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Unbiased Tests for Location and Scale Parameters
-Case of Cauchy Distribution-

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Unbiased Tests for Location and Scale Parameters
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Abstract.

In this paper we deal with Cauchy distribution with the density

$$f(x|\theta,\xi)=\xi\pi^{-1}\{\xi^2+(x-\theta)^2\}^{-1},\quad \text{for } -\infty<x<\infty$$

where $-\infty<\theta<\infty$ and $\xi>0$.

We first consider $\xi=1$. Based on a random sample of size n from $f(x|\theta,1)$ we consider the problem of testing the null hypothesis $H_0:\theta=\theta_0$ versus the alternative $H_1:\theta\neq\theta_0$ for some constant θ_0. We propose the test with the acceptance region derived from inverting the shortest confidence interval (C. I.) for θ_0 and check if this test is unbiased.

We secondly consider $\theta=0$. This time we consider the problem of testing $H_0:\xi=\xi_0$ versus $H_1:\xi\neq\xi_0$ for some constant ξ_0. We again propose the test with acceptance region derived from inverting the C. I. for ξ_0 and check if this test is unbiased.

§1. Introduction.

In this paper we deal with Cauchy distribution whose density is given as follows:

(1) $f(x|\theta,\xi) = \xi\pi^{-1}\{\xi^2 + (x-\theta)^2\}^{-1}$ for $-\infty<x<\infty$

provided that $-\infty<\theta<\infty$ and $\xi>0$.

Let $\doteq$ be the defining property. We first consider the density $f(x|\theta)\doteq f(x|\theta,1)$. Let $X_1, \ldots, X_n$ be a random sample of size n taken from the density $f(x|\theta)$. We find in Section 2 the confidence interval(C. I.) for the location parameter θ with the shortest length using Lagrange's method. In Section 3 we consider the problem of testing the null hypothesis $H_0:\theta=\theta_0$ versus the alternative hypothesis $H_1:\theta\neq\theta_0$ for some constant θ_0. We propose the test with the acceptance region derived from inverting the shortest C. I. for θ_0. Let α be a real number such that $0<\alpha<1$. When n=2m+1 with m a nonnegative integer, we show that our test is unbiased and of size α. But, when n=2m, because we use conventional method to get the C. I. for θ, we cannot show unbiasedness of our test. (However, for large m our test becomes almost unbiased as the test in case of n=2m+1 shows.)

In the second half We consider the density $f(x|\xi)\doteq f(x|0,\xi)$. Based on a random sample of size n from the density $f(x|\xi)$ we find in Section 4 the C. I. for the scale parameter ξ. In Section 5 we consider the problem of testing $H_0:\xi=\xi_0$ versus $H_1:\xi\neq\xi_0$ for some constant ξ_0. Again we propose the test with acceptance region derived from inverting the C. I. for ξ_0. When n=2m+1, we show that our test is unbiased and of size α. But, in the same reason as that for θ our test is not unbiased when n=2m. (However, for large m our test becomes almost unbiased as the test in case of n=2m+1 shows.)

§2. The Interval Estimation for θ.

In this section we deal with the density

(2) $$f(x|\theta)\doteq f(x|\theta,1)=\pi^{-1}\{1+(x-\theta)^2\}^{-1}, \qquad \text{for } -\infty<x<\infty$$

where $-\infty<\theta<\infty$. We find the shortest C. I. for θ using Lagrange's method.

