Semimetric Thresholds for Finite Posets

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Abstract

Let $\succ$ be a binary relation on a finite set X. This paper proves that $\succ$ is irreflexive and transitive if and only if there is a real valued function u on X and a semimetric Ω on X such that, for all $x, y \in X$, $x \succ y \Leftrightarrow u(x) > u(y) + \Omega(x, y)$.

1 Introduction

Let $\succ$ be an asymmetric binary relation on a set X with symmetric complement $\sim$: for all $x, y \in X$, $x \sim y$ if $\neg(x \succ y)$ and $\neg(y \succ x)$. When $\succ$ is acyclic (i.e., the transitive closure of $\succ$ is transitive), the simple relational system $(X, \succ)$ will be referred to as an *acyclic set*. When $\succ$ is irreflexive and transitive, $(X, \succ)$ will be referred to as a *poset* (partially ordered set).

It is proved by Bridges (1983) that if X is countable, then $(X, \succ)$ is an acyclic set if and only if the following numerical representation holds: there is a real valued function u on X such that, for all $x, y \in X$,

$$x \succ y \Rightarrow u(x) > u(y).$$

This "one-way" representation is undesirable because preferences are not recovered from the numerical representation u.

Several recent studies uncovered "two-way" representations for acyclic sets $(X, \succ)$, i.e., the numerical representations also reconstruct qualitative relation $\succ$. Abbas and Vincke (1993) and Agoev and Aleskerov (1993) considered finite acyclic sets and obtained the following two-way representation: there exist a real valued function u and a real valued bivariate function $\Omega \geq 0$ on $X \times X$ such that, for all $x, y \in X$,

$$x \succ y \Leftrightarrow u(x) > u(y) + \Omega(x, y).$$

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Since Ω can be interpreted as a threshold, this representation will be dubbed here a *bivariate threshold representation*. Rodríguez-Palmero (1997) provided sufficient qualitative conditions for the representation when X is a second countable topological space. A complete qualitative characterization of the representation for arbitrary X was obtained by Diaye (1999). Nakamura (2000) developed several necessary and sufficient qualitative conditions for the existence of the representation when X is the power set of a finite set. Other type of two-way representations for acyclic sets may be possible. For example, Subiza (1994) represents acyclicity by means of set-valued real functions.

The aim of the paper is to prove a similar bivariate threshold representation for finite posets. We show that $(X, \succ)$ is a poset if and only if $(X, \succ)$ has a bivariate threshold representation with Ω a semimetric on X, defined below, which is called a *semimetric threshold representation*. Posets may be more important than acyclic sets in many applications. However, there have been proposed and characterized no two-way representation of posets except Herrero and Subiza (1999), who represented arbitrary posets by means of set-valued real functions.

2 The Main Theorem

A *semimetric*[1] Ω on a set X is a real valued function on $X \times X$ that satisfies the following three properties, understood as applying to all $x, y, z \in X$,

(1) $\Omega(x, x) \geq 0$,
(2) $\Omega(x, y) = \Omega(y, x)$,
(3) $\Omega(x, y) + \Omega(y, z) \geq \Omega(x, z)$.

We note by (1) and (3) that $\Omega(x, y) \geq 0$ for all $x, y \in X$. The property (3) is called the triangle inequality.

Our main theorem is stated as follows.

Theorem 1 *Suppose that X is finite. Then $(X, \succ)$ is a poset if and only if there exist a real valued function u on X and a semimetric Ω on X such that, for all $x, y \in X$,*

$$x \succ y \Leftrightarrow u(x) > u(y) + \Omega(x, y).$$

When a semimetric Ω is additively separable, i.e., for all $x, y \in X$,

$$\Omega(x, y) = \omega(x) + \omega(y)$$

for a nonnegative real valued function ω on X, the semimetric threshold representation characterizes special posets known as interval ordered sets.

[1] A metric is a semimetric that has the property that $\Omega(x, y) = 0$ if and only if $x = y$.

Nakamura (2001) provided a complete qualitative characterization of the representation for arbitrary X.

