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Abstract

We consider the standard linear complementarity problem (LCP): Find $(x, y) \in R^{2n}$ such that $y = Mx + q$, $(x, y) \geq 0$ and $x_i y_i = 0$ $(i = 1, 2, \ldots, n)$, where M is an $n \times n$ matrix and q is an n-dimensional vector. Recently several smoothing methods have been developed for solving monotone and/or P_0 LCPs. The aim of this paper is to derive a complexity bound of smoothing methods using Chen-Harker-Kanzow-Smale functions in the case where the monotone LCP has an feasible interior point. After a smoothing method is provided, some properties of the CHKS-function are described. As a consequence, we show that the algorithm terminates in $O\left(\frac{\bar{\gamma}^6 n}{\epsilon^6} \log \frac{\bar{\gamma}^2 n}{\epsilon^2}\right)$ Newton iterations where $\bar{\gamma}$ is a number which depends on the problem and the initial point. We also discuss some relationships between the interior point methods and the smoothing methods.

1 Introduction

We consider the (standard) linear complementarity problem(LCP):

$$\begin{array}{ll} \text{Find} & (x, y) \in \mathbb{R}^{2n} \\ \text{such that} & y = Mx + q, \ (x, y) \geq 0, \ x_i y_i = 0 \ (i \in N). \end{array}$$

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where $N = \{1, 2, \ldots, n\}$, M is an $n \times n$ matrix and q is an n-dimensional vector. We assume that the following condition holds.

Condition 1.1. **(i)** *The LCP is monotone, i.e., the matrix M is positive semi-definite:*

$$(x^1 - x^2)^T M (x^1 - x^2) \geq 0$$

for every $x^1, x^2 \in \mathbb{R}^n$.

(ii) *The LCP has an interior feasible solution $(\overset{\circ}{x}, \overset{\circ}{y})$ satisfying*

$$\overset{\circ}{y} = M \overset{\circ}{x} + q, \quad (\overset{\circ}{x}, \overset{\circ}{y}) > 0$$

It is well known that both of the linear program and the convex quadratic problem can be modeled as monotone LCPs. Recently several smoothing methods have been developed for solving monotone and/or P_0 LCPs, e.g., Burke and Xu [1],Chen and Chen [3], Gowda and Tawhid [6],Chen and Ye [5], Song, Gowda and Ravindran [11],Tseng [13],etc. Among others, Tseng [13] showed globally linear and locally superlinear convergence of a smoothing method using Chen-Mangasarian functions under a condition which is milder than Condition 1.1.

Our approach is based on the use of Chen-Harker-Kanzow-Smale smoothing function

$$\phi(\mu, a, b) := a + b - \sqrt{(a-b)^2 + 4\mu^2} \tag{1}$$

with a positive number $\mu > 0$. This function was given by Chen and Harker[4] to construct the first non-interior path-following method for the LCP. Several properties of this function have been observed by Kanzow[8], Qi and Sun[9, 10], Burke and Xu [2], Sun and Qi[12], etc. The method proposed in this paper is very similar to the homotopy continuation method for the nonlinear complementarity problem provided by Hotta and Yoshise [7]. However, there are some differences among them. The first difference is that we use the function (1) instead of using the function $\tilde{\phi}(\mu, a, b) := a + b - \sqrt{(a-b)^2 + 4\mu}$ as in [7]. As observed in [9] (and also in Section 3 of the paper), the behavior of the function ϕ can be easily estimated compared to the function $\tilde{\phi}$. The second difference is that we generate a sequence in the affine space $\{(x, y) \in \mathbb{R}^{2n} : \ y = Mx + q\}$. The LCP can be regarded as a problem to find a solution which satisfies three categories of the constraints: (i) the equality constraints $y = Mx + q$, (ii) the nonnegativity constraints $(x, y) \geq 0$, and (iii) the complementarity conditions $x_i y_i = 0 (i \in N)$. In [7], it is supposed that all of these conditions do not satisfied at the initial point. However, in linear cases, it is not difficult to find a point satisfying one of three categories of constraints. In this paper, we start from an initial point satisfying the equality constraints in contrast to the infeasible interior point methods in which the nonnegativity constraints are always satisfied. The last difference is that we intend to reduce the value of μ and the value of ϕ to zero, separately, while the values are changed simultaneously in [7].

The main purpose of this paper is to derive a complexity bound of a smoothing method with the assumption that Condition 1.1 holds and without imposing any other additional assumption. The paper is organized as follows. In Section 2, we propose our smoothing algorithm for solving the monotone LCP. In Section 3, we collect several basic properties which will be used in the complexity analysis of our algorithm. As a by-product, we find a close relationship between the smoothing method and the interior point method in terms of the Newton direction. In Section 4, we derive that the computational complexity bound of our smoothing method results in $O\left(\frac{\bar{\gamma}^6 n}{\epsilon^6}\log\frac{\bar{\gamma}^2 n}{\epsilon^2}\right)$ Newton iterations where $\bar{\gamma}$ is a number which depends on the problem and the initial point. We also discuss some relationships between the interior point methods and the smoothing methods in Section 5.

To simplify the presentation, we will use the notation

$$\mathrm{vec}\,\{x_i\} := (x_1, x_2, \ldots, x_n)^T, \quad \Phi(\mu, x, y) := \mathrm{vec}\,\{\phi(\mu, x_i, y_i)\ (i \in N)\}.$$

The vector $\Phi(\mu, x, y)$ will be sometimes abbreviated by $\Phi := \mathrm{vec}\,\{\phi(\mu, x_i, y_i)\ (i \in N)\}$ and similarly, $\Phi(\bar{\mu}, \bar{x}, \bar{y})$ by $\bar{\Phi}$, $\Phi(\hat{\mu}, \hat{x}, \hat{y})$ by $\hat{\Phi}$ and $\Phi(\mu^k, x^k, y^k)$ by Φ^k. We also use $\mathrm{diag}\,\{x_i(i \in N)\}$ to denote the diagonal matrix whose ith diagonal element is x_i $(i \in N)$.

2 A smoothing algorithm using CHKS-function

As we have mentioned, the constraints in the LCP can be categorized into three groups:

(i) the equality constraints $y = Mx + q$,

(ii) the nonnegativity constraints $(x, y) \geq 0$,

(iii) the complementarity conditions $x_i y_i = 0 (i \in N)$.

While the first and second groups consist of linear smooth constraints, the CHKS-function defined by (1) gives us a way to approximate the non-smooth constraints in the third group.

