suppose $S' \geq U'$.

(a) Let $S > U$ and $S > U'$. Then DR-A is optimal for $t \geq 1$. ⇐ Lemma 5.2

(b) Let $U' \geq S$ and $U' \geq U$.

1 DR-B is optimal for $t = 1$. ⇐ Lemma 5.3(a)

2 If $U' > r/(1-\lambda)$, then DR-B is optimal for $t \geq 2$. ⇐ Lemmas 5.3(b) and 5.1

3 If $U' = r/(1-\lambda)$, then DR-B and DR-E are indifferent for $t \geq 2$. ⇐ Lemmas 5.3(c), (a), and 5.1

4 If $U' < r/(1-\lambda)$, then DR-E is optimal for $t \geq 2$. ⇐ Lemmas 5.3(d) and (a)

(c) Let $U \geq S$ and $U > U'$. Then DR-D is optimal for $t \geq 1$. ⇐ Lemmas 5.4

Optimal Decision Rule 6.2 Suppose $S' < U'$, $S > U$ and $S > U'$.

(a) DR-A is optimal for $t = 1$. ⇐ Lemma 5.7(a)

(b) If $S \geq A$, then DR-A is optimal for $t \geq 2$. ⇐ Lemma 5.7(b)

(c) If $U' < S < A$, there must exist a unique $t_S \geq 2$.

1 DR-A is optimal for $2 \leq t < t_S$. ⇐ Lemma 5.7(c2)

2 DR-C is optimal for $t = t_S$. ⇐ Lemmas 5.7(c3) and 5.5(a)

3 Let $\lambda' - \lambda \geq 0$. Then DR-C is optimal for $t > t_S$. ⇐ Lemmas 5.7(c4) and 5.5(a)

4 Let $\lambda' - \lambda < 0$.

i If $U' < S < A < \delta$, then DR-C is optimal for $t > t_S$. ⇐ Lemmas 5.7(c5i) and 5.5(a)

ii If $U' < S < \delta \leq A$, there must exist a unique $t_s > t_S$, hence DR-C is optimal for $t_S < t \leq t_s$ and DR-H is optimal for $t > t_s$. ⇐ Lemmas 5.7(c5ii) and 5.5(a)

iii If $\delta \leq S < A$, then DR-G is optimal for $t > t_S$. ⇐ Lemmas 5.7(c5iii) and 5.5(a)

Optimal Decision Rule 6.3 Suppose $S' < U'$, $U' \geq S$ and $U' \geq U$.

(a) DR-B is optimal for $t = 1$. ⇐ Lemma 5.8(a)

(b) Let $(\lambda' - \lambda)U' + r' - r \geq 0$.

1 Let $\lambda' - \lambda \geq 0$. Then DR-C is optimal for $t \geq 2$. ⇐ Lemmas 5.8(b1) and 5.5(a)

2 Let $\lambda' - \lambda < 0$.

i If $\delta \geq A$, then DR-C is optimal for $t \geq 2$. ⇐ Lemmas 5.8(b2ii) and 5.5(a)

ii If $U' < \delta < A$, there must exist a unique $t_s \geq 2$, hence DR-C is optimal for $2 \leq t \leq t_s$ and DR-H is optimal for $t > t_s$. ⇐ Lemmas 5.8(b2iii) and 5.5(a)

iii If $\delta = U'$, then DR-C and DR-E are indifferent for $t = 2$, and DR-E and DR-F are indifferent for $t \geq 3$. ⇐ Lemmas 5.8(b2iv) and 5.5(a)

(c) Let $(\lambda' - \lambda)U' + r' - r < 0$.

1 Let $\lambda' - \lambda \leq 0$. Then DR-E is optimal for $t \geq 2$. ⇐ Lemmas 5.8(c2) and (a)

2 Let $\lambda' - \lambda > 0$.

i If $\delta \geq A$, then DR-E is optimal for $t \geq 2$. ⇐ Lemmas 5.8(c3ii) and (a)

ii If $U' < \delta < A$, then DR-E is optimal for $2 \leq t < t_z$ and DR-C is optimal for $t \geq t_z$ where t_z may be infinite. ⇐ Lemmas 5.8(c3iii), (a), and 5.5(a)

(d) t_z can be found by using Lemma 5.10(a) if it is finite. ⇐ Lemma 5.10(a)

Optimal Decision Rule 6.4 Suppose $S' < U'$, $U \geq S$ and $U > U'$.