Let $n=2m+1$ with m a nonnegative integer, until (15). Let $X_{(i)}$ be the i-th smallest observation of $X_1, \ldots, X_n$. We estimate θ by $Y\doteq X_{(m+1)}$. To get the shortest C. I. for θ we first find the density of Y. Let $F(x|\theta)$ be the cumulative distribution function(c. d. f.) of X. Then, by (2) we get

(3) $$F(x)\doteq F(x|\theta)=\pi^{-1}\tan^{-1}(x-\theta)+2^{-1}, \qquad \text{for } -\infty<x<\infty.$$

Hence, the density of Y is of form

(4) $$g_Y(y|\theta)=k(F(y))^m(1-F(y))^m f(y|\theta), \qquad \text{for } -\infty<y<\infty$$

where

(5) $$k=\Gamma(2m+2)/(\Gamma(m+1))^2.$$

Let α be a real number such that $0<\alpha<1$. Let r_1 and r_2 be real numbers such that $r_1<r_2$. To find the shortest C. I. for θ at confidence coefficient $1-\alpha$ we want to minimize r_2-r_1 under the condition that

(6) $$P_\theta[r_1 < Y-\theta < r_2] = 1-\alpha.$$

But, it follows by a variable transformation $W=F(Y)$ that

$$\text{the left hand side of (6)}=P_\theta[r_1+\theta < Y < r_2+\theta]$$

(7) $$=P_\theta[F(r_1+\theta) < W < F(r_2+\theta)]=1-\alpha.$$

Hence, we want to minimize r_2-r_1 under the condition (7). To do so we use Lagrange's method. Let λ be a real number and define

(8) $$L \doteq L(r_1, r_2; \lambda) = r_2 - r_1 - \lambda\left\{\int_{F(r_1+\theta)}^{F(r_2+\theta)} h_W(w)\,dw - 1 + \alpha\right\}$$

where $h_W(w)$ is the density of W given by

(9) $$h_W(w) = kw^m(1-w)^m, \quad \text{for } 0<w<1$$

where k is given by (5). The right hand side of (9) is the probability density function (p.d.f.) of Beta distribution Beta(m+1,m+1) with (m+1,m+1) degrees of freedom. Then, by Lagrange's method we have that

(10) $$\begin{cases} \partial L/\partial r_1 = -1+\lambda h_W(F(r_1+\theta))f(r_1+\theta\,|\,\theta)=0 \\ \partial L/\partial r_2 = 1-\lambda h_W(F(r_2+\theta))f(r_2+\theta\,|\,\theta) = 0 \end{cases}$$

By (10) we get

(11) $$h_W(F(r_1+\theta))f(r_1+\theta\,|\,\theta)=h_W(F(r_2+\theta))f(r_2+\theta\,|\,\theta)\,(=\lambda^{-1}), \quad \forall\theta.$$

Taking

(12) $$F(r_1+\theta)=\beta(\alpha/2) \quad \text{and} \quad F(r_2+\theta)=1-\beta(\alpha/2)$$

where $\beta(\alpha/2)$ is given by

(13) $$\int_0^{\beta(\alpha/2)} h_W(w)\,dw=\alpha/2$$

we obtain by (3) that $r_1=-r_2 \doteq -r$ where

(14) $$r=F^{-1}(1-\beta(\alpha/2))-\theta = \tan[(2^{-1}-\beta(\alpha/2))\pi].$$

We also have that $h_W(F(-r+\theta))=h_W(F(r+\theta))$ and $f(-r+\theta\mid\theta)=f(r+\theta\mid\theta)$ with r given by (14). Thus, (11) and (7) are satisfied for $r_1=-r_2=-r$ with r given by (14). Therefore, the shortest C. I. for θ at confidence coefficient $1-\alpha$ is given by

(15) $(Y-r,\ Y+r)\doteq(Y-\tan[(2^{-1}-\beta(\alpha/2))\pi],\ Y+\tan[(2^{-1}-\beta(\alpha/2))\pi])$.

Let n=2m. This time we estimate θ by $Y\doteq X_{(m)}$. In the similar way to the above we get the density of Y as follows:

(16) $g_Y(y\mid\theta)=k_1(F(y))^{m-1}(1-F(y))^m f(y\mid\theta)$, for $-\infty<y<\infty$

where

(17) $k_1\doteq\Gamma(2m+1)/\{\Gamma(m)\Gamma(m+1)\}$.