Proposition 2.1. *Let $\phi(\mu, a, b) := a + b - \sqrt{(a-b)^2 + 4\mu^2}$. For every nonnegative number $\mu \geq 0$, the following equivalence results hold for every $a, b, c \in \mathbb{R}$.*

(i) *(Lemma 1.1 of [7])*

$$[\phi(\mu, a, b) = c] \iff [((a - c/2), (b - c/2)) \geq 0 \text{ and } (a - c/2)(b - c/2) = \mu^2 \geq 0].$$

(ii)

$$[\phi(\mu,a,b)\le 0] \iff [ab\le \mu^2 \ \textit{if}\ a+b\ge 0].$$

Specially, if $\mu>0$ then

$$[\phi(\mu,a,b)< 0] \iff [ab< \mu^2 \ \textit{if}\ a+b> 0].$$

(iii) *(Lemma 2 of [9])*

$$\nabla^2\phi(\mu,a,b) = -\frac{4}{(\sqrt{(a-b)^2+4\mu^2})^3}\begin{pmatrix}(a-b)\\ -\mu\\ \mu\end{pmatrix}((a-b),-\mu,\mu)$$

i.e., ϕ is a concave function and

$$\|\nabla^2\phi(\mu,a,b)\| \le \frac{4}{(\sqrt{(a-b)^2+4\mu^2})^3}\left\|\begin{pmatrix}(a-b)\\ -\mu\\ \mu\end{pmatrix}\right\|^2 \le \frac{4}{\sqrt{(a-b)^2+4\mu^2}} \le \frac{2}{\mu}$$

Suppose that $(\bar{x},\bar{y})\in \mathbb{R}^{2n}$ satisfies the equality constraints, i.e.,

$$\bar{y} = M\bar{x}+q$$

By the above lemma, we know that if the point $(\bar{x},\bar{y})\in\mathbb{R}^{2n}$ satisfies

$$\phi(\mu,\bar{x}_i,\bar{y}_i) = \bar{\phi}_i \ (i\in N)$$

for some $\mu>0$ and $\bar{\Phi}\in\mathbb{R}^n$, then

$$((\bar{x}_i-\bar{\phi}_i/2),(\bar{y}_i-\bar{\phi}_i/2))>0,\ (\bar{x}_i-\bar{\phi}_i/2)(\bar{y}_i-\bar{\phi}_i/2)=\mu^2>0,\ \bar{y}=M\bar{x}+q$$

which implies that the point $(\bar{x}-\bar{\Phi}/2,\bar{y}-\bar{\Phi}/2)\in\mathbb{R}^{2n}$ is an analytical center of the perturbed problem LCP$(\bar{\mu},\bar{x},\bar{y})$ given by

$$\begin{array}{ll}\text{Find} & (x',y')\in\mathbb{R}^{2n}\\ \text{such that} & y'=Mx'+q',\ (x',y')\ge 0,\ x_i'y_i'=0\ (i\in N)\end{array} \tag{2}$$

where $q' = M\bar{\Phi}/2-\bar{\Phi}/2+q$.

Suppose that we obtain a point $(\bar{\mu},\bar{x},\bar{y})\in\mathbb{R}_{++}\times\mathbb{R}^{2n}$ satisfying $\bar{y}=M\bar{x}+q$ and let $\bar{\phi}_i=\phi(\bar{\mu},\bar{x}_i,\bar{y}_i)$ $(i\in N)$. We employ the following system of equations on the triplet $(\mu,x,y)\in\mathbb{R}_+\times\mathbb{R}^{2n}$ to approximate the solution of the LCP at a point $(\bar{\mu},\bar{x},\bar{y})\in\mathbb{R}_{++}\times\mathbb{R}^{2n}$:

$$
\begin{aligned}
&y - Mx - q = 0, \\
&\mu = (1 - \sigma_\mu)\bar{\mu}, \\
&\phi(\mu, x_i, y_i) = (1 - \sigma_\phi)\bar{\phi}_i \; (i \in N).
\end{aligned} \tag{3}
$$

Note that the first and the second equations are linear and they are always satisfied as long as the Newton iteration for the above system goes. The Newton direction $(\Delta\mu, \Delta x, \Delta y) \in \mathbb{R}^{1+2n}$ to the system (3) should satisfy the following equations:

$$
-M\Delta x + \Delta y = 0, \tag{4a}
$$

$$
\Delta\mu = -(1 - \sigma_\mu)\bar{\mu}, \tag{4b}
$$

$$
d_\mu \Delta\mu + D_x \Delta x + D_y \Delta y = -(1 - \sigma_\phi)\bar{\Phi} \tag{4c}
$$

where

$$
d_\mu := \operatorname{vec}\left\{-\frac{4\bar{\mu}}{\sqrt{(\bar{x}_i - \bar{y}_i)^2 + 4\bar{\mu}^2}}\right\}, \tag{5a}
$$

$$
D_x := \operatorname{diag}\left\{1 - \frac{\bar{x}_i - \bar{y}_i}{\sqrt{(\bar{x}_i - \bar{y}_i)^2 + 4\bar{\mu}^2}}\right\}, \tag{5b}
$$

$$
D_y := \operatorname{diag}\left\{1 + \frac{\bar{x}_i - \bar{y}_i}{\sqrt{(\bar{x}_i - \bar{y}_i)^2 + 4\bar{\mu}^2}}\right\}. \tag{5c}
$$

Since the equation (4b) is trivial, the system of the Newton direction can be reduced to

$$
\begin{pmatrix} -M & I \\ D_x & D_y \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} 0 \\ (1 - \sigma_\mu)\bar{\mu} d_\mu - (1 - \sigma_\phi)\bar{\Phi} \end{pmatrix}. \tag{6}
$$

The following results are used in many papers on smoothing methods, hence we omit their proofs.

Proposition 2.2. **(i)** *For every $(\bar{\mu}, \bar{x}, \bar{y}) \in \mathbb{R}_{++} \times \mathbb{R}^{2n}$,*

$$
0 < 1 \pm \frac{\bar{x}_i - \bar{y}_i}{\sqrt{(\bar{x}_i - \bar{y}_i)^2 + 4\bar{\mu}^2}} < 2.
$$

(ii) *Thus D_x and D_y given by (5b) and (5c) are positive diagonal matrices and the system (6) has a unique solution $(\Delta x, \Delta y)$ whenever Condition 1.1 holds.*

In our algorithm, we first find an initial point (μ^0, x^0, y^0) so that it satisfies

$$
y^0 = Mx^0 + q \;\text{ and }\; \Phi^0 = \Phi(\mu^0, x^0, y^0) \le 0. \tag{7}
$$

In fact, the determination of the initial point in Step 0 and (ii) of Proposition 2.1 ensures that (μ^0, x^0, y^0) satisfies (7). Each iteration consists of two steps, Step 1.1 and Step 1.2. In Step 1.1, we fix the value μ^k and devote ourselves to reduce the value of $\|\Phi^k\|$ in the set $\{(x,y) \in \mathbb{R}^{2n} : \Phi(\mu^k, x, y) \leq 0\}$ until the inequalities $\|\Phi^k\| < \epsilon$ is satisfied with a sufficiently small ϵ. Throughout these steps, we choose $\sigma_\mu = 1$ and hence $\Delta\mu^p = 0$ (see (4)) the value μ^k is never changed. In Step 1.2, the value of μ is certainly updated by a half of it and also the vectors (x, y) are updated along the Newton direction.

