STABILITY AND INSTABILITY OF THE COURNOT EQUILIBRIUM

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I. INTRODUCTION

As far a lot of studies have explored conditions for the Cournot equilibrium to be stable. Earlier studies in the literature concerning the stability of the Cournot equilibrium such as Fisher (1961), Hahn (1962), and Okuguchi (1964, 1976, 1999) assumed that (1) for each firm, marginal cost should rise more rapidly than marginal revenue, with other firms expanding their 'collective' output, and that (2) for each firm, marginal cost should not fall more rapidly than market demand for total industry output. The first assumption was given as a sufficient condition for the profit maximization problem of the firms, which is known as the second-order condition for the optimum. The second assumption was given as a condition for the Cournot equilibrium to be stable, which we call *Fisher-Hahn-Okuguchi condition*.

Seade (1980) and Al-Nowaihi and Levine (1985) gave additional conditions for the Cournot equilibrium to be unstable. However, we find a counterexample such that their assumptions are no longer compatible with the second-order condition for the optimum. In this paper we explore alternative conditions for the Cournot equilibrium to be (locally asymptotically) stable or unstable which are still compatible with the second-order condition for the optimum.

Our findings are summarized as follows. The Cournot equilibrium may not be stable even if F-H-O condition is satisfied. However, as long as a game by symmetric firms (i.e., the firms which produce homogenous goods with identical cost functions) is concerned, the Cournot equilibrium is unstable whenever F-H-O condition is not satisfied, which is contrary to the results we have known in the literature on the stability of the Cournot equilibrium.

The remainder of this paper is organized as follows. In Section II we formulate a basic model of Cournot oligopoly with homogenous products. In Section III we discuss the existing conditions for the Cournot equilibrium to be stable or unstable. We also explore alternative stability or instability conditions being still compatible with the second-order condition for the optimum. In Section IV we provide a concluding remark.

II. A BASIC MODEL

We assume that there are n (fixed) firms in an industry, labeled $i = 1, \ldots, n$ respectively, each of which produces x_i (≥ 0) units of homogenous commodities to be sold in a single market. Denote by $X = \Sigma_{i=1}^{n} x_i$ the total output in the industry. Let $p = P(X)$ be the market demand, where p is the market price of the good. We assume that the function $P(X)$ is continuous, twice differentiable, and monotonically decreasing.

Consider the interval $0 \leq x_i \leq M$ for each i, where M is a number such that $P(X) = 0$ for $X \geq M$. We write $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ which consists of all components but x_i.

Let $C_i(x_i)$ be the cost of firm i to produce x_i unit of the output. We assume that the function $C_i(x_i)$ is continuous, twice differentiable, and monotonically increasing.

The profit of firm i is written in the form $\pi_i(x_i, x_{-i}) = P(X)x_i - C_i(x_i)$. Note that $\pi_i(x_i, x_{-i})$ is continuous since $P(X)$ and $C_i(x_i)$ are continuous by assumption. In order to guarantee the non-negative profit, we may assume that $C_i(0) = 0$ for all i.[1]

Firm i maximizes the profit $\pi_i(x_i, x_{-i}) = P(X)x_i - C_i(x_i)$ with respect to the output x_i, taking all the other components $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ as given. The first-order condition for the optimum is given by

$$P(\varphi_i(x_{-i}) + \Sigma_{j\neq i}\, x_j) + \varphi_i(x_{-i})P'\,(\varphi_i(x_{-i}) + \Sigma_{j\neq i}\, x_j) - C_i'\,(\varphi_i(x_{-i})) = 0, \tag{1}$$

where $x_i^{\mathrm{C}} = \varphi_i(x_{-i})$ is the reaction function of firm i.[2]

In order to guarantee that the solution x_i^{C} of equation (1) uniquely exists and is stable, we assume that $\pi_i(x_i, x_{-i})$ is strictly concave:

Condition 1. $\partial^2\pi_i/\partial x_i^2 = (P' + x_iP'') + (P' - C_i'') < 0$ at x_i^{C} for all i.

We see that given Condition 1, the second-order condition for the optimum is satisfied.

The following is what we call *Fisher-Hahn-Okuguchi condition*:

Condition 2. $P' - C_i'' < 0$ at x_i^{C} for all i.

Among the following assumptions, (A3) and (B2) are due to Seade (1980), and (A3), (H1), (B2) due to Al-Nowaihi and Levine (1985):

(A3) $(P' + x_iP'') + (P' - C_i'') < 0$ at x_i^C for all i,

(H1) $P' - C_i'' < 0$ at x_i^C for all i,

(B2) $(-1/n)(P' - C_i'') \leq P' + x_iP''$ at x_i^C for all i.

