Contributions to the Additive Theory of Numbers

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To my wife, Rie.

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Notation

The letter p stands for a prime number, $s = \sigma + it$ is a complex variable with $\sigma = \Re s$ and $t = \Im s$. The letter ε indicates a positive constant normally to be regarded as being small, and not necessarily the same at each occurence.

The parenthesises (,) denote the greatest common divisor. The square brackets [,] denote the least common multiple. (We also use square brackets to denote intervals as usual.)

For a set S, we denote the cardinality of S by $\sharp S$. For an interval I, we denote the length of I by $|I|$.

We also use the following symbols.

$\mathbb{N}$: the set of all natural numbers,

$\mathbb{Z}$: the integer ring,

$\mathbb{Q}$: the rational number field,

$\mathbb{R}$: the real number field,

$e(\alpha) := e^{2\pi i\alpha}$,

$\Gamma(s)$: the gamma function,

$\zeta(s)$: the Riemann zeta function,

$\Lambda(n)$: the von Mangoldt function, that is

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ with } m \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

$\varphi(n)$: Euler's totient function,

$\mu(n)$: the Möbius function,

$c_q(n)$: the Ramanujan sum, that is

$$c_q(n) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\Big(\frac{a}{q}n\Big),$$

$\tau_r(n)$: the number of ways of writing n as the product of r natural numbers, that is

$$\tau_r(n) = \Big\{ \ (m_1, \dots, m_r) \ ; \prod_{j=1}^{r} m_j = n, \quad m_j \in \mathbb{N} \Big\},$$

$\tau(n)$: the divisor function, namely $\tau(n) = \tau_2(n)$.

§1. Introduction

In this dissertation we treat several important problems in additive number theory.

In the first place, we consider the distribution of prime k-tuplets in arithmetic progressions. Let $k \ge 2$ be an integer, and let a_j, b_j $(j = 0, 1, \dots, k-1)$ be $2k$ integers. We call $\{a_0 n + b_0, \dots, a_{k-1} n + b_{k-1}\}$ as "prime k-tuplets", if all the numbers $a_j n + b_j$ $(j = 0, \dots, k-1)$ are primes. We can regard problems concerned with prime k-tuplets as generalization of the prime twins problem and the binary Goldbach problem.

In the study of the distribution of primes in arithmetic progressions, it is important to estimate the "error term"

$$E(x; q, a) = \sum_{\substack{n \le x \\ n \equiv a \pmod q}} \Lambda(n) \quad - \quad \frac{x}{\varphi(q)}$$

for co-prime integers q and a. In 1965 Bombieri established the important inequality (see [**7**, Ch 28] for example)

$$\sum_{q \le x^{1/2} (\log x)^{-B}} \max_{(a,q)=1} \max_{z \le x} |E(z; q, a)| \ll x (\log x)^{-A}$$

for any fixed constant $A > 0$ and a constant $B > 0$ depending only on A. (A.I. Vinogradov proved, independently, a slightly weaker result, and the above inequality is known as the Bombieri–Vinogradov theorem.) After a while, Barban [**3**] showed, in 1966,

$$\sum_{q \le x (\log x)^{-B}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} E(x; q, a)^2 \ll x^2 (\log x)^{-A}$$

for any fixed constant $A > 0$ and a constant $B > 0$ depending only on A. And this theorem was improved by Davenport–Halberstam [**8**], Gallagher [**11**], Montgomery [**26**] and Hooley [**16**].

The results we shall show about prime k-tuplets are analogues to these theorems. Now we fix an integer $k \ge 2$, non-zero integers $a_0, \dots, a_{k-1}$, and an integer b_0 with $(a_0, b_0) = 1$. The symbol $\mathbf{b}$ stands for a $(k-1)$-dimensional vector in $\mathbb{Z}^{k-1}$, and we set $\mathbf{b} = (b_1, \dots, b_{k-1})$. To count the number of n's in an arithmetic progression for which all $a_j n + b_j$ $(0 \le j \le k-1)$ are primes and $\le x$, we introduce the function

$$\Psi(x; \mathbf{b}; q, a) = \sum_{\substack{n \in N(x; \mathbf{b}) \\ n \equiv a \pmod q}} \prod_{j=0}^{k-1} \Lambda(a_j n + b_j),$$

where

$$N(x;\mathbf{b}) = \{t \in \mathbb{R}; 1 \le a_j t + b_j \le x \text{ for all } 0 \le j \le k-1\}.$$

On the other hand, for any prime p, let $\rho(p) = \rho(p,\mathbf{b})$ be the number of solutions of the congruence

$$\prod_{j=0}^{k-1}(a_j n + b_j) \equiv 0 \pmod{p},$$

and let

$$R(\mathbf{b}) = \prod_{0\le j\le k-1} |a_j| \prod_{0\le i<j\le k-1} |a_i b_j - a_j b_i|.$$

We see that $p \nmid R(\mathbf{b})$ implies $\rho(p) = k$. So the infinite product appearing in the definition of $\sigma(\mathbf{b};q)$, below, converges absolutely;

$$\sigma(\mathbf{b};q) = \begin{cases} \frac{1}{q}\prod_{p|q}\left(1-\frac{\rho(p)}{p}\right)^{-1}\prod_p\left(1-\frac{\rho(p)}{p}\right)(1-\frac{1}{p})^{-k} \\ \qquad\qquad (\text{if } \rho(p) < p \text{ for all } p \text{ and } R(\mathbf{b}) \ne 0), \\ 0 \qquad\qquad (\text{otherwise}), \end{cases}$$

Further we put

$$\sigma(\mathbf{b};q,a) = \begin{cases} \sigma(\mathbf{b};q) & (\text{if } (a_j a + b_j, q) = 1 \text{ for all } 0 \le j \le k-1), \\ 0 & (\text{otherwise}). \end{cases}$$

It follows at once that $\sigma(\mathbf{b};q,a) = 0$ holds, if, and only if

$$\rho(p) = p \ \text{ for some prime } p, \tag{1.1}$$

or

$$\left(\prod_{j=0}^{k-1}(a_j a + b_j)\ ,\ q\right) > 1, \tag{1.2}$$

or

$$R(\mathbf{b}) = 0. \tag{1.3}$$

Assume that (1.1) or (1.2) holds. Then, by an elementary argument, there exists an integer j, with $0 \le j \le k-1$, such that $a_j n + b_j$ takes a prime value for at most one integer n satisfying $n \equiv a \pmod{q}$, and we get easily

$$\Psi(x;\mathbf{b};q,a) \ll (\log x)^{k+1}. \tag{1.4}$$

Next, assume that (1.1) is false, and that (1.3) holds. Then, noticing that the former condition implies $(a_j, b_j) = 1$ for all j, there exist distinct integers i and

j such that we have either $a_i = a_j$, $b_i = b_j$ or $a_i = -a_j$, $b_i = -b_j$. In other words, two of k numbers $a_j n + b_j$ $(0 \le j \le k-1)$ are always the same, or at least one of these k numbers is not positive. Thus we have no interest in the case $\sigma(\mathbf{b}; q, a) = 0$.

By a heuristic evidence (see Bateman–Horn [4]), when $\sigma(\mathbf{b}; q, a) > 0$, it is expected that

$$\Psi(x; \mathbf{b}; q, a) \sim \sigma(\mathbf{b}; q, a)|N(x; \mathbf{b})|,$$

as x tends to infinity. We define

$$\mathcal{E}(x; \mathbf{b}; q, a) = \Psi(x; \mathbf{b}; q, a) - \sigma(\mathbf{b}; q, a)|N(x; \mathbf{b})|,$$

and

$$Z(x) = \{\mathbf{b} \in \mathbb{Z}^{k-1};\ |N(x; \mathbf{b})| \neq 0\}.$$

We estimate the following sums $\mathcal{E}_1(x; Q)$ and $\mathcal{E}_2(x; Q)$;

$$\mathcal{E}_1(x; Q) = \sum_{q \le Q} \max_{1 \le a \le q} \max_{z \le x} \sum_{\mathbf{b} \in Z(z)} |\mathcal{E}(z; \mathbf{b}; q, a)|,$$

and

$$\mathcal{E}_2(x; Q) = \sum_{q \le Q} \sum_{a=1}^{q} \sum_{\mathbf{b} \in Z(x)} \mathcal{E}(x; \mathbf{b}; q, a)^2.$$

As a generalization of the Bombieri–Vinogradov theorem, we expect

$$\mathcal{E}_1(x; Q) \ll x^k (\log x)^{-A} \tag{1.5}$$

for any fixed $A > 0$ and for the parameter Q in a certain range. In 1990, Maier and Pomerance [23] showed that the inequality (1.5) is valid for $k = 2$ and for $Q \le x^\delta$ with some (small) positive constant δ. And they applied this result to their argument about the difference between consecutive primes. Later, Balog [1] established (1.5) for the general case $k \ge 2$ and for $Q \le x^{1/3}(\log x)^{-B}$ with a constant $B > 0$ depending only on k and A. He also stated that this result is applicable to some interesting problems for primes (see [2]).

Recently, Mikawa [25] obtained (1.5) for $Q \le x^{1/2}(\log x)^{-B}$ in the case $k = 2$, by means of the dispersion method, where $B > 0$ is a constant depending only on A. We prove the same result for the general case $k \ge 2$ by the traditional circle method which is also called as the Hardy–Littlewood method.

THEOREM 1. *For any fixed $A > 0$, we have*

$$\mathcal{E}_1(x; Q) \ll x^k (\log x)^{-A}$$

providing

$$Q \le x^{1/2}(\log x)^{-B},$$

where the constant B depends only on k and A.

By contrast with the Bombieri–Vinogradov theorem, the range of Q in Mikawa's result and our Theorem 1 seems the best possible for the present.

We turn to $\mathcal{E}_2(x;Q)$. According to Montgomery [26] and Hooley [16], we estimate $\mathcal{E}_2(x;Q)$ asymptotically. We put

$$a_* = \max_{0\le j\le k-1} |a_j|,$$

and obtain the following Theorem 2.

THEOREM 2. *Let $A > 0$ and $B > 1$ be arbitrary constants.*

(I) For $Q < x/a_*$, we have

$$\mathcal{E}_2(x;Q) = \frac{1}{\varphi(|a_0|)} x^k Q(\log x - 1)^k - x^k Q \sum_{m=0}^{k} \xi_m \big(\log \frac{x}{Qa_*}\big)^m + \tag{1.6}$$
$$+ O\left(x^{k-\frac{1}{k+2}} Q^{\frac{k+3}{k+2}} + x^{k+1}(\log x)^{-A}\right),$$

where $\xi_0,\ \xi_1, \ldots, \xi_k$ are constants depending only on $a_0, \ldots, a_{k-1}$. Especially,

$$\xi_k = \frac{1}{\varphi(|a_0|)k!}. \tag{1.7}$$

(II) For $x/a_* \le Q \le x^B$, we have

$$\mathcal{E}_2(x;Q) = \frac{1}{\varphi(|a_0|)} x^k Q(\log x - 1)^k - \eta_0 x^{k+1} (\log \frac{Qa_*}{x}) +$$
$$+ \eta_1 x^{k+1} + O\left(x^k Q(\log x)^{-A}\right),$$

where η_0 and η_1 are constants depending only on $a_0, \ldots, a_{k-1}$. Further, we can write η_0 explicitly. Let $g(p)$ be the number of a_j's such that $a_j \equiv 0 \pmod p$,

$$f_1(p) = \big(1 - \frac{1}{p}\big)^{-k} \left\{ \big(1-\frac{1}{p}\big)^2 \big(\frac{p-2}{p-1}\big)^{k-g(p)} + \frac{2}{p} - \frac{1}{p^2} \right\},$$

and let

$$\Omega = \int_0^{1/a_*} \prod_{j=0}^{k-1} (1 - |a_j|u)\, du.$$

Then,

$$\eta_0 = \frac{2}{\varphi(|a_0|)}\Omega \prod_p f_1(p). \tag{1.8}$$

We note that the formula (1.6) yields a non-trivial bound for $\mathcal{E}_2(x;Q)$, namely

$$\mathcal{E}_2(x;Q) \ll x^{k+1}(\log x)^{-A}$$

providing $Q \ll x(\log x)^{-A-k}$.

It is easy to obtain the same results for short intervals. We just change the tools used in our proofs of Theorems 1 and 2.

For an interval I, we define

$$\begin{aligned} N(I;\mathbf{b}) &= \{t \in \mathbb{R};\ a_j t + b_j \in I \text{ for all } 0 \le j \le k-1\}, \\ Z(I) &= \{\mathbf{b} \in \mathbb{Z}^{k-1};\ |N(I;\mathbf{b})| \neq 0\ \}, \\ \Psi(I;\mathbf{b};q,a) &= \sum_{\substack{n \in N(I;\mathbf{b}) \\ n \equiv a \pmod{q}}} \prod_{j=0}^{k-1} \Lambda(a_j n + b_j), \end{aligned}$$

and

$$\mathcal{E}(I;\mathbf{b};q,a) = \Psi(I;\mathbf{b};q,a) - \sigma(\mathbf{b};q,a)|N(I;\mathbf{b})|.$$

Especially, we write

$$\begin{aligned} Z(x,y) &= Z([x-y,x]), \\ N(x,y;\mathbf{b}) &= N([x-y,x];\mathbf{b}), \\ \mathcal{E}(x,y;\mathbf{b};q,a) &= \mathcal{E}([x-y,x];\mathbf{b};q,a). \end{aligned}$$

And we introduce

$$\mathcal{E}_1(x,y;Q) = \sum_{q \le Q} \max_{1 \le a \le q} \max_{I \subset [x-y,x]} \sum_{\mathbf{b} \in Z(I)} |\mathcal{E}(I;\mathbf{b};q,a)|,$$

where I runs over all intervals in $[x-y,x]$, and

$$\mathcal{E}_2(x,y;Q) = \sum_{q \le Q} \sum_{a=1}^{q} \sum_{\mathbf{b} \in Z(x,y)} \mathcal{E}(x,y;\mathbf{b};q,a)^2.$$

Then we have

THEOREM 3. *Let $A > 0$ be fixed constant, and assume that*

$$x^{2/3}(\log x)^C < y \le x$$

with some positive constant C depending only on k and A. Then we have

$$\mathcal{E}_1(x, y; Q) \ll y^k (\log x)^{-A}$$

providing

$$Q \le y x^{-1/2} (\log x)^{-B},$$

where the constant B depends only on k and A.

THEOREM 4. *Let the constants A, B, η_0, η_1 and ξ_m's be the same as in Theorem 2. Assume that*

$$x^{2/3}(\log x)^C < y \le x,$$

where the constant C is the same as in Theorem 3.

(I) For $Q < y/a_*$, we have

$$(1.9)\quad \mathcal{E}_2(x, y; Q) = \frac{1}{\varphi(|a_0|)} y^k Q(\log x - 1)^k - y^k Q \sum_{m=0}^{k} \xi_m \left(\log \frac{y}{Q a_*}\right)^m + O\left(y^{k - \frac{1}{k+2}} Q^{\frac{k+3}{k+2}} + y^{k+1} (\log x)^{-A}\right).$$

(II) For $y/a_* \le Q \le x^B$, we have

$$(1.10)\quad \mathcal{E}_2(x, y; Q) = \frac{1}{\varphi(|a_0|)} y^k Q(\log x - 1)^k - \eta_0 y^{k+1} \left(\log \frac{Q a_*}{y}\right) + \eta_1 y^{k+1} + O\left(y^k Q (\log x)^{-A}\right),$$

By the formula (1.9), we obtain

$$\mathcal{E}(x, y; Q) \ll y^{k+1} (\log x)^{-A}$$

for $Q \le y(\log x)^{-A-k}$.

To compare with Theorems 3 and 4, we mention the corresponding results for primes. Putting

$$E(x, y; q, a) = \sum_{\substack{x-y<n\le x \\ n\equiv a \ (\mathrm{mod}\ q)}} \Lambda(n) - \frac{y}{\varphi(q)},$$

we have

$$\sum_{q\le Q} \max_{(a,q)=1} |E(x, y; q, a)| \ll y(\log x)^{-A}$$

providing

$$Q \leq yx^{-1/2}(\log x)^{-B} \quad \text{and} \quad x^{3/5}(\log x)^{C} < y \leq x,$$

where $A > 0$ is an arbitrary constant and where B and C are positive constants depending only on A (see Perelli, Pintz and Salerno [32]).

If we take $k = 1$, $a_0 = 1$ and $b_0 = 0$ formally in Theorem 4, then we get correct asymptotic formulae for the sum

$$\sum_{q \leq Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} E(x, y; q, a)^2,$$

and our proof for Theorem 4 still work correctly. But, using the large sieve method, we can prove that these asymptotic formulae are valid providing

$$x^{7/12+\varepsilon} < y \leq x.$$

We prove Theorem 3 in §§3–7, and Theorem 4 in §§8–10. Then Theorems 1 and 2 follow from Theorems 3 and 4, respectively, by taking $y = x$.

Next we consider problems on additive representation of natural numbers.

Let $k \geq 2$ be an integer, and, for a natural number n, let $r_k(n)$ be the number of representations of n as the sum of a prime number and a k-th power;

$$n = p + m^k.$$

Throughout this dissertation, a "k-th power" means k-th power of a positive integer.

We denote by $E_k(N)$ the number of natural numbers $n \leq N$ with $r_k(n) = 0$. In 1937, Davenport and Heilbronn [9] proved that

$$E_k(N) = O(N(\log N)^{-c_k})$$

with a positive constant c_k depending only on k, in other words, almost all natural numbers are representable as the sum of a prime and a k-th power. After their result, some articles established sharper bounds for $E_k(N)$, and, at present, the best result is

$$E_k(N) = O(N^{1-\delta_k})$$

with a positive constant δ_k depending only on k, which was proved by A.I. Vinogradov [41] and Brünner, Perelli, and Pintz [5] for $k = 2$, and by Plaksin [33] and Zaccagnini [44] for $k \geq 3$.

On the other hand, in the case $k = 2$, Hardy and Littlewood [**14**] conjectured that

$$r_2(n) \sim \mathfrak{S}_2(n) \frac{\sqrt{n}}{\log n}, \tag{1.11}$$

as n tends to infinity providing that n is not a perfect square, where

$$\mathfrak{S}_2(n) = \prod_{p>2} (1 - \frac{\left(\frac{n}{p}\right)}{p-1}),$$

with the Legendre symbol $(\frac{n}{p})$. If n is a persect square, say $n = n_1^2$, then we have $r_2(n) = 1$ or 0 according as $2n_1 - 1$ is a prime or not. In 1968, Miech [**24**] showed that the above asymptotic formula (1.11) is valid for almost all n. More precisely, he proved that

$$r_2(n) = \mathfrak{S}_2(n) \int_1^{\sqrt{n-2}} \frac{dx}{\log(n - x^2)} + O(\sqrt{n}(\log n)^{-B})$$

for all but $O(\ N(\log N)^{-A}\ )$ natural numbers $n \le N$ with any positive constants A and B. Here we see for $n \ge 3$

$$\int_1^{\sqrt{n-2}} \frac{dx}{\log(n - x^2)} = \frac{\sqrt{n}}{\log n} + O\left(\frac{\sqrt{n}\log\log n}{(\log n)^2} \right).$$

For each $k \ge 2$, we put

$$\mathfrak{S}_k(n) = \prod_p (1 - \frac{\rho_n(p) - 1}{p-1}),$$

where $\rho_n(d) = \rho_{n,k}(d)$ denotes the number of solutions m of the congruence

$$x^k - n \equiv 0 \ (\mathrm{mod}\ d)$$

with $1 \le m \le d$. And we define the set

$$\mathbf{E}_k = \{ n \in \mathbb{N}\,;\ \text{the polynomial } x^k - n \text{ is irreducible in } \mathbb{Q}[x]\ \}.$$

Then we can expect that

$$r_k(n) \sim \mathfrak{S}_k(n) \frac{n^{\frac{1}{k}}}{\log n}, \tag{1.12}$$

as n tends to infinity providing $n \in \mathbf{E}_k$. In the case $k = 2$, this conjecture coincides with the above Hardy–Littlewood conjecture, because $\rho_{n,2}(2) = 1$ and $\rho_{n,2}(p) = (\frac{n}{p}) + 1$ for $p > 2$. Our next purpose is to prove that, in the gengeral case $k \ge 3$, the asymptotic formula (1.12) is true for almost all n.

The essential difference between the cases $k=2$ and $k\geq 3$ occurs in the treatment of the singular series. The singular series in our problem is the sum of the following form;

$$\mathfrak{S}_k(n,Q)=\sum_{q\leq Q}\frac{\mu(q)}{\varphi(q)}\prod_{p|q}(\rho_n(p)-1). \tag{1.13}$$

In the articles [9], [33] and [44], the singular series $\mathfrak{S}_k(n,Q)$ is approximated, for almost all n, by a finite product of the form

$$\prod_{p\leq P}(1-\frac{\rho_n(p)-1}{p-1})$$

with a suitable parameter P, and it was derived from this approximation a good lower bound for $\mathfrak{S}_k(n,Q)$ which is sufficient to deduce $r_k(n)>0$. In contrast with this way, we shall show, by using Perron's formula, that $\mathfrak{S}_k(n,Q)$ is approximated by the infinite product $\mathfrak{S}_k(n)$ for almost all n.

To this end, we introduce the function

$$Z_n(s)=\prod_p\left(1-\frac{\rho_n(p)-1}{(p-1)p^{s-1}}\right),$$

for $\sigma>1$. We see in §13,

$$Z_n(s)=\frac{\zeta(s)}{\zeta_n(s)}\xi(s)\Xi(s),$$

where $\zeta(s)$ and $\zeta_n(s)$ are the Riemann zeta function and the Dedekind zeta function of the field $\mathbb{Q}(n^{1/k})$, respectively, and where $\xi_n(s)$ and $\Xi_n(s)$ are certain functions. Since the functions $\xi_n(s)$ and $\Xi_n(s)$ are quite easy to treat, we may regard essentially $Z_n(s)$ as $\zeta(s)/\zeta_n(s)$. We find here the most important difference between $k=2$ and $k\geq 3$. In the case $k=2$, the function $\zeta(s)/\zeta_n(s)$ equals to the reciprocal of the Dirichlet L function for a certain real primitive character. Therefore we can utilize known results on L functions to investigate $\mathfrak{S}_2(n,Q)$. Especially, Bombieri's zero density theorem for L functions plays essential role in Miech's treatment of $\mathfrak{S}_2(n,Q)$ (see [24]). For $k\geq 3$, however, there is not such a known result, so we need to study the zero density for $\zeta(s)/\zeta_n(s)$'s.

In §16, we obtain an estimate for the zero density, then our treatment of $\mathfrak{S}_k(n,Q)$ with $k\geq 3$ is achieved by standard application of Perron's formula. Combining this argument with the frame work of the circle method, we have the following result.

THEOREM 5. *Let $k\geq 3$ be a fixed integer, and let A, $B>0$ be arbitrary fixed* *constants. Then, for $n \leq N$, we have*

$$r_k(n) = \mathfrak{S}_k(n) \int_1^{n-2} \frac{x^{-1+\frac{1}{k}}\,dx}{k\log(n-x)} + O(n^{\frac{1}{k}}(\log n)^{-B})$$

with possible exception of $O(\ N(\log N)^{-A}\ )$ n's.

We remark that

$$\int_1^{n-2} \frac{x^{-1+\frac{1}{k}}\,dx}{k\log(n-x)} = \frac{n^{\frac{1}{k}}}{\log n} + O\left(\frac{n^{\frac{1}{k}}\log\log n}{(\log n)^2}\right),$$

for $n \geq 3$.

Because of the possible existence of the Siegel zeros, Miech's result [**24**] and our Theorem 5 seem the best possible for the present. We prove Theorem 5 in §§11–17.

Page [**29**], [**30**] and Hooley [**17**] investigated the asymptotic behaviour of the number of representations of a natural number as a sum of squares and products of two positive factors, and established an asymptotic formula for the number of representations in each case where it can exist. We consider here similar problems for cubes instead of squares. As is mentioned in Hooley [**17**, p.180], an asymptotic formula for the cases with at least five cubes and a product, or with at least two products and a cube is obtained by standard application of the circle method of Hardy and Littlewood. So we consider the number $R_k(N)$, say, of representations of a natural number N as the sum of four positive cubes and a product of k positive factors;

$$N = l_1 l_2 \ldots l_k + m_1^3 + m_2^3 + m_3^3 + m_4^3, \tag{1.14}$$

where $k \geq 2$ is a fixed integer. For the case $k = 2$, we can still apply the circle method plainly.

In 1981, Hooley [**17**] published a new approach to problems in additive number theory, different from the circle method. As a consequence of application of his method, he obtained an asymptotic formula for $R_3(N)$, namely,

$$R_3(N) = \frac{3}{8}\Gamma\left(\frac{4}{3}\right)^3 \mathfrak{S}(N) N^{\frac{4}{3}}(\log N)^2 + O\left(N^{\frac{4}{3}}(\log N)^{\frac{9}{5}}\right), \tag{1.15}$$

where $\mathfrak{S}(N)$ is what is called the singular series.

Five years later from the above memoir, Vaughan [**40**] established an asymptotic formula for the number of representations of a natural number as the sum of eight positive cubes, in a frame of the circle method. Vaughan's ingenious treatment of the minor arcs [**40**, Theorem B] enable us to apply the circle method to our $R_k(N)$ with $k \geq 3$.

For the sake of the successful use of Vaughan's method, we should treat separately those solutions of (1.14) for which m_1, m_2 and m_3 have no prime factor in a certain range. The estimation of the number of such solutions is accomplished by Wolke's result [**43**, Satz 1]. Via this way, we establish the following result which contains a refinement of (1.15).

THEOREM 6. *For $k \geq 3$, we have*

$$R_k(N) = N^{\frac{4}{3}} \sum_{j=0}^{2} \xi_k^{(j)}(N)(\log N)^{k-1-j} + O(N^{\frac{4}{3}}(\log N)^{k-4}(\log\log N)^{C_k}),$$

where

$$C_k = \frac{1}{6}k(k-1)(k+4) + 3,$$

the implied constant depends only on k, and the coefficients $\xi_k^{(j)}(N)$ are defined explicitly in §21 (see (21.8) below). In particular, we have $\xi_k^{(j)}(N) \ll 1$ for all $0 \leq j \leq 2$, and $\xi_k^{(0)}(N) \gg 1$.

We can apply our argument to a similar problem. As a direct consequence of the result due to Davenport [**6**] (see also [**40**, Theorem 4]), it follows that every sufficiently large number is representable as the sum of four positive cubes and a prime. We denote by $R_0(N)$ the number of such representations of a natural number N. In the same manner as for $R_k(N)$ with $k \geq 3$, we obtain an asymptotic formula for $R_0(N)$.

THEOREM 7. *We have*

$$R_0(N) = \Gamma\left(\frac{4}{3}\right)^3 \mathfrak{S}_0(N) \int_2^N \frac{(N-t)^{\frac{1}{3}}}{\log t}\, dt + O(N^{\frac{4}{3}}(\log N)^{-4}(\log\log N)^4),$$

where $\mathfrak{S}_0(N)$ is defined in §21 (see (21.7) below). In particular we have $1 \ll \mathfrak{S}_0(N) \ll 1$.

We note here that

$$\int_2^N \frac{(N-t)^{\frac{1}{3}}}{\log t}\, dt = \frac{3}{4}\frac{N^{\frac{4}{3}}}{\log N} + O\left(\frac{N^{\frac{4}{3}}}{(\log N)^2}\log\log N\right).$$

We prove Theorems 6 and 7 in §§18–22.

All the above Theorems come from the author's study in University of Tsukuba under guidance of Professor Saburô Uchiyama. As for the Theorems 1, 3 and 5, see [**18**] and [**19**].

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§2. Preliminaries for the proof of Theorem 3

We define three subsets in $Z(I)$;

$$Z_0(I;q,a) = \{\mathbf{b} \in Z(I);\ \sigma(\mathbf{b};q,a) \neq 0\ \},$$
$$Z_1(I;q,a) = \{\mathbf{b} \in Z(I);\ (1.1) \text{ or } (1.2) \text{ holds.}\ \},$$
$$Z_2(I;q,a) = \{\mathbf{b} \in Z(I);\ (1.3) \text{ holds.}\ \}.$$

As we note in §1, if $\sigma(\mathbf{b};q,a) = 0$ then $\mathbf{b} \in \bigcup_{\nu=1,2} Z_\nu(I;q,a)$. So we have

$$\mathcal{E}_1(x,y;Q) \le \sum_{\nu=0}^{2} \mathcal{E}_{1,\nu}(x,y;Q),$$

where

$$\mathcal{E}_{1,\nu}(x,y;Q) = \sum_{q\le Q} \max_{1\le a\le q} \max_{I\subset[x-y,x]} \sum_{\mathbf{b}\in Z_\nu(I;q,a)} |\mathcal{E}(I;\mathbf{b};q,a)|$$

for $\nu = 0,\ 1,\ 2$.

The inequality (1.4) and the trivial estimate $\sharp Z(I) \ll y^{k-1}$ yield

$$\mathcal{E}_{1,1}(x,y;Q) \ll y^{k-1}Q(\log x)^{k+1}.$$

We see easily $\sharp Z_2(I;q,a) \ll y^{k-2}$, and get

$$\mathcal{E}_{1,2}(x,y;Q) \ll y^{k-1}(\log x)^{k+1}.$$

Here we use the trivial bound $\Psi(I;\mathbf{b};q,a) \ll yq^{-1}(\log x)^k$. Therefore we obtain

$$\mathcal{E}_1(x,y;Q) \le \mathcal{E}_{1,0}(x,y;Q) + O\big(y^{k-1}Q(\log x)^{k+1}\big), \tag{2.1}$$

and we attend only to the interesting case $\sigma(\mathbf{b};q,a) \neq 0$ in the sequel.