Algorithm.

Step 0: Let $\epsilon > 0$ and $k := 0$. Let $x^0 \in \mathbb{R}^n$, $y^0 := Mx^0 + q$ and choose a μ^0 such that

$$(\mu^0)^2 > \max\{0, x_i^0 y_i^0 \ (i \in N)\}$$

and let $\phi_i^0 := \phi(\mu^0, x_i^0, y_i^0)$ $(i \in N)$.

Step 1: If $\mu^k < \epsilon$ then stop else let $p := 0$, $\hat{x}^0 := x^k$, $\hat{y}^0 := y^k$ and $\hat{\phi}_i^0 := \phi_i^k$ $(i \in N)$.

Step 1.1 If $\|\hat{\Phi}^p\| \geq \epsilon$ then let $\sigma_\mu := 1$ and $\sigma_\phi := 0$ else go to Step 1.2. Compute the Newton direction $(\Delta\mu^p, \Delta x^p, \Delta y^p)$ by solving the system (6). Define

$$g^p(\theta) := (1-\theta)\|\hat{\Phi}^p\| + \frac{\theta^2}{\mu^k}\{\|\Delta x^p\|^2 + \|\Delta y^p\|^2\} \tag{8}$$

and define θ^p as

$$\theta^p := \min\left\{1, \ \frac{\|\hat{\Phi}^p\|\mu^k}{2(\|\Delta x^p\|^2 + \|\Delta y^p\|^2)}\right\} \tag{9}$$

Let

$$(\hat{x}^{p+1}, \hat{y}^{p+1}) := (\hat{x}^p, \hat{y}^p) + \theta^p(\Delta x^p, \Delta^p), \quad \hat{\Phi}^{p+1} = \Phi(\mu^k, \hat{x}^{p+1}, \hat{y}^{p+1})$$

and let $p := p + 1$. Go to Step 1.1.

Step 1.2 Let $(\bar{x}, \bar{y}) := (\hat{x}^p, \hat{y}^p)$. Let $\sigma_\mu := 1/2$ and $\sigma_\phi := 1$. Compute the Newton direction $(\Delta\bar{\mu}, \Delta\bar{x}, \Delta\bar{y})$ by solving the system (6). Let

$$(\mu^{k+1}, x^{k+1}, y^{k+1}) := (\mu^k, \bar{x}, \bar{y}) + (\Delta\bar{\mu}, \Delta\bar{x}, \Delta\bar{y}) = (\mu^k/2, \bar{x} + \Delta\bar{x}, \bar{y} + \Delta\bar{y})$$

and let $k := k + 1$. Go to Step 1.

By Proposition 2.2, the Newton direction at each step can be calculated. Also, θ^p in Step 1.1 is well-defined since $(\Delta x^p, \Delta y^p) = 0$ means $\hat{\Phi}^p = 0$ by (6) which contradicts $\|\hat{\Phi}^p\| \geq \epsilon$. Also, since the value μ^k is always reduced to $\mu^k/2$ at each iteration k, the number of Newton iterations required by the outer steps is $\lceil \log \frac{\mu^0}{\epsilon} \rceil$ where $\lceil x \rceil$ denotes the smallest integer k satisfying $x \leq k$. Thus, what we should do is to give a bound of the number of Newton iterations required in Step 1.1 at each k. In Section 4, we will discuss it and derive a bound of the total number of iterations of the algorithm.

It may happen that the step size θ^p in the algorithm is too small for the practical use. In such a case, we can adopt a certain inexact linesearch procedure for the problem

$$\begin{array}{ll} \text{minimize} & \|\Phi(\hat{\mu}^p + \theta\Delta\mu^p, \hat{x}^p + \theta\Delta x^p, \hat{y}^p + \theta\Delta y^p)\|, \\ \text{subject to} & \Phi(\hat{\mu}^p + \theta\Delta\mu^p, \hat{x}^p + \theta\Delta x^p, \hat{y}^p + \theta\Delta y^p) \leq 0. \end{array}$$

3 Some properties of the function ϕ and the Newton directions

In this section, we collect some basic properties of the CHKS function ϕ and the Newton directions satisfying the system (4a) – (4c) and/or (6)

The following results will be used to show that the generated sequence lies in a bounded simplex whose volume depends on the problem. Similar result has been shown by Burke and Xu[2].

Proposition 3.1 (Lemmas 2.2 and 2.4 of [2]) : *Suppose that Condition 1.1 holds. Let us define the set*

$$\Lambda(\beta, \mu^0) := \{(x, y) \in \mathbb{R}^{2n} : \ y = Mx+q, \ \Phi(\mu, x, y) \leq 0, \ \|\Phi(\mu, x, y)\| \leq \beta, \ \mu \in (0, \mu^0]\} (1$$

for every $\beta > 0$ and $\mu^0 > 0$. Then for every $(x, y) \in \Lambda(\beta, \mu^0)$,

$$-(\beta/2)e \leq x \leq \gamma(\beta, \mu^0)e, \quad -(\beta/2)e \leq y \leq \gamma(\beta, \mu^0)e,$$

where γ is given by

$$\gamma(\beta, \mu^0) := \frac{n(\mu^0)^2 + (\mathring{x} + (\beta/2)e)^T(\mathring{y} + (\beta/2)e)}{\min_i\{\mathring{x}_i, \mathring{y}_i\}} > \beta/2. \tag{11}$$

Proof: Let $(x, y) \in \Lambda(\beta, \mu^0)$ and $\phi_i = \phi(\mu, x_i, y_i)$ $(i \in N)$. The lower bounds of x and y are directly follows from (i) of Proposition 2.1 and $\phi(\mu, x_i, y_i) \geq -\beta$ for every $i \in N$.