To prove the theorem, we use the following version of the familiar lemma for the existence of a solution to a finite system of linear inequalities (see Fishburn, 1970). Given two N dimensional vectors of real numbers, $a = (a_1, \ldots, a_N)$ and $b = (b_1, \ldots, b_N)$, we denote the inner product by $a \cdot b = \sum_{i=1}^{N} a_i b_i$. A real vector is called *rational* if each component is a rational number, and is called *integral* if each of its components is an integer.

Lemma 1 *Let $a^1, \ldots, a^M$ be N dimensional rational vectors and $1 \leq K \leq M$. Then either there is an N dimensional integral vector ρ such that*

$$\begin{aligned} \rho \cdot a^k &> 0 \quad \textit{for } k = 1, \ldots, K, \\ \rho \cdot a^k &\geq 0 \quad \textit{for } k = K+1, \ldots, M, \end{aligned}$$

or else there are nonnegative integers $\alpha_1, \ldots, \alpha_M$, with $\alpha_k > 0$ for some $k \leq K$, such that

$$\sum_{k=1}^{M} \alpha_k a_j^k = 0 \textit{ for } j = 1, \ldots, N.$$

Note that the last equations in the lemma are described in the vector form by

$$\sum_{k=1}^{M} \alpha_k a^k = 0,$$

where 0 is an N dimensional zero vector. Since this equation says that some of $a^1, \ldots, a^M$ are linearly dependent, we shall call it the *linearly dependent* (LD) equation.

Proof of Theorem 1 If $(X, \succ)$ has a semimetric threshold representation, then it easily follows that $(X, \succ)$ is a poset. We shall assume henceforth that $X = \{x_1, \ldots, x_n\}$ is a nonempty finite set and that $(X, \succ)$ is a poset.

To specify our system of linear inequalities, suppose that $(X, \succ)$ has a semimetric threshold representation with a real valued function u on X and a semimetric Ω on X satisfying

(1a) $u(x) - u(y) - \Omega(x, y) > 0$ for all $x, y \in X$ such that $x \succ y$.
(1b) $u(x) - u(y) + \Omega(x, y) \geq 0$ and $u(y) - u(x) + \Omega(x, y) \geq 0$ for all $x, y \in$X such that $x \sim y$.

For real valued functions, u on X and Ω on $X \times X$, we define an n dimensional row vector ρ_1 and a $\frac{1}{2}n(n+1)$ dimensional row vector ρ_2 by

$$\begin{aligned} \rho_1 &= (u(x_1), \ldots, u(x_n)), \\ \rho_2 &= (\Omega(x_1, x_1), \Omega(x_2, x_1), \Omega(x_2, x_2), \ldots, \\ &\qquad \Omega(x_n, x_1), \Omega(x_n, x_2), \ldots, \Omega(x_n, x_n)). \end{aligned}$$

For all $x, y \in X$, we define two column vectors, $\theta(x)$ with dimension n, and $\tau(x, y)$ with dimension $\frac{1}{2}n(n+1)$ as follows: for $k = 1, \ldots, n$, $\ell = 1, \ldots, n$, $i = 1, \ldots, n$ and $j = 1, \ldots, \frac{1}{2}n(n+1)$, the i-th component of $\theta(x_k)$ and the j-th component of $\tau(x_k, y_\ell)$ are given by

$$\theta_i(x_k) = \begin{cases} 1 & \text{if } i = k, \\ 0 & \text{otherwise,} \end{cases}$$

$$\tau_j(x_k, y_\ell) = \begin{cases} 1 & \text{if } j = \frac{1}{2}k(k-1) + \ell \text{ and } k \geq \ell, \\ 1 & \text{if } j = \frac{1}{2}\ell(\ell-1) + k \text{ and } k < \ell, \\ 0 & \text{otherwise.} \end{cases}$$

We note that θ and τ are unit vectors, and $\tau(x, y) = \tau(y, x)$ for all $x, y \in X$.

