

INSTITUTE OF POLICY AND PLANNING SCIENCES

Discussion Paper Series

No. 1103

**Customer selection problem with
multiple servers and profit from a sideline**

by

Jae-Dong Son

January 2005

UNIVERSITY OF TSUKUBA
Tsukuba, Ibaraki 305-8573
JAPAN

CUSTOMER SELECTION PROBLEM WITH MULTIPLE SERVERS AND PROFIT FROM A SIDELINE

Jae-Dong Son

*Doctoral Program in Systems and Information Engineering
University of Tsukuba, Tennodai, Tsukuba, Ibaraki, 305-8573, Japan*

Email: sonjd88@hotmail.com

(January 18, 2005)

Abstract

This paper deals with discrete-time version of problem as to selecting profitable orders out of customers sequentially arriving at companies operating in service industries which provide two classes of services. The first class of service is designed to meet the various needs of their customers, and the company 1) has an option to accept or reject a particular order (admission control), or 2) decides the price of a particular order to offer to an arriving customer (pricing control). The second classes of service is provided as a sideline to avoid server's being idle, and to yield extra income, referred to as the profit from a sideline; in other words, the second class of service is offered only when the number of orders for the first class of service is less than the number of servers. Further, a cost is paid to search for customers, called the search cost. The introduction of the search cost eventually yields the option as to whether or not to conduct the search. We discuss the admission control problem and pricing control problem in an identical framework and investigate the following problems:

1. a) When to accept or reject an arriving order (in admission control problem), or what price to offer to an arriving customer (in pricing control problem); and b) which type of service to provide?
2. When to enact the search or skip the search?
3. How many servers to assign to the sideline when the profit from the sideline is sufficiently large?

We examine and clarify the structure of the optimal decision rule maximizing the total expected present discounted net profit gained over an infinite planning horizon. Finally, we show that when the profit from the sideline is large, the optimal policies may not be monotone in the number of orders in the system.

Keywords: Queueing; Admission control; Pricing control; Profit from a sideline

1 Introduction

In [18] we posed and examined the problem of selecting profitable orders out of customers sequentially arriving at companies operating in service industries which provide two classes of services. The first class of service is designed to meet the various needs of their customers, and the company 1) has an option to accept or reject a particular order (admission control [1] [2] [6] [11] [19]), or 2) decides the price of a particular service to offer to an arriving customer (pricing control [8] [9] [10] [13] [20] [23]). The admission control problem and the pricing control problem have been separately formulated and examined so far [5] [7] [12] [21]. It was shown in [18] that

both problems can be discussed in an identical framework and that this result comes from the fact that the properties of the underlying functions are identical. The second class of service is provided as a sideline to avoid server's being idle, and to yield extra income, referred to as the profit from a sideline; in other words, the second class of service is offered only when the number of orders for the first class of service is less than the number of servers.

Our work is motivated by a man power company that places greater emphasis on providing specialized services (such as human resources recruitment, personnel training, and so on) than providing temporary staffing as a sideline. Temporary staffing is designed to connect candidates seeking open positions, who post their resumes in a talent bank in the company through the internet, with clients who are reluctant to make a long term staffing commitment. In a company, when there exist more orders for specialized services to be handled than the number of servers/teams will be dispatched or assigned to them. However, when the number of orders accepted so far is less than that of servers/teams, and thereby some of teams become to be idle, the servers/teams that are idle will take part in providing temporary staffing as a sideline during that period. The company charges a fee as a commission for providing this service: The fee is the profit from the sideline in this case. The strict definition of the subsidiary service is provided in A2 and A3 of the next section. When the profit obtained from subsidiary services as a sideline is larger than that from specialized services, the company will naturally place a higher priority on the former than the latter, or the company will assign some of servers/teams to the latter. Here, a problem arises in determining the level of profit from the sideline such that if the profit is higher than the level, 1) the company should give the sideline the topmost priority 2) the company should decide the number of servers/teams to assign to the sideline. In this paper we have succeeded in answering this problem through the conclusion that the optimal decision rule has a bimodal property in the number of orders in the system (see Section 8).

From the practical viewpoint that some costs must be paid in order for the company to find orders; the introduction of the search cost may be an inevitable requirement. The search cost has been introduced in almost all conventional models of optimal stopping problems [4] [15] [16] [22] but not in those of customer selection problems. Further, it should be noted that the introduction of the search cost eventually yields the option as to whether or not to conduct the search. However, thus far this new option has not been taken into consideration in the models of the customer selection problems. The decision on whether or not to enact the search may be influenced not only by the search cost but also the profit from the sideline. In this paper we clarify that if the search cost or the profit from the sideline is less than a given value, it is optimal to conduct the search for orders, or else to skip the search.

The rest of this paper is organized as follows. Section 2 provides a strict definition of the model of the problem treated in the paper. Section 3 describes the optimal equation of the model. In Section 4 we define some functions, called underlying function. Using these functions, in Section 5 we transform the optimal equation for convenience of discussion in the subsequent sections. In Section 6 the properties of the optimal decision rule are examined and summarized. Section 7 discusses some important aspects of the problem through numerical experiments, and Section 8

considers the practical implications of the results obtained in the above sections and summarizes the conclusions derived.

2 Model

The model examined in the paper is defined on the assumptions below:

- A1. The model is defined as a discrete-time sequential stochastic decision process with an infinite planning horizon. Let points in time be equally spaced on the axis of the planning horizon, and let the time interval between successive two points in time be called the period.
- A2. When the number of particular orders accepted so far is less than that of servers/teams, and thereby some of teams become to be idle, the servers/teams that are idle will take part in providing subsidiary services as a sideline during that period, and thereby yields the *profit from the sideline*. It is assumed that any subsidiary service can be completed within one period, implying that the orders of subsidiary service arriving during that period should be always accepted. Let the profit obtained from the sideline for one server during one period be denoted by $r \geq 0$.
- A3. Only when a search is enacted by paying a *search cost* $s \geq 0$ at a point in time, a particular order appears at the next point in time with a probability λ ($0 < \lambda \leq 1$). It is assumed that the orders of subsidiary service appear without paying search cost.
- A4. By $N > 1$ let us denote the maximum permissible number of orders which can be held in the system at any instance; a model with $N = 1$ is examined in [17].
- A5. By $n \geq 2$ let us denote the number of servers/teams available in the company where $n \leq N$.
- A6. Let the prices offered by subsequently appearing customers, w, w', \dots , in the admission control problem and the maximum permissible ordering prices of subsequently appearing customers, w, w', \dots , in the pricing control problem be both independent and identically distributed random variables having a known continuous distribution function $F(w)$ with a finite expectation μ . Then, in the pricing control problem, if the system offers a price z to an appearing customer, the probability of the customer placing the order with the system is given by

$$p(z) = \Pr\{z \leq w\}. \quad (2.1)$$

In both admission control and pricing control problems, for certain given numbers a and b ($0 < a < b < \infty$) let us define the probability density function as follows;

$$f(w) = 0, \quad w < a, \quad f(w) > 0, \quad a \leq w \leq b, \quad f(w) = 0, \quad b < w \quad (2.2)$$

where clearly $a < \mu < b$. Throughout the paper, for simplicity let us denote the expectation of a given function $g(w)$ as to w by $\mathbf{E}[g(w)]$.