Note that the assumptions (A3) and (H1) are exactly the same as Condition 1 and Condition 2, respectively. In particular, we find a counterexample such that (H1) and (B2) do not satisfy (A3):

Counterexample. Suppose that $(-1/n)(P' - C_i'') \leq P' + x_iP''$ at x_i^C for all i, as in (B2). For fixed $m \geq n$, if $0 < (-1/n)(P' - C_i'') \leq (1/m)(P' + x_iP'') \leq P' + x_iP''$ at x_i^C for all i, where (H1) is taken into account, then $0 < (-1/n)(P' - C_i'') \leq (1/m)(P' + x_iP'') \leq (1/n)(P' + x_iP'')$, whence $(1/n)\{(P' + x_iP'') + (P' - C_i'')\} > 0$. Hence (A3) is violated.

We address the following condition instead of Condition 2:

Condition 3. $(-1/n)(P' - C_i'') > \max\{0, P' + x_iP''\}$ at x_i^C for all i.

We observe that Condition 3 satisfies both Condition 1 and Condition 2. In fact, if (i) $0 > P' + x_iP''$, then $P' + x_iP'' < 0 < (-1/n)(P' - C_i'') < (-1)(P' - C_i'')$. Hence $(P' + x_iP'') + (P' - C_i'') < 0$ with $P' - C_i'' < 0$. On the other hand, if (ii) $0 < P' + x_iP''$, then $0 < P' + x_iP'' < (-1/n)(P' - C_i'') < (-1)(P' - C_i'')$. Hence $(P' + x_iP'') + (P' - C_i'') < 0$ with $P' - C_i'' < 0$.

III. STABILITY OF THE COURNOT EQUILIBRIUM

From now on we explore the (dynamic) stability of the Cournot oligopoly solution. We consider an output adjusting process in which once an actual output x_i diverges at time t from the Cournot equilibrium output x_i^C, that is,

$$dx_i/dt = K_i\Delta_i, \tag{2}$$

where $K_i > 0$ is a constant and $\Delta_i = x_i^C - x_i$ for $i = 1, \ldots, n$.[3]

Suppose a Lyapounov function W such that

$$W(t) = \Sigma_{i=1}^{n} K_i(\Delta_i)^2/2, \tag{3}$$

where $W(t) > 0$ for all t unless $\Delta_i = 0$ for all i. Differentiating W with respect to t, we have

$$dW/dt = -[\Sigma_i q_i K_i \Delta_i \Sigma_{j\neq i} K_j \Delta_j + \Sigma_i (K_i \Delta_i)^2]. \tag{4}$$

Taking the total derivative of equation (1), we have $dx_i^{\mathrm{C}}/dt = -q_i \Sigma_{j\neq i} dx_j/dt$, where

$$q_i = (P' + x_i P'')/\{(P' + x_i P'') + (P' - C_i'')\} \quad \text{at } x_i^{\mathrm{C}} \quad \text{for all } i, \tag{5}$$

which is equal to $-\partial\varphi_i/\partial x_j$, $i, j = 1, \ldots, n$, $j \neq i$, with $q_i < 1$ (resp. > 1) as $P' - C_i'' < 0$ (resp. > 0). Immediately we see that Condition 2 is compatible with Condition 1 if and only if $q_i < 1$ for all i.

Unless $\Delta_i = 0$ for all i, we have inductively

$$dW/dt = -(K_1\Delta_1, \ldots, K_n\Delta_n)A^T(K_1\Delta_1, \ldots, K_n\Delta_n), \tag{6}$$

where $A = (a_{ij})$ is a real symmetric $n \times n$ matrix such that $a_{ii} = 1$ for all i, $a_{ij} = (1/2)(q_i + q_j) = a_{ji}$ for all $j \neq i$, and ${}^T(K_1\Delta_1, \ldots, K_n\Delta_n)$ is a transpose of $(K_1\Delta_1, \ldots, K_n\Delta_n)$. From linear algebra we conclude that A is positive definite, and hence the sign of (6) is < 0 for all $K_i\Delta_i \neq 0$, if and only if all eigenvalues of A are positive.[4]

If all firms produce with identical cost functions $C_i = C(x_i)$ for all i, then we have $x_1^{\mathrm{C}} = \ldots = x_n^{\mathrm{C}}$. We write $x_i^{\mathrm{C}} = x^{\mathrm{C}}$ (constant) for all i. Hence $q_i = q$ (constant) at $x_i^{\mathrm{C}} = x^{\mathrm{C}}$ for all i. The characteristic polynomial of A with eigenvalues λ is expressed as

$$\begin{aligned} F_A(\lambda) &= \det(\lambda E - A) \\ &= \begin{vmatrix} \lambda - 1 & -q & \ldots -q \\ -q & \lambda - 1 & \ldots -q \\ \ldots & & \\ -q & -q & \ldots \lambda - 1 \end{vmatrix} \\ &= (\lambda - 1 + q)^{n-1}\{\lambda - 1 - (n-1)q\}, \end{aligned} \tag{7}$$

where E is the unit $n \times n$ matrix. We observe that all eigenvalues λ of A are positive if and only if $-1/(n-1) < q < 1$. Note that $-1/(n-1) < q < 1$ is equivalent to Condition 3. We have just proved:

Proposition 1. *Given Condition 3, the Cournot symmetric equilibrium is stable*.