Now we specify the system of linear inequalities for (1a) and (1b). Enumerate $\succ$ as $(x^1, y^1), \ldots, (x^{L_1}, y^{L_1})$, half of $\sim$ as $(z^1, w^1), \ldots, (z^{L_2}, w^{L_2})$ by using one of (x, y) and (y, x) when $x \sim y$, and $X \times X \times X$ as $(a^1, b^1, c^1), \ldots,$ $(a^{L_3}, b^{L_3}, c^{L_3})$. Then letting $\rho = (\rho_1, \rho_2)$ be a $\frac{1}{2}n(n+3)$ dimensional row vector, our system of linear inequalities are stated as follows:

(a)

$$\rho \cdot \begin{bmatrix} \theta(x^i) - \theta(y^i) \\ -\tau(x^i, y^i) \end{bmatrix} > 0 \qquad \text{for } i = 1, \ldots, L_1,$$

(b)

$$\rho \cdot \begin{bmatrix} \theta(z^i) - \theta(w^i) \\ \tau(z^i, w^i) \end{bmatrix} \geq 0 \qquad \text{and}$$

$$\rho \cdot \begin{bmatrix} \theta(w^i) - \theta(z^i) \\ \tau(z^i, w^i) \end{bmatrix} \geq 0 \qquad \text{for } i = 1, \ldots, L_2,$$

(c)

$$\rho \cdot \begin{bmatrix} 0 \\ \tau(a^i, b^i) + \tau(b^i, c^i) - \tau(a^i, c^i) \end{bmatrix} \geq 0 \quad \text{for } i = 1, \ldots, L_3.$$

Inequalities (a) and (b) follow from (1a) and (1b), respectively. The triangle inequality is reflected in (c). Nonnegativity of Ω follows from (b), (c), and irreflexivity of $\succ$. Symmetry of Ω is already reflected in definition of τ.

We are to establish that the system of linear inequalities (a), (b), and (c) has a ρ solution. Therefore, a poset $(X, \succ)$ has a semimetric threshold representation. Suppose on the contrary that there is no ρ solution. Then it follows from Lemma 1 that there are nonnegative integers α_1 for $i = 1, \ldots, L_1$, β_{i1} for $i = 1, \ldots, L_2$, β_{i2} for $i = 1, \ldots, L_2$, and γ_i for $i = 1, \ldots, L_3$ such that $\alpha_j > 0$ for some $1 \leq j \leq L_1$, and the following LD equation holds:

$$\sum_{i=1}^{L_1} \alpha_i \begin{bmatrix} \theta(x^i) - \theta(y^i) \\ -\tau(x^i, y^i) \end{bmatrix} + \sum_{i=1}^{L_2} \beta_{i1} \begin{bmatrix} \theta(z^i) - \theta(w^i) \\ \tau(z^i, w^i) \end{bmatrix} + \sum_{i=1}^{L_2} \beta_{i2} \begin{bmatrix} \theta(w^i) - \theta(z^i) \\ \tau(z^i, w^i) \end{bmatrix}$$

$$+ \sum_{i=1}^{L_3} \gamma_i \begin{bmatrix} 0 \\ \tau(a^i, b^i) + \tau(b^i, c^i) - \tau(a^i, c^i) \end{bmatrix} = 0.$$

Let $m = \sum \alpha_i$, $\ell = \sum \beta_{i1} + \sum \beta_{i2}$, and $k = \sum \gamma_i$. Then $m > 0$, $m = \ell + k$, and $0 \leq \ell \leq k$, because τs are unit vectors and, for all $x, y, z, w \in X$, $\tau(x, y) \neq \tau(z, w)$ if $x \succ y$ and $z \sim w$.