- A7. With a probability q ($0 < q < 1$) an order in the system at a certain point in time is completed and goes out of the system at the next point in time.

A8. Let the discount factor be denoted by $\beta < 1$.

Here, note that the decision on the problem is based on the following three rules:

- 1) The rule whether or not to accept an order from each arriving customer in the admission control problem.
- 2) The rule as to the ordering price to offer in the pricing control problem.
- 3) The rule whether to continue or to skip the search in both problems.

The objective is to find the optimal decision rule so as to maximize the total expected present discounted net profit gained over an infinite planning horizon, the total expected present discounted value of prices of orders accepted or placed *plus* the profits from a sideline *minus* the total expected present discounted value of search costs.

For expressional simplicity, by the notations **C**, **K**, **A**, and **R** let us denote the decisions of, respectively: continuing the search, skipping the search, accepting an order, and rejecting an order¹. Further, by the notations $\langle \mathbf{C} \rangle$, $\langle \mathbf{K} \rangle$, $\langle \mathbf{A} \rangle$, and $\langle \mathbf{R} \rangle$ let us imply that the corresponding decisions are optimal. $\langle \mathbf{A}(w) \rangle$ and $\langle \mathbf{R}(w) \rangle$ denote that it is optimal to accept an appearing order w and reject it, respectively, in the admission control problem. $\langle \mathbf{O}(z) \rangle$ denotes that it is optimal to offer the price z for an order in the pricing control problem.

Further, for convenience in later discussions let us define

$$\alpha = \lambda\beta T(0) - c, \quad (2.3)$$

$$\gamma = (1 - \beta(1 - q))^{-1} > 1 \quad (2.4)$$

where it can be easily shown that

$$1 - \gamma q\beta = \gamma(1 - \beta) > 0. \quad (2.5)$$

3 Optimality Equation

In the derivation of the system of the optimal equations of this problem, the following three points should be noted:

1. In both admission control problem and pricing control problem, by $u(\phi, i)$ we shall denote the maximum total expected present discounted net profits starting from a state of having the fictitious customer ϕ and i ($0 \leq i \leq N$) orders in the company; let us refer to such a situation as state (ϕ, i) . If $i \leq n$, then $n - i$ production lines will be idle, which implies that the profit from a sideline $(n - i)r$ is yielded. When in state (ϕ, N) , even if a customer appears, the order cannot be accepted due to the assumption of $i \leq N$; accordingly, the present state (ϕ, N) remains unchanged at the next point in time if no order in the company is completed with probability $1 - q$.
2. In the admission control problem, by $u(w, i)$ let us denote the maximum total expected present discounted net profits starting with i ($0 \leq i < N$) orders in the company and an arriving customer, who offers a price w .

¹We do not use **S** as a notation representing “skipping the search” because it is often used for representing “stop the search”

3. In the pricing control problem, by $u(1, i)$ let us denote the maximum total expected present discounted net profits starting with i ($0 \leq i < N$) orders in the company and an arriving customer, to whom the company proposes a price z for an order.

Since the expectation of immediate reward at any point in time is clearly finite, using the conventional way outlined in the discussion of the Markovian decision process [14, p29-30], we can easily show that $|u(\phi, i)| \leq M/(1 - \beta)$ for a sufficiently large $M > 0$, i.e., $u(\phi, i)$, $u(w, i)$, and $u(1, i)$ are bounded in i . Now, for convenience in the later discussions, let us define

$$h_i = u(\phi, i) - u(\phi, i + 1), \quad 0 \leq i < N. \quad (3.1)$$

Then the system of optimal equations can be described as follows:

1. **Admission control problem:**

$$u(\phi, 0) = \max \left\{ \begin{array}{l} \text{C} : \beta(\lambda \mathbf{E}[u(\xi, 0)] + (1 - \lambda)u(\phi, 0)) - s + nr, \\ \text{K} : \beta u(\phi, 0) + nr \end{array} \right\}, \quad (3.2)$$

$$u(\phi, i) = \max \left\{ \begin{array}{l} \text{C} : (1 - q)\beta(\lambda \mathbf{E}[u(\xi, i)] + (1 - \lambda)u(\phi, i)) \\ \quad + q\beta(\lambda \mathbf{E}[u(\xi, i - 1)] + (1 - \lambda)u(\phi, i - 1)) - s, \\ \text{K} : (1 - q)\beta u(\phi, i) + q\beta u(\phi, i - 1) \end{array} \right\} \quad (3.3)$$

$$+(n - i)rI(i \leq n)^3, \quad 1 \leq i < N,$$

$$u(\phi, N) = \max \left\{ \begin{array}{l} \text{C} : (1 - q)\beta u(\phi, N) \\ \quad + q\beta(\lambda \mathbf{E}[u(\xi, N - 1)] + (1 - \lambda)u(\phi, N - 1)) - s, \\ \text{K} : (1 - q)\beta u(\phi, N) + q\beta u(\phi, N - 1) \end{array} \right\}, \quad (3.4)$$

$$u(w, i) = \max \left\{ \begin{array}{l} \text{A} : w + u(\phi, i + 1), \\ \text{R} : u(\phi, i) \end{array} \right\}, \quad (3.5)$$

$$= \max\{w - h_i, 0\} + u(\phi, i), \quad 0 \leq i < N. \quad \square \quad (3.6)$$

2. **Pricing control problem:**

⁵ $I(\cdot)$ denotes the indicator function. For the given statement S if S is true, then $I(S) = 1$, or else $I(S) = 0$.