To show the upper bounds, we use the fact that Condition 1.1 holds. Since $y = Mx+q$, $\mathring{y} = M\mathring{x} + q$ and M is positive semidefinite, we have

$$\begin{aligned} 0 &\leq (x - \mathring{x})^T(y - \mathring{y}) \\ &= \{(x - \Phi/2) - (\mathring{x} - \Phi/2)\}^T\{(y - \Phi/2) - (\mathring{y} - \Phi/2)\}. \end{aligned}$$

By expanding the right hand side, we observe that

$$(\overset{\circ}{y} - \Phi/2)^T(x - \Phi/2) + (\overset{\circ}{x} - \Phi/2)^T(y - \Phi/2) \leq (x - \Phi/2)^T(y - \Phi/2) + (\overset{\circ}{x} - \Phi/2)^T(\overset{\circ}{y} - \Phi/2).($$

Let us show that the right hand side of the above inequality is bounded. The boundedness of the first term follows from (i) of Proposition 2.1, i.e.,

$$(x_i - \phi_i/2)(y_i - \phi_i/2) = \mu^2$$

and hence

$$(x - \Phi/2)^T(y - \Phi/2) = n\mu^2 \leq n(\mu^0)^2.$$

For the second term, the inequality

$$(\overset{\circ}{x} - \Phi/2)^T(\overset{\circ}{y} - \Phi/2) \leq (\overset{\circ}{x} + (\beta/2)e)^T(\overset{\circ}{y} + (\beta/2)e)$$

holds since

$$0 \geq \Phi \geq -\beta e \quad \text{and} \quad (\overset{\circ}{x}, \overset{\circ}{y}) > 0. \tag{13}$$

Combining the above two bounds with (12), we have

$$(\overset{\circ}{y} - \Phi/2)^T(x - \Phi/2) + (\overset{\circ}{x} - \Phi/2)^T(y - \Phi/2) \leq n(\mu^0)^2 + (\overset{\circ}{x} + (\beta/2)e)^T(\overset{\circ}{y} + (\beta/2)e). \tag{14}$$

For the right hand side of (12), the inequalities (13) also imply that

$$\overset{\circ}{y} - \Phi/2 \geq \overset{\circ}{y} > 0, \quad \overset{\circ}{x} - \Phi/2 \geq \overset{\circ}{x} > 0$$

and that

$$0 < (\overset{\circ}{y}_i - \phi_i/2)(x_i - \phi_i/2), \quad 0 < (\overset{\circ}{x}_i - \phi_i/2)(y_i - \phi_i/2)$$

for every $i \in N$. Thus, by (14), we see that

$$\begin{aligned}
x_i - \phi_i/2 &\leq \frac{n(\mu^0)^2 + (\overset{\circ}{x} + (\beta/2)e)^T(\overset{\circ}{y} + (\beta/2)e)}{\overset{\circ}{y}_i - \phi_i/2} \\
&\leq \frac{n(\mu^0)^2 + (\overset{\circ}{x} + (\beta/2)e)^T(\overset{\circ}{y} + (\beta/2)e)}{\min_i\{\overset{\circ}{x}_i, \overset{\circ}{y}_i\}}, \\
y_i - \phi_i/2 &\leq \frac{n(\mu^0)^2 + (\overset{\circ}{x} + (\beta/2)e)^T(\overset{\circ}{y} + (\beta/2)e)}{\overset{\circ}{x}_i - \phi_i/2} \\
&\leq \frac{n(\mu^0)^2 + (\overset{\circ}{x} + (\beta/2)e)^T(\overset{\circ}{y} + (\beta/2)e)}{\min_i\{\overset{\circ}{x}_i, \overset{\circ}{y}_i\}}.
\end{aligned}$$

Finally, the inequality $\gamma(\beta, \mu^0) > \beta/2$ follows from $(\overset{\circ}{x}, \overset{\circ}{y}) > 0$ and $\mu^0 > 0$. ∎

The proposition below is a collection of simple results on the Newton equation (6), but plays a key role in providing a complexity of the algorithm.

Proposition 3.2. *Let $(\bar{\mu}, \bar{x}, \bar{y}) \in \mathbb{R}_{++} \times \mathbb{R}^{2n}$ be a point satisfying $\bar{y} = M\bar{x} + q$ and let $\bar{\phi}_i = \phi(\bar{\mu}, \bar{x}_i, \bar{y}_i)$ $(i \in N)$. Define*

$$(x', y') := (\bar{x} - \bar{\Phi}/2, \bar{y} - \bar{\Phi}/2). \tag{15}$$

Then the following results are true.

(i) $\sqrt{(\bar{x}_i - \bar{y}_i)^2 + 4\bar{\mu}^2} = x_i' + y_i' \geq 2\bar{\mu} \quad (i \in N).$

(ii) *The solution of (6) is the unique solution of the system*

$$\begin{pmatrix} -M & I \\ Y' & X' \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{h} \end{pmatrix} \tag{16}$$

where

$$\bar{h} := -2(1 - \sigma_\mu)\bar{\mu}^2 e - \frac{1 - \sigma_\phi}{2}(X' + Y')\bar{\Phi}. \tag{17}$$

(iii) *Suppose that Condition 1.1 holds. If $(\bar{\mu}, \bar{x}, \bar{y}) \in \mathbb{R}_{++} \times \mathbb{R}^{2n}$ lies in the set $\Lambda(\beta, \mu^0)$ for some $\beta > 0$ and $\mu^0 > 0$ then*

$$0 < \frac{\bar{\mu}^2}{\bar{\gamma}(\beta, \mu^0)} \leq x_i' \leq 2\bar{\gamma}(\beta, \mu^0) \quad (i \in N),$$
$$0 < \frac{\bar{\mu}^2}{\bar{\gamma}(\beta, \mu^0)} \leq y_i' \leq 2\bar{\gamma}(\beta, \mu^0) \quad (i \in N).$$

where

$$\bar{\gamma}(\beta, \mu^0) := \max\{\gamma(\beta, \mu^0),\ \mu^0\} \tag{18}$$

(see (11) for the definition of γ).

Proof: (i): The inequalities are straightforward since $\sqrt{(\bar{x}_i - \bar{y}_i)^2 + 4\bar{\mu}^2} = (\bar{x}_i + \bar{y}_i) - \bar{\phi}_i$ by the definition of $\bar{\phi}_i$, and since $(x_i - y_i)^2 \geq 0$.

(ii): It follows from (6) and (5a) – (5c) that the direction $(\Delta x, \Delta y)$ satisfies

$$\left(1 - \frac{\bar{x}_i - \bar{y}_i}{\sqrt{(\bar{x}_i - \bar{y}_i)^2 + 4\bar{\mu}^2}}\right) \Delta x_i + \left(1 + \frac{\bar{x}_i - \bar{y}_i}{\sqrt{(\bar{x}_i - \bar{y}_i)^2 + 4\bar{\mu}^2}}\right) \Delta y_i$$
$$= (1 - \sigma_\mu) \left(\frac{4\bar{\mu}}{\sqrt{(\bar{x}_i - \bar{y}_i)^2 + 4\bar{\mu}^2}}\right) - (1 - \sigma_\phi)\bar{\Phi}_i$$

for every $i \in N$ Multiplying both sides of the above by $\sqrt{(\bar{x}_i - \bar{y}_i)^2 + 4\bar{\mu}^2} > 0$ and substituting $\sqrt{(\bar{x}_i - \bar{y}_i)^2 + 4\bar{\mu}^2} = x'_i + y'_i$, the assertion follows. (iii): By the definitions of y'_i and $\bar{\phi}_i$ ($i \in N$), we see that

$$\begin{aligned} y'_i &= \bar{y}_i - \bar{\phi}_i/2 \\ &= \bar{y}_i - \frac{1}{2}\{(\bar{x}_i + \bar{y}_i) - \sqrt{(\bar{x}_i - \bar{y}_i)^2 + 4\bar{\mu}^2}\} \\ &= \frac{1}{2}\{(\bar{y}_i - \bar{x}_i) + \sqrt{(\bar{y}_i - \bar{x}_i)^2 + 4\bar{\mu}^2}\}. \end{aligned}$$

