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SEQUENTIAL ESTIMATION PROCEDURES FOR END POINTS OF SUPPORT IN A NON-REGULAR DISTRIBUTION

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ABSTRACT

In this paper, we consider sequential estimation of the end points of the support based on the extreme values when the underlying distribution has a bound support. Some sequential fixed-width confidence intervals are proposed. Stopping rules based on the range are proposed and the estimation procedures based on them are shown to be asymptotically efficient. The results of numerical simulations are presented. Moreover, the sequential point estimation problem is considered under squared loss plus cost of sampling.

1. INTRODUCTION

In the case of the uniform distribution $U(0,\theta)$ on the interval $(0,\theta)$ $(\theta \in \mathbb{R})$, sequential estimation problems was studied by Graybill and Connell (1964), Cooke (1971), Govindarajulu (1997), and others. A sequential point estimation of θ of the uniform distribution $U(\theta-(1/2),\theta+(1/2))$ was also discussed by Wald (1950) and Akahira and Takeuchi (2003) (see also Ghosh et al. (1997)). Mukhopadhyay et al. (1983) considered a similar sequential point estimation problem in a power family distribution(see also Mukhopadhyay (1987) and Mukhopadhyay and Cicconetti (2002)).

Recently, Koike (2007a,b) considered the case of a location-scale parameter family of distributions with a bound support, obtained a sequential confidence interval with fixed width and a sequential point estimation procedure of θ, and showed their asymptotic efficiencies.

In this paper we consider sequential interval and point estimation problems of the end points of the support for a non-regular distribution. These estimation procedures might be applied to a truncated distribution. We can give the problem of the size selectivity of trawl gear as an example (see, Millar (1992) and Millar and Fryer (1999)). The size of the mesh of the net has a great influence on the size of fish captured, and the size of fish is distributed according to a truncated distribution (see also Section 4.4 of Gulland (1983)).

2. SEQUENTIAL INTERVAL ESTIMATION

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables according to the density function $f_0(x)$ ($\theta \in \mathbb{R}^1$) with respect to the Lebesgue measure. We assume throughout the paper that $f_0(x)$ has a bound support (θ_1, θ_2) ($\theta_1 < \theta_2$), i.e., $f_0(x) > 0$ for $\theta_1 < x < \theta_2$, and $f_0(x) = 0$ otherwise, and is twice continuously differentiable in (θ_1, θ_2).

We assume the following condition as non-regular distribution.

(A) $f_0(x)$ satisfies

$$\lim_{x \to \theta_1 + 0} (x - \theta_1)^{-\gamma_1} f_0(x) = g_1(\theta_2 - \theta_1), \quad \lim_{x \to \theta_2 - 0} (\theta_2 - x)^{-\gamma_2} f_0(x) = g_2(\theta_2 - \theta_1),$$

where $\gamma_i > -1$ ($i = 1, 2$) and $g_1(\theta_2 - \theta_1)$ and $g_2(\theta_2 - \theta_1)$ are strictly decreasing, continuous, positive value functions of $\theta_2 - \theta_1$.

Note that $f_0(x)$ satisfying (A) converges to 0 with the order of $(x - \theta_1)^{\gamma_1}$ and $|x - \theta_2|^{\gamma_2}$ as $x \to \theta_1 + 0$ and $x \to \theta_2 - 0$, respectively. So, the density changes sharply at the end points of the support if $-1 < \gamma_i < 1$ and changes smoothly if $\gamma_i > 1$ ($i = 1, 2$). This condition is essentially the same as those in Akahira (1975a, b), Akahira and Takeuchi (1981, p. 31; 1995, pp. 81, 148) and Koike (2007a, b). And note that the assumptions concerning g_1 and g_2 are satisfied for the uniform distribution $U(\theta_1, \theta_2)$ over (θ_1, θ_2) ($\theta_1 < \theta_2$). In fact, in this case, $\gamma_1 = \gamma_2 = 0$ and $g_1(\theta_2 - \theta_1) = g_2(\theta_2 - \theta_1) = 1/(\theta_2 - \theta_1)$.

Hereafter we assume the condition (A).

Put $X_{(1:n)} := \min_{1 \le i \le n} X_i$, $X_{(n:n)} := \max_{1 \le i \le n} X_i$. Defining $U := n^{1/(\gamma_1+1)}(X_{(1:n)} - \theta_1)$ and $V := n^{1/(\gamma_2+1)}(X_{(n:n)} - \theta_2)$, we can show by employing the same technique in Koike

(2007a) that the joint density $f_{U,V}^{(n)}(u,v)$ of (U,V) satisfies

$$f_{U,V}^{(n)}(u,v) \to \begin{cases} g_1 g_2 u^{\gamma_1}(-v)^{\gamma_2} \exp\left\{-\frac{g_2}{\gamma_2+1}(-v)^{\gamma_2+1} - \frac{g_1}{\gamma_1+1}u^{\gamma_1+1}\right\} & (v < 0 < u), \\ 0 & (\text{otherwise}) \end{cases} \tag{2.1}$$

as $n \to \infty$, where $g_1 = g_1(\theta_2 - \theta_1)$ and $g_2 = g_2(\theta_2 - \theta_1)$. Hence U and $-V$ are asymptotically, independently distributed according to Weibull distributions.