Seade (1980) asserted that, given (A3) and (B2), the Cournot equilibrium is unstable. Al-Nowaihi and Levine (1985) asserted that, given (A3), (H1), and (B2), the Cournot equilibrium is unstable. However, as mentioned above, neither (H1) nor (B2) is compatible with (A3).

Proposition 1 implies that provided that a Lyapounov function for the dynamic output adjustment process can be found, the Cournot symmetric equilibrium is *unstable* whenever Condition 3 is not satisfied, that is, if any of the following two cases occurs:

Case (*a*) $q \leq -1/(n-1)$, or equivalently, $(-1/n)(P' - C'') \leq P' + x^{C}P''$, which is the same as (B2).
Case (*b*) $q \geq 1$, or equivalently, $-(P' + x^{C}P'') > P' - C'' \geq 0$, which is the negation of (H1).

Case (*a*), together with Condition 1, leads to the same results as those in Seade (1980). Case (*b*) is another condition for the Cournot equilibrium to be unstable, which was not given in the existing literature. Consequently, we have:

Proposition 2. *If either (B2) is satisfied or (H1) is not satisfied, then the Cournot symmetric equilibrium is unstable*.

IV. CONCLUDING REMARKS

We have explored conditions for the Cournot equilibrium to be locally asymptotically stable or unstable which are still compatible with the second-order condition for the optimum.

The main results are summarized as follows.

The Cournot equilibrium may not be stable even if F-H-O condition is satisfied. However, as long as a game by symmetric firms is concerned, the Cournot equilibrium is unstable whenever F-H-O condition is not satisfied. In this sense, that F-H-O condition is not satisfied is sufficient for the Cournot equilibrium to be unstable. These results had not been pointed out in the existing literature.

Gal-Or (1985), considering a game in which two symmetric players move sequentially, showed that whether the first mover gains more or less than the second mover depends on whether the reaction functions of the players are downwards or upwards sloping. Okuguchi (1999), comparing a sequential-mover game (i.e., Stackelberg duopoly) and a simultaneous-mover game (i.e., Cournot duopoly), showed that which of the first mover, second mover, and simultaneous mover is more advantageous than the others depends on the slope of the reaction functions. The results of comparison among the first-mover, second-mover, and simultaneous-mover advantage may depend on whether the conditions discussed in this paper are satisfied or not. It is left to future studies to investigate this problem.

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FOOTNOTES

1. As to this point, see Okuguchi (1976).

2. A Cournot equilibrium is defined as a pair $(x_i^{\mathrm{C}}, x_{-i}^{\mathrm{C}})$ such that $\pi_i(x_i^{\mathrm{C}}, x_{-i}^{\mathrm{C}}) \geq \pi_i(x_i, x_{-i}^{\mathrm{C}})$ for all x_i. Consider the map φ: $\{x_1, \ldots, x_n\} \rightarrow \{x_1^{\mathrm{C}}, \ldots, x_n^{\mathrm{C}}\}$, in terms of its coordinates, $(\varphi_1, \ldots, \varphi_n)$, such that φ_i: $x_{-i} \mapsto x_i^{\mathrm{C}}$. If $x_i^{\mathrm{C}} \in \varphi_i(x_{-i})$, then φ_i is called the *reaction correspondence of firm i*. One could show the following properties of the function φ_i: (i) φ_i is continuous; (ii) φ_i is differentiable in an open neighborhood of x_{-i}^{C}. Using these properties, we can find that there exists a pair $(x_i^{\mathrm{C}}, x_{-i}^{\mathrm{C}})$ of the Cournot equilibrium. Proofs of the existence of the Cournot equilibrium are found in Burger (1963), Frank Jr. and Quandt (1963), Friedman (1977), and Okuguchi (1976), for example.

3. In the earlier literature on the stability of the Cournot oligopoly solution, Theocharis (1960), Fisher (1961), McManus and Quandt (1961), Hahn (1962), and Okuguchi (1964) considered output adjustment process.

4. Hahn (1962) assumed $q_i < 1$ for all i, which was the “worst” case where the term $\Sigma_i q_i K_i \Delta_i \Sigma_{j\neq i} K_j \Delta_j$ in the right-hand side of (4) might be negative. In the case of duopoly (i.e., i = 1, 2), we see that the sign of (6) is < 0 if and only if $-2 < q_1 + q_2 < 2$, so it suffices to assume that $|\, q_i \,| < 1$ for all i.