(a) DR-D is optimal for $t = 1$. ⇐ Lemma 5.9(a)

(b) Let $(\lambda' - \alpha\lambda)U' + r' - r \geq 0$.

1 Let $\lambda' - \lambda \geq 0$. Then DR-C is optimal for $t \geq 2$. ⇐ Lemmas 5.9(b2) and 5.5(a)

2 Let $\lambda' - \lambda < 0$.

i If $\delta \geq A$, then DR-C is optimal for $t \geq 2$. ⇐ Lemmas 5.9(b3ii) and 5.5(a)

ii If $U' < \delta < A$, there must exist a unique $t_s \geq 2$, hence DR-C is optimal for $2 \leq t \leq t_s$ and DR-H is optimal for $t > t_s$. ⇐ Lemmas 5.9(b3iii) and 5.5(a)

(c) Let $(\lambda' - \alpha\lambda)U' + r' - r < 0$.

1 Let $\lambda' - \lambda \geq 0$.

i If $\delta \geq A$, then DR-D is optimal for $t \geq 2$. ⇐ Lemma 5.9(c1i)

ii If $\delta < A$, then DR-D is optimal for $2 \leq t < t_z$ and DR-C is optimal for $t \geq t_z$ where t_z may be infinite. ⇐ Lemmas 5.9(c1ii) and 5.5(a)

2 Let $\lambda' - \lambda < 0$.

i If $\delta \geq A$, then DR-D is optimal for $2 \leq t < t_z$ and DR-C is optimal for $t \geq t_z$ where t_z may be infinite. ⇐ Lemmas 5.9(c2i) and 5.5(a)

ii If $U' < \delta < A$, there must exist a unique $t_s \geq 2$, hence DR-D is optimal for $2 \leq t < t_z$, DR-C is optimal for $t_z \leq t \leq t_s$, and DR-H is optimal for $t > t_s$ where t_z may be infinite. ⇐ Lemmas 5.9(c2ii) and 5.5(a)

iii If $\delta \leq U'$, then DR-D is optimal for $t \geq 2$. ⇐ Lemma 5.9(c2iii)

(d) t_z can be found by using Lemma 5.10(b) if it is finite. ⇐ Lemma 5.10(b)

The optimal decision rules prescribed above have a very complicated structure, so, in order to made them understandable, let us summarize them as in Table 6.1. In the table, we use the symbols DR-$A_{t\geq 1}$, DR-$C_{2\leq t\leq t_s}$, $\cdots$ where, in general, for a given statement $T(t)$ as to time t when the hostage event occurs, DR-$X_{T(t)}$ implies that DR-X is optimal for t satisfying $T(t)$.

Taking as examples the three cells, Ex.1, Ex.2 and Ex.3, let us demonstrate how to interpret the contents of the table.

Ex.1 (DR-$A_{t\geq 1}$) : DR-A is optimal when the hostage event occurs at time $t \geq 1$.

Ex.2 ([DR-C~DR-E]$_{t=2}$, [DR-E~DR-F]$_{t\geq 3}$) : Any of DR-C and DR-E is optimal when the hostage event occurs at time $t = 2$, and any of DR-E and DR-F is optimal when the hostage event occurs at time $t \geq 3$.

Ex.3 (DR-$D_{2\leq t<t_z}$, DR-$C_{t_z \leq t \leq t_s}$, DR-$H_{t>t_s}$) : When a hostage event occurs at time t, we can calculate t_z by using Lemma 5.10(b). If t_z can not be found on [2, t], then $t < t_z$, hence DR-D is optimal for $2 \leq t < t_z$. If t_z can be found on [2, t], then $t \geq t_z$, and further we can calculate t_s by using Eq. (5.2), $A_1 = U'$, and Lemma 4.2(a). If $t \leq t_s$, then DR-C is optimal for $t_z \leq t \leq t_s$, and if $t > t_s$, then DR-H is optimal for $t > t_s$.