In the first place, we construct a sequential confidence interval for θ_1. If $\theta_2 - \theta_1$ is known, we have

$$\begin{aligned} P\{X_{(1:n)} - d \le \theta_1 \le X_{(1:n)}\} =& P\left\{0 \le n^{1/(\gamma_1+1)}(X_{(1:n)} - \theta_1) \le n^{1/(\gamma_1+1)}d\right\} \\ \approx& \int_0^{n^{1/(\gamma_1+1)}d} f_U(u)du \\ =& 1 - \exp\left\{-\frac{g_1(\theta_2-\theta_1)}{\gamma_1+1}nd^{\gamma_1+1}\right\}, \end{aligned}$$

for $n \in \mathbb{N}$, where "$\approx$" means that the distribution of $n^{1/(\gamma_1+1)}(X_{(1:n)} - \theta_1)$ is approximated by the asymptotic distribution of U whose density is given by

$$f_U(u) = g_1(\theta_2 - \theta_1)u^{\gamma_1} \exp\left\{-\frac{g_1(\theta_2-\theta_1)}{\gamma_1+1}u^{\gamma_1+1}\right\} \qquad (u > 0) \tag{2.2}$$

from (2.1). Letting $n^* = -\frac{(\gamma_1+1)\log\alpha}{g_1(\theta_2-\theta_1)d^{\gamma_1+1}}$, we have for $n \ge n^*$

$$1 - \exp\left\{-\frac{g_1(\theta_2-\theta_1)}{\gamma_1+1}nd^{\gamma_1+1}\right\} \ge 1 - \alpha$$

for $0 < \alpha < 1$. n^* is referred as the asymptotically *optimal* size of samples if $\theta_2 - \theta_1$ is known.

Now we take as the stopping rule

$$\tau_1 := \inf\left\{n \ge n_0 \;\middle|\; n \ge -\frac{(\gamma_1+1)\log\alpha}{g_1(R_n)d^{\gamma_1+1}}\right\}, \tag{2.3}$$

where $n_0 (\ge 2)$ is the initial size of sample and $R_n := X_{(n:n)} - X_{(1:n)}$. Then we obtain the asymptotic properties of the sequential interval estimation procedure $(\tau_1, [X_{(1:\tau_1)} - d, X_{(1:\tau_1)}])$ for θ_1 as follows.

Theorem 2.1. *Under the condition (A), we have the following.*

(i) $\lim_{d\to 0+} P\{X_{(1:\tau_1)} - d \le \theta_1 \le X_{(1:\tau_1)}\} = 1 - \alpha$ (asymptotic consistency).

(ii) $\tau_1/n^* \xrightarrow{\text{a.s.}} 1 \quad (d \to 0+)$.

(iii) $E(\tau_1)/n^* \to 1$ $(d \to 0+)$ (asymptotic efficiency).

Proof. From Lemma 1 of Chow and Robbins (1965), the stopping rule τ_1 given by (2.3) satisfies

$$\lim_{d\to 0+} -\frac{\tau_1 d^{\gamma_1+1} g_1(\theta_2-\theta_1)}{(\gamma_1+1)\log\alpha} = \lim_{d\to 0+} \frac{\tau_1}{n^*} = 1 \quad \text{a.s.} \tag{2.4}$$

Hence (ii) follows. Since U converges in distribution to a distribution with the density given by (2.2) as $n \to \infty$, it follows from Theorem 1 of Anscombe (1952) that $\tau_1^{1/(\gamma_1+1)}(X_{(1:\tau_1)} - \theta_1)$ converges in distribution to the same distribution as $d \to 0+$. Hence, from (2.4), it follows that

$$\begin{aligned}\lim_{d\to 0+} P\{X_{(1:\tau_1)} - d \le \theta_1 \le X_{(1:\tau_1)}\} = &\lim_{d\to 0+} P\left\{0 \le \tau_1^{1/(\gamma_1+1)}(X_{(1:\tau_1)} - \theta_1) \le \tau_1^{1/(\gamma_1+1)} d\right\} \\ =&1-\alpha.\end{aligned}$$

To prove (iii), from Fatou's lemma, we have

$$\liminf_{d\to 0+} \frac{E(\tau_1)}{n^*} \ge E\left(\liminf_{d\to 0+} \frac{\tau_1}{n^*}\right) = 1. \tag{2.5}$$

