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An Unbiased Test for the Location Parameter of the Uniform Distribution

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Abstract.

In this paper the author considers the uniform distribution with the density

$$f(x|\theta)=\begin{cases}(\delta_2-\delta_1)^{-1} & \text{for } \theta+\delta_1\leq X<\theta+\delta_2\\ 0, & \text{otherwise}\end{cases}$$

where $-\infty<\theta<\infty$ and $\delta_i\,(i=1,2)$ are real numbers such that $\delta_1<\delta_2$. Based on a random sample $X_1, \ldots, X_n$ of size n from the density $f(x|\theta)$ she considers the problem of testing the null hypothesis $H_0:\theta=\theta_0$ versus the alternative hypothesis $H_1:\theta\neq\theta_0$ for some constant θ_0. She proposes the test with the acceptance region derived from inverting the shortest confidence interval for θ_0 and show that this test is unbiased.

§1. Introduction.

Let $I_A(x)$ be an indicator function such that $I_A(x)=1$ if $x \in A$; $=0$ if $x \notin A$, for a set A. Let $\doteq$ be the defining property. In this paper we deal with the uniform distribution over the set $[\theta+\delta_1, \theta+\delta_2)$ with the density

$$f(x|\theta)= c^{-1} I_{[\theta+\delta_1,\ \theta+\delta_2)}(x), \qquad \forall\theta \in (-\infty, \infty) \tag{1.1}$$

where $\delta_i (i=1,2)$ are real numbers such that $\delta_1 < \delta_2$ and $c \doteq \delta_2 - \delta_1 (>0)$.

Let $X_1, \ldots, X_n$ be a random sample of size n taken from the density (1.1). Let $X_{(i)}$ be the i-th smallest observation of $X_1, \ldots, X_n$. In this paper we use, in Section 2, the Lagrange's method to get the shortest confidence interval (C. I.) for the location parameter θ based on an unbiased estimator $Y=(X_{(1)}+X_{(n)}-\delta_0)/2$ with $\delta_0 = \delta_1 + \delta_2$. In Section 3 we consider the problem of testing the null hypothesis $H_0 : \theta=\theta_0$ versus the alternative hypothesis $H_1 : \theta \neq \theta_0$ for some constant θ_0. Let α be a real number such that $0 < \alpha < 1$. We propose the test with the acceptance region derived from inverting the shortest C. I. for θ_0 at confidence coefficient $1-\alpha$. We show that the proposed test is unbiased and of size α.

§2. The interval eatimation for θ.

Let $X_1, \ldots, X_n$ be a random sample of size n taken from the density (1.1). We find the shortest C. I. for θ at confidence coefficient $1-\alpha$ using the Lagrange's method.

We first estimate θ by $Y \doteq (X_{(1)}+X_{(n)}-\delta_0)/2$. To get the shortest C. I. for θ we find the probability density function (p. d. f.) of Y. Applying the variable transformation $Y=(X_{(1)}+X_{(n)}-\delta_0)/2$ and $Z=X_{(1)}$ to the joint p. d. f. of $(X_{(1)}, X_{(n)})$ and taking the marginal p. d. f. we obtain the p. d. f. of Y as follows:

$$g_Y(y|\theta)=\begin{cases} nc^{-n}(c-2|y-\theta|)^{n-1}, & \text{for } -c/2<y-\theta<c/2 \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

From (2.1) θ is also the location parameter of the distribution of Y. Hence, to get the shortest C. I. for θ at confidence coefficient $1-\alpha$ we shall find real numbers r_1 and r_2 $(r_1<r_2)$ which minimize r_2-r_1 under the condition that

$$P_\theta[r_1<Y-\theta<r_2] = \int_{\theta+r_1}^{\theta+r_2} g_Y(y|\theta)\,dy=1-\alpha. \tag{2.2}$$

Let λ be a real number and define

$$L\doteq L(r_1,r_2;\lambda)\doteq r_2 - r_1 - \lambda\{\int_{\theta+r_1}^{\theta+r_2} g_Y(y|\theta)\,dy -1+\alpha\}.$$