Table 6.1: Summary of Optimal Decision Rules

	$S' \geq U'$		$S' < U'$			
$S > U$, $S > U'$ (Case 1)	DR-$A_{t\geq 1}$ [Ex.1]					DR-$A_{t=1}$
			$S \geq A$			DR-$A_{t\geq 2}$
			$U' < S < A$			DR-$A_{2\leq t<t_S}$, DR-$C_{t=t_S}$
			$U' < S < A$	$\lambda' - \lambda \geq 0$		DR-$C_{t>t_S}$
			$U' < S < A$	$\lambda' - \lambda < 0$	$U' < S < A < \delta$	DR-$C_{t>t_S}$
			$U' < S < A$	$\lambda' - \lambda < 0$	$U' < S < \delta \leq A$	DR-$C_{t_S<t\leq t_\delta}$, DR-$H_{t>t_\delta}$
			$U' < S < A$	$\lambda' - \lambda < 0$	$\delta \leq S < A$	DR-$G_{t>t_S}$
$U' \geq S$, $U' \geq U$ (Case 2)	DR-$B_{t=1}$					DR-$B_{t=1}$
	$U' > r/(1-\lambda)$	DR-$B_{t\geq 2}$	$\rho_1 \geq 0$	$\lambda' - \lambda \geq 0$		DR-$C_{t\geq 2}$
			$\rho_1 \geq 0$	$\lambda' - \lambda < 0$	$\delta \geq A$	DR-$C_{t\geq 2}$
	$U' = r/(1-\lambda)$	[DR-B$\sim$DR-E]$_{t\geq 2}$	$\rho_1 \geq 0$	$\lambda' - \lambda < 0$	$U' < \delta < A$	DR-$C_{2\leq t\leq t_\delta}$, DR-$H_{t>t_\delta}$
			$\rho_1 \geq 0$	$\lambda' - \lambda < 0$	$\delta = U'$	[DR-C$\sim$DR-E]$_{t=2}$, [DR-E$\sim$DR-F]$_{t\geq 3}$ [Ex.2]
	$U' < r/(1-\lambda)$	DR-$E_{t\geq 2}$	$\rho_1 < 0$	$\lambda' - \lambda \leq 0$		DR-$E_{t\geq 2}$
			$\rho_1 < 0$	$\lambda' - \lambda > 0$	$\delta \geq A$	DR-$E_{t\geq 2}$
			$\rho_1 < 0$	$\lambda' - \lambda > 0$	$U' < \delta < A$	DR-$E_{2\leq t<t_Z}$, DR-$C_{t\geq t_Z}$
$U \geq S$, $U > U'$ (Case 3)	DR-$D_{t\geq 1}$					DR-$D_{t=1}$
			$\rho_\alpha \geq 0$	$\lambda' - \lambda \geq 0$		DR-$C_{t\geq 2}$
			$\rho_\alpha \geq 0$	$\lambda' - \lambda < 0$	$\delta \geq A$	DR-$C_{t\geq 2}$
			$\rho_\alpha \geq 0$	$\lambda' - \lambda < 0$	$U' < \delta < A$	DR-$C_{2\leq t\leq t_\delta}$, DR-$H_{t>t_\delta}$
			$\rho_\alpha < 0$	$\lambda' - \lambda \geq 0$	$\delta \geq A$	DR-$D_{t\geq 2}$
			$\rho_\alpha < 0$	$\lambda' - \lambda \geq 0$	$\delta < A$	DR-$D_{2\leq t<t_Z}$, DR-$C_{t\geq t_Z}$
			$\rho_\alpha < 0$	$\lambda' - \lambda < 0$	$\delta \geq A$	DR-$D_{2\leq t<t_Z}$, DR-$C_{t\geq t_Z}$
			$\rho_\alpha < 0$	$\lambda' - \lambda < 0$	$U' < \delta < A$	DR-$D_{2\leq t<t_Z}$, DR-$C_{t_Z\leq t\leq t_\delta}$, DR-$H_{t>t_\delta}$ [Ex.3]
			$\rho_\alpha < 0$	$\lambda' - \lambda < 0$	$\delta \leq U'$	DR-$D_{t\geq 2}$

Note: (1) $\rho_1 = (\lambda' - \lambda)U' + r' - r$, $\rho_\alpha = (\lambda' - \alpha\lambda)U' + r' - r$.
(2) The symbol "$\sim$" means "indifferent".
(3) t_S and t_δ can be calculated by using Eqs. (5.1) and (5.2), respectively, where A_t is given by $A_1 = U'$ and Lemma 4.2(a).
(4) t_Z in Case 2 and Case 3 can be calculated by using Lemma 5.10(a) and (b), respectively.