Customer Selection Problem with Profit from a Sideline

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January 2005

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Abstract

This thesis deals with the problem of selecting profitable orders to accept out of customers sequentially arriving at custom production companies such as a shipbuilding company, advertising agency, consulting company, design office, construction firm, and so on. We consider in particular the companies operating in service industries which provide specialized services designed to meet the various needs of their customers. The companies, taking their limited production capacity into consideration, make a decision on whether or not to accept orders from customers who need specialized services. When all the orders accepted so far are completed and delivered, they provide subsidiary services in order to prevent their system from being idle, yielding extra income, called the profit from a sideline. In addition to the profit from a sideline, we consider situations where a cost must be paid to search for customers, called the search cost. The introduction of profit from a sideline and search cost is the distinguished point in this thesis, and it has not been taken into consideration so far.

Further, as to which the customer or the company declares the price of the order, we consider two problems, usually called the admission control problem and the pricing control problem. In the former, a customer offers the price for his order, and judging from this, the company decides whether or not to accept it. In the latter, an arriving customer has the maximum permissible ordering price, called reservation price, that he is willing to pay for his order. In this case, to an arriving customer the company proposes a price for an order from the customer, and if and only if that price proposed by the company is lower than his reservation price, will the customer place the order with the company.

In this thesis we define five basic models, Models I to V. The objective in each of the models is to find the optimal decision rule so as to maximize the total expected present discounted net profit gained over an infinite planning horizon. The thesis consists of the nine chapters outlined below.

Chapter 1 states what the customer selection problem is all about, specifies the type of companies considered in the thesis, provides some reviews of the literature related to, and clarifies what our contributions are. In Chapter 2 we state assumptions common to all the models, sketch out each of the five models, and define symbols and notations used in the chapters that follow. In Chapter 3 we define some functions, called the underlying functions, and examine their properties. These functions play an important role in describing the optimal equations of these models and analyzing the properties of the optimal decision rule for every model. In Chapters 4 and 5 we examine Models I and II, in which it is assumed that only one backorder is allowed to be held and that a penalty is paid for delay of delivery. Further, in Model II an order undergoing processing is assumed to be cancelled with a known probability. In Chapter 6 and 7 we examine Models III and IV, in which multiple back orders are permitted to be held. While in Chapter 6 an order undergoing processing at the present time is assumed to be completed

with a known probability at the next point in time, in Chapter 7 an accepted order is assumed to be completed after fixed periods from the start of production. In Chapter 8 we propose and examine Model V. In all the models stated above except Model V it is implicitly assumed that only one production line is available; contrary to this, in Model V multiple production lines are assumed to be available. In Chapter 9 we summarize the overall conclusions obtained throughout the thesis and suggest the subjects of study to be tackled in the future.

Acknowledgements

I would like to express my first and deepest appreciation to Professor Seizo Ikuta of the University of Tsukuba for his valuable advice and warm-hearted encouragement. He is the person who first introduced me to the field of stochastic decision processes. I have learned various things from him such as the way of thinking, the way of conducting research, presentation skills, and so on.

I feel fortunate to have had the opportunity for completing my doctoral degree at the Doctoral Program in Graduate School of Systems and Information Engineering (University of Tsukuba). In particular, I wish to express my gratitude to Professor Yoshitsugu Yamamoto, Ryoh Fujihara, Hideaki Takagi, Masato Koda, Akiko Kuno (Yoshise), Ryo Sato, Ushio Sumita, Yongbing Zhang, and Hideo Suzuki of the University of Tsukuba for their valuable suggestions and comments.

I would like to also thank Professor Byung-Kook Kang of the University of Nanzan, Dr. Feng-Bo Shi, Mr. Jun-Suk Kwon, and Ms. Ee Mong Shan for their supports and hospitality. Mr. Michael Cullen gave me helpful advices in English.

I acknowledge the Hirose International Scholarship Foundation and the Nakamura Sekizen-kai Scholarship Foundation for providing scholarships to pursue doctoral studies.

Last but not the least, I would like to dedicate this dissertation to my parents in Heaven who must be most pleased with the completion of my doctoral degree. My greatest gratitude goes to my wife Ji-Yeon Lee who has shared with me not only the hardship of my research but also the joy of my success. Furthermore, a special word of appreciation goes to my brothers and sisters and parents-in-law for their affection and understanding. Their unquestioning faith in my ability and me has helped to make this dissertation possible.

Contents

Chapter 1

Introduction

This chapter specifies the types of companies considered in the thesis, provides some reviews of the literature related to, clarifies what our contributions are, and states the structure of this thesis.

Contents

1.1 What the Customer Selection Problem Is About?

Let us consider a custom production company such as a shipbuilding company, advertising agency, consulting company, design office, construction firm, and so on. The company, taking its limited production capacity into account, must make a decision on whether or not to accept orders from sequentially arriving customers. Every accepted order is registered in an order book, and the processing of the order cannot start until all the orders accepted so far are completed. When the accepted order is completed, it is delivered to its customer. The objective here is to maximize the expected long run profit. In order to attain the objective, the following two kinds of opportunity loss should be taken into consideration:

1. *Opportunity loss I*. Suppose orders from all arriving customers are accepted irrespective of their profitabilities. The production process soon becomes full; with the result that orders from customers arriving thereafter can not be accepted, however high their profitabilities may be, for the reason that even if they were accepted, they could not be processed by the appointed date of delivery. This leads to the opportunity loss that if adequate allowance were kept in the production lines by having rejected less profitable orders in advance, the company could have enjoyed accepting upcoming profitable orders. We shall refer to this loss as Opportunity loss I.

2. *Opportunity loss II*. Excessively refraining from accepting orders due to the apprehension that Opportunity loss I could occur causes a decrease in the back orders. This time, the production process may soon become idle, causing the opportunity loss that if more orders had been accepted in advance, profit could have been gained from them. We shall refer to this loss as Opportunity loss II.

Both Opportunity losses may cause diminishment in the long run profit. Accordingly, the objective is to find an optimal customer selection rule so as to maximize the expected long run profit through keeping an appropriate level of back orders by controlling the number of orders to accept in advance with the aim to avoid both Opportunity losses.

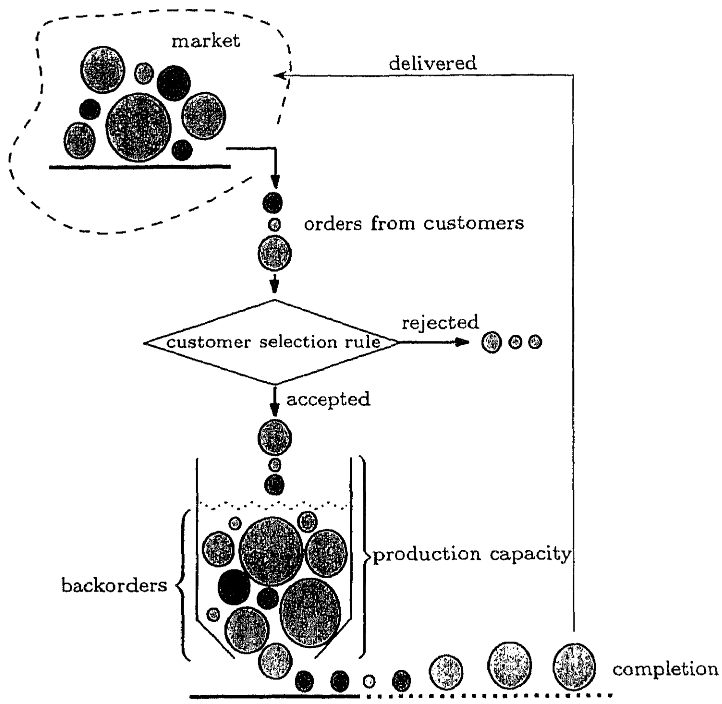

Figure 1.1.1: Graphical representation of the customer selection problem.

This problem is usually called the *customer selection problem* [12, Ikuta (1971)] [21, Kushner and Tetzlaff (1971)], and it can be graphically explained by Figure 1.1.1.

The customer selection problem is formulated as the *admission control problem* and the *pricing control problem*. In the former, a customer offers the price for his order, and judging from this, the company decides whether or not to accept. In the latter, an arriving customer has the maximum permissible ordering price, called the reservation price, which he is willing to pay for his order. In this case, to an arriving customer the company proposes a price for an order, and if and only if the price proposed by the company is lower than his reservation price, the customer is willing to place the order with the company.

1.2 Type of Companies Considered in the Thesis

In this thesis we consider in particular the customer selection problem of the companies operating in the service industries which provide the specialized services designed to meet the various needs of their customers. The companies, taking their limited production capacity into consideration, make a decision on whether or not to accept particular orders from customers who need specialized services. However, when all the orders accepted so far are completed and delivered, they provide *subsidiary services* in order to prevent their system from being idle, yielding an extra income. Let us refer to the extra income from subsidiary services as the *profit from a sideline*. In this thesis, as a sideline we only consider simple subsidiary business which can be done within one period. Although, of course, there exist a sideline which can not be completed within one period, let us leave the models of this case as the subject of future studies. Below, we present some examples such companies.

1. **Staff service providing company**

The crucial performance of a staff service providing company is characterized by placing greater emphasis on accepting orders of specialized services such as human resources recruitment, personnel training, and so on. Indeed, since the company's topmost priority is to take care of its clients asking for specialized services, it devotes itself to meet their needs. However, once the company has satisfied the diverse needs of clients asking for its specialized services, it offers the less preferred service for itself such as temporary staffing. Temporary staffing is designed to connect candidates seeking open positions, who post their resumes in a talent bank in the company through the internet, with clients who are reluctant to make a long term staffing commitment. The company charges a fee as a commission for providing the service: The fee is the profit from the sideline.

2. **Consulting company**

   A consulting company with a global reputation mainly focuses its efforts on developing professional advisory services along with miscellaneous services such as market research, commercial surveys, and so on. The first priority of the consulting company with a full foresight of its production capacity is to help its clients improve their performance and thereby bring the company a good amount of profits. Accordingly, it is more likely to aim to provide professional advisory services than miscellaneous services stated above. However, when the company has filled the needs of clients asking for professional advisory services, if, after that, clients ask for miscellaneous services, the company, accepting their requests, will be willing to make a relatively small amount of profit by providing the servies.

3. **Design office**

   Consider a design office providing its clients with strategic design services such as a logo design, an advertising design, brand identity, and so on. Sometimes some clients ask the office to make leaflets and handbills. However, when the company has requests from its clients for strategic design services which are important in order for it to survive in a competitive environment, it will hardly accept a client's request to make leaflets or handbills which would not be so profitable. However, when all the orders accepted so far are completed and delivered, the company will be willing to spend its time and resources in serving its clients by making leaflets and handbills.

These companies usually conduct a variety of business activities to get orders, incurring some costs, called the *search cost.* Business activities include, for example, the advertisement, the making of materials for presentation, the demonstration of sample products, and so on.

1.3 Literature Review

Admission and pricing control problems have been widely investigated to improve the performance of queueing systems in the telecommunications and manufacturing industries. Crabill et al. [4, (1977)] and Stidham and Weber [47, (1993)] give a comprehensive review of the research conducted on the dynamic control of queues, and provide an extensive list of the literature. The topics can be classified into two main categories: the arrival process control and service process control. It is only the arrival control process that both the admission control and pricing control problems in our work relate. From both the reviews of research above and the literature having appeared thereafter, we have taken only that literature concerning our study and clarify the difference between our studies and conventional ones.

1. **Admission Control Problem**

The optimal policies of the admission control problem were originally considered in [11, Heyman (1968)]. Many models of queueing systems have been proposed and examined to explain different cases such as a general arrival process, a multiple server system, or multiple customer classes: [34, Miller (1969)], [21, Kushner and Tetzlaff (1971)], [29, Lippman and Stidham (1971)], [12, Ikuta (1971)], [39, Rue and Rosenshine (1981)], and [45, Stidham (1985)]. While these papers provide the optimal decision rules based on information of the status of queues, Kuri and Kumar [20, (1995)] and Lin and Ross [25, (2003)] [26, (2004)] consider the problem where the number of customers in the system is not always known.

In the above literature the vacation of the server is not considered, contrary to this, much research on queueing systems with vacations of the server has been done. The optimal threshold policy in queues with removable service station was originally considered by Yadin and Naor [49, (1963)], and this was later applied to a variety of versions by Levy and Yechiali [22, (1975)], Doshi [6, (1986)], Kella [17, (1989)] [18, (1990)], and Altman and Nain [1, (1996)]. In the above works, it is assumed that the system/server is turned off as soon as the system gets idle and that fixed costs are incurred whenever the system/server is turned on or turned off. Of particular interest in these papers is the threshold policy that turns the system/server on only when the number of customers in the system is larger than or equal to a given value and turns it off when the system is idle.

2. **Pricing Control Problem**

Low [30, (1974)] [31, (1974)] considers the dynamic pricing policies for $M/M/s$ queue and shows that the optimal price is nondecreasing in the number of customers in the system. Lippman [27, (1975)] extends Low's monotonicity result to the finite and infinite horizon discounted problem. Mendelson [32, (1985)] treats service capacity as a long term decision variable and gives a methodology to find optimal pricing and did capacity decisions. In his model, the delay cost is assumed to be linear. Dewan and Mendelson [5, (1990)] extends Mendelson's model by assuming a nonlinear delay cost function. Mendelson and Whang [33, (1990)] consider $M/M/1$ with multiple customer classes where each class has a different delay cost and an expected service time. Stidham [46, (1992)] studies a slightly different version of Dewan and Mendelson' model by putting an upper bound on the arrival rate. Johansen [15, (1994)] studies optimal pricing in $M/G/1$ where the price charged depends on the sum of the remaining processing times of the jobs in the system. Paschalidis and Tsitsiklis [37, (2000)] studies pricing policies within a network service provider context. Zia et al. [53, (2002)] considers optimal pricing for a service facility and investigates relationships between the optimal price and the reservation price.

Now, we should note that pricing problems have been studies thus far in various models of operations research which are not directly related to the problems stated above. However, we see that the concepts and methodology used there are applicable to our models. The typical examples include [51, You (1998)], [9, Feng and Xiao(2000)], [52, Zhao and Zheng (2000)], and [48, Talluri and van Ryzin (2004)], and see [8, Elmaghraby and Keskinocak (2003)] for an extensive survey of the literature as to the pricing problem.

3. **Both Problems**

Noar [35, (1969)] considers an $M/M/1$ queue where a decision-maker decides whether or not to accept an arriving customer into the system based on the length of the queue and where tolls are imposed on an arriving customer, according to which customers are not accepted if the number of customers in the system is less than or equal to a given number. Lippman and Ross [28, (1971)] and Knudsen [19, (1972)] extend Noar's model and results. In Yoon and Lewis's [50, (2004)], the most recent work, they investigate the monotonicity of optimal policies in stationary queues and extend this result to the non-stationary problem.

1.4 Our Contributions

In the previous section we have reviewed the literature related to our studies. Although all of the previous works mentioned there have some similarities to our work, our contributions can be summarized in the following three points.

1. Many researches cited in the previous section deal with the problems where the system is turned off as soon as it gets idle. In this thesis, however, we assume that the system is never turned off even though all the orders accepted so far are completed, and thereby generates profit from offering services as a sideline during this period. As mentioned in Section 1.2, in the companies offering services as a sideline, when there are orders of specialized services to handle, employees (human resources) will be dispatched or assigned to them. Further, when all the orders accepted so far are completed and delivered, the employees will take part in providing subsidiary services as a sideline, (i.e., temporary staffing, market research, making of leaflets, and so on) which can be completed within one period*. However, when the profit obtained from subsidiary services as a sideline is larger than that from specialized services during the period, the company will naturally place higher priority on engaging in the former than the latter. Here, a problem arises in determining the level of profit from the sideline such that if the profit from the sideline is higher than the level, the company should give a sideline the topmost priority. In this thesis we succeeded in answering this problem through the conclusion that the optimal decision rule has a unimodal property in the number of back orders. It is not verified in any of conventional models without the profit from a sideline that the optimal decision rule has unimodality. In this thesis we clarified that if the profit from the sideline is neither sufficiently large nor sufficiently small, the unimodality of the optimal selection criterion plays a role in placing higher priority on the sideline. The managerial implications of a unimodal property will be discussed in detail in Chaps. 6 and 7.

2. As stated in Section 1.2, from the practical viewpoint that some costs must be paid for the company to be able to find orders; the introduction of the search cost is an inevitable requirement. The search cost has been introduced in almost all conventional models of optimal stopping problems [41, Sakaguchi (1961)] [14, Ikuta (2004)] [16, Kang (1999)] [40, Saito (1998)] [51, You (1998)] but not in those of customer selection problems. Further, it should be noted that the introduction of the search cost eventually yields the option as to whether or not to conduct the search. However, thus far this new option has not been taken into consideration in the models of the customer selection problems. The

*All the models in this thesis are defined as a discrete-time stochastic decision process, in which a period denotes the time interval between successive points in time.

decision on whether or not to enact the search may be influenced not only by the search cost but also the profit from a sideline. In this thesis we clarify that if the search cost or the profit from the sideline is less than a given value, it is optimal to conduct the search for orders, or else to skip the search. These properties are examined in every model proposed in the thesis.

3. Models of the admission control problem and the pricing control problem have been separately formulated and examined so far. In this thesis we show that both problems can be discussed in an identical framework. It will be proven later that this result comes from the fact that the properties of underlying function for both problems are identical (See Section 3.2). More detail discussions are shown in Section 3.3.

1.5 Structure of Thesis

The thesis consists of the nine chapters including this chapter. In Chapter 2 we state assumptions common to all the models, sketch out each of the five models, called Models I to V, examined in the thesis, and provide some symbols and notations used in the chapters that follow. In Chapter 3 we define some functions, called the underlying function, and examine their properties. These functions play an important role in analyzing the properties of the optimal decision rule of every model. In Chapters 4 to 8 we examine Models I to V and clarify the properties of the optimal decision rule in the model of each chapter, respectively. In Chapter 9, the final chapter, we summarize overall conclusions obtained throughout the thesis and state the subjects of study to be tackled in the future.

Chapter 2

Definition of Models

In this chapter we state the assumptions and the distribution function commonly used in all models, provide the strict definition of the five models dealt with in Chapters 4 to 8, and define some notations and symbols which will be used in the subsequent chapters.

Contents

2.1 Common Assumptions

The models examined in the thesis are defined on the following eight common assumptions. Assumptions that are further added in each model are stated in the definition of the model.

A1. Each model is defined as a discrete-time stochastic decision process with an infinite planning horizon. Let points in time be equally spaced on the axis of the planning horizon, and let the time interval between successive points in time be called the period.

A2. It is only when a search is conducted by paying a *search cost* $s \geq 0$ at a point in time that a customer arrives at the next point in time. Here a customer is assumed to arrive with a probability λ $(0 < \lambda \leq 1)$ if the search is enacted. The introduction of the search cost inevitably yields the option whether to *skip* the search or not.

A3. By $N \geq 1$ let us denote the maximum permissible number of orders that can be held in the company at any instance.

A4. In the admission control problem, prices offered by subsequently appearing customers, $w, w', \cdots$, are assumed to be independent identically distributed random variables having a known distribution function $F(w)$ with a finite expectation μ. See Section 2.2 for a strict definition of $F(w)$.

A5. In the pricing problem, the maximum permissible ordering prices of subsequently appearing customers, $w, w', \cdots$, are assumed to be independent identically distributed random variables having a known distribution function $F(w)$ with a finite expectation μ. When the price z is offered by the company, if and only if $z \leq w$, the customer is willing to place the order with the company. Accordingly, the probability of the customer placing the order with the company is given by

$$p(z) = \Pr\{z \leq w\}.$$

See Section 2.2 for a strict definition of $F(w)$.

A6. With the probability q $(0 < q < 1)$ an order held in the company at a certain point in time is completed up to the next point in time.

A7. When there exist no orders of specialized service in the system, the company provides the subsidiary services as a sideline, and thereby yields the *profit from a sideline* $r \geq 0$. Let the subsidiary services as a sideline be completed within one period.

A8. Let the discount factor be denoted by β; that is, a monetary value of one unit a period hence is equivalent to that of β units at the present point in time; let $\beta < 1$ throughout the thesis.

The decisions on the customer selection problems are based on the following three rules:

D1. The rule whether or not to accept an order from arriving customers in the admission control problem.

D2. The rule as to the price to propose in the pricing control problem.

D3. The rule whether to continue or to skip the search in both problems.

The objective is to find the optimal decision rule so as to maximize the total expected present discounted net profit gained over an infinite planning horizon, the total expected present discounted value of prices of orders accepted or placed *plus* the profits from a sideline *minus* the total expected present discounted value of search costs.

2.2 Distribution Function

Let distribution function $F(w)$ stated in the assumptions A4 and A5 of Section 2.1 be defined as follows. In general, random variables are represented by capital letters X, Y, ... , and their realized values by corresponding small letters x, y, ... In this thesis, however, for convenience of husbandry of symbols used as well as for avoidance of complication in descriptions, let us employ the same small letters for both, in cases where there is no possibility of confusion. Let $f(w)$ denote the probability (density) function of $F(w)$, which is truncated on both sides as seen in Figure 2.2.1 where $0 < a < b < \infty$; more precisely, let it be defined as follows.

1. If $F(w)$ is continuous, for certain given real numbers a and b let

$$F(w) = 0, \quad w \leq a, \qquad 0 < F(w) < 1, \quad a < w < b, \qquad F(w) = 1, \quad b \leq w, \tag{2.2.1}$$

where

$$f(w) = 0, \quad w < a, \qquad f(w) > 0, \quad a \leq w \leq b, \qquad f(w) = 0, \quad b < w. \tag{2.2.2}$$

Then $F(w)$ is strictly increasing on $[a, b]$.

2. If $F(w)$ is discrete, for certain given integers a and b let

$$F(w) = 0, \quad w < a, \qquad 0 < F(w) < 1, \quad a \leq w < b, \qquad F(w) = 1, \quad b \leq w, \tag{2.2.3}$$

where its mass points w are integers, i.e., $|w| = 0, 1, \cdots$ and where

$$f(w) = 0, \quad w < a, \qquad f(w) > 0, \quad a \leq w \leq b, \qquad f(w) = 0, \quad b < w. \tag{2.2.4}$$

Then $F(w)$ is strictly increasing on $\{a-1, a, \ldots, b\}$.

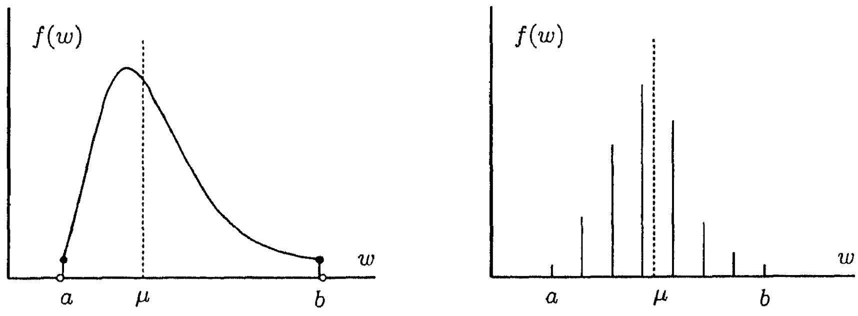

Figure 2.2.1: Probability (density) function $f(w)$.

Now, in the pricing control problem it can be easily seen that

$$p(z)\begin{cases} =1, & z \le a \quad \cdots(1), \\ <1, & a < z \quad \cdots(2), \end{cases} \qquad p(z)\begin{cases} >0, & z < b \quad \cdots(3), & z \le b, \cdots(3') \\ =0, & b \le z \quad \cdots(4), & b < z. \cdots(4'). \end{cases} \tag{2.2.5}$$

(In the second case, the first column is for continuous $F(w)$ and the second for discrete $F(w)$.)

Throughout the thesis, let us denote the expectation of a given function $g(w)$ as to w by $\mathbf{E}[g(w)]$.

2.3 Definition of Models

In this thesis the following five models are proposed and examined.

1. **Model I: Stochastic model with penalty cost** ($N = 1$) (Chapter 4)

   In this model we assume that only one customer is allowed to be held in the company, namely, $N = 1$, and that any accepted order must be delivered up to τ periods from when it was accepted. For an accepted order that cannot be delivered within τ periods, a penalty $\theta > 0$ must be paid for one period delayed.

2. **Model II: Stochastic model with cancellation** ($N = 1$) (Chapter 5)

   This model is the same as Model I except that an order undergoing processing may be canceled due to unavoidable circumstances with customer. With known probabilities ν and $\bar{\nu}$ let a customer cancel the contract, respectively, within τ periods and after more than τ periods. If a customer cancels the contract within the contracted period τ or after more than τ periods, he must pay a penalty, respectively, $\vartheta > 0$ or $\bar{\vartheta} > 0$ to the company.

3. **Model III: Stochastic model with multiple customers being allowed to be held** ($N \ge 2$) (Chapter 6)

   This model is the same as Model I except that (1) multiple customers can be held in the company, $N \ge 2$, and (2) the penalty $\theta = 0$. In this model it is allowed to have back orders, so that a new problem arises in clarifying the relationship of the optimal selection rule and the optimal pricing rule with the number of orders, denoted by i, in the company.

4. **Model IV: Deterministic model** (Chapter 7)

   While the completion of the order is stochastic in the above three models, in this model it is assumed that every accepted orders requires the fixed production periods d (deterministic) and that any order completed must be delivered to the customer within τ periods where $\tau \geq d$.

5. **Model V: Stochastic model with multiple production lines** (Chapter 8)
The model is the same as Model III except that the company holds multiple production lines. By $n \geq 2$ let us denote the number of production lines available in the company.

Above we defined five models of customer selection problem examined in the thesis. Table 2.3.1 below shows a map of the five models and those to be tackled in the future, represented X.

Table 2.3.1: Models in the thesis

	stochastic model		deterministic model
	$N=1$	$N \geq 2$	
no penalty / no cancellation	Model I	Model III	Model IV
penalty / no cancellation	Model I	X	X
penalty / cancellation	Model II	X	X
i-different arrival probability	nonexistence	Model III	X
multiple production lines	nonexistence	Model V	X

* X represents future studies to be tackled.

2.4 Notations

The following two expressions are commonly used throughout the thesis.

$$\eta = (1-q)\beta < 1, \tag{2.4.1}$$

$$\gamma = (1-\eta)^{-1} > 1, \tag{2.4.2}$$

where it can be easily seen that

$$1 - \gamma q \beta = \gamma(1-\beta) > 0. \tag{2.4.3}$$

Further, for explanatory convenience let us define the symbols as to decisions as in Table 2.4.2.

Table 2.4.2: Symbols as to decisions.

<table>
<tr><th>Symbol</th><th>Decision</th><th>Symbol</th><th>Decision</th><th>Symbol</th><th>Decision</th></tr>
<tr><td>C</td><td>continuing the search</td><td>⟨C⟩</td><td rowspan="4">Each corresponding decision is optimal</td><td rowspan="2">⟨O(z)⟩</td><td rowspan="2">It is optimal to offer the price z for an order in pricing control</td></tr>
<tr><td>K</td><td>skipping the search</td><td>⟨K⟩</td></tr>
<tr><td>A</td><td>accepting an order</td><td>⟨A⟩</td><td>⟨A(w)⟩</td><td>It is optimal to accept an appearing order w in admission control</td></tr>
<tr><td>R</td><td>rejecting an order</td><td>⟨R⟩</td><td>⟨R(w)⟩</td><td>It is optimal to reject an appearing order w in admission control</td></tr>
</table>

*We do not use S as a symbol representing "skipping the search" because it is often used for representing "stopping the search"

Chapter 3

Underlying Functions

In this chapter we define some functions, called the *underlying function*, and examine their properties. These functions play an important role in analyzing the properties of the optimal decision rule of every model.

Contents

3.1 Definitions

For any real number x let us define the following functions:

$$T(x) = \begin{cases} \mathbf{E}[\max\{w - x, 0\}] & \text{for the admission control problem,} \\ \max_z p(z)(z - x) & \text{for the pricing control problem,} \end{cases} \tag{3.1.1}$$

called the *T-function* [14, Ikuta] [51, You]. Here note that $T(0) > 0$ (See [14]).

In the pricing control problem, for a given x by $z(x)$ let us designate the z attaining the maximum of $p(z)(z-x)$ on $(-\infty, \infty)$ if it exists, i.e.,

$$T(x) = p(z(x))(z(x) - x).$$

Further, define

$$L(x) = \lambda\beta T(x) - s, \tag{3.1.2}$$

$$\alpha = \lambda\beta T(0) - s. \tag{3.1.3}$$

When regarding x as a function of the profit from a sideline r, i.e., $x = x(r)$, by r^* let us denote the solution of $L(x(r)) = 0$, if it exists, i.e.,

$$L(x(r^*)) = 0. \tag{3.1.4}$$

If multiple solutions exist in each of the above two equations, let us define the *smallest* of them as r^*. Finally, let us define

$$M(x) = L(x) + \beta x.$$

3.2 Properties

Lemma 3.2.1

(a) *$z(x)$ exists with $z(x) \geq a$.*

(b) *$a \leq z(x) \leq b$ for all x where $z(x) = b$ if $x \geq b$.*

(c) *If $x < b$, then $x < z(x) < b$.*

(d) *$z(x)$ is nondecreasing in x.*

(e) *There exists a finite $x^\star < a$ such that if $x < (>)\, x^\star$, then $z(x) = (>)\, a$.*

Proof. See [14, Ikuta] [51, You]† . ∎

Note. It is not yet proven in [14] which of $z(x^\star) > a$ or $z(x^\star) = a$ is true. If $F(w)$ is a uniform distribution on $[a, b]$ with $0 < a < b$, then $x^\star = 2a - b$ (See Section 3.4).

Lemma 3.2.2 *In both admission control and pricing control problems we have:*

(a) *$T(x)$ is continuous and nonincreasing on $(-\infty, \infty)$, strictly decreasing on $(-\infty, b]$, and convex on $(-\infty, \infty)$.*

(b) *$T(x) \geq 0$ on $(-\infty, \infty)$.*

(c) *$T(x) > 0$ on $(-\infty, b)$, and $T(x) = 0$ on $[b, \infty)$.*

(d) *If $T(x) = (>)\, 0$, then $b \leq (>)\, x$.*

(e) *For any given $y > 0$ the equation $T(x) = y$ has a unique solution, less than b.*

(f) *$\lim_{x\to\infty} T(x) = 0$ and $\lim_{x\to-\infty} T(x) = \infty$.*

(g) *$\varrho T(x) + x$ is nondecreasing in x for $0 \leq \varrho \leq 1$ and strictly increasing in x for $0 \leq \varrho < 1$.*

(h) *$\lim_{x\to\infty} \varrho T(x) + x = \infty$ for $0 \leq \varrho \leq 1$ and $\lim_{x\to-\infty} \varrho T(x) + x = -\infty$ for $0 \leq \varrho < 1$.*

Proof. See [14, Ikuta] [51, You]. ∎

Lemma 3.2.3

(a) *$L(x)$ is nonincreasing on $(-\infty, \infty)$, strictly decreasing on $(-\infty, b)$, and convex on $(-\infty, \infty)$.*

(b) *$L(x) \geq -s$ on $(-\infty, \infty)$.*

(c) *$L(x) \leq \lambda\beta T(0) - s$ on $[0, \infty)$, $L(x) > -s$ on $(-\infty, b)$, and $L(x) = -s$ on $[b, \infty)$.*

(d) *$\lim_{x\to\infty} L(x) = -s$ and $\lim_{x\to-\infty} L(x) = \infty$.*

(e) *If $L(x) > 0$, then $x < b$.*

(f) *If $L(x) = L(y) > 0$, then $x = y$, and if $L(x) > L(y)$, then $x < y$.*

Proof. See [14, Ikuta]. ∎

Lemma 3.2.4 *$M(x)$ is nondecreasing in x and $M(b) = \beta b - s$.*

Proof. $M(x) = L(x) + \beta x = \beta(\lambda T(x) + x) - s$, which is nondecreasing in x due to Lemma 3.2.2(g). Further, for $x \geq b$ we get $M(x) = \beta x - s$ due to Lemma 3.2.3(c), hence $M(b) = \beta b - s$. ∎

† As [14] includes the contents of [51], readers should refer to [14]

3.3 Role of T-function

In Eq. (3.1.1) we defined the two T-functions with the same function name, $T(x)$, which are used in the analyses of the admission control problem and the pricing control problem, respectively, and by using these two T-functions we showed that the two different problems have basically the same form of optimal equations. Noting this result and the fact that the two T-functions have identical properties as seen in Lemma 3.2.2, we succeeded in analyzing both problems in an identical framework. However, discussions as to the optimal prices, which are not seen in the admission control problem, are added to the pricing control problem (see Lemma 3.2.1). Fortunately this point can be discussed outside the identical framework.

3.4 Examples

Let $F(w)$ be a uniform distribution function on $[a, b]$ where $\mu = 0.5(a + b)$.

1. **Admission control problem:** We have

$$T(x) = \begin{cases} \mu - x & = 0.5(a+b) - x \quad \text{for} \quad x < a, \\ \displaystyle\int_x^b (w-x)/(b-a)dw & = 0.5(b-x)^2/(b-a) \quad \text{for} \quad a \le x < b, \\ 0 & = 0 \quad \text{for} \quad b \le x \end{cases} \tag{3.4.1}$$

   where $T(0) = 0.5(a + b)$.

2. **Pricing control problem:** It is evident from Eq. (2.2.5) that $p(z) = 1$ for $z \le a$, $p(z) = (b - z)/(b - a)$ for $a \le z \le b$, and $p(z) = 0$ for $b \le z$. For convenience, let $g(z, x) = p(z)(z - x)$, hence $T(x) = \max_z g(z, x)$. Further, for any real numbers z and x let us define $y(z, x) = (b-z)(z-x)/(b-a)$, and by $z^*(x)$ let us denote the z attaining the maximum of $y(z, x)$ for any given $x \in (-\infty, \infty)$. Then $g(z, x)$ can be expressed as follows.

$$g(z,x) = \begin{cases} z - x & \text{for} \quad z < a, \\ y(z,x) & \text{for} \quad a \le z < b, \\ 0 & \text{for} \quad b \le z. \end{cases} \tag{3.4.2}$$

   Here, note that $g(z, x)$ is maximized at $z = z(x)$ by the definition where $a \le z(x) \le b$ from Lemma 3.2.1(b). Since $\partial y(z, x)/\partial z = (-2z + x + b)/(b - a)$, we obtain $z^*(x) = (x + b)/2$. Then $g(z, x)$ can be depicted as any one of the three graphs in Figure 3.4.1, depending on a value that $z^*(x)$ takes on.

   From the figure it is immediately seen that

   i. if $z^*(x) < a$, hence $x < 2a - b$, then $z(x) = a$, so $T(x) = p(a)(a - x) = a - x$,

   ii. if $a \le z^*(x) < b$, hence $2a-b \le x < b$, then $z(x) = (x+b)/2$, so $T(x) = p((x+b)/2)((x+b)/2-x) = 0.25(b - x)^2/(b - a)$,

   iii. if $z^*(x) \ge b$, hence $x \ge b$, then $z(x) = b$, so $T(x) = p(b)(b - x) = 0$.

   The above can be summarized as follows.

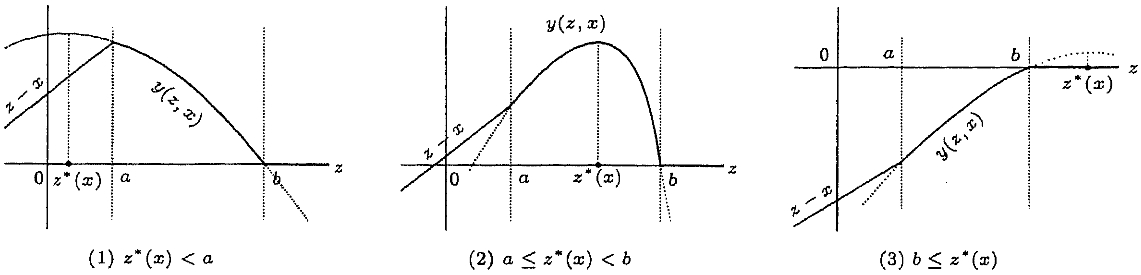

Figure 3.4.1: Graphs of $g(z, x)$.

$$T(x) = \begin{cases} a - x & \text{for} \quad x < 2a - b \quad \rightarrow \quad z(x) = a, \\ 0.25(b-x)^2/(b-a) & \text{for} \quad 2a - b \leq x < b \quad \rightarrow \quad z(x) = (x+b)/2, \\ 0 & \text{for} \quad b \leq x \quad \rightarrow \quad z(x) = b, \end{cases} \tag{3.4.3}$$

from which we obtain $x^\star = 2a - b$ (See Lemma 3.2.1(e)). Accordingly, if $x^\star = 2a - b > 0$, then $T(0) = a$, or else $T(0) = 0.25\, b^2/(b-a)$.

Chapter 4

Model I: Stochastic model with penalty ($N = 1$)

In this chapter we define and examine a model where only one customer is allowed to be held in the system and where any accepted order must be delivered up to τ periods from when it was accepted. For an accepted order that cannot be delivered within τ periods, a penalty $\theta > 0$ must be paid for one period delayed.

Contents

4.1 System of Optimal Equations

Either if the search was skipped at the previous point in time or if no customer has appeared with probability $1 - \lambda$ regardless of having conducted the search at the previous point in time, it follows that no customer appears at the present point in time. For convenience, we shall refer to such a situation as "the company has a *fictitious customer* ϕ", called the state (ϕ).

1. In both the admission control problem and pricing control problem, by $u(\phi)$ and $u(\phi, l)$ we shall denote the maximum total expected present discounted net profit starting from state (ϕ), provided, respectively, that there exists no order in the company and that there exists an order accepted l periods ago in the company. Suppose the company is in state (ϕ, l) at a certain point in time. If $l = \tau$, the contract is not yet violated although the order has not yet been completed as of the latest date signed, hence no penalty is needed to be paid at that time; however, since it is at the latest $\tau + 1$ periods after that the order is completed, the penalty θ must be paid at the next point in time. In general, if $l \geq \tau$, the penalty θ must be paid for one period delayed.

2. In the admission control problem, by $u(w)$ let us denote the maximum total expected present discounted net profit starting with no order in the company and an arriving customer w, who offers a price w for his order.

3. In the pricing control problem, by $u(1)$ let us denote the maximum total expected present discounted net profit starting with no order in the company and an arriving customer, to whom the company proposes a price z for an order.

Since the expectation of immediate reward at any point in time is clearly finite, using the conventional way outlined in the discussion of the Markovian decision process [38, Ross](p29-30), we can easily show that $|u(\phi)| \leq M/(1-\beta)$ and $|u(\phi,l)| \leq M/(1-\beta)$, $l \geq 0$, for a sufficiently large $M > 0$, i.e., $u(\phi)$ is finite and $u(\phi,l)$ is bounded in $l \geq 0$. For convenience in the discussions that follow, let us define

$$h = u(\phi) - u(\phi,0). \tag{4.1.1}$$

Then the optimal equations for both problems can be described as follows.

1. **Admission control problem:**

$$u(\phi) = \max\left\{\begin{array}{l} \mathrm{C}: \beta\big(\lambda\,\mathbf{E}[u(\xi)] + (1-\lambda)u(\phi)\big) - s + r, \\ \mathrm{K}: \beta u(\phi) + r, \end{array}\right\} \tag{4.1.2}$$

$$u(\phi,l) = \max\left\{\begin{array}{l} \mathrm{C}: (1-q)\beta\big(u(\phi,l+1) - \theta I(l \geq \tau)^{\ddagger}\big) \\ \qquad + q\beta\big(\lambda\,\mathbf{E}[u(\xi)] + (1-\lambda)u(\phi) - \theta I(l \geq \tau)\big) - s, \\ \mathrm{K}: (1-q)\beta\big(u(\phi,l+1) - \theta I(l \geq \tau)\big) + q\beta\big(u(\phi) - \theta I(l \geq \tau)\big), \end{array}\right\}, \; l \geq 0, \tag{4.1.3}$$

$$u(w) = \max\left\{\begin{array}{l} \mathrm{A}: w + u(\phi,0) \\ \mathrm{R}: u(\phi) \end{array}\right\}. \tag{4.1.4}$$

The last equation can be rearranged as

$$u(w) = \max\{w - h, 0\} + u(\phi). \qquad \square \tag{4.1.5}$$

2. **Pricing control problem:**

$$u(\phi) = \max\left\{\begin{array}{l} \mathrm{C}: \beta\big(\lambda u(1) + (1-\lambda)u(\phi)\big) - s + r, \\ \mathrm{K}: \beta u(\phi) + r, \end{array}\right. \tag{4.1.6}$$

$$u(\phi,l) = \max\left\{\begin{array}{l} \mathrm{C}: (1-q)\beta\big(u(\phi,l+1) - \theta I(l \geq \tau)\big) \\ \qquad + q\beta\big(\lambda u(1) + (1-\lambda)u(\phi) - \theta I(l \geq \tau)\big) - s, \\ \mathrm{K}: (1-q)\beta\big(u(\phi,l+1) - \theta I(l \geq \tau)\big) + q\beta\big(u(\phi) - \theta I(l \geq \tau)\big), \end{array}\right\}, \; l \geq 0, \tag{4.1.7}$$

$$u(1) = \max_z\{p(z)\big(z + u(\phi,0)\big) + (1-p(z))u(\phi)\}. \tag{4.1.8}$$

The last equation can be rearranged as

$$u(1) = \max_z p(z)(z-h) + u(\phi). \quad \square \tag{4.1.9}$$

See Lemma 4.2.1 for the proof of the unique existence of the solutions for the above system of equations.

4.2 Transformation

Let us define

‡ $I(\cdot)$ denotes the indicator function. For the given statement S if S is true, then $I(\mathrm{S}) = 1$, or else $I(\mathrm{S}) = 0$.

$$\dot{L}(h) = \lambda q\beta T(h) - s. \tag{4.2.1}$$

When regarding h as a function of r, i.e., $h = h(r)$, by $\dot{r}^*$ let us denote the solutions of $\dot{L}(h(r)) = 0$, if it exists, i.e.,

$$\dot{L}(h(\dot{r}^*)) = 0. \tag{4.2.2}$$

If multiple solutions exist in the above equation, let us define the *smallest* of them as $\dot{r}^*$. Further, let us define

$$v(0) = \begin{cases} \mathbf{E}[u(w)] & \text{for the admission control problem,} \\ u(1) & \text{for the pricing control problem.} \end{cases} \tag{4.2.3}$$

Then using Eq. (3.1.1), we can express both "Eqs. (4.1.2), (4.1.3), and (4.1.5)" and "Eqs. (4.1.6), (4.1.7), and (4.1.9)" by the identical equations below.

$$u(\phi) = \max\{\lambda\beta v(0) + (1-\lambda)\beta u(\phi) - s, \beta u(\phi)\} + r, \tag{4.2.4}$$

$$u(\phi, l) = \max\left\{ \begin{array}{l} (1-q)\beta u(\phi, l+1) + q\beta\big(\lambda v(0) + (1-\lambda)u(\phi)\big) - \beta\theta I(\tau \le l) - s \\ (1-q)\beta u(\phi, l+1) + q\beta u(\phi) - \beta\theta I(\tau \le l) \end{array} \right\}, \quad l \ge 0, \tag{4.2.5}$$

$$v(0) = T(h) + u(\phi). \tag{4.2.6}$$

Further, Eqs. (4.2.4) and (4.2.5) can be rearranged into, respectively,

$$u(\phi) = \beta u(\phi) + \max\{\lambda\beta\big(v(0) - u(\phi)\big) - s, 0\} + r, \tag{4.2.7}$$

$$u(\phi, l) = (1-q)\beta u(\phi, l+1) + q\beta u(\phi) + \max\{\lambda q\beta\big(v(0) - u(\phi)\big) - s, 0\} - \beta\theta I(\tau \le l), \quad l \ge 0. \tag{4.2.8}$$

Using Eq. (4.2.6), we can rearrange Eqs. (4.2.7) and (4.2.8) as follows, respectively,

$$u(\phi) = (\max\{L(h), 0\} + r)/(1-\beta) \ge 0, \tag{4.2.9}$$

$$u(\phi, l) = (1-q)\beta u(\phi, l+1) + q\beta u(\phi) + \max\{\dot{L}(h), 0\} - \beta\theta I(\tau \le l), \quad l \ge 0. \tag{4.2.10}$$

Lemma 4.2.1 *The system of* Eqs. (4.2.4) *to* (4.2.6) *has a unique solution.*

Proof. See App. A.1. ∎

Lemma 4.2.2 Eq. (4.2.10) *with* $l = 0$ *can be rewritten*

$$u(\phi, 0) = \gamma q\beta u(\phi) + \gamma \max\{\dot{L}(h), 0\} - \rho. \tag{4.2.11}$$

Proof. See App. A.2. ∎

Now, Eqs. (4.1.5) and (4.1.9) and Eqs. (4.2.9) and (4.2.10) tell us that the optimal decision rules can be prescribed as follows.

☐ *Optimal Decision Rule 4.2.1*

1. Admission control problem:
   i. If $L(h) > 0$ $(\dot{L}(h) > 0)$, then $\langle \mathtt{C}\rangle_\phi$ $(\langle \mathtt{C}\rangle_l)^\dagger$, or else $\langle \mathtt{K}\rangle_\phi$ $(\langle \mathtt{K}\rangle_l)$.
   ii. Let an order with value w appear after the search was enacted. Then if $w > h$, then $\langle \mathtt{A}(w)\rangle$, or else $\langle \mathtt{R}(w)\rangle$.
2. Pricing control problem:
   i. If $L(h) > 0$ $(\dot{L}(h) > 0)$, then $\langle \mathtt{C}\rangle_\phi$ $(\langle \mathtt{C}\rangle_l)$, or else $\langle \mathtt{K}\rangle_\phi$ $(\langle \mathtt{K}\rangle_l)$.
   ii. Let a customer appear after the search was enacted. Then $\langle \mathtt{O}(z(h))\rangle$.

Here note that the optimal decision rule is independent of l.

4.3 Analysis

In this section we shall prove many assertions related to r. Then it is often that h, $G(x)$, x_1^*, and x_2^* must be regarded as a function of r, i.e., $h = h(r)$, $G(x, r)$, $x_1^* = x_1^*(r)$, and $x_2^* = x_2^*(r)$. However, for explanatory simplicity let us employ the notations "h, $G(x)$, x_1^*, and x_2^*" in Lemmas 4.3.3 to 4.3.6 except Lemma 4.3.6(d) and "$h(r)$, $G(x,r)$, $x_1^*(r)$, and $x_2^*(r)$" in their proofs. For convenience in the later discussions let us define

$$\rho = \gamma\eta^\tau\beta\theta \geq 0, \tag{4.3.1}$$

$$\chi = s\gamma(1-q)/q + \gamma r + \rho. \tag{4.3.2}$$

For any real number x let us define the following three functions, which becomes necessary in the analyses of Section 4.3.

$$G(x) = \gamma\big(\max\{L(x), 0\} - \max\{\dot{L}(x), 0\}\big) - x + \gamma r + \rho, \tag{4.3.3}$$

$$B_1(x) = T(x) - (x - \gamma r - \rho)/\gamma\lambda(1-q)\beta, \tag{4.3.4}$$

$$B_2(x) = T(x) - (x + \gamma(s-r) - \rho)/\gamma\lambda\beta. \tag{4.3.5}$$

By x^*, x_1^*, and x_2^* let us denote the solution of, respectively, $G(x) = 0$, $B_1(x) = 0$, and $B_2(x) = 0$, if they exist, i.e.,

$$G(x^*) = 0, \qquad B_1(x_1^*) = 0, \qquad B_2(x_2^*) = 0. \tag{4.3.6}$$

Further, by $x_1^*(0)$ and $x_2^*(0)$ let us denote, respectively, x_1^* and x_2^* for $r = 0$.

Lemma 4.3.1

(a) *$G(x)$ is continuous and strictly decreasing in x.*

---

†The notation $\langle \mathtt{C}\rangle_\phi$ implies that conducting the search is optimal in state (ϕ), and $\langle \mathtt{K}\rangle_l$ implies that skipping the search is optimal in state (ϕ, l) for $l \geq 0$.

(b) $G(x) < (>) 0$ *for any sufficiently large* $x > 0$ ($x < 0$).

(c) $G(x)$ *is nonincreasing in* s *and strictly decreasing in* q *and* τ *for all* x.

(d) $G(x)$ *is nondecreasing in* λ *and strictly increasing in* β *and* θ *for all* x.

(e) $G(x)$ *is strictly increasing in* r *for all* x.

Proof. See App. A.3. ∎

Lemma 4.3.2

(a) $B_1(x)$ *and* $B_2(x)$ *are both strictly decreasing in* x *where* $B_1(x) > (<) 0$ *and* $B_2(x) > (<) 0$ *for any sufficiently small (large)* x.

(b) x_1^* *and* x_2^* *are both uniquely exist, which are positive if* $\lambda\beta T(\rho) > s$.

(c) $\chi > (= (<))\, x \Leftrightarrow B_1(x) > (= (<))\, B_2(x)$ *where* $B_1(\chi) = B_2(\chi)$.

(d) $x_2^* > \chi \Leftrightarrow x_1^* > \chi$ *and* $x_2^* \leq \chi \Leftrightarrow x_1^* \leq \chi$.