List the elements of $\succ$, $\sim$, and $X \times X \times X$ with α_i repeats for (x^i, y^i), β_{i1} repeats for (z^i, w^i), β_{i2} repeats for (w^i, z^i), and γ_i repeats for (a^i, b^i, c^i), and enumerate them as

$$\begin{array}{ll} (x_1^*, y_1^*), \ldots, (x_m^*, y_m^*) & \text{for } \succ, \\ (z_1^*, w_1^*), \ldots, (z_\ell^*, w_\ell^*) & \text{for } \sim, \\ (a_1^*, b_1^*, c_1^*), \ldots, (a_k^*, b_k^*, c_k^*) & \text{for } X \times X \times X. \end{array}$$

Then the LD equation is described as follows:

$$\sum_{i=1}^{m} \begin{bmatrix} \theta(x_i^*) - \theta(y_i^*) \\ -\tau(x_i^*, y_i^*) \end{bmatrix} + \sum_{i=1}^{\ell} \begin{bmatrix} \theta(z_i^*) - \theta(w_i^*) \\ \tau(z_i^*, w_i^*) \end{bmatrix}$$

$$+ \sum_{i=1}^{k} \begin{bmatrix} 0 \\ \tau(a_i^*, b_i^*) + \tau(b_i^*, c_i^*) - \tau(a_i^*, c_i^*) \end{bmatrix} = 0.$$

In what follows, we show that the LD equation contradicts transitivity of $\succ$. We have two cases to examine: $\ell = 0$; $0 < \ell \leq k$.

Case 1 $(\ell = 0)$ The first n rows of the LD equation is

$$\sum_{i=1}^{m} \theta(x_i^*) = \sum_{i=1}^{m} \theta(y_i^*),$$

which gives that the sequence $x_1^*, \ldots, x_m^*$ is a permutation of the sequence $y_1^*, \ldots, y_m^*$. Since $x_i^* \succ y_i^*$ for $i = 1, \ldots, m$, it is easily seen that transitivity of $\succ$ is violated.

Case 2 $(0 < \ell \leq k)$ With no loss of generality, we assume that $\tau(z_i^*, w_i^*) = \tau(a_i^*, c_i^*)$ for $i = 1, \ldots, \ell$. Let $I_0 = \{1, \ldots, \ell\}$. List the elements from the set $\{\tau(a_1^*, b_1^*), \ldots, \tau(a_\ell^*, b_\ell^*), \tau(b_1^*, c_1^*), \ldots, \tau(b_\ell^*, c_\ell^*)\}$ that have no identical vector in $\tau(x_1^*, y_1^*), \ldots, \tau(x_m^*, y_m^*)$, and enumerate them as $\tau^1, \ldots, \tau^{\ell_1}$. If there is an $1 \leq i' \leq \ell$ such that

$$\begin{array}{rcll} \tau(a_{i'}^*, b_{i'}^*) & = & \tau(x_{j'}^*, y_{j'}^*) & \text{for some } 1 \leq j' \leq m, \\ \tau(b_{i'}^*, c_{i'}^*) & = & \tau(x_{j''}^*, y_{j''}^*) & \text{for some } 1 \leq j'' \leq m, \end{array}$$

then $a_{i'}^* \succ b_{i'}^*$ and $b_{i'}^* \succ c_{i'}^*$, so by transitivity of $\succ$, $a_{i'}^* \succ c_{i'}^*$. Since $\tau(a_{i'}^*, c_{i'}^*) = \tau(z_{i'}, w_{i'}^*)$, we obtain a contradiction $a_{i'}^* \sim c_{i'}^*$. Hence $\ell \leq \ell_1 \leq 2\ell$.

By the LD equation, there is a sequence of ℓ_1 vectors from the set $\{\tau(a_{\ell+1}^*, c_{\ell+1}^*), \ldots, \tau(a_k^*, c_k^*)\}$ that is identical to the sequence $\tau^1, \ldots, \tau^{\ell_1}$. Thus $2\ell \leq k$. With no loss of generality, we assume that $\tau^i = \tau(a_{\ell+i}^*, c_{\ell+i}^*)$ for $i = 1, \ldots, \ell_1$. Thus let $I_1 = \{\ell + 1, \ldots, \ell + \ell_1\}$.