Putting $W\doteq F(Y)$ we minimize r_2-r_1 under the condition (7). However, since the density of W is now of form

(18) $h_1(w)=k_1w^{m-1}(1-w)^m$, for $0<w<1$

which is the p. d. f. of the Beta(m, m+1) distribution with k_1 defined by (17), it is difficult to get exact values for $F(r_i+\theta)$, i=1, 2 which satisfy

(19) $h_1(F(r_1+\theta))f(r_1+\theta\mid\theta)=h_1(F(r_2+\theta))f(r_2+\theta\mid\theta)$.

Hence, we use conventional values for $F(r_i+\theta)$, i=1, 2. Those are

(20) $F(r_1+\theta)=\beta_{m,\,m+1}(\alpha/2)$ and $F(r_2+\theta)=1-\beta_{m+1,\,m}(\alpha/2)$

where $\beta_{m,\,m+1}(\alpha/2)$ and $\beta_{m+1,\,m}(\alpha/2)$ are respectively determined by

(21) $$\int_0^{\beta_{m,\,m+1}(\alpha/2)} h_1(w)\,dw=\alpha/2=\int_0^{\beta_{m+1,\,m}(\alpha/2)} k_1w^m(1-w)^{m-1}\,dw.$$

Thus, by (3) r_1 and r_2 are respectively given by

$$(22)\qquad \begin{cases} r_1=F^{-1}(\beta_{m,m+1}(\alpha/2))-\theta = -\tan[(2^{-1}-\beta_{m,m+1}(\alpha/2))\pi], \\ r_2=F^{-1}(1-\beta_{m+1,m}(\alpha/2))-\theta = \tan[(2^{-1}-\beta_{m+1,m}(\alpha/2))\pi]. \end{cases}$$

Therefore, the C. I. for θ at confidence coefficient $1-\alpha$ is

$$(23)\qquad (Y-r_2, Y-r_1)\doteq(Y-\tan[(2^{-1}-\beta_{m+1,m}(\alpha/2))\pi],\ Y+\tan[(2^{-1}-\beta_{m,m+1}(\alpha/2))\pi]).$$

In the next section we check if the tests with the acceptance regions derived from inverting the C. I.'s (15) for $n=2m+1$ and (23) for $n=2m$ are unbiased and of size α.

§3. Two-Sided Test for θ.

In this section we consider the problem of testing the null hypothesis $H_0:\theta=\theta_0$ versus the alternative hypothesis $H_1:\theta\neq\theta_0$ for some constant θ_0. We propose the two-sided tests with the acceptance regions derived from inverting the (shortest) C. I.'s for θ_0 obtained in Section 2. When $n=2m+1$, we show that our test is unbiased and of size α. When $n=2m$, our test is not unbiased because of usage of conventional method for constructing the C. I. for θ.

Let $n=2m+1$. As in Section 2 we define $Y\doteq X_{(m+1)}$. By inverting the shortest C. I. (15) for θ_0 our test is to reject H_0 if $Y\in(-\infty,\theta_0-r]\cup[\theta_0+r,+\infty)$ and to accept H_0 if $Y\in(\theta_0-r,\theta_0+r)$ where r is given by (14). Now, we show that this test is unbiased and of size α.

Let y_1^0 and y_2^0 be real numbers depending on θ_0 such that $y_1^0<y_2^0$. Define $\psi(\theta)$ by

$$(24)\qquad \psi(\theta) \doteq P_\theta[Y<y_1^0 \quad\text{or}\quad y_2^0<Y] = 1 - \int_{y_1^0}^{y_2^0} g_Y(y|\theta)\,dy$$

where $g_Y(y|\theta)$ is defined by (4).