Proof. See App. A.4. ∎

Now, noting Eq. (2.4.3), from Eq. (4.1.1) we have

$$h = \gamma(1-\beta)u(\phi) - \gamma \max\{\dot{L}(h), 0\} + \rho. \tag{4.3.7}$$

Rearranging Eq. (4.3.7) by substituting Eq. (4.2.9) yields

$$h = \gamma\big(\max\{L(h), 0\} - \max\{\dot{L}(h), 0\}\big) + \gamma r + \rho, \tag{4.3.8}$$

which can be eventually rewritten

$$G(h) = 0; \tag{4.3.9}$$

in other words, the h, defined by Eq. (4.1.1), is given by the solution of $G(x) = 0$.

Lemma 4.3.3

(a) x^* *uniquely exists with* $x^* = h$, $x^* \geq r$, *and* $x^* \geq \rho$.

(b) *If* $r < (b-\rho)/\gamma$ (*, hence* $b > \rho$ *due to* $r \geq 0$), *then* $h < b$, *or else* $h \geq b$.

(c) h *is nonincreasing in* s *and strictly decreasing in* q *and* τ.

(d) h *is nondecreasing in* λ *and strictly increasing in* β *and* θ.

(e) h *is strictly increasing in* r *with* $\lim_{r\to\infty} h = \infty$ *and* $\lim_{r\to-\infty} h = -\infty$.

Proof. Below, note that $L(x) = \lambda\beta T(x) - s \geq \lambda q\beta T(x) - s = \dot{L}(x)$ for any x because $T(x) \geq 0$ due to Lemma 3.2.2(b).

(a) The unique existence of x^*, independent of r, is immediate from Lemma 4.3.1(a,b), hence $h = x^*$ from Eq. (4.3.9). Since $\gamma > 1$ due to Eq. (2.4.2), we have

$$G(r, r) = \gamma\big(\max\{L(r), 0\} - \max\{\dot{L}(r), 0\}\big) + (\gamma - 1)r + \rho \geq 0,$$

$$G(\rho, r) = \gamma\big(\max\{L(\rho), 0\} - \max\{\dot{L}(\rho), 0\}\big) + \gamma r \geq 0,$$

hence $x^* \geq r$ and $x^* \geq \rho$ due to Lemma 4.3.1(a).

(b) Since $L(b) = \dot{L}(b) = -s$ due to Eq. (3.1.2) and Lemma 3.2.2(c), we obtain $G(b, r) = -b + \gamma r + \rho$. If $r < (b - \rho)/\gamma$, then $G(b, r) < 0$, implying $h < b$, or else $G(b, r) \geq 0$, so $h \geq b$.

(c,d) Evident from Lemma 4.3.1(c,d), respectively.

(e) The former half is immediate from Lemma 4.3.1(e). Suppose $h(r)$ converges to a finite $\bar{h}$ as $r \to \infty$. Then since $h(r) < \bar{h}$ for any r, we have $G(\bar{h}, r) < G(h(r), r) = 0$ due to Lemma 4.3.1(a); accordingly, $\lim_{r\to\infty} G(\bar{h}, r) \leq 0$ due to Lemma 4.3.1(e). However, $\lim_{r\to\infty} G(\bar{h}, r) = \infty$ from Eq. (4.3.3), which is a contradiction. Thus $h(r)$ must diverge as $r \to \infty$. Similarly also proven $\lim_{r\to-\infty} h(r) = -\infty$. ∎

Below, by r_b let us denote the solution of $h(r) = b$ if it exists, i.e., $h(r_b) = b$.

Lemma 4.3.4

(a) *r_b uniquely exists.*

(b) *Both $L(h)$ and $\dot{L}(h)$ are strictly decreasing in $r < r_b$ and nonincreasing in r on $(-\infty, \infty)$.*

(c) *$L(h) > 0$ and $\dot{L}(h) > 0$ for any sufficiently small r.*

(d) *If $s > 0$, then $L(h) < 0$ and $\dot{L}(h) < 0$ for any sufficiently large r.*

(e) *$L(h) > \dot{L}(h)$ for $r < r_b$ and $L(h) = \dot{L}(h) = -s$ for $r \geq r_b$.*

Proof. (a) Immediate from Lemma 4.3.3(e).

(b) For any $r < r' < r_b$ since $h(r) < h(r') < h(r_b) = b$ due to Lemma 4.3.3(e), we get $L(h(r)) > L(h(r'))$ and $\dot{L}(h(r)) > \dot{L}(h(r'))$ from Lemma 3.2.2(a), hence the former half of the assertion holds. From Lemmas 4.3.3(e) and 3.2.2(a) it can be immediately seen that $L(h(r))$ and $\dot{L}(h(r))$ are nonincreasing in r.

(c) Let $s > 0$. Then from Lemmas 4.3.3(e) and 3.2.2(e) we easily see that there exists an $\bar{r}$ such that $T(h(\bar{r})) = s/\lambda\beta > 0$ and $h(\bar{r}) < b$. Now, for $r < \bar{r}$ we have $h(r) < h(\bar{r}) < b$ due to Lemma 4.3.3(e). Accordingly, from Lemma 3.2.2(a) we get $T(h(r)) > T(h(\bar{r})) = s/\lambda\beta$, hence $0 < \lambda\beta T(h(r)) - s = L(h(r))$. Let $s = 0$. Then for $r < r_b$ we obtain $h(r) < h(r_b) = b$, hence $L(h(r)) = \lambda\beta T(h(r)) > \lambda\beta T(b) = 0$ due to Lemma 3.2.2(b,c)). The proof of $\dot{L}(h(r)) > 0$ is also shown in quite the same way as the above.

(d) Let $s > 0$. Then for $r_b \leq r$ we have $b = h(r_b) \leq h(r)$ from Lemma 4.3.3(e), hence $L(h(r)) = \dot{L}(h(r)) = -s < 0$ due to Lemma 3.2.2(c).

(e) For any $r < r_b$ we have $h(r) < h(r_b) = b$ due to Lemma 4.3.3(e), hence $T(h(r)) > 0$ due to Lemma 3.2.2(c); accordingly, $L(h(r)) = \lambda\beta T(h(r)) - s > \lambda q\beta T(h(r)) - s = \dot{L}(h(r))$. For $r_b \leq r$ we get $b = h(r_b) \leq h(r)$, hence $L(h(r)) = \dot{L}(h(r)) = -s$ due to Lemma 3.2.2(c). ∎

From Lemma 4.3.4 we can depict Figure 4.3.1.

Lemma 4.3.5

(a) *Let $s = 0$. Then $L(h) \geq 0$ and $\dot{L}(h) \geq 0$.*

(b) *Let $s > 0$.*

1 *Both r^* and $\dot{r}^*$ uniquely exist with $\dot{r}^* < r^* < r_b$.*

2 *If $r^* \leq r$, then $L(h) \leq 0$ and $\dot{L}(h) \leq 0$.*

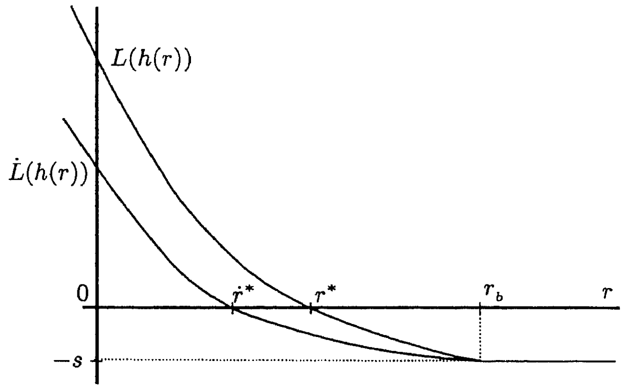

Figure 4.3.1: Graphs of $L(h(r))$ and $\dot{L}(h(r))$ with $s > 0$.

3 *If $r < r^*$, then $L(h) > 0$.*
4 *If $\dot{r}^* \leq r$, then $\dot{L}(h) \leq 0$.*
5 *If $r < \dot{r}^*$, then $L(h) > 0$ and $\dot{L}(h) > 0$.*

Proof. (a) If $s = 0$, then $L(h) = \lambda\beta T(h) \geq 0$ and $\dot{L}(h) = \lambda q\beta T(h) \geq 0$ due to Lemma 3.2.2(b).

(b) Let $s > 0$ (see Figure 4.3.1).

(b1) The unique existence of r^* and $\dot{r}^*$ are immediate from Lemma 4.3.4(b,c,d). The latter half is evident from Lemma 4.3.4(e).

(b2-b5) Evident from Lemma 4.3.4(b,e). ∎

Below, let us regard r^* and $\dot{r}^*$ as functions of s, i.e., $r^*(s)$ and $\dot{r}^*(s)$, and let s^* and $\dot{s}^*$ be the solution of $r^*(s) = 0$ and $\dot{r}^*(s) = 0$, respectively, if they exist, i.e.,

$$r^*(s^*) = 0, \qquad \dot{r}^*(\dot{s}^*) = 0. \tag{4.3.10}$$

Lemma 4.3.6

(a) *Both $r^*(s)$ and $\dot{r}^*(s)$ are strictly decreasing in s.*

(b) *If $s = 0$, then $r^* = \dot{r}^* = r_b = (b - \rho)/\gamma$.*

(c) $s^* = \lambda\beta T(\rho)$.

(d) $\dot{s}^* = q\big(x_1^*(0) - \rho\big)/\gamma(1 - q)$.

(e) *If $\rho < b$, then $\dot{s}^* \leq \lambda q\beta T(\rho) < s^*$.*

Proof. (a) Let $s < s'$. Then since $L\big(h(r^*(s))\big) = 0$ and $L\big(h(r^*(s'))\big) = 0$, we have $s = \lambda\beta T\big(h(r^*(s))\big)$ and $s' = \lambda\beta T\big(h(r^*(s'))\big)$; accordingly $\lambda\beta T\big(h(r^*(s))\big) = s < s' = \lambda\beta T\big(h(r^*(s'))\big)$. Hence $h\big(r^*(s)\big) > h\big(r^*(s')\big)$ because if not so, we have the contradiction of $T\big(h(r^*(s))\big) \geq T\big(h(r^*(s'))\big)$ due to Lemma 3.2.2(a). Therefore, we obtain $r^*(s) > r^*(s')$ from Lemma 4.3.3(e). Similarly proven for $\dot{r}^*(s)$.

(b) Let $s = 0$. Then since $L(x) = \lambda\beta T(x) \geq 0$ and $\dot{L}(x) = \lambda q\beta T(x) \geq 0$ for all x due to Lemma 3.2.2(b), we have $G(x, r) = \gamma\lambda(1 - q)\beta T(x) - x + \gamma r + \rho$. Hence letting $\bar{r} = (b - \rho)/\gamma$, we get $0 = G\big(h(\bar{r}), \bar{r}\big) = \gamma\lambda(1 - q)\beta T(h(\bar{r})) - h(\bar{r}) + b$ or equivalently

$$\gamma\lambda(1-q)\beta T(h(\bar{r})) = h(\bar{r}) - b. \tag{4.3.11}$$

Since $T(h(\bar{r})) \geq 0$ due to Lemma 3.2.2(b), we obtain $h(\bar{r}) \geq b$ or equivalently $h(\bar{r}) \geq b = h(r_b)$ due to the definition of r_b. Thus, $r_b \leq \bar{r}$ due to Lemma 4.3.3(e). Here, suppose $r_b < \bar{r}$. Then $h(\bar{r}) > h(r_b) = b$, hence $T(h(\bar{r})) > 0$ from Eq. (4.3.11), so that $h(\bar{r}) < b$ due to Lemma 3.2.2(d), which is a contradiction. Therefore it must be $\bar{r} = r_b$. Now, if $r < r_b$, then $h(r) < h(r_b)$ from Lemma 4.3.3(e), hence $L(h(r)) > L(h(r_b)) = 0$ due to Lemma 4.3.4(b) and (e) with $s = 0$, and if $r_b \leq r$, then $L(h(r)) = 0$ due to Lemma 4.3.4(e) with $s = 0$. Thus we have $r^* = r_b$ due to the definition of r^*. The proof of $\dot{r}^* = r_b$ is the same as the above.

(c) From Eqs. (3.1.2) and (3.1.4) we have, for any s,

$$s = \lambda\beta T(h(r^*)), \tag{4.3.12}$$

$$s = \lambda q\beta T(h(\dot{r}^*)). \tag{4.3.13}$$

Since $0 = L(h(r^*)) \geq \dot{L}(h(r^*))$ due to Lemma 3.2.2(b) and since $h(r^*)$ is the solution of $G(x, r^*) = 0$, from Eq. (4.3.3) we get

$$0 = G(h(r^*), r^*) = -h(r^*) + \gamma r^* + \rho,$$

hence $h(r^*) = \gamma r^* + \rho$. Accordingly, if $s = s^*$, then $r^* = r^*(s^*) = 0$, hence since $h(r^*) = \rho$, from Eq. (4.3.12) we have $s^* = \lambda\beta T(h(r^*)) = \lambda\beta T(\rho)$.

(d) Rearranging Eq. (4.3.3) with $x = h(\dot{r}^*)$ by substituting (4.3.13) yields

$$\begin{aligned}
0 = G(h(\dot{r}^*), \dot{r}^*) &= \gamma\big(\max\{\lambda\beta T(h(\dot{r}^*)) - s, 0\} - \max\{\lambda q\beta T(h(\dot{r}^*)) - s, 0\}\big) - h(\dot{r}^*) + \gamma\dot{r}^* + \rho \\
&= \gamma \max\{\lambda\beta T(h(\dot{r}^*)) - \lambda q\beta T(h(\dot{r}^*)), 0\} - h(\dot{r}^*) + \gamma\dot{r}^* + \rho \\
&= \gamma \max\{\lambda(1-q)\beta T(h(\dot{r}^*)), 0\} - h(\dot{r}^*) + \gamma\dot{r}^* + \rho \\
&= \gamma\lambda(1-q)\beta T(h(\dot{r}^*)) - h(\dot{r}^*) + \gamma\dot{r}^* + \rho
\end{aligned}$$

due to Lemma 3.2.2(b), from which

$$T(h(\dot{r}^*)) = (h(\dot{r}^*) - \gamma\dot{r}^* - \rho)/\gamma\lambda(1-q)\beta,$$

implying

$$B_1(h(\dot{r}^*)) = 0, \tag{4.3.14}$$

hence $x_1^*(\dot{r}^*) = h(\dot{r}^*)$ from Lemma 4.3.2(b). If $s = \dot{s}^*$, then $\dot{r}^* = \dot{r}^*(\dot{s}^*) = 0$ due to Eq. (4.3.10), thus $x_1^*(0) = h(0)$. Accordingly,

$$T(x_1^*(0)) = (x_1^*(0) - \rho)/\gamma\lambda(1-q)\beta. \tag{4.3.15}$$

Now, from Eq. (4.3.13) we have $\dot{s}^* = \lambda q\beta T(h(\dot{r}^*)) = \lambda q\beta T(h(0)) = \lambda q\beta T(x_1^*(0))$, from which $T(x_1^*(0)) = \dot{s}^*/\lambda q\beta$, hence from Eq. (4.3.15) $\dot{s}^*/\lambda q\beta = (x_1^*(0) - \rho)/\gamma\lambda(1-q)\beta$ or equivalently $\dot{s}^* = q(x_1^*(0) - \rho)/\gamma(1-q)$.

(e) Since $T(x_1^*(0)) \geq 0$ due to Lemma 3.2.2(b), we have $x_1^*(0) \geq \rho$ from Eq. (4.3.15), thus $T(\rho) \geq$

$T(x_1^*(0))$ from Lemma 3.2.2(a). Accordingly, from Eqs. (4.3.15) and (d) we have

$$T(\rho) \geq \frac{x_1^*(0) - \rho}{\gamma\lambda(1-q)\beta} = \frac{q(x_1^*(0) - \rho)/\gamma(1-q)}{\lambda q\beta} = \frac{\dot{s}^*}{\lambda q\beta},$$

hence $\dot{s}^* \leq \lambda q\beta T(\rho)$. If $\rho < b$, then since $T(\rho) > 0$ from Lemma 3.2.2(c), we obtain $\dot{s}^* \leq \lambda q\beta T(\rho) < \lambda\beta T(\rho) = s^*$ due to the assumption of $q < 1$ and (c). ∎

Lemma 4.3.7

(a) *Let $s^* > s$.*

1 $u(\phi) > 0$.

2 $r^* > 0$.

3 *Let $r < r^*$.*

i *If $\dot{r}^* \leq r$, then $h = x_2^* \leq \chi$.*

ii *If $r < \dot{r}^*$, then $h = x_1^* > \chi$.*

(b) *Let $s^* \leq s$. Then $\dot{L}(h) \leq L(h) \leq 0$.*

Proof. (a) Let $s^* > s$.

(a1) $u(\phi) \geq 0$ from Eq. (4.2.9). Suppose $u(\phi) = 0$. Then since $r = 0$ and $L(h) \leq 0 \cdots (*)$, from Eq. (4.3.8) we have $h = -\max\{\dot{L}(h), 0\} + \rho$, from which $h \leq \rho$. From this and Lemma 4.3.3(a) we get $h = \rho$. Accordingly, $L(h) = \lambda\beta T(h) - s = \lambda\beta T(\rho) - s = s^* - s > 0$ due to the assumption of $s^* > s$, which contradicts $(*)$. Thus $u(\phi) > 0$.

(a2) Let $r = 0$. Assume $\lambda\beta v(0) + (1-\lambda)\beta u(\phi) - s \leq \beta u(\phi)$. Then $u(\phi) = \beta u(\phi)$ from Eq. (4.2.4), leading to $\beta = 1$ due to $u(\phi) > 0$ from (a1), which contradicts the assumption of $\beta < 1$. Accordingly, it must be $\lambda\beta v(0) + (1-\lambda)\beta u(\phi) - s > \beta u(\phi)$, which can be rearranged into $0 < \lambda\beta(v(0) - u(\phi)) - s = \lambda\beta T(h) - s = L(h)$ from Eq. (4.2.6), implying $r^* > 0$ due to Lemma 4.3.4(b).

(a3) Let $r < r^*$. Then $L(h) > 0$ from Lemma 4.3.5(b3). Here note $\dot{r}^* < r^*$ from Lemma 4.3.5(b1).

(a3i) Let $\dot{r}^* \leq r$. Then $\dot{L}(h) \leq 0$ from Lemma 4.3.5(b4). Accordingly, from Eq. (4.3.8) we have $h = \gamma L(h) + \gamma r + \rho = \gamma\lambda\beta T(h) - \gamma(s - r) + \rho$, hence $T(h) - \big(h + \gamma(s-r) - \rho\big)/\gamma\lambda\beta = 0$, i.e., $B_2(h) = 0$ from Eq. (4.3.5), implying that h defined by Eq. (4.1.1) is given by x_2^*, which is the unique solution of $B_2(x) = 0$ due to Lemma 4.3.2(b), i.e., $h = x_2^*$. Hence

$$T(h) = T(x_2^*) = \big(x_2^* + \gamma(s-r) - \rho\big)/\gamma\lambda\beta. \tag{4.3.16}$$

Now, from the assumption of $\dot{r}^* \leq r$ we obtain $0 \geq \dot{L}(h) = \lambda q\beta T(h) - s$ due to Lemma 4.3.5(b4). Rearranging the inequality by substituting Eq. (4.3.16) produces

$$0 \geq qx_2^*/\gamma - s(1-q) - qr - q\rho/\gamma,$$

hence $x_2^* \leq c\gamma(1-q)/q + \gamma r + \rho = \chi$ from Eq. (4.3.2).

(a3ii) If $r < \dot{r}^*$, then $\dot{L}(h) > 0$ from Lemma 4.3.5(b5). Accordingly, from Eq. (4.3.8) we get $h = \gamma(L(h) - \dot{L}(h)) + \gamma r + \rho = \gamma\lambda(1-q)\beta T(h) + \gamma r + \rho$, hence $T(h) - (h - \gamma r - \rho)/\gamma\lambda(1-q)\beta = 0$, i.e.,

$B_1(h) = 0$ from Eq. (4.3.4). This implies that h defined by Eq. (4.1.1) is also given by x_1^* which is the unique solution of $B_1(x) = 0$ due to Lemma 4.3.2(b), i.e., $h = x_1^*$. Hence

$$T(h) = T(x_1^*) = (x_1^* - \gamma r - \rho)/\gamma\lambda(1-q)\beta. \tag{4.3.17}$$

Now, from the assumption of $r < \dot{r}^*$ and Lemma 4.3.5(b5) we have $0 < \dot{L}(h) = \lambda q\beta T(h) - s$. Rearranging this inequality by substituting Eq. (4.3.17) yields $q(x_1^* - \gamma r - \rho)/\gamma(1-q) - s > 0$ or equivalently $x_1^* > s\gamma(1-q)/q + \gamma r + \rho = \chi$ from Eq. (4.3.2).

(b) Let $s \geq s^*$. Then $s \geq \lambda\beta T(\rho)$ due to Lemma 4.3.6(c). Now, since $h \geq \rho$ from Lemmä 4.3.3(a), we have $T(h) \leq T(\rho)$ from Lemma 3.2.2(a). Hence, $0 \geq \lambda\beta T(\rho) - s \geq \lambda\beta T(h) - s \geq \lambda q\beta T(h) - s$ due to Lemma 3.2.2(b), i.e., $\dot{L}(h) \leq L(h) \leq 0$. ∎

4.4 Optimal Decision Rule

The following theorem prescribes the optimal decision rule.

Theorem 4.4.1

(a) *Let* $\rho \geq b$. *Then* $\langle \mathrm{K}\rangle_\phi$ *and* $\langle \mathrm{K}\rangle_l$.

(b) *Let* $\rho < b$.

  1 *Let* $s = 0$. *Then* $\langle \mathrm{C}\rangle_\phi$ *and* $\langle \mathrm{C}\rangle_l$.

  2 *Let* $s > 0$.

    i *If* $s^* \leq s$, *then* $\langle \mathrm{K}\rangle_\phi$ *and* $\langle \mathrm{K}\rangle_l$.

    ii *If* $s < s^*$, *then:*

      1 *If* $r^* \leq r$, *then* $\langle \mathrm{K}\rangle_\phi$ *and* $\langle \mathrm{K}\rangle_l$.

      2 *If* $r < r^*$, *then*

        i $\langle \mathrm{C}\rangle_\phi$.

        ii *If* $x_1^* \leq (>)\ \chi$, *then* $\langle \mathrm{K}\rangle_l(\langle \mathrm{C}\rangle_l)$, *or if* $x_2^* > (\leq)\ \chi$, *then* $\langle \mathrm{C}\rangle_l(\langle \mathrm{K}\rangle_l)$.

Proof. (a) Let $\rho \geq b$. Then $T(\rho) = 0$ due to Lemma 3.2.2(c); accordingly, $0 = \lambda\beta T(\rho) = s^* \leq s$ from Lemma 4.3.6(c) and the assumption of $s \geq 0$. Hence the assertion holds from Lemma 4.3.7(b).

(b1,b2i,b2ii1,b2ii2i) Immediate from Lemmas 4.3.5(a), 4.3.7(b), 4.3.5(b2), and 4.3.5(b3), respectively.

(b2ii2ii) Noting Lemmas 4.3.2(d), 4.3.5(b4,b5), and the contrapositions of Lemma 4.3.7(a3i,a3ii), we immediately obtain the following relationships.

$$x_1^* > \chi \overset{\text{Lemma 4.3.2(d)}}{\iff} x_2^* > \chi \overset{\text{Lemma 4.3.7(a3i)}}{\Longrightarrow} r < \dot{r}^* \overset{\text{Lemma 4.3.5(b5)}}{\Longrightarrow} \dot{L}(h) > 0 \Longrightarrow \langle \mathrm{C}\rangle_l$$

$$x_2^* \leq \chi \underset{\text{Lemma 4.3.2(d)}}{\iff} x_1^* \leq \chi \underset{\text{Lemma 4.3.7(a3ii)}}{\Longrightarrow} \dot{r}^* \leq r \underset{\text{Lemma 4.3.5(b4)}}{\Longrightarrow} \dot{L}(h)) \leq 0 \Longrightarrow \langle \mathrm{K}\rangle_l$$

∎

Theorem 4.4.2 *In the pricing control problem we have:*

(a) $z(h)$ *is nondecreasing in* λ, β, r, *and* θ *and nonincreasing in* q, s, *and* τ.

(b) *If* $r < (b-\rho)/\gamma$, *then* $h < z(h) < b$, *or else* $z(h) = b$.

Proof. (a) Immediate from Lemmas 4.3.3(c,d,e), and 3.2.1(d).

(b) Evident from Lemmas 4.3.3(b) and 3.2.1(b,c). ∎

4.5 Conclusions and Considerations

C1. **Optimal decision rules.**

Theorem 4.4.1 provides the most important conclusions obtained in this model, which can be summarized as in Table 4.5.1.

Table 4.5.1: Summary of optimal decision rules.

<table>
<tr><th>ρ</th><th>s</th><th>r</th><th>χ</th><th>State (ϕ)</th><th>State (ϕ, l)</th></tr>
<tr><td rowspan="5">$\rho < b$</td><td>$s = 0$</td><td colspan="2"></td><td rowspan="2">$\langle C \rangle_\phi$</td><td rowspan="2">$\langle C \rangle_l$</td></tr>
<tr><td rowspan="3">$0 < s < s^*$</td><td rowspan="2">$r < r^*$</td><td>$\chi < x_1^*$</td></tr>
<tr><td>$x_1^* \leq \chi$</td><td>$\langle C \rangle_\phi$</td><td>$\langle K \rangle_l$</td></tr>
<tr><td>$r^* \leq r$</td><td></td><td rowspan="3">$\langle K \rangle_\phi$</td><td rowspan="3">$\langle K \rangle_l$</td></tr>
<tr><td>$s^* \leq s$</td><td colspan="2"></td></tr>
<tr><td>$\rho \geq b$</td><td colspan="3"></td></tr>
</table>

Note that the optimal decisons in state (ϕ, l) is independent of l. The table shows the conditions on whether to continue the search for customers or not in both states $u(\phi)$ and $u(\phi, l)$. We in particular concern the conditions related to the parameters, θ, s, and r. If $b \leq \rho = \gamma\eta^\tau\theta$, or $s^* \leq s$, or $r^* \leq r$, it is optimal to skip the search when there exists no order in the system. This implies that if the delay cost or the search cost or the profit from a sideline is sufficiently large, it is optimal for the company to keep the system empty by not searching for customers and to enact subsidiary business, yielding the profit from a sideline.

C2. **Relationships of the optimal decision rules with parameters.**

If τ is sufficiently small or θ is sufficiently large, then $b \leq \rho = \gamma\eta^\tau\theta$, implying that skipping the search is optimal in both states (ϕ) and (ϕ, l). On the contrary, if τ is sufficiently large or θ is sufficiently small, then $b > \rho = \gamma\eta^\tau\theta$, hence we can depict the optimal decision rule in Table 4.5.1 as in Figure 4.5.2 where both $r^*(s)$ and $\dot{r}^*(s)$ are strictly decreasing in s with $s^* = \lambda\beta T(\rho)$, $\dot{s}^* = q(x_1^*(0) - \rho)/\gamma(1-q)$ (Lemma 4.3.6(a,c,d)), $\dot{r}^*(s) < r^*(s)$ for $s > 0$ (Lemma 4.3.5(b1)), and $\dot{r}^*(0) = r^*(0) = (b-\rho)/\gamma > 0$ (Lemma 4.3.6(b)). The three regions $\Omega(\mathtt{K},\mathtt{K})$, $\Omega(\mathtt{C},\mathtt{K})$, and $\Omega(\mathtt{C},\mathtt{C})$ in Figure 4.5.2 correspond to the optimal decisions as follows.

$\Omega(\mathtt{K},\mathtt{K}) \rightarrow$ skipping in both states (ϕ) and (ϕ, l)

$\Omega(\mathtt{C},\mathtt{K}) \rightarrow$ continuing in state (ϕ) and skipping in state (ϕ, l)

$\Omega(\mathtt{C},\mathtt{C}) \rightarrow$ continuing in both states (ϕ) and (ϕ, l)

Here note that the three regions are independent of l. The figure provides us with the following two points:

1. When the search cost s or the profit from a sideline r is sufficiently large, skipping the search becomes optimal in both states (ϕ) and (ϕ, l), implying, respectively, that it is profitable to avoid

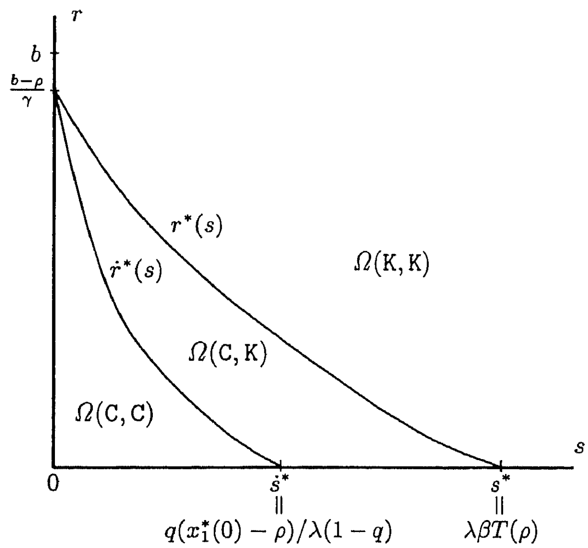

Figure 4.5.2: Three regions encircled by the functions $\dot{r}^*(s)$ and $r^*(s)$ and the axes s, r when $b > \rho$.

the search cost through skipping the search or that it becomes profitable to enjoy the profits from a sideline while emptying the process by skipping the search.

2. When the search cost s and the profit from a sideline r are sufficiently small, continuing the search becomes optimal in both states (ϕ) and (ϕ, l), implying, respectively, that it is reasonable to enjoy the profit from an order obtained through conducting the search and that it does not become profitable even when emptying the process by skipping the search.

C3. **Properties of h.**

1. In the admission control problem the optimal selection criterion, on which the system decides whether to accept an appearing customer or not, is given by h, and in the pricing control problem the optimal price, on which an appearing customer decides whether to place his order in the system or not, is given by the function $z(h)$. Further, it is only in the regions $\Omega(\mathtt{C},\mathtt{K}) \cup \Omega(\mathtt{C},\mathtt{C})$ that the decisions stated above are to be made.

2. The h is given by the unique solution x^* of the equation $G(x) = 0$, i.e., $h = x^*$; refer to Lemma 4.3.3 for the properties of h.

3. Let $(r, s) \in \Omega(\mathtt{C},\mathtt{C}) \cup \Omega(\mathtt{C},\mathtt{K})$, hence $h < b$ (Lemma 4.3.3(b)). Then the optimal decisions can be prescribed as follows.

   1) In the admission control problem, if $w > h$, an order with value w appearing after having conducted the search is accepted, or else rejected.

   2) In the pricing control problem, the optimal price $z(h)$ is in the interval (h, b) (Theorem 4.4.2(b)).

C4. **The monotonicities of h and $z(h)$ in the parameters** (Lemma 4.3.3(c,d,e) and Theorem 4.4.2(a)).

1. *Both h and $z(h)$ are nondecreasing in λ, β, r, and θ.* This implies that the larger the customer appearing probability λ, the discount factor β, the profit from a sideline r, and the penalty θ may be, it is reasonable to accept orders with higher values in the admission control problem and to propose higher prices in the pricing control problem.

2. *Both h and $z(h)$ are nonincreasing in q, s, and τ.* This implies that the larger the service completion probability q, the search cost s, and the date of delivery τ may be, the inverse of the above can be said. In other words, it is reasonable to accept orders in the admission control problem even if their values are smaller and to propose smaller prices in the pricing control problem.

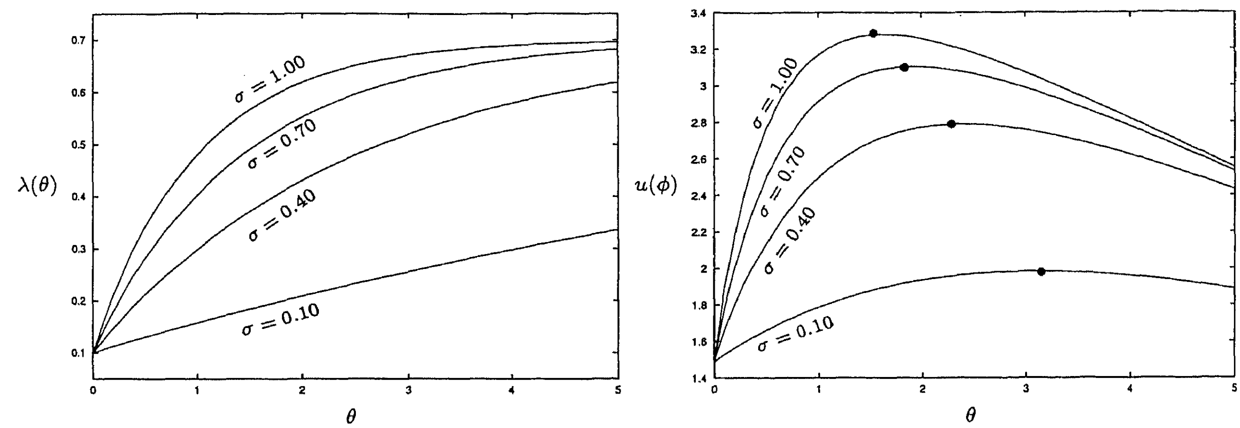

Figure 4.5.3: $\lambda(\theta)$ and $u(\phi)$.

C5. **Optimal penalty θ^*.**

From a practical viewpoint, if a penalty for delay of delivery, θ, is small enough, customers would be reluctant to place their orders with that company; on the contrary, if it is large enough, they are willing to place. This implies that the probability of customer arrival increases in the penalty θ, so let the probability be denoted by $\lambda(\theta)$ (See Figure 4.5.3). Then the total expected net profit may diminish due to few orders from the customer in the former case or due to too much penalty payment in the latter case. This consideration leads to the conjecture that there may exist a optimal penalty θ^* maximizing the total expected present discounted net profit $u(\phi)$. Below, let us examine the conjecture by numerical experiments. Here, let $\lambda(\theta) = -0.6\exp(-\sigma\theta) + 0.7$ ($\sigma = 0.10, 0.40, 0.70, 1.00$), $q = 0.30$, $\beta = 0.90$, $r = 0.10$, $s = 0.05$, and $\tau = 6.0$, and let $F(w)$ be the uniform distribution on $[1, 2]$. Then we obtain the results of numerical experiment, which are shown in Figure 4.5.3. Table 4.5.2 shows the θ^* maximizing $u(\phi)$.

Table 4.5.2: The optimal penalty θ^* and the maximum total expected present net profits $u(\phi)$.

	θ^*	$u(\phi)$
$\sigma = 0.10$	3.10	1.98
$\sigma = 0.40$	2.40	2.79
$\sigma = 0.70$	1.90	3.10
$\sigma = 1.00$	1.60	3.28

Chapter 5

Model II: Stochastic model with cancellation ($N = 1$)

The model defined and examined in this chapter is the same as Model I except that an order undergoing processing may be canceled with a known probability due to customer's unavoidable circumstance, and if it is canceled, the penalty is paid by the customer.

Contents

5.1 System of Optimal Equations

This model is the same as Model I except that the order undergoing processing may be cancelled due to the customer's unavoidable circumstance.

1. In both the admission control problem and pricing control problem, by $u(\phi)$ and $u(\phi, l)$ we shall denote the maximum total expected present discounted net profits starting from state (ϕ), provided, respectively, that there exists no order in the company and that there exists an order accepted l periods ago. Suppose the company is in state (ϕ, l) at a certain point in time. With known probabilities ν and $\bar{\nu}$ let a customer cancel the contract when, respectively, $l < \tau$ and $l \geq \tau$. If a customer cancels the order undergoing processing in the next point in time when $l < \tau$ or when $l \geq \tau$, whether it is completed in the next point in time or not, he must pay a penalty, respectively, $\vartheta > 0$ or $\bar{\vartheta} > 0$ to the company. Further, if $l \geq \tau$, then the penalty θ must be paid for one period delayed as introduced in Model I.
2. In the admission control problem, by $u(w)$ let us denote the maximum total expected present discounted net profit starting with no order in the company and an arriving customer w, who offers a price w for his order.
3. In the pricing control problem, by $u(1)$ let us denote the maximum total expected present discounted net profit starting with no order in the company and an arriving customer, to whom the company offers a price z for an order.

From the same reason as mentioned in Section 4.1 the maximum total expected present discounted net profits $u(\phi)$ and $u(\phi, l)$ are bounded in $l \geq 0$. For convenience in later discussions let us define

$$h = u(\phi) - u(\phi, 0). \tag{5.1.1}$$

Then the optimal equations for both problems can be described as follows.

1. **Admission control problem:**

$$u(\phi) = \max \left\{ \begin{array}{l} \mathtt{C}: \beta(\lambda \mathbf{E}[u(\xi)] + (1-\lambda)u(\phi)) - s + r, \\ \mathtt{K}: \beta u(\phi) + r, \end{array} \right\}, \tag{5.1.2}$$

$$u(\phi, l) = \max \left\{ \begin{array}{l} \mathtt{C}: (1-q)\beta\Big(\nu(\vartheta + \lambda \mathbf{E}[u(\xi)] + (1-\lambda)u(\phi)) \\ +(1-\nu)u(\phi, l+1)\Big) + q\beta(\nu\vartheta + \lambda \mathbf{E}[u(\xi)] + (1-\lambda)u(\phi)) - s, \\ \mathtt{K}: (1-q)\beta\Big(\nu(\vartheta + u(\phi)) + (1-\nu)u(\phi, l+1)\Big) + q\beta\Big(\nu\vartheta + u(\phi)\Big) \end{array} \right\}, \quad 0 \le l < \tau, \tag{5.1.3}$$

$$u(\phi, l) = \max \left\{ \begin{array}{l} \mathtt{C}: (1-q)\beta\Big(-\theta + \bar{\nu}(\bar{\vartheta} + \lambda \mathbf{E}[u(\xi)] + (1-\lambda)u(\phi)) \\ +(1-\bar{\nu})u(\phi, l+1)\Big) + q\beta(-\theta + \bar{\nu}\bar{\vartheta} + \lambda \mathbf{E}[u(\xi)] + (1-\lambda)u(\phi)) - s, \\ \mathtt{K}: (1-q)\beta\Big(-\theta + \bar{\nu}(\bar{\vartheta} + u(\phi)) + (1-\bar{\nu})u(\phi, l+1)\Big) \\ \qquad +q\beta\Big(-\theta + \bar{\nu}\bar{\vartheta} + u(\phi)\Big) \end{array} \right\}, \quad \tau \le l, \tag{5.1.4}$$

$$u(w) = \max \left\{ \begin{array}{l} \mathtt{A}: w + u(\phi, 0) \\ \mathtt{R}: u(\phi) \end{array} \right\}. \tag{5.1.5}$$

Eq. (5.1.5) can be rearranged as

$$u(w) = \max\{w - h, 0\} + u(\phi). \qquad \Box \tag{5.1.6}$$

2. **Pricing control problem:**

$$u(\phi) = \max \left\{ \begin{array}{l} \mathtt{C}: \beta(\lambda u(1) + (1-\lambda)u(\phi)) - s + r, \\ \mathtt{K}: \beta u(\phi) + r, \end{array} \right\}, \tag{5.1.7}$$

$$u(\phi, l) = \max \left\{ \begin{array}{l} \mathtt{C}: (1-q)\beta\Big(\nu(\vartheta + \lambda u(1) + (1-\lambda)u(\phi)) \\ +(1-\nu)u(\phi, l+1)\Big) + q\beta(\nu\vartheta + \lambda u(1) + (1-\lambda)u(\phi)) - s, \\ \mathtt{K}: (1-q)\beta\Big(\nu(\vartheta + u(\phi)) + (1-\nu)u(\phi, l+1)\Big) + q\beta(\nu\vartheta + u(\phi)) \end{array} \right\}, \quad 0 \le l < \tau, \tag{5.1.8}$$

$$u(\phi, l) = \max \left\{ \begin{array}{l} \mathtt{C}: (1-q)\beta\Big(-\theta + \bar{\vartheta} + \bar{\nu}(\lambda u(1) + (1-\lambda)u(\phi)) \\ +(1-\bar{\nu})u(\phi, l+1)\Big) + q\beta(-\theta + \bar{\nu}\bar{\vartheta} + \lambda u(1) + (1-\lambda)u(\phi)) - s, \\ \mathtt{K}: (1-q)\beta\Big(-\theta + \bar{\nu}(\bar{\vartheta} + u(\phi)) + (1-\bar{\nu})u(\phi, l+1)\Big) \\ \qquad +q\beta\Big(-\theta + \bar{\nu}\bar{\vartheta} + u(\phi)\Big) \end{array} \right\}, \quad \tau \le l, \tag{5.1.9}$$

$$u(1) = \max_z\{p(z)(z + u(\phi, 0)) + (1-p(z))u(\phi)\}. \tag{5.1.10}$$

Eq. (5.1.10) can be rearranged as

$$u(1) = \max_z p(z)(z-h) + u(\phi). \tag{5.1.11}$$

5.2 Transformation

Let us define

$$J(h) = \lambda\beta\big(q + (1-q)\nu\big)T(h) - s, \tag{5.2.1}$$

$$\bar{J}(h) = \lambda\beta\big(q + (1-q)\bar{\nu}\big)T(h) - s. \tag{5.2.2}$$

When regarding h as a function of r, i.e., $h = h(r)$, by r_J^* and $\bar{r}_J^*$ let us denote the solutions of $J(h(r)) = 0$, and $\bar{J}(h(r)) = 0$, respectively, if they exist, i.e.,

$$J(h(r_J^*)) = 0, \qquad \bar{J}(h(\bar{r}_J^*)) = 0. \tag{5.2.3}$$

If each of the above two equations has multiple solutions, let us define the *smallest* of them as r_J^* and $\bar{r}_J^*$, respectively. Further, let us define

$$v(0) = \begin{cases} \mathbf{E}[u(w)] & \text{for the admission control problem,} \\ u(1) & \text{for the pricing control problem.} \end{cases} \tag{5.2.4}$$

Then we can express "Eq. (5.1.2) to Eq. (5.1.4) and Eq. (5.1.6)" and "Eq. (5.1.7) to Eq. (5.1.9) and Eq. (5.1.11)" by the identical equations below.

$$u(\phi) = \max\{\lambda\beta v(0) + (1-\lambda)\beta u(\phi) - s, \beta u(\phi)\} + r, \tag{5.2.5}$$

$$u(\phi, l) = \max\left\{\begin{array}{l} (1-q)\beta\Big(\nu\big(\vartheta + \lambda v(0) + (1-\lambda)u(\phi)\big) \\ \quad +(1-\nu)u(\phi, l+1)\Big) + q\beta\big(\nu\vartheta + \lambda v(0) + (1-\lambda)u(\phi)\big) - s, \\ (1-q)\beta\Big(\nu\big(\vartheta + u(\phi)\big) + (1-\nu)u(\phi, l+1)\Big) + q\beta\big(\nu\vartheta + u(\phi)\big) \end{array}\right\}, \quad 0 \le l < \tau, \tag{5.2.6}$$

$$u(\phi, l) = \max\left\{\begin{array}{l} (1-q)\beta\Big(-\theta + \bar{\nu}\big(\bar{\vartheta} + \lambda v(0) + (1-\lambda)u(\phi)\big) \\ \quad +(1-\bar{\nu})u(\phi, l+1)\Big) + q\beta\big(\lambda v(0) + (1-\lambda)u(\phi)\big) - s, \\ (1-q)\beta\Big(-\theta + \bar{\nu}\big(\bar{\vartheta} + u(\phi)\big) + (1-\bar{\nu})u(\phi, l+1)\Big) \\ \quad +q\beta\big(-\theta + \bar{\nu}\bar{\vartheta} + u(\phi)\big) \end{array}\right\}, \quad \tau \le l, \tag{5.2.7}$$

$$v(0) = T(h) + u(\phi). \tag{5.2.8}$$

Further, Eq. (5.2.5) to Eq. (5.2.7) can be rearranged into, respectively,

$$u(\phi) = \beta u(\phi) + \max\{\lambda\beta(v(0) - u(\phi)) - s, 0\} + r, \tag{5.2.9}$$

$$u(\phi, l) = (1-q)\beta(\nu u(\phi) + (1-\nu)u(\phi, l+1)) + q\beta u(\phi) + \beta\nu\vartheta + \max\{\lambda\beta(q + (1-q)\nu)(v(0) - u(\phi)) - s, 0\}, \quad 0 \le l < \tau, \tag{5.2.10}$$

$$u(\phi, l) = (1-q)\beta(\bar{\nu} u(\phi) + (1-\bar{\nu})u(\phi, l+1)) + q\beta u(\phi) + \beta(\bar{\nu}\bar{\vartheta} - \theta) + \max\{\lambda\beta(q + (1-q)\bar{\nu})(v(0) - u(\phi)) - s, 0\}, \quad \tau \le l. \tag{5.2.11}$$

Noting Eq. (5.2.8), we can rearrange Eq. (5.2.9) into Eq. (5.2.11) as follows, respectively,

$$u(\phi) = (\max\{L(h), 0\} + r)/(1-\beta) \ge 0, \tag{5.2.12}$$

$$u(\phi, l) = (1-q)(1-\nu)\beta u(\phi, l+1) + (q + (1-q)\nu)\beta u(\phi) + \max\{J(h), 0\} + \beta\nu\vartheta, \quad 0 \le l < \tau, \tag{5.2.13}$$

$$u(\phi, l) = (1-q)(1-\bar{\nu})\beta u(\phi, l+1) + (q + (1-q)\bar{\nu})\beta u(\phi) + \max\{\bar{J}(h), 0\} + \beta(\bar{\nu}\bar{\vartheta} - \theta), \ \tau \le l. \tag{5.2.14}$$

Eqs. (5.1.6) and (5.1.11) and Eqs. (5.2.12) to (5.2.14) tell us that the optimal decision rules can be prescribed as follows.

□ *Optimal Decision Rule 5.2.1*

1. Admission control problem:
   i. If $L(h) > 0$, then $\langle \mathtt{C}\rangle_\phi$, or else $\langle \mathtt{K}\rangle_\phi$.
   ii. If $J(h) > 0$, then $\langle \mathtt{C}\rangle_l$, or else $\langle \mathtt{K}\rangle_l$ where $l < \tau$.
   iii. If $\bar{J}(h) > 0$, then $\langle \mathtt{C}\rangle_l$, or else $\langle \mathtt{K}\rangle_l$ where $l \ge \tau$.
   iv. Let an order with value w appear after the search was enacted. Then if $w > h$, then $\langle \mathtt{A}(w)\rangle$, or else $\langle \mathtt{R}(w)\rangle$.
2. Pricing control problem:
   i. If $L(h) > 0$, then $\langle \mathtt{C}\rangle_\phi$, or else $\langle \mathtt{K}\rangle_\phi$.
   ii. If $J(h) > 0$, then $\langle \mathtt{C}\rangle_l$, or else $\langle \mathtt{K}\rangle_l$ where $l < \tau$.
   iii. If $\bar{J}(h) > 0$, then $\langle \mathtt{C}\rangle_l$, or else $\langle \mathtt{K}\rangle_l$ where $l \ge \tau$.
   iv. Let a customer appear after the search was enacted. Then $\langle \mathtt{O}(z(h))\rangle$.

Lemma 5.2.1 *The system of* Eqs. (5.2.5) *to* (5.2.8) *has a unique solution.*
Proof. The same as the proofs of Lemma 4.2.1. ∎

5.3 Analysis

For convenience in the later discussions, let us define

$$\kappa = \eta(1-\nu), \quad (0 < \kappa < 1), \tag{5.3.1}$$

$$\bar{\kappa} = \eta(1-\bar{\nu}), \quad (0 < \bar{\kappa} < 1), \tag{5.3.2}$$

$$y = \kappa^\tau, \quad (0 < y < 1), \tag{5.3.3}$$

$$\rho = \beta(\theta - \bar{\nu}\bar{\vartheta})y/(1-\bar{\kappa}) - \beta\nu\vartheta(1-y)/(1-\kappa), \tag{5.3.4}$$

$$W = (1-y)/(1-\kappa) + y/(1-\bar{\kappa}), \tag{5.3.5}$$

$$H = (1-y)(q+(1-q)\nu)/(1-\kappa) + y(q+(1-q)\bar{\nu})/(1-\bar{\kappa}), \tag{5.3.6}$$

$$\chi_1 = Ws(1-q)(1-\nu)/\big(q+(1-q)\nu\big) + Wr + \rho, \tag{5.3.7}$$

$$\bar{\chi}_1 = Ws(1-q)(1-\bar{\nu})/\big(q+(1-q)\bar{\nu}\big) + Wr + \rho, \tag{5.3.8}$$

$$\chi_2 = (W-H)s/\big(q+(1-q)\bar{\nu}\big) + Wr + \rho, \tag{5.3.9}$$

$$\bar{\chi}_2 = (W-H)s/\big(q+(1-q)\nu\big) + Wr + \rho. \tag{5.3.10}$$

Lemma 5.3.1 *We have:*

$$\beta H = 1-(1-\beta)W, \tag{5.3.11}$$

$$W > 1 > H. \tag{5.3.12}$$

Proof. See App. A.5. ▮

Further, for any real number x let us define the following functions, which are used in the analyses conducted in Section 5.3.

$$G(x) = W\max\{L(x),0\} - (1-y)\max\{J(x),0\}/(1-\kappa) - y\max\{\bar{J}(x),0\}/(1-\bar{\kappa})$$
$$-x + Wr + \rho, \tag{5.3.13}$$

$$B_1(x) = T(x) - \big(x - Wr - \rho\big)/\lambda\beta W(1-q)(1-\nu), \tag{5.3.14}$$

$$\bar{B}_1(x) = T(x) - \big(x - Wr - \rho\big)/\lambda\beta W(1-q)(1-\bar{\nu}), \tag{5.3.15}$$

$$B_2(x) = T(x) - \big(x + ys/(1-\bar{\kappa}) - Wr - \rho\big)/\lambda\beta\Big(W - (q+(1-q)\nu)(1-y)/(1-\kappa)\Big), \tag{5.3.16}$$

$$\bar{B}_2(x) = T(x) - \big(x + (1-y)s/(1-\kappa) - Wr - \rho\big)/\lambda\beta\Big(W - (q+(1-q)\bar{\nu})y/(1-\bar{\kappa})\Big), \tag{5.3.17}$$

$$B_3(x) = T(x) - \big(x - Wr - \rho\big)\lambda\beta(W-H), \tag{5.3.18}$$

$$B_4(x) = T(x) - (x + Ws - Wr - \rho)/\lambda\beta W. \tag{5.3.19}$$

By x^*, x_1^*, $\bar{x}_1^*$, x_2^*, $\bar{x}_2^*$, x_3^*, and x_4^* let us denote the solutions of, respectively, $G(x) = 0$, $B_1(x) = 0$, $\bar{B}_1(x) = 0$, $B_2(x) = 0$, $\bar{B}_2(x) = 0$, $B_3(x) = 0$, and $B_4(x) = 0$, if they exist, i.e.,

$$G(x^*) = 0, \tag{5.3.20}$$

$$B_1(x_1^*) = 0, \qquad \bar{B}_1(\bar{x}_1^*) = 0, \tag{5.3.21}$$

$$B_2(x_2^*) = 0, \qquad \bar{B}_2(\bar{x}_2^*) = 0, \tag{5.3.22}$$

$$B_3(x_3^*) = 0, \qquad B_4(x_4^*) = 0. \tag{5.3.23}$$

Further, for convenience let us denote x_1^*, $\bar{x}_1^*$ and x_3^* for $r = 0$ by $x_1^*(0)$, $\bar{x}_1^*(0)$, and $x_3^*(0)$, respectively.