$$u(\phi, 0) = \max \left\{ \begin{array}{l} \mathbf{C} : \beta(\lambda u(1, 0) + (1 - \lambda)u(\phi, 0)) - s + nr, \\ \mathbf{K} : \beta u(\phi, 0) + nr \end{array} \right\}, \quad (3.7)$$

$$u(\phi, i) = \max \left\{ \begin{array}{l} \mathbf{C} : (1 - q)\beta(\lambda u(1, i) + (1 - \lambda)u(\phi, i)) \\ \quad + q\beta(\lambda u(1, i - 1) + (1 - \lambda)u(\phi, i - 1)) - s, \\ \mathbf{K} : (1 - q)\beta u(\phi, i) + q\beta u(\phi, i - 1) \end{array} \right\} \quad (3.8)$$

$$+(n - i)rI(i \leq n), \quad 1 \leq i < N,$$

$$u(\phi, N) = \max \left\{ \begin{array}{l} \mathbf{C} : (1 - q)\beta u(\phi, N) \\ \quad + q\beta(\lambda u(1, N - 1) + (1 - \lambda)u(\phi, N - 1)) - s, \\ \mathbf{K} : (1 - q)\beta u(\phi, N) + q\beta u(\phi, N - 1) \end{array} \right\}, \quad (3.9)$$

$$u(1, i) = \max_z \{p(z)(z + u(\phi, i + 1)) + (1 - p(z))u(\phi, i)\} \quad (3.10)$$

$$= \max_z p(z)(z - h_i) + u(\phi, i), \quad 0 \leq i < N. \quad \square \quad (3.11)$$

4 Underlying Functions

In this section we define some functions, called *underlying functions*, and state their properties. These functions play an important role in analyzing the properties of the optimal decision rule of the model. For any real number x let us define

$$T(x) = \begin{cases} \mathbf{E}[\max\{w - x, 0\}] & \text{for the admission control problem,} \\ \max_z p(z)(z - x) & \text{for the pricing control problem,} \end{cases} \quad (4.1)$$

$$L(x) = \lambda\beta T(x) - c, \quad (4.2)$$

called, respectively, the T - and L -functions where $T(0) > 0$. In the pricing control problem, by $z(x)$ let us designate the z attaining the maximum of $p(z)(z - x)$ on $(-\infty, \infty)$ for a given x if it exists; i.e., $T(z(x)) = p(z(x))(z(x) - x)$. By using the two T -functions with the same function name we will show that the two different optimal equations prescribed in the previous section can be reduced to the identical form of optimal equations in the next section. Noting this result and the fact that the two T -functions have identical properties (see lemma 6.1 of [18]), we succeeded in analyzing both problems in an identical framework. However, it should be noted that discussions as to the optimal prices, which are not seen in the admission control problem, are added to the pricing control problem (see Lemma 4.1).

Lemma 4.1 *For the pricing control problem we get:*

- (a) $z(x)$ is nondecreasing in x .
- (b) There exists a finite $x^* < a$ such that if $x < (>) x^*$, then $z(x) = (>) a$.

Proof. See [3, Lemmas 6.13, and 6.18]. \blacksquare

Note. It is not yet proven which of $z(x^*) > a$ or $z(x^*) = a$ is true in [3]. If $F(w)$ is a uniform distribution on $[a, b]$ with $0 < a < b$, then $x^* = 2a - b$ (See App. B. of [18]).

5 Transformation

Let us define

$$v(i) = \left\{ \begin{array}{ll} \mathbf{E}[u(w, i)] & \text{for the admission control problem} \\ u(1, i) & \text{for the pricing control problem} \end{array} \right\}, \quad 0 \leq i < N. \quad (5.1)$$

Then noting Eq. (4.1), from Eqs. (3.6) and (3.11) we have

$$v(i) = T(h_i) + u(\phi, i) \quad \text{or equivalently} \quad T(h_i) = v(i) - u(\phi, i), \quad 0 \leq i < N. \quad (5.2)$$

Now, we can immediately rearrange both Eqs. (3.2) to (3.4) and Eqs. (3.7) to (3.9) into the identical expressions below.

$$u(\phi, 0) = \max\{\lambda\beta v(0) + (1 - \lambda)\beta u(\phi, 0) - s, \beta u(\phi, 0)\} + nr, \quad (5.3)$$

$$u(\phi, i) = \max \left\{ \begin{array}{l} (1 - q)\beta(\lambda v(i) + (1 - \lambda)u(\phi, i)) \\ \quad + q\beta(\lambda v(i - 1) + (1 - \lambda)u(\phi, i - 1)) - s, \\ (1 - q)\beta u(\phi, i) + q\beta u(\phi, i - 1) \end{array} \right\}, \quad (5.4)$$

$$+(n - i)rI(i \leq n), \quad 1 \leq i < N, \quad (5.5)$$

$$u(\phi, N) = \max \left\{ \begin{array}{l} (1 - q)\beta u(\phi, N) + q\beta(\lambda v(N - 1) + (1 - \lambda)u(\phi, N - 1)) - s, \\ (1 - q)\beta u(\phi, N) + q\beta u(\phi, N - 1), \end{array} \right\}, \quad (5.6)$$

Further, Eqs. (5.3) to (5.6) can be rewritten as, respectively,

$$u(\phi, 0) = \beta u(\phi, 0) + \max\{\lambda\beta(v(0) - u(\phi, 0)) - s, 0\} + nr, \quad (5.7)$$

$$u(\phi, i) = (1 - q)\beta u(\phi, i) + q\beta u(\phi, i - 1) + (n - i)rI(i \leq n) + \max\{\lambda(1 - q)\beta(v(i) - u(\phi, i)) \\ + \lambda q\beta(v(i - 1) - u(\phi, i - 1)) - s, 0\}, \quad 1 \leq i < N, \quad (5.8)$$

$$u(\phi, N) = (1 - q)\beta u(\phi, N) + q\beta u(\phi, N - 1) + \max\{\lambda q\beta(v(N - 1) - u(\phi, N - 1)) - s, 0\}, \quad (5.9)$$

which can be immediately rearranged into

$$u(\phi, 0) = (\max\{\lambda\beta(v(0) - u(\phi, 0)) - s, 0\} + nr)/(1 - \beta), \quad (5.10)$$

$$u(\phi, i) = \gamma q\beta u(\phi, i - 1) + \gamma(n - i)rI(i \leq n) + \gamma \max\{\lambda(1 - q)\beta(v(i) - u(\phi, i)) \\ + \lambda q\beta(v(i - 1) - u(\phi, i - 1)) - s, 0\}, \quad 1 \leq i < N, \quad (5.11)$$

$$u(\phi, N) = \gamma q\beta u(\phi, N - 1) + \gamma \max\{\lambda q\beta(v(N - 1) - u(\phi, N - 1)) - s, 0\} \quad (5.12)$$

where γ is defined by Eq. (2.4). Hence, using Eq. (5.2), we can rewrite Eqs. (5.10) to (5.12) as follows.

