On Minimal Models

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Abstract

Let T be a complete theory in a first order language.

A model M of T is said to be **minimal** if there is no proper elementary submodel of M. In [17] Shelah showed that if T is totally transcendental then there is no infinite indiscernible set in a minimal model. On the other hand Marcus has constructed a minimal structure with an infinite indiscernible set ([11]). His structure is stable but not superstable. In chapter 1 we shall prove the following theorem:

Theorem A ([9]). *Let T be superstable and let A be any set. Then there is no minimal model over A which has an infinite set of indiscernibles over A.*

Let T be countable. It is still open whether there is T having exactly $\aleph_1$ non-isomorphic minimal models, or not. In [12] Marcus showed that if a theory with one unary function has a minimal non-prime model, then T has $2^{\aleph_0}$ minimal models. Our purpose is to generalize Marcus's result. In chapter 2 we will prove the following theorem:

Theorem B ([8]). *Let T be stable and trivial. Suppose that T has a model M such that (i) M is minimal and non-prime, and (ii) $U(a) \leq 1$, for all $a \in M$. Then T has $2^{\aleph_0}$ minimal models.*

In [16] Shelah has solved the central problem on the number of uncountable models. In his proof he introduced a dividing line, the dimensional order property (DOP). A theory said to have the **DOP** if there exist tuples a, b, a finite set F and a non-algebraic type p such that $p \not\perp abF$, $p \perp aF$, $p \perp bF$ and $a \downarrow_F b$. Let a theory S be interpretable in T and φ a formula in T which defines the universe of an S-model. Then we say that S is **fully interpreted** in T if for any S-model M there is a T-model N such that $\varphi^N \cong M$. In chapter 3 we show the following theorem:

Theorem C ([10]). *Let T be ω-categorical and ω-stable. Then T has the DOP if and only if S_1, S_2 or S_3^q (for some $q < \omega$) can be fully interpreted in T_F^{eq} for some finite set F.*

Chapter 0. Introduction

In this chapter we summarize some basic definitions and results necessary for proving our main theorems.

0.1. Structures, Languages and Satisfaction

0.1.1. Definition. (1) A **similarity type** τ is a 5-tuple (I, J, K, α, β) such that $\alpha : I \to \omega$ and $\beta : J \to \omega$.
(2) A **structure** $\mathcal{M}$ of similarity type τ consists of :

(i) A nonempty set M (called the universe of M);

(ii) A family $\{R_i^{\mathcal{M}} : i \in I\}$ such that for each $i \in I$, $R_i^{\mathcal{M}}$ is an $\alpha(i)$-place relation on M. An n-place relation on M is a subset of M^n. If $a_1, .., a_n \in M$, then it is customary to write $R(a_1, ..., a_n)$ instead of $(a_1, ..., a_n) \in R$;

(iii) A family $\{f_j^{\mathcal{M}} : j \in J\}$ such that for each $j \in J$, $f_j^{\mathcal{M}}$ is a $\beta(j)$-place function on M is a function from M^n into M;

(iv) A subset $\{c_k^{\mathcal{M}} : k \in K\}$ of M.

A useful form of notation for $\mathcal{M}$ is $(M, R_i^{\mathcal{M}}, f_j^{\mathcal{M}}, c_k^{\mathcal{M}})_{i\in I, j\in J, k\in K}$. The cardinality of $\mathcal{M}$ (in symbols $|\mathcal{M}|$)is defined to be the cardinality of M. We do not often distinguish between a structure $\mathcal{M}$ and its universe M.

Associated with each similarity type τ is a **first order language** L_τ.

0.1.2. Definition. The **primitive symbols** of L_τ are:

(i) variables $x_1, x_2, ...$;

(ii) logical connectives $\neg$ (not), $\wedge$ (and), $\exists$ (there exists), $=$ (equal);

(iii) $\alpha(i)$-place relation symbols R_i for $i \in I$;

(iv) $\beta(j)$-place function symbols f_j for $j \in J$;

(v) individual constants c_k for $k \in K$.