On the other hand, since $0 \le R_n \le \theta_2 - \theta_1$ with probability 1 for arbitrary $n \in \mathbb{N}$ and the assumption (A), $g_1(\theta_2-\theta_1) \ge g_1(R_n)$. Hence, $n > -(\gamma_1+1)\log\alpha/\{d^{\gamma_1+1}g_1(R_n)\}$ for n satisfying $n > -(\gamma_1+1)\log\alpha/\{d^{\gamma_1+1}g_1(\theta_2-\theta_1)\}+1$. So, we have

$$n_0 \le \tau_1 \le n^* + 1.$$

Dividing this by n^*, we have

$$\frac{E(\tau_1)}{n^*} \le \frac{n^*+1}{n^*} \to 1$$

as $d \to 0+$. Combining (2.5), we have the desired result. □

Remark 1. In a similar way to the above, we can construct a two-stage interval estimation procedure of θ_1. We denote

$$N_1 := \max\left\{m, \left[-\frac{(\gamma_1+1)\log\alpha}{g_1(\theta_2-\theta_1)d^{\gamma_1+1}}\right]^* + 1\right\}, \tag{2.6}$$

where $[x]^*$ means the largest integer smaller than x and $m = o(d^{-(\gamma_1+1)})$ $(0 < l < \gamma_1 + 1)$. Then we have the asymptotic consistency and efficiency of the two-stage procedure $(N_1, [X_{(1:N_1)} - d, X_{(1:N_1)}])$ for θ_1. The proof is the same as in Theorem 2.1. The asymptotic efficiencies of $(\tau_1, [X_{(1:\tau_1)} - d, X_{(1:\tau_1)}])$ and $(N, [X_{(1:N_1)} - d, X_{(1:N_1)}])$ are identical, but we have to "afford" to start with a larger sample size when d gets smaller in the latter, while the initial size of sample may be independent of d in the former (see, Mukhopadhyay (1980) and pp.156–157 of Ghosh et al. (1997)).

Next, we construct a sequential confidence interval for θ_2 in a similar way to the above. If $\theta_2 - \theta_1$ is known, we have

$$\begin{aligned} P\{X_{(n:n)} \le \theta_2 \le X_{(n:n)} + d\} =& P\left\{-n^{1/(\gamma_2+1)} d \le n^{1/(\gamma_2+1)}(X_{(n:n)} - \theta_2) \le 0\right\} \\ \approx& \int_{-n^{1/(\gamma_2+1)}d}^{0} f_V(v)dv \\ =& 1 - \exp\left\{-\frac{g_2(\theta_2-\theta_1)}{\gamma_2+1} n d^{\gamma_2+1}\right\}, \end{aligned}$$

for $n \in \mathbb{N}$, where "$\approx$" means that the distribution of $n^{1/(\gamma_2+1)}(X_{(n:n)} - \theta_2)$ is approximated by the asymptotic distribution of V whose density is given by

$$f_V(v) = g_2(\theta_2 - \theta_1)(-v)^{\gamma_2} \exp\left\{-\frac{g_2(\theta_2-\theta_1)}{\gamma_2+1}(-v)^{\gamma_2+1}\right\} \qquad (v < 0)$$

from (2.1). Letting $n^{**} = -\frac{(\gamma_2+1)\log\alpha}{g_2(\theta_2-\theta_1)d^{\gamma_2+1}}$, we have for $n \ge n^{**}$

$$1 - \exp\left\{-\frac{g_2(\theta_2-\theta_1)}{\gamma_2+1} n d^{\gamma_2+1}\right\} \ge 1 - \alpha$$

for $0 < \alpha < 1$. n^{**} is referred as the asymptotically *optimal* size of samples if $\theta_2 - \theta_1$ is known.

Now we take as the stopping rule

$$\tau_2 := \inf\left\{ n \ge n_0 \;\middle|\; n \ge -\frac{(\gamma_2+1)\log\alpha}{g_2(R_n)d^{\gamma_2+1}} \right\},$$

where $n_0 (\ge 2)$ is the initial size of sample. Then we obtain the following.

Theorem 2.2. *Under the condition (A), we have the following.*

(i) $\lim_{d\to 0+} P\{X_{(\tau_2:\tau_2)} \le \theta_2 \le X_{(\tau_2:\tau_2)} + d\} = 1 - \alpha$ (asymptotic consistency).

(ii) $\tau_2/n^{**} \xrightarrow{\text{a.s.}} 1 \quad (d \to 0+)$.

(iii) $E(\tau_2)/n^{**} \to 1$ $(d \to 0+)$ (asymptotic efficiency).

The proof is omitted since it is similar to the one of Theorem 2.1.

Example 2.1. If the parent distribution is $U(\theta_1, \theta_2)$, then $\gamma_1 = \gamma_2 = 0$, $g_1(\theta_2 - \theta_1) = g_2(\theta_2 - \theta_1) = 1/(\theta_2 - \theta_1)$. Hence $\tau_1 \approx n^* = -\{(\theta_2 - \theta_1)\log\alpha\}/d$ as $d \to 0+$. Note that this stopping rule is the same as the one given in Chaturvedi et al. (2001), in which they consider one-parameter case $U(0, \theta)$.