To get unbiased size-α test with the acceptance region (y_1^0, y_2^0) we choose y_1^0 and y_2^0 which satisfy

$$(25) \qquad \psi(\theta_0) = 1 - P_{\theta_0}[y_1^0 < Y < y_2^0] = \alpha$$

and minimize $\psi(\theta)$ at $\theta=\theta_0$; namely

$$(26) \qquad d\psi(\theta)/d\theta \Big|_{\theta=\theta_0} = g_Y(y_2^0|\theta_0) - g_Y(y_1^0|\theta_0) = 0.$$

We consider the test with the acceptance region $(\theta_0 - r, \theta_0 + r)$. Since from the construction the equality (11) with $r_1 = -r$, $r_2 = r$ and $\theta = \theta_0$ is satisfied, it follows from (4) and (9) that $g_Y(\theta_0 - r|\theta_0) = g_Y(\theta_0 + r|\theta_0)$; (26) is satisfied for y_1^0 and y_2^0 replaced by $\theta_0 - r$ and $\theta_0 + r$, respectively. (25) with y_1^0 and y_2^0 replaced by $\theta_0 - r$ and $\theta_0 + r$, respectively is the same as (6) except for θ, r_1 and r_2 replaced by θ_0, $-r$ and r, respectively. Therefore, our test with the acceptance region $(\theta_0 - r, \theta_0 + r)$ is unbiased and of size α.

Let n=2m. As in Section 2 we define $Y \doteq X_{(m)}$. Again, by inverting the C. I. (23) for θ_0 our test is to reject H_0 if $Y \in (-\infty, \theta_0 + r_1] \cup [\theta_0 + r_2, +\infty)$ and to accept H_0 if $Y \in (\theta_0 + r_1, \theta_0 + r_2)$ where r_1 and r_2 are given by (22). In this case our test depends on the conventional values for $F(r_i + \theta)$, i=1, 2. Hence, we have that $g_Y(\theta_0 + r_1|\theta_0) \neq g_Y(\theta_0 + r_2|\theta_0)$. Furthermore, (25) with y_1^0 and y_2^0 replaced by $\theta_0 + r_1$ and $\theta_0 + r_2$, respectively is the same as (6) except for θ replaced by θ_0. Thus, our test is of size α, but is not unbiased. However, for large m our test becomes almost unbiased as the test in the case of n=2m+1 shows.

In the next two sections we deal with the scale parameter ξ. In Section 4 we obtain the C. I. for ξ and in Section 5 we check if two-sided test with acceptance region derived from inverting the C. I. for ξ_0 is unbiased.

§4. The Interval Estimation for ξ.

In this section we consider the density (1) with $\theta=0$;

$$(27) \qquad f(x|\xi) \doteq f(x|0,\xi) = \xi\pi^{-1}\{\xi^2 + x^2\}^{-1}, \qquad \text{for } -\infty < x < \infty$$

provided that $\xi > 0$.

Let $X_1, \ldots, X_n$ be a random sample of size n taken from the population with density $f(x|\xi)$. Again, we first consider the case of n=2m+1 with m a nonnegative integer and secondly the case of n=2m. Putting $\xi^* \doteq \ln \xi$ we have

$$f(x|\xi)=\pi^{-1}e^{-\xi^*}\{1+e^{2(\ln|x|-\xi^*)}\}^{-1}, \quad \text{for } -\infty<x<\infty.$$

Thus, letting $Z \doteq \ln|X|$ and $Z_{(i)}$ be the i-th smallest observation of $Z_1, \ldots, Z_n$ we estimate ξ^* by $Y \doteq Z_{(m+1)}$ when n=2m+1 and by $Y \doteq Z_{(m)}$ when n=2m, respectively. We find the C. I.'s for ξ according to these estimates.