Lemma 5.3.2

(a) *$G(x)$ is continuous and strictly decreasing in x.*

(b) *$G(x) < (>)\,0$ for any sufficiently large (small) x.*

(c) *$G(x)$ is nonincreasing in s and nondecreasing in λ for all x.*

(d) *$G(x)$ is strictly increasing in r and ρ for all x.*

(e) *$G(x)$ is strictly increasing in θ and strictly decreasing in ϑ and $\bar{\vartheta}$ for all x.*

Proof. See App. A.6. ∎

Lemma 5.3.3

(a) *$B_1(x)$, $\bar{B}_1(x)$, $B_2(x)$ $\bar{B}_2(x)$, $B_3(x)$, and $B_4(x)$ are all strictly decreasing in x*

(b) *$B_1(x) > (<)\,0$, $\bar{B}_1(x) > (<)\,0$, $B_2(x) > (<)\,0$, $\bar{B}_2(x) > (<)\,0$, $B_3(x) > (<)\,0$, and $B_4(x) > (<)\,0$ for any sufficiently small (large) x.*

(c) *x_1^*, $\bar{x}_1^*$, x_2^*, $\bar{x}_2^*$, x_3^*, and x_4^* are all uniquely exist.*

(d) *Let $\nu > \bar{\nu}$.*

1 *If $r < (b-\rho)/W$, then $x_1^* < x_3^* < b$, or else $x_1^* = x_3^* \geq b$.*

2 $\chi_1 < \chi_2$.

3 $x > (= (<))\,\chi_1 \Leftrightarrow B_4(x) > (= (<))\,B_2(x)$.

4 $x > (= (<))\,\chi_2 \Leftrightarrow B_2(x) > (= (<))\,B_3(x)$.

(e) $\nu < \bar{\nu}$.

1 *If $r < (b-\rho)/W$, then $\bar{x}_1^* < x_3^* < b$, or else $\bar{x}_1^* = x_3^* \geq b$.*

2 $\bar{\chi}_1 < \bar{\chi}_2$.

3 $x > (= (<))\,\bar{\chi}_1 \Leftrightarrow B_4(x) > (= (<))\,\bar{B}_2(x)$.

4 $x > (= (<))\,\bar{\chi}_2 \Leftrightarrow \bar{B}_2(x) > (= (<))\,B_3(x)$.

(f) *If $\nu = \bar{\nu}$, then $x_1^* = x_3^* = \bar{x}_1^*$ and $\chi_1 = \chi_2 = \bar{\chi}_1 = \bar{\chi}_2$.*

Proof. See App. A.7. ∎

Lemma 5.3.4 Eq. (5.2.13) *with $l = 0$ can be rewritten as*

$$u(\phi, 0) = \big(1 - (1-\beta)W\big)u(\phi) + (1-y)\max\{\dot{L}(h), 0\}/(1-\kappa) + y\max\{\ddot{L}(h), 0\}/(1-\bar{\kappa}) - \rho. \quad (5.3.24)$$

Proof. See App. A.8. ∎

Now, from Eqs. (5.1.1) and (5.3.24) we have

$$h = (1-\beta)Wu(\phi) - (1-y)\max\{J(h), 0\}/(1-\kappa) - y\max\{\bar{J}(h), 0\}/(1-\bar{\kappa}) + \rho. \quad (5.3.25)$$

Rearranging Eq. (5.3.25) by substituting Eq. (5.2.12) yields

$$h = W\max\{L(h), 0\} - (1-y)\max\{J(h), 0\}/(1-\kappa) - y\max\{\bar{J}(h), 0\}/(1-\bar{\kappa}) + Wr + \rho. \quad (5.3.26)$$

Noting Eq. (5.3.13), we can eventually rewrite Eq. (5.3.26) as follows.

$$G(h) = 0; \quad (5.3.27)$$

in other words, the h is given by the solution of $G(x) = 0$.

Lemma 5.3.5

(a) *x^* uniquely exists with $x^* = h$ and $h \geq \rho$.*

(b) *If $r < (b-\rho)/W$ (, hence $b > \rho$ due to $r \geq 0$), then $h < b$, or else $h \geq b$.*

(c) *h is nonincreasing in s and nondecreasing in λ.*

(d) *h is strictly increasing in θ and strictly decreasing in ϑ and $\bar{\vartheta}$.*

(e) *h is strictly increasing in r with $\lim_{r\to\infty} h(r) = \infty$ and $\lim_{r\to-\infty} h(r) = -\infty$.*

Proof. Below, note that for any x

$$\begin{aligned} L(x) &= \lambda\beta T(x) - s \geq \lambda\beta(q + (1-q)\nu)T(x) - s = J(x), \\ L(x) &= \lambda\beta T(x) - s \geq \lambda\beta(q + (1-q)\bar{\nu})T(x) - s = \bar{J}(x) \end{aligned}$$

due to $1 > q + (1-q)\nu$, $1 > q + (1-q)\bar{\nu}$, and Lemma 3.2.2(b).

(a) The unique existence of x^* is immediate from Lemma 5.3.2(a,b), hence $h = x^*$ from Eq. (5.3.27). We have

$$\begin{aligned} G(\rho, r) &= \big((1-y)/(1-\kappa) + y/(1-\bar{\kappa})\big)\max\{L(\rho), 0\} \\ &\qquad - (1-y)\max\{J(\rho), 0\}/(1-\kappa) - y\max\{\bar{J}(\rho), 0\}/(1-\bar{\kappa}) + Wr \\ &= (1-y)\Big(\max\{L(\rho), 0\} - \max\{J(\rho), 0\}\Big)/(1-\kappa) \\ &\qquad + y\Big(\max\{L(\rho), 0\} - \max\{\bar{J}(\rho), 0\}\Big)/(1-\bar{\kappa}) + Wr \geq 0, \end{aligned}$$

implying $h \geq \rho$.

(b) Since $L(b) = J(b) = \bar{J}(b) = -s$ due to Eq. (5.2.2) and Lemma 3.2.2(c), we obtain $G(b, r) = -b + Wr + \rho$ from Eq. (5.3.13). If $r < (b-\rho)/W$, then $G(b, r) < 0$, implying $h < b$, or else $G(b, r) \geq 0$, implying $h \geq b$.

(c,d) Evident from Lemma 5.3.2(c,e).

(e) The same as the proofs of Lemma 4.3.3(e). ∎

Below, when regarding h as a function of r, i.e., $h = h(r)$, by r_b let us denote the solution of $h(r) = b$ if it exits.

Lemma 5.3.6

(a) *r_b uniquely exists.*

(b) *$L(h)$, $J(h)$, and $\bar{J}(h)$ are strictly decreasing in $r < r_b$ and nonincreasing in r.*

(c) *$L(h) > 0$, $J(h) > 0$, and $\bar{J}(h) > 0$ for any sufficiently small r.*

(d) *If $s > 0$, then $L(h) < 0$, $J(h) < 0$, and $\bar{J}(h) < 0$ for any sufficiently large r.*

(e) *If $\nu > \bar{\nu}$, then $L(h) > J(h) > \bar{J}(h)$ for $r < r_b$ and $L(h) = J(h) = \bar{J}(h) = -s$ for $r \geq r_b$.*

(f) *If $\nu < \bar{\nu}$, then $L(h) > \bar{J}(h) > J(h)$ for $r < r_b$ and $L(h) = J(h) = \bar{J}(h) = -s$ for $r \geq r_b$.*

(g) *If $\nu = \bar{\nu}$, then $L(h) > J(h) = \bar{J}(h)$ for $r < r_b$ and $L(h) = J(h) = \bar{J}(h) = -s$ for $r \geq r_b$.*

Proof. (a) Immediate from Lemma 5.3.5(e).

(b-d) The same as the proofs of Lemma 4.3.4(b-d).

(e) The same as the proof of Lemma 4.3.4(e).

(f) Let $\nu < \bar{\nu}$. Then noting $1 > q + (1-q)\bar{\nu} > q + (1-q)\nu$, we can easily prove the assertion in quite the same way as in the proof of Lemma 4.3.4(e).

(g) Evident. ∎

From Lemma 5.3.6 we can depict Figure 5.3.1.

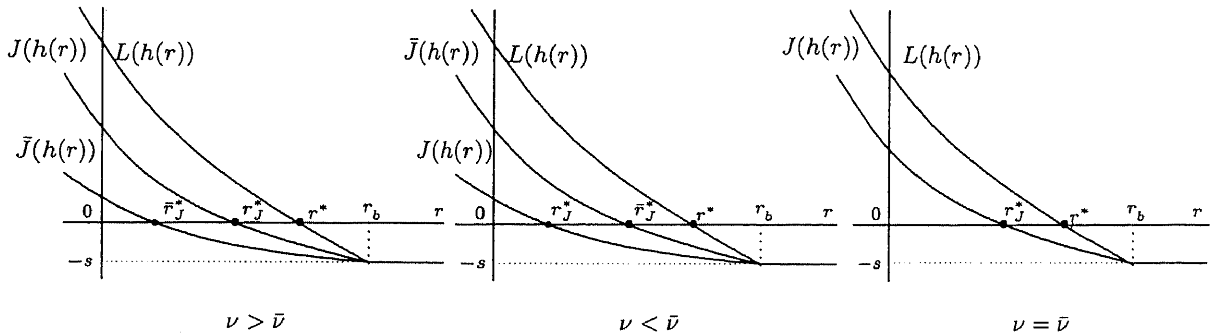

Figure 5.3.1: Graphs of $L(h(r))$, $J(h(r))$, and $\bar{J}(h(r))$ where $s > 0$.

Lemma 5.3.7

(a) *Let $s = 0$. Then $L(h) \geq 0$, $J(h) \geq 0$, and $\bar{J}(h) \geq 0$.*

(b) *Let $s > 0$. Then r^*, r_J^*, and $\bar{r}_J^*$ uniquely exist with $r^* < r_b$.*

1 *If $r^* \leq r$, then $L(h) \leq 0$, $J(h) \leq 0$, and $\bar{J}(h) \leq 0$.*

2 *Let $\nu > \bar{\nu}$. Then $\bar{r}_J^* < r_J^* < r^*$.*

i *If $r_J^* \leq r < r^*$, then $L(h) > 0$, $J(h) \leq 0$, and $\bar{J}(h) \leq 0$.*

ii *If $\bar{r}_J^* \leq r < r_J^*$, then $L(h) > 0$, $J(h) > 0$, and $\bar{J}(h) \leq 0$.*

iii *If $r < \bar{r}_J^*$, then $L(h) > 0$, $J(h) > 0$, and $\bar{J}(h) > 0$.*

3 *Let $\nu < \bar{\nu}$. Then $r_J^* < \bar{r}_J^* < r^*$.*

i *If $\bar{r}_J^* \leq r < r^*$, then $L(h) > 0$, $\bar{J}(h) \leq 0$, and $J(h) \leq 0$.*

ii *If $r_J^* \leq r < \bar{r}_J^*$, then $L(h) > 0$, $\bar{J}(h) > 0$, and $J(h) \leq 0$.*

iii *If $r < r_J^*$, then $L(h) > 0$, $\bar{J}(h) > 0$, and $J(h) > 0$.*

4 *Let $\nu = \bar{\nu}$. Then $r_J^* = \bar{r}_J^* < r^*$.*

Proof. (a) If $s = 0$, then $L(h) = \lambda\beta T(h) \geq 0$, $J(h) = \lambda\beta(q + (1-q)\nu)T(h) \geq 0$, and $\bar{J}(h) = \lambda\beta(q + (1-q)\bar{\nu})T(h) \geq 0$ due to Lemma 3.2.2(b).

(b) Let $s > 0$. The unique existence of r^*, r_J^*, and $\bar{r}_J^*$ are immediate from Lemma 5.3.6(b,c,d).

(b1-b4) Evident from Lemma 5.3.6(b-f). ∎

For convenience in the later discussions, let us regard r^*, r_J^*, and $\bar{r}_J^*$ as functions of s, i.e., $r^*(s)$, $r_J^*(s)$, and $\bar{r}_J^*(s)$. Further, by s^*, s_J^*, and $\bar{s}_J^*$ let us denote the solution of $r^*(s) = 0$, $r_J^*(s) = 0$, and $\bar{r}_J^*(s) = 0$ respectively, if they exist, i.e.,

$$r^*(s^*) = 0, \qquad r_J^*(s_J^*) = 0, \qquad \bar{r}_J^*(\bar{s}_J^*) = 0. \tag{5.3.28}$$

Lemma 5.3.8

(a) *$r^*(s)$, $r_J^*(s)$, and $\bar{r}_J^*(s)$ are strictly decreasing in s.*

(b) *If $s = 0$, then $r^* = r_J^* = \bar{r}_J^* = r_b = (b - \rho)/W$.*

(c) *$s^* = \lambda\beta T(\rho)$.*

(d) *Let $\nu > \bar{\nu}$.*

1 $s_J^* = \big(q + (1-q)\nu\big)\big(x_1^*(0) - \rho\big)/W(1-q)(1-\nu)$.

2 $\bar{s}_J^* = \big(q + (1-q)\bar{\nu}\big)(x_3^*(0) - \rho)/(W - H)$.

3 *If $r < (b - \rho)/W$, then $\bar{s}_J^* < s_J^* < s^*$.*

(e) *Let $\nu < \bar{\nu}$.*

1 $s_J^* = \big(q + (1-q)\nu\big)(x_3^*(0) - \rho\big)/(W - H)$.

2 $\bar{s}_J^* = \big(q + (1-q)\bar{\nu}\big)\big(\bar{x}_1^*(0) - \rho\big)/W(1-q)(1-\nu)$.

3 *If $r < (b - \rho)/W$, then $s_J^* < \bar{s}_J^* < s^*$.*

(f) *Let $\nu = \bar{\nu}$. Then $s_J^* = \bar{s}_J^* = \big(q + (1-q)\nu\big)(x_1^*(0) - \rho)/(W - H)$ and if $r < (b - \rho)/W$, then $s_J^* = \bar{s}_J^* < s^*$.*

Proof. (a) Let $s < s'$. Then since $L\big(h(r^*(s))\big) = 0$ and $L\big(h(r^*(s'))\big) = 0$, we have $s = \lambda\beta T\big(h(r^*(s))\big)$ and $s' = \lambda\beta T\big(h(r^*(s'))\big)$. Accordingly $\lambda\beta T\big(h(r^*(s))\big) = s < s' = \lambda\beta T\big(h(r^*(s'))\big)$. Hence $h\big(r^*(s)\big) > h\big(r^*(s')\big)$ because if not so, we have the contradiction of $T\big(h(r^*(s))\big) \geq T\big(h(r^*(s'))\big)$ due to Lemma 3.2.2(a). Therefore, we obtain $r^*(s) > r^*(s')$ from Lemma 4.3.3(e). Similarly also proven for $r_J^*(s)$ and $\bar{r}_J^*(s)$.

(b) Let $s = 0$. Then since $L(x) = \lambda\beta T(x) \geq 0$, $J(x) = \lambda\beta(q + (1-q)\nu)T(x) \geq 0$, and $\bar{J}(x) = \lambda\beta(q + (1-q)\bar{\nu})T(x) \geq 0$ for all x due to Lemma 3.2.2(b), from Eq. (5.3.13) we have

$$\begin{aligned} G(x, r) &= WL(x) - (1-y)J(x)/(1-\kappa) - y\bar{J}(x)/(1-\bar{\kappa}) - x + Wr + \rho \\ &= \lambda\beta\Big(W - (1-y)\big(q + (1-q)\nu\big)/(1-\kappa) - y\big(q + (1-q)\bar{\nu}\big)/(1-\bar{\kappa})\Big)T(x) - x + Wr + \rho \\ &= \lambda\beta(W - H)T(x) - x + Wr + \rho \quad \text{(due to Eq. (5.3.6))}. \end{aligned}$$

Hence letting $\dot{r} = (b - \rho)/W$, we get $0 = G\big(h(\dot{r}), \dot{r}\big) = \lambda\beta(W - H)T(h(\dot{r})) - h(\dot{r}) + b$. Thus

$$\lambda\beta(W - H)T(h(\dot{r})) = h(\dot{r}) - b. \tag{5.3.29}$$

Since $W > H$ due to Eq. (5.3.12) and $T(h(\dot{r})) \geq 0$ due to Lemma 3.2.2(b), we obtain $h(\dot{r}) \geq b$. Accordingly, in quite the same way as the proofs of Lemma 4.3.6(b) we can prove $r^* = r_J^* = \bar{r}_J^* = r_b$.

(c) From Eq. (5.2.3) we have

$$s = \lambda\beta T\big(h(r^*)\big), \tag{5.3.30}$$

$$s = \lambda\beta(q + (1-q)\nu)T\big(h(r_J^*)\big), \tag{5.3.31}$$

$$s = \lambda\beta(q + (1-q)\bar{\nu})T\big(h(\bar{r}_J^*)\big). \tag{5.3.32}$$

Since $1 > q + (1-q)\nu$ and $1 > q + (1-q)\bar{\nu}$ due to the assumption of $q < 1$, $\nu < 1$, and $\bar{\nu} < 1$, we have $0 = L(h(r^*)) \geq J(h(r^*))$ and $0 = L(h(r^*)) \geq \bar{J}(h(r^*))$ due to Lemma 3.2.2(b). Now, noting $h(r^*)$ is the solution of $G(x) = 0$ with $r = r^*$, we get

$$0 = G\big(h(r^*), r^*\big) = -h(r^*) + Wr^* + \rho,$$

hence $h(r^*) = Wr^* + \rho$ for any s. Accordingly, if $s = s^*$, then $r^* = r^*(s^*) = 0$, hence since $h\big(r^*(s^*)\big) = \rho$, we have $s^* = \lambda\beta T\big(h(r^*(s^*))\big) = \lambda\beta T(\rho)$ from Eq. (5.3.30).

(d) Let $\nu \geq \bar{\nu}$.

(d1) If $r = r^*_J$, then $L(h) > 0$, $J(h) = 0$, and $\bar{J}(h) \leq 0$ due to Lemma 5.3.7(b2,b2i). Hence from Eq. (5.3.13) we have, for $x = h(r^*_J)$,

$$0 = G\big(h(r^*_J), r^*_J\big) = W\big(\lambda\beta T(h(r^*_J)) - s\big) - h(r^*_J) + Wr^*_J + \rho.$$

Substituting Eq. (5.3.31) into the above equality yields

$$0 = W\lambda\beta(1-q)(1-\nu)T\big(h(r^*_J)\big) - h(r^*_J) + Wr^*_J + \rho,$$

implying

$$B_1\big(h(r^*_J)\big) = 0, \tag{5.3.33}$$

hence $x^*_1(r^*_J) = h(r^*_J)$ from Lemma 5.3.3(c). If $s = s^*_J$, then $r^*_J = r^*_J(s^*_J) = 0$ due to Eq. (5.3.28), thus $x^*_1(0) = h(0)$. Accordingly,

$$T\big(x^*_1(0)\big) = (x^*_1(0) - \rho)/W\lambda\beta(1-q)(1-\nu). \tag{5.3.34}$$

Now, from Eq. (5.3.31) we have

$$s^*_J = \lambda\beta(q + (1-q)\nu)T(h(r^*_J)) = \lambda\beta(q + (1-q)\nu)T(h(0)) = \lambda\beta(q + (1-q)\nu)T\big(x^*_1(0)\big),$$

from which $T\big(x^*_1(0)\big) = s^*_J/\lambda\beta(q + (1-q)\nu)$, hence from Eq. (5.3.34) we obtain

$$s^*_J/\lambda\beta(q + (1-q)\nu) = \big(x^*_1(0) - \rho\big)/W\lambda\beta(1-q)(1-\nu)$$

or equivalently $s^*_J = (q + (1-q)\nu)\big(x^*_1(0) - \rho\big)/W(1-q)(1-\nu)$.

(d2) If $r = \bar{r}^*_J$, then $L(h) > 0$, $J(h) \geq 0$, and $\bar{J}(h) = 0$ due to Lemma 5.3.7(b2,b2ii). Hence from Eq. (5.3.13) we have, for $x = h(r^*_J)$,

$$\begin{aligned}
0 &= G\big(h(\bar{r}^*_J), \bar{r}^*_J\big) \\
&= \lambda\beta\Big(W - (1-y)(q + (1-q)\nu)/(1-\kappa)\Big)T(h(\bar{r}^*_J)) \\
&\qquad\qquad -\big(W - (1-y)/(1-\kappa)\big)s - h(\bar{r}^*_J) + W\bar{r}^*_J + \rho \\
&= \lambda\beta\Big(W - (1-y)(q + (1-q)\nu)/(1-\kappa)\Big)T(h(\bar{r}^*_J)) - ys/(1-\bar{\kappa}) - h(\bar{r}^*_J) + W\bar{r}^*_J + \rho
\end{aligned}$$

due to Eq. (5.3.5). Substituting Eq. (5.3.32) into the above equations produces

$$\begin{aligned} 0 &= \lambda\beta\Big(W - (1-y)(q+(1-q)\nu)/(1-\kappa) - y(q+(1-q)\bar{\nu})/(1-\bar{\kappa})\Big)T(h(\bar{r}_J^*)) - h(\bar{r}_J^*) + W\bar{r}_J^* + \rho \\ &= \lambda\beta(W-H)T(h(\bar{r}_J^*)) - h(\bar{r}_J^*) + W\bar{r}_J^* + \rho \quad \text{(due to Eq. (5.3.6))} \\ &= T(h(\bar{r}_J^*)) - \big(h(\bar{r}_J^*) - W\bar{r}_J^* - \rho\big)/\lambda\beta(W-H), \end{aligned}$$

implying

$$B_3\big(h(\bar{r}_J^*)\big) = 0, \tag{5.3.35}$$

hence $x_3^*(\bar{r}_J^*) = h(\bar{r}_J^*)$ from Lemma 5.3.3(c). If $s = \bar{s}_J^*$, then $\bar{r}_J^* = \bar{r}_J^*(\bar{s}_J^*) = 0$ due to Eq. (5.3.28), thus $x_3^*(0) = h(0)$. Accordingly,

$$T\big(x_3^*(0)\big) = \big(x_3^*(0) - \rho\big)/\lambda\beta(W-H). \tag{5.3.36}$$

Now, from Eq. (5.3.32) we have

$$\bar{s}_J^* = \lambda\beta(q+(1-q)\bar{\nu})T(h(\bar{r}_J^*)) = \lambda\beta(q+(1-q)\bar{\nu})T(h(0)) = \lambda\beta(q+(1-q)\bar{\nu})T\big(x_3^*(0)\big),$$

from which $T\big(x_3^*(0)\big) = \bar{s}_J^*/\lambda\beta(q+(1-q)\bar{\nu})$, hence from Eq. (5.3.36) we get

$$\bar{s}_J^*/\lambda\beta(q+(1-q)\bar{\nu}) = \big(x_3^*(0)-\rho\big)/\lambda\beta(W-H)$$

or equivalently $\bar{s}_J^* = (q+(1-q)\bar{\nu})\big(x_3^*(0)-\rho\big)/(W-H)$.

(d3) Since $T(x_1^*(0)) \ge 0$ due to Lemma 3.2.2(b), we have $x_1^*(0) \ge \rho$ from Eq. (5.3.34), thus $T(\rho) \ge T(x_1^*(0))$ from Lemma 3.2.2(a). Accordingly, from Eq. (5.3.34) and (d1) we have

$$\begin{aligned} T(\rho) &\ge \frac{x_1^*(0)-\rho}{W\lambda\beta(1-q)(1-\nu)} \\ &= \frac{(q+(1-q)\nu)(x_1^*(0)-\rho)/W(1-q)(1-\nu)}{\lambda\beta(q+(1-q)\nu)} = \frac{s_J^*}{\lambda\beta(q+(1-q)\nu)}. \end{aligned}$$

Let $r < (b-\rho)/W$. Then $b > \rho$ due to the assumption of $r \ge 0$. Accordingly, since $T(\rho) > 0$ from Lemma 3.2.2(c), we obtain $s_J^* \le \lambda\beta(q+(1-q)\nu)T(\rho) < \lambda\beta T(\rho) = s^*$ due to (c) and the assumptions of $q < 1$ and $\nu < 1$. Further, $x_1^* < x_3^* < b$ from Lemma 5.3.3(d1), hence $T(x_3^*) < T(x_1^*)$ due to Lemma 3.2.2(a). Therefore, from Eqs. (5.3.34) and (5.3.36) we can immediately show that $\bar{s}_J^* < s_J^*$. Thus we eventually obtain $\bar{s}_J^* < s_J^* < s$.

(e) Let $\nu < \bar{\nu}$.

(e1) If $r = r_J^*$, then $L(h) > 0$, $\bar{J}(h) > 0$, and $J(h) = 0$ due to Lemma 5.3.7(b2,b2i). Hence from Eq. (5.3.13) we have, for $x = h(r_J^*)$,

$$\begin{aligned} 0 &= G\big(h(r_J^*), r_J^*\big) \\ &= \lambda\beta\Big(W - y(q+(1-q)\bar{\nu})/(1-\bar{\kappa})\Big)T(h(r_J^*)) - \big(W - y/(1-\bar{\kappa})\big)s - h(r_J^*) + Wr_J^* + \rho \\ &= \lambda\beta\Big(W - y(q+(1-q)\bar{\nu})/(1-\bar{\kappa})\Big)T(h(r_J^*)) - (1-y)s/(1-\kappa) - h(r_J^*) + Wr_J^* + \rho. \end{aligned}$$

due to Eq. (5.3.5). Substituting Eq. (5.3.31) into the above equality yields

$$\begin{aligned} 0 &= \lambda\beta\Big(W - y(q+(1-q)\bar{\nu})/(1-\bar{\kappa}) - (1-y)(q+(1-q)\nu)/(1-\kappa)\Big)T(h(\bar{r}_J^*)) - h(\bar{r}_J^*) + W\bar{r}_J^* + \rho \\ &= \lambda\beta(W-H)T(h(\bar{r}_J^*)) - h(\bar{r}_J^*) + W\bar{r}_J^* + \rho \quad \text{(due to Eq. (5.3.6))} \\ &= T(h(\bar{r}_J^*)) - \big(h(\bar{r}_J^*) - W\bar{r}_J^* - \rho\big)/\lambda\beta(W-H), \end{aligned}$$

implying

$$B_3\big(h(r_J^*)\big) \;=\; 0, \tag{5.3.37}$$

hence $x_3^*(r_J^*) = h(r_J^*)$ from Lemma 5.3.3(c). If $s = s_J^*$, then $r_J^* = r_J^*(s_J^*) = 0$ due to Eq. (5.3.28), thus $x_3^*(0) = h(0)$. Accordingly,

$$T\big(x_3^*(0)\big) \;=\; (x_3^*(0) - \rho)/\lambda\beta(W-H). \tag{5.3.38}$$

Now, from Eq. (5.3.31) we have

$$s_J^* = \lambda\beta(q+(1-q)\nu)T(h(r_J^*)) = \lambda\beta(q+(1-q)\nu)T(h(0)) = \lambda\beta(q+(1-q)\nu)T\big(x_3^*(0)\big),$$

from which $T\big(x_3^*(0)\big) = s_J^*/\lambda\beta(q+(1-q)\nu)$, hence from Eq. (5.3.38) we obtain

$$s_J^*/\lambda\beta(q+(1-q)\nu) = \big(x_3^*(0)-\rho\big)/\lambda\beta(W-H)$$

or equivalently $s_J^* = (q+(1-q)\nu)\big(x_3^*(0)-\rho\big)/(W-H)$.

(e2) If $r = \bar{r}_J^*$, then $L(h) > 0$, $J(h) \le 0$, and $\bar{J}(h) = 0$ due to Lemma 5.3.7(b2,b2i). Hence from Eq. (5.3.13) we have, for $x = h(\bar{r}_J^*)$,

$$0 = G\big(h(\bar{r}_J^*), \bar{r}_J^*\big) = W\big(\lambda\beta T(h(\bar{r}_J^*)) - s\big) - h(\bar{r}_J^*) + W\bar{r}_J^* + \rho.$$

Substituting Eq. (5.3.32) into the above equality yields

$$0 = W\lambda\beta(1-q)(1-\bar{\nu})T\big(h(\bar{r}_J^*)\big) - h(\bar{r}_J^*) + W\bar{r}_J^* + \rho,$$

implying

$$\bar{B}_1\big(h(\bar{r}_J^*)\big) \;=\; 0, \tag{5.3.39}$$

hence $\bar{x}_1^*(\bar{r}_J^*) = h(\bar{r}_J^*)$ from Lemma 5.3.3(c). If $s = \bar{s}_J^*$, then $\bar{r}_J^* = \bar{r}_J^*(s_J^*) = 0$ due to Eq. (5.3.28), thus $\bar{x}_1^*(0) = h(0)$. Accordingly,

$$T\big(\bar{x}_1^*(0)\big) \;=\; (\bar{x}_1^*(0) - \rho)/W\lambda\beta(1-q)(1-\bar{\nu}). \tag{5.3.40}$$

Now, from Eq. (5.3.31) we have

$$\bar{s}_J^* = \lambda\beta(q+(1-q)\bar{\nu})T(h(\bar{r}_J^*)) = \lambda\beta(q+(1-q)\bar{\nu})T(h(0)) = \lambda\beta(q+(1-q)\bar{\nu})T\big(\bar{x}_1^*(0)\big),$$

from which $T\big(\bar{x}_1^*(0)\big) = \bar{s}_J^*/\lambda\beta(q+(1-q)\bar{\nu})$, hence from Eq. (5.3.40) we get

$$\bar{s}_J^*/\lambda\beta(q+(1-q)\bar{\nu}) = \big(\bar{x}_1^*(0)-\rho\big)/W\lambda\beta(1-q)(1-\bar{\nu})$$

or equivalently $\bar{s}_J^* = (q+(1-q)\bar{\nu})\big(\bar{x}_1^*(0)-\rho\big)/W(1-q)(1-\bar{\nu})$.

(e3) Since $T(\bar{x}_1^*(0)) \geq 0$ due to Lemma 3.2.2(b), we have $\bar{x}_1^*(0) \geq \rho$ from Eq. (5.3.40), thus $T(\rho) \geq T(\bar{x}_1^*(0))$ from Lemma 3.2.2(a). Accordingly, from Eq. (5.3.40) and (e2) we have

$$\begin{aligned} T(\rho) &\geq \frac{\bar{x}_1^*(0) - \rho}{W\lambda\beta(1-q)(1-\bar{\nu})} \\ &= \frac{(q+(1-q)\bar{\nu})(\bar{x}_1^*(0)-\rho)/W(1-q)(1-\bar{\nu})}{\lambda\beta(q+(1-q)\bar{\nu})} = \frac{\bar{s}_J^*}{\lambda\beta(q+(1-q)\bar{\nu})}. \end{aligned}$$

Let $r < (b-\rho)/W$. Then $b > \rho$ due to the assumption of $r \geq 0$. Accordingly, since $T(\rho) > 0$ from Lemma 3.2.2(c), we obtain $\bar{s}_J^* \leq \lambda\beta(q+(1-q)\bar{\nu})T(\rho) < \lambda\beta T(\rho) = s^*$ due to (c) and the assumptions of $q < 1$ and $\nu < 1$. Further, $\bar{x}_1^* < x_3^* < b$ from Lemma 5.3.3(d1), hence $T(x_3^*) > T(\bar{x}_1^*)$ due to Lemma 3.2.2(a). Therefore, from Eqs. (5.3.38) and (5.3.40) we can immediately show that $s_J^* < \bar{s}_J^*$. Thus we eventually obtain $s_J^* < \bar{s}_J^* < s$.

(f) Immediate from the fact that $x_1^* = \bar{x}_1^* = x_3^*$ due to Lemma 5.3.3(f) and $W(1-q)(1-\nu) = W(1-q)(1-\bar{\nu}) = W - H$, which was shown in the proof of Lemma 5.3.3(f). ∎

Lemma 5.3.9

(a) *Let $s^* \leq s$. Then $L(h) \leq 0$, $J(h) \leq 0$, and $\bar{J}(h) \leq 0$.*

(b) *Let $s^* > s$.*

- 1 $u(\phi) > 0$.
- 2 $r^* > 0$.
- 3 *Let $r < r^*$.*
  - i *Let $\nu > \bar{\nu}$.*
    - 1 *If $r_J^* \leq r$, then $x_4^* \leq \chi_1$.*
    - 2 *If $\bar{r}_J^* \leq r < r_J^*$, then $\chi_1 < x_2^* \leq \chi_2$.*
    - 3 *If $r < \bar{r}_J^*$, then $h = x_3^* > \chi_2$.*
  - ii *Let $\nu < \bar{\nu}$.*
    - 1 *If $\bar{r}_J^* \leq r$, then $x_4^* \leq \bar{\chi}_1$.*
    - 2 *If $r_J^* \leq r < \bar{r}_J^*$, then $\bar{\chi}_1 < \bar{x}_2^* \leq \bar{\chi}_2$.*
    - 3 *If $r < r_J^*$, then $x_3^* > \bar{\chi}_2$.*
  - iii *Let $\nu = \bar{\nu}$. Then $r_J = \bar{r}_J^*$.*
    - 1 *If $r_J^* \leq r$, then $x_4^* \leq \chi_1$.*
    - 2 *If $r < r_J^*$, then $x_3^* > \chi_1$.*

Proof. (a) Let $s \geq s^*$. Then $s \geq \lambda\beta T(\rho)$ due to Lemma 5.3.8(c). Now, since $h \geq \rho$ from Lemma 5.3.5(a), we have $T(h) \leq T(\rho)$ from Lemma 3.2.2(a). Hence, noting Lemma 3.2.2(b), we get $0 \geq \lambda\beta T(\rho) - s \geq \lambda\beta T(h) - s \geq \lambda\beta(q+(1-q)\nu)T(h) - s$ and $0 \geq \lambda\beta T(h) - s \geq \lambda\beta(q+(1-q)\bar{\nu})T(h) - s$ due to $1 > q+(1-q)\nu$ and $1 > q+(1-q)\bar{\nu}$. Therefore, we obtain $L(h) \leq 0$, $J(h) \leq 0$, and $\bar{J}(h) \leq 0$.

(b) Let $s^* > s$.

(b1) $u(\phi) \geq 0$ from Eq. (5.2.12). Suppose $u(\phi) = 0$. Then since $r = 0$ and $L(h) \leq 0 \cdots (*)$, from Eq. (5.3.26) we have $h = -(1-y)\max\{J(h), 0\}/(1-\kappa) - y\max\{\bar{J}(h), 0\}/(1-\bar{\kappa}) + \rho$, from which $h \leq \rho$. From this and Lemma 5.3.5(a) we get $h = \rho$. Accordingly, $L(h) = \lambda\beta T(h) - s = \lambda\beta T(\rho) - s = s^* - s > 0$ due to the assumption of $s^* > s$, which contradicts $(*)$. Thus $u(\phi) > 0$.

(b2) Let $r = 0$. Assume $\lambda\beta v(0) + (1-\lambda)\beta u(\phi) - s \leq \beta u(\phi)$. Then $u(\phi) = \beta u(\phi)$ from Eq. (5.2.5), leading to $\beta = 1$ due to $u(\phi) > 0$ from (b1), which contradicts the assumption of $\beta < 1$. Accordingly, it must be $\lambda\beta v(0) + (1-\lambda)\beta u(\phi) - s > \beta u(\phi)$, which can be rearranged into $0 < \lambda\beta(v(0) - u(\phi)) - s = \lambda\beta T(h) - s = L(h(0))$ from Eq. (5.2.8), implying $r^* > 0$ due to Lemma 5.3.6(b).

(b3) Let $r < r^*$. Then $L(h) > 0$ from Lemma 5.3.7(b2ii). Here note $r_J^* < r^*$.

(b3i) Let $\nu > \bar{\nu}$.

(b3i1) Let $r_J^* \leq r$. Then $0 \geq J(h) \geq \bar{J}(h)$ from Lemma 5.3.7(b2i), hence from Eq. (5.3.26) we have

$$h = \lambda\beta W(T(h) - s) + Wr + \rho,$$

hence $T(h) - \big(h + W(s-r) - \rho\big)/W\lambda\beta = 0$, i.e., $B_4(h) = 0$ from Eq. (5.3.19), implying that h defined by Eq. (5.1.1) is given by x_4^*, which is the unique solution of $B_4(x) = 0$ due to Lemma 5.3.3(c), i.e., $h = x_4^*$. Consequently, we get

$$T(h) = T(x_4^*) = \big(x_4^* + W(s-r) - \rho\big)/W\lambda\beta. \tag{5.3.41}$$

Now, from the assumption of $r_J^* \leq r$ we obtain $0 \geq J(h) = \lambda\beta\big(q+(1-q)\nu\big)T(h) - s$ due to Lemma 5.3.7(b2i). Rearranging the inequality by substituting Eq. (5.3.41) produces

$$0 \;\geq\; \big(q + (1-q)\nu\big)\big(x_4^* + W(s-r) - \rho\big)/W - s,$$

from which

$$\begin{aligned} x_4^* \;&\leq\; \frac{Ws}{q+(1-q)\nu} - Ws + Wr + \rho \\ &=\; \frac{Ws}{q+(1-q)\nu}(1 - q - (1-q)\nu) + Wr + \rho \\ &=\; \frac{Ws(1-q)(1-\nu)}{q+(1-q)\nu} + Wr + \rho = \chi_1. \end{aligned}$$

(b3i2) Let $\bar{r}_J^* \leq r < r_J^*$. Then $J(h) > 0 \geq \bar{J}(h)$ from Lemma 5.3.7(b2ii), hence from Eq. (5.3.26) we have

$$\begin{aligned} h \;&=\; WL(h) - (1-y)J(h)/(1-\kappa) + Wr + \rho \\ &=\; \lambda\beta\big(W - (q+(1-q)\nu)(1-y)/(1-\kappa)\big)T(h) - \big(W - (1-y)/(1-\kappa)\big)s + Wr + \rho \\ &=\; \lambda\beta\big(W - (q+(1-q)\nu)(1-y)/(1-\kappa)\big)T(h) - ys/(1-\bar{\kappa}) + Wr + \rho. \end{aligned}$$

Hence

$$T(h) - \frac{h + ys/(1-\bar{\kappa}) - Wr - \rho}{\lambda\beta\big(W - (q+(1-q)\nu)(1-y)/(1-\kappa)\big)} = 0$$

or equivalently $B_2(h) = 0$ from Eq. (5.3.16). This implies that h defined by Eq. (5.1.1) is given by x_2^*, which is the unique solution of $B_4(x) = 0$ due to Lemma 5.3.3(c), i.e., $h = x_2^*$. Therefore,

$$T(h) = T(x_2^*) = \frac{x_2^* + ys/(1-\bar{\kappa}) - Wr - \rho}{\lambda\beta\big(W - (q+(1-q)\nu)(1-y)/(1-\kappa)\big)}. \tag{5.3.42}$$

Now, from the assumption of $\bar{r}_J^* \le r < r_J^*$ we obtain

$$0 \;<\; J(h) = \lambda\beta\big(q + (1-q)\nu\big)T(h) - s, \tag{5.3.43}$$

$$0 \;\ge\; \bar{J}(h) = \lambda\beta\big(q + (1-q)\bar{\nu}\big)T(h) - s \tag{5.3.44}$$

due to Lemma 5.3.7(b2ii). Rearranging Eq. (5.3.43) by substituting Eq. (5.3.42) produces

$$0 \;<\; \big(q + (1-q)\nu\big)\frac{x_2^* + ys/(1-\bar{\kappa}) - Wr - \rho}{W - (q+(1-q)\nu)(1-y)/(1-\kappa)} - s,$$

from which

$$\begin{aligned}
x_2^* \;&>\; \frac{W - (q+(1-q)\nu)(1-y)/(1-\kappa)}{q+(1-q)\nu}s - \frac{yc}{1-\bar{\kappa}} + Wr + \rho \\
&= \frac{s}{q+(1-q)\nu}\Big(W - \frac{(q+(1-q)\nu)(1-y)}{1-\kappa} - \frac{y(q+(1-q)\nu)}{1-\bar{\kappa}}\Big) + Wr + \rho \\
&= \frac{s}{q+(1-q)\nu}\big(W - (q+(1-q)\nu)W\big) + Wr + \rho \\
&= Ws(1-q)(1-\nu)/(q+(1-q)\nu) + Wr + \rho = \chi_1.
\end{aligned}$$

Further, rearranging Eq. (5.3.44) by substituting Eq. (5.3.42) produces

$$0 \;\ge\; \big(q + (1-q)\bar{\nu}\big)\frac{x_2^* + ys/(1-\bar{\kappa}) - Wr - \rho}{W - (q+(1-q)\nu)(1-y)/(1-\kappa)} - s,$$

from which

$$\begin{aligned}
x_2^* \;&\le\; \frac{W - (q+(1-q)\nu)(1-y)/(1-\kappa)}{q+(1-q)\bar{\nu}}s - \frac{yc}{1-\bar{\kappa}} + Wr + \rho \\
&= \frac{s}{q+(1-q)\bar{\nu}}\Big(W - \frac{(q+(1-q)\nu)(1-y)}{1-\kappa} - \frac{y(q+(1-q)\bar{\nu})}{1-\bar{\kappa}}\Big) + Wr + \rho \\
&= (W-H)s/(q+(1-q)\bar{\nu}) + Wr + \rho = \chi_2
\end{aligned}$$

due to Eqs. (5.3.6) and (5.3.9). Thus, $\chi_1 < x_2^* \le \chi_2$.

(b3i3) Let $r < \bar{r}_J^*$. Then $L(h) > J(h) \ge \bar{J}(h) > 0$ from Lemma 5.3.7(b2iii), hence from Eq. (5.3.26) we have

$$\begin{aligned}
h \;&=\; WL(h) - (1-y)J(h)/(1-\kappa) - y\bar{J}/(1-\bar{\kappa}) + Wr + \rho \\
&= \lambda\beta\big(W - \frac{(q+(1-q)\nu)(1-y)}{1-\kappa} - \frac{y(q+(1-q)\bar{\nu})}{1-\bar{\kappa}}\big)T(x) \\
&\qquad\qquad + \big(W - \frac{1-y}{1-\kappa} - \frac{y}{1-\bar{\kappa}}\big)s + Wr + \rho \\
&= \lambda\beta(W-H)T(h) + Wr + \rho.
\end{aligned}$$

Accordingly, we get

$$T(h) - \frac{h - Wr - \rho}{\lambda\beta(W-H)} = 0$$

or equivalently $B_3(h) = 0$ from Eq. (5.3.18), implying that h defined by Eq. (5.1.1) is given by x_3^*, which is the unique solution of $B_3(x) = 0$ due to Lemma 5.3.3(c), i.e., $h = x_3^*$. Hence

$$T(h) = T(x_3^*) = \frac{x_3^* - Wr - \rho}{\lambda\beta(W - H)}. \tag{5.3.45}$$

Now, from the assumption of $r < \bar{r}_J^*$ we obtain $0 < \bar{J}(h) = \lambda\beta\big(q+(1-q)\bar{\nu}\big)T(h)-s$ due to Lemma 5.3.7(b2iii). Rearranging the inequality by substituting Eq. (5.3.45) produces

$$0 \;<\; \frac{x_3^* - Wr - \rho}{W - H}\big(q + (1-q)\bar{\nu}\big) - s,$$

from which

$$x_3 \;>\; (W - H)s/(q + (1-q)\bar{\nu}) + Wr + \rho = \chi_2.$$

(b3ii) Let $\nu < \bar{\nu}$.

(b3ii1) Let $\bar{r}_J^* \leq r$. Then $0 \geq \bar{J}(h) \geq J(h)$ from Lemma 5.3.7(b3i), hence from Eq. (5.3.26) we have

$$h = \lambda\beta W T(h) - Ws + Wr + \rho,$$

hence $T(h) - \big(h + W(s-r) - \rho\big)/W\lambda\beta = 0$, i.e., $B_4(h) = 0$ from Eq. (5.3.19). This implies that h defined by Eq. (5.1.1) is given by x_4^*, which is the unique solution of $B_4(x) = 0$ due to Lemma 5.3.3(c), i.e., $h = x_4^*$. Thus we obtain

$$T(h) = T(x_4^*) = \big(x_4^* + W(s-r) - \rho\big)/W\lambda\beta. \tag{5.3.46}$$

Now, from the assumption of $\bar{r}_J^* \leq r$ we obtain $0 \geq \bar{J}(h) = \lambda\beta\big(q+(1-q)\bar{\nu}\big)T(h)-s$ due to Lemma 5.3.7(b3i). Rearranging the inequality by substituting Eq. (5.3.46) produces

$$0 \;\geq\; \big(q + (1-q)\bar{\nu}\big)\big(x_4^* + W(s-r) - \rho\big)/W - s,$$

from which

$$\begin{aligned} x_4^* &\leq \frac{Ws}{q + (1-q)\bar{\nu}} - Ws + Wr + \rho \\ &= \frac{Ws}{q + (1-q)\bar{\nu}}(1 - q - (1-q)\nu) + Wr + \rho \\ &= \frac{Ws(1-q)(1-\bar{\nu})}{q + (1-q)\bar{\nu}} + Wr + \rho = \bar{\chi}_1. \end{aligned}$$

(b3ii2) Let $r_J^* \leq r < \bar{r}_J^*$. Then $\bar{J}(h) > 0 \geq J(h)$ from Lemma 5.3.7(b3ii), hence from Eq. (5.3.26) we have

$$\begin{aligned} h &= WL(h) - y\bar{J}(h)/(1-\bar{\kappa}) + Wr + \rho \\ &= \lambda\beta\big(W - (q + (1-q)\bar{\nu})y/(1-\bar{\kappa})\big)T(h) - \big(W - y/(1-\bar{\kappa})\big)s + Wr + \rho \\ &= \lambda\beta\big(W - (q + (1-q)\bar{\nu})y/(1-\bar{\kappa})\big)T(h) - (1-y)s/(1-\kappa) + Wr + \rho. \end{aligned}$$

Accordingly, we get

$$T(h) - \frac{h + (1-y)s/(1-\kappa) - Wr - \rho}{\lambda\beta\big(W - (q+(1-q)\bar{\nu})y/(1-\bar{\kappa})\big)} = 0$$

or equivalently $\bar{B}_2(h) = 0$ from Eq. (5.3.16), implying that h defined by Eq. (5.1.1) is given by $\bar{x}_2^*$, which is the unique solution of $\bar{B}_2(x) = 0$ due to Lemma 5.3.3(c), i.e., $h = \bar{x}_2^*$. Hence

$$T(h) = T(\bar{x}_2^*) = \frac{\bar{x}_2^* + (1-y)s/(1-\kappa) - Wr - \rho}{\lambda\beta\big(W - (q+(1-q)\bar{\nu})y/(1-\bar{\kappa})\big)}. \tag{5.3.47}$$

Now, from the assumption of $r_J^* \leq r < \bar{r}_J^*$ we obtain

$$0 < J(h) = \lambda\beta\big(q + (1-q)\nu\big)T(h) - s \tag{5.3.48}$$

$$0 \geq \bar{J}(h) = \lambda\beta\big(q + (1-q)\bar{\nu}\big)T(h) - s \tag{5.3.49}$$

due to Lemma 5.3.7(b3ii). Rearranging Eq. (5.3.48) by substituting Eq. (5.3.47) produces

$$0 < \big(q + (1-q)\bar{\nu}\big)\frac{\bar{x}_2^* + (1-y)s/(1-\kappa) - Wr - \rho}{W - (q+(1-q)\bar{\nu})y/(1-\bar{\kappa})} - s,$$

from which

$$\begin{aligned}
\bar{x}_2^* &> \frac{W - (q+(1-q)\bar{\nu})y/(1-\bar{\kappa})}{q+(1-q)\bar{\nu}}s - \frac{(1-y)s}{1-\kappa} + Wr + \rho \\
&= \frac{s}{q+(1-q)\bar{\nu}}\Big(W - \frac{(q+(1-q)\bar{\nu})y}{1-\bar{\kappa}} - \frac{(q+(1-q)\bar{\nu})(1-y)}{1-\kappa}\Big) + Wr + \rho \\
&= \frac{s}{q+(1-q)\bar{\nu}}\big(W - (q+(1-q)\bar{\nu})W\big) + Wr + \rho \\
&= Ws(1-q)(1-\bar{\nu})/(q+(1-q)\bar{\nu}) + Wr + \rho = \bar{\chi}_1.
\end{aligned}$$

Further, rearranging Eq. (5.3.49) by substituting Eq. (5.3.47) produces

$$0 \geq \big(q + (1-q)\nu\big)\frac{x_2^* + (1-y)s/(1-\kappa) - Wr - \rho}{W - (q+(1-q)\bar{\nu})y/(1-\bar{\kappa})} - s,$$

from which

$$\begin{aligned}
x_2^* &\leq \frac{W - (q+(1-q)\bar{\nu})y/(1-\bar{\kappa})}{q+(1-q)\nu}s - \frac{(1-y)s}{1-\kappa} + Wr + \rho \\
&= \frac{s}{q+(1-q)\nu}\Big(W - \frac{(q+(1-q)\bar{\nu})y}{1-\bar{\kappa}} - \frac{(q+(1-q)\nu)(1-y)}{1-\kappa}\Big) + Wr + \rho \\
&= (W - H)s/(q+(1-q)\nu) + Wr + \rho = \bar{\chi}_2.
\end{aligned}$$

Thus we consequencly obtain $\bar{\chi}_1 < \bar{x}_2^* \leq \bar{\chi}_3$.