As mentioned in §1, our proof is in the frame work of the circle method. We utilize the functions

$$P(\alpha) = P(\alpha;I) = \sum_{n\in I} \Lambda(n)e(n\alpha),$$
$$P_{q,a}(\alpha) = P_{q,a}(\alpha;I) = \sum_{\substack{a_0n+b_0\in I \\ n\equiv a \ (\mathrm{mod}\ q)}} \Lambda(a_0n+b_0)e(n\alpha).$$

We take a constant $C_1 > 0$ such that is larger than a certain number depending only on k and A, and assume that

$$x^{2/3}(\log x)^{3C_1+657} < y \le x. \tag{2.2}$$

We put

$$Q_1 = (\log x)^{C_1} \quad \text{and} \quad \Delta = y^{-1}(\log x)^{2A+2C_1+4},$$

and define the major and minor arcs;

$$\mathfrak{M}(c,q) = \left[\frac{c}{q} - \Delta,\ \frac{c}{q} + \Delta\right],$$

$$\mathfrak{M} = \bigcup_{q \le Q_1} \bigcup_{\substack{1 \le c \le q \\ (c,q)=1}} \mathfrak{M}(c,q)\,,$$

$$\mathfrak{m} = \left[x^{-1/6},\ 1 + x^{-1/6}\right] \setminus \mathfrak{M}\,.$$

We note that $\mathfrak{M}(c,q)$'s are disjoint for $q \le Q_1$, $1 \le c \le q$, $(c,q) = 1$. We also note that if $\alpha \in \mathfrak{m}$ then there exist co-prime natural numbers q and c such that

$$q \le Q_1 \quad \text{and} \quad \Delta < \left|\alpha - \frac{c}{q}\right| \le q^{-1}x^{-1/6}$$

or

$$Q_1 < q \le x^{1/6} \quad \text{and} \quad \left|\alpha - \frac{c}{q}\right| \le q^{-1}x^{-1/6}.$$

Our proof is based on the following results.

LEMMA 2.1. *Assume that* $\alpha \in \mathfrak{M}(c,q)$, $q \le Q_1$, $1 \le c \le q$, $(c,q) = 1$, *and write* $\alpha = c/q + \beta$. *Then we have*

$$P(\alpha) = \frac{\mu(q)}{\varphi(q)}T(\beta) + O\big(y\exp(-\delta_0(\log x)^{1/5})\big),$$

where δ_0 *is a positive constant and* $T(\beta) = T(\beta; I) = \sum_{n \in I} e(n\beta)$.

LEMMA 2.2. *We have*

$$\max_{\alpha \in \mathfrak{m}} |P(\alpha)| \ll y(\log x)^{-C_1+1}.$$

LEMMA 2.3. *Let*

$$E^*(x,y;q) = \max_{\substack{1 \le a \le q \\ (a,q)=1}} \max_{I \subset [x-y,x]} \left| \sum_{\substack{n \in I \\ n \equiv a \ (\mathrm{mod}\ q)}} \Lambda(n) - \frac{|I|}{\varphi(q)} \right|,$$

where I *runs over all intervals in* $[x-y, x]$. *Then, for any constant* $A_1 > 0$, *we have*

$$\sum_{q \le yx^{-1/2}(\log x)^{-B_1}} E^*(x,y;q) \ll y(\log x)^{-A_1}, \tag{2.3}$$

where $B_1 > 0$ is a constant depending only on A_1.

Lemmata 2.1 and 2.2 are minor modifications of Pan and Pan [**31**, Theorem 3 and p.146]. Their proofs are based on the results about the zeros of Dirichlet's L functions, and Lemma 2.1 was showed for $x^{7/12+\varepsilon} < y \le x$. But Lemma 2.2 is justified only for y satisfying (2.2) hitherto.

Lemma 2.3 is a Bombieri–Vinogradov type theorem for short intervals, and Perelli, Pintz and Salerno [**32**] proved this result for $y > x^{3/5+\varepsilon}$.

We set $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{k-1})$, and

$$F(\boldsymbol{\alpha}) = P_{q,a}\left(-\sum_{j=1}^{k-1} a_j\alpha_j\right)\prod_{j=1}^{k-1} P(\alpha_j),$$

then we can write

$$\begin{aligned} \Psi(I;\mathbf{b};q,a) &= \int_0^1 \cdots \int_0^1 F(\boldsymbol{\alpha})e\left(-\sum_{j=1}^{k-1} b_j\alpha_j\right) d\alpha_1 \dots d\alpha_{k-1} \\ &= \mathbf{I}_{\mathfrak{M}} + \sum_{h=1}^{k-1} \mathbf{I}_{\mathfrak{m},h}\ , \end{aligned} \tag{2.4}$$

where

$$\mathbf{I}_{\mathfrak{M}} = \int_{\mathfrak{M}} \cdots \int_{\mathfrak{M}} F(\boldsymbol{\alpha})e\left(-\sum_{j=1}^{k-1} b_j\alpha_j\right) d\alpha_1 \dots d\alpha_{k-1},$$

and, for $1 \le h \le k-1$,

$$\mathbf{I}_{\mathfrak{m},h} = \int \cdots \int_{\mathfrak{m}^{(h)}} F(\boldsymbol{\alpha})e\left(-\sum_{j=1}^{k-1} b_j\alpha_j\right) d\alpha_1 \dots d\alpha_{k-1},$$

with

$$\begin{aligned} \mathfrak{m}^{(h)} = \big\{ \boldsymbol{\alpha}; \alpha_j &\in \mathfrak{M} \text{ for } 1 \le j < h, \\ \alpha_h &\in \mathfrak{m}, \\ \alpha_j &\in [0,1] \text{ for } h < j \le k-1 \,\big\}. \end{aligned}$$

§3. Integrals on minor arcs

The purpose of this section is to prove the following Lemma 3.1.

LEMMA 3.1. *Suppose that $C_1 > 2A+k+4$. Then, for each $1 \le h \le k-1$, we have*

$$\sum_{q\le Q} \max_a \max_{I\subset[x-y,x]} \sum_{\mathbf{b}\in Z_0(I,q,a)} |\mathbf{I}_{\mathfrak{m},h}| \ll y^k(\log x)^{-C_2},$$

providing $Q \le y(\log x)^{-2A-k-2}$.

We have by the Cauchy–Schwartz inequality

$$\begin{aligned}\sum_{q\le Q} \max_a \max_{I\subset[x-y,x]} \sum_{\mathbf{b}\in Z_0(I,q,a)} |\mathbf{I}_{\mathfrak{m},h}| &\ll \left(\sum_{q\le Q}\frac{1}{q}y^{k-1}\right)^{1/2} S_h^{1/2}\\ &\ll \left(S_h y^{k-1}\log x\right)^{1/2},\end{aligned}$$

where

$$S_h = \sum_{q\le Q} q \max_a \max_{I\subset[x-y,x]} \sum_{\mathbf{b}\in Z(I)} |\mathbf{I}_{\mathfrak{m},h}|^2.$$

So, it suffices to show that

$$S_h \ll y^{k+1}(\log x)^{-2A-1}$$

for $Q \le y(\log x)^{-2A-k-2}$.

We use Bessel's inequality repeatedly, and obtain

$$\begin{aligned}&\sum_{\mathbf{b}\in Z(I)} |\mathbf{I}_{\mathfrak{m}}|^2\\ &= \sum_{b_1}\cdots\sum_{b_{k-1}}\left|\int P(\alpha_1)\left(\int\cdots\int \ldots d\alpha_2\ldots d\alpha_{k-1}\right)e(-b_1\alpha_1)\,d\alpha_1\right|^2\\ &\le \int |P(\alpha_1)|^2 \sum_{b_2}\cdots\sum_{b_{k-1}}\left|\int P(\alpha_2)\Big(\int \ldots\ldots d\alpha_{k-1}\Big)e(-b_2\alpha_2)\,d\alpha_2\right|^2 d\alpha_1\\ &\le \ddots\\ &\le \int\cdots\int_{\mathfrak{m}(h)}\left(\prod_{j=1}^{k-1}|P(\alpha_j)|^2\right)\left|P_{q,a}\left(-\sum_{j=1}^{k-1}a_j\alpha_j\right)\right|^2 d\alpha_1\ldots d\alpha_{k-1}.\end{aligned}$$

We see

$$\left|P_{q,a}\left(-\sum_{j=1}^{k-1} a_j\alpha_j\right)\right|^2$$

$$= \sum_{\substack{a_0n_1+b_0\in I\\ n_1\equiv a \pmod q}} \sum_{\substack{a_0n_2+b_0\in I\\ n_2\equiv a \pmod q}} \Lambda(a_0n_1+b_0)\Lambda(a_0n_2+b_0)e\left(-\sum_{j=1}^{k-1} a_j\alpha_j(n_1-n_2)\right)$$

$$= \sum_{\substack{|l|\le y\\ l\equiv 0 \pmod q}} \prod_{j=1}^{k-1} e(a_j\alpha_j l) \sum_{\substack{a_0n+b_0\in I\\ a_0(n+l)+b_0\in I\\ n\equiv a \pmod q}} \Lambda(a_0n+b_0)\Lambda(a_0(n+l)+b_0),$$

thus it follows uniformly in a and $I\subset[x-y,x]$ that

$$\sum_{\mathbf{b}\in Z(I)} |\mathbf{I}_{\mathfrak{m}}|^2 \le \sum_{\substack{|l|\le y\\ l\equiv 0 \pmod q}} \left(\sum_{\substack{a_0n+b_0\in I\\ a_0(n+l)+b_0\in I\\ n\equiv a \pmod q}} \Lambda(a_0n+b_0)\Lambda(a_0(n+l)+b_0)\right)\times$$

$$\times \int\cdots\int_{\mathfrak{m}^{(h)}} \prod_{j=1}^{k-1}\left(|P(\alpha_j)|^2 e(a_j\alpha_j l)\right)\,d\alpha_1\ldots d\alpha_{k-1}$$

$$\ll \frac{y}{q}(\log x)^2 \prod_{\substack{j=1\\ j\ne h}}^{k-1}\left(\int_0^1 |P(\alpha_j)|^2\,d\alpha_j\right)\times$$

$$\times \sum_{\substack{|l|\le y\\ l\equiv 0 \pmod q}} \left|\int_{\mathfrak{m}} |P(\alpha_h)|^2 e(a_h\alpha_h l)\,d\alpha_h\right|$$

$$\ll \frac{y^{k-1}}{q}(\log x)^k \sum_{\substack{|l|\le y\\ l\equiv 0 \pmod q}} \left|\int_{\mathfrak{m}} |P(\alpha)|^2 e(a_h\alpha l)\,d\alpha\right|, \tag{3.1}$$

because

$$\int_0^1 |P(\alpha)|^2\,d\alpha = \sum_{x-y<n\le x} \Lambda(n)^2 \ll y\log x.$$

Dividing the summation over l in (3.1) according as $l=0$ or not, we obtain

$$\sum_{\mathbf{b}\in Z(I)} |\mathbf{I}_{\mathfrak{m}}|^2 \ll \frac{y^k}{q}(\log x)^{k+1} + \sum_{\substack{0<l\le y\\ l\equiv 0 \pmod q}} \left|\int_{\mathfrak{m},\,a_hl}\right|,$$

where

$$\int_{\mathfrak{m},\,r} = \int_{\mathfrak{m}} |P(\alpha)|^2 e(\alpha r)\,d\alpha.$$

Therefore

$$S_h \ll y^k Q(\log x)^{k+1} + y^{k-1}(\log x)^k \sum_{q\le Q} \sum_{\substack{0<l\le y \\ l\equiv 0 \ (\mathrm{mod}\ q)}} \left|\int_{\mathfrak{m},\,a_h l}\right|$$

$$\ll y^k Q(\log x)^{k+1} + y^{k-1}(\log x)^k \sum_{0<l\le y} \tau(l) \left|\int_{\mathfrak{m},\,a_h l}\right|. \tag{3.2}$$

Since

$$\sum_{0<l\le y} \tau(l)^2 \ll y(\log y)^3,$$

we get

$$\sum_{0<l\le y} \tau(l) \left|\int_{\mathfrak{m},\,a_h l}\right|$$

$$\ll \left(\sum_{0<l\le y} \tau(l)^2\right)^{1/2} \left(\sum_{0<l\le y} \left|\int_{\mathfrak{m}} |P(\alpha)|^2 e(-a_h \alpha l)\,d\alpha\right|^2\right)^{1/2}$$

$$\ll \left(y(\log x)^3 \int_{\mathfrak{m}} |P(\alpha)|^4\,d\alpha\right)^{1/2}$$

$$\ll \left(y(\log x)^3 \max_{\alpha\in\mathfrak{m}} |P(\alpha)|^2 \int_0^1 |P(\alpha)|^2\,d\alpha\right)^{1/2}$$

$$\ll y(\log x)^2 \max_{\alpha\in\mathfrak{m}} |P(\alpha)|,$$

and this is

$$\ll y^2(\log x)^{-C_1+3},$$

by Lemma 2.2. In view of (3.2), we have

$$\begin{aligned} S_h &\ll y^k Q(\log x)^{k+1} + y^{k+1}(\log x)^{-C_1+k+3} \\ &\ll y^{k+1}(\log x)^{-2A-1}, \end{aligned}$$

as required, providing $Q \le y(\log x)^{-2A-k-2}$ and $C_1 > 2A + k + 4$.

§4. Integrals on major arcs

In this section, we evaluate $\mathbf{I}_{\mathfrak{M}}$. We use the following notation for brevity's sake: The bold face letters $\mathbf{q}$, $\mathbf{c}$ and $\mathbf{d}$ denote elements in $\mathbb{N}^{k-1}$, and we set

$$\mathbf{q} = (q_1, \ldots, q_{k-1}), \quad \mathbf{c} = (c_1, \ldots, c_{k-1}), \quad \mathbf{d} = (d_1, \ldots, d_{k-1}).$$

The symbol $\mathbf{q} \le Q$ denotes the condition

$$q_j \le Q \quad \text{for all } 1 \le j \le k-1.$$

We write

$$[\mathbf{q}] = [q_1, \ldots, q_{k-1}] \quad \text{and} \quad [q, \mathbf{q}] = [q, q_1, \ldots, q_{k-1}],$$

and so on. The symbols

$$\sum_{\mathbf{d}}^{\mathbf{q}} \quad \text{and} \quad \sum_{\mathbf{c}}^{\mathbf{q}}{}^{*}$$

denote, respectively, the summation over all the $\mathbf{d}$'s with $1 \le d_j \le q_j$ for all $1 \le j \le k-1$, and the summation over all the $\mathbf{c}$'s satisfying

$$1 \le c_j \le q_j \quad \text{and} \quad (c_j, q_j) = 1 \quad \text{for all } 1 \le j \le k-1.$$

By the definition, we write

$$\mathbf{I}_{\mathfrak{M}} = \sum_{\mathbf{q} \le Q_1} \sum_{\mathbf{c}}^{\mathbf{q}}{}^{*} \int \cdots \int_{\prod_{j=1}^{k-1} \mathfrak{M}(q_j, c_j)} F(\boldsymbol{\alpha}) e\left(-\sum_{j=1}^{k-1} b_j \alpha_j\right) d\alpha_1 \ldots d\alpha_{k-1}.$$

For

$$\boldsymbol{\alpha} \in \prod_{j=1}^{k-1} \mathfrak{M}(q_j, c_j),$$

we put $\beta_j = \alpha_j - c_j/q_j$ and

$$\boldsymbol{\beta} = (\beta_1, \ldots, \beta_{k-1}).$$

Then, by Lemma 2.1, we have

$$\mathbf{I}_{\mathfrak{M}} = \sum_{\mathbf{q} \le Q_1} \sum_{\mathbf{c}}^{\mathbf{q}}{}^{*} \prod_{j=1}^{k-1} \left(\frac{\mu(q_j)}{\varphi(q_j)} e(-\frac{c_j}{q_j} b_j)\right) \mathbf{J}(\mathbf{b}; \mathbf{q}, \mathbf{c}) +$$

$$+ O\left(\frac{y}{q} \exp(-\frac{\delta_0}{2}(\log x)^{\frac{1}{5}})\right),$$

where

$$\mathbf{J}(\mathbf{b}; \mathbf{q}, \mathbf{c}) = \int_{-\Delta}^{\Delta} \cdots \int_{-\Delta}^{\Delta} \widetilde{F}(\boldsymbol{\beta}; \mathbf{b}; \mathbf{q}, \mathbf{c})\, d\beta_1 \ldots \beta_{k-1},$$

with

$$\widetilde{F}(\boldsymbol{\beta};\mathbf{b};\mathbf{q},\mathbf{c}) = \prod_{j=1}^{k-1}\big(T(\beta_j)e(-b_j\beta_j)\big)P_{q,a}\left(-\sum_{j=1}^{k-1}a_j\big(\frac{c_j}{q_j}+\beta_j\big)\right).$$

Let

$$\mathbf{J}_0(\mathbf{b};\mathbf{q},\mathbf{c}) = \int_0^1\cdots\int_0^1 \widetilde{F}(\boldsymbol{\beta};\mathbf{b};\mathbf{q},\mathbf{c})\,d\beta_1\ldots\beta_{k-1},$$

and let, for $1\le h\le k-1$

$$\mathbf{J}_h(\mathbf{b};\mathbf{q},\mathbf{c}) = \int\cdots\int_{\mathfrak{n}^{(h)}} \widetilde{F}(\boldsymbol{\beta};\mathbf{b};\mathbf{q},\mathbf{c})\,d\beta_1\ldots\beta_{k-1},$$

where

$$\begin{aligned}\mathfrak{n}^{(h)} = \big\{\boldsymbol{\beta}\,;\,&\beta_j\in[-\Delta,\Delta]\ \text{ for } 1\le j<h\,,\\ &\beta_h\in[\Delta,1-\Delta]\,,\\ &\beta_j\in[0,1]\ \text{ for } h<j\le k-1\,\big\}.\end{aligned}$$

Since $\mathbf{J}(\mathbf{b};\mathbf{q},\mathbf{c}) = \mathbf{J}_0(\mathbf{b};\mathbf{q},\mathbf{c}) - \sum_{h=1}^{k-1}\mathbf{J}_h(\mathbf{b};\mathbf{q},\mathbf{c})$, we have

$$\mathbf{I}_{\mathfrak{M}} = \mathbf{I}_{\mathfrak{M},0} - \sum_{h=1}^{k-1}\mathbf{I}_{\mathfrak{M},h} + O\left(\frac{y}{q}\exp\big(-\frac{\delta_0}{2}(\log x)^{\frac{1}{5}}\big)\right), \tag{4.1}$$

where

$$\mathbf{I}_{\mathfrak{M},h} = \sum_{\mathbf{q}\le Q_1}\sum_{\mathbf{c}}^{\mathbf{q}}{}^{*}\prod_{j=1}^{k-1}\left(\frac{\mu(q_j)}{\varphi(q_j)}e\big(-\frac{c_j}{q_j}b_j\big)\right)\mathbf{J}_h(\mathbf{b};\mathbf{q},\mathbf{c}),$$

for $0\le h\le k-1$.

As we did in the preceding section, we use Bessel's inequality repeatedly, and obtain for $1\le h\le k-1$

$$\begin{aligned}&\sum_{\mathbf{b}\in Z(I)}|\mathbf{J}_h(\mathbf{b};\mathbf{q},\mathbf{c})|^2\\
&\le \int\cdots\int_{\mathfrak{n}^{(h)}}\prod_{j=1}^{k-1}|T(\beta_j)|^2\left|P_{q,a}\left(-\sum_{j=1}^{k-1}a_j\big(\frac{c_j}{q_j}+\beta_j\big)\right)\right|^2 d\beta_1\ldots d\beta_{k-1}\\
&\ll \big(\frac{y}{q}\log x\big)^2\prod_{\substack{j=1\\ j\ne h}}^{k-1}\left(\int_0^1|T(\beta_j)|^2\,d\beta_j\right)\int_\Delta^{1-\Delta}|T(\beta_h)|^2\,d\beta_h\\
&\ll \frac{y^k}{q^2}(\log x)^2\int_\Delta^{1-\Delta}|T(\beta)|^2\,d\beta\end{aligned}$$

It follows from the well-known estimate $|T(\beta)| \ll ||\beta||^{-1}$ that

$$\int_{\Delta}^{1-\Delta} |T(\beta)|^2 \, d\beta \ll \int_{\Delta}^{\frac{1}{2}} \beta^{-2} \, d\beta \ll \Delta^{-1},$$

thus we get

$$\sum_{\mathbf{b} \in Z(I)} |\mathbf{J}_h(\mathbf{b}; \mathbf{q}, \mathbf{c})|^2 \ll \frac{y^k}{q^2} (\log x)^2 \Delta^{-1}$$

uniformly in $\mathbf{q}$, $\mathbf{c}$, a and $I \subset [x-y, x]$. Making use of this inequality, we have

$$\begin{aligned}
&\sum_{\mathbf{b} \in Z(I)} |\mathbf{I}_{\mathfrak{M}}|^2 \\
&\quad \ll \left(\sum_{\mathbf{q} \le Q_1} \sum_{\mathbf{c}}^{\mathbf{q}}{}^{*} \prod_{j=1}^{k-1} \frac{1}{\varphi(q_j)} \right) \left(\sum_{\mathbf{q} \le Q_1} \sum_{\mathbf{c}}^{\mathbf{q}}{}^{*} \prod_{j=1}^{k-1} \frac{1}{\varphi(q_j)} \sum_{\mathbf{b} \in Z(I)} |\mathbf{J}_h(\mathbf{b}; \mathbf{q}, \mathbf{c})|^2 \right) \\
&\quad \ll \frac{y^k}{q^2} (\log x)^2 \Delta^{-1} Q_1^2,
\end{aligned}$$

hence

$$\begin{aligned}
&\sum_{q \le Q} \max_a \max_{I \subset [x-y,x]} \sum_{\mathbf{b} \in Z_0(I; q, a)} |\mathbf{I}_{\mathfrak{M}}| \\
&\qquad \ll \left(\sum_{q \le Q} \frac{1}{q} y^{k-1} \right)^{\frac{1}{2}} \left(\sum_{q \le Q} q \max_a \max_{I \subset [x-y,x]} \sum_{\mathbf{b} \in Z(I)} |\mathbf{I}_{\mathfrak{M}}|^2 \right)^{\frac{1}{2}} \\
&\qquad \ll (y^{2k-1} \Delta^{-1})^{\frac{1}{2}} (\log x)^2 Q_1 \\
&\qquad \ll y^k (\log x)^{-A}, \qquad\qquad (4.2)
\end{aligned}$$

for each $1 \le h \le k-1$.

We turn to $\mathbf{I}_{\mathfrak{M},0}$. By simple calculation, we have

$$\mathbf{J}_0(\mathbf{b}; \mathbf{q}, \mathbf{c}) = \sum_{\substack{n \in N(I; \mathbf{b}) \\ n \equiv a \pmod{q}}} \Lambda(a_0 n + b_0) e \left(-\sum_{j=1}^{k-1} \frac{c_j}{q_j} a_j n \right).$$

We divide this sum on residue classes of n to moduli q_j's, and write

$$\mathbf{J}_0(\mathbf{b}; \mathbf{q}, \mathbf{c}) = \sum_{\mathbf{d}}^{\mathbf{q}} e \left(-\sum_{j=1}^{k-1} \frac{c_j}{q_j} a_j d_j \right) \Psi(\mathbf{q}, \mathbf{d}),$$

where

$$\Psi(\mathbf{q},\mathbf{d}) = \sum_{\substack{n\in N(I;\mathbf{b})\\ n\equiv a \pmod q\\ n\equiv d_j \pmod{q_j}\ (\text{for } 1\le j\le k-1)}} \Lambda(a_0 n + b_0).$$

Now we define the set

$$\begin{aligned} D(q,a;\mathbf{q}) = \{\mathbf{d}\,;\, a &\equiv d_j \pmod{(q,q_j)} \text{ for all } 1\le j\le k-1,\\ d_i &\equiv d_j \pmod{(q_i,q_j)} \text{ for all } 1\le i<j\le k-1,\\ (a_0 d_j &+ b_0, q_j) = 1 \text{ for all } 1\le j\le k-1\}. \end{aligned}$$

Then, for $\mathbf{d}\notin D(q,a;\mathbf{q})$, we see plainly

$$\Psi(\mathbf{q},\mathbf{d}) \ll (\log x)^2.$$

If $\mathbf{d}\in D(q,a;\mathbf{q})$ then there is an integer $n_0 = n_0(q,a;\mathbf{q},\mathbf{d})$ such that

$$n\equiv n_0 \pmod{[q,\mathbf{q}]} \Longleftrightarrow \begin{cases} n\equiv a \pmod q & \text{and}\\ n\equiv d_j \pmod{q_j} & \text{for all } 1\le j\le k-1. \end{cases}$$

Noticing $(a_0 n_0 + b_0, [q,\mathbf{q}]) = 1$ and $(a_0,b_0)=1$, we get

$$\begin{aligned} \Psi(\mathbf{q},\mathbf{d}) &= \sum_{\substack{n\in N(I;\mathbf{b})\\ n\equiv n_0 \pmod{[q,\mathbf{q}]}}} \Lambda(a_0 n + b_0)\\ &= \sum_{\substack{(m-b_0)/a_0\in N(I;\mathbf{b})\\ m\equiv a_0 n_0 + b_0 \pmod{|a_0|[q,\mathbf{q}]}}} \Lambda(m)\\ &= \frac{|a_0||N(I;\mathbf{b})|}{\varphi(|a_0|[q,\mathbf{q}])} + O\big(E^*(|a_0|x, |a_0|y; |a_0|[q,\mathbf{q}])\big). \end{aligned}$$

Therefore we obtain

$$\begin{aligned} \mathbf{J}_0(\mathbf{b};\mathbf{q},\mathbf{c}) = \frac{|a_0||N(I;\mathbf{b})|}{\varphi(|a_0|[q,\mathbf{q}])} \sum_{\mathbf{d}\in D(q,a;\mathbf{q})} e\left(-\sum_{j=1}^{k-1}\frac{c_j}{q_j}a_j d_j\right) +&\\ + O\Big(\big(E^*(|a_0|x, |a_0|y; |a_0|[q,\mathbf{q}]) + (\log x)^2\big)Q_1^{k-1}\Big)&, \end{aligned}$$

and

$$\mathbf{I}_{\mathfrak{M},0} = |a_0||N(I;\mathbf{b})| \sum_{\mathfrak{q}\le Q_1} \frac{1}{\varphi(|a_0|[q,\mathfrak{q}])} \prod_{j=1}^{k-1} \frac{\mu(q_j)}{\varphi(q_j)} \times$$

$$\times \sum_{\mathbf{d}\in D(q,a;\mathfrak{q})} \sum_{\mathbf{c}}^{\mathfrak{q}}{}^{*} \prod_{j=1}^{k-1} e(-\frac{c_j}{q_j}(a_j d_j + b_j)) +$$

$$+ O\left(Q_1^{k-1} \sum_{\mathfrak{q}\le Q_1} E^*(|a_0|x, |a_0|y; |a_0|[q,\mathfrak{q}]) + Q_1^{2(k-1)}(\log x)^2 \right)$$

$$= |a_0||N(I;\mathbf{b})| S(\mathbf{b};q,a) + \tag{4.3}$$

$$+ O\left(Q_1^{k-1} \sum_{\mathfrak{q}\le Q_1} E^*(|a_0|x, |a_0|y; |a_0|[q,\mathfrak{q}]) + Q_1^{2(k-1)}(\log x)^2 \right),$$

where

$$S(\mathbf{b};q,a) = \sum_{\mathfrak{q}\le Q_1} \frac{1}{\varphi(|a_0|[q,\mathfrak{q}])} \prod_{j=1}^{k-1} \frac{\mu(q_j)}{\varphi(q_j)} \sum_{\mathbf{d}\in D(q,a;\mathfrak{q})} \prod_{j=1}^{k-1} c_{q_j}(a_j d_j + b_j). \tag{4.4}$$

By virtue of Lemma 2.3, we have

$$\sum_{q\le Q} \sum_{\mathfrak{q}\le Q_1} E^*(|a_0|x, |a_0|y; |a_0|[q,\mathfrak{q}])$$

$$\le \sum_{r\le |a_0|QQ_1^{k-1}} E^*(|a_0|x, |a_0|y; r) \sum_{\substack{q\le Q \\ \mathfrak{q}\le Q_1 \\ |a_0|[q,\mathfrak{q}]=r}} 1$$

$$\ll Q_1^{2k} \sum_{r\le |a_0|QQ_1^{k-1}} E^*(|a_0|x, |a_0|y; r)$$

$$\ll y(\log x)^{-A-C_1(k-1)}, \tag{4.5}$$

providing that $|a_0|QQ_1^{k-1} \le yx^{-1/2}(\log x)^{-B_1}$, that is

$$Q \le yx^{-1/2}|a_0|^{-1}(\log x)^{-B_1-C_1(k-1)},$$

where B_1 is the constant in (2.3) of Lemma 2.3 corresponding to $A_1 = A + C_1(3k-1)$. The restriction of Q in our Theorem 3 comes from here.