We beforehand derive the distribution of Z. Since $x=e^z$ for $x>0$; $x=-e^z$ for $x<0$; $z=-\infty$ for $x=0$, by a variable transformation $Z=\ln|X|$ the density of Z is obtained as follows:

$$q_Z(z) \doteq q_Z(z|\xi)=f(e^z|\xi)|de^z/dz| + f(-e^z|\xi)|d(-e^z)/dz|$$

$$(28) \qquad = 2\pi^{-1}\frac{e^{z-\xi^*}}{1+e^{2(z-\xi^*)}}, \qquad -\infty<z<\infty$$

where $-\infty<\xi^*<\infty$. Since $q_Z(2\xi^*-z)=q_Z(z)$, $q_Z(z)$ is symmetric about $z=\xi^*$ and the unimodal function with the mode ξ^*.

Now, we let n=2m+1 until (37). We estimate ξ^* by $Y \doteq Z_{(m+1)}$. Letting $Q_Z(z)$ be the c.d.f. of Z we obtain by (28) that

$$(29) \qquad Q_Z(z) \doteq Q_Z(z|\xi)= 2\pi^{-1}\tan^{-1}(e^{z-\xi^*}), \text{ for } -\infty<z<\infty.$$

The p.d.f. $g_Y(y|\xi)$ of Y is derived as follows:

$$(30) \qquad g_Y(y|\xi)=k(Q_Z(y))^m(1-Q_Z(y))^m q_Z(y), \quad \text{for } -\infty<y<\infty.$$

Let α be a real number such that $0<\alpha<1$. Let r_1 and r_2 be real numbers such that $0<r_1<r_2$. To find the C. I. for ξ at confidence coefficient $1-\alpha$ we want to find r_1 and r_2 under the condition that

$$(31) \qquad P_\xi[r_1e^Y< \xi <r_2e^Y] = 1-\alpha.$$

But, it follows by a variable transformation $W=Q_Z(Y)$ that

$$\text{the left hand side of (31)}=P_\xi[-\ln r_2 < Y-\xi^* < -\ln r_1]$$

(32) $$=P_\xi[Q_Z(\xi^*-\ln r_2) < W < Q_Z(\xi^*-\ln r_1)]=1-\alpha.$$

Hence, we want to find r_1 and r_2 which minimize $Q_Z(\xi^*-\ln r_1)-Q_Z(\xi^*-\ln r_2)$ under the condition (32). To do so we use Lagrange's method. Let λ be a real number and define

$$L \doteq L(Q_Z(\xi^*-\ln r_1),Q_Z(\xi^*-\ln r_2);\lambda)$$

(33) $$\doteq Q_Z(\xi^*-\ln r_1)-Q_Z(\xi^*-\ln r_2)-\lambda\{\int_{Q_Z(\xi^*-\ln r_2)}^{Q_Z(\xi^*-\ln r_1)} h_W(w)\,dw -1 +\alpha\}$$

where $h_W(w)$ is defined by (9). Then, by Lagrange's method we have that

(34) $$\begin{cases} \partial L/\partial Q_Z(\xi^*-\ln r_1) = 1-\lambda\ h_W(Q_Z(\xi^*-\ln r_1)=0 \\ \partial L/\partial Q_Z(\xi^*-\ln r_2) = -1+\lambda\ h_W(Q_Z(\xi^*-\ln r_2)=0 \end{cases}$$

By (34) we get

(35) $$h_W(Q_Z(\xi^*-\ln r_1))=h_W(Q_Z(\xi^*-\ln r_2))\ (=\lambda^{-1}), \qquad \forall\xi.$$

Taking

$$Q_Z(\xi^*-\ln r_2)=\beta(\alpha/2) \quad \text{and} \quad Q_Z(\xi^*-\ln r_1)=1-\beta(\alpha/2)$$

where $\beta(\alpha/2)$ is given by (13), we obtain by (29) that

(36) $$\begin{cases} r_1=[\tan\{2^{-1}\pi(1-\beta(\alpha/2))\}]^{-1} \\ r_2=[\tan\{2^{-1}\pi\beta(\alpha/2)\}]^{-1} \end{cases}$$

and furthermore (35) and (32) are satisfied for r_1 and r_2 given by (36). Therefore, the C. I. for ξ is given by