(b3ii3) Let $r < r_J^*$. Then $\bar{J}(h) > J(h) > 0$ from Lemma 5.3.7(b3iii), hence from Eq. (5.3.26) we have

$$\begin{aligned}
h &= WL(h) - (1-y)J(h)/(1-\kappa) - y\bar{J}/(1-\bar{\kappa}) + Wr + \rho \\
&= \lambda\beta\big(W - H\big)T(h) + \big(W - (1-y)/(1-\kappa) - y/(1-\bar{\kappa})\big)s + Wr + \rho \\
&= \lambda\beta\big(W - H\big)T(h) + Wr + \rho.
\end{aligned}$$

Consequently, we get

$$T(h) - \frac{h - Wr - \rho}{\lambda\beta(W - H)} = 0$$

or equivalently $B_3(h) = 0$ from Eq. (5.3.18). This implies that h defined by Eq. (5.1.1) is given by x_3^*, which is the unique solution of $B_3(x) = 0$ due to Lemma 5.3.3(c), i.e., $h = x_3^*$. Hence

$$T(h) = T(x_3^*) = \frac{x_3^* - Wr - \rho}{\lambda\beta(W - H)}. \tag{5.3.50}$$

Now, from the assumption of $r < r_J^*$ we obtain $0 < J(h) = \lambda\beta\big(q+(1-q)\nu\big)T(h)-s$ due to Lemma 5.3.7(b3iii). Rearranging the former inequality by substituting Eq. (5.3.50) yields

$$0 < \frac{x_3^* - Wr - \rho}{W - H}\big(q + (1-q)\nu\big) - s,$$

from which

$$x_3 > (W - H)s/(q + (1-q)\nu) + Wr + \rho = \bar{\chi}_2.$$

(b3iii) Let $\nu = \bar{\nu}$. Then $r_J^* = \bar{r}_J^*$ due to Lemma 5.3.7(b4) and $\chi_1 = \chi_2 = \bar{\chi}_1 = \bar{\chi}_2$ due to Lemma 5.3.3(f).

(b3iii1) Immediate from (b3i1) and (b3i3).

(b3iii2) Evident from (b3ii1) and (b3ii3). ∎

Corollary 5.3.1

(a) *Let* $\nu > \bar{\nu}$.

1 *If* $x_4^* > \chi_1$, *then* $r < r_J^*$.

2 *If* $x_2^* \le \chi_1$ *or* $x_4^* > \chi_2$, *then* $r < \bar{r}_J^*$ *or* $r_J^* \le r$.

3 *If* $x_3^* \le \chi_2$, *then* $\bar{r}_J^* \le r$.

(b) *Let* $\nu < \bar{\nu}$.

1 *If* $x_4^* > \bar{\chi}_1$, *then* $r < r_J^*$.

2 *If* $\bar{x}_2^* \le \bar{\chi}_1$ *or* $\bar{x}_2^* > \bar{\chi}_2$, *then* $r < \bar{r}_J^*$ *or* $r_J^* \le r$.

3 *If* $x_3^* \le \bar{\chi}_2$, *then* $\bar{r}_J^* \le r$.

(c) *Let* $\nu = \bar{\nu}$.

1 *If* $x_4^* > \chi_1$, *then* $r < r_J^*$.

2 *If* $x_3^* \le \chi_1$, *then* $r_J^* \le r$.

Proof. Immediate from the contrapositions of Lemma 5.3.9(b3i,b3ii,b3iii), respectively. ∎

Lemma 5.3.10

(a) *Let* $\nu > \bar{\nu}$.

1 *If* $x_4^* \le \chi_1$, *then* $r_J^* \le r$.

2 *If* $\chi_1 < x_2^* \le \chi_2$, *then* $\bar{r}_J^* \le r < r_J^*$.

3 *If* $x_3^* > \chi_2$, *then* $r < \bar{r}_J^*$.

(b) *Let* $\nu < \bar{\nu}$.

1 *If* $x_4^* \le \bar{\chi}_1$, *then* $r_J^* \le r$.

2 *If* $\bar{\chi}_1 < \bar{x}_2^* \leq \bar{\chi}_2$, *then* $\bar{r}_J^* \leq r < r_J^*$.

3 *If* $x_3^* > \bar{\chi}_2$, *then* $r < \bar{r}_J^*$.

(c) *Let* $\nu = \bar{\nu}$. *Then* $\bar{r}_J^* = r_J^*$.

1 *If* $x_4^* \leq \chi_1$, *then* $r_J^* \leq r$.

2 *If* $x_3^* > \chi_1$, *then* $r < r_J^*$.

Proof. (a) Let $\nu > \bar{\nu}$.

(a1) Let $x_4^* \leq \chi_1$. Then since $0 = B_4(x_4^*) \geq B_4(\chi_1) = B_2(\chi_1)$ due to Eq. (5.3.23) and Lemma 5.3.3(a,d3), we have $B_2(x_2^*) = 0 \geq B_2(\chi_1)$, hence $x_2^* \leq \chi_1$. Thus $r < \bar{r}_J^*$ or $r_J^* \leq r$ due to Corollary 5.3.1(a2). Further, $0 \geq B_2(\chi_1) > B_2(\chi_2) = B_3(\chi_2)$ due to Lemma 5.3.3(d2,a,d4). Since $B_3(x_3^*) = 0 > B_3(\chi_2)$ due to Eq. (5.3.23), we have $x_3^* < \chi_2$ due to Lemma 5.3.3(a), hence $\bar{r}_J^* \leq r$ due to Corollary 5.3.1(a3). Accordingly, $r_J^* \leq r$.

(a2) Let $\chi_1 < x_2^*$. Then $0 = B_2(x_2^*) < B_2(\chi_1) = B_4(\chi_1)$ due to Eq. (5.3.23) and Lemma 5.3.3(a,d3). Since $B_4(x_4^*) = 0 < B_4(\chi_1)$ due to Eq. (5.3.23), we obtain $x_4^* > \chi_1$, hence $r < r_J^*$ due to Corollary 5.3.1(a1). Let $x_2 \leq \chi_2$. Then $0 = B_2(x_2^*) \geq B_2(\chi_2)$ due to Eq. (5.3.22) and Lemma 5.3.3(a). Since $B_3(x_3^*) = 0 \geq B_2(\chi_2)$ due to Eq. (5.3.22), we get $x_3^* \leq \chi_2$ due to Lemma 5.3.3(a), hence $\bar{r}_J^* \leq r$ due to Corollary 5.3.1(a3). Accordingly, $\bar{r}_J^* \leq r < r_J^*$.

(a3) Let $x_3^* > \chi_2$. Then $0 = B_3(x_3^*) < B_3(\chi_3) = B_2(\chi_2)$ due to Eq. (5.3.22) and Lemma 5.3.3(a,d4). Since $B_2(x_2^*) = 0 < B_2(\chi_2)$ due to Eq. (5.3.22), we have $x_2^* > \chi_2$ due to Lemma 5.3.3(a), hence $r < \bar{r}_J^*$ or $r_J^* \leq r$ due to Corollary 5.3.1(a2). Further, since $0 < B_2(\chi_2)$, we have $0 < B_2(\chi_2) < B_2(\chi_1) = B_4(\chi_1)$ due to Lemma 5.3.3(d2,a,d3). Hence $0 < B_4(\chi_1)$, implying $x_4^* > \chi_1$. Thus $r < r_J^*$ due to Corollary 5.3.1(a1). Therefore, $r < \bar{r}_J^*$.

(b) Only changing χ_1, χ_2, $B_2(x)$, and x_2^* into $\bar{\chi}_1$, $\bar{\chi}_2$, $\bar{B}_2(x)$, and $\bar{x}_2^*$, we can prove the assertions in the same way as the above.

(c) Let $\nu = \bar{\nu}$. Then $r_J^* = \bar{r}_J^*$ due to Evident from Lemma 5.3.7(b4, b2,b3) and $\chi_1 = \chi_2 = \bar{\chi}_1 = \bar{\chi}_2$ due to Lemma 5.3.3(f).

(c1) Obvious from (a1,b1).

(c2) Evident from (a3,b3). ∎

5.4 Optimal Decision Rule

The following theorem prescribes the optimal decision rule.

Theorem 5.4.1

(a) *Let* $\rho \geq b$. *Then* $\langle \mathrm{K} \rangle_\phi$ *and* $\langle \mathrm{K} \rangle_l$ *with* $l \geq 0$.

(b) *Let* $\rho < b$.

1 *Let* $s = 0$. *Then* $\langle \mathrm{C} \rangle_\phi$ *and* $\langle \mathrm{C} \rangle_l$ *with* $l \geq 0$.

2 *Let* $s > 0$.

i *If* $s^* \leq s$, *then* $\langle \mathrm{K} \rangle_\phi$ *and* $\langle \mathrm{K} \rangle_l$ *with* $l \geq 0$.

ii *If* $s < s^*$ *and* $r^* \leq r$, *then* $\langle \mathrm{K} \rangle_\phi$ *and* $\langle \mathrm{K} \rangle_l$ *with* $l \geq 0$.

iii *If* $s < s^*$ *and* $r < r^*$, *then*

1 $\langle \mathrm{C} \rangle_\phi$.

2 *Let* $\nu > \bar{\nu}$. *Then* $\chi_1 < \chi_2$.

i *If* $x_2^* \le \chi_1$, *then* $\langle \mathrm{K} \rangle_l$ *with* $l \ge 0$.

ii *If* $\chi_1 < x_2^* \le \chi_2$, *then* $\langle \mathrm{C} \rangle_l$ *with* $l < \tau$ *and* $\langle \mathrm{K} \rangle_l$ *with* $l \ge \tau$.

iii *If* $\chi_2 < x_2^*$, *then* $\langle \mathrm{C} \rangle_l$ *with* $l \ge 0$.

3 *Let* $\nu < \bar{\nu}$. *Then* $\bar{\chi}_1 < \bar{\chi}_2$.

i *If* $\bar{x}_2^* \le \bar{\chi}_1$, *then* $\langle \mathrm{K} \rangle_l$ *with* $l \ge 0$.

ii *If* $\bar{\chi}_1 < \bar{x}_2^* \le \bar{\chi}_2$, *then* $\langle \mathrm{C} \rangle_l$ *with* $l < \tau$ *and* $\langle \mathrm{K} \rangle_l$ *with* $l \ge \tau$.

iii *If* $\bar{\chi}_2 < \bar{x}_2^*$, *then* $\langle \mathrm{C} \rangle_l$ *with* $l \ge 0$.

4 *Let* $\nu = \bar{\nu}$. *If* $x_2^* \le \chi_1$, *then* $\langle \mathrm{K} \rangle_l$ *with* $l \ge 0$, *or else* $\langle \mathrm{C} \rangle_l$ *with* $l \ge 0$.

Proof. (a) Let $\rho \ge b$. Then $T(\rho) = 0$ due to Lemma 3.2.2(c). Accordingly, $0 = \lambda\beta T(\rho) = s^* \le s$ from Lemma 5.3.8(c) and the assumption of $s \ge 0$. Hence the assertion holds from Lemma 5.3.9(a).

(b1,b2i,b2ii) Immediate from Lemmas 5.3.7(a), 5.3.9(a), 5.3.7(b1), respectively.

(b2iii1) Immediate from Lemma 5.3.7(b2,b3,b4).

(b2iii2) Let $\nu > \bar{\nu}$. Then evident from Lemma 5.3.3(d2).

(b2iii2i) Let $x_2^* \le \chi_1$. Then $0 = B_2(x_2^*) \ge B_2(\chi_1) = B_4(\chi_1)$ due to Eq. (5.3.23) and Lemma 5.3.3(a,d3), hence $B_4(x_4^*) = 0 \ge B_4(\chi_2)$. Thus $x_4^* \le \chi_1$ due to Lemma 5.3.3(a,c). From this and Lemma 5.3.10(a1) we have $r_J^* < r$, hence $J(h) \le 0$ and $\bar{J}(h) \le 0$ due to Lemma 5.3.7(b2i).

(b2iii2ii) Immediate from Lemmas 5.3.10(a2) and 5.3.7(b2ii).

(b2iii2iii) Let $\chi_2 < x_2^*$. Then $0 = B_2(x_2^*) < B_2(\chi_2) = B_3(\chi_2)$ due to Eq. (5.3.22) and Lemma 5.3.3(a,d4), implying $x_3^* > \chi_2$ due to Lemma 5.3.3(a,c). From this and Lemma 5.3.10(a3) we have $r < \bar{r}_J^*$, hence $J(h) > 0$ and $\bar{J}(h) > 0$ due to Lemma 5.3.7(b2i).

(b2iii3) By only changing χ_1, χ_2, $B_2(x)$, and x_2^* into $\bar{\chi}_1$, $\bar{\chi}_2$, $\bar{B}_2(x)$, and $\bar{x}_2^*$, we can prove the assertions in the same way as the above.

(b2iii4) Let $\nu = \bar{\nu}$. Then $\chi_1 = \chi_2$ due to Lemma 5.3.3(f). If $x_2^* \le \chi_1$, then $\langle \mathrm{K} \rangle_l$ with $l \ge 0$ due to (b2iii2i). If $x_2^* > \chi_1$, i.e., $\chi_2 < x_2^*$, then $\langle \mathrm{C} \rangle_l$ with $l \ge 0$ due to (b2iii2iii). ∎

Theorem 5.4.2 *In the pricing control problem we have:*

(a) *$z(h)$ is nondecreasing in λ, r, and θ and nonincreasing in s, ϑ, and $\bar{\vartheta}$.*

(b) *If $b > \rho$ and $r < (b - \rho)/W$, then $h < z(h) < b$, or else $z(h) = b$.*

Proof. (a) Immediate from Lemmas 5.3.5(c-e), and 3.2.1(d).

(b) Evident from Lemmas 5.3.5(b) and 3.2.1(b,c). ∎

5.5 Conclusions and Considerations

C1. **Optimal decision rules.**

The most important conclusions obtained in the model are the statements in Theorem 5.4.1, which can be summarized as in Table 5.5.1.

Table 5.5.1: Summary of optimal decision rules.

<table>
<tr><th>ρ</th><th>s</th><th>r</th><th>ν, $\bar{\nu}$</th><th>x_2^*, $\bar{x}_2^*$</th><th>state (ϕ)</th><th>state (ϕ, l), $l < \tau$</th><th>state (ϕ, l), $l \geq \tau$</th></tr>
<tr><td rowspan="11">$\rho < b$</td><td>$s = 0$</td><td colspan="3"></td><td rowspan="2">$\langle C \rangle_\phi$</td><td rowspan="2">$\langle C \rangle_l$</td><td rowspan="2">$\langle C \rangle_l$</td></tr>
<tr><td rowspan="9">$0 < s < s^*$</td><td rowspan="8">$r < r^*$</td><td rowspan="3">$\nu > \bar{\nu}$</td><td>$\chi_2 < x_2^*$</td></tr>
<tr><td>$\chi_1 < x_2^* \leq \chi_2$</td><td>$\langle C \rangle_\phi$</td><td>$\langle C \rangle_l$</td><td>$\langle K \rangle_l$</td></tr>
<tr><td>$\chi_1 \geq x_2^*$</td><td>$\langle C \rangle_\phi$</td><td>$\langle K \rangle_l$</td><td>$\langle K \rangle_l$</td></tr>
<tr><td rowspan="3">$\nu < \bar{\nu}$</td><td>$\bar{\chi}_2 < \bar{x}_2^*$</td><td>$\langle C \rangle_\phi$</td><td>$\langle C \rangle_l$</td><td>$\langle C \rangle_l$</td></tr>
<tr><td>$\bar{\chi}_1 < \bar{x}_2^* \leq \bar{\chi}_2$</td><td>$\langle C \rangle_\phi$</td><td>$\langle C \rangle_l$</td><td>$\langle K \rangle_l$</td></tr>
<tr><td>$\bar{\chi}_1 \geq \bar{x}_2^*$</td><td>$\langle C \rangle_\phi$</td><td>$\langle K \rangle_l$</td><td>$\langle K \rangle_l$</td></tr>
<tr><td rowspan="2">$\nu = \bar{\nu}$</td><td>$\chi_1 < x_2^*$</td><td>$\langle C \rangle_\phi$</td><td>$\langle C \rangle_l$</td><td>$\langle C \rangle_l$</td></tr>
<tr><td>$\chi_1 \geq x_2^*$</td><td>$\langle C \rangle_\phi$</td><td>$\langle K \rangle_l$</td><td>$\langle K \rangle_l$</td></tr>
<tr><td>$r^* \leq r$</td><td colspan="2"></td><td rowspan="3">$\langle K \rangle_\phi$</td><td rowspan="3">$\langle K \rangle_l$</td><td rowspan="3">$\langle K \rangle_l$</td></tr>
<tr><td>$s^* \leq s$</td><td colspan="3"></td></tr>
<tr><td>$\rho \geq b$</td><td colspan="4"></td></tr>
</table>

C2. **Relationships of the optimal decision rules with parameters.**

When $\rho < b$, depicting the optimal decision rule prescribed in Table 5.5.1 in the relationship with the search cost s and the profit from a sideline r, we have Figure 5.5.2, in which $r^*(s)$, $r_J^*(s)$, and $\bar{r}_J^*(s)$ are strictly decreasing in s with $r^*(0) = r_J^*(0) = \bar{r}_J^*(0) = (b-\rho)/W$, $s^* = \lambda\beta T(\rho)$ (Lemma 5.3.8(a-c)), and

1. if $\nu > \bar{\nu}$, then:
   i. $s_J^* = \big(q + (1-q)\nu\big)\big(x_1^*(0) - \rho\big)/W(1-q)(1-\nu)$ (Lemma 5.3.8(d1)),
   ii. $\bar{s}_J^* = \big(q + (1-q)\bar{\nu}\big)\big(x_3^*(0) - \rho\big)/(W - H)$ (Lemma 5.3.8(d2)),
   iii. $\bar{r}_J^*(s) < r_J^*(s) < r^*(s)$ (Lemma 5.3.7(b2)),
2. if $\nu = \bar{\nu}$, then:
   i. $s_J^* = \bar{s}_J^* = \big(q + (1-q)\nu\big)\big(x_1^*(0) - \rho\big)/(W - H)$ (Lemma 5.3.8(f)),
   ii. $r_J^*(s) = \bar{r}_J^*(s) < r^*(s)$ (Lemma 5.3.7(b4)),
3. if $\nu < \bar{\nu}$, then:
   i. $s_J^* = \big(q + (1-q)\nu\big)\big(x_3^*(0) - \rho\big)/(W - H)$ (Lemma 5.3.8(e1)),
   ii. $\bar{s}_J^* = \big(q + (1-q)\bar{\nu}\big)\big(\bar{x}_1^*(0) - \rho\big)/W(1-q)(1-\nu)$ (Lemma 5.3.8(e1)),
   iii. $r_J^*(s) < \bar{r}_J^*(s) < r^*(s)$ (Lemma 5.3.7(b3)).

The regions $\Omega(\mathrm{K,K,K})$, $\Omega(\mathrm{C,K,K})$, $\Omega(\mathrm{C,C,K})$, $\Omega(\mathrm{C,C,C})$, and $\Omega(\mathrm{C,K,C})$ in Figure 5.5.2 correspond to the optimal decisions as follows.

$\Omega(\mathrm{K,K,K}) \to$ skipping in both states (ϕ) and (ϕ, l) with $l \geq 0$

$\Omega(\mathrm{C,K,K}) \to$ continuing in state (ϕ) and skipping in state (ϕ, l) with $l \geq 0$

$\Omega(\mathrm{C,C,K}) \to$ continuing in states (ϕ) and (ϕ, l) with $l < \tau$ and skipping in state (ϕ, l) with $l \geq \tau$

$\Omega(\mathrm{C,C,C}) \to$ continuing in any states (ϕ) and (ϕ, l) with $l \geq 0$

$\Omega(\mathrm{C,K,C}) \to$ continuing in states (ϕ) and (ϕ, l) with $l < \tau$ and skipping in state (ϕ, l) with $l \geq \tau$

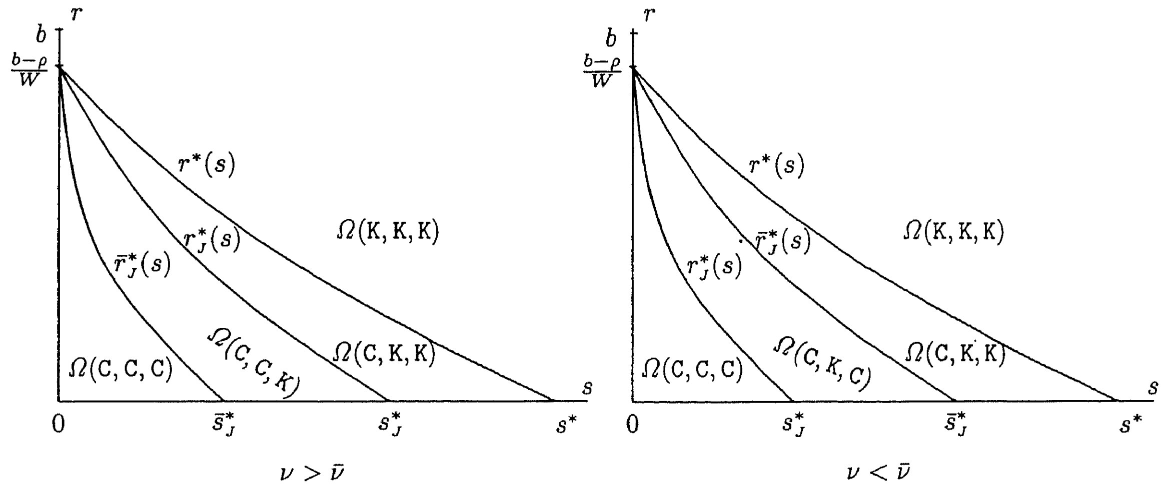

Figure 5.5.2: Four regions encircled by the functions $\bar{r}_J^*(s)$, $r_J^*(s)$, and $r^*(s)$ and the axes s, r when $b > \rho$. Note $\bar{r}_J^*(s) = r_J^*(s)$ when $\nu = \bar{\nu}$.

C3. **Properties of h and $z(h)$.**

1. In the admission control problem the optimal selection criterion, on which the company decides whether to accept an appearing customer or not, is given by h, and in the pricing control problem the optimal price, on which an appearing customer decides whether to place his order with the company or not, is given by the function $z(h)$. Further, it is only in the following regions that the decisions stated above are to be made:
   i. $\Omega(\mathrm{C,C,C}) \cup \Omega(\mathrm{C,C,K}) \cup \Omega(\mathrm{C,K,K})$ for $\nu > \bar{\nu}$
   ii. $\Omega(\mathrm{C,C,C}) \cup \Omega(\mathrm{C,K,K})$ for $\nu = \bar{\nu}$
   iii. $\Omega(\mathrm{C,C,C}) \cup \Omega(\mathrm{C,K,C}) \cup \Omega(\mathrm{C,K,K})$ for $\nu < \bar{\nu}$
2. The h is given by the unique solution x^* of the equation $G(x) = 0$, i.e., $h = x^*$; refer Lemma 5.3.5 for the properties of the h.
3. Let

$$(r,s) \in \Omega(\mathrm{C,C,C}) \cup \Omega(\mathrm{C,C,K}) \cup \Omega(\mathrm{C,K,K}) \quad \text{for } \nu > \bar{\nu},$$
$$(r,s) \in \Omega(\mathrm{C,C,C}) \cup \Omega(\mathrm{C,K,K}) \quad \text{for } \nu = \bar{\nu},$$
$$(r,s) \in \Omega(\mathrm{C,C,C}) \cup \Omega(\mathrm{C,K,C}) \cup \Omega(\mathrm{C,K,K}) \quad \text{for } \nu < \bar{\nu}.$$

Then $h < b$ (Lemma 5.3.5(b)). Accordingly, the optimal decisions can be prescribed as follows.

1) In the admission control problem, if $w > h$, an order with value w appearing after having conducted the search is accepted, or else rejected.

2) In the pricing control problem, the optimal price $z(h)$ is in the interval (h, b) (Lemma 5.4.2(b)).

C4. **The monotonicities of h and $z(h)$ in the parameters.**

See Lemmas 5.3.5(c-e) and 5.4.2(a) for the parameters λ, r, s, θ, ϑ, and $\bar{\vartheta}$. The monotonicities of h and $z(h)$ in the other parameters q, β, ν, $\bar{\nu}$, and τ are so difficult to theoretically prove that let us show those through the numerical experiments where let $\lambda = 0.90$, $r = 0.30$, $s = 0.20$, $\theta = 0.30$, $\vartheta = 0.50$, and $\bar{\vartheta} = 0.50$ and let $F(w)$ be a uniform distribution on $[0.01, 1.01]$. See Figures 5.5.3, 5.5.4, and 5.5.5.

1. *Both h and $z(h)$ are nondecreasing in λ, r, and θ.* This implies that the larger the customer appearing probability λ, the profit from a sideline r, and the penalty θ for delay of delivery may be, it is reasonable to accept orders with higher values in the admission control and to offer higher prices in the pricing control problem.
2. *Both h and $z(h)$ are nonincreasing in q, ν, $\bar{\nu}$, s, ϑ and $\bar{\vartheta}$.* This implies that the higher the service completion probability q, the cancellation probability ν and $\bar{\nu}$, the search cost s, and the penalty for cancellation ϑ and $\bar{\vartheta}$ may be, it is reasonable to accept orders, even if their values are smaller, in the admission control problem and to offer smaller prices in the pricing control problem.
3. *Monotonicity of h and $z(h)$ in β and τ (See* Figures 5.5.4 *and* 5.5.5, *respectively).* The figures demonstrate the three patterns related to β and the two patterns related to τ.

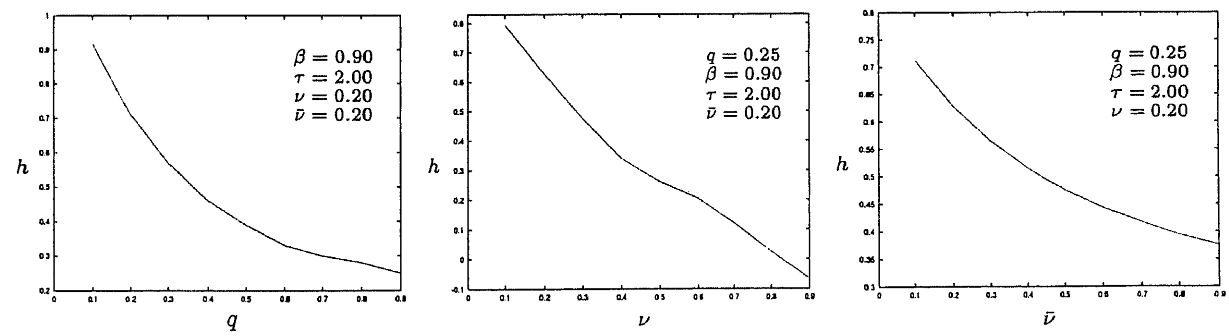

Figure 5.5.3: Monotonicities of h in q, ν, and $\bar{\nu}$.

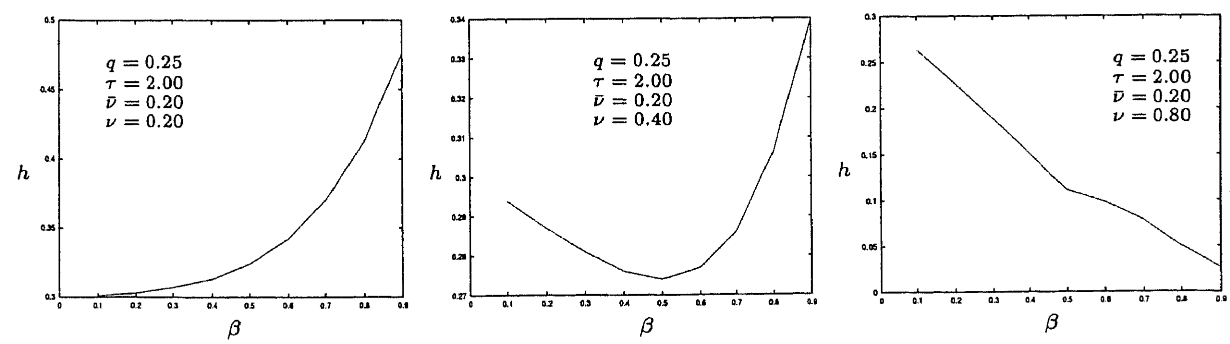

Figure 5.5.4: Monotonicities of h in β.

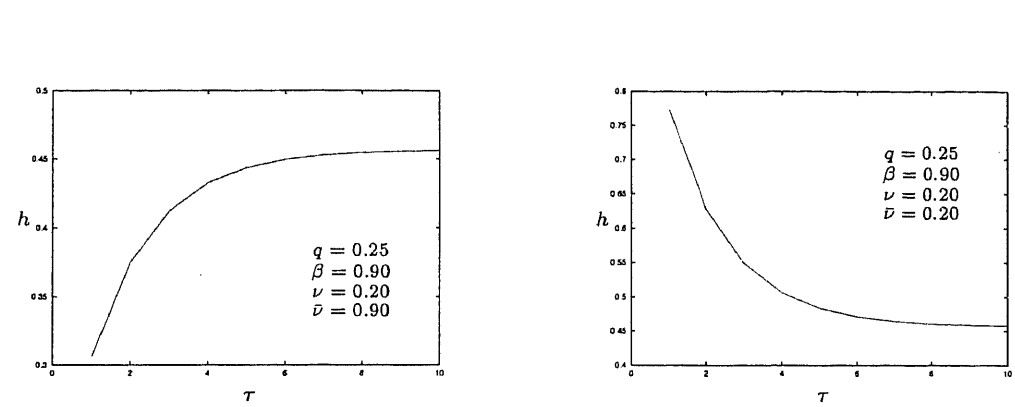

Figure 5.5.5: Monotonicities of h in τ.

C5. **Optimal ratio δ^*.**

In this model, we assumed two types of penalty; one is for the delay of delivery, θ, the other is for the cancellation of the contract due to unavoidable circumstances with the customer, $\bar{\vartheta}$, which are paid by, respectively, the company and the customer. Now, let $\delta = \bar{\vartheta}/\theta$. Then from a practical viewpoint, if δ is small enough, customers would be reluctant to place their orders with that company; on the contrary, if it is large enough, they are willing to place. This implies that the probability of customer arrival is decreasing in the ratio δ, so let this probability be denoted by $\lambda(\delta)$ (See Figure 5.5.6). Then the total expected net profit may diminish due to few orders from the customer in the former case or too much payment of penalty in the latter. This consideration leads to the conjecture that there exist an optimal ratio δ^* maximizing the total expected present discounted net profit $u(\phi)$. Below, let us examine the conjecture by numerical experiments. Here, let $\lambda(\delta) = 0.5\exp(-\sigma^2\delta^2)$ ($\sigma = 0.50, 0.80, 1.10, 1.40$), $q = 0.20$, $\beta = 0.95$, $\nu = 0.3$, $\bar{\nu} = 0.5$, $r = 0.10$, $s = 0.05$, $\tau = 3.00$, and $\vartheta = 1.50$, and let $F(w)$ be the uniform distribution on $[1, 2]$. Then we obtain the results of numerical experiment as shown in Figure 5.5.6. Table 5.5.2 shows the δ^* maximizing $u(\phi)$.

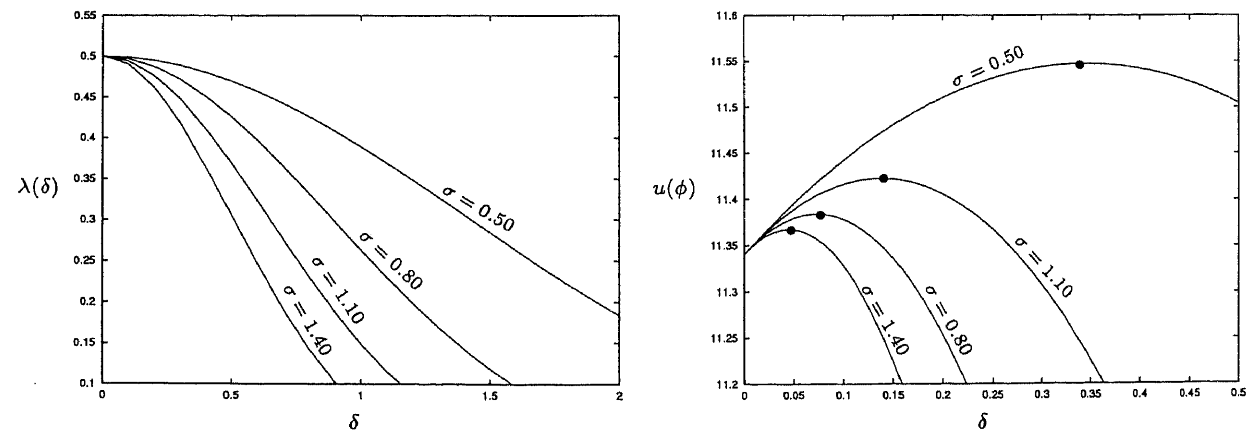

Figure 5.5.6: $\lambda(\delta)$ and $u(\phi)$.

Table 5.5.2: The optimal ratio δ^* and the maximum total expected present net profits $u(\phi)$.

	δ^*	$u(\phi)$
$\sigma = 0.50$	0.34	11.55
$\sigma = 0.80$	0.14	11.42
$\sigma = 1.10$	0.07	11.38
$\sigma = 1.40$	0.05	11.37

Chapter 6

Model III: Stochastic model with multiple customers being able to be held

The model defined and examined in this chapter is the same as Model I except that (1) multiple customers can be held in the company and (2) the penalty is equal to zero.

Contents

6.1 System of Optimal Equations

Either if the search was skipped at the previous point in time or if no customer has appeared with probability $1-\lambda$ regardless of having conducted the search at the previous point in time, it follows that no customer appears at the present point in time. For convenience, we shall refer to such a situation as "the company has a *fictitious customer* ϕ".

1. In both the admission control problem and pricing control problem, by $u(\phi, i)$ we shall denote the maximum total expected present discounted net profits starting from a state of having the fictitious customer ϕ and $i\,(0 \le i \le N)$ orders in the company; let us refer to such a situation as state (ϕ, i). When in state (ϕ, N), even if a customer appears, it cannot be accepted due to the assumption of $i \le N$. Accordingly, in this case, the present state (ϕ, N) remains unchanged at the next point in time if no order in the company is completed with probability $1-q$.

2. In the admission control problem, by $u(w, i)$ let us denote the maximum total expected present discounted net profit starting with $i\,(0 \le i < N)$ orders in the company and an arriving customer who offers a price w.

3. In the pricing control problem, by $u(1, i)$ let us denote the maximum total expected present discounted net profit starting with $i\,(0 \le i < N)$ orders in the company and an arriving customer, to whom the company proposes a price z for an order.

Since the expectation of immediate reward at any point in time is clearly finite, using the conventional way outlined in the discussion of the Markovian decision process [38, Ross](p29-30), we can easily show that $|u(\phi,i)| \leq M/(1-\beta)$ for a sufficiently large $M > 0$, i.e., $u(\phi,i)$ is bounded in i. Furthermore, $u(w,i)$ and $u(1,i)$ are also bounded in i for the reason as the above. Now, for convenience in the later discussions, let us define

$$h_i = u(\phi,i) - u(\phi,i+1), \quad 0 \leq i < N. \tag{6.1.1}$$

Then the optimal equations for both problems can be described as follows.

1. **Admission control problem:**

$$u(\phi,0) = \max\begin{cases} \mathtt{C}: \beta\big(\lambda\,\mathbf{E}[u(\xi,0)] + (1-\lambda)u(\phi,0)\big) - s + r, \\ \mathtt{K}: \beta u(\phi,0) + r, \end{cases} \tag{6.1.2}$$

$$u(\phi,i) = \max\begin{cases} \mathtt{C}: (1-q)\beta\big(\lambda\,\mathbf{E}[u(\xi,i)] + (1-\lambda)u(\phi,i)\big) \\ \qquad +q\beta\big(\lambda\,\mathbf{E}[u(\xi,i-1)] + (1-\lambda)u(\phi,i-1)\big) - s, \qquad 1 \leq i < N, \\ \mathtt{K}: (1-q)\beta u(\phi,i) + q\beta u(\phi,i-1), \end{cases} \tag{6.1.3}$$

$$u(\phi,N) = \max\begin{cases} \mathtt{C}: (1-q)\beta u(\phi,N) + q\beta\big(\lambda\,\mathbf{E}[u(\xi,N-1)] + (1-\lambda)u(\phi,N-1)\big) - s, \\ \mathtt{K}: (1-q)\beta u(\phi,N) + q\beta u(\phi,N-1), \end{cases} \tag{6.1.4}$$

$$u(w,i) = \max\begin{cases} \mathtt{A}: w + u(\phi,i+1) \\ \mathtt{R}: u(\phi,i) \end{cases} \tag{6.1.5}$$

$$= \max\{w - h_i, 0\} + u(\phi,i), \quad 0 \leq i < N. \qquad \square \tag{6.1.6}$$

2. **Pricing control problem:**

$$u(\phi,0) = \max\begin{cases} \mathtt{C}: \beta\big(\lambda u(1,0) + (1-\lambda)u(\phi,0)\big) - s + r, \\ \mathtt{K}: \beta u(\phi,0) + r, \end{cases} \tag{6.1.7}$$

$$u(\phi,i) = \max\begin{cases} \mathtt{C}: (1-q)\beta(\lambda u(1,i) + (1-\lambda)u(\phi,i)) \\ \qquad +q\beta\big(\lambda u(1,i-1) + (1-\lambda)u(\phi,i-1)\big) - s, \qquad 1 \leq i < N, \\ \mathtt{K}: (1-q)\beta u(\phi,i) + q\beta u(\phi,i-1), \end{cases} \tag{6.1.8}$$

$$u(\phi,N) = \max\begin{cases} \mathtt{C}: (1-q)\beta u(\phi,N) + q\beta\big(\lambda u(1,N-1) + (1-\lambda)u(\phi,N-1)\big) - s, \\ \mathtt{K}: (1-q)\beta u(\phi,N) + q\beta u(\phi,N-1), \end{cases} \tag{6.1.9}$$

$$u(1,i) = \max_z\{p(z)\big(z + u(\phi,i+1)\big) + (1-p(z))u(\phi,i)\} \tag{6.1.10}$$

$$= \max_z p(z)(z - h_i) + u(\phi,i), \quad 0 \leq i < N. \qquad \square \tag{6.1.11}$$

See Lemma 6.3.1 for the unique existence of the solution of the above equations.

6.2 Transformation

Let us define

$$v(i) = \left\{ \begin{array}{ll} \mathbf{E}[u(w,i)] & \text{for the admission control problem} \\ u(1,i) & \text{for the pricing control problem} \end{array} \right\}, \quad 0 \le i < N. \tag{6.2.1}$$

Then using Eqs. (6.2.1) and (3.1.1), we can immediately rearrange both Eqs. (6.1.2) to (6.1.5) and Eqs. (6.1.7) to (6.1.10) into the identical expression below.

$$u(\phi,0) = \max\{\lambda\beta v(0) + (1-\lambda)\beta u(\phi,0) - s, \beta u(\phi,0)\} + r, \tag{6.2.2}$$

$$u(\phi,i) = \max\left\{ \begin{array}{l} (1-q)\beta(\lambda v(i) + (1-\lambda)u(\phi,i)) \\ \qquad +q\beta\big(\lambda v(i-1) + (1-\lambda)u(\phi,i-1)\big) - s, \\ (1-q)\beta u(\phi,i) + q\beta u(\phi,i-1) \end{array} \right\}, \quad 1 \le i < N, \tag{6.2.3}$$

$$u(\phi,N) = \max\left\{ \begin{array}{l} (1-q)\beta u(\phi,N) + q\beta\big(\lambda v(N-1) + (1-\lambda)u(\phi,N-1)\big) - s, \\ (1-q)\beta u(\phi,N) + q\beta u(\phi,N-1), \end{array} \right\}, \tag{6.2.4}$$

$$v(i) = T(h_i) + u(\phi,i) \quad \text{or equivalently} \quad T(h_i) = v(i) - u(\phi,i), \quad 0 \le i < N. \tag{6.2.5}$$

Further, Eqs. (6.2.2) to (6.2.4) can be rewritten, respectively,

$$u(\phi,0) = \beta u(\phi,0) + \max\{\lambda\beta\big(v(0) - u(\phi,0)\big) - s, 0\} + r, \tag{6.2.6}$$

$$\begin{aligned} u(\phi,i) &= (1-q)\beta u(\phi,i) + q\beta u(\phi,i-1) \\ &\quad + \max\{\lambda(1-q)\beta(v(i) - u(\phi,i)) + \lambda q\beta\big(v(i-1) - u(\phi,i-1)\big) - s, 0\}, \quad 1 \le i < N, \end{aligned} \tag{6.2.7}$$

$$u(\phi,N) = (1-q)\beta u(\phi,N) + q\beta u(\phi,N-1) + \max\{\lambda q\beta\big(v(N-1) - u(\phi,N-1)\big) - s, 0\}, \tag{6.2.8}$$

which can be immediately rearranged into

$$u(\phi,0) = \big(\max\{\lambda\beta\big(v(0) - u(\phi,0)\big) - s, 0\} + r\big)/(1-\beta), \tag{6.2.9}$$

$$\begin{aligned} u(\phi,i) &= \gamma q\beta u(\phi,i-1) \\ &\quad +\gamma\max\{\lambda(1-q)\beta\big(v(i) - u(\phi,i)\big) + \lambda q\beta\big(v(i-1) - u(\phi,i-1)\big) - s, 0\}, \ 1 \le i < N, \end{aligned} \tag{6.2.10}$$

$$u(\phi,N) = \gamma q\beta u(\phi,N-1) + \gamma\max\{\lambda q\beta\big(v(N-1) - u(\phi,N-1)\big) - s, 0\} \tag{6.2.11}$$

where γ is defined by Eq. (2.4.2). Hence, using Eq. (6.2.5), we can rewrite Eqs. (6.2.9) to (6.2.11) as follows.

$$u(\phi,0) = (\max\{\lambda\beta T(h_0) - s, 0\} + r)/(1-\beta), \tag{6.2.12}$$

$$u(\phi,i) = \gamma q\beta u(\phi,i-1) + \gamma\max\{\lambda(1-q)\beta T(h_i) + \lambda q\beta T(h_{i-1}) - s, 0\}, \quad 1 \le i < N, \tag{6.2.13}$$

$$u(\phi,N) = \gamma q\beta u(\phi,N-1) + \gamma\max\{\lambda q\beta T(h_{N-1}) - s, 0\}. \tag{6.2.14}$$

Further, using the *L-function* defined by Eq. (3.1.2), we can rewrite Eqs. (6.2.12) to (6.2.14) as follows.

$$u(\phi,0) = (\max\{L(h_0),0\}+r)/(1-\beta), \tag{6.2.15}$$

$$u(\phi,i) = \gamma q\beta u(\phi,i-1)+\gamma\max\{(1-q)L(h_i)+qL(h_{i-1}),0\}, \quad 1\le i<N, \tag{6.2.16}$$

$$u(\phi,N) = \gamma q\beta u(\phi,N-1)+\gamma\max\{qL(h_{N-1})-(1-q)s,0\}. \tag{6.2.17}$$

Below, for convenience let

$$Q_0 = L(h_0), \tag{6.2.18}$$

$$Q_i = (1-q)L(h_i)+qL(h_{i-1}), \quad 1\le i<N, \tag{6.2.19}$$

$$Q_N = qL(h_{N-1})-(1-q)s. \tag{6.2.20}$$

Then Eq. (6.2.15) to Eq. (6.2.17) can be rewritten as follows.

$$u(\phi,0) = (\max\{Q_0,0\}+r)/(1-\beta), \tag{6.2.21}$$

$$u(\phi,i) = \gamma q\beta u(\phi,i-1)+\gamma\max\{Q_i,0\}, \quad 1\le i\le N. \tag{6.2.22}$$

Now, noting Eqs. (6.2.22) and (2.4.3), we can rewrite Eq. (6.1.1) with $i=0$ as follows.

$$h_0 = u(\phi,0)-u(\phi,1)=\gamma(1-\beta)u(\phi,0)-\gamma\max\{Q_1,0\}. \tag{6.2.23}$$

Rearranging Eq. (6.2.23) by substituting Eq. (6.2.21) yields

$$h_0 = \gamma\max\{Q_0,0\}-\gamma\max\{Q_1,0\}+\gamma s. \tag{6.2.24}$$

Similarly, we obtain

$$h_i = \gamma q\beta h_{i-1}+\gamma\max\{Q_i,0\}-\gamma\max\{Q_{i+1},0\}, \quad 1\le i<N, \tag{6.2.25}$$

Regarding h_i as a function of r, let us represent h_i and Q_i by, respectively, $h_i(r)$ and $Q_i(r)$, i.e.,

$$Q_0(r) = L(h_0(r)), \tag{6.2.26}$$

$$Q_i(r) = (1-q)L(h_i(r))+qL(h_{i-1}(r)), \quad 1\le i<N, \tag{6.2.27}$$

$$Q_N(r) = qL(h_{N-1}(r))-(1-q)s. \tag{6.2.28}$$

Here, by r_i let us denote the smallest solution of $Q_i(r)=0$, if it exists, i.e.,

$$r_i = \min\{r \mid Q_i(r)=0\}. \tag{6.2.29}$$

From all the above it can be easily seen that the optimal decision rules for any given i can be prescribed as follows.

☐ *Optimal Decision Rule 6.2.1*

1. Admission control problem:

   i. If $Q_i > 0$, then $\langle \mathtt{C} \rangle_i$[†] , or else $\langle \mathtt{K} \rangle_i$ for $0 \le i \le N$.

   ii. Let $0 \le i < N$ and an order with value w appear after the search was enacted. If $w > h_i$, then $\langle \mathtt{A}(w) \rangle_i$, or else $\langle \mathtt{R}(w) \rangle_i$.

2. Pricing control problem:

   i. If $Q_i > 0$, then $\langle \mathtt{C} \rangle_i$, or else $\langle \mathtt{K} \rangle_i$ for $0 \le i \le N$.

   ii. Let $0 \le i < N$ and a customer appear after the search was enacted. Then $\langle \mathtt{O}(z_i) \rangle$ where $z_i = z(h_i)$.

6.3 Analysis

Lemma 6.3.1 *The system of equations* Eq. (6.1.2) *to* Eq. (6.1.5) *and* Eq. (6.1.7) *to* Eq. (6.1.10) *has a unique solution.*

Proof. Quite the same as Lemma 4.2.1. ∎

Lemma 6.3.2

(a) $u(\phi, i)$ *and* $v(i)$ *are nonincreasing in* i *where* $u(\phi, i) \ge 0$ *for* $0 \le i \le N$.

(b) $h_i \ge 0$ *for* $0 \le i < N$.

Proof. See App. A.9. ∎

Noting $u(\phi, i) \ge 0$ due to Lemma 6.3.2, we obtain $u(w, i) \ge w$ and $u(1, i) \ge \max_z p(z)z$ from Eqs. (6.1.5) and (6.1.10), hence $\mathbf{E}[u(w, i)] \ge \mu = T(0)$ and $u(1, i) \ge \max_z p(z)z = T(0)$. Accordingly, from Eq. (6.2.1) we get

$$v(i) \ \ge \ T(0), \quad 0 \le i < N. \tag{6.3.1}$$

6.3.1 Case of $\alpha \le 0$

Lemma 6.3.3 $Q_i \le 0$ *for* $0 \le i \le N$.