$$u(\phi, 0) = (\max\{\lambda\beta T(h_0) - s, 0\} + nr)/(1 - \beta), \quad (5.13)$$

$$u(\phi, i) = \gamma q\beta u(\phi, i - 1) + \gamma(n - i)rI(i \leq n) \\ + \gamma \max\{\lambda(1 - q)\beta T(h_i) + \lambda q\beta T(h_{i-1}) - s, 0\}, \quad 1 \leq i < N, \quad (5.14)$$

$$u(\phi, N) = \gamma q\beta u(\phi, N - 1) + \gamma \max\{\lambda q\beta T(h_{N-1}) - s, 0\}. \quad (5.15)$$

Further, using the L -function defined by Eq. (4.2), we can rewrite Eqs. (5.13) to (5.15) as follows.

$$u(\phi, 0) = (\max\{L(h_0), 0\} + nr)/(1 - \beta), \quad (5.16)$$

$$u(\phi, i) = \gamma q\beta u(\phi, i - 1) \quad (5.17)$$

$$+ \gamma \max\{(1 - q)L(h_i) + qL(h_{i-1}), 0\} + \gamma(n - i)rI(i \leq n), \quad 1 \leq i < N,$$

$$u(\phi, N) = \gamma q\beta u(\phi, N - 1) + \gamma \max\{qL(h_{N-1}) - (1 - q)s, 0\}. \quad (5.18)$$

Below, for convenience let

$$Q_0 = L(h_0), \quad (5.19)$$

$$Q_i = (1 - q)L(h_i) + qL(h_{i-1}), \quad 1 \leq i < N, \quad (5.20)$$

$$Q_N = qL(h_{N-1}) - (1 - q)s. \quad (5.21)$$

Then Eqs. (5.16) to (5.18) can be rewritten as follows.

$$u(\phi, 0) = (\max\{Q_0, 0\} + nr)/(1 - \beta), \quad (5.22)$$

$$u(\phi, i) = \gamma q\beta u(\phi, i - 1) + \gamma \max\{Q_i, 0\} + \gamma(n - i)rI(i \leq n), \quad 1 \leq i \leq N. \quad (5.23)$$

Regarding h_i as a function of r , let us represent h_i and Q_i by, respectively, $h_i(r)$ and $Q_i(r)$, i.e.,

$$Q_0(r) = L(h_0(r)), \quad (5.24)$$

$$Q_i(r) = (1 - q)L(h_i(r)) + qL(h_{i-1}(r)), \quad 1 \leq i < N, \quad (5.25)$$

$$Q_N(r) = qL(h_{N-1}(r)) - (1 - q)s. \quad (5.26)$$

Here, by r_i let us denote the smallest solution of $Q_i(r) = 0$, if it exists, i.e.,

$$r_i = \min\{r \mid Q_i(r) = 0\}. \quad (5.27)$$

From all the above it can be easily seen that the optimal decision rules for any given i can be prescribed as follows.

□ *Optimal Decision Rule 5.1*

1. Admission control problem:

- i. Let $0 \leq i \leq N$. If $Q_i > 0$, then $\langle C \rangle_i$, or else $\langle K \rangle_i$.
- ii. Let $0 \leq i < N$ and an order with value w appear after the search was enacted. If $w > h_i$, then $\langle A(w) \rangle_i$, or else $\langle R(w) \rangle_i$.

2. Pricing control problem:

- i. Let $0 \leq i \leq N$. If $Q_i > 0$, then $\langle C \rangle_i$, or else $\langle K \rangle_i$.
- ii. Let $0 \leq i < N$ and a customer appear after the search was enacted. Then $\langle 0(z) \rangle$ where $z_i = z(h_i)$.

6 Analysis

Lemma 6.1 *Let $\alpha \leq 0$. Then $Q_i \leq 0$ for $0 \leq i \leq N$.*

Proof. Proven in the same way as in the proof of lemma 6.5 of [18]

■

Lemma 6.2 *For a given i such that $n \leq i < N$ we have:*

- (a) *If $Q_i \leq 0$, then $h_{i-1} > h_i$, hence $h_{i-1} \geq h_i$.*
- (b) *If $h_{i-1} < h_i$, then $h_{i-1} < h_i < \dots < h_{n-1} < b$ and $Q_j > 0$ for j with $i \leq j < N$.*
- (c) *If $Q_i > 0$, then $Q_j > 0$ for $i \leq j < N$.*

Proof. Proven in the same as in the proofs of, respectively, lemmas 6.6(b), 6.7, and 6.9 of [18].

■

Corollary 6.1 *If $h_{i-1} \leq h_i$, then $h_{i-1} \leq h_i \leq \dots \leq h_{M-1} < b$ and $Q_j > 0$ for j with $i \leq j < N$.*

Proof. Proven in the same way as in the proof of lemma 6.2(b). ■

Lemma 6.3

- (a) *$h_i(r)$ is nondecreasing in r for $i \geq 0$.*
- (b) *$\lim_{r \rightarrow \infty} h_i(r) = \infty$ and $\lim_{r \rightarrow -\infty} h_i(r) = -\infty$ for $i \geq 0$.*
- (c) *$Q_i(r)$ is nonincreasing in r for all $i \geq 0$.*
- (d) *For $0 \leq i \leq n$ we have:*
 - 1 *There exists $r_i > 0$.*
 - 2 *If $r < (\geq) r_i$, then $Q_i(r) > (\leq) 0$.*

Proof. Proven in the same way as in the proofs of, respectively, lemmas 6.10, 6.11, 6.12, and 6.13 of [18]. ■

Lemma 6.4

- (a) *Let $r = 0$.*
 - 1 *$Q_0(r) > 0$.*
 - 2 *If $h_0 = 0$, then h_i is nondecreasing in i .*
 - 3 *If $h_0 > 0$, then h_i is strictly increasing in i .*
- (b) *If $r_n \leq r$, then $h_{n-1} > h_n$.*

Proof. (a) Proven in the same way as in the proofs of lemma 6.14(a) of [18].