(37) $$(r_1 e^Y, r_2 e^Y) \doteq ([\tan\{2^{-1}\pi(1-\beta(\alpha/2))\}]^{-1}e^Y,\ [\tan\{2^{-1}\pi\beta(\alpha/2)\}]^{-1}e^Y).$$

We now consider the case of n=2m. In this case we estimate ξ^* by $Y \doteq Z_{(m)}$. Then, the p. d. f. of Y is given by

(38) $$g_Y(y|\xi)=k_1(Q_Z(y))^{m-1}(1-Q_Z(y))^m q_Z(y), \quad \text{for } -\infty<y<\infty$$

where k_1 is given by (17). To find the C. I. for ξ at confidence coefficient $1-\alpha$ we want to find r_1 and r_2 with $0<r_1<r_2$ under the condition that

(39) $$P_\xi[r_1 e^Y<\xi<r_2 e^Y]=1-\alpha.$$

But, it follows by a variable transformation $W=Q_Z(Y)$ that

$$\text{the left hand side of (39)}=P_\xi[-\ln r_2<Y-\xi^*<-\ln r_1]$$

(40) $$=P_\xi[Q_Z(\xi^*-\ln r_2)< W < Q_Z(\xi^*-\ln r_1)]=1-\alpha.$$

Hence, we want to find r_1 and r_2 which minimize $Q_Z(\xi^*-\ln r_1)-Q_Z(\xi^*-\ln r_2)$ under the condition (40). Going through the similar process to (33) through (35), we get

(41) $$h_1(Q_Z(\xi^*-\ln r_1))=h_1(Q_Z(\xi^*-\ln r_2)) \ (=\lambda^{-1}), \quad \forall\xi$$

where $h_1(w)$ is the density of W given by (18). However, again it is difficult to get exact values of $Q_Z(\xi^*-\ln r_i)$, i=1, 2 which satisfy (41) (and furthermore $q_Z(\xi^*-\ln r_1)=q_Z(\xi^*-\ln r_2)$). Hence, we use conventional values for $Q_Z(\xi^*-\ln r_i)$, i=1, 2. Those are

(42) $$Q_Z(\xi^*-\ln r_2)=\beta_{m,\,m+1}(\alpha/2) \quad \text{and} \quad Q_Z(\xi^*-\ln r_1)=1-\beta_{m+1,\,m}(\alpha/2)$$

where $\beta_{m,m+1}(\alpha/2)$ and $\beta_{m+1,m}(\alpha/2)$ are respectively determined by (21). Thus, by (29) we obtain

$$(43)\qquad \begin{cases} r_1=[\tan\{2^{-1}\pi(1-\beta_{m+1,m}(\alpha/2))\}]^{-1}, \\ \\ r_2=[\tan\{2^{-1}\pi\beta_{m,m+1}(\alpha/2)\}]^{-1}. \end{cases}$$

Therefore, the C. I. for ξ is

$$(44)\qquad (r_1e^Y,\ r_2e^Y)$$

where r_1 and r_2 are given by (43).

§5. Two-Sided Test for ξ.

In this section we consider the problem of testing the hypothesis $H_0:\xi=\xi_0$ versus the alternative hypothesis $H_1:\xi\neq\xi_0$ for some constant ξ_0. We propose the test with the acceptance region derived from inverting the C. I. for ξ_0. Let n be the size of the random sample $X_1, \ldots, X_n$. When n=2m+1 with m a nonnegative integer, we show that this test is unbiased and of size α. When n=2m, our test is of size α, but cannot be unbiased because we use the conventional device to determine the C. I. for ξ. However, it will be almost unbiased for large m.