Example 2.2. We generalize the power family distribution in Mukhopadhyay et al. (1983). as follows. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables according to the density function

$$f_0(x) = \begin{cases} \delta(x-\theta_1)^{\delta-1}(\theta_2-\theta_1)^{-\delta} & (\theta_1 < x < \theta_2), \\ 0 & (\text{otherwise}) \end{cases} \tag{2.7}$$

with known $\delta > 0$ and unknown $\theta_1 < \theta_2$. In this case, $(x-\theta_1)^{-\delta+1}f_0(x) \to \delta(\theta_2-\theta_1)^{-\delta}$ as $x \to \theta_1 + 0$ and $(\theta_2 - x)^0 f_0(x) \to \delta(\theta_2-\theta_1)^{-1}$ as $x \to \theta_2 - 0$. Hence the assumption (A) is satisfied. τ_1 in (2.3) is given by

$$\tau_1 = \inf\left\{ n \geq n_0 \;\middle|\; n \geq -\frac{R_n^\delta \log\alpha}{d^\delta} \right\},$$

and $\tau_1 \approx n^* = -\frac{(\theta_2-\theta_1)^\delta \log\alpha}{d^\delta}$.

3. SEQUENTIAL POINT ESTIMATION

In this section, at first, we construct an asymptotic sequential point estimation procedure for θ_1.

Since the asymptotic density of $U := n^{1/(\gamma_1+1)}(X_{(1:n)} - \theta_1)$ is given by (2.2), the asymptotic expectation of U^2 is

$$E(U^2) \approx \int_0^\infty g_1 u^{\gamma_1+2} \exp\left\{-\frac{g_1}{\gamma_1+1}u^{\gamma_1+1}\right\} du = \left(\frac{\gamma_1+1}{g_1}\right)^{2/(\gamma_1+1)} \Gamma\left(\frac{\gamma_1+3}{\gamma_1+1}\right),$$

where $g_1 = g_1(\theta_2 - \theta_1)$ and $\Gamma(\cdot)$ is the gamma function. In a similar way to Lemma 2.1 of Koike (2007b), we can show that there exists a constant C such that $E(U^2) \to C$ as $n \to \infty$.

In addition to this, we assume the following condition.

(B1) There exists a positive valued, increasing, continuous function $h_1(\theta_2 - \theta_1)$ of $\theta_2 - \theta_1$ satisfying $E(U^2) \to h_1(\theta_2 - \theta_1)$ as $n \to \infty$.

Note that (B1) is satisfied for the uniform distribution $U(\theta_1, \theta_2)$ over (θ_1, θ_2) $(\theta_1 < \theta_2)$. In fact, in this case, $\gamma_1 = \gamma_2 = 0$, and an easy computation yields $E(U^2) = 2n^2(\theta_2 - \theta_1)^2/\{(n+1)(n+2)\} \to 2(\theta_2 - \theta_1)^2$ as $n \to \infty$.

If θ_1 is estimated by $X_{(1:n)}$, then the risk is given by

$$r_n^{(1)} := E(X_{(1:n)} - \theta_1)^2 + dn,$$

where $d(> 0)$ is the cost per observation. From $U = n^{1/(\gamma_1+1)}(X_{(1:n)} - \theta_1)$, $r_n^{(1)}$ is approximated by $h_1(\theta_2 - \theta_1)n^{-2/(\gamma_1+1)} + dn$, which is minimized at the integer closest to $n = n^{***} := \left\{\frac{2h_1(\theta_2-\theta_1)}{(\gamma_1+1)d}\right\}^{(\gamma_1+1)/(\gamma_1+3)}$ and the minimized value is $r_{n^{***}}^{(1)*} := h_1(\theta_2 - \theta_1)\left\{\frac{d(\gamma_1+1)}{2h_1(\theta_2-\theta_1)}\right\}^{2/(\gamma_1+3)} \cdot \left(\frac{\gamma_1+3}{\gamma_1+1}\right)$. However, unless $\theta_2 - \theta_1$ is known, one can not attain this risk with a non-sequential procedure. Since the range $R_n = X_{(n)} - X_{(1)}$ converges to $\theta_2 - \theta_1$ almost surely as $n \to \infty$, therefore we consider the following stopping rule:

$$\tau_3 := \left\{ n \geq m_d^{(1)} \;\middle|\; n \geq \left\{ \frac{2h_1(R_n)}{(\gamma_1 + 1)d} \right\}^{(\gamma_1+1)/(\gamma_1+3)} \right\},$$

where $m_d^{(1)}$ is the initial size of samples with $d^{-l} \leq m_d^{(1)} = o(d^{-(\gamma_1+1)/(\gamma_1+3)})$ $(0 < l < (\gamma_1 + 1)/(\gamma_1 + 3))$. Then we have the (first order) asymptotic efficiency of the estimation procedure $[\tau_3, X_{(1:\tau_3)}]$ as follows.