By (4.1), (4.2), (4.3) and (4.5), we come to the following conclusion;

LEMMA 4.1. *Putting*

$$\mathbf{I}_{\mathfrak{M}} = |a_0||N(I;\mathbf{b})|S(\mathbf{b};q,a) + \widetilde{\mathcal{E}}(I;\mathbf{b};q,a),$$

we have

$$\sum_{q\le Q} \max_{a} \max_{I\subset[x-y,x]} \sum_{\mathbf{b}\in Z_0(I;q,a)} |\widetilde{\mathcal{E}}(I;\mathbf{b};q,a)| \ll y^k(\log x)^{-A},$$

providing

$$Q \le yx^{-\frac{1}{2}}(\log x)^{-B},$$

where B is a suitable positive constant depending on k and A.

§5. The Singular Series

In this section, we evaluate $S(\mathbf{b}; q, a)$. We put

$$W(r) = \sum_{[\mathbf{q}]=r} \prod_{j=1}^{k-1} \frac{\mu(q_j)}{\varphi(q_j)} \sum_{\mathbf{d}\in D(q,a;\mathbf{q})} \prod_{j=1}^{k-1} c_{q_j}(a_j d_j + b_j),$$

and

$$W_1(r) = \sum_{\substack{[\mathbf{q}]=r \\ \mathbf{q}\leq Q_1}} \prod_{j=1}^{k-1} \frac{\mu(q_j)}{\varphi(q_j)} \sum_{\mathbf{d}\in D(q,a;\mathbf{q})} \prod_{j=1}^{k-1} c_{q_j}(a_j d_j + b_j).$$

Then, by the definition (4.4),

$$\begin{aligned} S(\mathbf{b}; q, a) &= \sum_{r\leq Q_1} \frac{\mu(r)^2}{\varphi(|a_0|[q,r])} W(r) + \sum_{Q_1 < r \leq Q_1^{k-1}} \frac{\mu(r)^2}{\varphi(|a_0|[q,r])} W_1(r) \\ &= S_0 + S_1, \quad \text{say.} \end{aligned} \tag{5.1}$$

A simple arithmetical deduction shows that $W(r)$ is a multiplicative function. Indeed, suppose that $r = r_1 r_2$ and $(r_1, r_2) = 1$. For $\mathbf{q}$ satisfying $[\mathbf{q}] = r$, we put

$$q_j^{(i)} = (q_j, r_i) \quad \text{and} \quad \mathbf{q}_i = (q_1^{(i)}, \ldots, q_{k-1}^{(i)})$$

for $i = 1, 2$. And for $\mathbf{d} \in D(q, a; \mathbf{q})$, we write

$$d_j = e_j^{(1)} q_j^{(2)} + e_j^{(2)} q_j^{(1)},$$

where $e_j^{(i)}$ runs through residue classes of modulo $q_j^{(i)}$ for $i = 1, 2$ and $1 \leq j \leq k-1$. We note, for $1 \leq i < j \leq k-1$,

$$d_j \equiv a \pmod{(q, q_j)} \iff \begin{cases} e_j^{(1)} q_j^{(2)} \equiv a \pmod{(q_j^{(1)}, q)} & \text{and} \\ e_j^{(2)} q_j^{(1)} \equiv a \pmod{(q_j^{(2)}, q)}, \end{cases}$$

$$d_i \equiv d_j \pmod{(q_i, q_j)} \iff \begin{cases} e_i^{(1)} q_i^{(2)} \equiv e_j^{(1)} q_j^{(2)} \pmod{(q_i^{(1)}, q_j^{(1)})} & \text{and} \\ e_i^{(2)} q_i^{(1)} \equiv e_j^{(2)} q_j^{(1)} \pmod{(q_i^{(2)}, q_j^{(2)})}, \end{cases}$$

$$(a_0 d_j + b_0, q_j) = 1 \iff \begin{cases} (a_0 e_j^{(1)} q_j^{(2)} + b_0, q_j^{(1)}) = 1 & \text{and} \\ (a_0 e_j^{(2)} q_j^{(1)} + b_0, q_j^{(2)}) = 1. \end{cases}$$

Now we write $d_j^{(1)} = e_j^{(1)} q_j(2)$, $\quad d_j^{(2)} = e_j^{(2)} q_j^{(1)}$ and

$$\mathbf{d}_i = (d_1^{(i)}, \ldots, d_{k-1}^{(i)})$$

for $i=1,2$. Then we get

$$\begin{aligned}
W(r_1r_2) &= \sum_{[\mathbf{q}]=r_1r_2} \prod_{j=1}^{k-1} \frac{\mu(q_j)}{\varphi(q_j)} \sum_{\mathbf{d}\in D(q,a;\mathbf{q})} \prod_{j=1}^{k-1} c_{q_j}(a_jd_j+b_j) \\
&= \sum_{[\mathbf{q}_1]=r_1} \sum_{[\mathbf{q}_2]=r_2} \prod_{j=1}^{k-1} \left(\frac{\mu(q_j^{(1)})}{\varphi(q_j^{(1)})} \frac{\mu(q_j^{(2)})}{\varphi(q_j^{(2)})} \right) \times \\
&\quad \times \sum_{\mathbf{d}_1\in D(q,a;\mathbf{q}_1)} \sum_{\mathbf{d}_2\in D(q,a;\mathbf{q}_2)} \prod_{j=1}^{k-1} \left(c_{q_j^{(1)}}(a_jd_j^{(1)}+b_j)c_{q_j^{(2)}}(a_jd_j^{(2)}+b_j)\right) \\
&= W(r_1)W(r_2),
\end{aligned}$$

namely, $W(r)$ is multiplicative.

Because $W(r)$ occurs only for square-free r's, it suffices to examine $W(p)$ for a prime p. If $[\mathbf{q}]=p$ then $q_j=1$ or p for all $1\le j\le k-1$, and at least one q_j is p. We denote by M the set of all subscripts of q_j's such that $q_j=p$. Then,

$$\begin{aligned}
W(p) &= \sum_{\substack{M\subset\{1,\dots,k-1\} \\ \sharp M\ge 1}} \left(\frac{-1}{p-1}\right)^{\sharp M} \sum_{\substack{d=1 \\ d\equiv a \ (\mathrm{mod}\ (p,q)) \\ (a_0d+b_0,p)=1}}^{p} \prod_{j\in M} c_p(a_jd+b_j) \\
&= \sum_{\substack{d=1 \\ d\equiv a \ (\mathrm{mod}\ (p,q)) \\ (a_0d+b_0,p)=1}}^{p} \left(\sum_{M\subset\{1,\dots,k-1\}} \prod_{j\in M} \left(\frac{-c_p(a_jd+b_j)}{p-1}\right) - 1 \right) \\
&= \sum_{\substack{d=1 \\ d\equiv a \ (\mathrm{mod}\ (p,q)) \\ (a_0d+b_0,p)=1}}^{p} \left(\prod_{j=1}^{k-1} \left(1-\frac{c_p(a_jd+b_j)}{p-1}\right) - 1 \right).
\end{aligned}$$

We remember here that we should consider for $\mathbf{b}$ with $\sigma(\mathbf{b};q,a)\neq 0$, and that the last condition is equivalent to

$$\rho(p) < p \ \text{ for all prime } p,$$

and

$$(a_ja+b_j,q)=1 \ \text{ for all } 1\le j\le k-1, \tag{5.2}$$

and

$$R(\mathbf{b})\neq 0.$$

Noticing this, we have

$$W(p)=\begin{cases}\left(1-\frac{1}{p}\right)^{-k+1}-1 & (\text{if } p \mid q),\\ (p-\rho(p))\left(1-\frac{1}{p}\right)^{-k+1}-p & (\text{if } p\nmid q \text{ and } p\mid a_0),\\ (p-\rho(p))\left(1-\frac{1}{p}\right)^{-k+1}-p+1 & (\text{if } p\nmid q \text{ and } p\nmid a_0).\end{cases}$$

On writing

$$\begin{aligned}\varphi(|a_0|[q,r]) &= |a_0|[q,r]\prod_{p\mid(a_0qr)}\left(1-\frac{1}{p}\right)\\ &= |a_0|q\prod_{p\mid(a_0q)}\left(1-\frac{1}{p}\right)\prod_{\substack{p\mid r\\ p\nmid q\\ p\mid a_0}}p\prod_{\substack{p\mid r\\ p\nmid q\\ p\nmid a_0}}(p-1),\end{aligned}\tag{5.3}$$

we obtain

$$\frac{\mu(r)^2}{\varphi(|a_0|[q,r])}W(r)=\frac{1}{|a_0|q}\prod_{p\mid(a_0q)}\left(1-\frac{1}{p}\right)^{-1}\prod_{\substack{p\mid r\\ p\mid q}}\left(\left(1-\frac{1}{p}\right)^{-k+1}-1\right)\times\tag{5.4}$$

$$\times\prod_{\substack{p\mid r\\ p\nmid q\\ p\mid a_0}}\left(\left(1-\frac{\rho(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-k+1}-1\right)\prod_{\substack{p\mid r\\ p\nmid q\\ p\nmid a_0}}\left(\left(1-\frac{\rho(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-k}-1\right).$$

It follows easily

$$\left(1-\frac{1}{p}\right)^{-k+1}-1\ll\frac{1}{p},\qquad \left(1-\frac{\rho(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-k+1}-1\ll\frac{1}{p},$$

and

$$\left(1-\frac{\rho(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-k}-1\ll\frac{1}{p}.$$

Especially, if $p\nmid R(\mathbf{b})$ then

$$\left(1-\frac{\rho(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-k}-1\ll\frac{1}{p^2},$$

because $\rho(p)=k$ in this case. Thus, by (5.4), we get

$$\begin{aligned}\left|\frac{\mu(r)^2}{\varphi(|a_0|[q,r])}W(r)\right| &\le \frac{1}{|a_0|q}\prod_{p\mid(a_0q)}2\prod_{\substack{p\mid r\\ p\mid(a_0qR(\mathbf{b}))}}\frac{L}{p}\prod_{\substack{p\mid r\\ p\nmid(a_0qR(\mathbf{b}))}}\frac{L}{p^2}\\ &\le \frac{\tau(|a_0|q)}{|a_0|q}(r,a_0qR(\mathbf{b}))\frac{\mu(r)^2\tau_L(r)}{r^2},\end{aligned}\tag{5.5}$$

where L is a natural number which depends only on k. Since

$$\sum_{r\le z}\tau_L(r)\ll z(\log z)^{L-1}$$

for $z\ge 2$, we have by partial summation

$$\sum_{r>Q_1}\frac{\mu(r)^2}{\varphi(|a_0|[q,r])}W(r)\ll\frac{\tau_K(q)}{q}\tau_K(R(\mathbf{b}))Q_1^{-1}(\log Q_1)^{L-1},$$

with a natural number K depending only on k. Then, it follows from (5.4) that

$$\begin{aligned}
S_0&=\sum_{r=1}^{\infty}\frac{\mu(r)^2}{\varphi(|a_0|[q,r])}W(r)+O\Big(\frac{\tau_K(q)}{q}\tau_K(R(\mathbf{b}))Q_1^{-1}(\log Q_1)^{L-1}\Big)\\
&=\frac{1}{|a_0|q}\prod_{p|(a_0q)}\Big(1-\frac{1}{p}\Big)^{-1}\prod_{p|q}\Big(1-\frac{1}{p}\Big)^{-k+1}\times\\
&\quad\times\prod_{\substack{p\nmid q\\ p|a_0}}\Big(1-\frac{\rho(p)}{p}\Big)\Big(1-\frac{1}{p}\Big)^{-k+1}\prod_{\substack{p\nmid q\\ p\nmid a_0}}\Big(1-\frac{\rho(p)}{p}\Big)\Big(1-\frac{1}{p}\Big)^{-k}+\\
&\qquad+O\Big(\frac{\tau_K(q)}{q}\tau_K(R(\mathbf{b}))Q_1^{-1}(\log Q_1)^{L-1}\Big)\\
&=\frac{1}{|a_0|q}\prod_{p|q}\Big(1-\frac{1}{p}\Big)^{-k}\prod_{p\nmid q}\Big(1-\frac{\rho(p)}{p}\Big)\Big(1-\frac{1}{p}\Big)^{-k}+\\
&\qquad+O\Big(\frac{\tau_K(q)}{q}\tau_K(R(\mathbf{b}))Q_1^{-1}(\log Q_1)^{L-1}\Big)\\
&=\frac{1}{|a_0|}\sigma(\mathbf{b};q,a)+O\Big(\frac{\tau_K(q)}{q}\tau_K(R(\mathbf{b}))Q_1^{-1}(\log Q_1)^{L-1}\Big).
\end{aligned}\tag{5.6}$$

We next estimate S_1. Let

$$W_2(r)=\sum_{[\mathbf{q}]=r}\prod_{j=1}^{k-1}\frac{\mu(q_j)^2}{\varphi(q_j)}\left|\sum_{\mathbf{d}\in D(q,a;\mathbf{q})}\prod_{j=1}^{k-1}c_{q_j}(a_jd_j+b_j)\right|,$$

and let

$$S_2=\sum_{Q_1<r\le Q_1^{k-1}}\frac{\mu(r)^2}{\varphi(|a_0|[q,r])}W_2(r),$$

then,

$$|S_1|\le S_2.\tag{5.7}$$

By comparison with $W(r)$, we see at once that $W_2(r)$ is multiplicative. For a prime p, we write $W_2(p)$, similarly to $W(p)$, as follows.

$$\begin{aligned}
W_2(p) &= \sum_{\substack{M\subset\{1,\dots,k-1\}\\ \sharp M\geq 1}} \Big(\frac{-1}{p-1}\Big)^{\sharp M} \left| \sum_{\substack{d=1\\ d\equiv a \ (\mathrm{mod}\ (p,q))\\ (a_0d+b_0,p)=1}}^{p} \prod_{j\in M} c_p(a_jd+b_j) \right| \\
&= \sum_{\substack{M\subset\{1,\dots,k-1\}\\ \sharp M\geq 1}} \Big(\frac{-1}{p-1}\Big)^{\sharp M} |W_3(p,M)|, \quad \text{say.}
\end{aligned}$$

If $p \mid q$ then $|W_3(p,M)| \leq 1$ by (5.2). For $p \nmid q$, we see

$$\left|\prod_{j\in M} c_p(a_jd+b_j)\right| \leq 1 \quad \text{unless} \quad \prod_{j\in M} c_p(a_jd+b_j) \equiv 0 \pmod p,$$

and $|W_3(p,M)| \leq k(p-1)^{\sharp M}+p$. Next assume, $p \nmid (a_0qR(\mathbf{b}))$. If $\sharp M = 1$ then

$$W_3(p,M) = (p-1)+(-1)(p-2) = 1,$$

and if $\sharp M \geq 2$ then

$$|W_3(p,M)| \leq (p-1)\sharp M + 1\cdots(p-1-\sharp M) \leq kp.$$

Thus we have

$$W_2(p) \leq \begin{cases} 2^{k+1}kp^{-1} & (\text{if } p\nmid (a_0qR(\mathbf{b}))), \\ (k+2)2^{k-1} & (\text{otherwise}), \end{cases}$$

and

$$\begin{aligned}
\left|\frac{\mu(r)^2}{\varphi(|a_0|[q,r])}W_2(r)\right| &\leq \frac{1}{|a_0|q}\prod_{p\mid(a_0q)} 2 \prod_{\substack{p\mid r\\ p\mid(a_0qR(\mathbf{b}))}} \frac{(k+2)2^{k-1}}{p} \prod_{\substack{p\mid r\\ p\nmid(a_0qR(\mathbf{b}))}} \frac{2^{k+2}k}{p^2} \\
&\leq \frac{\tau(|a_0|q)}{|a_0|q}(r,a_0qR(\mathbf{b}))\frac{\mu(r)^2\tau_{2^{k+2}k}(r)}{r^2}.
\end{aligned}$$

Hence, by partial summation,

$$S_2 \ll \frac{\tau_K(q)}{q}\tau_K(R(\mathbf{b}))Q_1^{-1}(\log Q_1)^{2^{k+2}k-1}. \tag{5.8}$$

We conclude from (5.1), (5.6), (5.7) and (5.8) that

LEMMA 5.1. *There is a natural number K depending only on k such that*

$$S(\mathbf{b};q,a) = \frac{1}{|a_0|}\sigma(\mathbf{b};q,a) + O\Big(\frac{\tau_K(q)}{q}\tau_K(R(\mathbf{b}))(\log x)^{-C_1+1}\Big).$$

§6. Proof of Theorem 3

In view of (2.4), Lemmata 3.1, 4.1 and 5.1, we obtain

$$\begin{aligned}\mathcal{E}(I;\mathbf{b};q,a) &= \Psi(I;\mathbf{b};q,a)-\sigma(\mathbf{b};q,a)|N(I;\mathbf{b})|\\ &= |a_0||N(I;\mathbf{b})|\left(S(\mathbf{b};q,a)-\frac{1}{|a_0|}\sigma(\mathbf{b};q,a)\right)+\\ &\qquad +\widetilde{\mathcal{E}}(I;\mathbf{b};q,a)+\sum_{h=1}^{k-1}\mathbf{I}_{\mathrm{m},h}\\ &\ll y\frac{\tau_K(q)}{q}\tau_K(R(\mathbf{b}))(\log x)^{-C_1+1})+\widetilde{\mathcal{E}}(I;\mathbf{b};q,a)+\sum_{h=1}^{k-1}\mathbf{I}_{\mathrm{m},h}\ ,\end{aligned}$$

and

$$\begin{aligned}\mathcal{E}_{1,0}(x,y;Q) \ll y(\log x)^{-C_1+1}\sum_{q\le Q}\frac{\tau_K(q)}{q}\max_a\max_{I\subset[x-y,x]}\sum_{\mathbf{b}\in Z_0(I;q,a)}\tau_K(R(\mathbf{b}))+\\ +y^k(\log x)^{-A},\end{aligned}$$

providing $Q\le yx^{-1/2}(\log x)^{-B}$. It is known on the divisor functions that

$$\sum_{q\le Q}\frac{\tau_K(q)}{q}\ll(\log Q)^K,\quad\text{and}\quad\sum_{\mathbf{b}\in Z_0(I;q,a)}\tau_K(R(\mathbf{b}))\ll y^{k-1}(\log x)^{K_1},$$

where K_1 is a constant depending only on k. Therefore,

$$\mathcal{E}_{1,0}(x,y;Q)\ll y^k(\log x)^{-A}, \tag{6.1}$$

if we choose $C_1>A+K_1+K+1$. By (2.1) and (6.1), we complete our proof of Theorem 3.

§7. Calculation on the singular series

We turn to prove Theorem 4.

We put

$$U_1(q) = \sum_{a=1}^{q} \sum_{\mathbf{b}\in Z(x,y)} \sigma(\mathbf{b}; q, a)^2 |N(x, y; \mathbf{b})|^2,$$

and we calculate $\sum_{q\le Q} U_1(q)$ in this section. It follows from (5.6) that

$$\sigma(\mathbf{b}; q, a) = |a_0|S_0 + O\Big(\frac{\tau_K(q)}{q}\tau_K(R(\mathbf{b}))(\log x)^{-C_1+1}\Big),$$

providing that $R(\mathbf{b}) \ne 0$ and $(a_0 a + b_0, q) = 1$. And, by (5.5), we have

$$S_1 \ll \frac{\tau(q)}{q}(\log Q_1)^L$$

in all cases. Since the number of $\mathbf{b}$'s with $R(\mathbf{b}) = 0$ is $O(y^{k-2})$, we obtain
(7.1)

$$\sum_{q\le Q} U_1(q) = |a_0|^2 \sum_{q\le Q} \sum_{\substack{a=1\\(a_0a+b_0,q)=1}}^{q} \sum_{\mathbf{b}\in Z(x,y)} S_1^2 |N(x, y; \mathbf{b})|^2 + O(y^{k+1}(\log x)^{-A}).$$

We substitute S_1 by its definition and $|N(x, y; \mathbf{b})|^2$ by

$$\int \int_{\substack{x-y\le a_j t_i + b_j \le x\\ \text{for all } 0\le j\le k-1\\ \text{and for } i=1,2}} dt_1 dt_2,$$

then the calculation of the innermost sum can be reduced to that of the sum

$$\sum_{\substack{b_j\\ x-y\le a_j t_i + b_j\le x\\ \text{for } i=1,2}} e\left(b_j\Big(\frac{h}{q_j} + \frac{h'}{q'_j}\Big)\right)$$

with $(h, q_j) = (h', q'_j) = 1$ and q_j, $q'_j \le Q_1$. So far as $\|\frac{h}{q_j} + \frac{h'}{q'_j}\| \ne 0$, we have an estimate

$$\left\| \frac{h}{q_j} + \frac{h'}{q'_j} \right\| \ge \frac{1}{q_j q'_j} \ge Q_1^{-2},$$

then,

$$\sum_{\substack{b_j\\ x-y\le a_j t_i + b_j\le x\\ \text{for } i=1,2}} e\left(b_j\Big(\frac{h}{q_j} + \frac{h'}{q'_j}\Big)\right) \ll Q_1^2.$$

If $\|\frac{h}{q_j}+\frac{h'}{q'_j}\|=0$, then $q_j=q'_j$, $\quad h\equiv h' \pmod{q_j}$ and

$$\sum_{\substack{b_j\\ x-y\le a_jt_i+b_j\le x\\ \text{for } i=1,2}} e\left(b_j\left(\frac{h}{q_j}+\frac{h'}{q'_j}\right)\right)=\max\{y-|a_j(t_1-t_2)|,\ 0\}+O(1).$$

Taking account of these results, we have

$$\sum_{\mathbf{b}\in Z(x,y)} S_1^2|N(x,y;\mathbf{b})|^2=S(q,a)J+O\big(y^k\varphi(q)^{-2}Q_1^{2(k+1)}\big), \tag{7.2}$$

where

$$J=\int\underset{\substack{x-y\le a_0t_i+b_0\le x\\ \text{for } i=1,2\\ a_*|t_1-t_2|\le y}}{\int}\prod_{j=1}^{k-1}(y-|a_j(t_1-t_2)|)\,dt_1dt_2,$$

$$S(q,a)=\sum_{r\le Q_1}\frac{\mu(r)^2}{\varphi(|a_0[q,r]|)^2}S_0(r),$$

with

$$S_0(r)=\sum_{[\mathbf{q}]=r}\prod_{j=1}^{k-1}\frac{\mu(q_j)^2}{\varphi(q_j)^2}\sum_{\mathbf{d}_1\in D(q,a;\mathbf{q})}\sum_{\mathbf{d}_2\in D(q,a;\mathbf{q})}\prod_{j=1}^{k-1}c_{q_j}\big(a_j(d_j^{(1)}-d_j^{(2)})\big).$$

Here we use the notation $\mathbf{d}_i=(d_1^{(i)},\ldots,d_{k-1}^{(i)})$ for $i=1,\ 2$.

It is easily seen that

$$\begin{aligned}J&=\int_{a_*|t|\le y}\prod_{j=1}^{k-1}(y-|a_jt|)\left(\int_{\substack{x-y\le a_0t_1+b_0\le x\\ x-y\le a_0(t_1-t)+b_0\le x}}dt_1\right)dt\\ &=\frac{2}{|a_0|}\int_0^{\frac{y}{a_*}}\prod_{j=1}^{k-1}(y-|a_jt|)\,dt+O(y^k)\\ &=\frac{2}{|a_0|}y^{k+1}\Omega+O(y^k).\end{aligned} \tag{7.3}$$

Similar to $W(r)$ in §5, it is proved that $S_0(r)$ is a multiplicative function in r. So it suffices to calculate $S_0(p)$ only for each prime p. As in §5, we denote by M the set of subscripts of q_j's such that $q_j = p$, and get

$$\begin{aligned}
S_0(p) &= \sum_{\substack{M\subset\{1,\dots,k-1\}\\ \sharp M\geq 1}} \Big(\frac{1}{(p-1)^2}\Big)^{\sharp M} \sum_{\substack{d_1=1\\ d_i\equiv a \\ (a_0d_i+b_0,p)=1}}^{p} \sum_{\substack{d_2=1\\ (\mathrm{mod}\ (p,q))\\ \text{for } i=1,2}}^{p} \prod_{j\in M} c_p(a_j(d_1-d_2)) \\
&= \sum_{d_1}\sum_{d_2}\left(\sum_{M\subset\{1,\dots,k-1\}}\prod_{j\in M}\Big(\frac{c_p(a_j(d_1-d_2))}{(p-1)^2}\Big) - 1\right) \\
&= \sum_{d_1}\sum_{d_2}\left(\prod_{j=1}^{k-1}\Big(1+\frac{c_p(a_j(d_1-d_2))}{(p-1)^2}\Big) - 1\right)
\end{aligned}$$

$$= \begin{cases} (1-\frac{1}{p})^{-k+1} - 1 \quad (\text{if } p \mid q), \\ p(1-\frac{1}{p})^{-k+1} + (p^2-p)(1-\frac{1}{(p-1)^2})^{k-g(p)}(1-\frac{1}{p-1})^{g(p)-1} - p^2 \\ \qquad (\text{if } p\nmid q \text{ and } p\mid a_0), \\ (p-1)(1-\frac{1}{p})^{-k+1} + ((p-1)^2-(p-1))(1-\frac{1}{(p-1)^2})^{k-1-g(p)}\times \\ \qquad \times(1+\frac{1}{p-1})^{g(p)} - (p-1)^2 \\ \qquad (\text{if } p\nmid qa_0). \end{cases}$$

Noticing (5.3), we have for a square-free r

$$\begin{aligned}
\frac{\mu(r)^2S_0(r)}{\varphi(|a_0|[q,r])^2} = \frac{1}{|a_0|^2q^2}\prod_{p\mid a_0q}\Big(1-\frac{1}{p}\Big)^{-2}\prod_{\substack{p\mid r\\ p\mid q}}\Big(\Big(1-\frac{1}{p}\Big)^{-k+1}-1\Big)\times \\
\times\prod_{\substack{p\mid r\\ p\mid a_0\\ p\nmid q}}\Big(\Big(1-\frac{1}{p}\Big)f_2(p)-1\Big)\prod_{\substack{p\mid r\\ p\nmid a_0q}}(f_2(p)-1),
\end{aligned}$$

where

$$f_2(p) = \Big(1-\frac{1}{p}\Big)^{-k}\Big(\frac{1}{p}+\Big(1-\frac{1}{p}\Big)\Big(\frac{p-2}{p-1}\Big)^{k-g(p)}\Big).$$

Now it is clear that

$$\frac{\mu(r)^2S_0(r)}{\varphi(|a_0|[q,r])^2} \ll q^{-2}\prod_{p\mid q}\Big(1-\frac{1}{p}\Big)^{-2}\cdot\left(r, q\prod_{j=0}^{k-1}a_j\right)\tau_{K_1}(r)r^{-2},$$

hence

$$\begin{aligned}
S(q,a) &= \sum_{r=1}^{\infty} \frac{\mu(r)^2 S_0(r)}{\varphi(|a_0|[q,r])^2} + O\left(\sum_{r>Q_1} \frac{\mu(r)^2 S_0(r)}{\varphi(|a_0|[q,r])^2} \right) \\
&= \frac{1}{|a_0|^2 q^2} \prod_{p|a_0 q} (1-\frac{1}{p})^{-1} \prod_{p|q} f_3(p) \prod_{p} f_2(p) + \\
&\qquad + O\left(\frac{\tau_{K_2}(q)}{q^2} (\log x)^{-C_1+1} \right), \qquad (7.4)
\end{aligned}$$

where K_1 and K_2 are natural numbers depending only on k, and where

$$f_3(p) = \left(\frac{1}{p} + (1-\frac{1}{p})(\frac{p-2}{p-1})^{k-g(p)}\right)^{-1}.$$

Now we obtain from (7.1), (7.2), (7.3) and (7.4)

$$\begin{aligned}
\sum_{q \le Q} U_1(q) = \frac{2}{|a_0|} y^{k+1} \Omega \prod_{p} f_2(p) \sum_{q\le Q} \frac{1}{q} \prod_{p|q} f_3(p) \sum_{\substack{a=1 \\ (a_0 a + b_0, q)=1}}^{q} 1 + \\
+ O(y^{k+1} (\log x)^{-A}).
\end{aligned}$$

Since

$$\sum_{\substack{a=1 \\ (a_0 a + b_0, q)=1}}^{q} 1 = q \prod_{\substack{p|q \\ p\nmid a_0}} (1-\frac{1}{p}),$$

we have

Lemma 7.1. *For any fixed $A > 0$,*

$$\sum_{q\le Q} U_1(q) = \frac{2}{\varphi(|a_0|)} y^{k+1} \Omega \prod_{p} f_2(p) \sum_{q\le Q} \frac{1}{q} \prod_{p|q} f_3(p) + O(y^{k+1}(\log x)^{-A}).$$

For a square–free natural number d, we define

$$h(d) = \prod_{p|d} (f_3(p) - 1),$$

then

$$\begin{aligned}
\sum_{q\le Q} \frac{1}{q} \prod_{p|q} f_3(p) &= \sum_{q \le Q} \frac{1}{q} \sum_{d|q} \mu(d)^2 h(d) \\
&= \sum_{d\le Q} \frac{\mu(d)^2 h(d)}{d} \sum_{m \le Q/d} \frac{1}{m}.
\end{aligned}$$

We know the innermost sum equals to $\log Q - \log d + \gamma + O(d/Q)$ with the Euler constant γ. Since $f_3(p) \geq 1$, we see $h(d) \geq 0$ and

$$\sum_{d \leq D} \mu(d)^2 h(d) \leq \prod_{p \leq D} (1 + h(p)) = \prod_{p \leq D} f_3(p) \ll (\log D)^k.$$

So we have by partial summation

$$\sum_{q \leq Q} \frac{1}{q} \prod_{p|q} f_3(p) =$$

$$(\log Q + \gamma) \left(\prod_p (1 - \frac{1}{p} + \frac{1}{p} f_3(p)) - \sum_{d > Q} \frac{\mu(d)^2 h(d)}{d} \right)$$

$$- \left(\sum_{d=1}^{\infty} \frac{\mu(d)^2 h(d)}{d} (\log d) - \sum_{d > Q} \frac{\mu(d)^2 h(d)}{d} (\log d) \right)$$

$$+ O\left(Q^{-1} \sum_{q \leq Q} \mu(d)^2 h(d) \right)$$

$$= \log Q \prod_p (1 - \frac{1}{p} + \frac{1}{p} f_3(p)) + \gamma_1' + O(Q^{-1}(\log Q)^{k+1}), \tag{7.5}$$

where

$$\gamma_1' = \gamma \prod_p (1 - \frac{1}{p} + \frac{1}{p} f_3(p)) - \sum_{d=1}^{\infty} \frac{\mu(d)^2 h(d)}{d} (\log d),$$

which depends only on $a_0, a_1, \ldots, a_{k-1}$. Therefore we derive from Lemma 7.1 that

LEMMA 7.2. *For any fixed $A > 0$, we have*

$$\sum_{q \leq Q} U_1(q) = \frac{2}{\varphi(|a_0|)} y^{k+1} \Omega(\log Q) \prod_p f_1(p) + \gamma_1 y^{k+1} +$$
$$+ O\big(y^{k+1}(\log x)^{-A} + y^{k+1} Q^{-1} (\log Q)^{k+1}\big),$$

where γ_1 is a constant depending only on $a_0, a_1, \ldots, a_{k-1}$.