(b) Let $r_n \leq r$. Then from Lemma 6.3(d2) we have $Q_n(r) \leq 0$, hence $h_{n-1} > h_n$ due to Lemma 6.2(a). ■

Let us define

$$\hat{r} = \min\{r \mid h_{n-1}(r) > h_n(r)\}. \quad (6.1)$$

Lemma 6.5 We have $r_n \geq \hat{r} > 0$ where if $r \geq (<) \hat{r}$, then $h_{n-1} > (<=) h_n$.

Proof. From Lemma 6.4 we have $h_{n-1} \leq h_n$ for $r = 0$ and $h_{n-1} > h_n$ for $r \geq r_n$, implying that there exists a positive $\hat{r} \leq r_n$ such as $h_{n-1}(\hat{r}) > h_n(\hat{r})$. Accordingly, the latter half of the assertion is clearly true. ■

From the results obtained so far, we have the following theorem restating *Optimal Decision Rule 5.1*.

Theorem 6.1

(a) Let $\alpha \leq 0$. Then $\langle \mathbf{K} \rangle_{0 \leq i \leq N}$.

(b) Let $\alpha > 0$.

- 1 Let $r_n \leq r$. Then $\langle \mathbf{K} \rangle_{n \leq i < N}$ or there exists $i^*(n < i^* < N)$ such that $\langle \mathbf{K} \rangle_{n \leq i < i^*}$ and $\langle \mathbf{C} \rangle_{i^* \leq i < N}$.
- 2 Let $r < r_n$.
 - i $\langle \mathbf{C} \rangle_{n \leq i < N}$.
 - ii Let $\hat{r} \leq r$. Then h_i is not always nondecreasing in $i \geq n$.
 - iii Let $r = 0$.
 - 1 If $h_0 = 0$, then h_i is nondecreasing in i with $h_i < b$ for $0 \leq i < N$.
 - 2 If $h_0 > 0$, then h_i is strictly increasing in i with $h_i < b$ for $0 \leq i < N$.

Proof. (a) Evident from Lemma 6.1.

(b) Let $\alpha > 0$. Here note that $\hat{r} \leq r_n$ from Lemma 6.5.

(b1) Let $r_n \leq r$. Clearly $Q_n(r) \leq 0$ from Lemma 6.3(d2) with $i = n$, hence $\langle \mathbf{K} \rangle_n$. From this result and the fact that once continuing the search is optimal for a certain i , i.e., $\langle \mathbf{C} \rangle_i$, then it also is so for all i' with $i \leq i' < N$ due to Lemma 6.2(c). Accordingly, the assertion clearly holds.

(b2) Let $r < r_n$.

(b2i) Then $Q_N(r) > 0$ from Lemma 6.3(d2) with $i = n$, hence $Q_i(r) > 0$ for $n \leq i < N$ from Lemma 6.2(c), thus $\langle \mathbf{C} \rangle_{n \leq i < N}$.

(b2ii) Let $\hat{r} \leq r$. Then since $h_{n-1} > h_n$ from Lemma 6.4(b), it follows that h_i is not always nondecreasing in i .

(b2iii) Let $r = 0$.

(b2iii1,b2iii2) Immediate from Lemmas 6.4(a). ■

In the pricing control problem it should be noted that the monotonicity of h_i in i stated above is inherited to the optimal price z_i due to Lemma 4.1(a). Since $z_i = z(h_i)$, from Lemma 4.1(b) we see that $z_i = a$ if $h_i < x^*$.

7 Numerical Experiments

Let us examine the properties of the optimal decision rules through numerical experiments.

7.1 Admission Control Problem

Let $F(w)$ be the uniform distribution on $[0.01, 1.01]$ and let $\lambda = 0.95$, $q = 0.35$, $\beta = 0.99$, $s = 0.01$, and $N = 15$. In this case, $T(0) = 0.51$, hence $\alpha = \lambda\beta T(0) - s = 0.47 > 0$. Then for \hat{r} , r_n , and

$h_{n-1} \simeq h_n$, $n = 2, 3, 4, 5$, we obtain the results of numerical experiments shown in Table 7.1. Here note that it is only when $r = \hat{r}$ that $h_{n-1} \simeq h_n$ may occur due to the definition of \hat{r} given by Eq. (6.1).

Table 7.1: \hat{r} , r_n , and $h_{n-1} \simeq h_n$.

	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$\hat{r} \simeq$	0.019	0.007	0.005	0.005
$r_n \simeq$	0.121	0.064	0.041	0.030
$h_{n-1} \simeq h_n \simeq$	0.340	0.373	0.389	0.404

I. Relationship among \hat{r} , r_0 , and h_i .

Figure 7.1 depicts the relationships of h_i with the number of back orders i and the profit from a sideline r . The figure tells us that:

1. h_i is nondecreasing in the profit from a sideline r for all i .
2. If $r < \hat{r}$, then h_i is strictly increasing in $i \geq 0$.