Proof. Assume $\alpha \le 0$. Then from Lemmas 6.3.2(b) and 3.2.3(c) we have $0 \ge \alpha = \lambda\beta T(0) - s \ge L(h_i)$ for $0 \le i < N$. Hence

i. $0 \ge L(h_0) = Q_0$,

ii. $0 \ge \lambda\beta T(0) - s = (1-q)(\lambda\beta T(0) - s) + q(\lambda\beta T(0) - s) \ge (1-q)L(h_i) + qL(h_{i-1}) = Q_i$ for $1 \le i < N$,

iii. $0 \ge \lambda\beta T(0) - s > \lambda q\beta T(0) - s = q(\lambda\beta T(0) - s) - (1-q)s \ge qL(h_{N-1}) - (1-q)s = Q_N$. ∎

[†] The notation $\langle \mathtt{C} \rangle_i$ implies that continuing the search is optimal in state (ϕ, i).

6.3.2 Case of $\alpha > 0$

Lemma 6.3.4

(a) *$u(\phi, i) > 0$ for $0 \le i \le N$.*

(b) *If $Q_i \le 0$ for a given i such as $1 \le i < N$, then $h_{i-1} > h_i$, hence $h_{i-1} \ge h_i$.*

Proof. (a) First, note $u(\phi, i) \ge 0$ for all i from Lemma 6.3.2(a). Hence, from Eqs. (6.2.2) and (6.3.1) we have $u(\phi, 0) \ge \beta(\lambda v(0) + (1-\lambda)u(\phi, 0)) - s + r \ge \lambda\beta T(0) - s + r = \alpha + r > 0$. Suppose $u(\phi, i-1) > 0$. Then from Eqs. (6.2.3) and (6.2.4) we can get $u(\phi, i) \ge (1-q)\beta u(\phi, i) + q\beta u(\phi, i-1) > 0$ for $1 \le i \le N$.

(b) Let $Q_i \le 0$ for a given i such as $1 \le i < N$. Then from Eq. (6.2.22) we have $u(\phi, i) = \gamma q\beta u(\phi, i-1)$, hence

$$u(\phi, i+1) = \gamma q\beta u(\phi, i) + \gamma \max\{Q_{i+1}, 0\} = (\gamma q\beta)^2 u(\phi, i-1) + \gamma \max\{Q_{i+1}, 0\}.$$

Accordingly, we get

$$\begin{aligned} h_i - h_{i-1} &= 2u(\phi, i) - u(\phi, i-1) - u(\phi, i+1) \\ &= 2\gamma q\beta u(\phi, i-1) - u(\phi, i-1) - (\gamma q\beta)^2 u(\phi, i-1) - \gamma \max\{Q_{i+1}, 0\} \\ &= -(1-\gamma q\beta)^2 u(\phi, i-1) - \gamma \max\{Q_{i+1}, 0\} < 0 \end{aligned}$$

due to $u(\phi, i-1) > 0$ from (a) and $1 > \gamma q\beta$ from Eq. (2.4.3). Hence, $h_{i-1} > h_i$, thus $h_{i-1} \ge h_i$. ∎

Lemma 6.3.5 *Let $h_{i-1} < h_i$ for a given i such as $1 \le i < N$. Then $h_{i-1} < h_i < \cdots < h_{N-1} < b$ and $Q_j > 0$ for j with $i \le j < N$.*

Proof. Let $h_{i-1} < h_i$ for a given i such as $1 \le i < N$. Then $Q_i > 0$ from the contrapositions of Lemma 6.3.4(b). Hence from Lemma 3.2.3(a) we get

$$0 < Q_i = (1-q)L(h_i) + qL(h_{i-1}) \le (1-q)L(h_{i-1}) + qL(h_{i-1}) = L(h_{i-1}),$$

implying $h_{i-1} < b$ due to Lemma 3.2.3(e). Further, from Eq. (6.2.25) we have

$$\begin{aligned} h_i &= \gamma q\beta h_{i-1} + \gamma Q_i - \gamma \max\{Q_{i+1}, 0\} \\ &= \gamma q\beta h_{i-1} + \gamma(1-q)L(h_i) + \gamma qL(h_{i-1}) - \gamma \max\{Q_{i+1}, 0\} \\ &\le \gamma q\big(\beta h_{i-1} + L(h_{i-1})\big) + \gamma(1-q)L(h_i) \\ &= \gamma qM(h_{i-1}) + \gamma(1-q)L(h_i). \end{aligned} \tag{6.3.2}$$

Assume $h_i \ge b$. Then $L(h_i) = -s \le 0$ from Lemma 3.2.3(c), hence $h_i \le \gamma qM(h_{i-1})$. Since $h_{i-1} < b$, from Lemma 3.2.4 we get $h_i \le \gamma qM(b) = \gamma q(\beta b - s) \le \gamma q\beta b < b$ due to Eq. (2.4.3), which is a contradiction. Hence, it must be $h_{i-1} < h_i < b$. Here, let us assume $Q_{i+1} \le 0$. Then $h_{i+1} < h_i < b$ from Lemma 6.3.4(b) and the above result. Further, from Lemma 3.2.3(a) we have

$$0 \ge Q_{i+1} = (1-q)L(h_{i+1}) + qL(h_i) > (1-q)L(h_i) + qL(h_i) = L(h_i) \quad \cdots (1^*).$$

From Eq. (6.3.2) and Lemma 3.2.4 we have

$$\begin{aligned} h_i &\leq \gamma q M(h_i) + \gamma(1-q)L(h_i) \\ &= \gamma q\big(\beta h_i + L(h_i)\big) + \gamma(1-q)L(h_i) \\ &= \gamma q\beta h_i + \gamma L(h_i), \end{aligned}$$

from which we have $(1-\gamma q\beta)h_i \leq \gamma L(h_i)$. Using Eq. (2.4.3), we can rewrite the above inequality $\gamma(1-\beta)h_i \leq \gamma L(h_i)$, i.e., $(1-\beta)h_i \leq L(h_i)$. Since $h_i \geq 0$ from Lemma 6.3.2(b), we have $L(h_i) > 0$, which contradicts (1*), hence it must be $Q_{i+1} > 0$. From this and $Q_i > 0$ we can rewrite Eq. (6.2.25) as follows.

$$h_i = \gamma q\beta h_{i-1} + \gamma(Q_i - Q_{i+1}).$$

Since $\gamma(Q_i - Q_{i+1}) = \gamma q L(h_{i-1}) + \gamma(1-2q)L(h_i) - \gamma(1-q)L(h_{i+1})$, we obtain

$$\begin{aligned} h_i &= \gamma q\big(\beta h_{i-1} + L(h_{i-1})\big) + \gamma(1-2q)L(h_i) - \gamma(1-q)L(h_{i+1}) \\ &= \gamma q M(h_{i-1}) + \gamma(1-2q)L(h_i) - \gamma(1-q)L(h_{i+1}). \end{aligned}$$

Noting the assumption of $h_{i-1} < h_i$, from Lemma 3.2.4 we get

$$\begin{aligned} h_i &\leq \gamma q M(h_i) + \gamma(1-2q)L(h_i) - \gamma(1-q)L(h_{i+1}) \\ &= \gamma q(\beta h_i + L(h_i)) + \gamma(1-2q)L(h_i) - \gamma(1-q)L(h_{i+1}) \\ &= \gamma q\beta h_i + \gamma(1-q)\big(L(h_i) - L(h_{i+1})\big), \end{aligned}$$

from which we have

$$(1-\gamma q\beta)h_i \leq \gamma(1-q)\big(L(h_i) - L(h_{i+1})\big).$$

Here, since $h_{i-1} \geq 0$ from Lemma 6.3.2(b), we have $h_i > 0$ due to the assumption of $h_{i-1} < h_i$. From this result and $1 > \gamma q\beta$ due to Eq. (2.4.3) we obtain $(1-\gamma q\beta)h_i > 0$, so that $L(h_{i+1}) < L(h_i)$, implying $h_i < h_{i+1}$. Repeating the same procedure as the above leads to the completion of the proof. ∎

Lemma 6.3.6 *Let $h_{i-1} \leq h_i$ for a given i such as $1 \leq i < N$. Then $h_{i-1} \leq h_i \leq \cdots \leq h_{N-1} < b$ and $Q_j > 0$ for j with $i \leq j < N$.*
Proof. Almost the same as in the proof of Lemma 6.3.5. ∎

Lemma 6.3.7 *If $Q_i > 0$ for a given i such as $0 \leq i < N$, then $Q_j > 0$ for $i \leq j < N$.*
Proof. Let $Q_i > 0$ for a given i such as $1 \leq i < N$. First, let $h_{i-1} < h_i$. Then $Q_{i+1} > 0$ from Lemma 6.3.5. Next, let $h_{i-1} \geq h_i$. Then since $L(h_{i-1}) \leq L(h_i)$ due to Lemma 3.2.3(a), we get $0 < Q_i = (1-q)L(h_i) + qL(h_{i-1}) \leq L(h_i)$. Here, assume $Q_{i+1} \leq 0$, i.e., $(1-q)L(h_{i+1}) + qL(h_i) \leq 0$. Then $h_i \geq h_{i+1}$ due to Lemma 6.3.4(b). Noting $L(x)$ is convex on $(-\infty, \infty)$ from Lemma 3.2.3(a), we have

$$L((1-q)h_{i+1} + qh_i) \leq (1-q)L(h_{i+1}) + qL(h_i) \leq 0 < L(h_i),$$

from which we have $(1-q)h_{i+1} + qh_i > h_i$, hence $h_i < h_{i+1}$ due to the assumption of $q < 1$, which is a contradiction. Hence, it must be $Q_{i+1} > 0$. Repeating the same procedure leads to the completion of the induction. Let $Q_0 > 0$. Then since $L(h_0) > 0$ from Eq. (6.2.18). Here, assuming $Q_1 \leq 0$, we can also derive a contradiction in quite the same way as the above, hence it must be $Q_1 > 0$. ∎

Lemma 6.3.8 *$h_i(r)$ is nondecreasing in r for $i \geq 0$.*

Proof. See App. A.10. ∎

Lemma 6.3.9 $\lim_{r\to\infty} h_i(r) = \infty$ *and* $\lim_{r\to-\infty} h_i(r) = -\infty$ *for* $i \geq 0$.

Proof. Since $L(h_i) \leq \lambda\beta T(0) - s$ for $0 \leq i < N$ due to Lemmas 6.3.2(b) and 3.2.3(c), noting Eq. (6.2.24), we have

$$\begin{aligned} h_0(r) &\geq -\gamma \max\{Q_1, 0\} + \gamma r \\ &= -\gamma \max\{(1-q)L(h_1) + qL(h_0), 0\} + \gamma c \geq -\gamma \max\{\lambda\beta T(0) - s, 0\} + \gamma s, \\ h_0(r) &\leq \gamma \max\{Q_0, 0\} + \gamma r \\ &= \gamma \max\{L(h_0), 0\} + \gamma c \leq \gamma \max\{\lambda\beta T(0) - s, 0\} + \gamma c, \end{aligned}$$

from which $\lim_{r\to\infty} h_0(r) = \infty$ and $\lim_{r\to-\infty} h_0(r) = -\infty$. Let $\lim_{r\to\infty} h_{i-1}(r) = \infty$ and $\lim_{r\to-\infty} h_{i-1}(r) = -\infty$. Then noting Eq. (6.2.25), in the same way as the above we obtain

$$\begin{aligned} h_i(r) &\geq \gamma q\beta h_{i-1}(r) - \gamma \max\{\lambda\beta T(0) - s, 0\}, \\ h_i(r) &\leq \gamma q\beta h_{i-1}(r) - \gamma \max\{\lambda\beta T(0) - s, 0\}. \end{aligned}$$

Therefore, $\lim_{r\to\infty} h_i(r) = \infty$ and $\lim_{r\to-\infty} h_i(r) = -\infty$. Hence, by induction the assertion holds for $0 \leq i \leq N-1$. From this result and Eq. (6.2.28) we can prove $\lim_{r\to\infty} h_{N-1}(r) = \infty$ and $\lim_{r\to-\infty} h_{N-1}(r) = -\infty$ in the same way as the above. ∎

Lemma 6.3.10 *$Q_i(r)$ is nonincreasing in r for all $i \geq 0$.*

Proof. Since $h_i(r)$ is nondecreasing in r for $0 \leq i < N$ due to Lemma 6.3.8, from Lemma 3.2.3(a) we can easily see that $Q_i(r)$ is nonincreasing in r. ∎

Lemma 6.3.11 *For $0 \leq i \leq N$ we have:*

(a) *There exists $r_i > 0$.*

(b) *If $r < (\geq)\ r_i$, then $Q_i(r) > (\leq)\ 0$.*

Proof. (a) From Lemmas 6.3.9 and 3.2.3(d) we clearly have $\lim_{r\to-\infty} Q_i(r) = \infty$. Further, for a sufficiently large r we have $h_i(r) \geq b$ due to Lemma 6.3.9, hence $\lim_{r\to\infty} Q_i(r) = -s \leq 0$. Thus, it follows that r_i exists.

(b) Immediate from the definition of r_i and Lemma 6.3.10. ∎

Lemma 6.3.12

(a) *Let $r = 0$.*

1 $Q_0(r) > 0$.

2 *If* $h_0 = 0$, *then* $h_0 = h_1$.

3 *If* $h_0 > 0$, *then* $h_0 < h_1$.

(b) *If* $r_\circ \le r$, *then* $h_0 > h_1$.

Proof. (a) Let $r = 0$.

(a1) Assume $\lambda\beta v(0) + (1-\lambda)\beta u(\phi,0) - s \le \beta u(\phi,0)$ from Eq. (6.2.2). Then since $u(\phi,0) = \beta u(\phi,0)$, we have $\beta = 1$ due to $u(\phi,0) > 0$ from Lemma 6.3.4(a), which contradicts the assumption of $\beta < 1$. Accordingly, we have

$$\begin{aligned} u(\phi,0) &= \lambda\beta v(0) + (1-\lambda)\beta u(\phi,0) - s = \lambda\beta(v(0) - u(\phi,0)) + \beta u(\phi,0) - s \\ &= \lambda\beta T(h_0) + \beta u(\phi,0) - s \end{aligned}$$

due to Eq. (6.2.5), from which

$$0 < u(\phi,0) = (\lambda\beta T(h_0) - s)/(1-\beta) = L(h_0)/(1-\beta), \tag{6.3.3}$$

hence $L(h_0) > 0$, i.e., $Q_0(r) > 0$.

(a2) We have $Q_0 = L(h_0) > 0$ due to (a1), hence $Q_1 > 0$ due to Lemma 6.3.4(b). Accordingly, from Eq. (6.2.24) with $r = 0$ we have $h_0 = \gamma(Q_0 - Q_1)$, i.e.,

$$h_0 = \gamma\big(L(h_0) - (1-q)L(h_1) - qL(h_0)\big) = \gamma(1-q)\big(L(h_0) - L(h_1)\big), \tag{6.3.4}$$

from which we obtain that if $h_0 = 0$, then $L(h_1) = L(h_0) = L(0) = \lambda\beta T(0) - s = \alpha > 0$, hence, $h_1 = 0$ due to Lemma 3.2.3(f).

(a3) From Eq. (6.3.4), if $h_0 > 0$, then $L(h_0) > L(h_1)$, implying $h_0 < h_1$ due to Lemma 3.2.3(f).

(b) Let $r_\circ \le r$. Then from Lemma 6.3.11(b) we have $Q_0(r) = L(h_0(r)) \le 0$. Assume $h_0 \le h_1$. Then $Q_1 = (1-q)L(h_1) + qL(h_0) \le (1-q)L(h_0) + qL(h_0) = L(h_0) \le 0$, implying $h_0 > h_1$ due to Lemma 6.3.4(b). This is a contradiction, hence it must be $h_0 > h_1$. ∎

Let us define

$$\hat{r} = \min\{r \mid h_0(r) > h_1(r)\}.$$

Lemma 6.3.13 *We have* $r_\circ \ge \hat{r} > 0$ *where if* $r \ge (<)\ \hat{r}$, *then* $h_0 > (\le)\ h_1$.

Proof. From Lemma 6.3.12 we have $h_0 \le h_1$ for $r = 0$ and $h_0 > h_1$ for $r \ge r_\circ$, implying that there exists a positive $\hat{r} \le r_\circ$ such as $h_0(r) > h_1(r)$. Accordingly, the latter half of the assertion is clearly true. ∎

Lemma 6.3.14 *If* $r = 0$, *then* h_i *is nondecreasing in* i *and* $Q_i > 0$ *for* $0 \le i < N$.

Proof. The former half is immediate from Lemmas 6.3.12(a2) and 6.3.6 with $i = 1$. The latter half is evident from Lemmas 6.3.12(a1) and 6.3.6. ∎

Theorem 6.3.1

(a) *Let* $\alpha \le 0$. *Then* $\langle\mathrm{K}\rangle_{0\le i\le N}$.

(b) *Let* $\alpha > 0$.

1 *Let* $r_0 \leq r$. *Then* $\langle \mathrm{K} \rangle_{0 \leq i < N}$ *or there exists* $i^*(0 < i^* < N)$ *such that* $\langle \mathrm{K} \rangle_{0 \leq i < i^*}$ *and* $\langle \mathrm{C} \rangle_{i^* \leq i < N}$.

2 *Let* $r < r_0$.

i $\langle \mathrm{C} \rangle_{0 \leq i < N}$.

ii *Let* $\hat{r} \leq r$. *Then* h_i *is not always nondecreasing in* i.

iii *Let* $r < \hat{r}$.

1 $h_0 \leq h_1$.

2 *If* $h_0 = h_1$, *then* h_i *is nondecreasing in* i *with* $h_i < b$ *for* $0 \leq i < N$.

3 *If* $h_0 < h_1$, *then* h_i *is strictly increasing in* i *with* $h_i < b$ *for* $0 \leq i < N$.

Proof. (a) Evident from Lemma 6.3.3.

(b) Let $\alpha > 0$. Here note that $\hat{r} \leq r_0$ from Lemma 6.3.13.

(b1) Let $r_0 \leq r$. Clearly $Q_0(r) \leq 0$ from Lemma 6.3.11(b with $i = 0$), hence $\langle \mathrm{K} \rangle_0$. From this result and the fact that once continuing the search is optimal for a certain i, i.e., $\langle \mathrm{C} \rangle_i$, then so also is for all i' with $i \leq i' < N$ due to Lemma 6.3.7. Accordingly, the assertion clearly holds.

(b2) Let $r < r_0$.

(b2i) Then $Q_0(r) > 0$ from Lemma 6.3.11(b with $i = 0$), hence $Q_i(r) > 0$ for $0 \leq i < N$ from Lemma 6.3.7, thus $\langle \mathrm{C} \rangle_{0 \leq i < N}$.

(b2ii) Let $\hat{r} \leq r$. Then since $h_0 > h_1$ from Lemma 6.3.13, it follows that h_i is not always nondecreasing in i.

(b2iii) Let $r < \hat{r}$.

(b2iii1-b2iii3) Immediate from, respectively, Lemmas 6.3.13, 6.3.6, and 6.3.5. ∎

6.4 Optimal Decision Rule

For explanatory convenience, let us define the two assertions below:

Assertion SB: Engaging only a subsidiary business without searching for new customers is always better than doing a custom production with searching for customers.

Assertion CP: Conducting custom production and doing subsidiary business when all orders are completed is always better than only enacting subsidiary business without searching for new customers.

Let us assume that the process starts without orders in the system, i.e., $i = 0$. Then if skipping the search is optimal, i.e., $\langle \mathrm{K} \rangle_0$, then since no customer appears, it follows that the number of back orders remains forever zero, i.e., $i = 0$ over the entire planning horizon. Accordingly, it eventually follows that Assertion SB holds. Consequently, the Optimal Decision Rule *6.2.1* can be restated as follows.

☐ *Optimal Decision Rule 6.4.1*

(a) Let $\alpha \leq 0$ or "$\alpha > 0$ and $r_0 \leq r$". Then $\langle \mathrm{K} \rangle_0$ (Theorem 6.3.1(a,b1)), hence Assertion SB holds for the reason stated above.

(b) Let $\alpha > 0$ and $r < r_0$. Then since $\langle \mathrm{C} \rangle_{0 \leq i < N}$ (Lemma 6.3.1(b2i)), it is optimal to conduct the search by paying a search cost s, implying that Assertion CP holds for $0 \leq i < N$. If $i = N$, any of continuing the search and skipping the search may be optimal; more precisely, if $r < r_N$, then $\langle \mathrm{C} \rangle_N$, or else $\langle \mathrm{K} \rangle_N$ (Lemma 6.3.11(b) with $i = N$).

1 Let $\hat{r} \le r$. Then h_i is not always nondecreasing in i (Theorem 6.3.1(b2ii)); in other words, as seen in Figure 6.5.2, there exists a $i^*(r) \ge 1$ such that h_i is decreasing in $i \le i^*(r)$ and increasing in $i > i^*(r)$.

2 Let $r < \hat{r}$. Then h_i is nondecreasing in i with $h_0 \le h_1$ where if $h_0 < h_1$, then h_i is strictly increasing in i (Theorem 6.3.1(b2iii2,b2iii3)).

In the pricing control problem it should be noted that the monotonicity of h_i in i stated above is inherited to the optimal price z_i due to Lemma 3.2.1(d). Since $z_i = z(h_i)$, from Lemma 3.2.1(e) we see that $z_i = a$ if $h_i < x^\star$.

6.5 Numerical Examples

In the first two subsections below, let us show some numerical examples of the optimal decision rule summarized in Section 6.4. Further, in the last subsection let us show the numerical examples that the probability of customer arrival is dependent on the number of back orders.

6.5.1 Admission Control Problem

Let $F(w)$ be the uniform distribution on $[0.01, 1.01]$, i.e., $a = 0.01$ and $b = 1.01$, and let $\lambda = 0.95$, $q = 0.35$, $\beta = 0.99$ and $s = 0.01$. Then from Eq. (3.4.1) we have

$$T(x) = \begin{cases} 0.51 - x & \text{for} \quad x < 0.01, \\ 0.5(1.01 - x)^2 & \text{for} \quad 0.01 \le x < 1.01, \\ 0 & \text{for} \quad 1.01 \le x. \end{cases}$$

In this case, $T(0) = 0.51$, hence $\alpha = \lambda\beta T(0) - s = 0.47 > 0$. Performing numerical calculations, we obtain $\hat{r} \simeq 0.133$ and $r_0 \simeq 0.326$. Accordingly, if the profit from a sideline $r \ge 0.326$, the assertion SB holds, and if $r < 0.326$, the assertion CP holds.

I. **Relationship of $\hat{r}$ and r_0 with related parameters λ, q, β, and s.**

1. Figure 6.5.1 illustrates the relationships of $\hat{r}$ and r_0 with the four related parameters λ, q, β, and s where the calculations are made by setting one of the four parameters as a variable with all the others being fixed. Here, it is to be noted that each of the coordinates planes of the four graphs is divided into the three regions;

   $$\mathcal{R}\langle \mathtt{K}\rangle \quad \text{for } r_0 \le r, \qquad \breve{\mathcal{R}}\langle \mathtt{C}\rangle \quad \text{for } \hat{r} \le r < r_0, \qquad \acute{\mathcal{R}}\langle \mathtt{C}\rangle \quad \text{for } r < \hat{r}.$$

   In the region $\mathcal{R}\langle \mathtt{K}\rangle$, not conducting the search, i.e., skipping the search is always optimal, and in both regions $\breve{\mathcal{R}}\langle \mathtt{C}\rangle$ and $\acute{\mathcal{R}}\langle \mathtt{C}\rangle$, conducting the search is always optimal where h_i is unimodal in i on $\breve{\mathcal{R}}\langle \mathtt{C}\rangle$ and nondecreasing in i on $\acute{\mathcal{R}}\langle \mathtt{C}\rangle$ (see Figure 6.5.2).

2. From Figure 6.5.1 it can be seen that:
   i. $\hat{r}$ is nonincreasing in s and nondecreasing in λ and β.
   ii. r_0 is nonincreasing in s and nondecreasing in λ and q.
   iii. $\hat{r}$ and r_0 are unimodal in, respectively, q and β. That $\hat{r}$ is unimodal in q implies that for a certain given r there exists q' and q'' with $q' < q''$ such that if $q \le q'$, then $(q, r) \in \breve{\mathcal{R}}\langle \mathtt{C}\rangle$, if $q' < q \le q''$, then $(q, r) \in \acute{\mathcal{R}}\langle \mathtt{C}\rangle$, and if $q'' \le q$, then *again* $(q, r) \in \breve{\mathcal{R}}\langle \mathtt{C}\rangle$; in other words, there exists two critical values of q such that the shape of h_i changes from "unimodal" to "nondecreasing" at $q = q'$ and from "nondecreasing" to "unimodal" at $q = q''$.

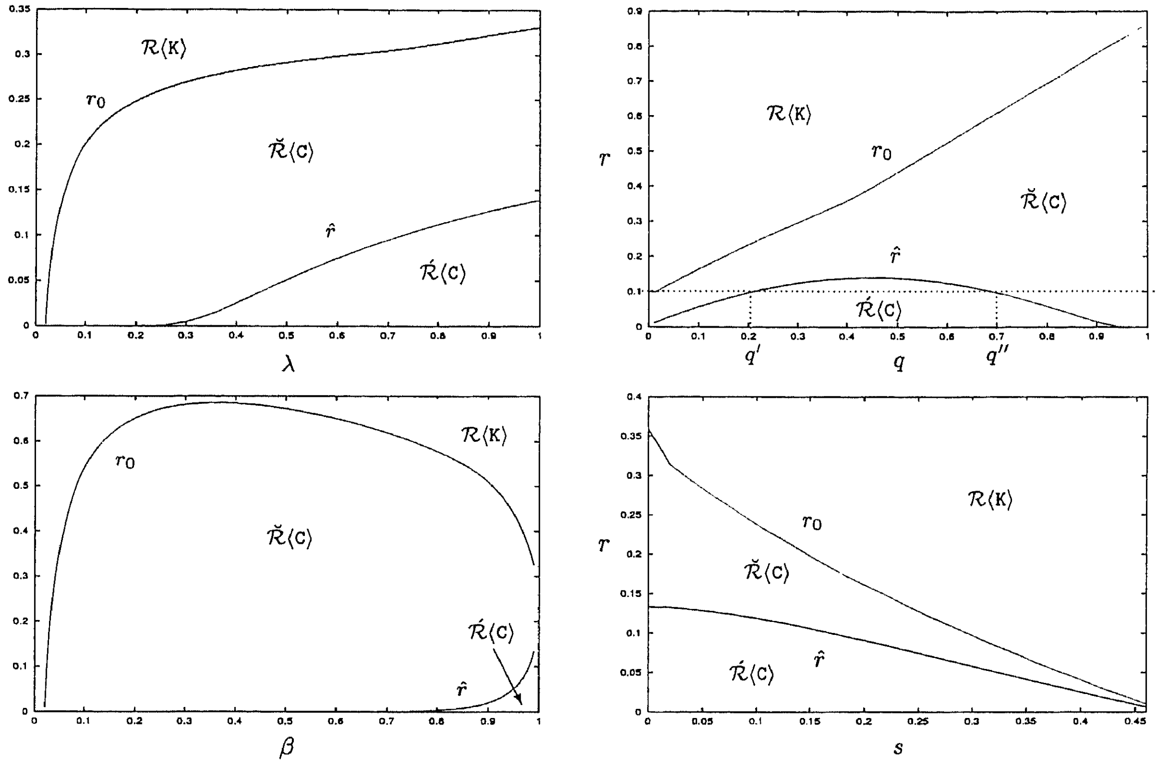

Figure 6.5.1: Relationships of $\hat{r}$ and r_0 with related parameters λ, q, β, and s.

II. **Optimal selection criterion h_i.**

Figure 6.5.2 depicts the relationships of h_i with the number of back orders i and the profit from a sideline r. The figure tells us that:

1. If $r < 0.133$, then h_i is strictly increasing in $i \geq 0$.
2. If $r = 0.133$, then $h_0 \simeq h_1 \simeq 0.373$ and h_i is strictly increasing in $i \geq 1$.
3. If $0.133 < r < 0.326$, then h_i is unimodal, i.e., there exists a $i' \geq 1$ such that h_i is strictly decreasing in $i \leq i'$ and strictly increasing in $i \geq i'$.
4. h_i is nondecreasing in r for all i (Lemma 6.3.8). Further, if i is sufficiently large, then h_i coincides with h_i with $r = 0.000$. The latter finding reflects the fact that as the number of back orders becomes larger, since the possibility of the backorder being exhausted gets smaller, the effect of r on h_i is gradually diminished.

6.5.2 Pricing Control Problem

Let $F(w)$ be the uniform distribution on $[2, 3]$, i.e., $a = 2$ and $b = 3$, and let $\lambda = 0.75$, $q = 0.55$, $\beta = 0.99$ and $s = 0.05$. Then from Eq. (3.4.3) we get

$$T(x) = \begin{cases} 2 - x & \text{for } x < 1 \quad \rightarrow \quad z(x) = 2, \\ 0.25(3-x)^2 & \text{for } 1 \leq x < 3 \quad \rightarrow \quad z(x) = (x+3)/2, \\ 0 & \text{for } 3 \leq x \quad \rightarrow \quad z(x) = 3. \end{cases}$$

In this case, $T(0) = 2$, hence $\alpha = \lambda\beta T(0) - s = 1.435 > 0$. Further, $x^* = 2a - b = 1$. Performing numerical calculations, we obtain $\hat{r} \simeq 0.499$ and $\simeq 1.383$. Accordingly, if the profit from a sideline $r \geq 1.383$, the

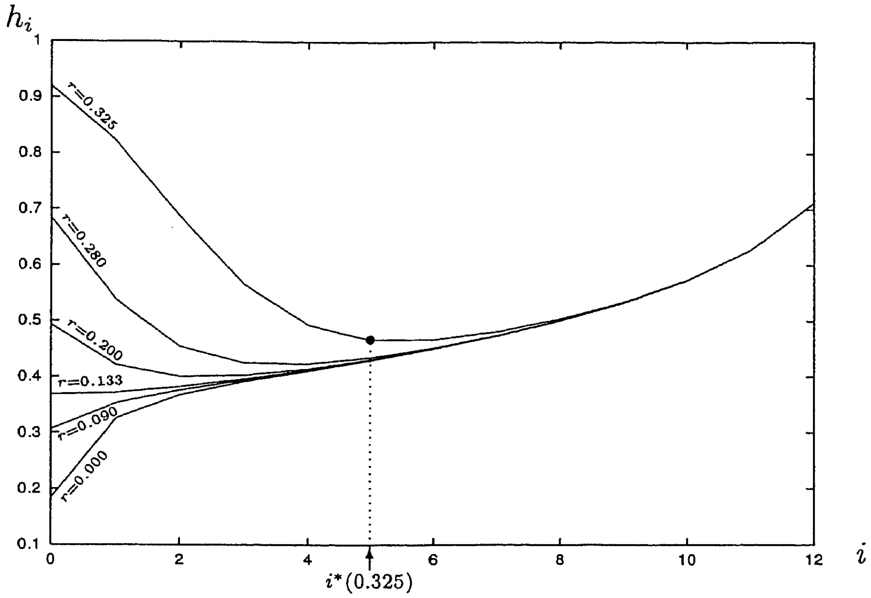

Figure 6.5.2: Graph of h_i where $\hat{r} = 0.133$ and $r_0 = 0.326$. Here, note that if $r < 0.133$, then h_i is strictly increasing in i and if $0.133 \le r < 0.326$, then h_i is unimodal in i.

assertion SB holds, and if $r < 1.383$, the assertion CP holds. In this case we obtain almost the same graphs as Figure 6.5.1. Figure 6.5.3 depicts the relationships of h_i and $z_i (= z(h_i))$ with the number of back orders i and the profit from a sideline r.

1. The graph on the left tells us that:
   i. If $r < 0.499$, then h_i is strictly increasing in $i \ge 0$.
   ii. If $r = 0.499$ then $h_0 \simeq h_1 \simeq 0.899$ and h_i is strictly increasing in $i \ge 1$.
   iii. If $0.499 \le r < 1.383$, there exists a $i' \ge 1$ such that h_i is strictly decreasing in $i \le i'$ and strictly increasing in $i \ge i'$.
2. The graph on the right shows the optimal ordering price z_i. Here note that there exists i such that $h_i < x^* = 2a - b = 1$ in the graph of h_i. Since $z_i = z(h_i) = a$ for $h_i < x^* = 1$ due to Lemma 3.2.1(e), it follows that $z_i = z(h_i)$ for such i becomes equal to $a = 2$; in other words, $z_i = z(h_i)$ is truncated by a, the low bound of the distribution. Further, it should be noted that there exists $h_i < a$ such that its corresponding optimal ordering price z_i becomes greater than a, i.e., $z_i = z(h_i) > a$.

6.5.3 i-dependent Customer Arrival Probability

In general, the information on the current number of back orders has an influence on customer's decision as to whether or not to place an order with the company. Hence, if the company holds many back orders, the customer may be reluctant to place an order with that company due to the possibility of having to wait a long time or suffering the delay in delivery. Accordingly, it is natural to assume that the probability of customer arrival, λ, is nonincreasing in the current number of back orders i, i.e., $\lambda = \lambda_i$ is nonincreasing in i. In order to evaluate how the i-dependence of customer arrival probability influences the optimal decision rule, let us here demonstrate some numerical examples under the same conditions as those in Section 6.5.1 except that $\lambda_i = 0.7\exp(-0.03i^2)$. The results obtained are as follows:

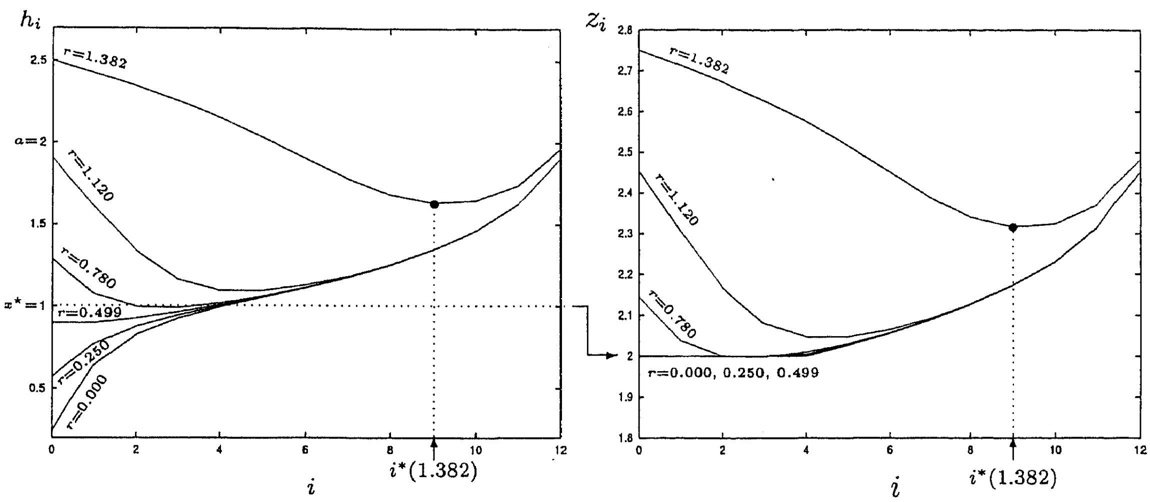

Figure 6.5.3: Graphs of h_i and z_i.

1. $\hat{r} = 0.215$ and $r_0 \simeq 0.301$.

2. Table 6.5.1 shows the optimal decision rules on continuing the search or skipping the search in states (ϕ, i) with $0 \leq i \leq 13$ where $r = 0.000$, $r = 0.300$, $r = 0.308$, and $r = 0.311$. The table tells us that:

   (i) Let $r < r_0 = 0.301$. Then it is not always optimal to continue the search, while for i-independent customer arrival probability it is always optimal to continue the search as seen in Theorem 6.3.1(b2i) except state (ϕ, N). Further, there exists i' such that $\langle \mathrm{C} \rangle_{0 \leq i < i'}$ and $\langle \mathrm{K} \rangle_{i' \leq i \leq N}$; $i' = 7$ for $r = 0.000$ and $i' = 4$ for $r = 0.300$.

Table 6.5.1: Optimal decision rules on continuing or skipping the search.

r \ i	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0.000	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩
0.300	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩
0.308	⟨K⟩	⟨K⟩	⟨C⟩	⟨C⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩
0.311	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩

   (ii) Let $r \geq r_0$. Then it is always optimal to skip the search for $r = 0.311$, a sufficiently large profit from a sideline. Further, there exists $i' < i''$ such that if $i \leq i'$, skipping the search is optimal, if $i' < i < i''$, continuing the search is optimal, and if $i'' \leq i$, *again* skipping the search is optimal; that is, there exist *double critical values* in terms of i; $i' = 1$ and $i'' = 4$ for $r = 0.308$.

3. The left graph of Figure 6.5.4 illustrates $\lambda_i = 0.7\exp(-0.03i^2)$ and the right graph depicts the relationship of h_i with the number of back orders i and the profit from a sideline r, implying that:

   i. h_i is nondecreasing in r for all i.

   ii. Let $\hat{r} \leq r < r_0$. Then h_i is strictly decreasing in i.

   iii. Let $0 \leq r < \hat{r}$. Then h_i is unimodal for a sufficiently small r; in other words, there exists a $i'(r) \geq 1$ such that h_i is strictly decreasing in $i \leq i'(r)$ and strictly increasing in $i \geq i'(r)$. This

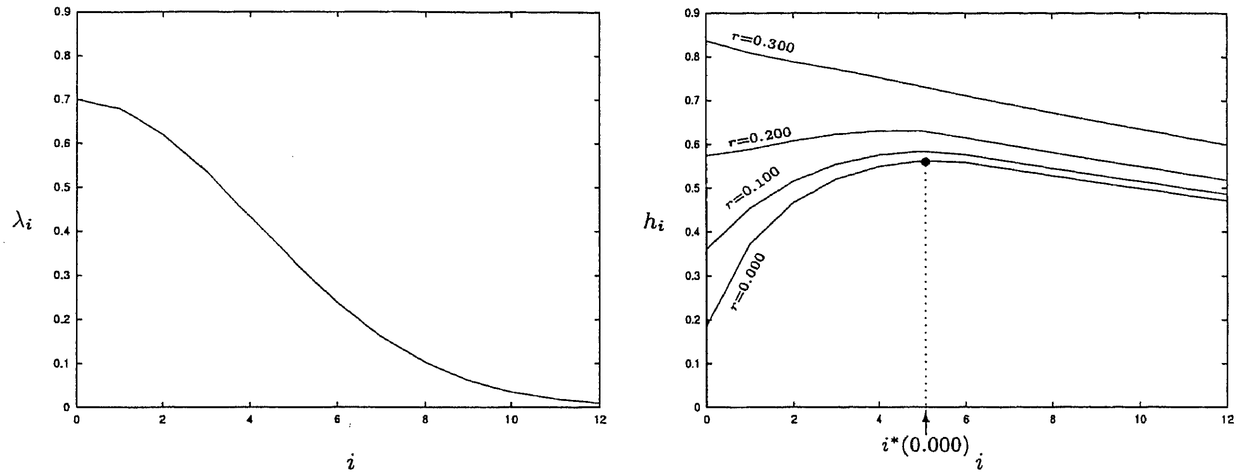

Figure 6.5.4: Graphs of h_i where $\lambda_i = 0.7\exp(-0.03i^2)$.

finding reflects that there exists $i' < i''$ such that if $i \leq i'$, accepting the order of customer is optimal, if $i' < i \leq i''$, rejecting it is optimal, and if $i'' < i$, *again* accepting it is optimal; that is to say, there exist *double critical values* in terms of i at both of which rejecting and accepting become indifferent.

In the pricing control problem, we obtain almost the same table and graph as Table 6.5.1 and Figure 6.5.4.

6.6 Conclusions and Considerations

Now, let us examine the practical implications of the optimal decision rule described in Section 6.4.

C1. Let $\alpha \leq 0$ or equivalently $\lambda\beta T(0) \leq s$, implying that the search cost s is sufficiently large to be greater than or equal to $\lambda\beta T(0)$. Then not conducting the search or equivalently skipping the search always becomes optimal, i.e., $\langle \mathtt{K} \rangle_0$. In this case, the assertion SB holds.

C2. Let $\alpha > 0$ or equivalently $\lambda\beta T(0) > s$, implying that the search cost s is sufficiently small to be smaller than $\lambda\beta T(0)$, including $s = 0$. Then it can be conjectured that conducting the search is always optimal, i.e., $\langle \mathtt{C} \rangle_{0 \leq i < N}$; Is this always the case? Unfortunately the answer is negative for the reasons stated below.

1. Let $r_0 \leq r$. In this case, even though the search cost is sufficiently small, if the profit from a sideline is sufficiently large to be greater than or equal to r_0, it becomes optimal to skip the search in order to enjoy the profit from a sideline, i.e., $\langle \mathtt{K} \rangle_0$. Accordingly, the conjecture stated above is false. In this case, the number of back orders remains forever zero, hence it follows that the subsidiary business is always operated, i.e., the assertion SB holds.
2. Let $r < \hat{r}$, that is, both search cost s and profit from a sideline r are sufficiently small. In the case, it is optimal to conduct the search, i.e., $\langle \mathtt{C} \rangle_{0 \leq i < N}$, with the resultant conclusion that the assertion CP holds. Accordingly, the conjecture stated above is true. Further, in this case the optimal selection criterion h_i in the admission control problem and the optimal ordering price z_i

in the pricing control problem both increase in the number of back orders i as seen in Figures 6.5.2 and 6.5.3 (Theorem 6.3.1(b2iii)). Below, let us consider the implication of the monotonicity of h_i and z_i in i.

i. Assume that the number of back orders i is sufficiently small. Then in order to avoid Opportunity loss II, the company should accept any order however low in price it may be; of course, although there exists a low bound. This implies that the optimal selection criterion h_i in the admission control problem and the optimal ordering price z_i in the pricing control problem must be set to be low.

ii. Assume that the number of back orders i is sufficiently large. Then in order to avoid Opportunity loss I, the company should reject orders with low price by setting the high selection criterion in the admission control problem and proposing the high price in the pricing control problem; with the result that only orders with a high price are accepted in the admission control problem and that a high price is offered in the pricing control problem.

iii. The above two considerations imply that the optimal selection criterion h_i in the admission control problem and the optimal ordering price z_i in the pricing control problem should be set to be increasing in the number of back orders i. The monotonicity of h_i and z_i brings about the following dynamic behavior for the movement of the number of back orders i. First, let us consider the admission control problem. When the number of back orders is small, since the selection criterion is low, the number of orders accepted becomes large; accordingly, the number of back orders increases. Since the selecting criterion becomes high as the number of back orders increases, the number of orders accepted becomes small; therefore, the number of back orders becomes small, hence it follows that the number of back orders decreases. The above fact can be restated as follows. The smaller the number of back orders may become, the stronger the force making itself large may become; on the contrary, the larger the number of back orders may become, the stronger the force making itself small may become. Such a movement in the number of back orders looks just like the free oscillation of a pendulum, always moving toward the vertical, the most stable position. The above consideration leads us to the implication that the number of back orders fluctuates while all the time it is being pulled toward the equilibrium point in the stochastic sense. Stabilization of the number of back orders is also what management desires. In the pricing control problem the same consideration as the above can be given.

3. Let $\hat{r} \leq r < r_0$, i.e., the profit from a sideline be neither sufficiently large nor sufficiently small. For example, if $0.133 \leq r < 0.326$ in Figure 6.5.2 and if $0.499 \leq r < 1.383$ in Figure 6.5.3, there exists a $r' \geq 1$ such that h_i is decreasing in $i \leq r'$ and h_i is increasing in $i \geq r'$; in other words, neither the optimal selection criterion h_i nor the optimal ordering price z_i are always increasing in the number of back orders i, i.e., h_i and z_i are both unimodal in i. This implies the following. Let the number of back orders i be sufficiently small. Then if orders are rejected by setting the high selection criterion in the admission control problem, the probability of production process becoming idle is large; as a result, the company can enjoy the profit from a sideline. Further, as the number of back orders increases until $i = r'$ and goes cross r', since the influence of profit from a sideline on the selection criterion and the ordering price get weaker, they become nondecreasing

in i as in the case of $r = 0.000$. Now, that h_i takes a shape such as that stated above in the admission control problem first tells us the following. For an appearing customer with a certain value w there exists $i' < i''$ such that if $i \leq i'$, rejecting the order of customer is optimal, if $i' < i \leq i''$, accepting it is optimal, and if $i'' < i$, *again* rejecting it is optimal; that is, it follows that there exist *double critical values* in terms of i at both of which rejecting and accepting become indifferent. In the pricing control problem the same consideration as the above can be also given.

C3. For explanatory convenience let us refer to the model with an i-independent (i-dependent) customer arrival probability λ (λ_i) as the i-independent (i-dependent) model. Then the optimal decision rules on continuing or skipping the search for both models can be summarized as in the table below.

Table 6.6.2: Optimal decision rules on continuing or skipping the search.

	i-independent model	i-dependent model
$r < r_0$	$\langle C \rangle_{0 \leq i < N}$, and either $\langle C \rangle_N$ or $\langle K \rangle_N$	There exists i' such that $\langle C \rangle_{0 \leq i < i'}$ and $\langle K \rangle_{i' \leq i \leq N}$
$r_0 \leq r$	$\langle K \rangle_{0 \leq i \leq N}$ or there exists i' such that $\langle K \rangle_{0 \leq i < i'}$ and $\langle C \rangle_{i' \leq i < N}$	$\langle K \rangle_{0 \leq i \leq N}$ or there exists $i' < i''$ such that $\langle K \rangle_{0 \leq i \leq i'}$, $\langle C \rangle_{i' < i < i''}$, and $\langle K \rangle_{i'' \leq i \leq N}$

Further, the monotonicity of h_i and z_i in i can be summarized as in the table below.

Table 6.6.3: Monotonicity of h_i and z_i in i.

	i-independent model	i-dependent model
$0 \leq r < \hat{r}$	h_i and z_i are nondecreasing in i	h_i and z_i are concave and unimodal in i
$\hat{r} \leq r < r_0$	h_i and z_i are convex and unimodal in i	h_i and z_i are nonincreasing in i

Noting that h_i and z_i are convex (concave) and unimodal in i for $\hat{r} \leq r < r_0$ $(0 \leq r < \hat{r})$ in the i-independent (i-dependent) model from the above table, we see that there exists a I such that the optimal selection criterion h_i and the optimal price z_i are both nondecreasing (nonincreasing) in $i \geq I$ for $r < r_0$. This fact can be explained as follows. First, in the i-independent model the company should reject orders with low value in the admission control problem and offer low price for an order in the pricing control problem by setting high h_i and z_i in order to avoid Opportunity loss I. On the other hand, in i-dependent model, the larger the number of back orders may be, the smaller the probability of customer arrival may be, hence the number of orders accepted becomes small, causing the decrease of back orders and the occurrence of Opportunity loss II. Accordingly, to avoid the opportunity loss, the company should accept even orders with low value in the admission control problem and offer low price in the pricing control problem. This implies that the optimal selection criterion h_i and the optimal price z_i in both problems are nonincreasing in i (See Figure 6.5.4).

Chapter 7

Model IV: Deterministic model

While the completion of order is assumed to be stochastic in all the models in the previous three chapters, in this chapter we assume that every accepted orders requires a fixed production periods d (deterministic) and that any order accepted must be delivered to the customer within τ periods since it was accepted.

Contents

7.1 System of Optimal Equations

While the completion of the order is assumed to be stochastic in the models of the previous three chapters, in the model in this chapter we assume that every accepted order requires a fixed production period d (deterministic) and that any order accepted must be delivered to the customer within τ periods since it was accepted where $\tau \geq d$. In the derivation of the system of the optimal equations of this model, the following three points should be noted:

1. In both admission control and pricing control problems, by $u(\phi, i)$ we shall denote the maximum total expected present discounted net profits starting from a state of having the fictitious customer ϕ and orders of $i\,(0 \leq i \leq \tau)$ periods in the company; let us refer to such a situation as the state (ϕ, i). When in state (ϕ, i) with $\tau - d + 1 < i \leq \tau$, even if a customer appears, it can not be accepted because if the order of the customer is accepted, it can not be completed within τ periods due to $\tau - d + 1 + d = \tau + 1$; in other words, the search must be skipped. Accordingly, in this case the state (ϕ, i) is changed into $u(\phi, i-1)$ at the next point in time.
2. In the admission control problem, by $u(w, i)$ let us denote the maximum total expected present discounted net profits starting with orders of $i\,(0 \leq i \leq \tau)$ periods in the company and an arriving customer who offers a price w.
3. In the pricing control problem, by $u(1, i)$ let us denote the maximum total expected present discounted net profits starting with orders of $i\,(0 \leq i \leq \tau)$ periods in the company and an arriving customer to whom the company proposes a price z for an order.