§8. Montgomery's argument

We put

$$U_2(q) = \sum_{\mathbf{b}\in Z(x,y)} \sum_{\substack{n,m\in \\ n\neq m \\ n\equiv m \pmod q}} \sum_{N(x,y;\mathbf{b})} \prod_{j=0}^{k-1} \Lambda(a_j n + b_j)\Lambda(a_j m + b_j),$$

and evaluate $\sum_{q\leq Q} U_2(q)$ in this section, according to Montgomery [26]. His argument is based on the following lemma:

LEMMA 8.1. *Let*

$$\mathcal{F}(x,y;q,a;h) = \sum_{\substack{x-y\leq n\leq x \\ x-y\leq n+h\leq x \\ n\equiv a \pmod q}} \Lambda(n)\Lambda(n+h) - 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\times$$

$$\times \prod_{\substack{p|qh \\ p>2}} \left(\frac{p-1}{p-2}\right) \cdot \frac{(y-|h|)}{\varphi(q)},$$

if $h\equiv 0$ (mod 2) and $(a,q)=(a+h,q)=1$, and, otherwise, let

$$\mathcal{F}(x,y;q,a;h) = \sum_{\substack{x-y\leq n\leq x \\ x-y\leq n+h\leq x \\ n\equiv a \pmod q}} \Lambda(n)\Lambda(n+h).$$

Then, for y satisfying (2.1), and for any A, $B>0$, we have

$$\sum_{0<|h|\leq y} \tau(|h|)\mathcal{F}(x,y;q,a;h) \ll y^2(\log x)^{-A},$$

uniformly in $q\leq(\log x)^B$ and a.

This lemma follows by taking $k=2,\ a_0=a_1=1,\ b_0=0$ in our Theorem 3. In the case $y=x$, this lemma is due to Lavrik [22].

In the definition of $U_2(q)$, we write $r = m - n$ and obtain

$$U_2(q) = \sum_{\substack{0<|r|y/a_* \\ r\equiv 0 \pmod q}} \sum_{\substack{n \\ x-y\leq a_0 n+b_0\leq x \\ x-y\leq a_0(n+r)+b_0\leq x}} \Lambda(a_0 n + b_0)\Lambda(a_0(n+r)+b_0)\times$$

$$\times \prod_{j=1}^{k-1} \left(\sum_{\substack{b_j \\ x-y\leq a_j n+b_j\leq x \\ x-y\leq a_j(n+r)+b_j\leq x}} \Lambda(a_j n + b_j)\Lambda(a_j(n+r)+b_j) \right).$$

By virtue of Lemma 8.1, we have

$$\sum_{q\leq Q} U_2(q) = \sum_{q\leq Q} \sum_{\substack{0<|r|y/a_* \\ r\equiv 0 \pmod q}} \frac{2^k}{\varphi(|a_0|)} \prod_{p>2} (1 - \frac{1}{(p-1)^2})^k \times$$

$$\times \prod_{j=0}^{k-1} \left(\prod_{\substack{p|a_j r \\ p>2}} (\frac{p-1}{p-2}) \cdot (y - |a_j r|) \right) +$$

$$+O\left((y\log x)^{k-1} \sum_{0<|r|y/a_*} \tau(|r|)\Big(\mathcal{F}(x,y;a_0,b_0;a_0 r) + \sum_{j=1}^{k-1} \mathcal{F}(x,y;1,1;a_j r)\Big)\right)$$

$$(8.1) \qquad = \frac{2^{k+1}}{\varphi(|a_0|)} \prod_{p>2} \left((1 - \frac{1}{(p-1)^2})^k (\frac{p-1}{p-2})^{g(p)} \right) \cdot H_0(Q,y) +$$

$$+ O(y^{k+1}(\log x)^{-A}),$$

where

$$H_0(Q,y) = \sum_{q\leq Q} \sum_{\substack{0<m\leq \frac{y}{a_* q} \\ a_j qm\equiv 0 \pmod 2 \\ \text{for all } 0\leq j\leq k-1}} \prod_{\substack{p|mq \\ p>2}} (\frac{p-1}{p-2})^{k-g(p)} \prod_{j=0}^{k-1} (y - |a_j qm|).$$

We decompose $H_0(Q,y)$ into

$$H(Q,z) = \sum_{q\leq Q} \sum_{0<m\leq \frac{z}{a_* q}} \prod_{\substack{p|mq \\ p>2}} (\frac{p-1}{p-2})^{k-g(p)} \prod_{j=0}^{k-1} (y - |a_j qm|).$$

If $g(2) = k$, that is, all a_j's are even, then we have

$$(8.2) \qquad H_0(Q,y) = H(Q,y).$$

If $g(2) < k$, at least one a_j is odd, then we have

$$(8.3) \qquad H_0(Q,y) = 2^k H(Q,\frac{y}{2}) + 2^k H(\frac{Q}{2},\frac{y}{2}) - 2^{2k} H(\frac{Q}{2},\frac{y}{4}).$$

As for $H(Q,z)$, we shall prove the following Lemma 8.2. Here we define two more functions;

$$f_4(p) = 1 - \frac{1}{p} + \frac{1}{p}(\frac{p-1}{p-2})^{k-g(p)},$$
$$f_5(p) = (1-\frac{1}{p})^2 + \frac{1}{p}(2-\frac{1}{p})(\frac{p-1}{p-2})^{k-g(p)}.$$

LEMMA 8.2. *We have for* $Q \le 2z/a_*$

$$(8.4) \qquad H(Q,z) = z^{k+1}\Omega \prod_{p>2} f_4(p) \sum_{q \le Q} \frac{1}{q} \prod_{\substack{p|q \\ p>2}} f_3(p) + O\left(z^k Q (\log 3\frac{z}{Qa_*})^k\right),$$

and, for $Q \ge z/a_*$,

$$(8.5) \qquad H(Q,z) = z^{k+1}\Omega(\log\frac{z}{a_*}) \prod_{p>2} f_5(p) + \gamma_2 z^{k+1} + O(z^{k+\frac{1}{2}}(\log z)^{k+2}),$$

where γ_2 *is a constant depending only on* $a_0, a_1, \ldots, a_{k-1}$.

PROOF: Let $w(d)$ be a completely multiplicative function defined by, for a prime p,

$$w(p) = \begin{cases} (\frac{p-1}{p-2})^{k-g(p)} - 1 & (p \nmid 2q), \\ 0 & (p \mid 2q). \end{cases}$$

For $z \geq 1$, we have

$$\sum_{m \leq z} \prod_{\substack{p|m \\ p \nmid 2q}} \left(\frac{p-1}{p-2}\right)^{k-g(p)} = \sum_{m \leq z} \sum_{d|m} \mu(d)^2 w(d)$$

$$= z \sum_{d=1}^{\infty} \frac{\mu(d)^2 w(d)}{d} +$$

$$+ O\left(z \sum_{d>z} \frac{\mu(d)^2 w(d)}{d} + \sum_{d \leq z} \mu(d)^2 w(d) \right)$$

$$= z \prod_{p>2} f_4(p) \prod_{\substack{p|q \\ p>2}} f_4(p)^{-1} + O((\log z)^k). \tag{8.6}$$

Then, for $Q \leq z/a_*$, the formula (8.4) follows by partial summation. And the formula (8.4) is still valid for $z/a_* < Q \leq 2z/a_*$ because

$$\sum_{\frac{z}{a_*} < q \leq Q} \frac{1}{q} \prod_{\substack{p|q \\ p>2}} f_3(p) \ll 1.$$

Next we assume $Q \geq z/a_*$. We have

$$H(Q, z) = H(\frac{z}{a_*}, z) = \sum_{qm \leq z/a_*} \sum \prod_{\substack{p|qm \\ p>2}} \left(\frac{p-1}{p-2}\right)^{k-g(p)} \prod_{j=0}^{k-1} (y - |a_j qm|)$$

$$= \sum_{q \leq \sqrt{\frac{z}{a_*}}} \sum_{m \leq \frac{z}{a_* q}} + \sum_{m \leq \sqrt{\frac{z}{a_*}}} \sum_{q \leq \frac{z}{a_* m}} - \sum_{q \leq \sqrt{\frac{z}{a_*}}} \sum_{m \leq \sqrt{\frac{z}{a_*}}}$$

$$= 2H\left(\sqrt{\frac{z}{a_*}}, z\right) - H_1, \quad \text{say.} \tag{8.7}$$

It follows by (8.6) that

$$H_1 = \sum_{q \leq \sqrt{\frac{z}{a_*}}} \sum_{m \leq \sqrt{\frac{z}{a_*}}} \prod_{\substack{p|qm \\ p>2}} \left(\frac{p-1}{p-2}\right)^{k-g(p)} \prod_{j=0}^{k-1} (y - |a_j|qm)$$

$$= z^{k+1} \prod_{p>2} f_4(p) \times$$

$$\times \int\limits_{(za_*)^{-\frac{1}{2}}}^{a_*^{-1}} \sum_{u(za_*)^{\frac{1}{2}} \leq q \leq \left(\frac{z}{a_*}\right)^{\frac{1}{2}}} \frac{1}{q} \prod_{\substack{p|q \\ p>2}} f_3(p) \prod_{j=0}^{k-1} (1 - |a_j|u)\, du +$$

$$+ O\left(z^{k+\frac{1}{2}} (\log z)^k\right).$$

We calculate the integrand by (7.5), then we get, by partial summation, that

$$H_1 = z^{k+1} \prod_{p>2} f_5(p) \int_0^{a_*^{-1}} \frac{1}{u} \int_0^u \prod_{j=0}^{k-1} (1 - |a_j|v)\, dv\, du +$$

$$+ O\left(z^{k+\frac{1}{2}} (\log z)^k\right).$$

We see this double integral depends only on $a_0, \ldots, a_{k-1}$. Thus we obtain (8.5) by applying (8.4) and (7.5) to the first term of the right–hand side of (8.7). Now we complete the proof of Lemma 8.2.

Noticing that $f_1(2) = 2^k$ or $3 \cdot 2^{k-2}$ according as $g(2) = k$ or not, we conclude from (8.1), (8.2), (8.3) and (8.5) that

LEMMA 8.3. *For $Q \geq y/a_*$, we have*

$$\sum_{q \leq Q} U_1(q) = \frac{2}{\varphi(|a_0|)} y^{k+1} \Omega(\log \frac{y}{a_*}) \prod_p f_1(p) + \gamma_3 y^{k+1} +$$

$$+ O(y^{k+1} (\log x)^{-A}),$$

where γ_3 is a constant depending only on $a_0, a_1, \ldots, a_{k-1}$.

Next we assume $Q \leq y/a_*$. If $g(2) < k$ then $f_3(p) = 2$, so we have by (8.3) and (8.4)

$$H_0(Q, y) = y^{k+1} \Omega \prod_{p>2} f_4(p) \times$$

$$\times \left\{ \frac{1}{2} \sum_{\substack{q \leq Q \\ 2 \nmid q}} \frac{1}{q} \prod_{\substack{p|q \\ p>2}} f_3(p) + \sum_{\substack{q \leq Q \\ 2|q}} \frac{1}{q} \prod_{\substack{p|q \\ p>2}} f_3(p) \right\} +$$

$$+ O\left(y^k Q \left(\log \frac{2y}{Q a_*}\right)^k\right)$$

$$= \frac{1}{2} y^{k+1} \Omega \prod_{p>2} f_4(p) \cdot \sum_{q \le Q} \frac{1}{q} \prod_{p|q} f_3(p) + O\left(y^k Q \Big(\log \frac{2y}{Qa_*} \Big)^k \right). \tag{8.8}$$

And if $g(2) = k$ then $f_3(2) = 1$, so we have by (8.2) and (8.4)

$$H_0(Q, y) = y^{k+1} \Omega \prod_{p>2} f_4(p) \cdot \sum_{q \le Q} \frac{1}{q} \prod_{p|q} f_3(p) + O\left(y^k Q \Big(\log \frac{2y}{Qa_*} \Big)^k \right). \tag{8.9}$$

Since $f_2(2) = 2^k$ or 2^{k-1} according as $g(2) = k$ or not, we come to the following conclusion by using (8.1), (8.8) and (8.9).

LEMMA 8.4. *For $Q \le y/a_*$, we have*

$$\sum_{q \le Q} U_1(q) = \frac{2}{\varphi(|a_0|)} y^{k+1} \Omega \prod_{p} f_2(p) \cdot \sum_{q \le Q} \frac{1}{q} \prod_{p|q} f_3(p) +$$
$$+ O\left(y^k Q \Big(\log \frac{2y}{Qa_*} \Big)^k + y^{k+1} (\log x)^{-A} \right).$$

§9. Hooley's argument

Hereafter we set

$$Q_0 = y(\log x)^{-A-k},$$

and, for $Q_0 \le Q \le y/a_*$, we put

$$V(Q) = \sum_{Q<q\le y/a_*} U_2(q).$$

The purpose of this section is to prove the following Lemma 9.1 along Hooley's way [**16**].

LEMMA 9.1. *For $Q_0 \le Q \le y/a_*$, and for any fixed $A > 0$,*

$$V(Q) = \frac{2}{\varphi(|a_0|)} y^{k+1} \Omega(\log \frac{y}{Qa_*}) \prod_p f_1(p) + \eta_1 y^{k+1} + $$
$$+ y^k Q \sum_{m=0}^{k} \xi_m \big(\log \frac{y}{Qa_*}\big)^m + O\left(y^{k-\frac{1}{k+2}} Q^{\frac{k+3}{k+2}} + y^{k+1}(\log x)^{-A}\right).$$

PROOF: Putting $m = n + hq$,

$$V(Q) = 2 \sum_{\mathbf{b}\in Z(x,y)} \sum_{Q<q\le \frac{y}{a_*}} \sum_{\substack{n,m\in \\ n\equiv m \\ n<m}} \sum_{\substack{N(x,y;\mathbf{b}) \\ (\mathrm{mod}\ q)}} \prod_{j=0}^{k-1} \Lambda(a_j n + b_j)\Lambda(a_j m + b_j)$$
$$= 2 \sum_{h\le \frac{y}{Qa_*}} \sum_{\mathbf{b}\in Z(x,y)} \sum_{n\in N(x,y;\mathbf{b})} \prod_{j=0}^{k-1} \Lambda(a_j n + b_j) \sum_{\substack{m\in N(x,y;\mathbf{b}) \\ m>n+hQ \\ m\equiv n\ (\mathrm{mod}\ h)}} \prod_{j=0}^{k-1} \Lambda(a_j m + b_j).$$

Since $y/(Qa_*) \ll (\log x)^{A+k}$, we can apply Theorem 3 to the above, and obtain

$$V(Q) = 2 \sum_{h\le \frac{y}{Qa_*}} \sum_{\mathbf{b}\in Z(x,y)} \sigma(\mathbf{b};h)\sigma(\mathbf{b};1) \int\int_{\substack{t_1,t_2\ \in N(x,y;\mathbf{b}) \\ t_2<\ \ t_1-hQ}} dt_1 dt_2 +$$
$$+ O\big(y^{k+1}(\log x)^{-A}\big)$$

$$= 2 \sum_{h \le \frac{y}{Qa_*}} \sum_{\substack{a=1 \\ (a_0 a + b_0, h)=1}} \int \int_{\substack{x-y \le a_0 t_i + b_0 \le x \\ \text{for } i=1,2 \\ t_2 < t_1 - hQ}} \sum_{\substack{\mathbf{b} \\ x-y \le a_j t_i + b_j \le x \\ \text{for } 1 \le j \le k-1 \\ \text{and for } i=1,2}} \sigma(\mathbf{b}; h, a)^2 \, dt_1 dt_2 + $$

$$+ O\big(y^{k+1} (\log x)^{-A}\big).$$

By the same argument we used for the calculation of

$$\sum_{\mathbf{b}} S_1^2 |N(x, y; \mathbf{b})|^2$$

in §5, we can obtain

$$V(Q) = \frac{2}{\varphi(|a_0|)} y^{k+1} \prod_p f_2(p) \cdot V_1, \tag{9.1}$$

where

$$V_1 = \sum_{h \le \frac{y}{Qa_*}} \frac{1}{h} \prod_{p|h} f_3(p) \int_{hQ/y}^{a_*^{-1}} \prod_{j=0}^{k-1} (1 - |a_j| u) \, du.$$

We put $v = 1 - a_* u$, and define $r_1, \ldots, r_k$ as

$$\prod_{j=0}^{k-1} (1 - |a_j| u) \, du = \prod_{j=0}^{k-1} (1 - \frac{|a_j|}{a_*} + \frac{|a_j|}{a_*} v) = \sum_{m=1}^{k} r_m v^m. \tag{9.2}$$

Then simple calculation shows

$$V_1 = \frac{1}{a_*} \sum_{m=1}^{k} \frac{r_m}{m+1} \sum_{h \le \frac{y}{Qa_*}} (1 - \frac{hQa_*}{y})^{m+1} \frac{1}{h} \prod_{p|h} f_3(p)$$

$$= \frac{1}{a_*} \sum_{m=1}^{k} \frac{r_m}{m+1} \Xi(\frac{y}{Qa_*}, m+1), \tag{9.3}$$

where

$$\Xi(z, m) = \sum_{h \le z} (1 - \frac{h}{z})^m \frac{1}{h} \prod_{p|h} f_3(p).$$

Next we examine $\Xi(z,m)$ for $m \geq 2$. For $\sigma > 1$, we define

$$\Theta(s) = \sum_{h=1}^{\infty} h^{-s} \prod_{p|h} f_3(p),$$

where $s = \sigma + it$ is a complex variable. And we define the function $\theta(s)$ by

$$\Theta(s) = \zeta(s)\zeta(s+1)^k \theta(s). \tag{9.4}$$

In the half plane $\sigma \geq -1/2+\varepsilon$, it is easily seen that $\theta(s)$ is analytic and $\theta(s) \ll 1$. Thus, the relation (9.4) gives the analytic continuation of $\Theta(s)$ over $\sigma \geq -1/2+\varepsilon$. For $|t| > 1$, we have (see Tichmarsh's book [**37**], for example)

$$|\zeta(s)| \ll \begin{cases} |t|^{\frac{1}{2}-\sigma+\varepsilon} & (\sigma \leq 0), \\ |t|^{\frac{1}{2}(1-\sigma)+\varepsilon} & (0 \leq \sigma \leq 1). \end{cases}$$

Therefore, for $-1/2+\varepsilon \leq \sigma \leq 0$,

$$|\Theta(s)| \ll |t|^{-(\frac{k}{2}+1)\sigma+\frac{1}{2}+\varepsilon}. \tag{9.5}$$

We note here that the exponent of $|t|$ is less than 2 providing $\sigma > -3/(k+2)$.

It is known that, for $c > 0$, $u > 0$ and $m \geq 1$,

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{u^{-s}\, ds}{\prod_{j=0}^{m}(s+j)} = \begin{cases} \frac{1}{m!}(1-u)^m & (0 < u \leq 1), \\ 0 & (u > 1). \end{cases}$$

Making use of this formula, we have

$$\begin{aligned} \Xi(z,m) &= \frac{m!}{2\pi i} \int_{1-i\infty}^{1+i\infty} \Theta(s+1) \frac{z^s\, ds}{\prod_{j=0}^{m}(s+j)} \\ &= m!R_{0,m} + m!R_{1,m} + \frac{m!}{2\pi i} \int_{-\frac{k+3}{k+2}-i\infty}^{-\frac{k+3}{k+2}+i\infty} \Theta(s+1) \frac{z^s\, ds}{\prod_{j=0}^{m}(s+j)}, \end{aligned}$$

where $R_{0,m}$ and $R_{1,m}$ are the residue of the integrand at $s=0$ and $s=-1$, respectively. The inequality (9.5) shows, for $m \geq 2$, that the last integral converges absolutely and is bounded by $O(z^{-(k+3)/(k+2)})$.

On the other hand, we see by (9.4) that the integrand has poles of order 2 and $k+1$ at $s=0$ and $s=-1$, respectively. And we find that

$$m!R_{0,m} = \zeta(2)^k\theta(1)\log z \ + \ \gamma_4,$$
$$m!R_{1,m} = z^{-1}\sum_{j=0}^{k}\gamma_{j,m}(\log z)^j,$$

where γ_4 and $\gamma_{j,m}$'s are constants depending only on $a_0,\ldots,a_{k-1}$. We note that

$$\zeta(2)^k\theta(1) = \lim_{s\to 1}\frac{\Theta(s)}{\zeta(s)} = \prod_p(1-\frac{1}{p}+\frac{1}{p}f_3(p)),$$

and that

$$\gamma_{k,m} = -\frac{m}{k!}\zeta(0)\theta(0) = \frac{m}{2k!}\prod_p f_2(p)^{-1}. \tag{9.6}$$

Taking account of these results, we have

$$\Xi(z,m) = \prod_p(1-\frac{1}{p}+\frac{1}{p}f_3(p))\cdot(\log z)+\eta_2+ \\ +z^{-1}\sum_{j=0}^{k}\gamma_{j,m}(\log z)^j + O\left(z^{-\frac{k+3}{k+2}}\right). \tag{9.7}$$

Since

$$\sum_{m=1}^{k}\frac{r_m}{m+1} = \int_0^1\sum_{m=1}^{k} r_m v^m\,dv = a_*\int_0^1\prod_{j=0}^{k-1}(1-|a_j|u)\,du = a_*\Omega,$$

our Lemma 9.1 follows from (9.1), (9.3) and (9.7). Further, by taking $v=1$ in (9.2), we see $\sum_{m=1}^{k} r_m = 1$, and, combining this with (9.6), we obtain (1.7).

§10. Proof of Theorem 4

By the definition of $\mathcal{E}(x,y;\mathbf{b};q,a)$, we see

$$\begin{aligned}\mathcal{E}_2(x,y;Q) &= \sum_{q\le Q}\sum_{a=1}^{q}\sum_{\mathbf{b}\in Z(x,y)} \mathcal{E}(x,y;\mathbf{b};q,a)^2 \\ &= T + \sum_{q\le Q} U_2(q) - 2\sum_{q\le Q} U_3(q) - \sum_{q\le Q} U_1(q),\end{aligned} \tag{10.1}$$

where

$$T = Q\sum_{\mathbf{b}\in Z(x,y)}\sum_{n\in N(x,y;\mathbf{b})}\prod_{j=0}^{k-1}\Lambda(a_jn+b_j)^2,$$

$$\begin{aligned}U_3(q) = \sum_{a=1}^{q}\sum_{\mathbf{b}\in Z(x,y)} &\sigma(\mathbf{b};q,a)|N(x,y;\mathbf{b})|\times \\ &\times\left\{\Psi([x-y,x];\mathbf{b};q,a) - \sigma(\mathbf{b};q,a)|N(x,y;\mathbf{b})|\right\},\end{aligned}$$

and the definitions of $U_1(q)$ and $U_2(q)$ are in §7 and §8, respectively.

Making use of the prime number theorem for short arithmetic progressions, we obtain

$$\begin{aligned}T &= Q\sum_{x-y\le a_0n+b_0\le x}\Lambda(a_0n+b_0)^2\cdot\prod_{j=1}^{k-1}\left(\sum_{\substack{b_j \\ x-y\le a_jn+b_j\le x}}\Lambda(a_jn+b_j)^2\right) \\ &= Q\left(\sum_{\substack{x-y\le m\le x \\ m\equiv b_0 \pmod{a_0}}}\Lambda(m)^2\right)\left(\sum_{x-y\le m\le x}\Lambda(m)^2\right)^{k-1} \\ &= \frac{1}{\varphi(|a_0|)}y^kQ(\log x-1)^k + O(y^kQ(\log x)^{-A}).\end{aligned} \tag{10.2}$$

Next we see plainly

$$\sum_{\substack{a=1 \\ \left(\prod_{j=0}^{k-1}(a_ja+b_j),q\right)=1}}^{q} 1 \quad = q\prod_{p|q}\left(1-\frac{\rho(p)}{p}\right),$$

and

$$\sum_{\substack{a=1 \\ (\prod_{j=0}^{k-1}(a_j a+b_j),q)=1}}^{q} \{\Psi([x-y,x];\mathbf{b};q,a)-\sigma(\mathbf{b};q,a)|N(x,y;\mathbf{b})|\} =$$

$$= \Psi([x-y,x];\mathbf{b};1,1)-\sigma(\mathbf{b};1,1)|N(x,y;\mathbf{b})|+O((\log x)^{k+1}),$$

so we have

$$\sum_{q\le Q} U_3(q) = \sum_{q\le Q}\sum_{\mathbf{b}\in Z(x,y)} \sigma(\mathbf{b};q)|N(x,y;\mathbf{b})|\times$$
$$\times\left\{\Psi([x-y,x];\mathbf{b};1,1)-\sigma(\mathbf{b};1,1)|N(x,y;\mathbf{b})|+O((\log x)^{k+1})\right\}$$

$$\ll y(\log x)^2 \sum_{\mathbf{b}\in Z(x,y)} |\Psi([x-y,x];\mathbf{b};1,1)-\sigma(\mathbf{b};1,1)|N(x,y;\mathbf{b})||+$$
$$+ y^k(\log x)^{k+3}.$$

Then, applying Theorem 3, we get

$$\sum_{q\le Q} U_3(q) \ll y^{k+1}(\log x)^{-A}. \tag{10.3}$$

By (10.1), (10.2) and (10.3), we obtain

$$\mathcal{E}_2(x,y;Q) = \frac{1}{\varphi(|a_0|)} y^k Q(\log x-1)^k + \sum_{q\le Q} U_2(q) - \sum_{q\le Q} U_1(q)+ \tag{10.4}$$
$$+O(y^k Q(\log x)^{-A}+y^{k+1}(\log x)^{-A}).$$

Firstly, we suppose $Q\ge y/a_*$. Lemmata 7.2 and 8.4 yield

$$\sum_{q\le Q} U_2(q) - \sum_{q\le Q} U_1(q) = -\eta_0 y^{k+1}(\log\frac{Qa_*}{y}) + \eta_1 y^{k+1} + O(y^k Q(\log x)^{-A}),$$

then the formula (1.10) follows at once from (10.4).

Secondary, we suppose $Q\le y/a_*$. We have by Lemmata 7.1 and 8.3

$$\sum_{q\le Q} U_2(q) - \sum_{q\le Q} U_1(q) \ll y^k Q\big(\log 2\frac{y}{Qa_*}\big)^k + y^{k+1}(\log x)^{-A}, \tag{10.5}$$

and, in view of (10.4), this estimate shows the formula (1.9) providing $Q\le Q_0 = y(\log x)^{-A-k}$.

Finally, we suppose $Q_0 \leq Q < y/a_*$. By (10.5),

$$\sum_{q \leq Q_0} U_2(q) - \sum_{q \leq Q_0} U_1(q) \ll y^{k+1}(\log x)^{-A},$$

therefore

$$\mathcal{E}_2(x,y;Q) = \frac{1}{\varphi(|a_0|)} y^k Q(\log x - 1)^k + \sum_{Q_0<q\leq Q} U_2(q) - \sum_{Q_0<q\leq Q} U_1(q) + O(y^{k+1}(\log x)^{-A}). \tag{10.6}$$

By virtue of Lemma 9.1, it follows that

$$\begin{aligned} \sum_{Q_0<q\leq Q} U_2(q) &= V(Q_0) - V(Q) \\ &= \eta_0 y^{k+1}(\log \frac{Q}{Q_0}) - y^k Q \sum_{m=1}^{k} \xi_m (\log \frac{y}{Qa_*})^m + \\ &\quad + O\big(y^{k-\frac{1}{k+2}} Q^{\frac{k+3}{k+2}} + y^{k+1}(\log x)^{-A}\big). \end{aligned} \tag{10.7}$$

And we see by Lemma 7.2

$$\sum_{Q_0<q\leq Q} U_1(q) = \eta_0 y^{k+1}(\log \frac{Q}{Q_0}) + O(y^{k+1}(\log x)^{-A}). \tag{10.8}$$

Hence we obtain the formula (1.9) in this case, by (10.6), (10.7) and (10.8). Now we complete our proof of Theorem 4.