Let n=2m+1. As in Section 4 we let $Z\doteq\ln|X|$ and $Z_{(i)}$ be the i-th smallest observation of $Z_1, \ldots, Z_n$. Let $\xi_0{}^*\doteq\ln\xi_0$ and define $Y\doteq Z_{(m+1)}$. By inverting the C. I. (37) for ξ_0 our test is to reject H_0 if $Y\in(-\infty,\xi_0{}^*-\ln r_2]\cup[\xi_0{}^*-\ln r_1,+\infty)$ and to accept H_0 if $Y\in(\xi_0{}^*-\ln r_2,\ \xi_0{}^*-\ln r_1)$ where r_1 and r_2 are given by (36). Now, we show that this test is unbiased and of size α.

Let $y_1{}^0$ and $y_2{}^0$ be real numbers depending on ξ_0 such that $y_1{}^0<y_2{}^0$. Define $\psi(\xi)$ by

$$\psi(\xi)\doteq P_\xi[Y<y_1{}^0 \text{ or } y_2{}^0<Y]$$

$$(45)\qquad = 1-\int_{y_1{}^0}^{y_2{}^0} g_Y(y|\xi)\,dy$$

where $g_Y(y|\xi)$ is given by (30). To get unbiased size-α test with acceptance region $(y_1{}^0, y_2{}^0)$ we choose $y_1{}^0$ and $y_2{}^0$ which satisfy

$$(46)\qquad \psi(\xi_0) = 1 - P_{\xi_0}[y_1{}^0 < Y < y_2{}^0] = \alpha$$

and minimize $\psi(\xi)$ at $\xi=\xi_0$; namely

$$(47)\qquad d\psi(\xi)/d\xi\Big|_{\xi=\xi_0} = \xi_0{}^{-1}g_Y(y_2{}^0|\xi_0) - \xi_0{}^{-1}g_Y(y_1{}^0|\xi_0) = 0$$

Let $y_1{}^* = \xi_0{}^* - \ln r_2$ and $y_2{}^* = \xi_0{}^* - \ln r_1$. Then, since $q_Z(y_1{}^*|\xi_0) = \pi^{-1}\sin\{\pi\beta(\alpha/2)\}$ $= \pi^{-1}\sin\{\pi(1-\beta(\alpha/2))\} = q_Z(y_2{}^*|\xi_0)$, and since, from construction and (35), $h_W(Q_Z(y_1{}^*)) = h_W(Q_Z(y_2{}^*))$, we obtain by (30) and (9) that $g_Y(y_1{}^*|\xi_0) = g_Y(y_2{}^*|\xi_0)$. Therefore, $(y_1{}^*, y_2{}^*)$ satisfies (47). On the other hand, (46) with $y_1{}^0$ and $y_2{}^0$ replaced by $y_1{}^*$ and $y_2{}^*$, respectively is the same as (40) except for ξ replaced by ξ_0. Therefore, our test with the acceptance region $(y_1{}^*, y_2{}^*)$ is unbiased and of size α.

Let $n=2m$. As in Section 4 we define $Y \doteq Z_{(m)}$. Again, by inverting the C. I. (44) for ξ_0 our test is to reject H_0 if $Y \in (-\infty,\ \xi_0{}^* - \ln r_2] \cup [\xi_0{}^* - \ln r_1, +\infty)$ and to accept H_0 if $Y \in (\xi_0{}^* - \ln r_2,\ \xi_0{}^* - \ln r_1)$ where r_1 and r_2 are determined by (43). In this case our test depends on the conventional values for $Q_Z(\xi^* - \ln r_i)$, $i=1,2$. So, we have $g_Y(\xi_0{}^* - \ln r_2|\xi_0) \neq g_Y(\xi_0{}^* - \ln r_1|\xi_0)$. Furthermore, (46) with $y_1{}^0$ and $y_2{}^0$ replaced by $\xi_0{}^* - \ln r_2$ and $\xi_0{}^* - \ln r_1$, respectively is the same as (40) except for ξ replaced by ξ_0. Thus, our test is still of size-α, but is not unbiased. However, for large m our test becomes almost unbiased as the test in case of $n=2m+1$ shows.