Theorem 3.1. *Under the conditions (A) and (B1), as $d \to 0+$, we have*

$$\text{(i)} \quad \tau_3/n^{***} \overset{\text{a.s.}}{\to} 1. \quad \text{(ii)} \quad E(\tau_3)/n^{***} \to 1, \quad \text{(iii)} \quad r_{\tau_3}^{(1)}/r_{n^{***}}^{(1)*} \to 1.$$

Proof. The proof is similar to the one of Theorem 2.1 in Koike (2007b)(see also Lai (1996)). At first, we note that

$$m_d^{(1)} \leq \tau_3 \leq n^{***} + 1 \quad \text{with probability 1.} \tag{3.1}$$

In fact, since $0 \leq R_n \leq \theta_2 - \theta_1$ with probability 1, we have

$$0 \leq \left\{ \frac{2h_1(R_n)}{(\gamma_1+1)d} \right\}^{(\gamma_1+1)/(\gamma_1+3)} \leq \left\{ \frac{2h_1(\theta_2-\theta_1)}{(\gamma_1+1)d} \right\}^{(\gamma_1+1)/(\gamma_1+3)}$$

with probability 1. Hence, $n > \{2h_1(R_n)/((\gamma_1+1)d)\}^{(\gamma_1+1)/(\gamma_1+3)}$ for n satisfying $n > \{2h_1(\theta_2-\theta_1)/((\gamma_1+1)d)\}^{(\gamma_1+1)/(\gamma_1+3)}$. Therefore (3.1) holds. Since $\tau_3 \stackrel{\text{a.s.}}{\to} \infty$ and $R_n \stackrel{\text{a.s.}}{\to} \theta_2 - \theta_1$, $R_{\tau_3} \stackrel{\text{a.s.}}{\to} \theta_2 - \theta_1$. By the definition of τ_3,

$$\left\{ \frac{2h_1(R_{\tau_3})}{(\gamma_1+1)d} \right\}^{(\gamma_1+1)/(\gamma_1+3)} \leq \tau_3 < m_d^{(1)} + \left\{ \frac{2h_1(R_{\tau_3-1})}{(\gamma_1+1)d} \right\}^{(\gamma_1+1)/(\gamma_1+3)}.$$

Dividing this by n^{***}, we have (i) as $d \to 0+$ since $d^{-l} \leq m_d^{(1)} = o(d^{-(\gamma_1+1)/(\gamma_1+3)})$. To prove (ii), we have from (i) that

$$\liminf_{d \to 0+} E\left(\tau_3/n^{***}\right) \geq 1.$$

by Fatou's lemma. On the other hand, by (3.1),

$$\frac{E\left(\tau_3\right)}{n^{***}} \leq \frac{n^{***}+1}{n^{***}} \to 1 \quad (d \to 0+),$$

hence $E\left(\tau_3\right)/n^{***} \to 1$ as $d \to 0+$. So, we have (ii).

To prove (iii), we may assume $\theta_1 = 0$ without loss of generality. Putting $S_{k,n} := (k+n)^{1/(\gamma_1+1)} X_{(1:k+n)} - k^{1/(\gamma_1+1)} X_{(1:k)}$ $(k \geq 1, n \geq 0)$, we have by Minkowski's inequality, that

$$\begin{aligned} 0 &\leq \left(E|S_{k,n}|^4\right)^{1/4} = \left(E|(k+n)^{1/(\gamma_1+1)} X_{(1:k+n)} - k^{1/(\gamma_1+1)} X_{(1:k)}|^4\right)^{1/4} \\ &\leq \left(E|(k+n)^{1/(\gamma_1+1)} X_{(1:k+n)}|^4\right)^{1/4} + \left(E|k^{1/(\gamma_1+1)} X_{(1:k)}|^4\right)^{1/4} = O(1) \end{aligned} \tag{3.2}$$

from the condition (B1) and Lemma 2.2 in Koike (2007b). Taking η and λ satisfying $0 < \lambda < \{h_1(\theta_2-\theta_1)\}^{(\gamma_1+1)/(\gamma_1+3)} < \eta$, we have $P\big(\{d(\gamma_1+1)/2\}^{(\gamma_1+1)/(\gamma_1+3)} \tau_3 \geq \eta\big) \to 0$ as $d \to 0+$ from (i). By (3.2) and Theorem B of Serfling (1980),

$$E \max_{1 \leq i \leq n} |S_{k,i}|^4 = O(1) \quad \text{for } k \geq k_0, n \geq 1. \tag{3.3}$$

Since $\tau_3 \geq m_d^{(1)}$ with probability 1, we have by denoting $l_0 := \{d(\gamma_1+1)/2\}^{(\gamma_1+1)/(\gamma_1+3)}$,

$$
\begin{aligned}
&\eta^{-2/(\gamma_1+1)}\{d(\gamma_1+1)/2\}^{2/(\gamma_1+3)} E\left\{\tau_3^{2/(\gamma_1+1)} X^2_{(1:\tau_3)} I\left(\lambda \leq l_0\tau_3 \leq \eta\right)\right\} \\
\leq& E\left(X^2_{(1:\tau_3)}\right) \\
\leq& E\left\{X^2_{(1:\tau_3)} I\left(\tau_3 \leq \lambda/l_0\right)\right\} \\
&\quad + \lambda^{-2/(\gamma_1+1)}\left\{d(\gamma_1+1)/2\right\}^{2/(\gamma_1+3)} E\left[\tau_3^{2/(\gamma_1+1)} X^2_{(1:\tau_3)} I\left\{\lambda \leq l_0\tau_3 \leq \eta\right\}\right] \\
&\quad + E\left\{X^2_{(1:\tau_3)} I\left(\tau_3 \geq \eta/l_0\right)\right\}, \qquad (3.4)
\end{aligned}
$$