§11. The circle method

We start here the proof of Theorem 5.

We introduce functions

$$P(\alpha) = \sum_{p \le N} e(p\alpha) , \qquad T(\beta) = \sum_{2 \le m \le N} \frac{e(m\beta)}{\log m} ,$$

$$F(\alpha) = \sum_{m \le N^{1/k}} e(m^k\alpha) , \qquad v(\beta) = \sum_{m \le N} \frac{1}{k} m^{-1+\frac{1}{k}} e(m\beta) ,$$

$$V(a,q) = \sum_{r=1}^{q} e(\frac{a}{q} r^k) ,$$

and, for positive constants B and B_1 (which should be chosen later), we set

$$Q = N(\log N)^{-B}, \qquad Q_1 = (\log N)^{B_1}$$

$$\mathfrak{M}(a,q) = [\frac{a}{q} - \frac{1}{qQ}, \frac{a}{q} + \frac{1}{qQ}] ,$$

$$\mathfrak{M} = \bigcup_{q \le Q_1} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathfrak{M}(a,q) , \qquad \mathfrak{m} = [\frac{1}{Q}, 1 + \frac{1}{Q}] \setminus \mathfrak{M} .$$

Then, for $n \le N$, we have

$$\begin{aligned} r_k(n) &= \int_{Q^{-1}}^{1+Q^{-1}} P(\alpha)F(\alpha)e(-n\alpha)\, d\alpha \\ &= \int_{\mathfrak{M}} P(\alpha)F(\alpha)e(-n\alpha)\, d\alpha + \int_{\mathfrak{m}} P(\alpha)F(\alpha)e(-n\alpha)\, d\alpha \\ &= r_{k,0}(n) + r_{k,1}(n), \text{ say.} \end{aligned} \tag{11.1}$$

In order to calculate the integral on the "major arcs" $\mathfrak{M}$, we require the following known results.

LEMMA 11.1. *Let $q \le Q_1$, $0 < a \le q$, $(a,q) = 1$, $\alpha \in \mathfrak{M}(a,q)$ and let $\alpha = \frac{a}{q} + \beta$. Then, there exist a positive constant c_0 such that*

$$P(\alpha) = \frac{\mu(q)}{\varphi(q)} T(\beta) \; + \; O\left(N \exp(-c_0 \sqrt{\log N}\,)\right) .$$

LEMMA 11.2. *Let $0 < a \le q \le Q$, $(a,q) = 1$, $\alpha \in \mathfrak{M}(a,q)$ and let $\alpha = \frac{a}{q} + \beta$. Then we have*

$$F(\alpha) = \frac{V(a,q)}{q} v(\beta) \; + \; O\left(q^{\frac{1}{2}+\epsilon}(1 + \frac{N}{qQ})\right) .$$

LEMMA 11.3. *Let $||\beta||$ denote the distance from β to the nearest integer. Then we have*

$$|T(\beta)| \ll \min\left(\frac{N}{\log N}, ||\beta||^{-1}\right), \tag{11.2}$$

$$|v(\beta)| \ll \min\left(N^{1/k}, ||\beta||^{-1/k}\right). \tag{11.3}$$

Lemma 11.1 is proved by the Siegel-Walfisz theorem, and, for the proof, see Prachar [**34**, VI, Satz 3.2, p.181]. Lemma 11.2 and the inequality (11.3) of Lemma 11.3 are proved in Vaughan's book [**39**], see Theorem 4.1 and Lemma 2.8 of [**39**], respectively. The inequality (11.2) of Lemma 11.3 follows at once from the well-known estimate $\sum_{m\le N} e(m\beta) \ll \min(N, ||\beta||^{-1})$ by partial summation.

As for the integral on the "minor arcs" $\mathfrak{m}$, we use the following lemma. We postpone the proof of Lemma 11.4 until the end of this section.

LEMMA 11.4. *Let B_2 be a positive constant, and we assume that $B > B_1 \ge 2^k B_2 + k^2$. Then,*

$$\max_{\alpha\in\mathfrak{m}} |F(\alpha)| \ll N^{1/k}(\log N)^{-B_2}.$$

By Lemma 11.3, we have

$$\begin{aligned}\int_{|\beta|\le\frac{1}{qQ}} T(\beta)v(\beta)e(-n\beta)\,d\beta &= \int_0^1 T(\beta)v(\beta)e(-n\beta)\,d\beta + O\left(\int_{\frac{1}{qQ}}^{1/2} \beta^{-1-\frac{1}{k}}\,d\beta\right)\\ &= \sum_{m\le n-2} \frac{1}{k} m^{-1+\frac{1}{k}} \frac{1}{\log(n-m)} + O\left((qQ)^{1/k}\right)\\ &= I(n) + O\left((qQ)^{1/k}\right),\end{aligned}$$

where

$$I(n) = \int_1^{n-2} \frac{x^{-1+\frac{1}{k}}\,dx}{k\log(n-x)}. \tag{11.4}$$

Thus, by Lemmata 11.1 and 11.2, we obtain

$$\begin{aligned}r_{k,0}(n) &= \sum_{q\le Q_1} \frac{\mu(q)}{q\varphi(q)} \sum_{\substack{a=1\\(a,q)=1}}^{q} V(a,q)e(-\frac{a}{q}n) \int_{|\beta|\le\frac{1}{qQ}} T(\beta)v(\beta)e(-n\beta)\,d\beta + \\ &\qquad + O\left(N^{1+\frac{1}{k}}Q^{-1}Q_1\exp(-c_0\sqrt{\log N}\,)\right)\\ &= \sum_{q\le Q_1} \frac{\mu(q)}{q\varphi(q)} \sum_{r=1}^{q} c_q(r^k - n)\cdot I(n) + O\left(N^{1/k}(\log N)^{-\frac{1}{k}B+\frac{k+1}{k}B_1}\right).\end{aligned}$$

Since $c_q(m) = \sum_{\substack{d|m \\ d|q}} d\mu(q/d)$, we see

$$\begin{aligned}
\sum_{q\le Q_1} \frac{\mu(m)}{q\varphi(q)} \sum_{r=1}^{q} c_q(r^k - n) &= \sum_{q\le Q_1} \frac{\mu(m)}{q\varphi(q)} \sum_{d|q} d\mu(\frac{q}{d}) \sum_{\substack{r=1 \\ r^k - n\equiv 0 (\bmod d)}}^{q} 1 \\
&= \sum_{q\le Q_1} \frac{\mu(q)}{\varphi(q)} \sum_{d|q} \mu(\frac{q}{d}) \rho_n(d) \\
&= \mathfrak{S}_k(n, Q_1) ,
\end{aligned}$$

here $\mathfrak{S}_k(n, Q)$ is defined in (1.13). Therefore, we get

$$r_{k,0}(n) = \mathfrak{S}_k(n, Q_1) I(n) + O\left(N^{1/k} (\log N)^{-\frac{1}{k}B + \frac{k+1}{k}B_1}\right) . \tag{11.5}$$

On the other hand, it follows from Bessel's inequality and Lemma 11.4 that

$$\begin{aligned}
\sum_{n\le N} |r_{k,1}(n)|^2 &\ll \int_{\mathfrak{m}} |P(\alpha)|^2 \, |F(\alpha)|^2 \, d\alpha \\
&\ll N^{2/k} (\log N)^{-2B_2} \int_0^1 |P(\alpha)|^2 \, d\alpha \\
&\ll N^{1+\frac{2}{k}} (\log N)^{-2B_2 - 1} .
\end{aligned} \tag{11.6}$$

Hence, by (11.1), (11.5) and (11.6) with suitable choices of constants B, B_1 and B_2, we obtain the following lemma.

LEMMA 11.5. *For any constant $A_1 > 0$, we have*

$$\sum_{n\le N} |r_k(n) - \mathfrak{S}(n, Q_1) I(n)|^2 \ll N^{1+\frac{2}{k}} (\log N)^{-A_1} .$$

Now we are in the position to prove Lemma 11.4. By well-known Dirichlet's theorem, for any $\alpha \in \mathfrak{m}$, there exist integers a and q such that

$$\alpha \in \mathfrak{M}(a, q), \quad 0 < a \le q, \quad (a, q) = 1 \quad \text{and} \quad Q_1 < q \le Q . \tag{11.7}$$

Then, Weyl's inequality (see [**39**, Lemma 2.4]) gives

$$|F(\alpha)| \ll N^{\frac{1}{k}+\varepsilon} \left(q^{-1} + N^{-1/k} + qN^{-1}\right)^{1/2^{k-1}} , \tag{11.8}$$

but when q is near to Q_1 or Q, (11.8) gives only a trivial bound. If q is near to Q_1, the required bound follows from Lemma 11.2 with the estimate $V(a, q) \ll q^{1-\frac{1}{k}}$ ([**39**, Theorem 4.2]). For q near to Q, we modify the proof of (11.8) slightly to reduce the factor "N^ε" in (11.8) which arise from estimating the divisor function.

By Weyl's method, we have (see [**39**, pp.11-12, Lemma 2.3 and the proof of Lemma 2.4]),

$$|F(\alpha)|^{2^{k-1}} \ll N^{\frac{1}{k}(2^{k-1}-1)} + \\ + N^{\frac{1}{k}(2^{k-1}-k)} \sum_{0<h\le k!N^{1-\frac{1}{k}}} \tau_k(h) \min\left(N^{1/k},\ \|\alpha h\|^{-1}\right) . \tag{11.9}$$

We take a constant C, and divide the last summation over h into two parts according as $\tau_k(h) \le (\log N)^C$ or not. And we get

$$\sum_{h\le k!N^{1-\frac{1}{k}}} \tau_k(h) \min\left(N^{1/k},\ \|\beta\|^{-1}\right) \ll$$

$$\ll (\log N)^C \sum_{\substack{h\le k!N^{1-\frac{1}{k}} \\ \tau_k(h)\le(\log N)^C}} \min\left(N^{1/k},\ \|\beta\|^{-1}\right) +$$

$$+ (\log N)^{-C} \sum_{\substack{h\le k!N^{1-\frac{1}{k}} \\ \tau_k(h)>(\log N)^C}} \tau_k(h)^2 N^{1/k}$$

$$\ll N\left(q^{-1} + (\log N)^{-2C+k^2-2} + qN^{-1}\right)(\log N)^{C+1} , \tag{11.10}$$

here we use Lemma 2.2 of [**39**] and the well-known estimate $\sum_{h\le x} \tau_k(h)^2 \ll x(\log x)^{k^2-1}$. By (11.7), (11.9) and (11.10), we have Lemma 11.4 by choosing $C = (B_1 + k^2 - 2)/2$.

§12. Preliminary lemmata

First of all, we note that

LEMMA 12.1. *$\zeta_n(s)/\zeta(s)$ is an entire function.*

Lemma 12.1 is a consequence of Uchida [**38**] and van der Waall [**42**], because the Galois group of the extension $\mathbb{Q}(n^{1/k}, e(1/k))/\mathbb{Q}$ is solvable.

We define the set

$$E_k(N) = \left\{ n \le N; \text{the polynomial } x^k - n \text{ is irreducible in } \mathbb{Q}[x] \right\}.$$

We have easily

$$\sharp \{ n \le N;\ \ n \notin E_k(N) \} \ll \sqrt{N}, \tag{12.1}$$

where $\sharp$ denotes the cardinality of the indicated set.

Let D_n be the discriminant of $\mathbb{Q}(n^{1/k})$, and let D_n' be the discriminant of the polynomial $x^k - n$. It is known that $D_n' = (-1)^{k-1}k^k n^{k-1}$, and that there exist an integer D_n'' such that $D_n' = D_n(D_n'')^2$. Especially, we have

$$|D_n| \ll n^{k-1}. \tag{12.2}$$

By the functional equations of $\zeta(s)$ and $\zeta_n(s)$, we get, for $n \in E_k(N)$,

$$\frac{\zeta_n(s)}{\zeta(s)} = \left(2^{-r_2} \pi^{-\frac{k-1}{2}} \sqrt{|D_n|} \right)^{1-2s} \left(\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{1}{2}s)} \right)^{r_1 - 1} \left(\frac{\Gamma(1-s)}{\Gamma(s)} \right)^{r_2} \frac{\zeta_n(1-s)}{\zeta(1-s)},$$

where, as usual, r_1 and $2r_2$ are the numbers of real and complex conjugate fields of $\mathbb{Q}(n^{1/k})/\mathbb{Q}$, respectively. Then, by the Phragmen-Lindelöf theorem with (12.2), we have

LEMMA 12.2. *Let*

$$\kappa(\sigma) = \begin{cases} \frac{k-1}{2}(1-2\sigma) & (\sigma \le 0), \\ \frac{k-1}{2}(1-\sigma) & (0 \le \sigma \le 1), \\ 0 & (\sigma \ge 1), \end{cases}$$

and suppose that $n \in E_k(N)$, $T \ge 1$ and that $|t| \le T$. Then we have

$$\left| \frac{\zeta_n(s)}{\zeta(s)} \right| \ll (NT)^{\kappa(\sigma)+\varepsilon}$$

where the implied constant depends only on k and ε.

LEMMA 12.3. *We have*

$$\frac{\zeta(1)}{\zeta_n(1)} \gg (\log N)^{k-1}.$$

Lemma 12.3 follows from a known estimate for the residue of $\zeta_n(s)$ at $s=1$ (see, for example, [**21**, XVI, §1, p.322, proof of lemma 1 with $\alpha = \log N$]).

LEMMA 12.4. *Let $T \geq 2$ and $n \in E_k(N)$. Suppose that $s = \sigma + it$ is not a zero of $\zeta_n(s)/\zeta(s)$, and that $-1 \leq \sigma \leq 2$, $|t| \leq T$. Then we have*

$$\frac{\zeta_n'(s)}{\zeta_n(s)} - \frac{\zeta'(s)}{\zeta(s)} = \sum_{\rho:|\gamma-t|<1} \frac{1}{s-\rho} + O(\log NT) \ ,$$

where $\rho = \beta + i\gamma$ runs through all nontrivial zeros (that means $0 < \beta < 1$) of $\zeta_n(s)/\zeta(s)$ satisfying $|\gamma - t| < 1$.

Lemma 12.4 is proved by similar argument to Davenport's book [**7**, Ch.15, p.99 (4)]. We also use the results in Landau [**20**, Satz 180 and (161)]. In these results in [**20**], the dependency of n is not written explicitly, but we easily clarify it from the proofs in [**20**] and the inequality (12.2).

Let $\mathcal{N}(n;\alpha,T)$ be the number of zeros of $\zeta_n(s)/\zeta(s)$ in the region $\sigma \geq \alpha$ and $|t| \leq T$. Then Satz 181 in [**20**] states that

LEMMA 12.5.

$$\mathcal{N}(n;\frac{1}{2},T+1) - \mathcal{N}(n;\frac{1}{2},T-1) \ll \log NT \ .$$

LEMMA 12.6. *For $n \in E_k(N)$, $\delta > 0$ and $T \geq 2$, assume that $\mathcal{N}(n;1-\delta,T) = 0$ and that*

$$\frac{2}{T} \leq \eta \leq \frac{\delta}{4} < \frac{1}{8}.$$

Then there exist a positive constant c_0 such that

$$\max_{\substack{1-\eta\leq\sigma\leq 1+\eta \\ |t|\leq T/2}} \left| \frac{\zeta(s)}{\zeta_n(s)} \right| \ll \exp\left(\frac{c_0}{\eta} (\log NT)^{8\eta/\delta} \right) \ .$$

PROOF: Let $\log(\zeta(s)/\zeta_n(s))$ be the branch of the logarithm of $\zeta(s)/\zeta_n(s)$ that is zero at $s = \sigma = +\infty$. And let $a_n(f,p)$ be the number of prime ideals $\mathfrak{p}$ in $\mathbb{Q}(n^{1/k})$ such that the norm of $\mathfrak{p}$ is equal to p^f. By the Euler product for $\zeta(s)/\zeta_n(s)$, we get that if $\sigma \geq 1+\eta$ then

$$\begin{aligned}\log \frac{\zeta(s)}{\zeta_n(s)} &= \sum_p \log(1-\frac{1}{p^s})^{a_n(1,p)-1} + \sum_p \sum_{2\leq f\leq k} \log(1-\frac{1}{p^{fs}})^{a_n(f,p)} \\ &\ll \sum_p \frac{1}{p^{1+\eta}} + 1 \\ &\ll \frac{1}{\eta}\,. \end{aligned} \tag{12.3}$$

If $\sigma \geq 1-\frac{\delta}{2}$ and $|t| \leq T/2$ then we have, by Lemma 12.4,

$$\begin{aligned}\log \frac{\zeta(s)}{\zeta_n(s)} &= \int_{2+it}^{\sigma+it} \left(\frac{\zeta'(s)}{\zeta(s)} - \frac{\zeta_n'(s)}{\zeta_n(s)}\right) ds + \log \frac{\zeta(2+it)}{\zeta_n(2+it)} \\ &= \sum_{|\gamma-t|<1} \int_{\sigma+it}^{2+it} \frac{ds}{s-\rho} + O(\log NT). \end{aligned}$$

Since $\beta < 1-\delta$, we see $|s-\rho| \geq |\sigma-\beta| \geq \delta/2$, and, with Lemma 12.5, we get

$$\log \frac{\zeta(s)}{\zeta_n(s)} \ll \delta^{-1} \log NT\,. \tag{12.4}$$

Now we suppose $|1-\sigma| \leq \eta$ and $|t| \leq T/2$. For $j=1,2,3$ let C_j be the circle centered at $s_0 = \eta^{-1}+it$ of radius r_j, where $r_1 = \eta^{-1}-(1+\eta)$, $r_2 = \eta^{-1}-\sigma$, $r_3 = \eta^{-1}-(1+\eta-\frac{\delta}{2})$, so that $s=\sigma+it$ is on C_2. We set, for $j=1,2,3$,

$$M_j = \max_{s\in C_j} \left|\log \frac{\zeta(s)}{\zeta_n(s)}\right|\,.$$

The inequalities (12.3) and (12.4) imply

$$M_1 \ll \eta^{-1} \qquad \text{and} \qquad M_3 \ll \delta^{-1}\log NT\,,$$

respectively. Then it follows from Hadamard's three circle theorem that

$$M_2 \leq M_1^{1-a} M_3^a \ll \eta^{-1}(\log NT)^a\,,$$

with $a = (\log(r_2/r_1))/(\log(r_3/r_1))$. Therefore, our proof of Lemma 12.6 is completed by observing $a \leq 8\eta/\delta$.

Next we define arithmetical functions which are concerned with $\zeta_n(s)/\zeta(s)$. For $\sigma > 1$, we write

$$\frac{\zeta_n(s)}{\zeta(s)} = \sum_{m=1}^{\infty} \frac{\alpha_n(m)}{m^s}\,, \qquad \frac{\zeta(s)}{\zeta_n(s)} = \sum_{m=1}^{\infty} \frac{\alpha_n'(m)}{m^s}\,,$$

and put

$$\beta_n(m) = \mu(m)^2 \prod_{p|m} (\rho_n(p) - 1) .$$

We get at once $|\beta_n(m)| \leq \tau_{k-1}(m)$.

LEMMA 12.7. *For $n \in E_k(N)$, there exist arithmetical functions $\{\gamma_n(m)\}$, $\{\delta_n(m)\}$, $\{\gamma'_n(m)\}$ and $\{\delta'_n(m)\}$ which satisfy the following conditions;*

$$\alpha_n(m) = \sum_{m_1 m_2 m_3 = m} \beta_n(m_1)\gamma_n(m_2)\delta_n(m_3) , \tag{12.5}$$

$$\alpha'_n(m) = \sum_{m_1 m_2 m_3 = m} \mu(m_1)\beta_n(m_1)\gamma'_n(m_2)\delta'_n(m_3) , \tag{12.6}$$

$$|\delta_n(m)| \leq \tau_k(m) \quad \text{and} \quad |\delta'_n(m)| \leq \tau_k(m) \quad \text{for all } n \in E_k(N) \text{ and } m, \tag{12.7}$$

$$\sum_{\substack{m \leq M \\ |\delta_n(m)|>0}} 1 \ll (NM)^\varepsilon \quad \text{and} \quad \sum_{\substack{m \leq M \\ |\delta'_n(m)|>0}} 1 \ll (NM)^\varepsilon \quad \begin{array}{c}\text{for all } n \in E_k(N) \\ \text{and } M \geq 2.\end{array} \tag{12.8}$$

Further, there exists an arithmetical function $\{\gamma(m)\}$ satisfying the following conditions;

$$|\gamma_n(m)| \leq \gamma(m) \quad \text{and} \quad |\gamma'_n(m)| \leq \gamma(m) \quad \text{for all } n \in E_k(N) \text{ and } m, \tag{12.9}$$

$$\sum_{m \leq M} \frac{\gamma(m)}{m^{\sigma_0}} \ll (\log M)^{C_0} \quad \text{for any } M \geq 2, \tag{12.10}$$

where C_0 is a constant depending only on k, and $\sigma_0 = \frac{\log(k-1)}{\log(k+1)}$.

PROOF: Using the numbers $a_n(f,p)$ defined in the proof of Lemma 12.6, the Euler product for $\zeta_n(s)/\zeta(s)$ is written as follows;

$$\frac{\zeta_n(s)}{\zeta(s)} = \prod_p (1 - p^{-s})^{-a_n(1,p)+1} \prod_p \prod_{2 \leq f \leq k} (1 - p^{-fs})^{-a_n(f,p)} \quad (\sigma > 1).$$

For $n \in E_k(N)$, assume that

$$x^k - n \equiv \prod_{j=1}^{g} q_j(x)^{e_j} \pmod p,$$

where $q_j(x)$'s are distinct irreducible polynomials (mod p). It is known in algebraic number theory that, if $p \nmid D''_n$ then $a_n(f,p)$ is equal to the number of

$q_j(x)$'s such that the degree of $q_j(x)$ is equal to f. In particular, we have

$$a_n(1,p) = \rho_n(p),$$

providing that $n \in E_k(N)$ and $p \nmid kn$.

Now we put, for $\sigma > 1$,

$$\xi_{n,1}(s) = \prod_p \left(1 + \beta_n(p)p^{-s}\right)^{-1} \left(1 - p^{-s}\right)^{-\beta_n(p)} = \sum_{m=1}^{\infty} \frac{\gamma_n^{(1)}(m)}{m^s} ,$$

$$\xi_{n,2}(s) = \prod_p \prod_{2 \le f \le k} \left(1 - p^{-fs}\right)^{-a_n(f,p)} = \sum_{m=1}^{\infty} \frac{\gamma_n^{(2)}(m)}{m^s} ,$$

$$\xi_{n,3}(s) = \prod_p \left(1 - \beta_n(p)p^{-s}\right)^{-1} \left(1 - p^{-s}\right)^{\beta_n(p)} = \sum_{m=1}^{\infty} \frac{\gamma_n^{(3)}(m)}{m^s} ,$$

$$\Xi_n(s) = \prod_{p|kn} \left(1 - p^{-s}\right)^{-a_n(1,p)+\rho_n(p)} = \sum_{m=1}^{\infty} \frac{\delta_n(m)}{m^s} ,$$

$$\frac{1}{\xi_{n,2}(s)} = \sum_{m=1}^{\infty} \frac{\gamma_n^{(4)}(m)}{m^s} , \qquad \frac{1}{\Xi_n(s)} = \sum_{m=1}^{\infty} \frac{\delta_n'(m)}{m^s} ,$$

(12.11)

$$\gamma_n(m) = \sum_{m_1 m_2 = m} \gamma_n^{(1)}(m_1)\gamma_n^{(2)}(m_2), \quad \gamma_n'(m) = \sum_{m_1 m_2 = m} \gamma_n^{(3)}(m_1)\gamma_n^{(4)}(m_2) ,$$

then

$$\frac{\zeta_n(s)}{\zeta(s)} = \prod_p \left(1 + \beta_n(p)p^{-s}\right) \cdot \xi_{n,1}(s)\xi_{n,2}(s)\Xi_n(s) ,$$

and

(12.12)

$$\frac{\zeta(s)}{\zeta_n(s)} = \prod_p \left(1 - \beta_n(p)p^{-s}\right) \cdot \xi_{n,3}(s)\xi_{n,2}(s)^{-1}\Xi_n(s)^{-1} .$$

By these relations, we have (12.5) and (12.6) at once. Noticing $|a_n(1,p) - \rho_n(p)| \le k$, we get (12.7). The first estimate of (12.8) follows from

$$\sum_{\substack{m \le M \\ |\delta_n(m)|>0}} 1 \le M^\varepsilon \sum_{\substack{m \le M \\ |\delta_n(m)|>0}} \frac{1}{m^\varepsilon} \le M^\varepsilon \prod_{p|kn} (1 - p^{-\varepsilon})^{-1},$$

with the well-known bound

$$\nu(kn) \ll \frac{\log N}{\log\log N}, \tag{12.13}$$

where $\nu(m)$ denotes the number of distinct prime factors of m. The second estimate of (12.8) is obtained similarly.

We define the arithmetical function $\gamma^{(2)}(m)$ by

$$\prod_p \prod_{2\le f\le k} (1-p^{-fs})^{-k} = \left(\prod_{2\le f\le k} \zeta(fs)\right)^k = \sum_{m=1}^{\infty} \frac{\gamma^{(2)}(m)}{m^s} \qquad (\sigma > 1).$$

We see plainly that

$$|\gamma_n^{(j)}(m)| \le \gamma^{(2)}(m) \quad \text{for } j = 2,\ 4\ , \tag{12.14}$$

and that

$$\sum_{m\le M} \frac{\gamma^{(2)}(m)}{m^{1/2}} \le \prod_{p\le M} \prod_{2\le f\le k} \left(1-p^{-f/2}\right)^{-k} \ll (\log M)^k\ . \tag{12.15}$$

Next, we define the multiplicative function $\gamma^{(1)}(m)$ by the following 3 conditions;

(a) $\gamma^{(1)}(p) = 0$ for all prime p,

(b) $\gamma^{(1)}(p^l) = (\min(k-1,\ p-2))^l\,(l+k-2)^{k-1}$
for all odd primes p and for $l \ge 2$,

(c) $\gamma^{(1)}(2^l) = 1$ for $l \ge 2$.

Since $|\beta_n(p)| \le \min(k-1,\ p-2)$, simple calculation shows

$$|\gamma_n^{(j)}(m)| \le \gamma^{(1)}(m) \quad \text{for } j = 1,\ 3\ . \tag{12.16}$$

For $p \ge k+2$, there is a constant C_1, depending only on k, such that

$$\sum_{2\le l\le \log M} \gamma^{(1)}(p^l)p^{-l\sigma_0} \le C_1 p^{-2\sigma_0}.$$

Noticing that $p^{-\sigma_0}\min(k-1,\ p-2) \le 1$ and $\sigma_0 \ge \frac{1}{2}$, we get

$$\begin{aligned}
\sum_{m\le M} \frac{\gamma^{(1)}(m)}{m^{\sigma_0}} &\ll \prod_{2<p\le M} \left(1 + \sum_{2\le l\le \log M} \gamma^{(1)}(p^l)p^{-l\sigma_0}\right) \\
&\ll \prod_{2<p<k+2} \left(1 + \sum_{2\le l\le \log M} (l+k-2)^{k-1}\right) \cdot \prod_{k+2\le p\le M} (1 + C_1 p^{-2\sigma_0}) \\
&\ll (\log M)^{C_2}\ , \qquad (12.17)
\end{aligned}$$

where C_2 is a constant depending only on k.

Now, let

$$\gamma(m) = \sum_{m_1 m_2 = m} \gamma^{(1)}(m_1)\gamma^{(2)}(m_2) ,$$

then (12.9) and (12.10) follow from (12.11), (12.14), (12.15), (12.16) and (12.17), and our proof of Lemma 12.7 is complete.

For a squarefree integer m, we define two sets C_m and C_m^* of Dirichlet characters as follows;

$$\begin{aligned} C_m &= \{\chi (\mathrm{mod}\ m);\ \chi^k = \chi_{0,m} \text{ and } \chi \neq \chi_{0,m}\ \} , \\ C_m^* &= \{\chi \in C_m;\ \chi \text{ is primitive. }\} , \end{aligned}$$

where $\chi_{0,m}$ denotes the principal character (mod m). As for the cardinalities $\sharp C_m$ and $\sharp C_m^*$, it is easily seen that

$$\sharp C_m^* \leq \sharp C_m \leq \tau_k(m) - 1. \tag{12.18}$$

LEMMA 12.8. *For a squarefree integer m, we have*

$$\beta_n(m) = \sum_{\chi \in C_m^*} \chi(n) .$$

PROOF: This is the equation (10) in [**33**]. It suffices to show that, for any prime p,

$$\beta_n(p) = \sum_{\chi \in C_p^*} \chi(n) , \tag{12.19}$$

because of multiplicity. If $p \mid n$ then $\rho_n(p) = 1$ and (12.19) is true. If $p \nmid n$ then

$$\begin{aligned} \rho_n(p) &= \frac{1}{\varphi(p)} \sum_{\chi \ (\mathrm{mod}\ p)} \chi(n) \sum_{l=1}^{p} \overline{\chi}^k(l) \\ &= \sum_{\chi^k = \chi_{0,p}} \chi(n) \ = \ 1 + \sum_{\chi \in C_p^*} \chi(n) . \end{aligned}$$

Thus, in both cases, we have (12.19) as required.