0.1.3. Definition. The **terms** of L_τ are generated by two rules:

(i) all variables and individual constants are terms;

(ii) if f is an n-place function symbol and $t_1, t_2, ..., t_n$ are terms, then $f(t_1, t_2, ..., t_n)$ is a term.

0.1.4. Definition. The **atomic formulas** of L_τ are given by:

(i) if t_1 and t_2 are terms, then $t_1 = t_2$ is an atomic formula.

(ii) if R is an n-place relation symbol and $t_1, t_2, ..., t_n$ are terms, then $R(t_1, t_2, ..., t_n)$ is an atomic formula.

0.1.5. Definition. The **formulas** of L_τ are generated by two rules:

(i) every atomic formula is a formula;

(ii) if ϕ and ψ are formulas, then $\neg\phi$, $\phi \wedge \psi$ and $\exists x\phi$ are formulas

0.1.6. Note. The symbols $\vee$ (or), $\rightarrow$ (implies) and $\forall$ (for all) are useful abbreviations:

(i) $\phi \vee \psi$ for $\neg(\neg\phi \wedge \neg\psi)$;

(ii) $\phi \rightarrow \psi$ for $\neg\phi \vee \psi$;

(iii) $\forall\phi$ for $\neg(\exists x(\neg\phi))$.

0.1.7. Definition. The notion of **free variable** of a formula is defined inductively. The induction is on the number of steps needed to generate the formula:

(i) if ϕ is atomic and x occurs in ϕ, then x is a free variable of ϕ;

(ii) if x is a free variable of ϕ, then x is a free variable of $\exists y\phi$;

(iii) if x is free variable of ϕ, then x is free variable of $\neg\phi$ and of $\phi \wedge \psi$.

A **sentence** is a formula without free variables. The cardinality of L_τ (in symbols $|L_\tau|$) is the cardinality of the set of all formulas of L_τ. Clearly $|L_\tau| = max(\aleph_0, |I|, |J|, |K|)$.

If M is an L-structure and $A \subset M$, then $L(A)$ denotes the language obtained by adjoining to L new individual constants c_a for $a \in A$.

0.1.8. Definition (Satisfaction). Let M be a structure of similarity type τ.
(1) For a term t of $L_\tau(M)$ without free variables, we define t^M as follows:

(i) $c_a^M = a$;

(ii) $(f(t_1, ..., t_n))^M = f^M(t_1^M, ..., t_n^M)$.

(2) Let ϕ be a sentence of $L(M)$. Then relation "$M \models \phi$" (read ϕ is **true** in M) is defined by induction on the number of steps needed to generate ϕ.

(i) $M \models t_1 = t_2$ iff $t_1^M = t_2^M$.

(ii) $M \models R(t_1, ..., t_n)$ iff $R^M(t_1^M, ..., t_n^M)$.

(iii) $M \models \phi \wedge \psi$ iff $M \models \phi$ and $M \models \psi$.

(iv) $M \models \neg\phi$ iff it is not true that $M \models \phi$.

(v) $M \models \exists x\phi(x)$ iff $M \models \phi(c_a)$ for some $a \in M$.

0.1.9 Definition. Let $\phi(x_1, ..., x_n)$ be a formula of L, and let $a_1, ..., a_n \in M$.
(1) Then $a_1 ... a_n$ is said to **satisfy** (or **realize**) $\phi(x_1, ..., x_n)$ in M if $M \models$

$\phi(a_1, ..., a_n)$. Let ϕ^M denote the set of realizations of ϕ in M.
(2)$\phi(x_1, ..., x_n)$ is said to be **valid** in M if $(\forall x_1 ... \forall x_n)\phi(x_1, ..., x_n)$ is true in M.
(3) M is **elementarily equivalent** to N (in symbols $M \equiv N$) means: $M \models \phi$ iff $N \models \phi$ for every sentence ϕ of L.