Since the function $g(a) := a + \sqrt{a^2 + b}$ with some $b > 0$ is strictly increasing function w.r.t. $a \in \mathbb{R}$, a lower bound of $\bar{y}_i - \bar{x}_i$ in the set $\Lambda(\beta, \mu^0)$ gives the one of $(\bar{y}_i - \bar{x}_i) + \sqrt{(\bar{y}_i - \bar{x}_i)^2 + 4\bar{\mu}^2}$. Here, Proposition 3.1 and the inequality (11) ensure that

$$-2\gamma(\beta, \mu^0) \leq \bar{y}_i - \bar{x}_i \leq 2\gamma(\beta, \mu^0)$$

and hence we obtain that

$$y'_i \geq -\gamma(\beta, \mu^0) + \sqrt{\gamma(\beta, \mu^0)^2 + 4\bar{\mu}^2}.$$

By the definition (18) of $\bar{\gamma}$, the lower bound of y'_i is given as follows:

$$-\gamma(\beta, \mu^0) + \sqrt{\gamma(\beta, \mu^0)^2 + 4\bar{\mu}^2} = \frac{4\bar{\mu}^2}{\gamma(\beta, \mu^0) + \sqrt{\gamma(\beta, \mu^0)^2 + 4\bar{\mu}^2}} \geq \frac{\bar{\mu}^2}{\bar{\gamma}(\beta, \mu^0)}.$$

The upper bound of $y'_i = \bar{y}_i - \bar{\phi}_i/2$ follows from the three facts: $\bar{y}_i \leq \bar{\gamma}(\beta, \mu^0)$ (see Proposition 3.1), $\bar{\phi}_i/2 \leq \beta/2$ and $\beta/2 < \bar{\gamma}(\beta, \mu^0)$ (see (11) and (18)). By a similar discussion, the lower and the upper bounds of x'_i ($i \in N$) are obtained. ∎

The type of the system (16) often appears in the field of interior point algorithms and the following results are well-known (see, for example, [15]).

Proposition 3.3. *Suppose that M is an $n \times n$ positive semi-definite matrix. For every $(x', y') > 0$ and $\bar{h} \in \mathbb{R}^n$, the system (16) has the unique solution $(\Delta x, \Delta y)$ which satisfies the following inequalities:*

$$\begin{aligned} &0 \leq \Delta x^T \Delta y \leq \|(X'Y')^{-1/2}\bar{h}\| \\ &\|(X'Y')^{-1/2}Y'\Delta x\|^2 + \|(X'Y')^{-1/2}X'\Delta y\|^2 \leq \|(X'Y')^{-1/2}\bar{h}\|^2 \end{aligned} \tag{19}$$

In our analysis, by Proposition 2.1 and the definition (15) of (x', y'), we see that $X'Y' = \bar{\mu}^2 I$ with $\bar{\mu} > 0$. Thus the inequality (19) can be rewritten as

$$\|Y'\Delta x\|^2 + \|X'\Delta y\|^2 \le \|\bar{h}\|^2$$

and hence

$$\|Y'\Delta x\| \le \|\bar{h}\|, \quad \|X'\Delta y\| \le \|\bar{h}\|.$$

In addition, if $(\bar{x}, \bar{y}) \in \Lambda(\beta, \mu^0)$ as in (iii) of Proposition 3.2, then

$$\|(X')^{-1}\| \le \frac{\bar{\gamma}(\beta, \mu^0)}{\bar{\mu}^2}, \quad \|(Y')^{-1}\| \le \frac{\bar{\gamma}(\beta, \mu^0)}{\bar{\mu}^2}.$$

This leads us to the following corollary.

Corollary 3.4. *Suppose that the assumptions in (iii) of Proposition 3.2 are satisfied. Then*

$$\|\Delta x\| \le \|(X')^{-1}\|\|X'\Delta x\| \le \frac{\bar{\gamma}(\beta, \mu)}{\bar{\mu}^2}\|\bar{h}\|,$$
$$\|\Delta y\| \le \|(Y')^{-1}\|\|Y'\Delta y\| \le \frac{\bar{\gamma}(\beta, \mu)}{\bar{\mu}^2}\|\bar{h}\|.$$

Here $\bar{h}$ is given by (17).

The last proposition gives a second order approximation of the behavior of Φ along the Newton direction $(\Delta\mu, \Delta x, \Delta y)$.

Proposition 3.5. *Let $(\bar{\mu}, \bar{x}, \bar{y}) \in \mathbb{R}^{1+2n}$ such that $\Phi(\bar{\mu}, \bar{x}, \bar{y}) = \bar{\Phi} \le 0$ and let $(\Delta\mu, \Delta x, \Delta y)$ be the solution of the system (4a) – (4c).*

(i) *For every $i \in N$ and $\theta \in [0, 1]$,*

$$\begin{aligned} 0 &\ge \{1 - \theta(1 - \sigma_\phi)\}\bar{\phi}_i \\ &\ge \phi_i(\bar{\mu} + \theta\Delta\mu, \bar{x} + \theta\Delta x, \bar{y} + \theta\Delta y) \\ &\ge \{1 - \theta(1 - \sigma_\phi)\}\bar{\phi}_i - \frac{\theta^2}{\sigma_\mu\bar{\mu}}\{(1 - \sigma_\mu)^2\bar{\mu}^2 + \Delta x_i^2 + \Delta y_i^2\}. \end{aligned}$$

(ii)

$$\begin{aligned} &\|\Phi(\bar{\mu} + \theta\Delta\mu, \bar{x} + \theta\Delta x, \bar{y} + \theta\Delta y)\| \\ &\quad \le \{1 - \theta(1 - \sigma_\phi)\}\|\bar{\Phi}\| + \frac{\theta^2}{\sigma_\mu\bar{\mu}}\{(1 - \sigma_\mu)^2\sqrt{n}\bar{\mu}^2 + \|\Delta x\|^2 + \|\Delta y\|^2\}. \end{aligned}$$