where $I(A)$ is the indicator function of an event A. By Schwarz's inequality and (3.3),

$$
\begin{aligned}
&E\left\{X^2_{(1:\tau_3)} I\left(\tau_3 \geq \eta/l_0\right)\right\} \\
\leq& \eta^{-2} l_0^2 \sum_{j=0}^{\infty} 2^{-2j}\left[E\left\{\max{}^* |n^{1/(\gamma_1+1)} X_{(1:n)}|^4\right\}\right]^{1/2}\left[P\left\{2^j\eta/l_0 \leq \tau_3 \leq 2^{j+1}\eta/l_0\right\}\right]^{1/2} \\
=& o\left(d^{(2\gamma_1+2)/(\gamma_1+3)} \sum_{j=0}^{\infty} 2^{-2j} 2^j d^{-(\gamma_1+1)/(\gamma_1+3)}\right) = o\left(d^{(\gamma_1+1)/(\gamma_1+3)}\right)
\end{aligned}
$$

since $P\left(\tau_3 \geq \eta/l_0\right) \to 0$ as $d \to 0+$, where $\max^*$ means taking the maximum over $2^j\eta/l_0 \leq n \leq 2^{j+1}\eta/l_0$. For an $\varepsilon > 0$ satisfying $\lambda^{(\gamma_1+3)/(\gamma_1+1)} < h_1(\theta_2-\theta_1) - \varepsilon$,

$$
\begin{aligned}
&P\left\{\tau_3 \leq \lambda/l_0\right\} \\
\leq& P\left\{\lambda/l_0 \geq \left(\frac{2h_1(R_n)}{d(\gamma_1+1)}\right)^{(\gamma_1+1)/(\gamma_1+3)} \text{ for some } m_d^{(1)} \leq n \leq \lambda/l_0\right\} \\
=& P\left\{\lambda^{(\gamma_1+3)/(\gamma_1+1)} \geq h_1(R_n) \ \text{ for some } \ m_d^{(1)} \leq n \leq \lambda/l_0\right\} \\
\leq& P\left\{\lambda^{(\gamma_1+3)/(\gamma_1+1)} \geq h_1(R_{m_d^{(1)}})\right\} \qquad \text{(by the monotonicity of } R_n \text{ w.r.t. } n) \\
\leq& P\left(h_1(\theta_2-\theta_1) - \varepsilon \geq h_1(R_{m_d^{(1)}})\right) \\
=& O\left(\alpha^{m_d^{(1)}}\right), \qquad (3.5)
\end{aligned}
$$

where $\alpha \in (0,1)$ is a constant. By Schwarz's inequality and (3.5),

$$
\begin{aligned}
& E\left\{X^2_{(1:\tau_3)} I(\tau_3 \le \lambda/l_0)\right\} \\
\le & \left\{E|X_{(1:\tau_3)}|^4\right\}^{1/2} P^{1/2}\left(\tau_3 \le \lambda/l_0\right) \\
\le & \sum_{j: 2^j \ge m_d^{(1)}} 2^{-2j} \left\{E\left(\max_{2^j \le n \le 2^{j+1}} |n^{1/(\gamma_1+1)} X_{(1:n)}|^4\right)\right\}^{1/2} P^{1/2}\left(\tau_3 \le \lambda/l_0\right) \\
= & D \sum_{j: 2^j \ge m_d^{(1)}} 2^{-2j} \left(O\left(\alpha^{m_d^{(1)}}\right)\right)^{1/2} = O\left({m_d^{(1)}}^{-1} \alpha^{m_d^{(1)}/2}\right),
\end{aligned}
$$

where D is some constant. On the other hand, since $|a^2 - b^2| \le |a-b|^2 + 2|b||a-b|$ for $a, b \in \mathbb{R}$,

$$
\begin{aligned}
& \left| E\left\{\tau_3^{2/(\gamma_1+1)} X^2_{(1:\tau_3)} I(\lambda \le l_0\tau_3 \le \eta)\right\} - E\left\{\left([\lambda/l_0]^{1/(\gamma_1+1)} X_{(1:[\lambda/l_0])}\right)^2\right\}\right| \\
\le & E\left\{\max_{\lambda/l_0 \le n \le \eta/l_0} \left| n^{2/(\gamma_1+1)} X^2_{(1:n)} - \left([\lambda/l_0]^{1/(\gamma_1+1)} X_{(1:[\lambda/l_0])}\right)^2\right|\right\} \\
& + E\left[\left([\lambda/l_0]^{1/(\gamma_1+1)} X_{(1:[\lambda/l_0])}\right)^2 \left\{I\left(l_0\tau_3 < \lambda\right) + I\left(l_0\tau_3 > \eta\right)\right\}\right] \\
\le & \left\{E\left(\max_{\lambda/l_0 \le n \le \eta/l_0} \left|n^{1/(\gamma_1+1)} X_{(1:n)} - [\lambda/l_0]^{1/(\gamma_1+1)} X_{(1:[\lambda/l_0])}\right|^4\right)\right\}^{1/2} \\
& + 2\left[E\left\{\left([\lambda/l_0]^{1/(\gamma_1+1)} X_{(1:[\lambda/l_0])}\right)^2\right\}\right]^{1/2} \\
& \cdot \left\{E\left(\max_{\lambda/l_0 \le n \le \eta/l_0} \left|n^{1/(\gamma_1+1)} X_{(1:n)} - [\lambda/l_0]^{1/(\gamma_1+1)} X_{(1:[\lambda/l_0])}\right|^4\right)\right\}^{1/4} \\
& + \left\{E\left|[\lambda/l_0]^{1/(\gamma_1+1)} X_{(1:[\lambda/l_0])}\right|^4\right\}^2 \left\{P^{1/2}\left(l_0\tau_3 < \lambda\right) + P^{1/2}\left(l_0\tau_3 > \eta\right)\right\}
\end{aligned}
$$