Next we construct a set $I_2 = \{\ell + \ell_1 + 1, \dots, \ell + \ell_1 + \ell_2\}$ of indices as follows. List the elements from the set

$$\{\tau(a^*_{\ell+1}, b^*_{\ell+1}), \dots, \tau(a^*_{\ell+\ell_1}, b^*_{\ell+\ell_1}), \tau(b^*_{\ell+1}, c^*_{\ell+1}), \dots, \tau(b^*_{\ell+\ell_1}, c^*_{\ell+\ell_1})\}$$

that have no identical vector in $\tau(x^*_1, y^*_1), \dots, \tau(x^*_m, y^*_m)$, and enumerate them as $\tau^{\ell_1+1}, \dots, \tau^{\ell_1+\ell_2}$. With no loss of generality, we assume that $\tau^{\ell_1+i} = \tau(a^*_{\ell+\ell_1+i}, c^*_{\ell+\ell_1+i})$ for $i = 1, \dots, \ell_2$. Of course, we may have $I_2 = \emptyset$, i.e., $\ell_2 = 0$. If this is the case, we stop. Otherwise, we continue the recursive construction of $I_3, \dots, I_{m'}$ in a similar manner until $I_{m'}$ becomes empty. Since X is finite, m' is also finite.

Now we have that, for $i = 1, \dots, m'$,

$$I_i = \{\ell + \ell_1 + \cdots + \ell_{i-1} + 1, \dots, \ell + \ell_1 + \cdots + \ell_i\},$$

where $\ell_0 = \ell_{m'} = 0$. We observe that, for $i = 1, \dots, m' - 1$, there is a distinct $j' \in I_{i-1}$ for every $j \in I_i$ such that either $\tau(a^*_{j'}, b^*_{j'}) = \tau(a^*_j, c^*_j)$ or $\tau(b^*_{j'}, c^*_{j'}) = \tau(a^*_j, c^*_j)$.

Since $I_{m'} = \emptyset$, we obtain that, for all $i \in I_{m'-1}$, there are $1 \leq j' \leq m$ and $1 \leq j'' \leq m$ such that

$$\begin{aligned} \tau(a^*_i, b^*_i) &= \tau(x^*_{j'}, y^*_{j'}), \\ \tau(b^*_i, c^*_i) &= \tau(x^*_{j''}, y^*_{j''}), \end{aligned}$$

so that $a^*_i \succ b^*_i$ and $b^*_i \succ c^*_i$. Thus, by transitivity of $\succ$, $a^*_i \succ c^*_i$ for all $i \in I_{m'-1}$, so that, for all $i \in I_{m'-2}$,

$$\begin{array}{llcll} \text{either} & \tau(a^*_i, b^*_i) & = & \tau(a^*_{i'}, c^*_{i'}) & \text{for some } i' \in I_{m'-1}, \\ \text{or} & \tau(a^*_i, b^*_i) & = & \tau(x^*_{j'}, y^*_{j'}) & \text{for some } 1 \leq j' \leq m, \end{array}$$

and

$$\begin{array}{llcll} \text{either} & \tau(b^*_i, c^*_i) & = & \tau(a^*_{i''}, c^*_{i''}) & \text{for some } i'' \in I_{m'-1}, \\ \text{or} & \tau(b^*_i, c^*_i) & = & \tau(x^*_{j''}, y^*_{j''}) & \text{for some } 1 \leq j'' \leq m. \end{array}$$

Therefore, $a^*_i \succ c^*_i$ for all $i \in I_{m'-2}$. This process continues up to I_0 backwardly, so that we can conclude that $a^*_i \succ c^*_i$ for all $i \in I_0 \cup I_1 \cup \cdots I_{m'-1}$. However, $a^*_i \sim c^*_i$ for all $i \in I_0$. This is a contradiction. This completes the proof. □

3 Conclusion

This paper proved that a finite poset has a semimetric threshold representation. However, our proof of the representation theorem is not constructive. It remains an open problem to give a constructive proof, which may also answer a question whether arbitrary posets have semimetric threshold representations.

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