We state here three well-known results without proof.

LEMMA 12.9. *Let $0 < y < 1$ or $y > 1$ and let*

$$\delta(y) = \begin{cases} 0 & (0 < y < 1), \\ 1 & (y > 1). \end{cases}$$

Then, for $c > 0$, $T > 0$, we have

$$\frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{y^s}{s}\, ds = \delta(y) + O\left(y^c \min(1,\ T^{-1}|\log y|^{-1}) \right).$$

LEMMA 12.10. *Suppose $0 < \theta < 1/2$ and $x \geq 2$. Then we have*

$$\sum_{x-y<m\leq x} \tau_k(m) \ll y(\log x)^{k-1},$$

uniformly in y providing $x^\theta < y \leq x$.

LEMMA 12.11. *For any sequence $\{a_n\}$ of complex numbers, and for $Q \geq 1$, we have*

$$\sum_{q\leq Q} \frac{q}{\varphi(q)} \sideset{}{^*}\sum_{\chi \pmod q} \left| \sum_{x-y<n\leq x} a_n\chi(n) \right|^2 \ll (y+Q^2) \sum_{x-y<n\leq x} |a_n|^2 .$$

Here, $\sideset{}{^*}\sum_{\chi \pmod q}$ indicates a sum over all primitive characters $\chi \pmod q$.

For a proof of Lemma 12.9, see, for example, Davenport [7, p.105, Lemma]. Lemma 12.10 is a special case of Shiu [36, Theorem 2]. Lemma 12.11 is the well known large sieve inequality due to Gallagher [11].

§13. An asymptotic formula for $\mathfrak{S}(n,Q)$

In this section, we treat our singular series

$$\mathfrak{S}_k(n,Q) = \sum_{q\le Q} \frac{\mu(q)}{\varphi(q)}\beta_n(q)\,,$$

and our goal is to prove the following lemma.

LEMMA 13.1. *Let M be a real number satisfying*

$$\exp\left((\log N)^{c_1}\right) \le M \le N$$

with some constant $0 < c_1 \le 1$. We assume $n \in E_k(N)$ and

$$\mathcal{N}\big(n; 1-\delta,\, 2\exp(2\sqrt{\log M})\big) = 0$$

with some constant $\delta > 0$. Then we have

$$\mathfrak{S}_k(n,M) = \mathfrak{S}_k(n) \;+\; O\Big(\exp(-(\log N)^{-c_1/2})\Big)\,, \tag{13.1}$$

and

$$\mathfrak{S}_k(n) \gg (\log N)^{-k+1}(\log\log N)^{-k}\,. \tag{13.2}$$

PROOF: We put, for $\sigma > 1$,

$$Z_n(s) = \sum_{m=1}^{\infty} \frac{\mu(m)}{\varphi(m)m^{s-1}}\beta_n(m) = \prod_p \left(1 - \frac{\beta_n(p)}{(p-1)p^{s-1}}\right).$$

Simple calculation with (12.12) gives

$$\begin{aligned} Z_n(s) &= \prod_p \left(1-\frac{\beta_n(p)}{p^s}\right)\cdot\prod_p\left(1-\frac{\beta_n(p)}{(p-1)p^{s-1}}\right)\left(1-\frac{\beta_n(p)}{p^s}\right)^{-1} \\ &= \frac{\zeta(s)}{\zeta_n(s)}\Xi_n(s)\frac{\xi_{n,2}(s)}{\xi_{n,3}(s)}\prod_p\left(1-\frac{\beta_n(p)}{(p^s-\beta_n(p))(p-1)}\right) \\ &= \frac{\zeta(s)}{\zeta_n(s)}\Xi_n(s)\xi_n(s) \quad \text{, say.} \end{aligned}$$

We may assume $1-\delta > \sigma_0$, and set $T = \exp\left(2\sqrt{\log M}\right)$. By (12.14), (12.15), (12.16), (12.17) and the fact $p^{\sigma_0} \ge \min(k-1, p-2) \ge |\beta_n(p)|$ for all primes p, we see that $\xi_n(s)$ is analytic in the half plane $\sigma > \sigma_0$, and that

$$1 \ll |\xi_n(s)| \ll 1\,, \tag{13.3}$$

for $\sigma > \sigma_0 + \varepsilon$. Noticing Lemma 12.1 and the assumption $\mathcal{N}(n; 1-\delta, 2T) = 0$, we see that $Z_n(s)$ is analytic in the region $\sigma > 1-\delta$, $|t| \le 2T$.

For $\sigma \geq 1-\eta$ with some $0<\eta<1$, we get easily

$$(13.4) \qquad |\Xi_n(s)| \ll \exp\left(k\sum_{p|kn}\frac{1}{p^{1-\eta}}+O(1)\right) \ll \exp((\log N)^{\eta}),$$

and

$$(13.5) \qquad |\Xi_n(1)^{-1}| \ll (\log\log N)^k,$$

because of (12.13). From (13.3), (13.5) and Lemma 12.3, we have (13.2).

Now we suppose $2/T \leq \eta \leq \delta/4$, then it follows from (13.3), (13.4) and Lemma 12.6 that

$$\max_{\substack{1-\eta\leq\sigma\leq 1+\eta \\ |t|\leq T}} |Z_n(s)| \ll \exp\left(\frac{c_0}{\eta}(\log NT)^{8\eta/\delta}+(\log N)^{\eta}\right)$$

$$(13.6) \qquad \ll \exp\left(\frac{2c_0}{\eta}(\log N)^{8\eta/\delta}\right).$$

Without loss of generality, we suppose that M is not an integer. Making use of Lemma 12.9, we obtain, with $b=1+(\log N)^{-1}$,

$$\mathfrak{S}(n,M) = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} Z_n(s)\frac{M^{s-1}}{s-1}\,ds + O\left(\sum_{m=1}^{\infty}R_m\right)$$

(13.7)

$$= Z_n(1) + \frac{1}{2\pi i}\left(\int_{b-iT}^{1-\eta-iT}+\int_{1-\eta-iT}^{1-\eta+iT}+\int_{1-\eta+iT}^{b+iT}\right) Z_n(s)\frac{M^{s-1}}{s-1}\,ds + O\left(\sum_{m=1}^{\infty}R_m\right),$$

where

$$R_m = \tau_k(m)m^{-b}\min\left(1,\ T^{-1}|\log\frac{M}{m}|^{-1}\right),$$

here we note that $\frac{p}{p-1}|\beta_n(p)| \leq k$ for all primes p. It is easily obtained by (13.6) that

(13.8)

$$\left|\left(\int_{b-iT}^{1-\eta-iT}+\int_{1-\eta+iT}^{b+iT}\right) Z_n(s)\frac{M^{s-1}}{s-1}\,ds\right| \ll T^{-1}\exp\left(\frac{2c_0}{\eta}(\log N)^{8\eta/\delta}\right),$$

and that

$$(13.9) \qquad \left|\int_{1-\eta-iT}^{1-\eta+iT} Z_n(s)\frac{M^{s-1}}{s-1}\,ds\right| \ll M^{-\eta}(\log T)\exp\left(\frac{2c_0}{\eta}(\log N)^{8\eta/\delta}\right).$$

We turn to estimate $\sum_{m=1}^{\infty} R_m$. We see clearly

$$\left(\sum_{m\le\frac{1}{2}M} + \sum_{m\ge\frac{3}{2}M}\right) R_m \ll T^{-1}\sum_{m=1}^{\infty}\frac{\tau_k(m)}{m^b} = T^{-1}\zeta(b)^k$$

$$\ll T^{-1}(\log N)^k\ . \tag{13.10}$$

For $\sqrt{M}\le U\le M/2$, if $U<|m-M|\le 2U$ then

$$|\log\frac{M}{m}| = |\log\left(1+\frac{M-m}{m}\right)| \gg \frac{U}{m}$$

and, using Lemma 12.10,

$$\sum_{U<|m-M|\le 2U} R_m \ll T^{-1}U^{-1}\sum_{U<|m-M|\le 2U}\tau_k(m) \ll T^{-1}(\log M)^{k-1}.$$

So that we have

$$\left(\sum_{\frac{1}{2}M<m<M-\sqrt{M}} + \sum_{M+\sqrt{M}<m<\frac{3}{2}M}\right) R_m \ll T^{-1}(\log M)^k \tag{13.11}$$

We use Lemma 12.10 again to get

$$\sum_{M-\sqrt{M}\le m\le M+\sqrt{M}} R_m \ll M^{-1}\sum_{M-\sqrt{M}\le m\le m+\sqrt{M}}\tau_k(m) \ll M^{-1/2}(\log M)^{k-1}. \tag{13.12}$$

We choose $\eta=\frac{c_1\delta}{32}$, then, with (13.8)-(13.12), the formula (13.7) shows (13.1), and which completes our proof of Lemma 13.1.

§14. Fundamental Lemmata

In the study of zero density estimates for Dirichlet L-functions, the large sieve inequality (Lemma 12.11) is one of main tools. The following Lemma 14.1 takes similar play as the large sieve inequality in our estimate for $\mathcal{N}(n;\sigma,T)$.

LEMMA 14.1. *Let $\{a_m\}$ be any sequence of complex numbers, and let $0 < y \le x$. Then we have*

$$\sum_{n\le N}\left|\sum_{x-y<m\le x} a_m\beta_n(m)\right|^2 \ll (N + xy\log x)\sum_{x-y<m\le x}\mu(m)^2\tau_k(m)^2|a_m|^2 .$$

PROOF: By Lemma 12.8, we get

$$\sum_{n\le N}\left|\sum_{x-y<m\le x} a_m\beta_n(m)\right|^2 =$$

$$= \sum_{x-y<m_1}\sum_{,\, m_2\le x}\mu(m_1)^2\mu(m_2)^2 a_{m_1}\overline{a_{m_2}}\sum_{\chi_1\in C^*_{m_1}}\sum_{\chi_2\in C^*_{m_2}}\sum_{n\le N}\chi_1\overline{\chi_2}(n).$$

Since χ_1 and χ_2 are primitive characters, if $\chi_1\overline{\chi_2}$ is principal, then $m_1 = m_2$ and $\chi_1 = \chi_2$. Thus, by (12.18),

$$\sum_{n\le N}\left|\sum_{x-y<m\le x} a_m\beta_n(m)\right|^2 \ll N\sum_{x-y<m\le x}\mu(m)^2\tau_k(m)|a_m|^2 + S_{x,y} \ , \tag{14.1}$$

where

$$S_{x,y} = \sum_{x-y<m_1,}\sum_{m_2\le x}\mu(m_1)^2\mu(m_2)^2|a_{m_1}a_{m_2}|\times$$

$$\times\sum_{\chi_1\in C^*_{m_1}}\sum_{\substack{\chi_2\in C^*_{m_2}\\ \chi_1\overline{\chi_2}\text{ is non-principal.}}}\left|\sum_{n\le N}\chi_1\overline{\chi_2}(n)\right| .$$

Making use of the Pólya-Vinogradov inequality, we have

$$\begin{aligned} S_{x,y} &\ll x\log x\sum_{x-y<m_1}\sum_{,\, m_2\le x}\mu(m_1)^2\mu(m_2)^2\tau_k(m_1)\tau_k(m_2)|a_{m_1}a_{m_2}| \\ &\ll xy\log x\sum_{x-y<m\le x}\mu(m)^2\tau_k(m)^2|a_m|^2 , \end{aligned}\tag{14.2}$$

Lemma 14.1 follows from (14.1) and (14.2).

When $x > N$, Lemma 14.1 gives only a trivial bound. In this case, we need, instead of the Pólya-Vinogradov inequality, a non-trivial bound for the sum

$$\sum_{\substack{m \le M \\ \mu(m)^2=1}} \sum_{\chi \in C_m} \left| \sum_{n \le N} \chi(n) \right| ,$$

which is trivially $\ll NM(\log M)^{k-1}$. We shall prove

LEMMA 14.2. *Let h be a natural number, and let r be a natural number satisfying*

$$N^{r(r-1)} \ge M^{r+1}. \tag{14.3}$$

Then we have

$$\sum_{\substack{m \le M \\ \mu(m)^2=1}} \tau_h(m) \sum_{\chi \in C_m} \left| \sum_{n \le N} \chi(n) \right| \ll N^{1-\frac{1}{r+1}} M (\log N)^{\frac{r}{2}+2h^2k-1}. \tag{14.4}$$

PROOF: We denote the conductor of a character χ by cond.χ. If $\chi \in C_m$ is induced by $\chi_1 \pmod{l}$, then we have, by the Pólya-Vinogradov inequality,

$$\begin{aligned} \sum_{n \le N} \chi(n) = \sum_{\substack{n \le N \\ (n, \frac{m}{l})=1}} \chi_1(n) &= \sum_{d | \frac{m}{l}} \mu(d)\chi_1(d) \sum_{n' \le \frac{N}{d}} \chi_1(n') \\ &\ll \tau(m)\sqrt{l} \log l \, . \end{aligned}$$

Thus,

$$\sum_{\substack{m \le M \\ \mu(m)^2=1}} \tau_h(m) \sum_{\substack{\chi \in C_m \\ \text{cond.}\chi \le L}} \left| \sum_{n \le N} \chi(n) \right| \ll M\sqrt{L}(\log LM)^{hk}, \tag{14.5}$$

where L is a parameter chosen later.

We estimate the sum over χ with cond.$\chi > L$ by the method indicated in Elliott [10]. For a natural number r, we evaluate the sum

$$S = \sum_{m \le M} \sum_{\substack{\chi \pmod{m} \\ \text{cond.}\chi > L}} \left| \sum_{n \le N} \chi(n) \right|^{2r}$$

by the large sieve. We denote, by $\tau_{r,N}(n)$, the cardinality of the set

$$\{(n_1, n_2, \cdots, n_r)\,;\ n_j \le N \ \text{ for all } 1 \le j \le r, \quad \text{and } \prod_{j=1}^{r} n_j = n\},$$

so that

$$\left|\sum_{n\le N} \chi(n)\right|^{2r} = \left|\sum_{n\le N^r} \tau_{r,N}(n)\chi(n)\right|^2. \tag{14.6}$$

When $\chi \pmod{m}$ is induced $\chi_1 \pmod{l}$, we have

$$\begin{aligned} \sum_{n\le N^r} \tau_{r,N}(n)\chi(n) &= \sum_{\substack{n\le N^r\\ (n,\frac{m}{l})=1}} \tau_{r,N}(n)\chi_1(n) \\ &= \sum_{d\mid \frac{m}{l}} \mu(d) \sum_{\substack{n\le N^r\\ n\equiv 0 \pmod{d}}} \tau_{r,N}(n)\chi_1(n)\,. \end{aligned} \tag{14.7}$$

It follows from (14.6), (14.7) and Cauchy's inequality that

$$\begin{aligned} S &\le \sum_{L<l\le M} \sideset{}{^*}\sum_{\chi \pmod{l}} \sum_{\substack{m\le M\\ m\equiv 0 \pmod{l}}} \tau\left(\frac{m}{l}\right) \sum_{d\mid \frac{m}{l}} \left|\sum_{\substack{n\le N^r\\ n\equiv 0 \pmod{d}}} \tau_{r,N}(n)\chi(n)\right|^2 \\ &\ll M\log M \sum_{d\le M} \frac{\tau(d)}{d} \sum_{L<l\le \frac{M}{d}} \frac{1}{l} \sideset{}{^*}\sum_{\chi \pmod{l}} \left|\sum_{\substack{n\le N^r\\ n\equiv 0 \pmod{d}}} \tau_{r,N}(n)\chi(n)\right|^2. \end{aligned} \tag{14.8}$$

Making use of Lemma 12.11, we have, for $L \le U \le \frac{M}{d}$,

$$\sum_{U<l\le 2U} \frac{1}{l} \sideset{}{^*}\sum_{\chi \pmod{l}} \left|\sum_{\substack{n\le N^r\\ n\equiv 0 \pmod{d}}} \tau_{r,N}(n)\chi(n)\right|^2 \ll$$

$$\ll \left(U + \frac{N^r}{U}\right) \tau_r(d)^2 \frac{N^r}{d} (\log N)^{r^2-1}.$$

Summing this inequality over $U = L\cdot 2^j$ for an appropriate range of j, we obtain, from (14.8),

$$S \ll M \log M \sum_{d \le M} \frac{\tau(d)\tau_r(d)^2}{d^2} \left(\frac{M}{d} + \frac{N^r}{L}\right) N^r (\log N)^{r^2-1}$$

$$\ll MN^{2r}L^{-1}(\log N)^{r^2}, \tag{14.9}$$

providing that

$$N^r \ge ML. \tag{14.10}$$

We deduce from (14.9) and Hölder's inequality that

$$\sum_{\substack{m \le M \\ \mu(m)^2=1}} \tau_h(m) \sum_{\substack{\chi \in C_m \\ \mathrm{cond}.\chi > L}} \left| \sum_{n \le N} \chi(n) \right| \ll$$

$$\ll \left(\sum_{\substack{m \le M \\ \mu(m)^2=1}} \tau_k(m)\tau_h(m)^{2r/(2r-1)} \right)^{1-\frac{1}{2r}} \cdot S^{\frac{1}{2r}}$$

$$\ll NML^{-\frac{1}{2r}}(\log N)^{\frac{r}{2}+h^2k-1}. \tag{14.11}$$

Now we take

$$L = N^{\frac{2r}{r+1}},$$

then (14.10) is equivalent to (14.3), and the required bound (14.4) follows from (14.5) and (14.11). Hence we get Lemma 14.2.

We use Lemma 14.2 to prove the following lemma.

LEMMA 14.3. *Let $\{a_m\}$ be any sequence of complex numbers, and let r be a natural number satisfying $N^{r(r-1)} \ge x^{2(r+1)}$. Then we have*

$$\sum_{n \le N} \left| \sum_{m \le x} a_m \beta_n(m) \right|^2 \ll N \sum_{m \le x} \mu(m)^2 \tau_k(m) |a_m|^2 +$$

$$+ N^{1-\frac{1}{r+1}} \max_{M \le x} \left(M \max_{M < m \le 2M} |a_m| \right)^2 \cdot (\log N)^{\frac{r}{2}+18k^5+1}.$$

PROOF: In view of (14.1) with $x = y$, it suffices to show that

$$S_{x,x} \ll N^{1-\frac{1}{r+1}} \max_{M \le x} \left(M^2 \max_{M < m \le 2M} |a_m|^2 \right) (\log N)^{\frac{r}{2}+18k^5+1}.$$

We put, for $M_1,\ M_2 \le x$,

$$S(M_1,\, M_2) =$$

$$= \sum_{\substack{M_1<m_1\le 2M_1\\ \mu(m_1)^2=1}} \sum_{\substack{M_2<m_2\le 2M_2\\ \mu(m_2)^2=1}} |a_{m_1}a_{m_2}| \sum_{\chi_1\in C^*_{m_1}} \sum_{\substack{\chi_2\in C^*_{m_2}\\ \chi_1\overline{\chi_2} \text{ is non-principal.}}} \left|\sum_{n\le N} \chi_1\overline{\chi_2}(n)\right| ,$$

then, because $\chi_1\overline{\chi_2} \in C_m$, we have

$$S(M_1,\, M_2) \ll \max_{\substack{M_1<m_1\le 2M_1\\ M_2<m_2\le 2M_2}} |a_{m_1}a_{m_2}| \times$$

$$\times \sum_{\substack{m\le 4M_1M_2\\ \mu(m)^2=1}} \tau_k(m)^2 \left(\sum_{[m_1,\, m_2]=m} 1\right) \sum_{\chi\in C_m} \left|\sum_{n\le N} \chi(n)\right| .$$

Since $(M_1M_2)^{r+1} \le x^{2(r+1)} \le N^{r(r-1)}$, we obtain, by virtue of Lemma 14.2,

$$S(M_1,\, M_2) \ll N^{1-\frac{1}{r+1}} M_1M_2 \max_{\substack{M_1<m_1\le 2M_1\\ M_2<m_2\le 2M_2}} |a_{m_1}a_{m_2}| \cdot (\log N)^{\frac{r}{2}+18k^5-1},$$

and

$$\begin{aligned} S_{x,x} &\ll (\log x)^2 \max_{\substack{M_1\le x\\ M_2\le x}} S(M_1,\, M_2) \\ &\ll N^{1-\frac{1}{r+1}} \left(\max_{M\le x} M^2 \max_{M<m\le 2M} |a_m|^2\right) (\log N)^{\frac{r}{2}+18k^5+1}, \end{aligned}$$

as required.

§15. Mean value estimate for $\zeta_n(s)/\zeta(s)$

LEMMA 15.1. *For a natural number r, let*

$$\sigma_1 = 1 - \frac{1}{r(r-1)}\,.$$

We suppose $\sigma_1 > \sigma_0$ and

$$(NT)^{(r+1)(k-1)(3-2\sigma_1)} \le N^{r(r-1)}. \tag{15.1}$$

Then we have

$$\sum_{n\in E_k(N)} \left|\frac{\zeta_n(s)}{\zeta(s)}\right|^2 \ll N^{1+\varepsilon}, \tag{15.2}$$

uniformly for s satisfying $|t| \le T$ and $\sigma \ge \sigma_1 - (\log NT)^{-1}$.

PROOF: If $\sigma \ge 1$ then (15.2) is true by Lemma 12.2, so we assume $\sigma < 1$. We apply the method of Ramachandra [**35**].

We put $X = (NT)^{(k-1)(3-2\sigma)/2}$ and $Y = X(\log NT)^{-2}$. Making use of a well-known Mellin transform, we have

$$\sum_{m=1}^{\infty} \frac{\alpha_n(m)}{m^s} e^{-m/Y} = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \frac{\zeta_n(s+w)}{\zeta(s+w)}\Gamma(w)Y^w\,dw\,. \tag{15.3}$$

We shift the contour to the line $\Re w = -1 + (\log NT)^{-1}$. By Lemma 12.2 and a known estimate $\Gamma(w) \ll e^{-|\Im w|}$, the right-hand side of (15.3) becomes

$$\frac{\zeta_n(s)}{\zeta(s)} + \frac{1}{2\pi i}\int\limits_{-1+(\log NT)^{-1}-i\infty}^{-1+(\log NT)^{-1}+i\infty} \frac{\zeta_n(s+w)}{\zeta(s+w)}\Gamma(w)Y^w\,dw = \frac{\zeta_n(s)}{\zeta(s)} + O(N^\varepsilon).$$

On the other hand, the left-hand side of (15.3) turns into

$$\sum_{m\le X} \frac{\alpha_n(m)}{m^s} e^{-m/Y} + O(1) =$$

$$= \sum_{m_2m_3 \le X}\sum \frac{\gamma_n(m_2)}{m_2^s}\frac{\delta_n(m_3)}{m_3^s} \sum_{m_1 \le \frac{X}{m_2m_3}} \frac{\beta_n(m_1)}{m_1^s} e^{-m_1m_2m_3/Y} + O(1),$$

by (12.5). Accordingly we get, by (12.7), (12.8), (12.9), (12.10) and Cauchy's inequality,

$$\sum_{n\in E_k(N)}\left|\frac{\zeta_n(s)}{\zeta(s)}\right|^2 \ll N^{1+\varepsilon}+\sum_{n\in E_k(N)}\left(\sum_{m_2m_3\le X}\sum\frac{\gamma(m_2)}{m_2^{\sigma_0}}|\delta_n(m_3)|^2\right)\times$$

$$\times\left(\sum_{m_2m_3\le X}\sum\frac{\gamma(m_2)}{m_2^{2\sigma-\sigma_0}}\frac{1}{m_3^{2\sigma}}\left|\sum_{m_1\le\frac{X}{m_2m_3}}\frac{\beta_n(m_1)}{m_1^s}e^{-m_1m_2m_3/Y}\right|^2\right)$$

$$\ll N^{1+\varepsilon}+N^{\varepsilon}\sum_{m_2m_3\le X}\sum\frac{\gamma(m_2)}{m_2^{2\sigma-\sigma_0}}\frac{1}{m_3^{2\sigma}}\sum_{n\le N}\left|\sum_{m_1\le\frac{X}{m_2m_3}}\frac{\beta_n(m_1)}{m_1^s}e^{-m_1m_2m_3/Y}\right|^2.$$

Since $2\sigma-\sigma_0>\sigma_0$ and $2\sigma>1$, in view of (12.10), it suffices to show that

$$\sum_{n\le N}\left|\sum_{m\le X/l}\frac{\beta_n(m)}{m^s}e^{-ml/Y}\right|^2 \ll N^{1+\varepsilon}, \tag{15.4}$$

for any $l\le X$. We note here $X^{2(r+1)}\le N^{r(r-1)}$ by (15.1). Therefore, by virtue of Lemma 14.3, the left-hand side of (15.4) is

$$\ll N+N^{1-\frac{1}{r+1}+\varepsilon}X^{2(1-\sigma)}\ll N+N^{1-\frac{1}{r+1}(1-r(r-1)(1-\sigma))+\varepsilon},$$

and which is $\ll N^{1+\varepsilon}$ providing that $\sigma\ge\sigma_1-(\log NT)^{-1}$. Now we obtain Lemma 15.1.

§16. Zero density estimate

From Lemmas 14.1 and 15.1 we obtain the following two lemmata which we use to estimate the zero density for $\zeta_n(s)/\zeta(s)$'s.

LEMMA 16.1. *Let $T \geq \delta > 0$. And let $\mathcal{S}_n$ be a finite set of complex numbers $s = \sigma + it$ with the following properties;*

$$(a)\ T_0 + \frac{\delta}{2} \leq t \leq T_0 + T - \frac{\delta}{2} \quad \text{and} \quad \sigma \geq c \quad \text{for all } s = \sigma + it \in \mathcal{S}_n\,,$$
$$(b)\ |t - t'| \geq \delta \quad \text{for all distinct } s = \sigma + it \text{ and } s' = \sigma' + it' \text{ in } \mathcal{S}_n\,.$$

Then we have

$$\sum_{n \leq N} \sum_{s \in \mathcal{S}_n} \left| \sum_{m \leq M} a_m \beta_n(m) m^{-s} \right|^2 \ll$$
$$\ll (NT + M^2 \log M)(\delta^{-1} + \log M) \sum_{m \leq M} |a_m|^2 \tau_k(m)^2 m^{-2c} (1 + \log \frac{\log 2M}{\log 2m}).$$

LEMMA 16.2. *Let $T \geq \delta > 0$. And let $\mathcal{T}_n$ be a finite set of real numbers in the interval $[T_0 + \frac{\delta}{2}, T_0 + T - \frac{\delta}{2}]$ with the property;*

$$|t - t'| \geq \delta \qquad \text{for distinct } t,\ t' \in \mathcal{T}_n.$$

Assume that σ_1 and r satisfy the conditions in Lemma 15.1. Then we have

$$\sum_{n \in E_k(N)} \sum_{t \in \mathcal{T}_n} \left| \frac{\zeta_n(\sigma + it)}{\zeta(\sigma + it)} \right|^2 \ll N^{1+\varepsilon} T(\delta^{-1} + \log NT),$$

for $\sigma \geq \sigma_1$.

Theorem 7.5 of Montgomery's lecture note [**27**, Ch.7] is derived from the large sieve inequality (Lemma 12.11), using two important lemmata [**27**, Lemma 1.2 and Lemma 1.10], both of which are due to Gallagher. The proof of Lemma 16.1 proceeds on the same lines, except that we use Lemma 14.1 instead of the large sieve inequality.

The proof of Lemma 16.2 is almost the same to that of Theorem 10.3 of [**27**], except for using Lemma 15.1 instead of Theorem 10.1 of [**27**].

The purpose of this section is to prove the following Lemma 16.3.