from Schwarz's inequality. Therefore, since $E\left\{\left([\lambda/l_0]^{1/(\gamma_1+1)} X_{(1:[\lambda/l_0])}\right)^2\right\} \sim h_1(\theta_2 - \theta_1)$ as $d \to 0$. η and λ can be taken arbitrary close to $\{h_1(\theta_2-\theta_1)\}^{(\gamma_1+1)/(\gamma_1+3)}$,

$$
E\left(X_{(1:\tau_3)} - \theta_1\right)^2 \sim h_1(\theta_2 - \theta_1)\left\{d(\gamma_1+1)/(2h_1(\theta_2-\theta_1))\right\}^{2/(\gamma_1+3)}. \tag{3.6}
$$

By (ii) and (3.6), we have (iii). □

Remark 2. In a similar way to the above, we can construct a two-stage point estimation procedure of θ_1. We denote

$$
N_2 := \max\left\{m, \left[\left\{\frac{2h_1(R_m)}{d(\gamma_1+1)}\right\}^{(\gamma_1+1)/(\gamma_1+3)}\right]^* + 1\right\}, \tag{3.7}
$$

where $d^{-l} \leq m = o(d^{-(\gamma_1+1)/(\gamma_1+3)})$ $(0 < l < (\gamma_1+1)/(\gamma_1+3))$. Then we have the (first order) asymptotic efficiency of the two-stage procedure $(N_2, [X_{(1:N_2)} - d, X_{(1:N_2)}])$ for θ_1. The proof is the same as in Theorem 3.1. The asymptotic efficiencies of $(\tau_3, [X_{(1:\tau_3)} - d, X_{(1:\tau_3)}])$ and $(N_2, [X_{(1:N_2)} - d, X_{(1:N_2)}])$ are identical up to the first oder (see, Ghosh and Mukhopadhyay (1981)).

We may consider a sequential point estimation procedure for θ_2 in the same way. In that case we assume the following instead of (B1).

(B2) There exists a positive valued, increasing, continuous function $h_2(\theta_2 - \theta_1)$ of $\theta_2 - \theta_1$ satisfying $E(V^2) \to h_2(\theta_2 - \theta_1)$ as $n \to \infty$, where $V = n^{1/(\gamma_2+1)}(X_{(n:n)} - \theta_2)$.

If θ_2 is estimated by $X_{(n:n)}$, then the risk is given by

$$r_n^{(2)} := E(X_{(n:n)} - \theta_2)^2 + dn,$$

where $d(> 0)$ is the cost per observation. From $V = n^{1/(\gamma_2+1)}(X_{(n:n)}-\theta_2)$, $r_n^{(2)}$ is approximated by $h_2(\theta_2 - \theta_1)n^{-2/(\gamma_2+1)} + dn$, which is minimized at the integer closest to $n = n^{****} := \left\{\frac{2h_2(\theta_2-\theta_1)}{(\gamma_2+1)d}\right\}^{(\gamma_2+1)/(\gamma_2+3)}$ and the minimized value is $r^{(2)*}_{n^{****}} := h_2(\theta_2 - \theta_1)\left\{\frac{d(\gamma_2+1)}{2h_2(\theta_2-\theta_1)}\right\}^{2/(\gamma_2+3)}$ $\cdot \left(\frac{\gamma_2+3}{\gamma_2+1}\right)$. However, unless $\theta_2 - \theta_1$ is known, one can not attain this risk with a non-sequential procedure. Since the range R_n converges to $\theta_2 - \theta_1$ almost surely as $n \to \infty$, therefore we consider the following stopping rule:

$$\tau_4 := \left\{ n \geq m_d^{(2)} \;\middle|\; n \geq \left\{\frac{2h_2(R_n)}{(\gamma_2+1)d}\right\}^{(\gamma_2+1)/(\gamma_2+3)} \right\},$$

where $m_d^{(2)}$ is the initial size of samples with $d^{-l} \leq m_d^{(2)} = o(d^{-(\gamma_2+1)/(\gamma_2+3)})$ $(0 < l < (\gamma_2 + 1)/(\gamma_2 + 3))$. Then we have the asymptotic efficiency of the estimation procedure $[\tau_4, X_{(\tau_4:\tau_4)}]$ as follows.