From the same reason as mentioned in Section 6.1 the maximum total expected present discounted net profits $u(\phi, i)$, $u(w, i)$, and $u(1, i)$ are also bounded in i. For convenience in later discussions, let us define

$$h_i = u(\phi, i) - u(\phi, i+d), \quad 0 \le i \le \tau - d. \tag{7.1.1}$$

Then the system of optimal equations can be described as follows:

1. **Admission control problem:**

$$u(\phi, 0) = \max\left\{ \begin{array}{l} \mathtt{C} : \beta(\lambda\, \mathbf{E}[u(\xi, 0)] + (1-\lambda)u(\phi, 0)) - s + r \\ \mathtt{K} : \beta u(\phi, 0) + r \end{array} \right\}, \tag{7.1.2}$$

$$u(\phi, i) = \max\left\{ \begin{array}{l} \mathtt{C} : \beta(\lambda\, \mathbf{E}[u(\xi, i)] + (1-\lambda)u(\phi, i-1)) - s \\ \mathtt{K} : \beta u(\phi, i-1) \end{array} \right\}, \quad 1 \le i \le \tau - d + 1, \tag{7.1.3}$$

$$u(\phi, i) = \beta u(\phi, i-1), \quad \tau - d + 1 < i \le \tau, \tag{7.1.4}$$

$$u(w, i) = \max\left\{ \begin{array}{l} \mathtt{A} : w + u(\phi, i+d) \\ \mathtt{R} : u(\phi, i), \end{array} \right\} \tag{7.1.5}$$

$$= \max\{w - h_i, 0\} + u(\phi, i), \quad 0 \le i \le \tau - d. \quad \square \tag{7.1.6}$$

2. **Pricing control problem:**

$$u(\phi, 0) = \max\left\{ \begin{array}{l} \mathtt{C} : \beta(\lambda u(1, 0) + (1-\lambda)u(\phi, 0)) - s + r \\ \mathtt{K} : \beta u(\phi, 0) + r \end{array} \right\}, \tag{7.1.7}$$

$$u(\phi, i) = \max\left\{ \begin{array}{l} \mathtt{C} : \beta(\lambda u(1, i-1) + (1-\lambda)u(\phi, i-1)) - s \\ \mathtt{K} : \beta u(\phi, i-1) \end{array} \right\}, \quad 1 \le i \le \tau - d + 1, \tag{7.1.8}$$

$$u(\phi, i) = \beta u(\phi, i-1), \quad \tau - d + 1 < i \le \tau, \tag{7.1.9}$$

$$u(1, i) = \max_z\{p(z)\big(z + u(\phi, i+1)\big) + (1 - p(z))u(\phi, i)\} \tag{7.1.10}$$

$$= \max_z p(z)(z - h_i) + u(\phi, i), \quad 0 \le i \le \tau - d. \quad \square \tag{7.1.11}$$

The unique existence of the solution of the above equations can be proven in quite the same way as in Lemma 6.3.1.

7.2 Transformation

Let us define

$$v(i) = \left\{ \begin{array}{ll} \mathbf{E}[u(w, i)] & \text{for the admission control problem} \\ u(1, i) & \text{for the pricing control problem} \end{array} \right\}, \quad 0 \le i \le \tau - d. \tag{7.2.1}$$

Then, using Eqs. (7.2.1) and (3.1.1), we can immediately rearrange both Eqs. (7.1.2) to (7.1.6) and Eqs. (7.1.7) to (7.1.11) into the identical expressions below.

$$u(\phi,0) = \max\left\{ \begin{array}{l} \text{C}: \beta(\lambda v(0) + (1-\lambda)u(\phi,0)) - s \\ \text{K}: \beta u(\phi,0) \end{array} \right\} + r, \qquad (7.2.2)$$

$$u(\phi,i) = \max\left\{ \begin{array}{l} \text{C}: \beta(\lambda v(i-1) + (1-\lambda)u(\phi,i-1)) - s \\ \text{K}: \beta u(\phi,i-1) \end{array} \right\}, \quad 1 \le i \le \tau - d + 1, \qquad (7.2.3)$$

$$u(\phi,i) = \beta u(\phi,i-1), \quad \tau - d + 1 < i \le \tau, \qquad (7.2.4)$$

$$v(i) = T(h_i) + u(\phi,i), \quad 0 \le i \le \tau - d. \qquad (7.2.5)$$

Further, Eqs. (7.2.2) and (7.2.3) can be rewritten as, respectively,

$$u(\phi,0) = \beta u(\phi,0) + \max\{\lambda\beta(v(0) - u(\phi,0)) - s, 0\} + r, \qquad (7.2.6)$$

$$u(\phi,i) = \beta u(\phi,i-1) + \max\{\lambda\beta(v(i-1) - u(\phi,i-1)) - s, 0\}, \quad 1 \le i \le \tau - d + 1. \qquad (7.2.7)$$

Noting Eqs. (7.2.5), we can rewrite Eqs. (7.2.6) and (7.2.7) as follows.

$$u(\phi,0) = \beta u(\phi,0) + \max\{\lambda\beta T(h_0) - s, 0\} + r, \qquad (7.2.8)$$

$$u(\phi,i) = \beta u(\phi,i-1) + \max\{\lambda\beta T(h_{i-1}) - s, 0\}, \quad 1 \le i \le \tau - d + 1, \qquad (7.2.9)$$

where clearly

$$u(\phi,1) = u(\phi,0) - r. \qquad (7.2.10)$$

Accordingly, using the *L-function* defined in Eq. (3.1.2), we can rewrite Eqs. (7.2.8) and (7.2.9) as, respectively,

$$u(\phi,0) = \big(\max\{L(h_0),0\} + r\big)/(1-\beta), \qquad (7.2.11)$$

$$u(\phi,i) = \beta u(\phi,i-1) + \max\{L(h_{i-1}),0\}, \quad 1 \le i \le \tau - d + 1. \qquad (7.2.12)$$

Lemma 7.2.1 $u(\phi,i) \ge 0$ *for all* i

Proof. From Eq. (7.2.11) we have $u(\phi,0) \ge r/(1-\beta) \ge 0$. Further, for $1 \le i \le \tau$ we get $u(\phi,i) \ge \beta u(\phi,i-1)$ from Eqs. (7.2.12) and (7.2.4), hence $u(\phi,i) \ge 0$ for all i. ∎

Here noting $u(\phi,i) \ge 0$ due to Lemma 7.2.1, we obtain $u(w,i) \ge w$ and $u(1,i) \ge \max_z p(z)z$ from Eqs. (7.1.5) and (7.1.10), hence $\mathbf{E}[u(w,i)] \ge \mu = T(0)$ and $u(1,i) \ge \max_z p(z)z = T(0)$. Accordingly, from Eq. (7.2.1) we get

$$v(i) \ge T(0), \quad 0 \le i \le \tau - d. \qquad (7.2.13)$$

From all the above it can be easily seen that the optimal decision rules for any given i can be prescribed as follows.

☐ *Optimal Decision Rule 7.2.1*

1. Admission control problem:

i. Let $\tau - d + 1 < i \leq \tau$. Then $\langle \mathtt{K} \rangle_i$.

ii. Let $1 \leq i \leq \tau - d + 1$. If $L(h_{i-1}) > 0$, then $\langle \mathtt{C} \rangle_i$, or else $\langle \mathtt{K} \rangle_i$ where $\langle \mathtt{C} \rangle_0 \Leftrightarrow \langle \mathtt{C} \rangle_1$ and $\langle \mathtt{K} \rangle_0 \Leftrightarrow \langle \mathtt{K} \rangle_1$.

iii. Let $0 \leq i \leq \tau - d$ and an order with value w appear after the search was enacted. If $w > h_i$, then $\langle \mathtt{A}(w) \rangle_i$, or else $\langle \mathtt{R}(w) \rangle_i$.

2. Pricing control problem:

i. Let $\tau - d + 1 < i \leq \tau$. Then $\langle \mathtt{K} \rangle_i$.

ii. Let $1 \leq i \leq \tau - d + 1$. If $L(h_{i-1}) > 0$, then $\langle \mathtt{C} \rangle_i$, or else $\langle \mathtt{K} \rangle_i$ where $\langle \mathtt{C} \rangle_0 \Leftrightarrow \langle \mathtt{C} \rangle_1$ and $\langle \mathtt{K} \rangle_0 \Leftrightarrow \langle \mathtt{K} \rangle_1$.

iii. Let $0 \leq i \leq \tau - d$ and a customer appear after the search was enacted. Then $\langle \mathtt{O}(z_i) \rangle$ where $z_i = z(h_i)$.

7.3 Analysis

7.3.1 Case of $\tau = d$

In this case, from Eqs. (7.2.2) to (7.2.4) we have

$$u(\phi, 0) = \max\{\beta(\lambda v(0) + (1 - \lambda)u(\phi, 0)) - s, \beta u(\phi, 0)\} + r, \tag{7.3.1}$$

$$u(\phi, 1) = \max\{\beta(\lambda v(0) + (1 - \lambda)u(\phi, 0)) - s, \beta u(\phi, 0)\}, \tag{7.3.2}$$

$$u(\phi, i) = \beta u(\phi, i - 1), \quad 2 \leq i \leq d. \tag{7.3.3}$$

For convenience in the later discussions, for any given x let us define

$$G(x, r, d) = \max\{L(x), 0\} - \frac{1 - \beta}{1 - \beta^{d-1}} x + \frac{(1 - \beta^d)r}{1 - \beta^{d-1}}. \tag{7.3.4}$$

By h^* let us denote the solution of $G(x, r, d) = 0$ if it exists, i.e.,

$$G(h^*, r, d) = 0. \tag{7.3.5}$$

From Eq. (7.3.3) we get

$$u(\phi, d) = \beta^{d-1} u(\phi, 1). \tag{7.3.6}$$

Accordingly, from Eq. (7.1.1) with $i = 0$ we have

$$h_0 = u(\phi, 0) - \beta^{d-1} u(\phi, 1). \tag{7.3.7}$$

Substituting Eq. (7.2.10) into Eq. (7.3.7) yields

$$h_0 = (1 - \beta^{d-1})u(\phi, 0) + r\beta^{d-1} \geq 0 \tag{7.3.8}$$

or equivalently

$$u(\phi, 0) = \left(h_0 - r\beta^{d-1}\right)/(1 - \beta^{d-1}). \tag{7.3.9}$$

From Eqs. (7.2.11) and (7.3.9) we have

$$\frac{h_0}{1-\beta^{d-1}} = \frac{\max\{L(h_0), 0\} + r}{1-\beta} + \frac{r\beta^{d-1}}{1-\beta^{d-1}}. \tag{7.3.10}$$

Then, the above equation can be rewritten as

$$G(h_0, r, d) = 0. \tag{7.3.11}$$

Lemma 7.3.1

(a) *$G(x, r, d)$ is strictly decreasing in x.*

(b) *$G(x, r, d) < (>) \, 0$ for any sufficiently large (small) x.*

(c) *$G(x, r, d)$ is strictly increasing in r.*

Proof. (a) Immediate from Lemma 3.2.2(a) and the fact that $-(1-\beta)x/(1-\beta^{d-1})$ is strictly decreasing in x due to the assumption of $\beta < 1$.

(b) Applying Lemma 3.2.3(d) to Eq. (7.3.4) leads to

$$\lim_{x\to\infty} G(x, r, d) = \max\{-s, 0\} - \lim_{x\to\infty} \frac{1-\beta}{1-\beta^{d-1}}x + \frac{r(1-\beta^d)}{1-\beta^{d-1}} = -\infty.$$

Further, we have

$$\lim_{x\to-\infty} G(x, r, d) \geq - \lim_{x\to-\infty} \frac{1-\beta}{1-\beta^{d-1}}x + \frac{r(1-\beta^d)}{1-\beta^{d-1}} = \infty.$$

(c) Evident from Eq. (7.3.4). ∎

Lemma 7.3.2 *h^* uniquely exists with $h^* = h_0$.*

Proof. Evident from Lemma 7.3.1(a,b) and Eq. (7.3.11). ∎

Lemma 7.3.3 *$h_0(r)$ is strictly increasing in r with $\lim_{r\to\infty} h_0(r) = \infty$ and $\lim_{r\to-\infty} h_0(r) = -\infty$.*

Proof. The same as the proof in Lemma 4.3.3(e). ∎

Let r_b be the solution of $h_0(r) = b$ if it exists.

Lemma 7.3.4

(a) *r_b uniquely exists.*

(b) *$L(h_0(r))$ is strictly decreasing in $r \in (-\infty, r_b)$.*

(c) *$L(h_0(r)) > 0$ for any sufficiently small r and $L(h_0(r)) = -s$ for $r_b \leq r$.*

Proof. Proven in the same way as in the proofs of Lemma 4.3.4. ∎

Lemma 7.3.5 *Let $\alpha \leq 0$. Then $\langle K \rangle_0$ and $\langle K \rangle_1$.*

Proof. Assume $\alpha \leq 0$, i.e., $\lambda\beta T(0) - s \leq 0$. Then since $h_0 \geq 0$ from Eq. (7.3.8), we have $T(0) \geq T(h_0)$ due to Lemma 3.2.2(a), hence $0 \geq \alpha = \lambda\beta T(0) - s \geq \lambda\beta T(h_0) = L(h_0)$, implying $\langle K \rangle_0$ and $\langle K \rangle_1$. ∎

Lemma 7.3.6 *Let $\alpha > 0$.*

(a) *$u(\phi, i) > 0$ for all i.*

(b) *$h_0 = h^* > r \geq 0$.*

(c) *If* $s > 0$, *then* r^* *exists with* $0 < r^* \leq r_b$, *or else* $r^* > r_b$.

(d) *If* $r^* \leq r$, *then* $\langle \mathrm{K} \rangle_0$ *and* $\langle \mathrm{K} \rangle_1$.

(e) *If* $r^* > r$, *then* $\langle \mathrm{C} \rangle_0$ *and* $\langle \mathrm{C} \rangle_1$.

(f) h_0 *is strictly increasing in* d.

Proof. (a) Noting $u(\phi, 0) \geq 0$ due to Lemma 7.2.1, from Eqs. (7.3.2) and (7.2.13) we have

$$u(\phi, 1) \geq \beta(\lambda v(0) + (1 - \lambda)u(\phi, 0)) - s \geq \lambda\beta T(0) - s = \alpha > 0.$$

Accordingly, from Eq. (7.2.10) we get $u(\phi, 0) > r \geq 0$. Hence, $u(\phi, i) > 0$ for all i from Eq. (7.3.3).

(b) If $r = 0$, then $G(0, 0, d) = \max\{\alpha, 0\} = \alpha > 0$, implying $h^* > 0 = r$. If $r > 0$, then from Eq. (7.3.4) we obtain $G(r, r, d) = \max\{\lambda\beta T(r) - s, 0\} + \beta r > 0$. Thus, $h^* > r \geq 0$.

(c) Let $r = 0$. Assume $\lambda\beta v(0) + (1-\lambda)\beta u(\phi, 0) - s \leq \beta u(\phi, 0)$. Then $u(\phi, 0) = \beta u(\phi, 0)$ from Eq. (7.3.1), leading to $\beta = 1$ due to $u(\phi, 0) > 0$ from (a), which contradicts the assumption of $\beta < 1$. Accordingly, it must be $\lambda\beta v(0) + (1 - \lambda)\beta u(\phi, 0) - s > \beta u(\phi, 0)$, which can be rearranged into $0 < \lambda\beta(v(0) - u(\phi, 0)) - s = \lambda\beta T(h_0(0)) - s = L(h_0(0))$ from Eq. (4.2.6). This implies $r^* > 0$ due to Lemma 7.3.4(b). Now, let $s = 0$. Then if $r < r_b$, since $h_0(r) < h_0(r_b) = b$ due to Lemma 7.3.3, we have $L(h_0(r)) > 0$. Further, if $r_b \leq r$, since $b = h_0(r_b) \leq h_0(r)$ due to Lemma 7.3.3, we get $L(h_0(r)) = 0$. Accordingly, from the definition of r^* it must be $r^* = r_b$. Let $s > 0$. Then $L(h_0(r_b)) = -s < 0$ due to Lemma 7.3.4(c), implying $r^* < r_b$ due to Lemma 7.3.4(b). Moreover, r^* uniquely exists due to Lemma 7.3.4(b,c).

(d) If $r^* \leq r$, then $h_0(r^*) \leq h_0(r)$ from Lemma 7.3.3. Therefore, $L(h_0(r)) \leq L(h_0(r^*)) = 0$, implying $\langle \mathrm{K} \rangle_0$ and $\langle \mathrm{K} \rangle_1$.

(e) If $r^* > r$, then $h_0(r^*) > h_0(r)$ from Lemma 7.3.3. Accordingly, $L(h_0(r)) > L(h_0(r^*)) = 0$, implying $\langle \mathrm{C} \rangle_0$ and $\langle \mathrm{C} \rangle_1$.

(f) The differential of $G(h^*, r, d)$ given by Eq. (7.3.4) with respect to d becomes

$$\begin{aligned} G'(h^*, r, d) &= \frac{(-r\beta^d + r - (1 - \beta)h^*)(\beta^{d-1} \log \beta) - r\beta^d \log \beta(1 - \beta^{d-1})}{(1 - \beta^{d-1})^2} \\ &= \frac{\beta^{d-1}(1 - \beta)(r - h^*) \log \beta}{(1 - \beta^{d-1})^2} > 0 \end{aligned}$$

due to $\beta < 1$, $\log \beta < 0$, and $r < h^*$ from (b). Accordingly, $G(x, r, d)$ is strictly increasing in d. Hence, from the above result and the fact that $G(x, r, d)$ is strictly increasing in x it follows that h_0 is strictly increasing in d. ▮

7.3.2 Case of $\tau > d$

Lemma 7.3.7

(a) $u(\phi, i)$ *is nonincreasing in* i.

(b) $h_i \geq 0$ *for* $0 \leq i \leq \tau - d$.

Proof. (a) To begin with, for $t \geq 1$ let us define the following recurrent relations corresponding to Eqs. (7.2.2) to (7.2.4).

$$u_t(\phi,0) = \max\{\lambda\beta v_{t-1}(0) + (1-\lambda)\beta u_{t-1}(\phi,0) - s, \beta u_{t-1}(\phi,0)\} + r, \tag{7.3.12}$$

$$u_t(\phi,i) = \max\{\lambda\beta v_{t-1}(i-1) + (1-\lambda)\beta u_{t-1}(\phi,i-1) - s, \beta u_{t-1}(\phi,i-1)\},\ 1 \le i \le \tau - d + 1, \tag{7.3.13}$$

$$u_t(\phi,i) = \beta u_{t-1}(\phi,i-1), \quad \tau - d + 1 < i \le \tau, \tag{7.3.14}$$

where $u_0(\phi,i) = 0$ for all i. Further, as expressions corresponding to Eqs. (7.1.5) and (7.1.10) for $0 \le i \le n$ let us define, respectively,

$u_t(w,i) = \max\{w + u_t(\phi,i+d), u_t(\phi,i)\}$, for the admission control problem,
$u_t(1,i) = \max_z\{p(z)\big(z + u_t(\phi,i+d)\big) + (1-p(z))u_t(\phi,i)\}$, for the pricing control problem.

Further, let us define $v_t(i) = \mathbf{E}[u_t(w,i)]$ for the admission control problem and $v_t(i) = u_t(1,i)$ for the pricing control problem. Then we have

$$v_t(i) = \left\{ \begin{array}{l} \mathbf{E}[\max\{w + u_t(\phi, idl), u_t(\phi,i)\}], \\ \max_z\{p(z)\big(z + u_t(\phi,i+d) + (1-p(z))u_t(\phi,i)\} \end{array} \right\}, \quad 0 \le i \le \tau - d. \tag{7.3.15}$$

Now, clearly $u_0(\phi,i)$ is nonincreasing in i, hence $v_0(i)$ is also nonincreasing in i due to Eq. (7.3.15). Assume that $u_{t-1}(\phi,i)$ is nonincreasing in i, hence $v_{t-1}(i)$ is also nonincreasing in i. Then it is immediate that $u_t(\phi,i)$ is nonincreasing in i from Eqs. (7.3.12) to (7.3.14). Thus, $u(\phi,i)$ is nonincreasing in i.

(b) Immediate from Eq. (7.1.1) and (a). ∎

Lemma 7.3.8 *Let $\alpha \le 0$. Then $L(h_i) \le 0$ for $0 \le i \le \tau - d$.*

Proof. Let $\alpha \le 0$. Then from Lemmas 7.3.7(b) and 3.2.3(c) we have $L(h_i) \le \lambda\beta T(0) - s = \alpha \le 0$ for $0 \le i \le \tau - d$. ∎

Lemma 7.3.9

(a) *If $L(h_{i-1}) \le 0$ for a given i such as $1 \le i \le \tau - d + 1$, then $h_{i-1} \ge h_i$.*

(b) *If $L(h_{i-1}) > 0$ for a given i such that $1 \le i \le \tau - d + 1$, then $L(h_j) > 0$ for $i \le j \le \tau - d + 1$.*

Proof. Note that $u(\phi,i) \ge \beta u(\phi,i-1)$ for $1 \le i \le \tau$ from Eqs. (7.2.3) and (7.2.4), hence $u(\phi,i+d) \ge \beta u(\phi,i+d-1)$ for $1 \le i \le \tau - d$.

(a) Let $L(h_{i-1}) \le 0$ for a given i such as $1 \le i \le \tau - d + 1$. Then from Eq. (7.2.12) we have $u(\phi,i) = \beta u(\phi,i-1)$. Accordingly, we get

$$h_i = u(\phi,i) - u(\phi,i+d) \le \beta u(\phi,i-1) - \beta u(\phi,i+d-1) = \beta h_{i-1} \le h_{i-1}.$$

(b) Let $L(h_{i-1}) > 0$ for a given i such that $1 \le i \le \tau - d + 1$. Then from Eq. (7.2.12) we have

$$u(\phi,i) = \beta u(\phi,i-1) + L(h_{i-1}). \tag{7.3.16}$$

Accordingly, we obtain

$$\begin{aligned} h_i &= u(\phi,i) - u(\phi,i+d) \\ &\le \beta u(\phi,i-1) + L(h_{i-1}) - \beta u(\phi,i+d-1) = \beta h_{i-1} + L(h_{i-1}) = M(h_{i-1}). \end{aligned} \tag{7.3.17}$$

First, let $h_{i-1} \geq h_i$. Then clearly $0 < L(h_{i-1}) \leq L(h_i)$ due to Lemma 3.2.3(a). Next, let $h_{i-1} < h_i$. Then since $M(h_{i-1}) \leq M(h_i)$ due to Lemma 3.2.4, from Eq. (7.3.17) we get $h_i \leq M(h_{i-1}) \leq M(h_i) = \beta h_i + L(h_i)$, hence $(1-\beta)h_i \leq L(h_i)$. Since $0 \leq h_{i-1} < h_i$ due to Lemma 7.3.7(b), we have $L(h_i) > 0$. Repeating the same procedure yields $L(h_j) > 0$ for $i \leq j \leq \tau - d$. This completes the proof. ∎

Lemma 7.3.10 *Let $h_{i-1} < h_i$ for a given $i\,(1 \leq i \leq \tau - d - 1)$.*

(a) *$h_j < b$ for $i \leq j \leq \tau - d$.*

(b) *Let $\tau - 2d < i \leq \tau - d - 1$. Then $h_i < h_{i+1}$.*

(c) *Let $1 \leq i \leq \tau - 2d$.*

  1 *$h_i < h_{i+d-1}$.*

  2 *If $h_i \geq h_{i+1}$, then $h_{i+d-1} \geq h_{i+d}$.*

Proof. Let $h_{i-1} < h_i$ for a given $i\,(1 \leq i \leq \tau - d - 1)$. Then $L(h_{i-1}) > 0$ from the contraposition of Lemma 7.3.9(a), hence $L(h_j) > 0$ for $i \leq j \leq \tau - d$ due to Lemma 7.3.9(b). Now, from Eqs. (7.2.12) and (7.2.4) we have

$$u(\phi, j) = \beta u(\phi, j-1) + L(h_{j-1}), \quad i \leq j \leq \tau - d + 1, \tag{7.3.18}$$

$$u(\phi, j) = \beta u(\phi, j-1), \quad \tau - d + 1 < j \leq \tau. \tag{7.3.19}$$

(a) Immediate from the above fact of $L(h_j) > 0$ for $i \leq j \leq \tau - d$ and Lemma 3.2.4.

(b) First, let $i = \tau - 2d + 1$. Then since $i + d = \tau - d + 1$ and $i + d + 1 = \tau - d + 2 > \tau - d + 1$, from Eqs. (7.3.18) and (7.3.19) we obtain $u(\phi, i+d) = \beta u(\phi, i+d-1) + L(h_{i+d-1})$ and $u(\phi, i+d+1) = \beta u(\phi, i+d)$. Accordingly, we get

$$\begin{aligned} h_i - h_{i+1} &= u(\phi, i) - u(\phi, i+d) - u(\phi, i+1) + u(\phi, i+d+1) \\ &= \beta u(\phi, i-1) + L(h_{i-1}) - \beta u(\phi, i+d-1) - L(h_{i+d-1}) \\ &\qquad -\beta u(\phi, i) - L(h_i) + \beta u(\phi, i+d) \\ &= \beta h_{i-1} + L(h_{i-1}) - \beta h_i - L(h_i) - L(h_{i+d-1}) \\ &= M(h_{i-1}) - M(h_i) - L(h_{i+d-1}) < 0 \end{aligned}$$

due to the assumption of $h_{i-1} < h_i$ and Lemma 3.2.4. Next, let $\tau - 2d + 1 < i \leq \tau - d - 1$. Then $u(\phi, i+d) = \beta u(\phi, i+d-1)$ and $u(\phi, i+d+1) = \beta u(\phi, i+d)$ from Eqs. (7.3.18) and (7.3.19). Hence

$$h_i - h_{i+1} = u(\phi, i) - u(\phi, i+d) - u(\phi, i+1) + u(\phi, i+d+1) = M(h_{i-1}) - M(h_i) < 0.$$

(c) Let $1 \leq i \leq \tau - 2d$, so that $i \leq i + d \leq \tau - d \leq \tau - d + 1$ and $i \leq i + d + 1 \leq \tau - d + 1$. Hence $u(\phi, i+d) = \beta u(\phi, i+d-1) + L(h_{i+d-1})$ and $u(\phi, i+d+1) = \beta u(\phi, i+d) + L(h_{i+d})$ from Eq. (7.3.18).

(c1) From the above we have

$$\begin{aligned} h_i &= u(\phi,i) - u(\phi,i+d) \\ &= \beta u(\phi,i-1) + L(h_{i-1}) - \beta u(\phi,i+d-1) - L(h_{i+d-1}) \\ &= \beta h_{i-1} + L(h_{i-1}) - L(h_{i+d-1}) = M(h_{i-1}) - L(h_{i+d-1}) \\ &\le M(h_i) - L(h_{i+d-1}) = \beta h_i + L(h_i) - L(h_{i+d-1}) \end{aligned}$$

due to $M(h_{i-1}) \le M(h_i)$ from the assumption of $h_{i-1} < h_i$ and Lemma 3.2.4. Accordingly, $(1-\beta)h_i \le L(h_i) - L(h_{i+d-1})$. Since $h_i > h_{i-1} \ge 0$ due to Lemma 7.3.7(b), we have $(1-\beta)h_i > 0$, hence $L(h_i) > L(h_{i+d-1})$, implying $h_i < h_{i+d-1}$ due to Lemma 3.2.3(a).

(c2) Let $h_i \ge h_{i+1}$. Then we have

$$\begin{aligned} 0 \le h_i - h_{i+1} &= u(\phi,i) - u(\phi,i+d) - u(\phi,i+1) + u(\phi,i+d+1) \\ &= \beta u(\phi,i-1) + L(h_{i-1}) - \beta u(\phi,i+d-1) - L(h_{i+d-1}) \\ &\quad -\beta u(\phi,i) - L(h_i) + \beta u(\phi,i+d) + L(h_{i+d}) \\ &= \beta h_{i-1} + L(h_{i-1}) - \beta h_i - L(h_i) - L(h_{i+d-1}) + L(h_{i+d}) \\ &= M(h_{i-1}) - M(h_i) - L(h_{i+d-1}) + L(h_{i+d}), \end{aligned} \tag{7.3.20}$$

hence $L(h_{i+d}) - L(h_{i+d-1}) \ge M(h_i) - M(h_{i-1})$. Now $M(h_i) - M(h_{i-1}) \ge 0$ from the assumption of $h_{i-1} < h_i$ and Lemma 3.2.4, hence $L(h_{i+d}) \ge L(h_{i+d-1})$, implying $h_{i+d} \le h_{i+d-1}$ due to Lemma 3.2.4 and the fact $h_j < b$ for $i \le j \le \tau - d$. ▮

Lemma 7.3.11 *Let $h_{i-1} < h_i$ for a given i such that $1 \le i \le \tau - d - 1$. Then $h_{i-1} < h_i < \cdots < h_{\tau-d} < b$.*

Proof.

(1) Let $\tau - 2d + 1 \le i \le \tau - d - 1$. Then the assertion is immediately proven from Lemma 7.3.10(a,b). Accordingly, if $h_{\tau-2d} < h_{\tau-2d+1}$, then $h_{\tau-2d} < h_{\tau-2d+1} < h_{\tau-2d+2} < \cdots < h_{\tau-d} < b$.

(2) Let $h_{\tau-2d-1} < h_{\tau-2d}$. Then $h_{\tau-2d} < h_{\tau-d-1} \cdots (a^*)$ from Lemma 7.3.10(c1). Suppose $h_{\tau-2d} \ge h_{\tau-2d+1}$. Then $h_{\tau-d-1} \ge h_{\tau-d}$ from Lemma 7.3.10(c2). In this case, however, there must exist at least one j such that $h_j < h_{j+1}$ for $\tau - 2d + 1 \le j < j + 1 \le \tau - d - 1$ because if not so, we have $h_{\tau-2d} \ge h_{\tau-2d+1} \ge \cdots \ge h_{\tau-d-1}$, which contradicts (a^*). Therefore, $h_j < h_{j+1} < \cdots < h_{\tau-d-1} < h_{\tau-d}$ from (1), which is a contradiction. Hence it must be $h_{\tau-2d} < h_{\tau-2d+1}$, thus if $h_{\tau-2d-1} < h_{\tau-2d}$, then $h_{\tau-2d-1} < h_{\tau-2d} < h_{\tau-2d+1} < \cdots < h_{\tau-d} < b$.

(3) Assume that if $h_1 < h_2$, then $h_1 < h_2 < \cdots < h_{\tau-d} < b$. Let $h_0 < h_1$. Then $h_1 < h_d \cdots (b^*)$ from Lemma 7.3.10(c1). Suppose $h_1 \ge h_2$. Then $h_d \ge h_{d+1}$ from Lemma 7.3.10(c2). In this case, however, there must exist at least one j such that $h_j < h_{j+1}$ for $2 \le j < j+1 \le d$ because if not so, we have $h_1 \ge h_2 \ge \cdots \ge h_d$, which contradicts (b^*). Accordingly, $h_j < h_{j+1} < \cdots < h_d < h_{d+1}$ from the assumption, which is a contradiction. Hence it must be $h_1 < h_2$, thus if $h_0 < h_1$, then $h_0 < h_1 < h_2 < \cdots < h_{\tau-d} < b$. ▮

Corollary 7.3.1 *Let $h_{i-1} \leq h_i$ for a given i such that $1 \leq i \leq \tau - d - 1$. Then $h_{i-1} \leq h_i \leq \cdots \leq h_{\tau-d} < b$.*

Proof. Letting $h_{i-1} \leq h_i$ for a given i such that $1 \leq i \leq \tau - d - 1$, we can prove the assertion in the same way as in the proof of Lemmas 7.3.11. ∎

Lemma 7.3.12 *Let $r = 0$.*

(a) $L(h_0) > 0$.

(b) $h_0 \leq h_1$.

Proof. (a) Since both Eqs. (6.2.2) and (7.2.2) are the identical equation, we can prove the assertion in the same way as in the proofs of Lemma 6.3.12(a).

(b) Since $u(\phi, 0) = u(\phi, 1)$ for $r = 0$ from Eq. (7.2.10), we have $h_0 - h_1 = -\big(u(\phi, d) - u(\phi, d+1)\big) \leq 0$ due to Lemma 7.3.7(a), hence $h_0 \leq h_1$. ∎

Theorem 7.3.1

(a) *Let $\alpha \leq 0$. Then $\langle \mathrm{K} \rangle_{0 \leq i \leq \tau - d + 1}$.*

(b) *Let $\alpha > 0$.*

1 *If $\langle \mathrm{C} \rangle_i$ for a given i such that $0 \leq i \leq \tau - d + 1$, then $\langle \mathrm{C} \rangle_j$ for $i \leq j \leq \tau - d + 1$.*

2 *If $h_{i-1} < h_i$ for a given i such that $1 \leq i \leq \tau - d$, then $h_{i-1} < h_i < \cdots < h_{\tau-d} < b$.*

3 *Let $r = 0$.*

i $\langle \mathrm{C} \rangle_{0 \leq i \leq \tau - d + 1}$.

ii *h_i is nondecreasing in i for $0 \leq i \leq \tau - d$.*

Proof. (a) Evident from Lemma 7.3.5.

(b1,b2) Evident from, respectively, Lemmas 7.3.11 and Lemmas 7.3.9(b).

(b3i) Immediate from Lemmas 7.3.12(a) and 7.3.9(b).

(b3ii) Obvious from Lemma 7.3.12(b) and Corollary 7.3.1. ∎

7.4 Optimal Decision Rule

I. Case of $\tau = d$. The theorem below prescribes the optimal decision rule for the case of $\tau = d$.

Theorem 7.4.1 *In both admission control and pricing control problems, we have:*

(a) $\langle \mathrm{K} \rangle_{2 \leq i \leq \tau}$.

(b) *Let $\alpha \leq 0$. Then $\langle \mathrm{K} \rangle_0$ and $\langle \mathrm{K} \rangle_1$.*

(c) *Let $\alpha > 0$.*

1 *If $r^* \leq r$, then $\langle \mathrm{K} \rangle_0$ and $\langle \mathrm{K} \rangle_1$.*

2 *If $r^* > r$, then $\langle \mathrm{C} \rangle_0$ and $\langle \mathrm{C} \rangle_1$.*

Proof. (a) When in state (ϕ, i) with $2 \le i \le \tau$, even if a customer appears, it cannot be accepted because if the order of the customer is accepted, it cannot be completed within τ periods; in other words, it is optimal to skip the search.

(b-c2) Evident from Lemmas 7.3.5, 7.3.6(d), and 7.3.6(e). ∎

II. **Case of $\tau > d$.** For explanatory convenience, the two assertions *Assertion SB* and *Assertion CP* defined in Section 6.4 are also introduced. Let us assume that the production starts with no back order, i.e., $i = 0$. If skipping the search is optimal, i.e., $\langle \mathtt{K} \rangle_0$, then no customer appears, hence there exist no back orders ($i = 0$) over the entire planning horizon. Accordingly, it eventually follows that Assertion SB holds. Consequently, the Optimal Decision Rule *7.2.1* can be restated as follows.

□ *Optimal Decision Rule 7.4.1*

(a) Let $\alpha \le 0$. Then $\langle \mathtt{K} \rangle_0$ (Theorem 7.3.1(a)), hence Assertion SB holds for the reason stated above.

(b) Let $\alpha > 0$.

1 If $\langle \mathtt{C} \rangle_i$ for a given i such that $0 \le i \le \tau - d + 1$, then $\langle \mathtt{C} \rangle_j$ for $i \le j \le \tau - d + 1$ (Theorem 7.3.1(b1)), and $\langle \mathtt{K} \rangle_j$ for $\tau - d + 2 \le i \le \tau$ from the same reason stated in the proof of Theorem 7.4.1(a).

2 Let $r = 0$. Then since $\langle \mathtt{C} \rangle_{0 \le i \le \tau - d + 1}$ (Theorem 7.3.1(b3i)), it is optimal to enact the search by paying a search cost s, implying that Assertion CP holds. If $i > \tau - d + 2$, then $\langle \mathtt{K} \rangle_i$.

3 In the admission control problem, when there exists i number of back orders at the present point in time and an order with value w appears, if $w > h_i$, then $\langle \mathtt{A}(w) \rangle_i$, or else $\langle \mathtt{R}(w) \rangle_i$. See Theorem 7.3.1(b2 for $r > 0$, b3ii for $r = 0$).

In the pricing control problem it should be noted that the monotonicity of h_i in i stated above is inherited to the optimal price z_i due to Lemma 3.2.1(d). Since $z_i = z(h_i)$, from Lemma 3.2.1(e) we see that $z_i = a$ if $h_i < x^\star$.

7.5 Numerical Examples

In this section, let us show numerical examples of the optimal decision rule clarified in Section 7.4. Let us define

$$\hat{r} = \min\{r \mid h_0(r) > h_1(r)\}, \qquad r_0 = \min\{r \mid L(h_0(r)) = 0\}.$$

In the admission control problem, let $F(w)$ be the uniform distribution on $[0.01, 1.01]$, i.e., $a = 0.01$ and $b = 1.01$, and let $\lambda = 0.95$, $\beta = 0.95$, $d = 2.00$, $\tau = 14.00$, and $s = 0.01$. Then the following results are obtained from the numerical experiments.

I. **Optimal selection criterion h_i.**

We obtain $\hat{r} \simeq 0.082$ and $r_0 \simeq 0.456$. Figure 7.5.1 depicts the relationships of h_i with the number of backorders i and the profit from a sideline r. The figure tells us that:

1. h_i is nondecreasing in r for all i.
2. If $r < 0.082$, then h_i is strictly increasing in $i \ge 0$.

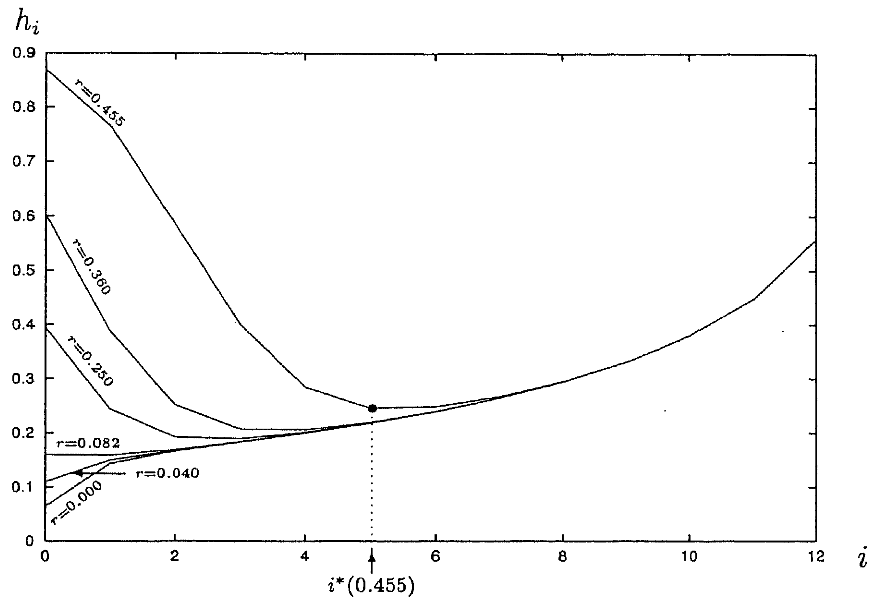

Figure 7.5.1: Graph of h_i where $\hat{r} = 0.082$ and $r_0 = 0.456$. Here, note that if $r < 0.082$, then h_i is strictly increasing in i and if $0.082 \le r < 0.456$, then h_i is unimodal in i.

3. If $r = 0.082$, then $h_0 \simeq h_1 \simeq 0.158\cdots$ and h_i is strictly increasing in $i \ge 1$.
4. If $0.082 < r < 0.456$, then h_i is unimodal, i.e., there exists a $i^*(r) \ge 1$ such that h_i is strictly decreasing in $i \le i^*(r)$ and strictly increasing in $i \ge i^*(r)$.
5. If i is sufficiently large, then h_i coincides with h_i with $r = 0.000$. This finding reflects the fact that the larger the number of backorders may become, the smaller the possibility of the backorder being exhausted may get; as a result, the effect of r on h_i is gradually diminished.

II. **Relationship of $\hat{r}$ and r_0 with related parameters λ, d, β, and s.**

1. Figure 7.5.2 illustrates the relationships of $\hat{r}$ and r_0 with the four related parameters λ, q, β, and s where the calculations are made by setting one of the four parameters as a variable with all the others being fixed. Here, it is to be noted that each of the coordinates planes of the four graphs is divided into the three regions:

$$\mathcal{R}\langle\mathrm{K}\rangle \quad \text{for } r_0 \le r, \qquad \breve{\mathcal{R}}\langle\mathrm{C}\rangle \quad \text{for } \hat{r} \le r < r_0, \qquad \acute{\mathcal{R}}\langle\mathrm{C}\rangle \quad \text{for } r < \hat{r}.$$

In the region $\mathcal{R}\langle\mathrm{K}\rangle$, not conducting the search, i.e., skipping the search is always optimal, and in both regions $\breve{\mathcal{R}}\langle\mathrm{C}\rangle$ and $\acute{\mathcal{R}}\langle\mathrm{C}\rangle$, conducting the search is always optimal where h_i is unimodal in i on $\breve{\mathcal{R}}\langle\mathrm{C}\rangle$ and nondecreasing in i on $\acute{\mathcal{R}}\langle\mathrm{C}\rangle$ (see Figure 7.5.2).

2. From Figure 7.5.2 it can be seen that:
   i. $\hat{r}$ is nonincreasing in s and nondecreasing in λ and β.
   ii. r_0 is nonincreasing in s and nondecreasing in λ and d.
   iii. $\hat{r}$ and r_0 are unimodal in, respectively, d and β. That $\hat{r}$ is unimodal in d implies that for a certain given r there exists d' and d'' with $d' < d''$ such that if $d \le d'$, then $(d, r) \in \breve{\mathcal{R}}\langle\mathrm{C}\rangle$, if $d' < d \le d''$, then $(d, r) \in \acute{\mathcal{R}}\langle\mathrm{C}\rangle$, and if $d'' \le d$, then *again* $(d, r) \in \breve{\mathcal{R}}\langle\mathrm{C}\rangle$; in other words, there exists two critical values of d such that the shape of h_i changes from "unimodal" to "nondecreasing" at $d = d'$ and from "nondecreasing" to "unimodal" at $d = d''$.

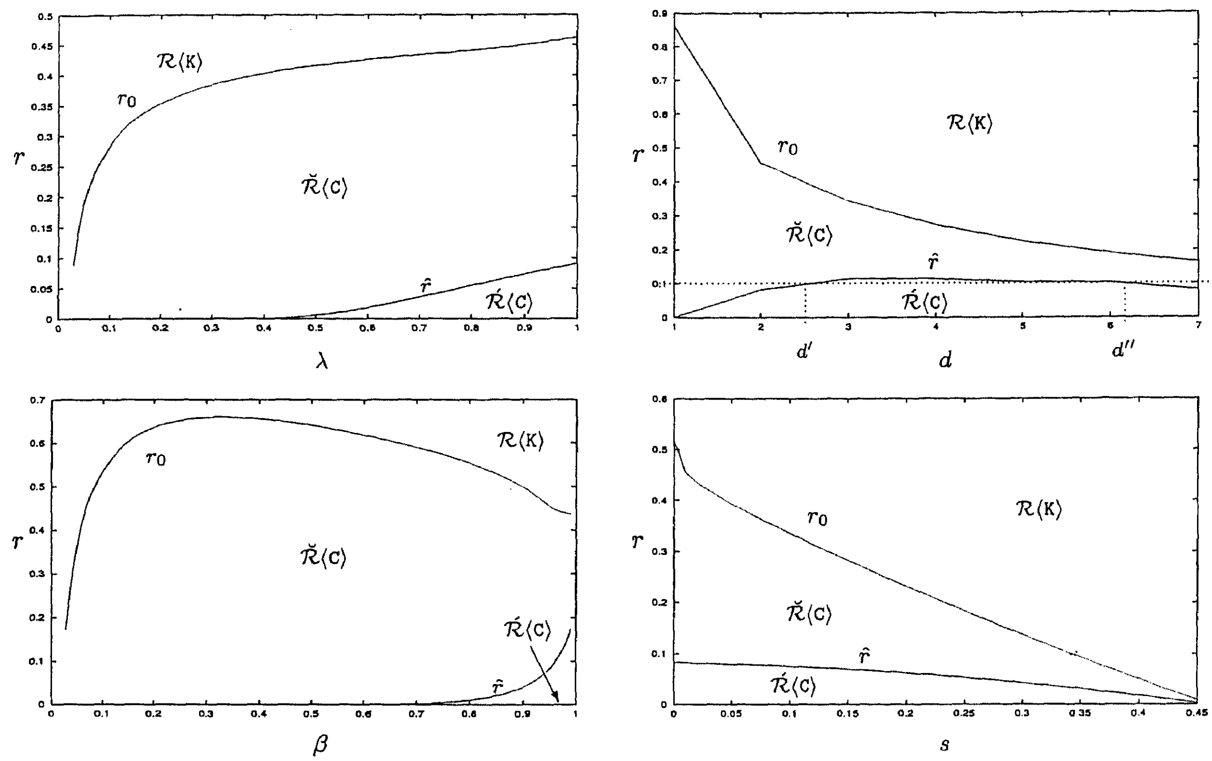

Figure 7.5.2: Relationships of $\hat{r}$ and r_0 with related parameters λ, d, β, and s.

III. **Optimal decision rules on continuing or skipping the search.**

Table 7.5.1 illustrates the optimal decision rules on continuing or skipping the search in each state (ϕ, i) with $0 \le i \le \tau - d + 1 = 13$ where $r > 0$ ($r = 0.050$, 0.350, 0.600).

1. If $r = 0.050 < 0.082 = \hat{r}$ and $r = 0.350 < 0.456 = r_0$, then $\langle C\rangle_{0 \le i \le 13}$, i.e., it is optimal to enact the search by paying a search cost s, implying that Assertion CP holds.
2. If $r_0 = 0.456 \le 0.600 = r$, the following two points can be said:
   i. $\langle K\rangle_0$, implying that Assertion SB holds.
   ii. There exists $i^* = 5$ such that $\langle K\rangle_{0 \le i < i^*}$ and $\langle C\rangle_{i^* \le i \le \tau - d + 1 = 13}$ (Theorem 7.3.1(b1)).

Table 7.5.1: Optimal decision rules on continuing or skipping the search.

r \ i	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0.050	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩
0.350	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩
0.600	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩

In the pricing control problem, we made some numerical experiments where $F(w)$ is the uniform distribution on $[2, 3]$ and $\lambda = 0.75$, $\beta = 0.99$, $d = 2.00$, $\tau = 14.00$, and $s = 0.01$. The graphs and table obtained are almost the same as those of the admission control problem excepting the following points. There exists i such that $h_i < x^\star = 2a - b = 1$. Since $z_i = z(h_i) = a$ for $h_i < x^\star = 1$ due to Lemma 3.2.1(e), it follows that $z_i = z(h_i) = 2$ for such i; in other words, $z_i = z(h_i)$ is truncated by a, the low bound of the distribution function $F(w)$. Further, it should be noted that there exists $h_i < a$ such that its corresponding optimal ordering price z_i becomes greater than a, i.e., $z_i = z(h_i) > a$.

7.6 Conclusions and Considerations

I. **Case of $\tau = d$**

C1. Let $\alpha \leq 0$ or equivalently $\lambda\beta T(0) \leq s$, implying that the search cost s is sufficiently large to be greater than or equal to $\lambda\beta T(0)$. Then not conducting the search, or skipping the search becomes always optimal, i.e., $\langle \mathrm{K} \rangle_{0 \leq i \leq \tau}$.

C2. Let $\alpha > 0$ or equivalently $\lambda\beta T(0) > s$, implying that the search cost s is sufficiently small to be smaller than $\lambda\beta T(0)$, including $s = 0$. Then it can be conjectured that conducting the search is always optimal; Is it always the case? Unfortunately the answer is negative as being stated below.

1. Let $r^* \leq r$. Then, even though the search cost is sufficiently small, if the profit from a sideline is sufficiently large to be greater than or equal to r^*, it becomes optimal to skip the search in order to enjoy the profit from a sideline, i.e., $\langle \mathrm{K} \rangle_{0 \leq i \leq \tau}$. Accordingly, the above conjecture is false.
2. Let $r < r^*$, that is both the search cost s and profit from a sideline r are sufficiently small. In the case, it is optimal to conduct the search in state $(\phi, 0)$ and $(\phi, 1)$, i.e., $\langle \mathrm{C} \rangle_{0,1}$, and it is optimal to skip the search in state (ϕ, i) with $2 \leq i \leq \tau$, i.e., $\langle \mathrm{K} \rangle_{2 \leq i \leq \tau}$.

II. **Case of $\tau > d$**

The practical implications of the optimal decision rules described in Section 7.4 are almost the same as C1 and C2 of Section 6.6. However, although the existence of $\hat{r}$ and r_0 was successfully proven in Model III which is a stochastic model, in Model V of this chapter which is a deterministic model it is difficult to theoretically prove their existence.

Chapter 8

Model V: Stochastic model with multiple production lines

In all models so far it has been implicitly assumed that the company holds only one production line. In this chapter we assume that the company holds multiple production lines.

Contents

8.1 System of Optimal Equations

In this model, by $n \geq 2$ let us denote the number of production lines available in the company where $n \leq N$. In the derivation of the system of the optimal equations of this model, the following three points should be noted:

1. In both admission control problem and pricing control problem, by $u(\phi, i)$ we shall denote the maximum total expected present discounted net profits starting from a state of having the fictitious customer ϕ and $i\,(0 \leq i \leq N)$ orders in the company; let us refer to such a situation as state (ϕ, i). If $i \leq n$, then $n - i$ production lines will be idle, which implies that the profit from a sideline $(n-i)r$ is yielded. When in state (ϕ, N), even if a customer appears, the order cannot be accepted due to the assumption of $i \leq N$; accordingly, the present state (ϕ, N) remains unchanged at the next point in time if no order in the company is completed with probability $1 - q$.