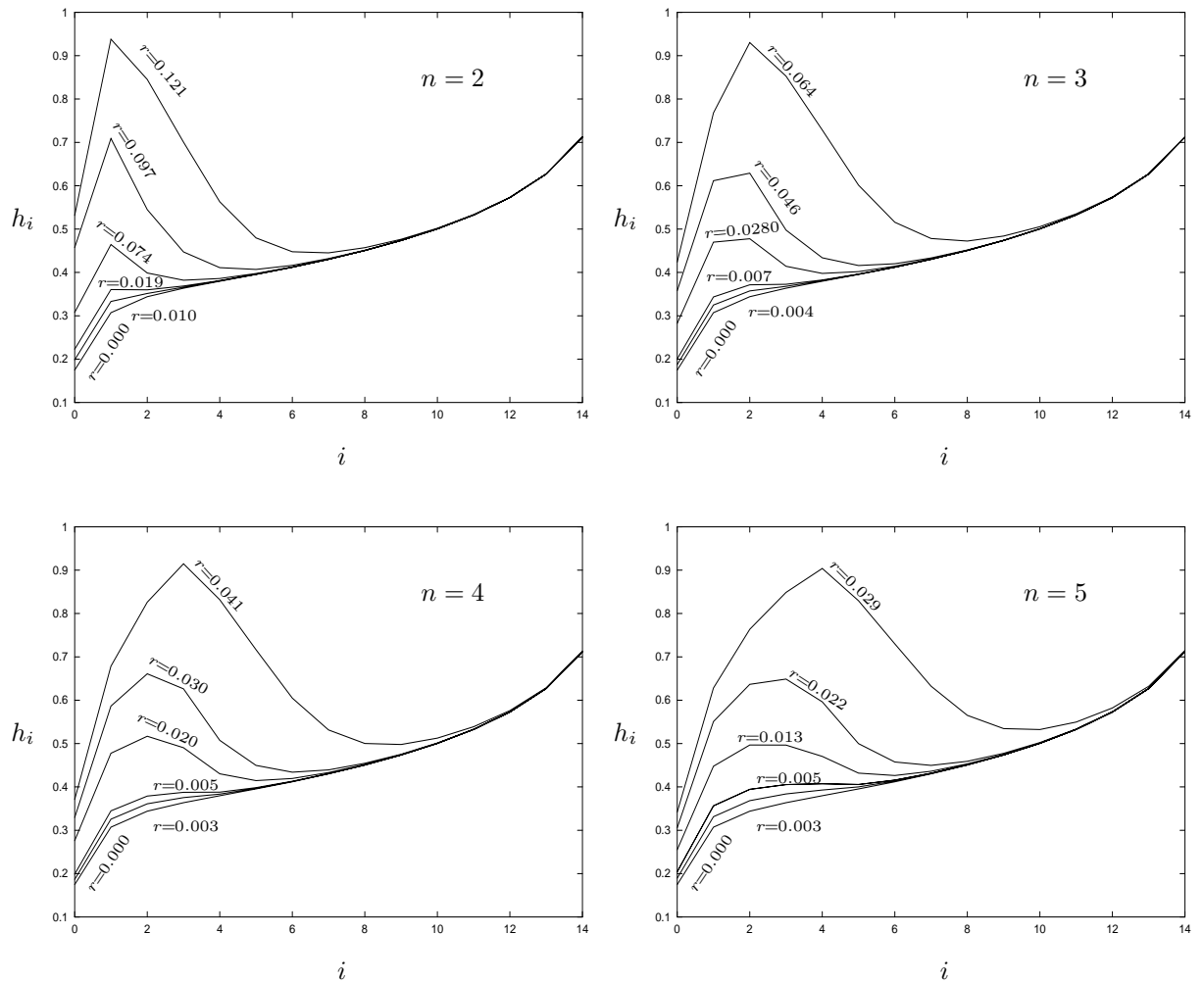


Figure 7.1: Graphs of the selection criterion h_i in the number of backorders i .

3. If $r = \hat{r}$, then $h_{n-1} \simeq h_n$ and h_i is strictly increasing in $i \geq n$.
4. If $\hat{r} \leq i < r_n$, then there exist i' and i'' such that h_i is strictly increasing in $i \leq i'$, strictly decreasing in $i' < i \leq i''$, and *again* strictly increasing in $i \geq i''$.
5. If i is sufficiently large, then h_i coincides with h_i for $r = 0.000$. This reflects the fact that the larger the number of back orders may become, the smaller the possibility of the back orders being exhausted may get; as a result, the effect of r on h_i is gradually diminished.

II. The optimal decision rules on continuing or skipping the search.

Table 7.2 represents the optimal decision rules on continuing the search or skipping the search in each state for each given r . Table 7.2 tells us that:

Table 7.2: Optimal decision rules on continuing or skipping the search.

n, r_n		i	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$n = 2$ $r_n = 0.121$	$r = 0.000$	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)
	$r = 0.121$	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)
	$r = 0.136$	(C)	(C)	(K)	(K)	(K)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)
	$r = 0.150$	(C)	(C)	(K)	(K)	(K)	(K)	(K)	(K)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)
$n = 3$ $r_n = 0.064$	$r = 0.000$	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)
	$r = 0.063$	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)
	$r = 0.065$	(C)	(C)	(C)	(K)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)
	$r = 0.100$	(C)	(C)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(C)	(C)	(C)	(K)
$n = 4$ $r_n = 0.041$	$r = 0.000$	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)
	$r = 0.040$	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)
	$r = 0.071$	(C)	(C)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(C)	(C)	(K)
	$r = 0.100$	(C)	(C)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)
$n = 5$ $r_n = 0.030$	$r = 0.000$	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)
	$r = 0.029$	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)	(C)
	$r = 0.071$	(C)	(C)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)
	$r = 0.310$	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)	(K)

1. If $r < r_n$, then it is always optimal to continue the search as seen Theorem 6.1(b2i) except for the state $(\phi, 15)$. When $i = 15$, any of continuing the search and skipping may be optimal,
2. If $r \geq r_n$, implying that if the profit from a sideline r is sufficiently large, it can be seen that it is always optimal to skip the search (case of $n = 5$ and $r = 0.310$) or that there exists $i' < i''$ such that if $i < i'$, continuing the search is optimal, if $i' \leq i \leq i''$, skipping the search is optimal, and if $i'' < i$, *again* continuing the search is optimal; that is, there exist *double critical values* in terms of i . In case of $n = 2$ and $r = 0.150$ we have $i' = 2$ and $i'' = 7$.

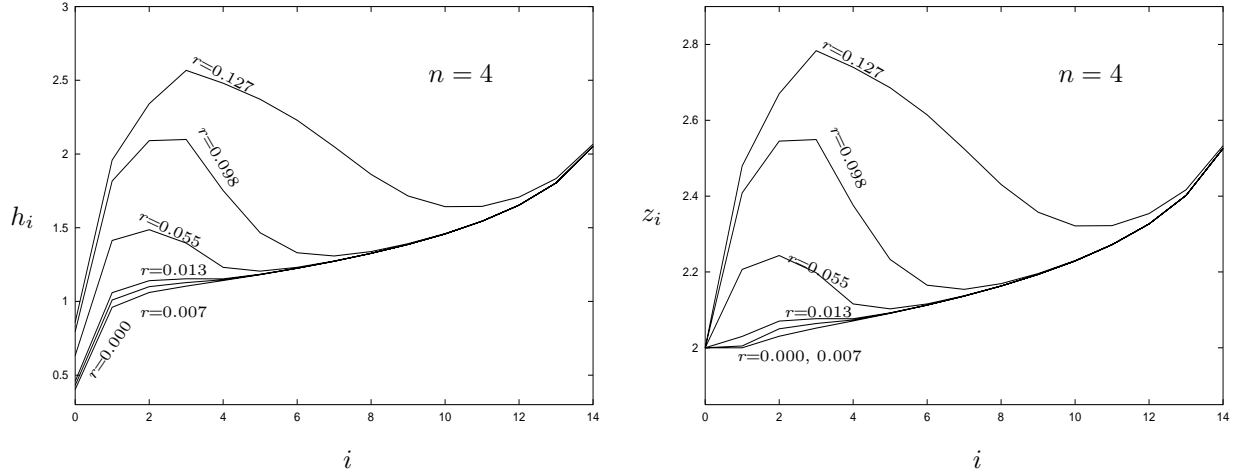


Figure 7.2: Graphs of h_i and z_i .