Proof: Using Taylor's expansion, we have

$$\begin{aligned}
& \phi_i(\bar{\mu}+\theta\Delta\mu, \bar{x}+\theta\Delta x, \bar{y}+\theta\Delta y) \\
= \quad & \bar{\phi}_i + \theta\nabla\phi(\bar{\mu},\bar{x}_i,\bar{y}_i)\begin{pmatrix}\Delta\mu \\ \Delta x_i \\ \Delta y_i\end{pmatrix} + \frac{\theta^2}{2}(\Delta\mu, \Delta x_i, \Delta y_i)\,\nabla^2\phi(\bar{\mu}',\bar{x}_i',\bar{y}_i')\begin{pmatrix}\Delta\mu, \\ \Delta x_i, \\ \Delta y_i\end{pmatrix}
\end{aligned}$$

where $(\bar{\mu}',\bar{x}_i',\bar{y}_i') = (\bar{\mu},\bar{x}_i,\bar{y}_i) + \theta'(\Delta\mu,\Delta x_i,\Delta y_i)$ and $\theta' \in [0,\theta] \subset [0,1]$. For the first derivative, by (4c) and a simple calculation, we can see that

$$\nabla\phi(\bar{\mu},\bar{x}_i,\bar{y}_i)\begin{pmatrix}\Delta\mu \\ \Delta x_i \\ \Delta y_i\end{pmatrix} = -(1-\sigma_\phi)\bar{\phi}_i.$$

For the second one, (iii) of Proposition 2.1 gives the bound

$$0 \geq (\Delta\mu, \Delta x_i, \Delta y_i)\,\nabla\phi(\bar{\mu}',\bar{x}_i',\bar{y}_i')\begin{pmatrix}\Delta\mu, \\ \Delta x_i, \\ \Delta y_i\end{pmatrix} \geq -\frac{\theta^2}{\bar{\mu}'}\{\Delta\mu^2 + \Delta x_i^2 + \Delta y_i^2\}.$$

Since $\Delta\mu = -(1-\sigma_\mu)\bar{\mu}$ and $\bar{\mu}' \geq \bar{\mu} - (1-\sigma_\mu)\bar{\mu} = \sigma_\mu\bar{\mu}$, we obtain the assertion (i).

Now we have that

$$\begin{aligned}
0 \quad & \geq \quad \{1-\theta(1-\sigma_\phi)\}\bar{\Phi} \\
& \geq \quad \Phi(\bar{\mu}+\theta\Delta\mu, \bar{x}+\theta\Delta x, \bar{y}+\theta\Delta y) \\
& \geq \quad \{1-\theta(1-\sigma_\phi)\}\bar{\Phi} - \frac{\theta^2}{\sigma_\mu\bar{\mu}}\left\{(1-\sigma_\mu)^2\bar{\mu}^2 e + \operatorname{vec}\{\Delta x_i^2\} + \operatorname{vec}\{\Delta y_i^2\}\right\}
\end{aligned}$$

By a similar discussion as in the proof of Theorem 4.1 in [2], the assertion (ii) follows. ∎

4 Complexity analysis of the algorithm

As we have mentioned in Section 2, we have to show the finite termination of Step 1.1 to derive a complexity bound of the algorithm. To do this, we prepare three lemmas below. Throughout the discussions below, we assume that Condition 1.1 holds.

Lemma 4.1. *At each iteration k, the inequality*

$$\|\hat{\Phi}^{p+1}\| \leq \max\left\{1 - \frac{\|\hat{\Phi}^p\|\mu^k}{4(\|\Delta x^p\|^2 + \|\Delta y^p\|^2)}, \frac{1}{2}\right\}\|\hat{\Phi}^p\|. \tag{20}$$

holds for $p = 0, 1, 2 \ldots$ in Step 1.1.

Proof: Throughout Step 1.1, we set the parameters σ_μ and σ_ϕ as $\sigma_\mu = 1$ and $\sigma_\phi = 0$. Thus, by (ii) of Proposition 3.5, we have

$$\|\hat{\Phi}^{p+1}\| \leq (1-\theta^p)\|\hat{\Phi}^p\| + \frac{(\theta^p)^2}{\mu^k}\{\|\Delta x^p\|^2 + \|\Delta y^p\|^2\} = g^p(\theta^p).$$

By the definition of (9), if

$$\frac{\|\hat{\Phi}^p\|\mu^k}{2(\|\Delta x^p\|^2 + \|\Delta y^p\|^2)} < 1$$

then

$$g^p(\theta^p) = \|\hat{\Phi}^p\| \left(1 - \frac{\|\hat{\Phi}^p\|\mu^k}{4(\|\Delta x^p\|^2 + \|\Delta y^p\|^2)}\right),$$

and if

$$\frac{\|\hat{\Phi}^p\|\mu^k}{2(\|\Delta x^p\|^2 + \|\Delta y^p\|^2)} \geq 1$$

then

$$g^p(\theta^p) = g(1) = \frac{\|\Delta x^p\|^2 + \|\Delta y^p\|^2}{\mu^k} \leq \frac{1}{2}\|\hat{\Phi}^p\|.$$

In both cases, we can check that the inequality (20) holds. ∎

Lemma 4.2. **(i)** *Let $(\tilde{\mu}, \tilde{x}, \tilde{y})$ be any point generated in the algorithm. Then*

$$\Phi(\tilde{\mu}, \tilde{x}, \tilde{y}) \leq 0.$$

(ii) *Let $(\tilde{\mu}^k, \tilde{x}^k, \tilde{y}^k)$ be any point generated at the iteration k. Then*

$$\|\Phi(\tilde{\mu}^k, \tilde{x}^k, \tilde{y}^k)\| \leq \beta^k$$

and hence

$$(\tilde{\mu}^k, \tilde{x}^k, \tilde{y}^k) \in \Lambda(\beta^k, \mu^0)$$

where

$$\beta^k := \frac{6\bar{\gamma}(\epsilon,\mu^0)^2 n}{\mu^k} \tag{21}$$

Proof: By the construction of the algorithm, the initial point (μ^0, x^0, y^0) satisfies $\Phi(\mu^0, x^0, y^0) \leq 0$ and each direction used in the algorithm is a solution of the system (4a) – (4c) for some $\sigma_\mu \in [0,1]$ and $\sigma_\phi \in [0,1]$. Since the step size θ never exceeds 1 by (i) of Proposition 3.5, we can see that at every next iterate, the value of ϕ_i is always nonpositive for every $i \in N$.