2. In the admission control problem, by $u(w, i)$ let us denote the maximum total expected present discounted net profits starting with $i\,(0 \leq i < N)$ orders in the company and an arriving customer, who offers a price w.

3. In the pricing control problem, by $u(1, i)$ let us denote the maximum total expected present discounted net profits starting with $i\,(0 \leq i < N)$ orders in the company and an arriving customer, to whom the company proposes a price z for an order.

From the same reason as mentioned in Section 6.1 the maximum total expected present discounted net profits $u(\phi, i)$, $u(w, i)$, and $u(1, i)$ are bounded in i. Now, for convenience in the later discussions, let us define

$$h_i = u(\phi, i) - u(\phi, i+1), \quad 0 \le i < N. \tag{8.1.1}$$

Then the system of optimal equations can be described as follows:

1. **Admission control problem**:

$$u(\phi, 0) = \max \left\{ \begin{array}{l} \mathtt{C} : \beta\big(\lambda\, \mathbf{E}[u(\xi, 0)] + (1-\lambda) u(\phi, 0)\big) - s + nr, \\ \mathtt{K} : \beta u(\phi, 0) + nr \end{array} \right\}, \tag{8.1.2}$$

$$u(\phi, i) = \max \left\{ \begin{array}{l} \mathtt{C} : (1-q)\beta\big(\lambda\, \mathbf{E}[u(\xi, i)] + (1-\lambda) u(\phi, i)\big) \\ \qquad\qquad + q\beta\big(\lambda\, \mathbf{E}[u(\xi, i-1)] + (1-\lambda) u(\phi, i-1)\big) - s, \\ \mathtt{K} : (1-q)\beta u(\phi, i) + q\beta u(\phi, i-1) \end{array} \right\} \tag{8.1.3}$$

$$+ (n-i) r I(i \le n)^{\ddagger}, \quad 1 \le i < N,$$

$$u(\phi, N) = \max \left\{ \begin{array}{l} \mathtt{C} : (1-q)\beta u(\phi, N) + q\beta\big(\lambda\, \mathbf{E}[u(\xi, N-1)] + (1-\lambda) u(\phi, N-1)\big) - s, \\ \mathtt{K} : (1-q)\beta u(\phi, N) + q\beta u(\phi, N-1) \end{array} \right\}, \tag{8.1.4}$$

$$u(w, i) = \max \left\{ \begin{array}{l} \mathtt{A} : w + u(\phi, i+1), \\ \mathtt{R} : u(\phi, i) \end{array} \right\}, \tag{8.1.5}$$

$$= \max\{w - h_i, 0\} + u(\phi, i), \quad 0 \le i < N. \quad \square \tag{8.1.6}$$

2. **Pricing control problem**:

$$u(\phi, 0) = \max \left\{ \begin{array}{l} \mathtt{C} : \beta\big(\lambda u(1, 0) + (1-\lambda) u(\phi, 0)\big) - s + nr, \\ \mathtt{K} : \beta u(\phi, 0) + nr \end{array} \right\}, \tag{8.1.7}$$

$$u(\phi, i) = \max \left\{ \begin{array}{l} \mathtt{C} : (1-q)\beta(\lambda u(1, i) + (1-\lambda) u(\phi, i)) \\ \qquad\qquad + q\beta\big(\lambda u(1, i-1) + (1-\lambda) u(\phi, i-1)\big) - s, \\ \mathtt{K} : (1-q)\beta u(\phi, i) + q\beta u(\phi, i-1) \end{array} \right\} \tag{8.1.8}$$

$$+ (n-i) r I(i \le n), \quad 1 \le i < N,$$

$$u(\phi, N) = \max \left\{ \begin{array}{l} \mathtt{C} : (1-q)\beta u(\phi, N) + q\beta\big(\lambda u(1, N-1) + (1-\lambda) u(\phi, N-1)\big) - s, \\ \mathtt{K} : (1-q)\beta u(\phi, N) + q\beta u(\phi, N-1) \end{array} \right\}, \tag{8.1.9}$$

$$u(1, i) = \max_z \{p(z)\big(z + u(\phi, i+1)\big) + (1 - p(z)) u(\phi, i)\} \tag{8.1.10}$$

$$= \max_z p(z)(z - h_i) + u(\phi, i), \quad 0 \le i < N. \quad \square \tag{8.1.11}$$

‡ $I(\cdot)$ denotes the indicator function. For the given statement S if S is true, then $I(\mathrm{S}) = 1$, or else $I(\mathrm{S}) = 0$.

The unique existence of the solution of the above equations can be proven in the same way as in Lemma 6.3.1.

8.2 Transformation

Let us define

$$v(i) = \left\{ \begin{array}{ll} \mathbf{E}[u(w,i)] & \text{for the admission control problem} \\ u(1,i) & \text{for the pricing control problem} \end{array} \right\}, \quad 0 \le i < N. \tag{8.2.1}$$

Then using Eq. (3.1.1), we can immediately rearrange both Eqs. (8.1.2) to (8.1.5) and Eqs. (8.1.7) to (8.1.10) into the identical expressions below.

$$u(\phi,0) = \max\{\lambda\beta v(0) + (1-\lambda)\beta u(\phi,0) - s, \beta u(\phi,0)\} + nr, \tag{8.2.2}$$

$$u(\phi,i) = \max \left\{ \begin{array}{l} (1-q)\beta(\lambda v(i) + (1-\lambda)u(\phi,i)) \\ \qquad + q\beta\big(\lambda v(i-1) + (1-\lambda)u(\phi,i-1)\big) - s, \\ (1-q)\beta u(\phi,i) + q\beta u(\phi,i-1) \end{array} \right\} \tag{8.2.3}$$

$$+(n-i)rI(i \le n), \quad 1 \le i < N, \tag{8.2.4}$$

$$u(\phi,N) = \max \left\{ \begin{array}{l} (1-q)\beta u(\phi,N) + q\beta\big(\lambda v(N-1) + (1-\lambda)u(\phi,N-1)\big) - s, \\ (1-q)\beta u(\phi,N) + q\beta u(\phi,N-1), \end{array} \right\}, \tag{8.2.5}$$

$$v(i) = T(h_i) + u(\phi,i) \quad \text{or equivalently} \quad T(h_i) = v(i) - u(\phi,i), \quad 0 \le i < N. \tag{8.2.6}$$

Further, Eqs. (8.2.2) to (8.2.5) can be rewritten as, respectively,

$$u(\phi,0) = \beta u(\phi,0) + \max\{\lambda\beta\big(v(0) - u(\phi,0)\big) - s, 0\} + nr, \tag{8.2.7}$$

$$u(\phi,i) = (1-q)\beta u(\phi,i) + q\beta u(\phi,i-1) + (n-i)rI(i \le n)$$

$$+ \max\{\lambda(1-q)\beta(v(i) - u(\phi,i)) + \lambda q\beta\big(v(i-1) - u(\phi,i-1)\big) - s, 0\}, \ 1 \le i < N, \tag{8.2.8}$$

$$u(\phi,N) = (1-q)\beta u(\phi,N) + q\beta u(\phi,N-1) + \max\{\lambda q\beta\big(v(N-1) - u(\phi,N-1)\big) - s, 0\}, \tag{8.2.9}$$

which can be immediately rearranged into

$$u(\phi,0) = \big(\max\{\lambda\beta\big(v(0) - u(\phi,0)\big) - s, 0\} + nr\big)/(1-\beta), \tag{8.2.10}$$

$$u(\phi,i) = \gamma q\beta u(\phi,i-1) + \gamma(n-i)rI(i \le n)$$

$$+\gamma \max\{\lambda(1-q)\beta\big(v(i) - u(\phi,i)\big) + \lambda q\beta\big(v(i-1) - u(\phi,i-1)\big) - s, 0\}, \ 1 \le i < N, \tag{8.2.11}$$

$$u(\phi,N) = \gamma q\beta u(\phi,N-1) + \gamma \max\{\lambda q\beta\big(v(N-1) - u(\phi,N-1)\big) - s, 0\} \tag{8.2.12}$$

where γ is defined by Eq. (2.4.2). Hence, using Eq. (8.2.6), we can rewrite Eqs. (8.2.10) to (8.2.12) as follows.

$$\begin{aligned} u(\phi,0) &= (\max\{\lambda\beta T(h_0)-s,0\}+nr)/(1-\beta), \qquad (8.2.13)\\ u(\phi,i) &= \gamma q\beta u(\phi,i-1)+\gamma(n-i)rI(i\le n)\\ &\quad +\gamma\max\{\lambda(1-q)\beta T(h_i)+\lambda q\beta T(h_{i-1})-s,0\}, \quad 1\le i<N, \qquad (8.2.14)\\ u(\phi,N) &= \gamma q\beta u(\phi,N-1)+\gamma\max\{\lambda q\beta T(h_{N-1})-s,0\}. \qquad (8.2.15)\end{aligned}$$

Further, using the *L- function* defined by Eq. (3.1.2), we can rewrite Eqs. (8.2.13) to (8.2.15) as follows.

$$\begin{aligned} u(\phi,0) &= (\max\{L(h_0),0\}+nr)/(1-\beta), \qquad (8.2.16)\\ u(\phi,i) &= \gamma q\beta u(\phi,i-1)+\gamma\max\{(1-q)L(h_i)+qL(h_{i-1}),0\}+\gamma(n-i)rI(i\le n),\ 1\le i<N, \qquad (8.2.17)\\ u(\phi,N) &= \gamma q\beta u(\phi,N-1)+\gamma\max\{qL(h_{N-1})-(1-q)s,0\}. \qquad (8.2.18)\end{aligned}$$

Below, for convenience let

$$\begin{aligned} Q_0 &= L(h_0), \qquad (8.2.19)\\ Q_i &= (1-q)L(h_i)+qL(h_{i-1}), \quad 1\le i<N, \qquad (8.2.20)\\ Q_N &= qL(h_{N-1})-(1-q)s. \qquad (8.2.21)\end{aligned}$$

Then Eqs. (8.2.16) to (8.2.18) can be rewritten as follows.

$$\begin{aligned} u(\phi,0) &= (\max\{Q_0,0\}+nr)/(1-\beta), \qquad (8.2.22)\\ u(\phi,i) &= \gamma q\beta u(\phi,i-1)+\gamma\max\{Q_i,0\}+\gamma(n-i)rI(i\le n), \quad 1\le i\le N. \qquad (8.2.23)\end{aligned}$$

Regarding h_i as a function of r, let us represent h_i and Q_i by, respectively, $h_i(r)$ and $Q_i(r)$, i.e.,

$$\begin{aligned} Q_0(r) &= L(h_0(r)), \qquad (8.2.24)\\ Q_i(r) &= (1-q)L(h_i(r))+qL(h_{i-1}(r)), \quad 1\le i<N, \qquad (8.2.25)\\ Q_N(r) &= qL(h_{n-1}(r))-(1-q)s. \qquad (8.2.26)\end{aligned}$$

Here, by r_i let us denote the smallest solution of $Q_i(r)=0$, if it exists, i.e.,

$$r_i = \min\{r \mid Q_i(r)=0\}. \qquad (8.2.27)$$

From all the above it can be easily seen that the optimal decision rules for any given i can be prescribed as follows.

□ *Optimal Decision Rule 8.2.1*

1. Admission control problem:
   i. Let $0 \le i \le N$. If $Q_i > 0$, then $\langle \texttt{C} \rangle_i$, or else $\langle \texttt{K} \rangle_i$.
   ii. Let $0 \le i < N$ and an order with value w appear after the search was enacted. If $w > h_i$, then $\langle \texttt{A(w)} \rangle_i$, or else $\langle \texttt{R(w)} \rangle_i$.
2. Pricing control problem:
   i. Let $0 \le i \le N$. If $Q_i > 0$, then $\langle \texttt{C} \rangle_i$, or else $\langle \texttt{K} \rangle_i$.
   ii. Let $0 \le i < N$ and a customer appear after the search was enacted. Then $\langle \texttt{O}(z_i) \rangle$ where $z_i = z(h_i)$.

8.3 Analysis

Lemma 8.3.1

(a) *$u(\phi, i)$ and $v(i)$ are nonincreasing in i where $u(\phi, i) \ge 0$ for $0 \le i \le N$.*

(b) *$h_i \ge 0$ for $0 \le i < N$.*

Proof. Proven in the same way as in the proofs of Lemma 6.3.2. ∎

Lemma 8.3.2 *Let $\alpha \le 0$. Then $Q_i \le 0$ for $0 \le i \le N$.*

Proof. Noting Lemmas 6.3.2(b) and 3.2.3(c), in the same way as in the proof of Lemma 6.3.3 we can prove the assertion. ∎

Lemma 8.3.3 *For a given i such that $n \le i < N$ we have:*

(a) *If $Q_i \le 0$, then $h_{i-1} > h_i$, hence $h_{i-1} \ge h_i$.*

(b) *If $h_{i-1} < h_i$, then $h_{i-1} < h_i < \cdots < h_{n-1} < b$ and $Q_j > 0$ for j with $i \le j < N$.*

(c) *If $Q_i > 0$, then $Q_j > 0$ for $i \le j < N$.*

Proof. Proven in the same as in the proofs of Lemmas 6.3.4(b), 6.3.5, and 6.3.7. ∎

Corollary 8.3.1 *If $h_{i-1} \le h_i$, then $h_{i-1} \le h_i \le \cdots \le h_{M-1} < b$ and $Q_j > 0$ for j with $i \le j < N$.*

Proof. Proven in the same way as in the proof of Lemma 8.3.3(b). ∎

Lemma 8.3.4

(a) *$h_i(r)$ is nondecreasing in r for $i \ge 0$.*

(b) *$\lim_{r\to\infty} h_i(r) = \infty$ and $\lim_{r\to-\infty} h_i(r) = -\infty$ for $i \ge 0$.*

(c) *$Q_i(r)$ is nonincreasing in r for all $i \ge 0$.*

(d) *For $0 \le i \le n$ we have:*
   1 *There exists $r_i > 0$.*
   2 *If $r < (\ge)\ r_i$, then $Q_i(r) > (\le)\ 0$.*

Proof. Proven in the same way as in the proofs of Lemmas 6.3.8, 6.3.10, and 6.3.11. ∎

Lemma 8.3.5

(a) *Let $r = 0$.*

1 $Q_0(r) > 0$.

2 *If* $h_0 = 0$, *then* h_i *is nondecreasing in* i.

3 *If* $h_0 > 0$, *then* h_i *is strictly increasing in* i.

(b) *If* $r_n \leq r$, *then* $h_{n-1} > h_n$.

Proof. (a) Proven in the same way as in the proofs of Lemmas 6.3.12(a).

(b) Let $r_n \leq r$. Then from Lemma 8.3.4(d2) we have $Q_n(r) \leq 0$, hence $h_{n-1} \gtrsim h_n$ due to Lemma 8.3.3(a). ▮

Let us define

$$\hat{r} = \min\{r \mid h_{n-1}(r) > h_n(r)\}. \tag{8.3.1}$$

Lemma 8.3.6 *We have* $r_n \geq \hat{r} > 0$ *where if* $r \geq (<)\ \hat{r}$, *then* $h_{n-1} > (\leq)\ h_n$.

Proof. From Lemma 8.3.5 we have $h_{n-1} \leq h_n$ for $r = 0$ and $h_{n-1} > h_n$ for $r \geq r_n$, implying that there exists a positive $\hat{r} \leq r_n$ such as $h_{n-1}(r) > h_n(r)$. Accordingly, the latter half of the assertion is clearly true. ▮

8.4 Optimal Decision Rule

The following theorem restates the Optimal Decision Rule *8.2.1*.

Theorem 8.4.1

(a) *Let* $\alpha \leq 0$. *Then* $\langle \mathrm{K} \rangle_{0 \leq i \leq N}$.

(b) *Let* $\alpha > 0$.

1 *Let* $r_n \leq r$. *Then* $\langle \mathrm{K} \rangle_{n \leq i < N}$ *or there exists* $i^*(n < i^* < N)$ *such that* $\langle \mathrm{K} \rangle_{n \leq i < i^*}$ *and* $\langle \mathrm{C} \rangle_{i^* \leq i < N}$.

2 *Let* $r < r_n$.

i $\langle \mathrm{C} \rangle_{n \leq i < N}$.

ii *Let* $\hat{r} \leq r$. *Then* h_i *is not always nondecreasing in* $i \geq n$.

iii *Let* $r = 0$.

1 *If* $h_0 = 0$, *then* h_i *is nondecreasing in* i *with* $h_i < b$ *for* $0 \leq i < N$.

2 *If* $h_0 > 0$, *then* h_i *is strictly increasing in* i *with* $h_i < b$ *for* $0 \leq i < N$.

Proof. (a) Evident from Lemma 8.3.2.

(b) Let $\alpha > 0$. Here note that $\hat{r} \leq r_n$ from Lemma 8.3.6.

(b1) Let $r_n \leq r$. Clearly $Q_n(r) \leq 0$ from Lemma 8.3.4(d2) with $i = n$, hence $\langle \mathrm{K} \rangle_n$. From this result and the fact that once continuing the search is optimal for a certain i, i.e., $\langle \mathrm{C} \rangle_i$, then it also is so for all i' with $i \leq i' < N$ due to Lemma 8.3.3(c). Accordingly, the assertion clearly holds.

(b2) Let $r < r_n$.

(b2i) Then $Q_N(r) > 0$ from Lemma 8.3.4(d2 with $i = n$), hence $Q_i(r) > 0$ for $n \leq i < N$ from Lemma 8.3.3(c), thus $\langle \mathrm{C} \rangle_{n \leq i < N}$.

(b2ii) Let $\hat{r} \leq r$. Then since $h_{n-1} > h_n$ from Lemma 8.3.5(b), it follows that h_i is not always nondecreasing in i.

(b2iii) Let $r = 0$.

(b2iii1,b2iii2) Immediate from Lemmas 8.3.5(a). ∎

In the pricing control problem it should be noted that the monotonicity of h_i in i stated above is inherited to the optimal price z_i due to Lemma 3.2.1(d). Since $z_i = z(h_i)$, from Lemma 3.2.1(e) we see that $z_i = a$ if $h_i < x^\star$.

8.5 Numerical Examples

Let us examine the properties of the optimal decision rules through numerical experiments.

8.5.1 Admission Control Problem

Let $F(w)$ be the uniform distribution on $[0.01, 1.01]$ and let $\lambda = 0.95$, $q = 0.35$, $\beta = 0.99$, $s = 0.01$, and $N = 15$. In this case, $T(0) = 0.51$, hence $\alpha = \lambda\beta T(0) - s = 0.47 > 0$. Then for $\hat{r}$, r_n, and $h_{n-1} \simeq h_n$, n = 2,3,4,5, we obtain the results of numerical experiments shown in Table 8.5.1. Here note that it is only when $r = \hat{r}$ that $h_{n-1} \simeq h_n$ may occurs due to the definition of $\hat{r}$ given by Eq. (8.3.1).

Table 8.5.1: $\hat{r}$, r_n, and $h_{n-1} \simeq h_n$.

	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$\hat{r} \simeq$	0.019	0.007	0.005	0.005
$r_n \simeq$	0.121	0.064	0.041	0.030
$h_{n-1} \simeq h_n \simeq$	0.340	0.373	0.389	0.404

I. **Relationship among $\hat{r}$, r_0, and h_i.**

Figure 8.5.1 depicts the relationships of h_i with the number of back orders i and the profit from a sideline r. The figure tells us that:

1. h_i is nondecreasing in the profit from a sideline r for all i.
2. If $r < \hat{r}$, then h_i is strictly increasing in $i \geq 0$.
3. If $r = \hat{r}$, then $h_{n-1} \simeq h_n$ and h_i is strictly increasing in $i \geq n$.
4. If $\hat{r} \leq i < r_n$, then there exist i' and i'' such that h_i is strictly increasing in $i \leq i'$, strictly decreasing in $i' < i \leq i''$, and *again* strictly increasing in $i \geq i''$.
5. If i is sufficiently large, then h_i coincides with h_i for $r = 0.000$. This reflects the fact that the larger the number of back orders may become, the smaller the possibility of the back orders being exhausted may get; as a result, the effect of r on h_i is gradually diminished.

II. **The optimal decision rules on continuing or skipping the search.**

Table 6.5.1 represents the optimal decision rules on continuing the search or skipping the search in each state for each given r. Table 6.5.1 tells us that:

1. If $r < r_n$, then it is always optimal to continue the search as seen Theorem 8.4.1(b2i) except for the state $(\phi, 15)$. When $i = 15$, any of continuing the search and skipping may be optimal,

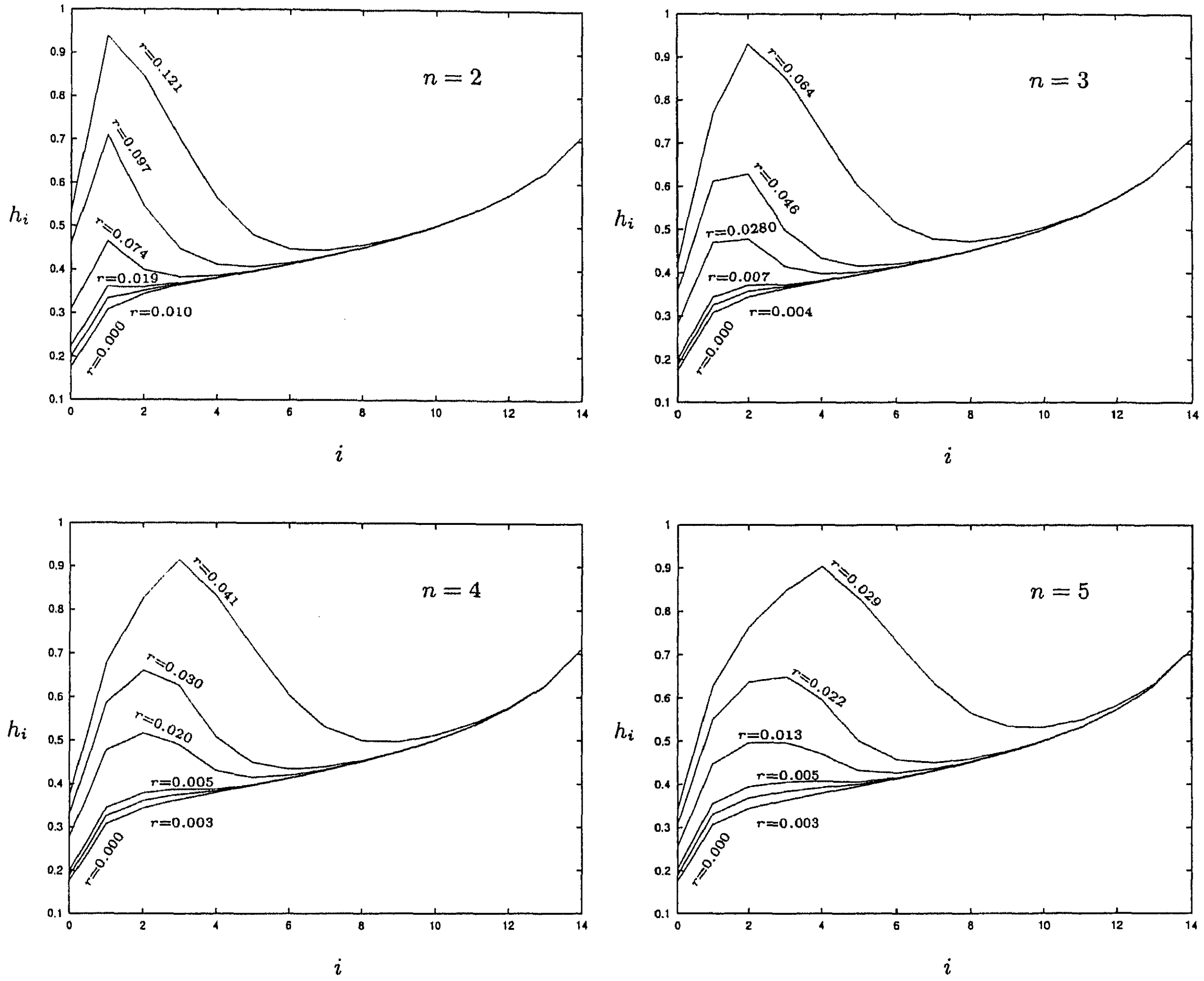

Figure 8.5.1: Graphs of the selection criterion h_i in the number of backorders i.

2. If $r \geq r_n$, implying that if the profit from a sideline r is sufficiently large, it can be seen that it is always optimal to skip the search (case of $n = 5$ and $r = 0.310$) or that there exists $i' < i''$ such that if $i < i'$, continuing the search is optimal, if $i' \leq i \leq i''$, skipping the search is optimal, and if $i'' < i$, *again* continuing the search is optimal; that is, there exist *double critical values* in terms of i. In case of $n = 2$ and $r = 0.150$ we have $i' = 2$ and $i'' = 7$.

8.5.2 Pricing Control Problem

Let $F(w)$ be the uniform distribution on $[2, 3]$, i.e., $a = 2$ and $b = 3$ and let $\lambda = 0.90$, $q = 0.55$, $\beta = 0.99$ and $s = 0.05$. In this case, we have $x^* = 2a - b = 1$. Since $x^* > 0$, we obtain $T(0) = a = 2$, hence $\alpha = \lambda\beta T(0) - s = 1.732 > 0$.

Figure 8.5.2 depicts the relationships of h_i and $z_i(= z(h_i))$ with the number of back orders i and the profit from a sideline r. The figures tell us that:

1. h_i is nondecreasing in the profit from a sideline r for all i.
2. If $r < \hat{r}$, then h_i is strictly increasing in $i \geq 0$.
3. If $r = \hat{r}$, then $h_{n-1} \simeq h_n$ and z_i is strictly increasing in $i \geq n$.

Table 8.5.2: Optimal decision rules on continuing or skipping the search.

n, r_n \ i		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$n = 2$, $r_n = 0.121$	$r = 0.000$	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩
	$r = 0.121$	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩
	$r = 0.136$	⟨C⟩	⟨C⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩
	$r = 0.150$	⟨C⟩	⟨C⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩
$n = 3$, $r_n = 0.064$	$r = 0.000$	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩
	$r = 0.063$	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩
	$r = 0.065$	⟨C⟩	⟨C⟩	⟨C⟩	⟨K⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩
	$r = 0.100$	⟨C⟩	⟨C⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨K⟩
$n = 4$, $r_n = 0.041$	$r = 0.000$	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩
	$r = 0.040$	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩
	$r = 0.071$	⟨C⟩	⟨C⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨C⟩	⟨C⟩	⟨K⟩
	$r = 0.100$	⟨C⟩	⟨C⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩
$n = 5$, $r_n = 0.030$	$r = 0.000$	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩
	$r = 0.029$	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩	⟨C⟩
	$r = 0.071$	⟨C⟩	⟨C⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩
	$r = 0.310$	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩	⟨K⟩

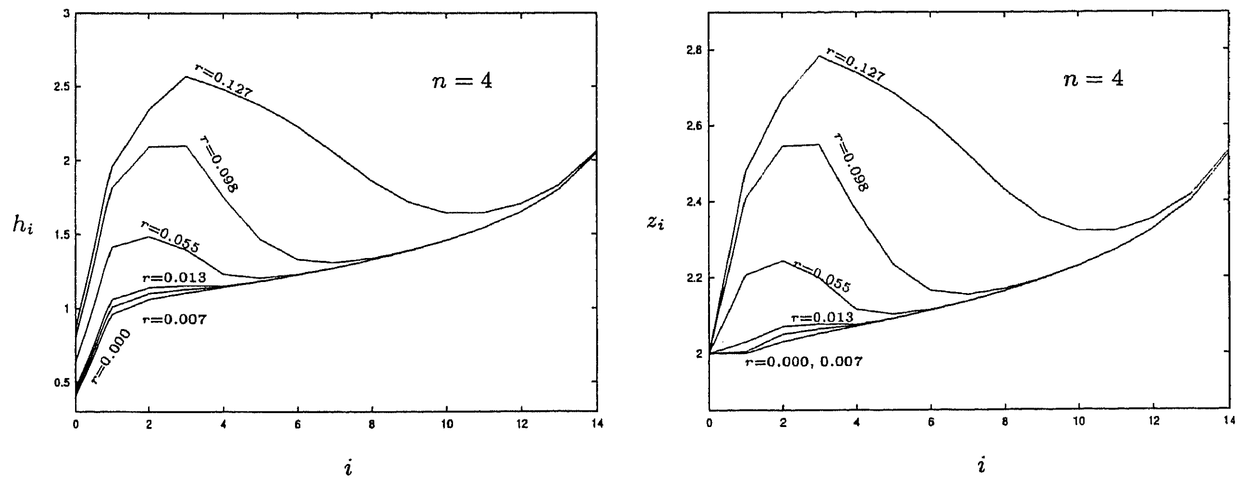

Figure 8.5.2: Graphs of h_i and z_i.

4. If $\hat{r} \leq i < r_n$, then there exist i' and i'' such that h_i and z_i is strictly increasing in $i \leq i'$, strictly decreasing in $i' \leq< i \leq i''$, and *again* strictly increasing in $i \geq i''$. We can notice that i' is given by $n - 1$.

5. The graph on the right shows the optimal ordering price z_i. Here note that there exists i such that $h_i < x^\star = 2a - b = 1$ in the graph of h_i. Since $z_i = z(h_i) = a$ for $h_i < x^\star = 1$ due to Lemma 3.2.1(e), it follows that $z_i = z(h_i)$ for such i becomes equal to $a = 2$; in other words, $z_i = z(h_i)$ is truncated by a, the low bound of the distribution function $F(w)$. Further, it should be noted that there exists $h_i < a$ such that its corresponding optimal ordering price z_i becomes greater than a, i.e., $z_i = z(h_i) > a$.

8.6 Conclusions and Considerations

To begin with, below let us state the two types of oscillations as to the number of back orders i.

1. On the range of i over which the optimal selection criterion h_i is increasing in i, the number of back orders i oscillates with an equilibrium point for the same reason as that stated in Section 6.6 ($r < \hat{r}$ of C2); let us refer to such behavior of i as the *stable oscillation.*

2. On the range of i over which the optimal selection criterion h_i is decreasing in i, the number of back orders i oscillates as follows: (1) The smaller the number of back orders may become, the higher the optimal selection criterion h_i becomes; as a result, the number of back orders is prompted to become further small and (2) The larger the number of back orders i may become, the lower the optimal selection criterion h_i gets; as a result, the number of back orders is prompted to become further large. This fact suggests that once the i enters this range, it behaves as if it is escaping from the region. Let us refer to such behavior as the *unstable oscillation.*

The optimal decision rules described in Section 8.4 are almost similar to those of Model III. However, the conclusions obtained from the model in this chapter, Model V, are different from those in Model III in the sense below.

First, it should be noted that there exist $\hat{r}$ and r_n with $\hat{r} < r_n$ $(2 \leq n \leq N)$, which provides thresholds implying that: (1) If the profit from a sideline r is less than $\hat{r}$, the optimal selection criterion h_i is increasing in the number of back orders i, or else it is bimodal in i and (2) If the profit from a sideline r is less than r_n, it is optimal to conduct the search for orders, or else it is not always optimal to enact the search. Below, let us explain the implications of the above two thresholds:

1. Let $r < \hat{r}$, i.e., the profit from a sideline is sufficiently small. Then the optimal selection criterion h_i is increasing in the number of back orders i. Hence, the behavior of the number of back orders shows the stable oscillation.

2. Let $\hat{r} \leq r < r_n$. In this case, as seen in Figure 8.5.1, there exist i' and i'' $(i' < i'')$ such that h_i is strictly increasing on $[0, i']$, strictly decreasing on $(i', i'']$, and *again* strictly increasing on $(i'', N]$. In other words, the optimal selection criterion h_i is bimodal in the number of back orders i over $[0, N]$. Below, let us state the implications of the bimodal property.

   i. Let $i \leq i'$. Then the optimal selection criterion h_i is increasing in the number of back orders i. This fact implies the following. If there are few orders in the system, all the production lines will become soon empty, and if they become empty, the company has to engage in the sideline with a relatively small amount of profit. This yields Opportunity loss II. Accordingly, in order to avoid this loss, the optimal selection criterion h_i should be set low to accept orders even though their profitabilities may not be so high. However, as the number of back orders i increases, all the production lines come to be filled with orders, leading to the possibility of obtaining an income from a sideline is small. This yields Opportunity loss I. Therefore, in order to avoid this loss and prevent all the production lines from being full with orders, the optimal selection criterion should be set high; as a result, the number of back orders i becomes small, hence the company can enjoy the profit from a sideline.

ii. Let $i' < i$. Then the optimal selection criterion h_i is unimodal in the number of back orders i over $(i', N]$; the managerial implication of this unimodality is the same as that stated in Section 6.6 ($\hat{r} \leq r < r_0$ of C2).

iii. The fact that the optimal selection criterion h_i is increasing on each of the two ranges, $[0, i']$ and $(i'', N]$, implies that there exists a stable point of oscillation on each of the two ranges. Once the number of back orders i enters the range $(i', i'']$, a dynamics starts operating to prompt the number of back orders i to move to one of the two ranges $[0, i']$ and $(i'', N]$ since the behavior of the number of back orders shows unstable oscillation. Here, it is to be noted that the stable points on $[0, i']$ and $(i'', N]$ are related to, respectively, the number of production lines to be fulled with orders and the number of back orders to be held in the system.

iv. The fact that h_i is a bimodal function of i suggests us the following. For an order with certain price w there exists $i' < i'' < i'''$ such that if $i \leq i'$, it is optimal to reject the order, if $i' < i \leq i''$, it is optimal to accept it, if $i'' < i \leq i'''$, *again* it is optimal to reject it, and $i''' < i$, *again* it is optimal to accept it. In other words, there exist the *triple critical values* in terms of i at which rejecting and accepting an order become indifferent.

3. Let $r_n \leq r$, i.e., the profit from a sideline is sufficiently large. Then the optimal selection criterion h_i may be monotone, unimodal, or bimodal in i. However, since no order appears on the range where it is optimal to skip the search (see Figure 8.5.2), the h_i has no practical meaning as a selection criterion. Now, as seen in Figure 6.6, there exist two critical values i' and i'' ($i' < i''$). The i' provides the number of production lines to be assigned to the custom production, so that the number of production lines to be assigned to the sideline will be given as $n - i'$. The i'' provides the number of back orders up to which skipping the search is optimal. Further, we see that:

   i. If $i \leq i'$, then i' production lines are all available for handling orders and it is optimal to conduct the search for orders until the number of back orders becomes i'.

   ii. If $i' < i < i''$, it is optimal to skip the search; as a result, the number of back orders decreases up to i'. Hence, it becomes possible to assign $n - i'$ production lines to the sideline, and thereby yields a profit from the sideline.

   iii. If $i'' \leq i$, the company shoul *again* conduct the search for orders to make profit.

The above stated considerations are related to the admission control problem. The same considerations as those stated above are also obtained for the pricing control problem.

Chapter 9

Overall Conclusions and Considerations

This chapter summarizes the results obtained throughout the thesis and then proposes some subjects of study that should be tackled in the future.

C1. First, the following three points should be emphasized.

1. The search cost has been introduced in almost all conventional models of optimal stopping problems. The introduction of the search cost eventually yields the option on whether or not to conduct the search. However, this new option has never been taken into consideration in the customer selection problems. In all the models of the thesis we clarified the existence of the optimal threshold as to the search cost such that if the search cost is less than the threshold, conduct the search, or else not.
2. In the conventional customer selection problems with vacation of the system, it has been assumed that the system is turned off as soon as it gets empty. In this thesis, however, the system is never turned off because the company starts to provide the subsidiary services as a sideline as soon as it becomes empty, and thereby generates profit from the sideline. This profit from a sideline is the most distinguished point of this thesis, which has never been introduced in the conventional customer selection problems. We have clarified that the decision on whether or not to conduct the search for orders is influenced by not only the search cost but also the profit from a sideline. In other words, we have proven that there exists a threshold as to the profit from a sideline at which whether or not to conduct the search become indifferent. Further, we showed that if the profit from a sideline is neither sufficiently small nor sufficiently large, the optimal selection criterion in the admission control problem and the optimal price in the pricing control problem become unimodal or bimodal in the number of back orders. The managerial implications of the unimodal and the bimodal properties were stated in Sections 6.6, 7.6, and 8.6. These properties can not be seen in any past papers.
3. The admission control problem and the pricing control problem have been separately discussed so far in the conventional customer selection problems. In this thesis, however, we clarified that both problems can be discussed in an identical framework and demonstrated that the properties of optimal decision rules obtained in admission control problem are inherited to the pricing control problem.

C2. In this thesis we have proposed five basic models for a customer selection problem with profit from a sideline and search cost. In order to make the models more practical, the following subjects should be necessarily investigated.

1. Penalty for delay of delivery is introduced only in Models I and II; it should be introduced in other models.
2. It was assumed that an order held in the system may be canceled in model II where only one order is allowed to be held. This assumption should be introduced in the models where multiple orders can be held .
3. Model IV should be also examined on the assumption that each order has a different appointed date of delivery.
4. Orders may be processed in a series of processes; in this case, a sequencing problem of orders to be processed arises.
5. Models where the probability of customer arrival is assumed to be dependent on the number of back orders should be also examined; simple examinations by numerical experiments are made for Model III.
6. Throughout the thesis we have assumed that any sideline can be completed within one period. However, there may exist a sideline which needs more than one period for its completion. In this case, the optimal decision rule may be naturally influenced by engaging in the sideline. Accordingly, a new decision problem arises on whether or not to be engaged in the sideline when all order are completed. This problem should be tackled in the future.
7. Thus far we have implicitly assumed that a customer once turned away cannot be solicited in the future. The future availability of once rejected customers, which is assumed in some models of conventional optimal stopping problems [13] [16] [40], should be also introduced in our model.

Appendix: Proofs of Lemmas

A.1. Lemma 4.2.1

For any given vector $\boldsymbol{x} = (x_\phi, x_0, x_1, \cdots)'$ let us define the norm $\|\boldsymbol{x}\| = \max\{|x_\phi|, |x_0|, |x_1|, \cdots\}$ where $\|\boldsymbol{x}\| \geq |x_i|$ for $i = \phi, 0, 1, \cdots$. Further, by $D_\phi u$ and $D_l u$ let us denote the right hand sides of Eqs. (4.2.4) and (4.2.5), and let $\boldsymbol{D}u = (D_\phi u, D_0 u, D_1 u, \cdots)'$ and $\boldsymbol{u} = (u(\phi), u(\phi, 0), u(\phi, 1), \cdots)'$. Then noting Eq. (4.2.3), from Eq. (4.1.4) we have

$$\begin{aligned} |v(0) - \hat{v}(0)| &\leq |\,\mathbf{E}[\max\{w + u(\phi, 0), u(\phi)\} - \max\{w + \hat{u}(\phi, 0), \hat{u}(\phi)\}]| \\ &\leq \max\{|u(\phi) - \hat{u}(\phi)|, |u(\phi, 0) - \hat{u}(\phi, 0)|\} \leq \|\boldsymbol{u} - \hat{\boldsymbol{u}}\|, \end{aligned}$$

and from Eq. (4.1.8) we get

$$\begin{aligned} |v(0) - \hat{v}(0)| &\leq \max_z\{p(z)|u(\phi, 0) - \hat{u}(\phi, 0)| + (1 - p(z))|u(\phi) - \hat{u}(\phi)|\} \\ &\leq \max_z\{p(z)\|\boldsymbol{u} - \hat{\boldsymbol{u}}\| + (1 - p(z))\|\boldsymbol{u} - \hat{\boldsymbol{u}}\|\} = \|\boldsymbol{u} - \hat{\boldsymbol{u}}\|. \end{aligned}$$

Accordingly, from Eq. (4.2.4) we obtain

$$|D_\phi u - D_\phi \hat{u}| \leq \max \left\{ \begin{array}{l} \lambda\beta|v(0) - \hat{v}(0)| + (1 - \lambda)\beta|u(\phi) - \hat{u}(\phi)|, \\ \beta|u(\phi) - \hat{u}(\phi)| \end{array} \right\} \leq \beta\|\boldsymbol{u} - \hat{\boldsymbol{u}}\|.$$

Similarly, from Eq. (4.2.5) we get $|D_l u - D_l \hat{u}| \leq \beta\|\boldsymbol{u} - \hat{\boldsymbol{u}}\|$ for $l \geq 0$. Thus $\|\boldsymbol{D}u - \boldsymbol{D}\hat{u}\| \leq \beta\|\boldsymbol{u} - \hat{\boldsymbol{u}}\|$, implying that $\boldsymbol{D}u$ is a contraction mapping; accordingly, the assertion holds. ∎

A.2. Lemma 4.2.2

For simplicity, let $R = q\beta u(\phi) + \max\{\dot{L}(h), 0\}$. Then Eq. (4.2.10) can be rewritten

$$u(\phi, l) = \eta u(\phi, l + 1) + R - \beta\theta I(\tau \leq l), \quad l \geq 0, \tag{A.2.1}$$

from which we can develop $u(\phi, 0)$ as follows.

$$\begin{aligned} u(\phi, 0) &= \eta u(\phi, 1) + R \\ &= \eta(\eta u(\phi, 2) + R) + R = \eta^2 u(\phi, 2) + (1 + \eta)R \\ &\quad \vdots \\ &= \eta^\tau u(\phi, \tau) + (1 + \eta + \cdots + \eta^{\tau-1})R \\ &= \eta^\tau u(\phi, \tau) + R(1 - \eta^\tau)/(1 - \eta).. \end{aligned} \tag{A.2.2}$$

Further, from Eq. (A.2.1) with $l = \tau$ we get

$$\begin{aligned} u(\phi,\tau) &= \eta u(\phi,\tau+1) + R - \beta\theta \\ &= \eta(\eta u(\phi,\tau+2) + R - \beta\theta) + R - \beta\theta = \eta^2 u(\phi,\tau+2) + (1+\eta)(R-\beta\theta) \\ &\vdots \\ &= \eta^j u(\phi,\tau+j) + (1+\eta+\cdots+\eta^{j-1})(R-\beta\theta) \\ &= \eta^j u(\phi,\tau+j) + (R-\beta\theta)(1-\eta^j)/(1-\eta). \end{aligned}$$

Accordingly, since $\eta^j \to 0$ as $j \to \infty$ due to $\eta < 1$, we obtain $\lim_{j\to\infty} \eta^j u(\phi,\tau+j) = 0$, hence

$$u(\phi,\tau) = (R-\beta\theta)/(1-\eta). \tag{A.2.3}$$

Rearranging Eq. (A.2.2) by substituting Eq. (A.2.3) produces

$$u(\phi,0) = \eta^\tau(R-\beta\theta)/(1-\eta) + R(1-\eta^\tau)/(1-\eta) = \big(R-\eta^\tau\beta\theta\big)/(1-\eta). \tag{A.2.4}$$

Noting Eqs. (2.4.2) and (4.3.1), we can arrange Eq. (A.2.4) as follows.

$$u(\phi,0) = \gamma R - \gamma\eta^\tau\beta\theta = \gamma q\beta u(\phi) + \gamma\max\{\dot{L}(h),0\} - \rho. \quad \blacksquare$$

A.3. Lemma 4.3.1

(a) Eq. (4.3.3) can be rearranged into

$$G(x) = \gamma\max\{\lambda\beta T(x) - s, 0\} - \max\{x + \gamma\lambda q\beta T(x) - \gamma s, x\} + \gamma r + \rho, \tag{A.3.1}$$

the first term of the right hand side of which is nonincreasing in x from Lemma 3.2.2(a). Since $1 > \gamma q\beta \geq \gamma\lambda q\beta$ from Eq. (2.4.3), it follows that $x + \gamma\lambda q\beta T(x) - \gamma s$ is strictly increasing in x from Lemma 3.2.2(g), hence the entire right hand side of Eq. (A.3.1) is strictly decreasing in x.

(b) Applying Lemma 3.2.2(f) to Eq. (4.3.3) leads to

$$\lim_{x\to\infty} G(x) = \gamma\big(\max\{-s,0\} - \max\{-s,0\}\big) - \lim_{x\to\infty} x + \gamma r + \rho = -\infty$$

Since $G(x) \geq -\max\{x + \gamma\lambda q\beta T(x) - \gamma s, x\} + \gamma r + \rho$ from Eq. (A.3.1), applying Lemma 3.2.2(h) to this inequality yields

$$\lim_{x\to-\infty} G(x) \geq -\max\{\lim_{x\to-\infty}\big(x + \gamma\lambda q\beta T(x)\big) - \gamma s, \lim_{x\to-\infty} x\} + \gamma r + \rho = \infty.$$

(c) For convenience, let us rewrite Eq. (4.3.3) as follows.

$$G(x) = \gamma\big(\lambda\beta\max\{T(x) - s/\lambda\beta, 0\} - \lambda q\beta\max\{T(x) - s/\lambda q\beta, 0\}\big) - x + \gamma r + \rho.$$

For any given $s > 0$ let $x_1(s)$ and $x_2(s)$ be the solution of $T(x) = s/\lambda q\beta$ and $T(x) = s/\lambda\beta$, respectively.

Then since $s/\lambda q\beta > s/\lambda\beta > 0$, clearly both $x_1(s)$ and $x_2(s)$ uniquely exist from Lemma 3.2.2(e) where $x_1(s) < b$ and $x_2(s) < b$. In addition, since $T(x_1(s)) = s/\lambda q\beta > s/\lambda\beta = T(x_2(s))$, we have $x_1(s) < x_2(s)$ due to Lemma 3.2.2(a). It is evident that $x_1(s)$ and $x_2(s)$ are both strictly decreasing in s. Now, let $s' = s + \varepsilon$ for any infinitesimal $\varepsilon > 0$, hence $s' > s$. Then $x_1(s') < x_1(s) < x_2(s') < x_2(s)$ (see Figure 1.3.1). Below, describing $G(x)$ as $G(x,s)$, let us examine the relationship of $G(x,s)$ and s. First, Eq. (4.3.3) for each s and s' can be rewritten as follows, respectively.

$$G(x,s) = \begin{cases} \gamma\big(\lambda(1-q)\beta T(x) + r\big) - x + \rho, & \text{on} \quad \text{I} \cup \text{II} \quad \cdots \ (1), \\ \gamma(\lambda\beta T(x) - s + r) - x + \rho, & \text{on} \quad \text{III} \cup \text{IV} \quad \cdots \ (2), \\ -x + \gamma r + \rho, & \text{on} \quad \text{V} \quad \cdots \ (3), \end{cases} \tag{A.3.2}$$

$$G(x,s') = \begin{cases} \gamma\big(\lambda(1-q)\beta T(x) + r\big) - x + \rho, & \text{on} \quad \text{I} \quad \cdots \ (1'), \\ \gamma(\lambda\beta T(x) - s' + r) - x + \rho, & \text{on} \quad \text{II} \cup \text{III} \quad \cdots \ (2'), \\ -x + \gamma r + \rho, & \text{on} \quad \text{IV} \cup \text{V} \quad \cdots \ (3'). \end{cases} \tag{A.3.3}$$

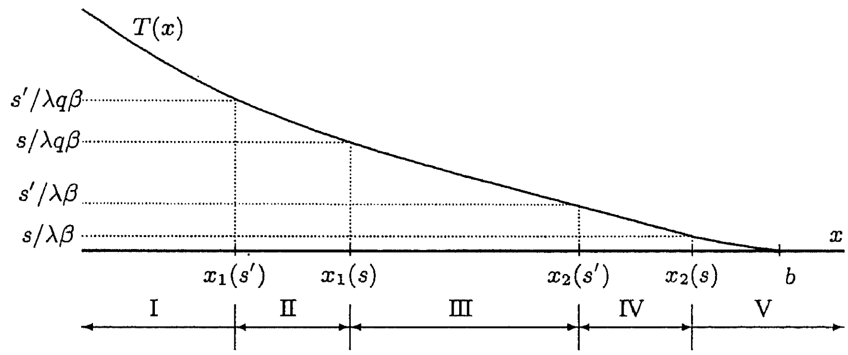

Figure 1.3.1: The relationship between $x_1(s'), x_1(s), x_2(s')$ and $x_2(s)$ where I $= (-\infty, x_1(s')]$, II $= \big(x_1(s'), x_1(s)\big]$, III $= \big(x_1(s), x_2(s')\big)$, IV $= [x_2(s'), x_2(s)\big)$, and V $= [x_2(s), \infty)$.

1. On the intervals I and V, we have $G(x,s) = G(x,s')$, respectively, from Eqs. (A.3.2 (1)) and (A.3.3 (1')) and Eqs. (A.3.2 (3)) and (A.3.3 (3')).

2. On the interval II, from Eqs. (A.3.2 (1)) and (A.3.3 (2')) we get

$$\begin{aligned} G(x,s) - G(x,s') &= \gamma\lambda(1-q)\beta T(x) - \gamma\lambda\beta T(x) + \gamma s' \\ &= -\gamma\lambda q\beta T(x) + \gamma s' = -\gamma(\lambda q\beta T(x) - s') > 0 \end{aligned}$$

   due to $\lambda q\beta T(x) - s' < 0$ on $x_1(s') < x$, hence $G(x,s) > G(x,s')$.