7.2 Pricing Control Problem

Let $F(w)$ be the uniform distribution on $[2, 3]$, i.e., $a = 2$ and $b = 3$ and let $\lambda = 0.90$, $q = 0.55$, $\beta = 0.99$ and $s = 0.05$. In this case, we have $x^* = 2a - b = 1$. Since $x^* > 0$, we obtain $T(0) = a = 2$, hence $\alpha = \lambda\beta T(0) - s = 1.732 > 0$.

Figure 7.2 depicts the relationships of h_i and $z_i (= z(h_i))$ with the number of back orders i and the profit from a sideline r . The figures tell us that:

1. h_i is nondecreasing in the profit from a sideline r for all i .
2. If $r < \hat{r}$, then h_i is strictly increasing in $i \geq 0$.
3. If $r = \hat{r}$, then $h_{n-1} \simeq h_n$ and z_i is strictly increasing in $i \geq n$.
4. If $\hat{r} \leq i < r_n$, then there exist i' and i'' such that h_i and z_i is strictly increasing in $i \leq i'$, strictly decreasing in $i' \leq i \leq i''$, and *again* strictly increasing in $i \geq i''$. We can notice that i' is given by $n - 1$.
5. The graph on the right shows the optimal ordering price z_i . Here note that there exists i such that $h_i < x^* = 2a - b = 1$ in the graph of h_i . Since $z_i = z(h_i) = a$ for $h_i < x^* = 1$ due to Lemma 4.1(b), it follows that $z_i = z(h_i)$ for such i becomes equal to $a = 2$; in other words, $z_i = z(h_i)$ is truncated by a , the low bound of the distribution function $F(w)$. Further, it should be noted that there exists $h_i < a$ such that its corresponding optimal ordering price z_i becomes greater than a , i.e., $z_i = z(h_i) > a$.

8 Conclusions and Considerations

First, it should be noted that the following two opportunity losses are closely related to the customer selection problem.

1. *Opportunity loss I.* Suppose there are great deal of particular orders in the system. Then the service capacity soon becomes full; with the result that orders from customers arriving thereafter can not be accepted, however high their profitabilities may be. This leads to the

opportunity loss that if adequate allowance were kept in the service capacity by having rejected less profitable orders in advance, the company could have enjoyed accepting upcoming profitable orders. We shall refer to this loss as Opportunity loss I.

2. *Opportunity loss II.* Suppose the profit from a sideline is sufficiently small. Then excessively refraining from accepting particular orders due to the apprehension that Opportunity loss I could occur causes a decrease in the number of orders in the system. This time, many servers may soon become idle and they should be assigned to the sideline with relatively small profit, causing the opportunity loss that if more particular orders had been accepted in advance, profit could have been gained from them. We shall refer to this loss as Opportunity loss II.

Next, below let us state the two types of oscillations as to the number of particular orders i in the system.

1. On the range of i over which the optimal selection criterion h_i is increasing in i , the number of particular orders i in the system oscillates with an equilibrium point for the same reason as that stated in Section 9 of [18] ($r < \hat{r}$ of C2); let us refer to such behavior of i as the *stable oscillation*.
2. On the range of i over which the optimal selection criterion h_i is decreasing in i , the number of particular orders i in the system oscillates as follows: (1) The smaller the number of particular orders in the system may become, the higher the optimal selection criterion h_i becomes; as a result, the number of particular orders in the system is prompted to become further small and (2) The larger the number of particular orders i in the system may become, the lower the optimal selection criterion h_i gets; as a result, the number of particular orders in the system is prompted to become further large. This fact suggests that once the i enters this range, it behaves as if it is escaping from the region. Let us refer to such behavior as the *unstable oscillation*.

The optimal decision rules described in theorem 6.1 are almost similar to those of [18]. However, the conclusions obtained from this problem are different from those in [18] in the sense below.

First, it should be noted that there exist \hat{r} and r_n with $\hat{r} < r_n$ ($2 \leq n \leq N$), which provides thresholds implying that: (1) If the profit from a sideline r is less than \hat{r} , the optimal selection criterion h_i is increasing in the number of particular orders i in the system, or else it is bimodal in i and (2) If the profit from a sideline r is less than r_n , it is optimal to conduct the search for orders, or else it is not always optimal to enact the search. Below, let us explain the implications of the above two thresholds:

1. Let $r < \hat{r}$, i.e., the profit from a sideline is sufficiently small. Then the optimal selection criterion h_i is increasing in the number of particular orders i in the system. Hence, the behavior of the number of particular orders in the system shows the stable oscillation.
2. Let $\hat{r} \leq r < r_n$. In this case, as seen in Figure 7.1, there exist i' and i'' ($i' < i''$) such that h_i is strictly increasing on $[0, i']$, strictly decreasing on $(i', i'']$, and *again* strictly increasing on $(i'', N]$. In other words, the optimal selection criterion h_i is bimodal in the number of particular orders i in the system over $[0, N]$. Below, let us state the implications of the bimodal property.