Lemma 4.1 ensures that the sequence $\{\hat{\Phi}^p\}$ generated in Step 1.1 is strictly decreasing. Since we assume that $\|\Phi(\mu^k, x^k, y^k)\| < \epsilon$ at the first stage of Step 1.2, we only have to consider how the value $\|\Phi(\mu^{k+1}, x^{k+1}, y^{k+1})\|$ is increased in Step 1.2. In Step 1.2, we choose $\sigma_\mu = 1/2$, $\sigma_\phi = 1$ and $\theta = 1$. Thus, it follows from (ii) of Proposition 3.5 that

$$\|\Phi(\mu^{k+1}, x^{k+1}, y^{k+1})\| \leq \epsilon + \frac{2}{\mu^k}\{\sqrt{n}(\mu^k)^2/4 + \|\Delta x\|^2 + \|\Delta y\|^2\}. \tag{22}$$

Since $(\mu^k, x^k, y^k) \in \Lambda(\epsilon, \mu^0)$, by Corollary 3.4, we see that

$$\|\Delta x\|^2 + \|\Delta y\|^2 \leq 2\frac{\bar{\gamma}(\epsilon,\mu^0)^2}{(\mu^k)^4}\|h^k\|^2$$

where $h^k = -(\mu^k)^2 e$ and

$$\|h^k\| = (\mu^k)^2\sqrt{n}.$$

Thus we can see that

$$\begin{aligned}\|\Delta x\|^2 + \|\Delta y\|^2 &\leq \frac{2\bar{\gamma}(\epsilon,\mu^0)^2}{(\mu^k)^4}(\mu^k)^4 n \\ &= 2\bar{\gamma}(\epsilon,\mu^0)^2 n.\end{aligned}$$

Since the definition (18) of $\bar{\gamma}$ and $\epsilon \leq \mu^k \leq \mu^0$ imply that

$$\frac{\sqrt{n}(\mu^k)^2}{4} \leq \frac{\bar{\gamma}(\epsilon,\mu^0)^2 n}{4} \quad \text{and} \quad \epsilon \leq \mu^0 \leq \frac{\bar{\gamma}(\epsilon,\mu^0)^2}{\mu^0} \leq \frac{\bar{\gamma}(\epsilon,\mu^0)^2}{\mu^k},$$

we obtain the bound

$$\begin{aligned}\|\Phi(\mu^{k+1}, x^{k+1}, y^{k+1})\| &\leq \epsilon + \frac{2}{\mu^k}\left\{\frac{\sqrt{n}(\mu^k)^2}{4} + 2\bar{\gamma}(\epsilon,\mu^0)^2 n\right\} \\ &\leq \epsilon + \frac{2}{\mu^k}\frac{9}{4}\bar{\gamma}(\epsilon,\mu^0)^2 n \\ &\leq \frac{6\bar{\gamma}(\epsilon,\mu^0)^2 n}{\mu^k}\end{aligned}$$

∎

Now we are ready to show our main theorem.

Theorem 4.3. **(i)** *At each iteration k, Step 1.1 terminates after*

$$P^k := \left\lceil 2 \max\left\{ \frac{16\bar{\gamma}(\beta^k, \mu^0)^4 \|\Phi^k\|}{(\mu^k)^5},\ 1 \right\} \log \frac{\|\Phi^k\|}{\epsilon} \right\rceil$$

Newton iterations.

(ii) *The total number of Newton iterations in the algorithm is bounded by*

$$\left\lceil \log \frac{\bar{\beta}}{\epsilon} \right\rceil \cdot \left\{ \left\lceil \frac{2^{11}\bar{\gamma}(\epsilon, \mu^0)^4 \bar{\beta}}{\epsilon^5} \right\rceil + 3\left(\left\lceil \log \frac{\mu^0}{\epsilon} \right\rceil + 1 \right) \right\}.$$

Here $\bar{\gamma}$ and β^k are given by (18) and (21), and

$$\bar{\beta} := \frac{6\bar{\gamma}(\epsilon, \mu^0)^2 n}{\epsilon} \tag{23}$$

Proof: (i): (ii) of Lemma 4.2 and Corollary 3.4 ensure that the inequality

$$\|\Delta x^p\|^2 + \|\Delta y^p\|^2 \le 2\frac{\bar{\gamma}(\beta^k, \mu^0)^2}{(\mu^k)^4} \|h^p\|^2$$

holds with

$$h^p = -\frac{1}{2}(X' + Y')\Phi^p.$$

Since the bound

$$\|h^p\| \le \frac{1}{2}\|X' + Y'\| \|\hat{\Phi}^p\| \le 2\bar{\gamma}(\beta^k, \mu^0)\|\hat{\Phi}^p\|$$

follows from (iii) of Proposition 3.2, we find that

$$\|\Delta x^p\|^2 + \|\Delta y^p\|^2 \le 2\frac{\bar{\gamma}(\beta^k, \mu^0)^2}{(\mu^k)^4} \left\{ 2\bar{\gamma}(\beta^k, \mu^0)\|\hat{\Phi}^p\| \right\}^2 = \frac{8\bar{\gamma}(\beta^k, \mu^0)^4}{(\mu^k)^4}\|\hat{\Phi}^p\|^2.$$

This implies that

$$\begin{aligned} \frac{\|\hat{\Phi}^p\|\mu^k}{4(\|\Delta x^p\|^2 + \|\Delta y^p\|^2)} &\ge \|\hat{\Phi}^p\|\mu^k \frac{(\mu^k)^4}{4 \cdot 8\bar{\gamma}(\beta^k, \mu^0)^4 \|\hat{\Phi}^p\|^2} \\ &= \frac{(\mu^k)^5}{32\bar{\gamma}(\beta^k, \mu^0)^4 \|\hat{\Phi}^p\|} \\ &\ge \frac{(\mu^k)^5}{32\bar{\gamma}(\beta^k, \mu^0)^4 \|\Phi^k\|}. \end{aligned}$$

Here the last inequality follows from Lemma 4.1, i.e.,

$$\|\hat{\Phi}^p\| \leq \|\hat{\Phi}^0\| = \|\Phi^k\|.$$

Thus $\|\hat{\Phi}^p\|$ will be reduced at least by the factor $1 - \delta^k$ where

$$\delta^k := \min\left\{\frac{(\mu^k)^5}{32\bar{\gamma}(\beta^k,\mu^0)^4\|\Phi^k\|}, \frac{1}{2}\right\} \tag{24}$$

at each iteration p. Now let us consider the inequality w.r.t. the number of iteration p

$$(1-\delta^k)^p\|\Phi^k\| < \epsilon. \tag{25}$$

By taking logarithms in both sides above and using the inequality

$$\log(1-\delta^k) \leq -\delta^k < 0$$

we can derive a lower bound of p

$$p \geq \left\lceil \frac{1}{\delta^k}\log\frac{\|\Phi^k\|}{\epsilon}\right\rceil$$

as a sufficient condition of p to satisfy (25). Substituting (24), we obtain the assertion (i) from Lemma 4.1.