Theorem 3.2. *Under the conditions (A) and (B2), as $d \to 0+$, we have*

$$\text{(i)}\quad \tau_4/n^{****} \overset{\text{a.s.}}{\to} 1. \quad \text{(ii)}\quad E(\tau_4)/n^{****} \to 1, \quad \text{(iii)}\quad r^{(2)}_{\tau_4}/r^{(2)*}_{n^{****}} \to 1.$$

The proof is omitted since it is similar to the one of Theorem 3.1.

Example 3.1. If the parent distribution is $U(\theta_1, \theta_2)$, then $\gamma_1 = \gamma_2 = 0$, $h_1(\theta_2 - \theta_1) = 2(\theta_2 - \theta_1)^2$. Hence $\tau_3 \approx n^{***} = \{4(\theta_2 - \theta_1)^2/c\}^{1/3}$ and $r_{\tau_3} \approx r_{n^{***}} = 2^{-1/3}\{d(\theta_2 - \theta_1)\}^{2/3}$ as $d \to 0+$.

Example 3.2. If the parent distribution is the power family distribution in (2.7), an easy computation yields

$$\begin{aligned} E\left\{n^{2/\delta}(X_{(1:n)} - \theta_1)^2\right\} =& (\theta_2 - \theta_1)^2 n^{(2/\delta)+1}\Gamma\left(\frac{2}{\delta} + 1\right)\Gamma(n) \Big/ \Gamma\left(\frac{2}{\delta} + n + 1\right) \\ \to& (\theta_2 - \theta_1)^2\Gamma\left(\frac{2}{\delta} + 1\right)\exp\{-(\delta/2) - 1\} \end{aligned}$$

as $n \to \infty$. Hence the assumption (B1) is satisfied. In this case, the stopping rule τ_3 is given by

$$\tau_3 = \left\{ n \geq m_d^{(1)} \;\middle|\; n \geq \left\{ \frac{2R_n^2\Gamma\left(\frac{2}{\delta} + 1\right)\exp\{-(\delta/2) - 1\}}{\delta d} \right\}^{\delta/(\delta+2)} \right\},$$

where $m_d^{(1)}$ is the initial size of samples with $d^{-l} \leq m_d^{(1)} = o(d^{-\delta/(\delta+2)})$ $(0 < l < \delta/(\delta+2))$, and $\tau_3 \approx n^{***} = \left\{ \frac{2(\theta_2-\theta_1)^2\Gamma\left(\frac{2}{\delta}+1\right)\exp\{-(\delta/2)-1\}}{\delta d} \right\}^{\delta/(\delta+2)}$ as $d \to 0+$.

Remark 3. Similarly, we may construct sequential interval and point estimation procedures of the range $\theta_2 - \theta_1$ based on the R_n which are asymptotically efficient. The stopping rules depend on the magnitude of γ_1 and γ_2.

4. NUMERICAL EXAMPLE

In this section we examine the coverage probability of the procedure $(\tau_1, [X_{(1:\tau_1)} - d, X_{(1:\tau_1)}])$ in Theorem 2.1 by simulation based on 10000 repetitions. Suppose that $X_1, X_2, \ldots, X_n, \ldots$ is a sequence of i.i.d. random variables according to the uniform distribution $U(\theta_1, \theta_2)$ $(\theta_1 < \theta_2)$. We may assume $\theta_1 = 0$ without loss of generality.

When $\alpha = 0.05$, $d = 0.01(0.01)0.05$, $\theta_2 = 1(1)5$ and $n_0 = 5$, Tables 1 and 2 show the values of coverage probabilities and the average sample sizes of the sequential estimation procedure $(\tau_1, [X_{(1:\tau_1)} - d, X_{(1:\tau_1)}])$, respectively. The result suggests that the estimation procedure is almost consistent for this case.

Table 1. Coverage probabilities of $[X_{(1:\tau_1)} - d, X_{(1:\tau_1)}]$

$\theta_2 \setminus d$	0.01	0.02	0.03	0.04	0.05
1	0.9485	0.9488	0.9452	0.9470	0.9484
2	0.9486	0.9457	0.9461	0.9462	0.9475
3	0.9473	0.9515	0.9470	0.9480	0.9477
4	0.9483	0.9472	0.9441	0.9510	0.9455
5	0.9505	0.9449	0.9453	0.9422	0.9493

Table 2. Average sample sizes of $[X_{(1:\tau_1)} - d, X_{(1:\tau_1)}]$

$\theta_2 \setminus d$	0.01	0.02	0.03	0.04	0.05
1	299.786	150.285	100.453	75.5243	60.58
2	598.378	299.757	200.02	150.168	120.211
3	896.152	449.261	299.342	224.829	180.331
4	1194.86	598.556	398.941	299.875	239.867
5	1493.92	747.616	499.066	374.182	299.727

5. SUMMARY

Sequential interval and point estimation procedures of the end points of the support were presented for a non-regular distribution with a bound support. And the asymptotic efficiencies were shown. Moreover, numerical simulations were presented.

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