3. On the interval III, from Eqs. (A.3.2 (2)) and (A.3.3 (2')) we have

$$G(x,s) = \gamma(\lambda\beta T(x) - s + r) - x + \rho > \gamma(\lambda\beta T(x) - s' + r) - x + \rho = G(x,s').$$

4. On the interval IV, from Eqs. (A.3.2 (2)) and (A.3.3 (3'))we obtain

$$G(x,s) - G(x,s') = \gamma\big(\lambda\beta T(x) - s + r\big) - x + \rho - (-x + \gamma r + \rho) = \gamma(\lambda\beta T(x) - s) > 0$$

due to $\lambda\beta T(x) - s > 0$ on $x < x_2(s)$.

From all the above, it eventually follows that $G(x,s) \geq G(x,s')$ for all x, that is $G(x,s)$ is nonincreasing in s for all x. Monotonicity of $G(x)$ in q can be proven in almost the same way as the above where it is to be noted that γ and ρ are strictly decreasing in q. The last assertion is evident from the fact that ρ is strictly decreasing in τ due to $\eta < 1$ and that $G(x)$ is strictly increasing in ρ.

(d) Proven in a similar way to that in the proof of (c) where it is to be noted that γ and ρ are strictly increasing in β. The last assertion is evident from the fact that ρ is strictly increasing in θ and that $G(x)$ is strictly increasing in ρ.

(e) Immediate from Eq. (4.3.3) since $\gamma > 0$. ∎

A.4. Lemma 4.3.2

(a) The former half is immediate from the facts that $T(x)$ is nonincreasing in x due to Lemma 3.2.2(a) and that both of $-x/\gamma\lambda(1-q)\beta$ and $-x/\gamma\lambda\beta$ are strictly decreasing in x. The latter half is evident from Lemma 3.2.2(f).

(b) The former half is evident from (a). Let $\lambda\beta T(\rho) > s$. Then since $T(0) \geq T(\rho)$ from Lemma 3.2.2(a) due to $\rho \geq 0$, noting $T(0) > 0$, we have

$$\begin{aligned} B_1(0) &= T(0) + (\gamma r + \rho)/\gamma\lambda(1-q)\beta > 0, \\ B_2(0) &= T(0) - s/\lambda\beta + (\gamma r + \rho)/\gamma\lambda\beta \geq T(\rho) - s/\lambda\beta + (\gamma r + \rho)/\gamma\lambda\beta \\ &= (\lambda\beta T(\rho) - s)/\lambda\beta + (\gamma r + \rho)/\gamma\lambda\beta > 0. \end{aligned}$$

Hence, x_1^* and x_2^* are positive for any $r \geq 0$.

(c) Clear from

$$\begin{aligned} B_1(x) - B_2(x) &= -(x - \gamma r - \rho)/\gamma\lambda(1-q)\beta + \big(x + \gamma(s-r) - \rho\big)/\gamma\lambda\beta \\ &= \big(-qx + \gamma q r + q\rho + \gamma(1-q)s\big)/\gamma\lambda(1-q)\beta = -q(x-\chi)/\gamma\lambda(1-q)\beta. \end{aligned}$$

(d) Let $x_2^* > \chi$. Then $0 = B_2(x_2^*) < B_2(\chi) = B_1(\chi)$ due to (a,c), hence $0 < B_1(\chi)$. Accordingly, since $B_1(x_1^*) = 0 < B_1(\chi)$, we obtain $x_1^* > \chi$ due to (a). Let $x_1^* > \chi$. Then $0 = B_1(x_1^*) < B_1(\chi) = B_2(\chi)$ due to (a,c), hence $0 < B_2(\chi)$. Accordingly, since $B_2(x_2^*) = 0 < B_2(\chi)$, we obtain $x_2^* > \chi$ due to (a). The latter half is proven by the contrapositions of the above results. ∎

A.5. Lemma 5.3.1

To begin with, we have

$$\nu\bar{\kappa} - \bar{\nu}\kappa = \eta(\nu - \bar{\nu}) = \bar{\kappa} - \kappa, \tag{A.5.1}$$

Now, from Eq. (5.3.6) we obtain

$$
\begin{aligned}
\beta H &= \beta\Big(\frac{(1-y)(q+(1-q)\nu)}{1-\kappa} + \frac{y(q+(1-q)\bar{\nu})}{1-\bar{\kappa}}\Big) \\
&= \beta\Big(\frac{q+(1-q)\nu}{1-\kappa} + \frac{q(1-\kappa)-q(1-\bar{\kappa})-(1-q)\nu(1-\bar{\kappa})+(1-q)\bar{\nu}(1-\kappa)}{(1-\kappa)(1-\bar{\kappa})}y\Big) \\
&= \beta\Big(\frac{q+(1-q)\nu}{1-\kappa} + \frac{q(\bar{\kappa}-\kappa)-(1-q)(\nu-\bar{\nu})+(1-q)(\nu\bar{\kappa}-\bar{\nu}\kappa)}{(1-\kappa)(1-\bar{\kappa})}y\Big) \\
&= \frac{q\beta+(1-q)\beta\nu}{1-\kappa} + \frac{q\beta(\bar{\kappa}-\kappa)-(1-q)\beta(\nu-\bar{\nu})+(1-q)\beta(\nu\bar{\kappa}-\bar{\nu}\kappa)}{(1-\kappa)(1-\bar{\kappa})}y \\
&= \frac{q\beta+(1-q)\beta\nu}{1-\kappa} + \frac{q\beta(\bar{\kappa}-\kappa)-\eta(\nu-\bar{\nu})+(1-q)\beta(\nu\bar{\kappa}-\bar{\nu}\kappa)}{(1-\kappa)(1-\bar{\kappa})}y \\
&= 1+\frac{-1+\kappa+q\beta+(1-q)\beta\nu}{1-\kappa} + \frac{q\beta(\bar{\kappa}-\kappa)-(\bar{\kappa}-\kappa)+(1-q)\beta(\bar{\kappa}-\kappa)}{(1-\kappa)(1-\bar{\kappa})}y \quad \text{(from Eq. (A.5.1))} \\
&= 1+\frac{-1+(1-q)\beta(1-\nu)+q\beta+(1-q)\beta\nu}{1-\kappa} - \frac{(\bar{\kappa}-\kappa)(1-q\beta-(1-q)\beta)}{(1-\kappa)(1-\bar{\kappa})}y \\
&= 1-\frac{1-\beta}{1-\kappa} - \frac{(1-\beta)(\bar{\kappa}-\kappa)}{(1-\kappa)(1-\bar{\kappa})}y \\
&= 1-(1-\beta)\Big(\frac{1}{1-\kappa} - \frac{(1-\kappa)-(1-\bar{\kappa})}{(1-\kappa)(1-\bar{\kappa})}y\Big) \\
&= 1-(1-\beta)\Big(\frac{1}{1-\kappa} - \frac{y}{1-\kappa} + \frac{y}{1-\bar{\kappa}}\Big) \\
&= 1-(1-\beta)\Big(\frac{1-y}{1-\kappa} + \frac{y}{1-\bar{\kappa}}\Big) \\
&= 1-(1-\beta)W \quad \text{(from Eq. (5.3.5))}.
\end{aligned}
$$

From Eqs. (5.3.5) and (5.3.6) we get

$$
\begin{aligned}
W-H &= \frac{1-y}{1-\kappa}(1-q-(1-q)\nu) + \frac{y}{1-\bar{\kappa}}(1-q-(1-q)\bar{\nu}) \\
&= \frac{(1-y)(1-q)(1-\nu)}{1-\kappa} + \frac{y(1-q)(1-\bar{\nu})}{1-\bar{\kappa}} > 0.
\end{aligned}
$$

Further, regarding H as a function of y, $H(y)$, with $0 < y < 1$, we obtain $H(0) = (q+(1-q)\nu)/(1-\kappa)$. Hence $H(0) < 1$ because

$$
\begin{aligned}
1-\kappa-(q+(1-q)\nu) &= 1-(1-q)\beta(1-\nu)-q-(1-q)\nu \\
&= (1-q)\big(1-\nu-\beta(1-\nu)\big) = (1-q)(1-\beta)(1-\nu) > 0
\end{aligned}
$$

Further, $H(1) = (q+(1-q)\bar{\nu})/(1-\bar{\kappa})$. Hence $H(1) < 1$, which can be proven in quite the same way as the above. Since $H(y)$ is a linear function of y due to Eq. (5.3.6), it follows that $H(y) < 1$ for $0 \le y \le 1$ or equivalently $H < 1$. ∎

A.6. Lemma 5.3.2

(a) For any $x' < x$ from Eq. (5.3.13) we have

$$
\begin{aligned}
G(x)-G(x') &= W\Big(\max\{L(x),0\}-\max\{L(x'),0\}\Big)+(1-y)\Big(\max\{\dot{L}(x'),0\}-\max\{\dot{L}(x),0\}\Big)/(1-\kappa)\\
&\quad +y\Big(\max\{\ddot{L}(x'),0\}-\max\{\ddot{L}(x),0\}\Big)/(1-\bar{\kappa})+(x'-x)\\
&\le W\Big(\max\{L(x)-L(x'),0\}\Big)+(1-y)\Big(\max\{\dot{L}(x')-\dot{L}(x),0\}\Big)/(1-\kappa)\\
&\quad +y\Big(\max\{\ddot{L}(x')-\ddot{L}(x),0\}\Big)/(1-\bar{\kappa})+(x'-x).
\end{aligned}
$$

Noting $L(x)$, $\dot{L}(x)$, and $\ddot{L}(x)$ are all nonincreasing in x due to Lemma 3.2.2(a) and Eq. (5.2.2), we get

$$
\begin{aligned}
G(x)-G(x') &\le (1-y)\Big(\dot{L}(x')-\dot{L}(x)\Big)/(1-\kappa)+y\Big(\ddot{L}(x')-\ddot{L}(x)\Big)/(1-\bar{\kappa})+(x'-x)\\
&= (1-y)\lambda\beta\big((1-q)\nu+q\big)\big(T(x')-T(x)\big)/(1-\kappa)\\
&\qquad +y\lambda\beta\big((1-q)\bar{\nu}+q\big)\big(T(x')-T(x)\big)/(1-\bar{\kappa})+(x'-x)\\
&= \lambda\beta\Big((1-y)\big((1-q)\nu+q\big)/(1-\kappa)+y\big((1-q)\bar{\nu}+q\big)/(1-\bar{\kappa})\Big)\big(T(x')-T(x)\big)+(x'-x)\\
&= \lambda\beta H\big(T(x')-T(x)\big)+(x'-x), \quad \text{due to Eq. (5.3.6)}\\
&= \lambda\beta HT(x')+x'-\big(\lambda\beta HT(x)+x\big)<0
\end{aligned}
$$

due to Eq. (5.3.12) and Lemma 3.2.2(g).

(b) Applying Lemma 3.2.2(f) to Eq. (5.3.13) leads to

$$
\begin{aligned}
\lim_{x\to\infty} G(x) &= W\max\{-s,0\}-(1-y)\max\{-s,0\}/(1-\kappa)\\
&\qquad -y\max\{-s,0\}/(1-\bar{\kappa})-\lim_{x\to\infty} x+Wr+\rho=-\infty.
\end{aligned}
$$

Now, from Eq. (5.3.13) we get

$$
\begin{aligned}
G(x) &\ge -(1-y)\max\{\lambda\beta(q+(1-q)\nu)T(x)-s,0\}/(1-\kappa)\\
&\qquad -y\max\{\lambda\beta(q+(1-q)\bar{\nu})T(x)-s,0\}/(1-\bar{\kappa})-x+Wr+\rho. \qquad \text{(A.6.1)}
\end{aligned}
$$

Let $s=0$. Then

$$
\begin{aligned}
G(x) &\ge -\lambda\beta\Big((1-y)\big((1-q)\nu+q\big)/(1-\kappa)+y\big((1-q)\bar{\nu}+q\big)/(1-\bar{\kappa})\Big)T(x)-x+Wr+\rho\\
&= -\big(\lambda\beta HT(x)+x\big)+Wr+\rho.
\end{aligned}
$$

Hence, since $H<1$ due to Eq. (5.3.12), applying Lemma 3.2.2(h) into the above inequality produces

$$
\lim_{x\to-\infty} G(x) \ge -\lim_{x\to-\infty}\big(\lambda\beta HT(x)+x\big)+Wr+\rho=\infty.
$$

Let $s>0$. Then there exists an $\bar{x}<b$ such that if $\bar{\nu}>\nu$, then $T(\bar{x})=s/\lambda\beta(q+(1-q)\nu)>s/\lambda\beta(q+(1-q)\bar{\nu})$, or else $T(\bar{x})=s/\lambda\beta(q+(1-q)\bar{\nu})>s/\lambda\beta(q+(1-q)\nu)$ due to Lemma 3.2.2(e). For $x<\bar{x}$ we have $T(x)>T(\bar{x})=s/\lambda\beta(q+(1-q)\nu)>s/\lambda\beta(q+(1-q)\bar{\nu})$ or $T(x)>T(\bar{x})=s/\lambda\beta(q+(1-q)\bar{\nu})>s/\lambda\beta(q+(1-q)\nu)$, hence Eq. (A.6.1) becomes

$$
\begin{aligned}
G(x) &\geq -\lambda\beta\Big((1-y)\big((1-q)\nu+q\big)/(1-\kappa)+y\big((1-q)\bar{\nu}+q\big)/(1-\bar{\kappa})\Big)T(x)-x \\
&\qquad\qquad +\big((1-y)/(1-\kappa)+y/(1-\bar{\kappa})s+Wr+\rho \\
&= -\big(\lambda\beta HT(x)+x\big)+Ws+Wr+\rho.
\end{aligned}
$$

Accordingly, we can also have $\lim_{x\to-\infty} G(x)=\infty$ in quite the same way as when $s=0$.

(c) For convenience, let us rewrite Eq. (5.3.13) as follows.

$$
\begin{aligned}
G(x) &= W\lambda\beta\max\{T(x)-s/\lambda\beta,0\} \\
&\qquad -\frac{\lambda\beta(1-y)\big(q+(1-q)\nu\big)}{1-\kappa}\max\{T(x)-s/\lambda\beta\big(q+(1-q)\nu\big),0\} \\
&\qquad -\frac{\lambda\beta y\big(q+(1-q)\bar{\nu}\big)}{1-\bar{\kappa}}\max\{T(x)-s/\lambda\beta\big(q+(1-q)\bar{\nu}\big),0\}-x+Wr+\rho.
\end{aligned}
$$

Let $\nu\geq\bar{\nu}$. For any given $s>0$ let $x_1(s)$, $x_2(s)$, and $x_3(s)$ be the solution of, respectively, $T(x)=s/\lambda\beta\big(q+(1-q)\bar{\nu}\big)$, $T(x)=s/\lambda\beta\big(q+(1-q)\nu\big)$, and $T(x)=s/\lambda\beta$. Then since $s/\lambda\beta\big(q+(1-q)\bar{\nu}\big)\geq s/\lambda\beta\big(q+(1-q)\nu\big)>s/\lambda\beta>0$ due to $q+(1-q)\bar{\nu}\leq q+(1-q)\nu<1$, clearly $x_1(s)$, $x_2(s)$, and $x_3(s)$ uniquely exist from Lemma 3.2.2(e) where $x_1(s)<b$, $x_2(s)<b$, and $x_3(s)<b$. In addition, we have $x_1(s)<x_2(s)<x_3(s)$ due to Lemma 3.2.2(a). It is evident that $x_1(s)$, $x_2(s)$, and $x_3(s)$ are strictly decreasing in s. Now, let $s'=s+\varepsilon$ for any infinitesimal $\varepsilon>0$, hence $s'>s$. Then $x_1(s')<x_1(s)<x_2(s')<x_2(s)$ (see Figure 1.6.2). Below, describing $G(x)$ as $G(x,s)$, let us examine the relationship of $G(x,s)$ and s. First, Eq. (5.3.13) for each s and s' can be rewritten as follows, respectively.

$$
G(x,s)=\begin{cases}
\lambda\beta(W-H)T(x)-x+Wr+\rho, & \text{on I} \cup \text{II} \quad \cdots (1),\\
\lambda\beta\Big(W-(1-y)\big(q+(1-q)\nu\big)/(1-\kappa)\Big)T(x) & \\
\qquad -ys/(1-\bar{\kappa})-x+Wr+\rho, & \text{on III} \cup \text{IV} \quad \cdots (2),\\
\lambda\beta WT(x)-x-Ws+Wr+\rho, & \text{on V} \cup \text{VI} \quad \cdots (3),\\
-x+Wr+\rho, & \text{on VII} \quad \cdots (4),
\end{cases}
\qquad (A.6.2)
$$

$$
G(x,s')=\begin{cases}
\lambda\beta(W-H)T(x)-x+Wr+\rho, & \text{on I} \quad \cdots (1'),\\
\lambda\beta\Big(W-(1-y)\big(q+(1-q)\nu\big)/(1-\kappa)\Big)T(x) & \\
\qquad -ys'/(1-\bar{\kappa})-x+Wr+\rho, & \text{on II} \cup \text{III} \quad \cdots (2'),\\
\lambda\beta WT(x)-x-Ws'+Wr+\rho, & \text{on IV} \cup \text{V} \quad \cdots (3'),\\
-x+Wr+\rho, & \text{on VI} \cup \text{VII} \quad \cdots (4').
\end{cases}
\qquad (A.6.3)
$$

1. On the intervals I and VII, we have $G(x,s)=G(x,s')$ from Eqs. (A.6.2 (1)) and (A.6.3 (1′)) and from Eqs. (A.6.2 (4)) and (A.6.3 (4′)), respectively.

2. On the interval II, from Eqs. (A.6.2 (1)) and (A.6.3 (2′)) we get

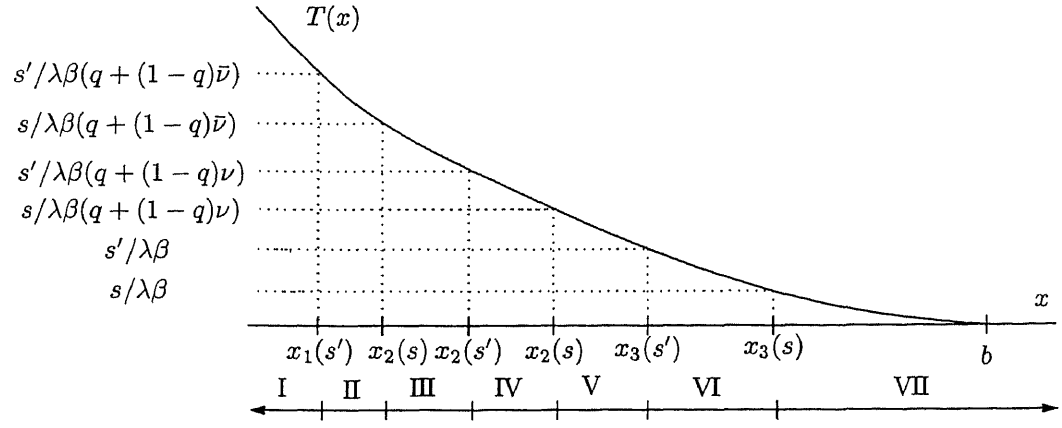

Figure 1.6.2: The relationship between $x_1(s')$, $x_1(s)$, $x_2(s')$, $x_2(s)$, $x_3(s)$, and $x_3(s')$.

$$\begin{aligned}
G(x,s) - G(x,s') &= \lambda\beta\big((1-y)(q+(1-q)\nu)/(1-\kappa) - H\big)T(x) - ys'/(1-\bar{\kappa}) \\
&= -y\lambda\beta\big(q+(1-q)\bar{\nu}\big)T(x)/(1-\bar{\kappa}) - ys'/(1-\bar{\kappa}) \quad \text{(due to Eq. (5.3.6))} \\
&= -y\big(\lambda\beta\big(q+(1-q)\bar{\nu}\big)T(x) - s'\big)/(1-\bar{\kappa}) > 0
\end{aligned}$$

due to $\lambda\beta\big(q+(1-q)\bar{\nu}\big)T(x) - s' < 0$ on $x_1(s') < x$, hence $G(x,s) > G(x,s')$.

3. On the interval III, from Eqs. (A.6.2 (2)) and (A.6.3 (2′)) we have

$$\begin{aligned}
G(x,s) &= \lambda\beta\Big(W - (1-y)\big(q+(1-q)\nu\big)/(1-\kappa)\Big)T(x) - ys/(1-\bar{\kappa}) - x + Wr + \rho \\
&> \lambda\beta\Big(W - (1-y)\big(q+(1-q)\nu\big)/(1-\kappa)\Big)T(x) - ys'/(1-\bar{\kappa}) - x + Wr + \rho = G(x,s').
\end{aligned}$$

4. On the interval IV, from Eqs. (A.6.2 (2)) and (A.6.3 (3′)) we obtain

$$\begin{aligned}
G(x,s) - G(x,s') &= -\lambda\beta(1-y)(q+(1-q)\nu)T(x) - ys/(1-\bar{\kappa}) + Ws' \\
&> -\lambda\beta(1-y)(q+(1-q)\nu)T(x) - ys'/(1-\bar{\kappa}) + Ws' \\
&= -(1-y)\lambda\beta(q+(1-q)\nu)T(x)/(1-\kappa) - (1-y)s'/(1-\kappa) \\
&= -(1-y)\Big(\lambda\beta\big(q+(1-q)\nu\big)T(x) - s'\Big)/(1-\kappa) > 0
\end{aligned}$$

due to $\lambda\beta\big(q+(1-q)\nu\big)T(x) - s' < 0$ on $x_2(s') < x$.

5. On the interval V, from Eqs. (A.6.2 (3)) and (A.6.3 (3′)) we have

$$G(x,s) = \lambda\beta WT(x) - x - Ws + Wr + \rho > \lambda\beta WT(x) - x - Ws' + Wr + \rho = G(x,s').$$

6. On the interval VI, from Eqs. (A.6.2 (3)) and (A.6.3 (4′)) we have

$$G(x,s) - G(x,s') = \lambda\beta WT(x) - Ws = W\big(\lambda\beta T(x) - s\big) > 0.$$

From all the above, it eventually follows that $G(x,s) \geq G(x,s')$ for all x, that is $G(x,s)$ is nonincreasing in s for all x. When $\nu < \bar{\nu}$, in the almost same way as the above we can prove that $G(x)$ is nonincreasing in s. The latter half can be proven in the almost same way as the above.

(d) Immediate from Eq. (5.3.13) since $W > 0$.

(e) Evident from the fact that $G(x)$ is strictly increasing in ρ and that the ρ is strictly decreasing in ϑ and $\bar{\vartheta}$ and strictly increasing in θ from Eq. (5.3.4). ∎

A.7. Lemma 5.3.3

(a) The former half is immediate from the facts that $T(x)$ is nonincreasing in x due to Lemma 3.2.2(a) and that $-x/Z$ for any real number $Z > 0$ is strictly decreasing in x.

(b) Evident from Lemma 3.2.2(f).

(c) Immediate from (a,b).

(d) Let $\nu > \bar{\nu}$.

(d1) From Eqs. (5.3.14) and (5.3.18) we get

$$\begin{aligned} B_1(x) - B_2(x) &= \lambda\beta\big(W(1-q)(1-\nu) - (W-h)\big)T(x) \\ &= \lambda\beta\big(W(1-(q+(1-q)\nu) - W + H\big)T(x) \\ &= \lambda\beta\big(-W(q+(1-q)\nu) + H\big)T(x) \\ &= \lambda\beta\Big(-(1-y)(q+(1-q)\nu)/(1-\kappa) - y(q+(1-q)\nu)/(1-\bar{\kappa}) \\ &\qquad +(1-y)(q+(1-q)\nu)/(1-\kappa) + y(q+(1-q)\bar{\nu})/(1-\bar{\kappa})\Big)T(x) \\ &= -\lambda\beta y(1-q)(\nu-\bar{\nu})T(x)/(1-\bar{\kappa}). \end{aligned}$$

If $x < b$, then $T(x) > 0$ due to Lemma 3.2.2(c), hence $B_1(x) < B_2(x)$, or else $T(x) = 0$, hence $B_1(x) = B_2(x)$. Now, if $r < (b-\rho)/W$, then $B_1(b) = B_2(b) = -b + Wr + \rho < 0$. Accordingly, x_1^* and x_2^* are less than b, implying $x_1^* < x_2^* < b$ due to $B_1(x) < B_2(x)$ for $x < b$. If $r \geq (b-\rho)/W$, then $B_1(b) = B_2(b) = -b + Wr + \rho \geq 0$. Therefore, x_1^* and x_2^* are greater than or equal to b, implying $x_1^* = x_2^* \geq b$ due to $B_1(x) = B_2(x)$ for $x \geq b$.

(d2) From Eqs. (5.3.7) and (5.3.9) we have

$$\begin{aligned} \chi_1 - \chi_2 &= \frac{Ws(1-q)(1-\nu)}{q+(1-q)\nu} - \frac{s(W-H)}{q+(1-q)\bar{\nu}} \\ &< \frac{Ws(1-q)(1-\nu)}{q+(1-q)\bar{\nu}} - \frac{s(W-H)}{q+(1-q)\bar{\nu}} \\ &= \frac{s}{q+(1-q)\bar{\nu}}\left(W(1-q)(1-\nu) - W + \frac{(1-y)(q+(1-q)\nu)}{1-\kappa} + \frac{y(q+(1-q)\bar{\nu})}{1-\bar{\kappa}}\right) \\ &< \frac{s}{q+(1-q)\bar{\nu}}\left(W(1-q)(1-\nu) - W + \frac{(1-y)(q+(1-q)\nu)}{1-\kappa} + \frac{y(q+(1-q)\bar{\nu})}{1-\bar{\kappa}}\right) \\ &= \frac{s}{q+(1-q)\bar{\nu}}\Big(W(1-q)(1-\nu) - W + W(q+(1-q)\nu)\Big) \\ &= \frac{s}{q+(1-q)\bar{\nu}}\Big(W(1-q)(1-\nu) - W(1-q)(1-\nu)\Big) = 0. \end{aligned}$$

(d3) For convenience let $A = (q + (1-q)\nu)(1-y)/(1-\kappa)$. Then we get

$$\begin{aligned} B_3(x) - B_4(x) &= \frac{-x - Ws + Wr + \rho}{\lambda\beta W} - \frac{-x - ys/(1-\bar{\kappa}) + Wr + \rho}{\lambda\beta(W-A)} \\ &= \frac{Ax - Ws(W - y/(1-\bar{\kappa}) - A) + r(W(W-A) - W^2) + \rho(W - A - W)}{\lambda\beta W(W-A)} \\ &= \frac{Ax - Ws\big((1-y)/(1-\kappa) - A\big) - WAr - A\rho}{\lambda\beta W(W-A)} \\ &= \frac{Ax - Ws(1-q)(1-\nu)(1-y)/(1-\kappa) - WAr - A\rho}{\lambda\beta W(W-A)} \\ &= \frac{A\big(x - Ws(1-q)(1-\nu)(1-y)/A(1-\kappa) - Wr - \rho\big)}{\lambda\beta W(W-A)} \\ &= \frac{A\big(x - Ws(1-q)(1-\nu)(1-y)/(q + (1-q)\nu) - Wr - \rho\big)}{\lambda\beta W(W-A)} \\ &= \frac{(q + (1-q)\nu)(x - \chi_1)}{\lambda\beta(1-\kappa)W(W-A)}. \end{aligned}$$

(d4) Let $A = (q + (1-q)\nu)(1-y)/(1-\kappa)$. Then immediately $H - A = (q + (1-q)\bar{\nu})y/(1-\bar{\kappa})$ from Eq. (5.3.6). Now we have

$$\begin{aligned} B_4(x) - B_2(x) &= \frac{-x - ys/(1-\bar{\kappa}) + Wr + \rho}{\lambda\beta(W-A)} - \frac{-x + Wr + \rho}{\lambda\beta(W-H)} \\ &= \frac{x(H-A) - ys(W-H)/(1-\bar{\kappa}) + Wr(W - H - W + A) + \rho(W - H - W + A)}{\lambda\beta(W-A)(W-H)} \\ &= \frac{x(H-A) - ys(W-H)/(1-\bar{\kappa}) - Wr(H-A) - \rho(H-A)}{\lambda\beta(W-A)(W-H)} \\ &= \frac{(H-A)\big(x - ys(W-H)/(H-A)(1-\bar{\kappa}) - Wr - \rho\big)}{\lambda\beta(W-A)(W-H)} \\ &= \frac{(q + (1-q)\bar{\nu})\big(x - s(W-H)/(q + (1-q)\bar{\nu}) - Wr - \rho\big)}{\lambda\beta(1-\bar{\kappa})(W-A)(W-H)} \\ &= \frac{(q + (1-q)\bar{\nu})(x - \chi_2)}{\lambda\beta(1-\bar{\kappa})(W-A)(W-H)}. \end{aligned}$$

(e) Let $\nu < \bar{\nu}$.

(e1) From Eqs. (5.3.15) and (5.3.18) we have $\bar{B}_1(x) - B_2(x) = \lambda\beta(1-y)(1-q)(\nu - \bar{\nu})T(x)/(1-\kappa)$ in the same way as in (d1). Since if $x<(\geq)b$, then $T(x) > (=) 0$ due to Lemma 3.2.2(c), we have $\bar{B}_1(x) < (=) B_2(x)$. Now, if $r < (b-\rho)/W$, then $\bar{B}_1(b) = B_2(b) = -b + Wr + \rho < 0$. Accordingly, $\bar{x}_1^*$ and x_2^* are less than b, implying $\bar{x}_1^* < x_2^* < b$ due to $\bar{B}_1(x) < B_2(x)$ for $x < b$. If $r \geq (b-\rho)/W$, then $\bar{B}_1(b) = B_2(b) = -b + Wr + \rho \geq 0$. Therefore, $\bar{x}_1^*$ and x_2^* are greater than or equal to b, implying $\bar{x}_1^* = x_2^* \geq b$ due to $\bar{B}_1(x) = B_2(x)$ for $x \geq b$.

(e2,e3,e4) It can be proven in the same way as in (d).

(f) Let $\nu = \bar{\nu}$. Then since $\kappa = \bar{\kappa}$, we immediately have $W = 1/(1-\kappa)$ and $H = (q + (1-q)\nu)/(1-\kappa)$ from Eqs. (5.3.5) and (5.3.6), hence $W - H = W(1-q)(1-\nu)$. Accordingly, $B_1(x) = \bar{B}_1(x) = B_2(x)$ due to Eq. (5.3.14) to Eq. (5.3.18) and $\chi_1 = \chi_2 = \bar{\chi}_1 = \bar{\chi}_2$ due to Eq. (5.3.7) to Eq. (5.3.10). ∎

A.8. Lemma 5.3.4

For simplicity, let $R = (q+(1-q)\nu)\beta u(\phi) + \max\{\dot{L}(h),0\}$ and $\bar{R} = (q+(1-q)\bar{\nu})\beta u(\phi) + \max\{\ddot{L}(h),0\}$. Then from Eq. (5.2.13) with $l=0$ we have

$$\begin{aligned} u(\phi,0) &= \kappa u(\phi,1) + R + \beta\nu\theta \\ &= \kappa(\kappa u(\phi,2) + R + \beta\theta) + R + \beta\nu\theta \\ &= \kappa^2 u(\phi,2) + (1+\kappa)(R+\beta\nu\theta) \\ &\vdots \\ &= \kappa^\tau u(\phi,\tau) + (1-\kappa^\tau)(R+\beta\nu\theta)/(1-\kappa).. \end{aligned} \tag{A.8.1}$$

The above equation can be rewritten as follows. Noting Eq. (5.3.3), we can rewrite the above equation as follows.

$$u(\phi,0) = yu(\phi,\tau) + (1-y)\big((q+(1-q)\nu)\beta u(\phi) + \max\{\dot{L}(h),0\} + \beta\nu\theta\big)/(1-\kappa). \tag{A.8.2}$$

Further, from Eq. (5.2.14) with $l = \tau - 1$ we get

$$\begin{aligned} u(\phi,\tau) &= \bar{\kappa} u(\phi,\tau+1) + \bar{R} - \beta(\vartheta - \bar{\nu}\bar{\theta}) \\ &= \bar{\kappa}\big(\bar{\kappa} u(\phi,\tau+2) + \bar{R} - \beta(\vartheta - \bar{\nu}\bar{\theta})\big) + R - \beta(\vartheta - \bar{\nu}\bar{\theta}) \\ &= \bar{\kappa}^2 u(\phi,\tau+2) + (1+\bar{\kappa})\big(\bar{R} - \beta(\vartheta - \bar{\nu}\bar{\theta})\big) \\ &\vdots \\ &= \bar{\kappa}^j u(\phi,\tau+j) + (1-\bar{\kappa}^j)\big(\bar{R} - \beta(\vartheta - \bar{\nu}\bar{\theta})\big)/(1-\bar{\kappa}). \end{aligned}$$

Accordingly, since $\bar{\kappa}^j \to 0$ as $j \to \infty$ due to $\bar{\kappa} < 1$ and $u(\phi,l)$ is bounded in $l \geq 0$, we obtain

$$u(\phi,\tau-1) = \big((q+(1-q)\bar{\nu})\beta u(\phi) + \max\{\ddot{L}(h),0\} - \beta(\vartheta - \bar{\nu}\bar{\theta})\big)/(1-\bar{\kappa}). \tag{A.8.3}$$

Substituting Eq. (A.8.3) to Eq. (A.8.2) produces

$$\begin{aligned} u(\phi,0) &= y\Big((q+(1-q)\bar{\nu})\beta u(\phi) + \max\{\ddot{L}(h),0\} - \beta(\vartheta - \bar{\nu}\bar{\theta})\Big)/(1-\bar{\kappa}) \\ &\qquad +(1-y)\big((q+(1-q)\nu)\beta u(\phi) + \max\{\dot{L}(h),0\} + \beta\nu\theta\big)/(1-\kappa) \\ &= \beta\Big(\frac{(1-y)(q+(1-q)\nu)}{1-\kappa} + \frac{y(q+(1-q)\bar{\nu})}{1-\bar{\kappa}}\Big)u(\phi) \\ &\qquad +\frac{1-y}{1-\kappa}\max\{\dot{L}(h),0\} + \frac{y}{1-\bar{\kappa}}\max\{\ddot{L}(h),0\} - \frac{\beta(\vartheta - \bar{\nu}\bar{\theta})y}{1-\bar{\kappa}} + \frac{\beta\nu\theta(1-y)}{1-\kappa}. \end{aligned}$$

From Eqs. (5.3.4) and (5.3.6) the above equation can be rewritten

$$u(\phi,0) = \beta H u(\phi) + (1-y)\max\{\dot{L}(h),0\}/(1-\kappa) + y\max\{\ddot{L}(h),0\}/(1-\bar{\kappa}) - \rho,$$

hence from Eq. (5.3.11) we obtain

$$u(\phi,0) = (1-(1-\beta)W)u(\phi) + (1-y)\max\{\dot{L}(h),0\}/(1-\kappa) + y\max\{\ddot{L}(h),0\}/(1-\ddot{\kappa}) - \rho. \quad \blacksquare$$

A.9. Lemma 6.3.2

(a) To begin with, let us define the following recurrent relations corresponding to Eq. (6.2.2) to Eq. (6.2.5).

$$u_t(\phi,0) = \max\{\lambda\beta v_{t-1}(0) + (1-\lambda)\beta u_{t-1}(\phi,0) - s, \beta u_{t-1}(\phi,0)\} + r, \quad t \ge 1, \tag{A.9.1}$$

$$u_t(\phi,i) = \max\left\{\begin{array}{l} (1-q)\beta(\lambda v_{t-1}(i) + (1-\lambda)u_{t-1}(\phi,i)) \\ \quad +q\beta(\lambda v_{t-1}(i-1) + (1-\lambda)u_{t-1}(\phi,i-1)) - s, \\ (1-q)\beta u_{t-1}(\phi,i) + q\beta u_{t-1}(\phi,i-1) \end{array}\right\}, \quad 1 \le i < n, \quad t \ge 1, \tag{A.9.2}$$

$$u_t(\phi,n) = \max\left\{\begin{array}{l} (1-q)\beta u_{t-1}(\phi,n) \\ \quad +q\beta(\lambda v_{t-1}(n-1) + (1-\lambda)u_{t-1}(\phi,n-1)) - s, \\ (1-q)\beta u_{t-1}(\phi,n) + q\beta u_{t-1}(\phi,n-1) \end{array}\right\}, \quad t \ge 1 \tag{A.9.3}$$

where $u_0(\phi,0) = 0$ for all i. Further, as expressions corresponding to Eqs. (6.1.5) and (6.1.10) for $0 \le i < n$ let us define, respectively,

$u_t(w,i) = \max\{w + u_t(\phi,i+1), u_t(\phi,i)\}$, for the admission control problem,
$u_t(1,i) = \max_z\{p(z)(z + u_t(\phi,i+1)) + (1-p(z))u_t(\phi,i)\}$, for the pricing control problem.

Further define $v_t(i) = \mathbf{E}[u_t(w,i)]$ for the admission control problem and $v_t(i) = u_t(1,i)$ for the pricing control problem. Then we have

$$v_t(i) = \left\{\begin{array}{l} \mathbf{E}[\max\{w + u_t(\phi,i+1), u_t(\phi,i)\}], \\ \max\limits_z\{p(z)(z + u_t(\phi,i+1)) + (1-p(z))u_t(\phi,i)\} \end{array}\right\}, \quad 0 \le i < n. \tag{A.9.4}$$

Accordingly, letting

$$h_{it} = u_t(\phi,i) - u_t(\phi,i+1), \quad 0 \le i < n, \quad t \ge 0, \tag{A.9.5}$$

from Eqs. (A.9.4) and (3.1.1) we have

$$v_t(i) = T(h_{it}) + u_t(\phi,i), \quad 0 \le i < n, \quad t \ge 0. \tag{A.9.6}$$

Clearly $u_0(\phi,i)$ in nonincreasing in i, hence $v_0(i)$ is nonincreasing in i from Eq. (A.9.4). Assume that $u_{t-1}(\phi,i)$ is nonincreasing in i, hence $v_{t-1}(i)$ is nonincreasing in i. Then we have

$$\begin{aligned} u_t(\phi,0) &\ge u_t(\phi,0) - s \\ &= \max\left\{\begin{array}{l} (1-q)\beta(\lambda v_{t-1}(0) + (1-\lambda)u_{t-1}(\phi,0)) + q\beta(\lambda v_{t-1}(0) + (1-\lambda)u_{t-1}(\phi,0)) - s \\ (1-q)\beta u_{t-1}(\phi,0) + q\beta u_{t-1}(\phi,0) \end{array}\right\} \\ &\ge u_t(\phi,1). \end{aligned}$$

In almost the same way as above, for $2 \le i \le n-1$ we get $u_t(\phi,i-1) \ge u_t(\phi,i)$. Now, noting that

$v_{t-1}(i) \geq u_{t-1}(\phi, i)$ from Eq. (A.9.6) due to Lemma 3.2.2(b) and that $u_{t-1}(\phi, i) \geq u_{t-1}(\phi, i+1)$ due to the induction hypothesis, from Eq. (A.9.2) with $i = n-1$ we obtain

$$u_t(\phi, n-1) \geq \max \left\{ \begin{array}{l} (1-q)\beta(\lambda u_{t-1}(\phi, n) + (1-\lambda)u_{t-1}(\phi, n)) \\ \qquad\qquad +q\beta(\lambda v_{t-1}(n-1) + (1-\lambda)u_{t-1}(\phi, n-1)) - s \\ (1-q)\beta u_{t-1}(\phi, n) + q\beta u_{t-1}(\phi, n-1) \end{array} \right\} = u_t(\phi, n).$$

Hence, since $u_t(\phi, i)$ is nonincreasing in $i \in [0, n]$, from Eq. (A.9.4) we immediately have $v_t(i-1) \geq v_t(i)$ for $1 \leq i \leq n-1$. Thus, $u(\phi, i)$ and $v(i)$ are nonincreasing in, respectively, $i \in [0, n]$ and $i \in [0, n-1]$. Now, since we can easily show that $u(\phi, 0) \geq c/(1-\beta) \geq 0$ from Eq. (6.2.12), we have $u(\phi, i) \geq \gamma q\beta u(\phi, i-1) \geq 0$ for $1 \leq i \leq n$ from Eq. (6.2.22), hence by induction $u(\phi, i) \geq 0$ for $0 \leq i \leq n$.

(b) Immediate from Eq. (6.1.1) and (a). ∎

A.10. Lemma 6.3.8

Let $u_0(\phi, i) = 0$ for all i. Then $h_{i0}(r) = 0$ for all i, which can be regarded as nondecreasing in $r \geq 0$. Assume that $h_{i,t-1}(r)$ is nondecreasing in $r \geq 0$ for all i.

1. *Proof of the monotonicity of $h_0(r)$ in r.* Let us define $\mathcal{R}_0^- = \{r \mid Q_0(r) \leq 0, r \geq 0\}$ and $\mathcal{R}_0^+ = \{r \mid Q_0(r) > 0, r \geq 0\}$. Then let us consider the two cases of $r \in \mathcal{R}_0^-$ and $r \in \mathcal{R}_0^+$.

   i. Case of $r \in \mathcal{R}_0^-$. Then since the optimal decision in state $(\phi, 0)$ is to skip the search, from Eqs. (6.2.2) and (6.2.3) with $i = 1$ the optimal equations become

   $$\begin{aligned} u(\phi, 0) &= \beta u(\phi, 0) + r, \\ u(\phi, 1) &= \max \left\{ \begin{array}{l} (1-q)\beta(\lambda v(1) + (1-\lambda)u(\phi, 1)) + q\beta(\lambda v(1) + (1-\lambda)u(\phi, 0)) - s, \\ (1-q)\beta u(\phi, 1) + q\beta u(\phi, 0). \end{array} \right. \end{aligned}$$

   Here, let us define the following recurrent relations corresponding to the above equations.

   $$\begin{aligned} u_t(\phi, 0) &= \beta u_{t-1}(\phi, 0) + r, \\ u_t(\phi, 1) &= (1-q)\beta u_{t-1}(\phi, 1) + q\beta u_{t-1}(\phi, 0) \\ &\quad + \max\{\lambda(1-q)\beta(v_{t-1}(1) - u_{t-1}(\phi, 1)) + \lambda q\beta(v_{t-1}(0) - u_{t-1}(\phi, 0)) - s, 0\}. \end{aligned}$$

   Accordingly, noting $T(h_{it}(r)) = v_t(i) - u_t(\phi, i)$ in Eq. (A.9.6), from Eq. (A.9.5) with $i = 0$ we have

   $$h_{0t}(r) = (1-q)\beta h_{0,t-1}(r) + r - \max\{(1-q)L(h_{1,t-1}(r)) + qL(h_{0,t-1}(r)), 0\}.$$

   Since $L(h_{i,t-1}(r))$ with $i = 0, 1$ are both nonincreasing in r from the induction hypothesis and Lemma 3.2.3(a), we immediately obtain that $h_{0t}(r)$ is nodecreasing in $r \in \mathcal{R}_0^-$.

   ii. Case of $r \in \mathcal{R}_0^+$. Then $Q_1 > 0$ due to Lemma 6.3.7. Accordingly, since the optimal decisions are to continue the search in both states $u(\phi, 0)$ and $(\phi, 0)$, the optimal equations Eqs. (6.2.2) and (6.2.3) with $i = 1$ become

$$u(\phi,0) = \lambda\beta v(0) + (1-\lambda)\beta u(\phi,0) - s + r,$$

$$u(\phi,1) = (1-q)\beta(\lambda v(1) + (1-\lambda)u(\phi,1)) + q\beta(\lambda v(0) + (1-\lambda)u(\phi,0)) - s.$$

Here, let us define the following recurrent relations corresponding to the above equations.

$$u_t(\phi,0) = \beta u_{t-1}(\phi,0) + \lambda\beta(v_{t-1}(0) - u_{t-1}(\phi,0)) - s + r,$$

$$\begin{aligned} u_t(\phi,1) &= (1-q)\beta u_{t-1}(\phi,1) + q\beta u_{t-1}(\phi,0) \\ &\quad + \lambda(1-q)\beta(v_{t-1}(1) - u_{t-1}(\phi,1)) + \lambda q\beta(v_{t-1}(0) - u_{t-1}(\phi,0)) - s. \end{aligned}$$

Accordingly, noting $T(h_{it}(r)) = v_t(i) - u_t(\phi,i)$ in Eq. (A.9.6), from Eq. (A.9.5) with $i = 0$ we have

$$\begin{aligned} h_{0t}(r) &= (1-q)\beta h_{0,t-1}(r) - \lambda(1-q)\beta T(h_{1,t-1}(r)) + \lambda(1-q)\beta T(h_{0,t-1}(r)) + r \\ &= (1-q)M(h_{0,t-1}(r)) - (1-q)L(h_{1,t-1}(r)) + r, \end{aligned}$$

which is nondecreasing in $r \in \mathcal{R}_0^+$ from Lemmas 3.2.4 and 3.2.3(a).

Now, since $h_{0t}(r)$ is continuous in $r \in \mathcal{R}_0^- \cup \mathcal{R}_0^+$, it eventually follows that $h_{0t}(r)$ is nondecreasing in $r \geq 0$.

2. *Proof of the monotonicity of $h_i(r)$ in r for $1 \leq i \leq n-2$.* From Eqs. (A.9.2) and (A.9.5) for $1 \leq i \leq n-2$ we have

$$\begin{aligned} h_{it}(r) &= (1-q)\beta u_{t-1}(\phi,i) + q\beta u_{t-1}(\phi,i-1) \\ &\quad + \max\{\lambda(1-q)\beta(v_{t-1}(i) - u_{t-1}(\phi,i)) + \lambda(1-q)\beta(v_{t-1}(i-1) - u_{t-1}(\phi,i-1)) - s, 0\} \\ &\quad -(1-q)\beta u_{t-1}(\phi,i+1) - q\beta u_{t-1}(\phi,i) \\ &\quad - \max\{\lambda(1-q)\beta(v_{t-1}(i+1) - u_{t-1}(\phi,i+1)) + \lambda(1-q)\beta(v_{t-1}(i) - u_{t-1}(\phi,i)) - s, 0\} \\ &= (1-q)\beta h_{i,t-1}(r) + q\beta h_{i-1,t-1}(r) + \max\{(1-q)L(h_{i,t-1}(r)) + qL(h_{i-1,t-1}(r)), 0\} \\ &\quad - \max\{(1-q)L(h_{i+1,t-1}(r)) + qL(h_{i,t-1}(r)), 0\}. \end{aligned} \tag{A.10.1}$$

Accordingly, from Eq. (A.10.1) we obtain, for any $r < r'$,

$$\begin{aligned} & h_{it}(r) - h_{it}(r - ') \\ &\leq (1-q)\beta\big(h_{i,t-1}(r) - h_{i,t-1}(r - ')\big) + q\beta\big(h_{i-1,t-1}(r) - h_{i-1,t-1}(r - ')\big) \\ &+ \max\{(1-q)\big(L(h_{i,t-1}(r)) - L(h_{i,t-1}(r - '))\big) + q\big(L(h_{i-1,t-1}(r)) - L(h_{i-1,t-1}(r - '))\big), 0\} \\ &+ \max\{(1-q)\big(L(h_{i+1,t-1}(r - ')) - L(h_{i+1,t-1}(r))\big) + q\big(L(h_{i,t-1}(r - ')) - L(h_{i,t-1}(r))\big), 0\} \end{aligned}$$

Since $L(h_{i,t-1}(r - ')) \leq L(h_{i,t-1}(r))$ for all i due to the induction hypothesis and Lemma 3.2.3(a), we have

$$
\begin{aligned}
h_{it}\ (r)\ & -h_{it}(r-\ell) \\
&\le (1-q)\Big(\big(\beta h_{i,t-1}(r)+L(h_{i,t-1}(r))\big)-\big(\beta h_{i,t-1}(r-\ell)+L(h_{i,t-1}(r-\ell))\big)\Big) \\
&\quad +q\Big(\big(\beta h_{i-1,t-1}(r)+L(h_{i-1,t-1}(r))\big)-\big(\beta h_{i-1,t-1}(r-\ell)+L(h_{i-1,t-1}(r-\ell))\big)\Big) \\
&= (1-q)\Big(M(h_{i,t-1}(r))-M(h_{i,t-1}(r-\ell))\Big)+q\Big(M(h_{i-1,t-1}(r))-M(h_{i-1,t-1}(r-\ell))\Big)\le 0
\end{aligned}
$$

due to Lemma 3.2.4. Hence, $h_{it}(r)$ in nondecreasing in r for $1 \le i \le n-2$.

3. *Proof of the monotonicity of $h_{n-1}(r)$ in r.* From Eqs. (A.9.2) with $i = n-1$, (A.9.3), and (A.9.6) we obtain

$$
\begin{aligned}
h_{n-1,t}(r) &= (1-q)\beta h_{n-1,t-1}(r)+q\beta h_{n-2,t-1}(r) \\
&\quad +\max\{(1-q)L(h_{n-1,t-1}(r))+qL(h_{n-2,t-1}(r)),0\} \\
&\quad -\max\{qL(h_{n-1,t-1}(r))-(1-q)s,0\}.
\end{aligned}
$$

Accordingly, in almost the same way as the proof of $1 \le i < n-1$, from Eq. (A.10.2) we can also prove $h_{n-1,t}(r) \le h_{n-1,t}(r-\ell)$, i.e., $h_{n-1,t}(r)$ is nondecreasing in r.

From all the above it eventually follows that $h_i(r)$ is nondecreasing in r for $0 \le i < n$. ∎

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Index

Terminology

Parameter and Variable

Function

Solution

Decision and Action

Optimal Equation

Set