(ii): By the definitions (18) of $\bar{\gamma}$, (21) of β^k and (23) of $\bar{\beta}$, and by the relation $\mu^k \geq \epsilon$, we have

$$\beta^k \leq \bar{\beta} \quad \text{and} \quad \bar{\gamma}(\beta^k,\mu^0) \leq \bar{\gamma}(\bar{\beta},\mu^0).$$

for every k. Since $\|\Phi^k\| \leq \beta^k$ by (ii) of Lemma 4.2, we have that

$$\begin{aligned} P^k &\leq \left\lceil 2\max\left\{\frac{16\bar{\gamma}(\bar{\beta},\mu^0)^4\bar{\beta}}{(\mu^k)^5}, 1\right\}\log\frac{\bar{\beta}}{\epsilon}\right\rceil \\ &\leq \left\lceil 2\left(\frac{16\bar{\gamma}(\bar{\beta},\mu^0)^4\bar{\beta}}{(\mu^k)^5}+1\right)\right\rceil\left\lceil\log\frac{\bar{\beta}}{\epsilon}\right\rceil. \end{aligned}$$

for every k. Thus the total number of Newton iterations is bounded by

$$\begin{aligned} \sum_{k=0}^{K} P^k &\leq \left\lceil\log\frac{\bar{\beta}}{\epsilon}\right\rceil\cdot\sum_{k=0}^{K}\left\lceil 2\left(\frac{16\bar{\gamma}(\bar{\beta},\mu^0)^4\bar{\beta}}{(\mu^k)^5}+1\right)\right\rceil \\ &\leq \left\lceil\log\frac{\bar{\beta}}{\epsilon}\right\rceil\cdot\left\{\left\lceil\sum_{k=0}^{K}2\left(\frac{16\bar{\gamma}(\bar{\beta},\mu^0)^4\bar{\beta}}{(\mu^k)^5}+1\right)\right\rceil+(K+1)\right\} \\ &= \left\lceil\log\frac{\bar{\beta}}{\epsilon}\right\rceil\cdot\left\{\left\lceil 32\bar{\gamma}(\bar{\beta},\mu^0)^4\bar{\beta}\sum_{k=0}^{K}\left(\frac{1}{(\mu^k)^5}\right)\right\rceil+3(K+1)\right\} \end{aligned}$$

where K satisfies $\epsilon/2 < (1/2)^K\mu^0 < \epsilon$ and is bounded by $\lceil\log(\mu^0/\epsilon)\rceil$. Since $\mu^k = (1/2)^k\mu^0$ and

$$\begin{aligned}
\sum_{k=0}^{K}\left(\frac{1}{\mu^k}\right)^5 &= \frac{1}{(\mu^0)^5}\sum_{k=0}^{K}(2^5)^k \\
&= \frac{1}{(\mu^0)^5}\frac{(2^5)^{K+1}-1}{(2^5-1)} \\
&= \frac{1}{(1/2)^{5(K+1)}(\mu^0)^5}\frac{1-(1/2)^{5(K+1)}}{2^5-1} \\
&\le \frac{1}{(1/2)^{5(K+1)}(\mu^0)^5}\frac{1}{31} \\
&= \frac{1}{(1/2)^5\{(1/2)^K(\mu^0)\}^5}\frac{1}{31} \\
&\le \frac{1}{(1/2)^5(\epsilon/2)^5}\frac{1}{31} \\
&= \frac{2^{10}}{31\epsilon^5} \\
&\le \frac{34}{\epsilon^5},
\end{aligned}$$

we complete the proof of the theorem. ∎

5 Concluding remarks

In this paper, we provide a computational complexity of a smoothing method for the LCP under Condition 1.1 holds. A feature of our analysis is that we need not impose any other additional assumptions on the problem. Consequently, we can show that the algorithm terminates in

$$O\left(\frac{\bar{\gamma}^6 n}{\epsilon^6}\log\frac{\bar{\gamma}^2 n}{\epsilon^2}\right)$$

Newton iterations where γ is a number which depends on the problem and the initial point.

In Section 3, we collect some properties of the smoothing method. These results give us not only the tools for complexity analysis but also information on the relationship between interior point method and smoothing method. Let (x', y') be an feasible interior point of the LCP' (2). In many interior point algorithms based on the primal-dual algorithm (see, for example, [14], etc.), the unique solution $(\Delta x^{\mathrm{int}}, \Delta y^{\mathrm{int}})$ of the system

$$\begin{pmatrix} -M & I \\ Y' & X' \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} 0 \\ -X'y' + \mu e \end{pmatrix}$$

for some $\mu > 0$ is often used as a search direction. Let us consider the Newton equation (16) with $\sigma_\mu = 0$ and $\sigma_\phi = 1$. As we have mentioned in Section 2, $(x'y')$ is an analytical center of the perturbed problem (2) satisfying

$$y' = Mx' + q', \ (x'y') > 0, \ y'_i x'_i = \mu > 0.$$

Thus, $X'y' = \mu e$ and the solution $(\Delta x^{\mathrm{sm}}, \Delta y^{\mathrm{sm}})$ coincides with $(\Delta x^{\mathrm{int}}, \Delta y^{\mathrm{int}})$. That is the direction $(\Delta x^{\mathrm{sm}}, \Delta y^{\mathrm{sm}})$ can be regarded as the search direction used in the interior point method at an analytical center (x', y').

At the end of Step 1.1 of the iteration 0, we obtain a point $(\bar{x}, \bar{y})$ satisfying

$$\|\Phi(\mu^0, \bar{x}, \bar{y})\| = \|\bar{\Phi}\| < \epsilon, \ y = Mx + q$$

after

$$\left\lceil 2 \max\left\{ \frac{16\bar{\gamma}(\beta^0, \mu^0)^4 \|\Phi^0\|}{(\mu^0)^5}, 1 \right\} \log \frac{\|\Phi^0\|}{\epsilon} \right\rceil$$

Newton iterations ((i) of Theorem 4.3. Let us consider the perturbed problem LCP$(\mu^0, \bar{x}, \bar{y})$ (2). Since $\|\bar{\Phi}\| = \|\Phi(\mu^0, \bar{x}, \bar{y})\|$ is sufficiently small, we may consider the problem as an ϵ-approximated problem in which the axes are shifted by $\bar{\phi}_i \in [-\epsilon/2, \epsilon/2](i \in N)$. Since $(\bar{x} - \bar{\Phi}/2, \bar{y} - \bar{\Phi}/2) > 0$ is an analytical center of a perturbed problem (2), we can apply an $O(\sqrt{n}L)$ feasible interior point algorithm for solving (2) with the initial point $(\bar{x}, \bar{y})$. Thus, if the number

$$\left\lceil 2 \max\left\{ \frac{16\bar{\gamma}(\beta^0, \mu^0)^4 \|\Phi^0\|}{(\mu^0)^5}, 1 \right\} \log \frac{\|\Phi^0\|}{\epsilon} \right\rceil$$

of iterations required in Step 1.1 does not exceed $O(\sqrt{n}L)$, then we can obtain an ϵ-approximated solution of the LCP in $O(\sqrt{n}L)$ iterations.

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