- i. Let $i \leq i'$. Then the optimal selection criterion h_i is increasing in the number of particular orders i in the system. This fact implies the following. If there are few particular orders in the system, all the servers will become soon idel, and the company has to assign these all servers to the sideline with a relatively small amount of profit. This yields Opportunity loss II. Accordingly, in order to avoid this loss, the optimal selection criterion h_i should be set low to accept orders even though their profitabilities may not be so high. However, as the number of particular orders i increases, the service capacity comes to be filled with particular orders, leading to the possibility of obtaining an income from a sideline is small. This yields Opportunity loss I. Therefore, in order to avoid this loss and prevent all the production lines from being full with orders, the optimal selection criterion should be set high; as a result, the number of particular orders i becomes small, hence the company can enjoy the profit from a sideline.
 - ii. Let $i' < i$. Then the optimal selection criterion h_i is unimodal in the number of particular orders i over $(i', N]$; the managerial implication of this unimodality is the same as that stated in Section 9 of [18] ($\hat{r} \leq r < r_0$ of C2).
 - iii. The fact that the optimal selection criterion h_i is increasing on each of the two ranges, $[0, i']$ and $(i'', N]$, implies that there exists a stable point of oscillation on each of the two ranges. Once the number of particular orders i enters the range $(i', i'']$, a dynamics starts operating to prompt the number of particular orders i to move to one of the two ranges $[0, i']$ and $(i'', N]$ since the behavior of the number of particular orders shows unstable oscillation. Here, it is to be noted that the stable points on $[0, i']$ and $(i'', N]$ are related to, respectively, the number of production lines to be fulfilled with orders and the number of particular orders to be held in the system.
 - iv. The fact that h_i is a bimodal function of i suggests us the following. For an order with certain price w there exists $i' < i'' < i'''$ such that if $i \leq i'$, it is optimal to reject the order, if $i' < i \leq i''$, it is optimal to accept it, if $i'' < i \leq i'''$, *again* it is optimal to reject it, and $i''' < i$, *again* it is optimal to accept it. In other words, there exist the *triple critical values* in terms of i at which rejecting and accepting an order become indifferent.
3. Let $r_n \leq r$, i.e., the profit from a sideline is sufficiently large. Then the optimal selection criterion h_i may be monotone, unimodal, or bimodal in i . However, since no order appears on the range where it is optimal to skip the search (see Table 7.2), the h_i has no practical meaning as a selection criterion. Now, as seen in Table 7.2, there exist two critical values i' and i'' ($i' < i''$). The i' provides the number of servers to be assigned to provide the specialized services (particular orders), so that the number of servers to be assigned to the sideline will be given as $n - i'$. The i'' provides the number of particular orders up to which skipping the search is optimal. Further, we see that:
 - i. If $i \leq i'$, then i' servers are all available for handling particular orders and it is optimal to conduct the search for orders until the number of particular orders becomes i' .
 - ii. If $i' < i < i''$, it is optimal to skip the search; as a result, the number of particular orders in the system decreases up to i' . Hence, it becomes possible to assign $n - i'$ servers to the

sideline, and thereby yields a profit from the sideline.

- iii. If $i'' \leq i$, the company should *again* conduct the search for orders to make profit.

The above stated considerations are related to the admission control problem. The same considerations as those stated above are also obtained for the pricing control problem.

References

- [1] Heyman, D. P.: Optimal operating policies for $M/G/1$ queueing systems, *Operations Research*, Vol. 16 (1968), 362–382.
- [2] Ikuta, S.: A basic theory of customers selection problem (Japanese), *Journal of Industrial Management Association*, Vol. 1 (1971).
- [3] Ikuta, S.: An integration of the optimal stopping problem and the optimal pricing problem - Model with no recall -, *Discussion paper*, No. 1084, University of Tsukuba, Institute of Policy and Planning Sciences, (2004).
- [4] Kang, B. K.: Optimal stopping problem with recall cost, *European Journal of Operational Research*, Vol. 117 (1999), 222–238.
- [5] Knudsen, N. C.: Individual and social optimization in a multi-server queue with a general cost-benefit structure, *Econometrica*, Vol. 40(3) (1972), 515–528.
- [6] Kushner, H. J. and Tetzlaff, U. A. W.: The determination of optimal customer selection and allocation policies for finite queues in parallel, *Working Papers, University of Michigan Business Administration*, Vol. 34 (1971).
- [7] Lippman, S. A. and Ross, S. M.: The streetwalker’s dilemma: a job shop model, *SIAM J. Appl. Math.*, Vol. 20(3) (1971), 336–342.
- [8] Low, D. W.: Optimal dynamic pricing policies for an $M/M/s$ queue, *Operations Research*, Vol. 22(3) (1974), 545–561.
- [9] Mendelson, H.: Pricing computer services: queueing effects, *Communications of the ACM*, Vol. 28(3) (1985), 312–321.
- [10] Mendelson, H. and Whang, S.: Optimal incentive-compatible priority pricing for the $M/M/1$ queue, *Operations Research*, Vol. 38 (1990), 870–883.
- [11] Miller, B.: A queueing reward system with several customer classes, *Management Science*, Vol. 16 (1969), 234–245.
- [12] Naor, P.: The regulation of queue size by levying tolls, *Econometrica*, Vol. 37(1) (1969), 15–24.
- [13] Paschalidis, I. C. and Tsitsiklis, J. N.: Congestion-dependent pricing of network services, *IEEE/ACM Transactions on Networking*, Vol. 8 (2000), 171–184.
- [14] Ross, S. M.: *Introduction to stochastic dynamic programming*, Academic Press, 1983.
- [15] Saito, T.: Optimal stopping problem with controlled recall, *Probability in the Engineering and Informational Sciences*, Vol. 12 (1998), 91–108.
- [16] Sakaguchi, M.: Dynamic programming of some sequential sampling design, *Journal of Mathematical Analysis and Applications*, Vol. 2 (1961), 446–466.
- [17] Son, J. D.: Customers selection problem with idling profit where only one customer is allowed to be held - revision of discussion paper No. 1032, *Discussion paper*, No.1086, University of Tsukuba, Institute of Policy and Planning Sciences, (2004).
- [18] Son, J. D.: Customers selection problem with idling profit, *Discussion paper*, No.1087, University of Tsukuba, Institute of Policy and Planning Sciences, (2004).
- [19] Stidham, S.: Optimal control of admission to a queueing system, *IEEE Transactions on Automatic Control*, Vol. 30(8) (1985), 705–713.
- [20] Stidham, S.: Pricing and capacity decisions for a service facility: stability and multiple local optima, *Management Science*, Vol. 38(8) (1992), 1121–1139.
- [21] Yoon, S. H. and Lewis, M. E.: Optimal pricing and admission control in a queueing system with periodically varying parameters, *Queueing Systems*, Vol. 47(3) (2004).

- [22] You, P. S.: Dynamic pricing policy in a selling problem, *Journal of Japan Society of Business Mathematics*, Vol. 20 (1998), 9–21.
- [23] Ziya, S., Ayhan, H. and Foley, R. D.: Optimal pricing for a service facility, *Working Paper, School of Industrial and Systems Engineering ,Georgia Institute of Technology*, Vol. 8 (2002), 171–184.