LEMMA 16.3. *Let $T \geq 1$ and assume that σ_1 and r satisfy the conditions in*

Lemma 15.1. Then we have, for $1/2 \le \sigma < 1$,

$$\sum_{n \in E_k(N)} \mathcal{N}(n;\, \sigma,\, T) \ll (NT)^{1-\frac{\sigma-\sigma_1}{3-\sigma-\sigma_1}+\varepsilon}.$$

PROOF: We prove Lemma 16.3 by standard argument (see [**27**, Ch.12, pp.103–110]). Let X and Y be parameters satisfying $2 \le X \le Y$ and $\log XY \ll \log NT$. We put

$$H_n(s) = \sum_{m \le X} \frac{\alpha_n'(m)}{m^s},$$

and

$$J_n(s) = \frac{\zeta_n(s)}{\zeta(s)} H_n(s).$$

For $\sigma > 1$, it is clear that

$$J_n(s) = 1 + \sum_{m > X} \frac{\alpha_n''(m)}{m^s},$$

where

$$\alpha_n''(m) = \sum_{\substack{d \mid m \\ d \le X}} \alpha_n\Big(\frac{m}{d}\Big)\alpha_n'(d). \tag{16.1}$$

Now we suppose

$$\sigma_1 + \log NT \le \sigma < 1,$$

because, otherwise, Lemma 16.3 is trivial from Lemma 12.5. We get, using a Mellin transform,

$$e^{-1/Y} + \sum_{m > X} \alpha_n''(m) m^{-s} e^{-m/Y} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} J_n(s+w)\Gamma(w)Y^w \, dw\,.$$

We shift the contour to the line $\Re w = \sigma_1 - \sigma$, then noticing $-1 < \sigma_1 - \sigma < 0$, the last integral turns into

$$\frac{\zeta_n(s)}{\zeta(s)} H_n(s) + J,$$

where

$$J = \frac{1}{2\pi} \int_{-\infty}^{\infty} J_n(\sigma_1 + it + iu)\Gamma(\sigma_1 - \sigma + iu)Y^{\sigma_1-\sigma+iu} \, du\,.$$

We set $Z = (\log NT)^2$. It is easily seen that

$$\begin{aligned} |J| &\le \frac{1}{2\pi}\left(\left|\int_{|u|\le Z}\right| + \left|\int_{|u|\ge Z}\right|\right) \\ &\le B_3 Y^{\sigma_1-\sigma} \log NT \cdot \max_{|u|\le Z} |J_n(\sigma_1 + i(t+u))| + \frac{1}{4}e^{-1/2}, \end{aligned}$$

where B_3 is an absolute constant.

It is also easily observed that

$$\left|\sum_{m>X} \alpha_n''(m) m^{-s} e^{-m/Y}\right| \leq |G_n(s)| + \frac{1}{4}e^{-1/2},$$

where

$$G_n(s) = \sum_{X<m\leq YZ} \alpha_n''(m) m^{-s} e^{-m/Y}.$$

Taking account of these results, when $s = \rho = \beta + i\gamma$ is a zero of $\zeta_n(s)/\zeta(s)$, we get

$$e^{-1/2} \leq |G_n(\rho)| + Y^{\sigma_1-\beta} Z \max_{|t-\gamma|\leq Z} |J_n(\sigma_1 + it)| + \frac{1}{2}e^{-1/2}.$$

Thus, we have whether

$$|G_n(\rho)| \geq \frac{1}{4}e^{-1/2}, \tag{16.2}$$

or

$$Y^{\sigma_1-\beta} Z |J_n(\sigma_1 + it_\rho)| \geq \frac{1}{4}e^{-1/2},$$

for some $t_\rho \in [\gamma - Z,\ \gamma + Z]$.

Then, in view of Lemma 12.5, for each $n \in E_k(N)$, we can define the sets $\mathcal{R}_n^{(1)}$ and $\mathcal{R}_n^{(2)}$ of zeros $\rho = \beta + i\gamma$ of $\zeta_n(s)/\zeta(s)$ which satisfy the following 5 conditions;

(a) $\rho \in \mathcal{R}_n^{(j)}$ $(j = 1, 2)$ implies $\beta \geq \sigma$ and $|\gamma| \leq T$.

(b) For any distinct $\rho = \beta + i\gamma$, $\rho' = \beta' + i\gamma' \in \mathcal{R}_n^{(j)}$ $(j = 1, 2)$ we have
$$|\gamma - \gamma'| \geq 3Z.$$

(c) If $\rho \in \mathcal{R}_n^{(1)}$ then we have (16.2).

(d) If $\rho \in \mathcal{R}_n^{(2)}$ then there exists a real number $t_\rho \in [\gamma - Z,\ \gamma + Z]$ such that
$$Y^{\sigma_1-\sigma} Z |J_n(\sigma_1 + it_\rho)| \geq \frac{1}{4}e^{-1/2}.$$

(e) Let $R_n^{(1)}$ and $R_n^{(2)}$ be the cardinality of $\mathcal{R}_n^{(1)}$ and $\mathcal{R}_n^{(2)}$, respectively. Then
$$\mathcal{N}(n;\ \sigma,\ T) \ll (R_n^{(1)} + R_n^{(2)})(\log NT)^3.$$

We put, for $j = 1, 2$, $R^{(j)} = \sum_{n\in E_k(N)} R_n^{(j)}$. Then, by the condition (e), we have

$$\sum_{n\in E_k(N)} \mathcal{N}(n;\ \sigma,\ T) \ll \left(R^{(1)} + R^{(2)}\right)(\log NT)^3. \tag{16.3}$$

Now we estimate $R^{(2)}$. Obviously, we get

$$R^{(2)} \ll Y^{\sigma_1-\sigma} Z \sum_{n\in E_k(N)} \sum_{\rho\in\mathcal{R}_n^{(2)}} |J_n(\sigma_1+it_\rho)|$$

$$(16.4) \qquad \ll Y^{\sigma_1-\sigma} Z \left(\sum_{n\in E_k(N)} \sum_{\rho\in\mathcal{R}_n^{(2)}} \left| \frac{\zeta_n(\sigma_1+it_\rho)}{\zeta(\sigma_1+it_\rho)} \right|^2 \right)^{\frac{1}{2}} \times$$

$$\times \left(\sum_{n\in E_k(N)} \sum_{\rho\in\mathcal{R}_n^{(2)}} |H_n(\sigma_1+it_\rho)|^2 \right)^{\frac{1}{2}}.$$

We note here that if ρ and ρ' are distinct elements in $\mathcal{R}_n^{(2)}$ then $|t_\rho - t_{\rho'}| \geq Z$.

Making use of Lemma 12.7, we have

$$|H_n(\sigma_1+it_\rho)|^2 \ll N^\varepsilon \sum_{m_2 m_3} \sum_{\leq X} \frac{\gamma(m_2)}{m_2^{2\sigma_1-\sigma_0}} \frac{1}{m_3^{2\sigma_1}} \left| \sum_{m_1 \leq \frac{X}{m_2 m_3}} \frac{\mu(m_1)\beta_n(m_1)}{m_1^{\sigma_1+it_\rho}} \right|^2,$$

and, by Lemma 16.1,

$$(16.5) \qquad \sum_{n\in E_k(N)} \sum_{\rho\in\mathcal{R}_n^{(2)}} |H_n(\sigma_1+it_\rho)|^2 \ll (NT+X^2)N^\varepsilon.$$

By (16.4), (16.5) and Lemma 16.2, we obtain

$$(16.6) \qquad R^{(2)} \ll Y^{\sigma_1-\sigma}(NT)^{1+\varepsilon},$$

providing that $X \leq \sqrt{NT}$.

Next we treat $R^{(1)}$. By (12.5), (12.6) and (16.1),

$$\alpha_n''(m) = \sum_{\substack{m_1 m_2 m_3 m_4 m_5 m_6 = m \\ m_2 m_4 m_6 \leq X}} \beta_n(m_1)\beta_n(m_2)\mu(m_2)\gamma_n(m_3)\gamma_n'(m_4)\delta_n(m_5)\delta_n'(m_6).$$

We put $l_1 = (m_1, m_2)$, $m_1 = l_1 m_1'$, $m_2 = l_1 m_2'$ and $l = m_1' m_2'$. Then

$$\alpha_n''(m) = \sum_{\substack{l l_1^2 m_3 m_4 m_5 m_6 = m \\ (l,l_1)=1}} \beta_n(l)\beta_n(l_1)^2 \mu(l_1)\gamma_n(m_3)\gamma_n'(m_4) \times$$

$$\times \delta_n(m_5)\delta_n'(m_6) \sum_{\substack{m_2' | l \\ l_1 m_2' m_4 m_6 \leq X}} \mu(m_2'),$$

accordingly, it follows from Lemma 12.7 that

$$(16.7)\quad |G_n(\rho)|^2 \ll N^\varepsilon \sum_{l_1^2 m_3 m_4 m_5 m_6 \le YZ} \frac{1}{l_1^{4\sigma-1}} \frac{\gamma(m_3)\gamma(m_4)}{(m_3 m_4)^{2\sigma-\sigma_0}} \frac{1}{(m_5 m_6)^{2\sigma}} \times$$

$$\times \left| \sum_{\frac{X}{l_1^2 m_3 m_4 m_5 m_6} < l \le \frac{YZ}{l_1^2 m_3 m_4 m_5 m_6}} \frac{\beta_n(l)}{l^\rho} \sum_{\substack{m_2' | l \\ m_2' \le \frac{X}{l_1 m_4 m_6}}} \mu(m_2') e^{-l l_1^2 m_3 m_4 m_5 m_6 / Y} \right|^2 .$$

For $U < U' \le 2U$, we obtain, by Lemma 16.1,

$$\sum_{n \in E_k(N)} \sum_{\rho \in \mathcal{R}_n^{(1)}} \left| \sum_{U < l \le U'} \frac{\beta_n(l)}{l^\rho} \sum_{\substack{m_2' | l \\ m_2' \le \frac{X}{l_1 m_4 m_6}}} \mu(m_2') e^{-l l_1^2 m_3 m_4 m_5 m_6 / Y} \right|^2 \ll$$

$$\ll (NT + U^2 \log U) \sum_{U < l \le U'} \frac{\tau(l)^2 \tau_k(l)}{l^{2\sigma}}$$
$$\ll (NTU^{1-2\sigma} + U^{3-2\sigma}) U^\varepsilon .$$

Therefore, by (16.7), we have

$$R^{(1)} \ll \sum_{n \in E_k(N)} \sum_{\rho \in \mathcal{R}_n^{(1)}} |G_n(\rho)|^2$$

$$\ll N^\varepsilon \sum_{l_1^2 m_3 m_4 m_5 m_6 \le YZ} \frac{1}{l_1^{4\sigma-1}} \frac{\gamma(m_3)\gamma(m_4)}{(m_3 m_4)^{2\sigma-\sigma_0}} \frac{1}{(m_5 m_6)^{2\sigma}} \times$$
$$\times \left\{ NT \min\left(1,\ \left(\frac{X}{l_1^2 m_3 m_4 m_5 m_6}\right)^{1-2\sigma}\right) + \left(\frac{Y}{l_1^2 m_3 m_4 m_5 m_6}\right)^{3-2\sigma} \right\}$$
$$\ll N^\varepsilon (NTW + Y^{3-2\sigma}),$$

where

$$W = \sum_{m_7 m_8 \le YZ} \sum \frac{\gamma'(m_7)}{m_7^{2\sigma-\sigma_0}} \frac{\gamma''(m_8)}{m_8^{2\sigma}} \min\left(1,\ \left(\frac{X}{m_7 m_8}\right)^{1-2\sigma}\right),$$

with

$$\gamma'(m) = \sum_{m_3 m_4 = m} \gamma(m_3)\gamma(m_4) \qquad \text{and} \qquad \gamma''(m) = \sum_{l_1^2 m_5 m_6 = m} l_1 .$$

By (12.10), we have

$$\sum_{m\le M}\frac{\gamma'(m)}{m^{\sigma_0}}\le\left(\sum_{m\le M}\frac{\gamma(m)}{m^{\sigma_0}}\right)^2\ll(\log M)^{2C_0}, \tag{16.8}$$

and, plainly,

$$\sum_{m\le M}\frac{\gamma''(m)}{m}\ll(\log M)^3. \tag{16.9}$$

Straightforward calculation with (16.8) and (16.9) shows that

$$\begin{aligned} W &\ll X^{1-2\sigma}\sum_{m_7m_8\le X}\sum\frac{\gamma'(m_7)}{m_7^{1-\sigma_0}}\frac{\gamma''(m_8)}{m_8}+\sum_{m_7\le YZ}\frac{\gamma'(m_7)}{m_7^{2\sigma-\sigma_0}}\sum_{\frac{X}{m_7}<m_8\le\frac{YZ}{m_7}}\frac{\gamma''(m_8)}{m_8^{2\sigma}}\\ &\ll X^{-2(\sigma-\sigma_0)}(\log NT)^{2C_0+3}, \end{aligned}$$

and

$$R^{(1)}\ll(NTX^{-2(\sigma-\sigma_0)}+Y^{3-2\sigma})N^{\varepsilon}. \tag{16.10}$$

Now we take

$$X=\sqrt{NT}\qquad\text{and}\qquad Y=(NT)^{\frac{1}{3-\sigma-\sigma_1}},$$

then Lemma 16.3 follows from (16.3), (16.6) and (16.10).

§17. Proof of Theorem 5

By virtue of Lemma 14.1, we get

$$\sum_{n\le N}\left|\mathfrak{S}_k(n,\sqrt{N})-\mathfrak{S}_k(n,Q_1)\right|^2 \ll N\log N\sum_{Q_1<m\le\sqrt{N}}\frac{\mu(m)^2\tau_k(m)^2}{\varphi(m)^2}$$
$$\ll N(\log N)^{-B_1+k^2+1}.$$

This inequality with Lemma 11.5, we have

$$\sum_{n\le N}\left|r_k(n)-\mathfrak{S}_k(n,\sqrt{N})I(n)\right| \ll$$
$$\ll \sum_{n\le N}|r_k(n)-\mathfrak{S}_k(n,Q_1)I(n)|^2+$$
$$+\sum_{n\le N}\left|\mathfrak{S}_k(n,\sqrt{N})-\mathfrak{S}_k(n,Q_1)\right|^2|I(n)|^2$$

$$\ll N^{1+\frac{2}{k}}(\log N)^{-A_1}. \tag{17.1}$$

On the other hand, we apply Lemma 16.3 with

$$T=2\exp(2\sqrt{\log N}),\quad r=k+1,\quad \sigma_1=1-\frac{1}{k(k+1)},$$

and obtain

$$\sharp\left\{n\in E_k(N);\ \mathcal{N}\left(n;1-\frac{1}{2k(k+1)},\,2\exp(2\sqrt{\log N})\right)>0\right\}\ll N^\theta,$$

for a certain $\theta<1$. Therefore, by (12.1), (17.1) and Lemma 13.1, we obtain, for any $A>0$,

$$r_k(n)=\prod_p\left(1-\frac{\rho_n(p)-1}{p-1}\right)\cdot\left\{I(n)+O\left(N^{1/k}(\log N)^{-(A_1-A-2k)/2}\right)\right\}$$

with at most $O(N(\log N)^{-A})$ exception of natural numbers $n\le N$. Hence, our proof of the Theorem 5 is competed. And we see easily, for $N(\log N)^{-A}<n\le$

N,

$$
\begin{aligned}
I(n) &= \left(\int_1^{n-n(\log N)^{-2}} + \int_{n-n(\log N)^{-2}}^{n-2}\right) \frac{x^{-1+\frac{1}{k}}\,dx}{k\log(n-x)} \\
&\leq \frac{1}{\log(n(\log N)^{-2})}\int_1^n \frac{1}{k}x^{-1+\frac{1}{k}}\,dx + \frac{1}{\log 2}\int_{n-n(\log N)^{-2}}^{n-2} \frac{1}{k}x^{-1+\frac{1}{k}}\,dx \\
&= \frac{n^{1/k}}{\log n}\left(1+O(\frac{\log\log N}{\log N})\right),
\end{aligned}
$$

and

$$
I(n) \geq \frac{1}{\log n}\int_1^{n-2} \frac{1}{k}x^{-1+\frac{1}{k}}\,dx = \frac{n^{1/k}}{\log n} + O(\frac{1}{\log N}),
$$

therefore

$$
I(n) = \frac{n^{1/k}}{\log n}\left(1+O(\frac{\log\log N}{\log N})\right).
$$

§18. Preliminaries for the proof of Theorems 6 and 7

Next we prove Theorems 6 and 7.

We denote by $\rho_n(d)$ the number of solutions of the congruence

$$x^3 \equiv n \pmod{d},$$

with $1 \le x \le d$. It is easily seen that

$$\rho_n(p) = \begin{cases} 1 & (p \not\equiv 1 \pmod 3), \text{ or } p \mid n), \\ 0 \text{ or } 3 & (p \equiv 1 \pmod 3) \text{ and } p \nmid n). \end{cases} \tag{18.1}$$

For $k \ge 1$, we define $d_k(m)$ as the number of ways of writing m as the product of k positive factors, in other words, $\sum_{m=1}^{\infty} d_k(m)m^{-s} = \zeta(s)^k$, for $\sigma > 1$. And for $k = 0$, we define

$$d_0(m) = \begin{cases} 1 & (\text{ if } m \text{ is a prime }), \\ 0 & (\text{ otherwise }). \end{cases}$$

Then, for $k \ge 0$,

$$R_k(N) = \sum_{m_1^3+m_2^3+m_3^3+m_4^3<N} d_k(N - m_1^3 - m_2^3 - m_3^3 - m_4^3) .$$

Let $\mathfrak{J}$ be the set of all natural numbers $\le N^{\frac{1}{3}}$, and let

$$\mathfrak{A} = \left\{ m \in \mathfrak{J};\ m \text{ has no prime factor } p \text{ such that } (\log N)^{4B_0} < p \le N^{\frac{1}{21}} \right\},$$

$$\overline{\mathfrak{A}} = \mathfrak{J} \setminus \mathfrak{A},$$

where $B_0 = B_0(k) = 2B + k^2 + 2$ with a constant $B > 5$. Making use of Selberg's upper bound sieve (see [**13**, Theorem 3.3]), it follows that

$$\sharp \mathfrak{A} \ll \frac{N^{\frac{1}{3}} \log\log N}{\log N}. \tag{18.2}$$

Here, the symbol $\sharp$ denotes the cardinality of the indicated set.

Now we put $e(x) = e^{2\pi i x}$, and, for a subset $\mathfrak{B} \subset \mathfrak{J}$, we introduce the function

$$F(\alpha; \mathfrak{B}) = \sum_{m \in \mathfrak{B}} e(m^3 \alpha).$$

In particular, for $\mathfrak{B} = \mathfrak{J}$, we write

$$F(\alpha) = F(\alpha; \mathfrak{I}) = \sum_{m \le N^{\frac{1}{3}}} e(m^3\alpha).$$

And, for $k \ge 0$, we define

$$D_k(\alpha) = \sum_{n \le N} d_k(n) e(n\alpha).$$

Next, we put $B_1 = B_1(k) = 6B + 3k^2$ and

$$Q_1 = Q_1(k) = (\log N)^{2B_1}, \quad Q_2 = Q_2(k) = N(\log N)^{-B_1}.$$

For $1 \le a \le q \le Q_1$, $(a, q) = 1$, let

$$\mathfrak{M}_0(q, a) = \left\{ \alpha;\ \left|\alpha - \frac{a}{q}\right| \le \frac{1}{Q_2} \right\},$$
$$\mathfrak{M}_1(q, a) = \left\{ \alpha;\ \frac{1}{Q_2} < \left|\alpha - \frac{a}{q}\right| \le \frac{1}{qN^{\frac{3}{4}}} \right\},$$

and, for $Q_1 < q \le N^{\frac{1}{4}}$, $1 \le a \le q$, $(a, q) = 1$, let

$$\mathfrak{M}_2(q, a) = \left\{ \alpha;\ \left|\alpha - \frac{a}{q}\right| \le \frac{1}{qN^{\frac{3}{4}}} \right\}.$$

Further, we set

$$\mathfrak{M}_j = \bigcup_{q \le Q_1} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathfrak{M}_j(q, a) \qquad (\text{for } j = 0, 1),$$
$$\mathfrak{M}_2 = \bigcup_{Q_1 < q \le N^{\frac{1}{4}}} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathfrak{M}_2(q, a),$$

and

$$\mathfrak{m} = \left[N^{-\frac{3}{4}},\ 1 + N^{-\frac{3}{4}}\right] \setminus \left(\mathfrak{M}_0 \bigcup \mathfrak{M}_1 \bigcup \mathfrak{M}_2\right).$$

Then we have, for $k \geq 0$,

$$R_k(N) = \int_{N^{-\frac{3}{4}}}^{1+N^{-\frac{3}{4}}} D_k(\alpha)F(\alpha)^4 e(-N\alpha)\, d\alpha$$

$$= \int_{N^{-\frac{3}{4}}}^{1+N^{-\frac{3}{4}}} D_k(\alpha)F(\alpha)\{F(\alpha)^3 - F(\alpha;\mathfrak{A})^3\}e(-N\alpha)\, d\alpha +$$

$$+ \int_{N^{-\frac{3}{4}}}^{1+N^{-\frac{3}{4}}} D_k(\alpha)F(\alpha)F(\alpha;\mathfrak{A})^3 e(-N\alpha)\, d\alpha$$

$$= I_k(\mathfrak{M}_0) + I_k(\mathfrak{M}_1) + I_k(\mathfrak{M}_2) + I_k(\mathfrak{m}) + \widetilde{R_k}(N)\,, \tag{18.3}$$

where

$$\widetilde{R_k}(N) = \sum_{\substack{m_1,\, m_2,\, m_3 \in \mathfrak{A} \\ m_4 < (N - m_1^3 - m_2^3 - m_3^3)^{\frac{1}{3}}}} d_k(N - m_1^3 - m_2^3 - m_3^3 - m_4^3)\ ,$$

and

$$I_k(\mathfrak{N}) = \int_{\mathfrak{N}} D_k(\alpha)F(\alpha)\{F(\alpha)^3 - F(\alpha;\mathfrak{A})^3\}e(-N\alpha)\, d\alpha$$

for $\mathfrak{N} = \mathfrak{M}_0,\ \mathfrak{M}_1,\ \mathfrak{M}_2$ and $\mathfrak{m}$.

19. Estimation of $I_k(\mathfrak{m})$

Our estimate for $I_k(\mathfrak{m})$ is based on Vaughan's work [40] on Waring's problem for cubes. By his method, we have, for any $\mathfrak{B} \subset \mathfrak{J}$,

$$\int_{\mathfrak{m}} |F(\alpha)|^2 |F(\alpha;\overline{\mathfrak{A}})|^2 |F(\alpha;\mathfrak{B})|^4 \, d\alpha \ll N^{\frac{5}{3}} (\log N)^{-B_0+3}. \tag{19.1}$$

In fact, we write

$$\begin{aligned}
|F(\alpha;\overline{\mathfrak{A}})|^2 = & \sum_{(\log N)^{4B_0} < d \leq N^{\frac{1}{3}}} \left(\sum_{\substack{m_1, m_2 \in \mathfrak{A} \\ (m_1, m_2) = d}} e((m_1^3 - m_2^3)\alpha) \right) + \\
& + \sum_{\substack{m_1, m_2 \in \mathfrak{A} \\ (m_1, m_2) \leq (\log N)^{4B_0}}} e((m_1^3 - m_2^3)\alpha) \\
= & \sum_{(\log N)^{4B_0} < d \leq N^{\frac{1}{3}}} F_d^{(1)}(\alpha) + F^{(2)}(\alpha), \quad \text{say,}
\end{aligned}$$

and note that each $m \in \overline{\mathfrak{A}}$ has a prime factor p satisfying $(\log N)^{4B_0} < p \leq N^{1/21}$. Then Vaughan's methods for estimating $I(\mathfrak{C})$ and $I(\mathfrak{D})$ in [40, pp. 137–138] yield the required bounds for

$$\sum_{(\log N)^{4B_0} < d \leq N^{\frac{1}{3}}} \int_{\mathfrak{m}} |F(\alpha)|^2 F_d^{(1)}(\alpha) \, |F(\alpha;\mathfrak{B})|^4 \, d\alpha$$

and

$$\int_{\mathfrak{m}} |F(\alpha)|^2 F^{(2)}(\alpha) \, |F(\alpha;\mathfrak{B})|^4 \, d\alpha,$$

respectively.

Now we can estimate $I_k(\mathfrak{m})$ easily. Since

$$\begin{aligned}
F(\alpha)^3 - F(\alpha;\mathfrak{A})^3 &= (F(\alpha) - F(\alpha;\mathfrak{A}))(F(\alpha)^2 + F(\alpha)F(\alpha;\mathfrak{A}) + F(\alpha;\mathfrak{A})^2) \\
&\ll |F(\alpha;\overline{\mathfrak{A}})|(|F(\alpha)|^2 + |F(\alpha;\mathfrak{A})|^2),
\end{aligned}$$

we have, using the Cauchy-Schwartz inequality,

$$\begin{aligned}
I_k(\mathfrak{m}) \ll \left(\int_0^1 |D_k(\alpha)|^2 \, d\alpha \right)^{\frac{1}{2}} \times & \\
\times \left(\int_{\mathfrak{m}} |F(\alpha)|^2 |F(\alpha;\overline{\mathfrak{A}})|^2 (|F(\alpha)|^4 + |F(\alpha;\mathfrak{A})|^4) \, d\alpha \right)^{\frac{1}{2}}. &
\end{aligned}$$

By (19.1) and the well-known fact

$$\int_0^1 |D_k(\alpha)|^2 \, d\alpha = \sum_{n\le N} d_k(n)^2 \ll N(\log N)^{k^2-1}, \tag{19.2}$$

for $k \geq 0$, we obtain

$$I_k(\mathfrak{m}) \ll N^{\frac{4}{3}}(\log N)^{-B}. \tag{19.3}$$

20. Estimations of $I_k(\mathfrak{M}_1)$ and $I_k(\mathfrak{M}_2)$

Essentially, it is not so difficult to treat $I_k(\mathfrak{M}_0), I_k(\mathfrak{M}_1)$, and $I_k(\mathfrak{M}_2)$. In this section, we shall estimate $I_k(\mathfrak{M}_1)$ and $I_k(\mathfrak{M}_2)$. We start with summarizing known results on the function $F(\alpha)$.

Let

$$V(q,a) = \sum_{r=1}^{q} e\Big(\frac{a}{q}r^3\Big),$$

$$V^*(q) = \sum_{\substack{a=1\\(a,q)=1}}^{q} \left|\frac{V(q,a)}{q}\right|^6,$$

$$\widetilde{V}_j(q,h) = \sum_{\substack{a=1\\(a,q)=1}}^{q} \Big(\frac{V(q,a)}{q}\Big)^j e\Big(-\frac{a}{q}h\Big),$$

and let

$$v(\beta) = \frac{1}{3}\sum_{m\le N} m^{-\frac{2}{3}} e(m\beta).$$

We note that $V^*(q)$ and $\widetilde{V}_j(q,h)$ are multiplicative functions of q.

LEMMA 20.1.

(i) *Assume that* $\alpha = (a/q)+\beta,\quad (a,q)=1$ *and* $|\beta| \le (6qN^{2/3})^{-1}$. *Then we have*

$$F(\alpha) = \frac{V(q,a)}{q} v(\beta) + O(q^{\frac{1}{2}+\epsilon}).$$

(ii) *For* $|\beta| \le \frac{1}{2}$, *we have*

$$v(\beta) \ll \min(\, N^{\frac{1}{3}},\ |\beta|^{-\frac{1}{3}}\,).$$

(iii) *For* $Q \ge 1$, *we have*

$$\sum_{q\le Q} q^{\frac{1}{2}} V^*(q) \ll 1.$$

(iv) *For* $Q\ge 2,\quad k\ge 1$ *and for any integer* h, *we have*

$$\sum_{q\le Q} \frac{d_k(q)}{q} |\widetilde{V}_1(q,h)| \ll (\log Q)^{C_k'},$$

where $C_k' = k(k+1)(k+5)/6$.

(v) *For $Q \geq 2$ and for any integer h, we have*

$$\sum_{q \leq Q} \frac{\mu(q)^2}{\varphi(q)} |\widetilde{V}_1(q,h)| \ll \log Q\,.$$

(vi) *Suppose $\delta > \epsilon > 0$ and $\Theta(q) \ll q^{-\delta}$. Then the series $\sum_{q=1}^{\infty} \Theta(q)\widetilde{V}_4(q,N)$ converges absolutely, and we have, for $Q \geq 1$,*

$$\sum_{q>Q} \Theta(q)\widetilde{V}_4(q,N) \ll Q^{-\delta+\epsilon}.$$

PROOF: As for (i) and (ii), see Theorem 4.1 and Lemma 2.8 in Vaughan's book [39].

Now we prove (iv). First, for $l \geq 1$, we write $l = 3u + v$ with integers u and $1 \leq v \leq 3$. By Lemma 4.7 of [39], we get

$$\widetilde{V}_4(p^l,h) \ll \begin{cases} p^{-u} & \text{(when } v = 1 \text{ and } p \nmid h), \\ p^{-u-1} & \text{(when } v \neq 1 \text{ and } p \nmid h), \\ p^{-u-\frac{1}{2}+l} & \text{(when } v = 1 \text{ and } p \mid h), \\ p^{-u-1+l} & \text{(when } v \neq 1 \text{ and } p \mid h). \end{cases}$$

Secondly, by simple calculation, we see $\widetilde{V}_1(p^l,h) = \rho_h(p^l) - \rho_h(p^{l-1})$, therefore we have by (18.1)

$$\widetilde{V}_1(p,h) = \begin{cases} 0 & (p \not\equiv 1 \pmod 3\text{, or } p \mid h), \\ 2 \text{ or } -1 & (p \equiv 1 \pmod 3 \text{ and } p \nmid h), \end{cases}$$

and if $p \mid h$ then $\widetilde{V}_1(p^v,h) \leq p^{v-1}$ for $v = 2,3$.