By the Lagrange's method, we find r_1 and r_2 which satisfy (2.2) and

$$\begin{cases} \partial L/\partial r_1 = -1+\lambda g_Y(\theta+r_1|\theta) = 0 \\ \partial L/\partial r_2 = 1-\lambda g_Y(\theta+r_2|\theta) = 0 \end{cases} \tag{2.3}$$

Since by (2.3) we obtain that

$$g_Y(\theta+r_1|\theta) = g_Y(\theta+r_2|\theta) \ (=\lambda^{-1}), \qquad \forall\theta, \tag{2.4}$$

we merely obtain r_1 and r_2 which satisfy (2.4) and (2.2) for any $\theta\in(-\infty,\infty)$. From (2.4) and (2.1) we obtain $r_2=-r_1(\doteq r)$. Substituting these into (2.2), making a variable change $u=y-\theta$ and performing further calculations leads to

$$\text{the left hand side of (2.2)}= 2\int_0^r nc^{-n}(c-2u)^{n-1}\,du = 1-(1-(2r/c))^n.$$

Solving $\alpha=(1-(2r/c))^n$ we get

$$r=c(1-\alpha^{1/n})/2. \tag{2.5}$$

Hence, the shortest C. I. for θ at confidence coefficient $1-\alpha$ is

$$(Y-r, Y+r) \tag{2.6}$$

where r is given by (2.5).

§3. The two-sided test for θ.

In this section we consider the problem of testing the null hypothesis H_0: $\theta=\theta_0$ versus the alternative hypothesis $H_1: \theta \neq \theta_0$ for some constant θ_0.

As in Section 2 we define $Y \doteq (X_{(1)}+X_{(n)}-\delta_0)/2$. By inverting the shortest C. I. (2.6) for θ_0 our test is to reject H_0 if $Y \in (-\infty, \theta_0-r] \cup [\theta_0+r, \infty)$ and to accept H_0 if $Y \in (\theta_0-r, \theta_0+r)$ where r is given by (2.5). Now, we show that this test is unbiased and of size α.

Let $y_1{}^0$ and $y_2{}^0$ be real numbers depending on θ_0 such that $y_1{}^0 < y_2{}^0$. We define $\psi(\theta)$ by

$$\psi(\theta) \doteq P_\theta[Y < y_1{}^0 \text{ or } y_2{}^0 < Y] = 1 - \int_{y_1{}^0}^{y_2{}^0} g_Y(y|\theta)\, dy$$

where $g_Y(y|\theta)$ is defined by (2.1). To get unbiased size-α test with the acceptance region $(y_1{}^0, y_2{}^0)$ we choose $y_1{}^0$ and $y_2{}^0$ which satisfy

$$\psi(\theta_0) = 1 - P_{\theta_0}[y_1{}^0 < Y < y_2{}^0] = \alpha \tag{3.1}$$

and minimize $\psi(\theta)$ at $\theta=\theta_0$; namely

$$d\psi(\theta)/d\theta \Big|_{\theta=\theta_0} = g_Y(y_2{}^0|\theta_0) - g_Y(y_1{}^0|\theta_0) = 0. \tag{3.2}$$

We consider the test with the acceptance region (θ_0-r, θ_0+r). Since from the construction the equality (2.4) with $r_1=-r$, $r_2=r$ and $\theta=\theta_0$ is satisfied, (3.2) with $y_1{}^0$ and $y_2{}^0$ replaced by θ_0-r and θ_0+r, respectively holds. On the other hand, (3.1) with $y_1{}^0$ and $y_2{}^0$ replaced by θ_0-r and θ_0+r, respectively is the same as (2.2) except for θ, r_1 and r_2 replaced by θ_0, $-r$ and r, respectively. Therefore, our test with the acceptance region (θ_0-r, θ_0+r) is unbiased and of size α.