Taking account of these results, we obtain

$$\begin{aligned} \sum_{q \leq Q} \frac{d_k(q)}{q} |\widetilde{V}_1(q,h)| &\leq \prod_{p \leq Q} \left(1 + \sum_{l=1}^{\infty} \frac{d_k(p^l)}{p^l} |\widetilde{V}_1(p^l,h)|\right) \\ &\leq \prod_{\substack{p \leq Q \\ p \nmid h}} \left(1 + \frac{2k}{p} + O(p^{-3+\epsilon})\right) \times \\ &\qquad \times \prod_{\substack{p \leq Q \\ p \mid h}} \left(1 + \frac{C'_k}{p} + O(p^{-\frac{3}{2}+\epsilon})\right) \\ &\ll \prod_{p \leq Q} \left(1 + \frac{C'_k}{p}\right) \ll (\log Q)^{C'_k}. \end{aligned}$$

Similarly, we obtain (v) and

$$\sum_{q=1}^{\infty} q^{-\epsilon}|\widetilde{V}_4(q,N)| \ll \prod_{p\nmid N}\left(1+O(p^{-\frac{3}{2}})\right)\prod_{p|N}\left(1+O(p^{-1-\epsilon})\right) \ll 1,$$

which gives (vi).

We turn to (iii). As is mentioned in [**39**, p.50], we get

$$V^*(p^l) \ll \begin{cases} p^{-6u-3+l} & (v=1), \\ p^{-6u-6+l} & (v=2,3), \end{cases}$$

for $l = 3u+v, \;\; 1 \le v \le 3$, and we have

$$\begin{aligned}\sum_{q\le Q} q^{\frac{1}{2}} V^*(q) &\le \prod_{p\le Q}\left(1+\sum_{l=1}^{\infty} p^{\frac{l}{2}} V^*(p^l)\right) \\ &\ll \prod_{p\le Q}\left(1+O(p^{-\frac{3}{2}})\right) \ll 1.\end{aligned}$$

Now we use Lemma 20.1, and obtain

$$\begin{aligned}\int_{\mathfrak{M}_1} |F(\alpha)|^6\, d\alpha &\ll \sum_{q\le Q_1}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{\frac{1}{Q_2}<|\beta|\le q^{-1}N^{-\frac{3}{4}}}\left(\left|\frac{V(q,a)}{q}\right|^6 |v(\beta)|^6 \;+\; q^{3+6\epsilon}\right) d\beta \\ &\ll \sum_{q\le Q_1} V^*(q) \int_{\frac{1}{Q_2}<|\beta|\le\frac{1}{2}} |\beta|^{-2}\, d\beta \;+\; N^{-\frac{3}{4}+\epsilon} \\ &\ll Q_2 = N(\log N)^{-B_1}, \qquad (20.1)\end{aligned}$$

and,

$$\begin{aligned}\int_{\mathfrak{M}_2} |F(\alpha)|^6\, d\alpha &\ll \sum_{Q_1<q\le N^{\frac{1}{4}}}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{|\beta|\le q^{-1}N^{-\frac{3}{4}}}\left(\left|\frac{V(q,a)}{q}\right|^6 |v(\beta)|^6 \;+\; q^{3+6\epsilon}\right) d\beta \\ &\ll \frac{1}{\sqrt{Q_1}}\sum_{q\le N^{\frac{1}{4}}} \sqrt{q}\, V^*(q) \int_{|\beta|\le\frac{1}{2}} \min(N^2,\, |\beta|^{-2})\, d\beta \;+\; N^{\frac{1}{4}+2\epsilon} \\ &\ll \frac{N}{\sqrt{Q_1}} = N(\log N)^{-B_1}. \qquad (20.2)\end{aligned}$$

On the other hand, for any $\mathfrak{B} \subset \mathfrak{I}$, we have trivially $|F(\alpha;\mathfrak{B})| \ll N^{\frac{1}{3}}$, and, by [**40**, Theorem 2],

$$\int_0^1 |F(\alpha;\mathfrak{B})|^8\,d\alpha \ll N^{\frac{5}{3}}. \tag{20.3}$$

These estimates with (19.2), (20.1) and (20.2) yield, for $j=1,2$,

$$\begin{aligned} I_k(\mathfrak{M}_j) &\ll \max_\alpha(|F(\alpha)|+|F(\alpha;\mathfrak{A})|)^{\frac{1}{3}}\left(\int_0^1 |D_k(\alpha)|^2\,d\alpha\right)^{\frac{1}{2}}\times \\ &\qquad\times\left(\int_{\mathfrak{M}_j}|F(\alpha)|^6\,d\alpha\right)^{\frac{1}{6}}\left(\int_0^1(|F(\alpha)|^8+|F(\alpha;\mathfrak{A})|^8)\,d\alpha\right)^{\frac{1}{3}} \\ &\ll N^{\frac{4}{3}}(\log N)^{-B}. \end{aligned} \tag{20.4}$$

21. On the integral $I_k(\mathfrak{M}_0)$

In order to evaluate $I_k(\mathfrak{M}_0)$, we need an appropriate approximation of $D_k(\alpha)$ for $\alpha \in \mathfrak{M}_0$. As for $k=0$, we know the following result.

LEMMA 21.1. *Let*

$$T_0(\beta;q) = \frac{\mu(q)}{\varphi(q)} \sum_{2\le n\le N} \frac{e(n\beta)}{\log n}.$$

(i) *Suppose that* $1 \le a \le q \le Q_1$, $(a,q)=1$, $\alpha \in \mathfrak{M}_0(q,a)$ *and* $\alpha = (a/q)+\beta$. *Then we have*

$$D_0(\alpha) = T_0(\beta;q) + O\left(N\exp(-c_1\sqrt{\log N})\right),$$

where c_1 *is a positive constant depending only on* B.

(ii) *For* $|\beta| \le 1/2$, *we have*

$$T_0(\beta;q) \ll \frac{\mu(q)^2}{\varphi(q)} \min\left(\frac{N}{\log N}, |\beta|^{-1}\right).$$

And for $|\beta| \le 1/\sqrt{N}$, *we have*

$$T_0(\beta;q) \ll \frac{\mu(q)^2}{\varphi(q)\log N} \min(N, |\beta|^{-1}).$$

For the proof of (i), see Prachar [**34**, VI, Satz 3.2, p.180]. The inequalities in (ii) follow easily from the well-known estimate $\sum_{n\le x} e(n\beta) \ll \min(x, |\beta|^{-1})$ for $|\beta| \le 1/2$, by partial summation.

As for the case $k \ge 2$, Motohashi [**28**] showed a sufficient result for our aim. Though he confined his attention within square free q's, his argument [**28**, Lemmata 5, 6 and p.60] still work for all q's with slight differences. Here we follow his way.

For $s = \sigma + it$, $\sigma > 1$, and for $k \ge 1$, we introduce the functions

$$\Delta_k\Big(s;\frac{a}{q}\Big) = \sum_{n=1}^{\infty} e\Big(\frac{a}{q}n\Big) d_k(n) n^{-s},$$

$$\Psi_k(s;q) = \frac{1}{\varphi(q)} \sum_{\substack{a=1\\(a,q)=1}}^{q} \Delta_k\Big(s;\frac{a}{q}\Big) = \frac{1}{\varphi(q)} \sum_{n=1}^{\infty} c_q(n) d_k(n) n^{-s},$$

and

$$\Phi_k(s;q)=\prod_{p|q}\left\{1-(1-p^{-s})^k\left(\sum_{\nu=0}^{g_p-2}\frac{d_k(p^\nu)}{p^{\nu s}}\right)-\right.$$
$$\left.(1-p^{-1})^{-1}(1-p^{-s})^k\frac{d_k(p^{g_p-1})}{p^{(g_p-1)s}}\right\},$$

where $g_p=g_p(q)$ is the number such that p^{g_p} is the highest power of p dividing q. After some calculation, we obtain

$$\Psi_k(s;q)=\zeta(s)^k\Phi_k(s;q). \tag{21.1}$$

If $k\geq 2$ and $(a,q)=1$ then we have

$$\Delta_k\left(s;\frac{a}{q}\right)=\sum_{b=1}^{q}\sum_{l=1}^{\infty}\sum_{\substack{m=1\\ am\equiv b\ (\mathrm{mod}\ q)}}^{\infty}e\Big(\frac{a}{q}lm\Big)d_{k-1}(m)(lm)^{-s}$$
$$=\zeta(s)^k\prod_{p|q}\left(1-(1-p^{-s})^{k-1}\left(\sum_{\nu=0}^{g_p-1}\frac{d_k(p^\nu)}{p^{\nu s}}\right)\right)+$$
$$+q^{-1}\sum_{b=1}^{q-1}\Delta_1\left(s;\frac{a}{q}\right)\sum_{h|q}\sum_{\substack{l=1\\ (l,h)=1}}^{h}e\Big(-\frac{bl}{h}\Big)\Delta_{k-1}\left(s;\frac{al}{q}\right).$$

Then we see the following facts by induction on k with known results on the Lerch zeta functions $\Delta_1(s;a/q)$ and the Riemann zeta function $\zeta(s)$.

(i) $\Delta_k(s;a/q)$ can be analytically continued to a meromorphic function over the whole complex plane, which is holomorphic save a possible pole at $s=1$.

(ii) If $(a,q)=1$ then the meromorphic part of $\Delta_k(s;a/q)$ at $s=1$ does not depend on a, therefore, $\Delta_k(s;a/q)$ has the same meromorphic part as $\Psi_k(s;q)$ at $s=1$.

(iii) For any fixed $\delta>0$, we have

$$\Delta_k\left(s;\frac{a}{q}\right)\ll q^{k-1}\left(1+|t|^{k(1-\sigma+\epsilon)/2}\right),$$

uniformly for $|s-1|\geq 1/2,\ \ \sigma\geq 1/2$.

Now we suppose $q\leq Q_1,\ \ (a,q)=1$, and $x\geq\sqrt{N}$, and put $T=x^{4\delta}$ with $\delta=\delta(k)=(10k)^{-1}$. Applying Perron's formula with the facts listed above, we obtain

$$\sum_{n\leq x} d_k(n)e\Big(\frac{a}{q}n\Big) = \frac{1}{2\pi i}\int_{1+\delta-iT}^{1+\delta+iT} \Delta_k\Big(s;\frac{a}{q}\Big)\frac{x^s}{s}\,ds + O(x^{1+\delta}T^{-1})$$

$$= \Upsilon_k(q) + O(x^{1-2\delta}), \tag{21.2}$$

where $\Upsilon_k(q)$ is the residue of $\Psi_k(s;q)x^s/s$ at $s=1$. For $k\geq 2$, it follows from (21.1) that $\Psi_k(s;q)$ has a pole of order k at $s=1$. To write down $\Upsilon_k(q)$, we use the value $\Phi_k^{(j)}(q)$, say, of the j-th derivative of $\Phi_k(s;q)$ at $s=1$, and the numbers $\eta_k(j)$ defined by

$$(s-1)^k\zeta(s)^k s^{-1} = \sum_{j=0}^{\infty}\eta_k(j)(s-1)^j \quad (\text{as } s\to 1).$$

Putting

$$\Theta_k^{(j)}(q) = \frac{1}{(k-1-j)!}\sum_{h=0}^{j}\frac{1}{h!}\eta_k(j-h)\Phi_k^{(h)}(q),$$

we have, for $k\geq 2$,

$$\Upsilon_k(q) = x\sum_{j=0}^{k-1}\Theta_k^{(j)}(q)(\log x)^{k-1-j}.$$

Then, Lemma 21.2 (i) below is derived from (21.2) by partial summation.

Next we estimate $\Theta_k^{(j)}(q)$. After some computation, we get

$$\Phi_k(s;q) = q^{-s}\prod_{p|q}\Bigg\{\sum_{u=0}^{k-2}\binom{g_p+k-2}{u}(1-p^{-s})^u p^{-s(k-2-u)} - \binom{g_p+k-2}{k-1}p^{s-1}(1-p^{-s})^{k-1}(1-p^{-1})^{-1}(1-p^{-s+1})\Bigg\},$$

and

$$\Theta_k^{(0)}(q) = \frac{1}{(k-1)!}\Phi_k(1;q) \ll \frac{d_{k-1}(q)}{q}.$$

For $1\leq j\leq k-1$, we have

$$\Phi_k^{(j)}(q) = \frac{j!}{2\pi i}\int_{|s-1|=(\log 2q)^{-1}}\frac{\Phi_k(s;q)}{(s-1)^{j+1}}\,ds \ll q^{-1+\epsilon},$$

and $\Theta_k^{(j)}(q) \ll q^{-1+\epsilon}$. From these estimates, we deduce Lemma 21.2 (ii) below.

We come to a conclusion;

LEMMA 21.2. *For $k \ge 2$, let*

$$T_k(\beta;q) = \sum_{j=0}^{k-1} \Theta_k^{(j)}(q) \sum_{n\le N} \{(\log n)^{k-1-j} + (k-1-j)(\log n)^{k-2-j}\} e(n\beta).$$

(i) *Suppose that* $1 \le a \le q \le Q_1$, $(a,q)=1$, $\alpha \in \mathfrak{M}_0(q,a)$ *and* $\alpha = (a/q)+\beta$. *Then we have*

$$D_k(\alpha) = T_k(\beta;q) + O(N^{1-(10k)^{-1}}).$$

(ii) *For* $|\beta| \le 1/2$ *and for* $q \le Q_1$, *we have*

$$T_k(\beta;q) \ll \frac{d_{k-1}(q)}{q}(\log N)^{k-1} \min(N,\, |\beta|^{-1}).$$

We write

$$\begin{aligned} I_k(\mathfrak{M}_0) &= \int_{\mathfrak{M}_0} D_k(\alpha) F(\alpha)^4 e(-N\alpha)\, d\alpha - \int_{\mathfrak{M}_0} D_k(\alpha)F(\alpha)F(\alpha;\mathfrak{A})^3 e(-N\alpha)\, d\alpha \\ &= I_k^{(0)}(\mathfrak{M}_0) - I_k^{(1)}(\mathfrak{M}_0), \quad \text{say.} \end{aligned} \tag{21.3}$$

By virtue of Lemma 20.1 (iv), Lemma 21.2 and the estimate (18.2), we have, for $k \ge 3$,

$$\begin{aligned} I_k^{(1)}(\mathfrak{M}_0) &= \sum_{\substack{m_j\in\mathfrak{A}\\(j=1,2,3)}} \sum_{q\le Q_1} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{|\beta|\le Q_2^{-1}} T_k(\beta;q)\frac{V(q,a)}{q}v(\beta)\times \\ &\qquad \times e\left((m_1^3+m_2^3+m_3^3-N)\left(\frac{a}{q}+\beta\right)\right) d\beta + O(N^{\frac43}(\log N)^{-B}) \\ &\ll (\log N)^{k-1} \sum_{\substack{m_j\in\mathfrak{A}\\(j=1,2,3)}} \sum_{q\le Q_1} \frac{d_{k-1}(q)}{q} |\widetilde{V}_1(q,\, N-m_1^3-m_2^3-m_3^3)| \times \\ &\qquad \times \int_{|\beta|\le Q_2^{-1}} \min(N^{\frac43},\, |\beta|^{-\frac43})\, d\beta \;+\; N^{\frac43}(\log N)^{-B} \\ &\ll N^{\frac43}(\log N)^{k-4}(\log\log N)^{\frac16 k(k-1)(k+4)+3}. \end{aligned} \tag{21.4}$$

Similarly, by Lemma 20.1 (v), Lemma 21.1 and (18.2), we have

$$I_0^{(1)}(\mathfrak{M}_0) \ll N^{\frac43}(\log N)^{-4}(\log\log N)^4. \tag{21.5}$$

We can also evaluate $I_k^{(0)}(\mathfrak{M}_0)$ straightforwardly by Lemmata 1, 2, 3 and

[**39**, Theorem 2.3]. For $k=0$, we have

$$I_0^{(0)}(\mathfrak{M}_0)=\Gamma\Big(\frac{3}{4}\Big)^3\mathfrak{S}_0(N)\int_2^N\frac{(N-t)^{\frac{1}{3}}}{\log t}\,dt+O(N^{\frac{4}{3}}(\log N)^{-B}), \tag{21.6}$$

where

$$\mathfrak{S}_0(N)=\sum_{q=1}^{\infty}\frac{\mu(q)}{\varphi(q)}\widetilde{V}_4(q,N)=\prod_p\left(1-\frac{\widetilde{V}_4(p,N)}{p-1}\right). \tag{21.7}$$

For $k\geq 3$, we put

$$\mathfrak{S}_k^{(j)}(N)=\sum_{q=1}^{\infty}\Theta_k^{(j)}(q)\widetilde{V}_4(q,N)$$

and

$$\xi_k^{(j)}(N)=\frac{1}{3}\Gamma\Big(\frac{3}{4}\Big)^3\sum_{h=0}^{j}\binom{k-1-h}{j-h}\mathfrak{S}_k^{(j)}(N)\int_0^1 t(\log t)^{j-h}(1-t)^{-\frac{2}{3}}\,dt, \tag{21.8}$$

then we have

$$I_k^{(0)}(\mathfrak{M}_0)=N^{\frac{4}{3}}\sum_{j=0}^{k-1}\xi_k^{(j)}(N)(\log N)^{k-1-j}\;+\;O(N^{\frac{4}{3}}(\log N)^{-B}), \tag{21.9}$$

We note that

$$\begin{aligned}\xi_k^{(0)}(N)&=\frac{3}{4}\Gamma\Big(\frac{4}{3}\Big)^3\mathfrak{S}_k^{(0)}(N)\\&=\frac{3}{4(k-1)!}\Gamma\Big(\frac{4}{3}\Big)^3\prod_p\left\{1+\sum_{l=1}^{\infty}\frac{\widetilde{V}_4(p^l,N)}{p^l}\times\right.\\&\qquad\left.\times\sum_{u=0}^{k-2}\binom{l+k-2}{u}(1-p^{-1})^u p^{-(k-2-u)}\right\}.\end{aligned}$$

By Lemma 20.1 (vi), we see at once $\mathfrak{S}_0(N)\ll 1$ and $\xi_k^{(j)}(N)\ll 1$ for $0\leq j<k$. In order to observe $\mathfrak{S}_0(N)\gg 1$ and $\xi_k^{(0)}(N)\gg 1$ for $k\geq 3$, it suffices to show

$$1-\frac{\widetilde{V}_4(p,N)}{p-1}>0 \tag{21.10}$$

and

$$1+\sum_{l=1}^{\infty}\widetilde{V}_4(p^l,N)\Phi_k^{(0)}(p^l)>0, \tag{21.11}$$

for all primes p. Since $|\widetilde{V}_4(p,N)|\le 2$ and $\widetilde{V}_4(2,N)=\widetilde{V}_4(3,N)=0$, we get (21.10). Next, for $k\ge 3$ we put $S_m=\sum_{l=0}^{m}\widetilde{V}_4(p^l,N)$. Then, by [**39**, Lemma 2.12], we have $S_m\ge 0$ and

$$\begin{aligned}1+\sum_{l=1}^{m}\widetilde{V}_4(p^l,N)\Phi_k^{(0)}(p^l)&=1-\Phi_k^{(0)}(p)+\sum_{l=1}^{m-1}p^{-l}(1-p^{-1})^{k-1}\binom{l+k-2}{k-2}S_l+\\&\qquad\qquad+\Phi_k^{(0)}(p^m)S_m\\&\ge(1-p^{-1})^{k-1},\end{aligned}$$

which yields (21.11).

22. Treatment for $\widetilde{R_k}(N)$

In view of (18.3), (19.3), (20.4), (21.3), (21.4), (21.5), (21.6) and (21.9), it suffices to prove

$$\widetilde{R_k}(N) \ll N^{\frac{4}{3}}(\log N)^{k-4}(\log\log N)^{C_k''}, \tag{22.1}$$

with some constant $C_k'' \leq C_k$ depending only on k.

We put, for $k \geq 0$,

$$W_k(n) = \sum_{m<n^{1/3}} d_k(n-m^3),$$

$$r(n;\mathfrak{A}) = \sum_{\substack{m_1,m_2,m_3\in\mathfrak{A}\\ m_1^3+m_2^3+m_3^3=n}} 1\,,$$

then

$$\begin{aligned}\widetilde{R_k}(N) &= \sum_{m_1,m_2,m_3\in\mathfrak{A}} W_k(N-m_1^3-m_2^3-m_3^3)\\ &= \sum_{n<N} r(N-n;\mathfrak{A})W_k(n). \end{aligned}\tag{22.2}$$

By (18.2), we get

$$\sum_{n\leq N} r(n;\mathfrak{A}) \ll \frac{N(\log\log N)^3}{(\log N)^3}. \tag{22.3}$$

And, by Hua's lemma (see [**39**, Lemma 2.5]), we have

$$\begin{aligned}\sum_{n\leq N} r(n;\mathfrak{A})^2 &\leq \int_0^1 |F(\alpha)|^6\,d\alpha\\ &\ll \left(\int_0^1 |F(\alpha)|^4\,d\alpha\right)^{\frac{1}{2}}\left(\int_0^1 |F(\alpha)|^8\,d\alpha\right)^{\frac{1}{2}}\\ &\ll N^{\frac{7}{6}+\epsilon}. \end{aligned}\tag{22.4}$$

We first consider the case $k \geq 3$. It follows from Wolke's result [**43**, Satz 1] that, for $n \leq N$,

$$W_k(n) \ll N^{\frac{1}{3}}\exp\left((k-1)\sum_{p\leq N}\frac{\rho_n(p)}{p}\right). \tag{22.5}$$

Since

$$\sum_{p\le x}\frac{\rho_n(p)}{p}\le\sum_{\substack{p\le x\\ p\not\equiv 1 \pmod 3}}\frac{1}{p}+\sum_{\substack{p\le x\\ p\equiv 1 \pmod 3}}\frac{3}{p}=2\log\log x+O(1),$$

we have $W_k(n)\ll N^{1/3}(\log N)^{2(k-1)}$, but this bound is not sufficient to obtain (22.1).

Now we put $m_0=[\epsilon\log N]$ and define the sets

$$\mathcal{E}(N,M)=\left\{n<N\,;\;\left|\sum_{M<p\le 2M}\frac{\rho_n(p)-1}{p}\right|\ge\frac{1}{m_0}\right\},$$

$$\mathcal{E}(N)=\bigcup_{m=1}^{m_0}\mathcal{E}(N,\,2^{m-1}(\log N)^{1/\epsilon}).$$

Since $N^{\epsilon/2}<2^{m_0}(\log N)^{1/\epsilon}<N^{\epsilon}$, we see that if $n\notin\mathcal{E}(N)$ and $n<N$ then

$$\begin{aligned}\sum_{(\log N)^{1/\epsilon}<p\le 2^{m_0}(\log N)^{1/\epsilon}}\frac{\rho_n(p)}{p}&=\sum_{(\log N)^{1/\epsilon}<p\le 2^{m_0}(\log N)^{1/\epsilon}}\frac{1}{p}+O(1)\\&=\log\log N-\log\log\log N+O(1),\end{aligned}$$

whence

$$\begin{aligned}\sum_{p\le N}\frac{\rho_n(p)}{p}&=\sum_{p\le(\log N)^{1/\epsilon}}\frac{\rho_n(p)}{p}+\sum_{(\log N)^{1/\epsilon}<p\le 2^{m_0}(\log N)^{1/\epsilon}}\frac{\rho_n(p)}{p}+O(1)\\&\le\log\log N+\log\log\log N+O(1).\end{aligned}$$

Thus, by (22.3) and (22.5), we have

$$\sum_{\substack{n<N\\ n\notin\mathcal{E}(N)}}r(N-n;\mathfrak{A})W_k(n)\ll N^{\frac{4}{3}}(\log N)^{k-4}(\log\log N)^{k+2}. \tag{22.6}$$

Next, we estimate the cardinality of $\mathcal{E}(N)$ through the way indicated in Plaksin's paper [**33**]. For $(\log N)^{1/\epsilon}\le M\le N^{\epsilon}$ and a natural number ν, we put

$$S(N,M,\nu)=\sum_{n\le N}\left|\sum_{M<p\le 2M}\frac{\rho_n(p)-1}{p}\right|^{2\nu}$$

And let C_p be the set of non–principal characters χ modulo p for which χ^3 is principal. It is easily observed that $\sharp C_p=2$ or 0 according as $p\equiv 1 \pmod 3$ or not, and that

$$\rho_n(p) - 1 = \sum_{\chi \in \mathcal{C}_p} \chi(n).$$

Making use of the Pólya–Vinogradov inequality, we get

$$\begin{aligned} S(N,M,\nu) &= \sum_{\substack{M<p_j\le 2M \\ (1\le j\le 2\nu)}} \prod_{j=1}^{2\nu} \frac{1}{p_j} \sum_{\substack{\chi_j\in\mathcal{C}_{p_j} \\ (1\le j\le 2\nu)}} \sum_{n\le N} \prod_{j=1}^{2\nu} \chi_j(n) \\ &\ll NM^{-2\nu} S_1(M,\nu) + 2^{5\nu} M^{\nu} \log((2M)^{2\nu}), \end{aligned} \tag{22.7}$$

where

$$S_1(M,\nu) = \sum_{\substack{M<p_j\le 2M \\ (1\le j\le 2\nu)}} \sum_{\substack{\chi_j\in\mathcal{C}_{p_j} \\ (1\le j\le 2\nu) \\ \prod_{j=1}^{2\nu}\chi_j \text{ is principal.}}} 1\,.$$

We note that if $\chi_j \in \mathcal{C}_{p_j}$ $(1 \le j \le 2\nu)$ and $\prod_{j=1}^{2\nu} \chi_j$ is principal, then $\prod_{j=1}^{2\nu} p_j$ is a powerful number. A natural number l is called "powerful" if $p^2 \mid l$ for all prime factors p of l. The number of powerful numbers not exceeding x is $O(\sqrt{x})$. (See Golomb [**12**].) So we have

$$S_1(M,\nu) \le 2^{2\nu}(2\nu)! \sum_{\substack{l\le(2M)^{2\nu} \\ l \text{ is powerful.}}} 1 \ll 2^{3\nu}(2\nu)! M^{\nu}.$$

Then, by (22.7) and the definition of $\mathcal{E}(N,M)$, we obtain

(22.8)

$$\sharp\mathcal{E}(N,M) \le m_0^{2\nu} S(N,M,\nu) \ll (\log N)^{3\nu} \left((2\nu)! NM^{-\nu} + M^{\nu}\log((2M)^{2\nu})\right).$$

We choose ν so as to satisfy $M^{\nu} \le \sqrt{N} < M^{\nu+1}$. Since $(\log N)^{1/\epsilon} \le M \le N^{\epsilon}$, it follows that

$$N^{\frac{1}{2}-\epsilon} < M^{\nu} \le N^{\frac{1}{2}} \quad \text{and} \quad \nu \le \frac{\epsilon}{2} \frac{\log N}{\log\log N},$$

therefore, $(2\nu)! \le \exp(2\nu\log(2\nu)) \le N^{\epsilon}$, and $(\log N)^{3\nu} \le N^{2\epsilon}$. So (22.8) gives

$$\sharp\mathcal{E}(N,M) \ll N^{\frac{1}{2}+4\epsilon}.$$

Hence,

$$\sharp\mathcal{E}(N) \le \sum_{m=1}^{m_0} \sharp\mathcal{E}(N,\, 2^{m-1}(\log N)^{1/\epsilon}) \ll N^{\frac{1}{2}+5\epsilon}. \tag{22.9}$$

By (22.4), (22.9) and a trivial bound $W_k(n) \ll N^{(1/3)+\epsilon}$, we have

$$\sum_{n \in \mathcal{E}(N)} r(N-n;\mathfrak{A}) W_k(n) \ll N^{\frac{1}{3}+\epsilon} \left(\sum_{n \in \mathcal{E}(N)} 1 \right)^{\frac{1}{2}} \left(\sum_{n \le N} r(n;\mathfrak{A})^2 \right)^{\frac{1}{2}}$$

$$\ll N^{\frac{7}{6}+4\epsilon}. \tag{22.10}$$

The inequalities (22.6) and (22.10) yield (22.1) with $C_k'' = k+2$ for $k \ge 3$, and which completes our proof of Theorem 6.

We proceed to the case $k=0$. By (22.3), we have at once

$$\widetilde{R_0}(N) \ll N^{\frac{4}{3}} (\log N)^{-3} (\log\log N)^3,$$

and which is sufficient to obtain an asymptotic formula for $R_0(N)$. To show (22.1), we should improve this trivial bound slightly.

We use Selberg's upper bound sieve. Taking $z = N^{1/7}$ and $\kappa = 3$ in Theorem 4.1 of [**13**], we obtain

$$W_0(n) \ll N^{\frac{1}{3}} \prod_{p \le N^{\frac{1}{7}}} \left(1 - \frac{\rho_n(p)}{p}\right) \ll N^{\frac{1}{3}} \exp\left(- \sum_{p \le N} \frac{\rho_n(p)}{p} \right),$$

for $n \le N$. It follows from the definition of the set $\mathcal{E}(N)$ that if $n \notin \mathcal{E}(N)$ and $n \le N$ then

$$\sum_{p \le N} \frac{\rho_n(p)}{p} \ge \sum_{\substack{p \le (\log N)^{1/\epsilon} \\ p \not\equiv 1 \pmod 3}} \frac{1}{p} + \sum_{(\log N)^{1/\epsilon} < p \le 2^{m_0} (\log N)^{1/\epsilon}} \frac{\rho_n(p)}{p}$$

$$= \log\log N - \frac{1}{2} \log\log\log N + O(1),$$

whence

$$W_0(n) \ll N^{\frac{1}{3}} (\log N)^{-1} (\log\log N)^{\frac{1}{2}}.$$

Therefore, in the same manner as for the case $k \ge 3$, we conclude that the inequality (22.1) holds for $k=0$ as well, with $C_0'' = 7/2$. Now we obtain Theorem 7.

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