0.2. The Number of Models

0.2.1. Definition. Let M and N be L-structures.
(1) Let m be a map from M into N. m is an **elementary mapping** means: $M \models \phi(a_1, ..., a_n)$ iff $N \models \phi(m(a_1), ..., m(a_n))$ for every formula ϕ of L and every $a_1, ..., a_n \in M$. Note that m is an elementary mapping iff $(M, a)_{a \in M} \equiv (N, m(a))_{a \in M}$.
(2) M is an **elementary substructure** of N (in symbols $M \prec N$) if the identity map $i_M : M \subset N$ is an elementary mapping. If $M \prec N$, then N is said to be an **elementary extension** of M.

0.2.2. Remark. (Tarski-Vaught) We can see that $M \prec N$ if and only if, for every $\phi(x) \in L(M)$ if $N \models \exists x \phi(x)$ then there is a element $a \in M$ such that $N \models \phi(a)$.

0.2.3. Definition. Let T be a set of sentences of some language L, and let ϕ a formula of L. ϕ is a **logical consequence** of T (in symbols $T \vdash \phi$), if ψ is among the formulas generated from T as follows:

(i) if $\phi \in T$, then $T \vdash \phi$;

(ii) if ϕ is a logical axiom, then $T \vdash \phi$;

(iii) if $T \vdash \phi_i$ when $1 \leq i \leq n$, and if ϕ is the result of applying some logical rule of inference to the sequence $\phi_1, ..., \phi_n$, then $T \vdash \phi$.

0.2.4. Definition. (1) T is **consistent** if no sentence of the form $\phi \wedge \neg\phi$ is a logical consequence of T.
(2) M is said to be a **model** of T (in symbols $M \models T$), if every member of T is true in M.

It is clear that if $T \vdash \phi$, then $T_0 \vdash \phi$ for some finite $T_0 \subset T$. Exploiting repeatedly this finitary character of the consequence relation $\vdash$, we can prove the following theorem:

0.2.5. Theorem (Completeness). *If T is a consistent set of sentences, then T has a model of cardinality $\leq max(\aleph_0, |T|)$.*

0.2.6. Definition. (1) T is said to be a **theory** in L if it is a consistent set of L-sentences.
(2) T is said to be **complete** if either $T \vdash \phi$ or $T \vdash \neg\phi$ for every sentence ϕ of T.
(3) Let $Th(M)$ denote the set of all sentences of L true in M.

By 0.2.5, T is complete iff all models of T are elementarily equivalent.

0.2.7. Theorem (Compactness). *Let T be a theory such that every finite subset of T has infinite model. Then T has a model of cardinality κ for every $\kappa \geq max(\aleph_0, |T|)$.*

Proof. Let $\{c_\alpha\}_{\alpha<\kappa}$ be a set of individual constants, none of which occur in the language of T. Let $T^* = T \cup \{c_\alpha \neq c_\beta : \alpha < \beta < \kappa\}$. If $S \subset T^*$ is finite, then S is consistent, since $S \cap T$ has an infinite model. By 0.2.5, T^* has a model M of cardinality $\leq \kappa$. But M must have cardinality at least κ, since c_α's must name distinct elements of M. □

Consider the number of models. By 0.2.5, any theory has at least one model. If a complete theory has a finite model, then it has exactly one model, up to isomorphism. But if not, the situation changes. By 0.2.7 we have:

0.2.8. Theorem (Upward Löwenheim-Skolem). *Let M be an infinite structure. Then M has an elementary extension N of cardinality κ for every $\kappa \geq max(|M|, |T|)$.*

It is clear that, if two structures have different cardinality, then they are non-isomorphic. Therefore, by the above theorem, if T has an infinite model then T has infinitely many models.

So we consider the number of models with cardinality λ, for fixed λ. $I(\lambda, T)$ denotes the number of models of T of cardinality λ, up to isomorphism. If $I(\lambda, T) = 1$, we say that T is **λ-categorical.**

0.3. Countable Models

We consider the number of countable models. Let T be a countable complete theory in L. First observe the following easy example.

0.3.1. Example. Let L be the language with no relation other than equality and T the theory which says that there are infinitely many elements. Then clearly $I(\aleph_0, T) = 1$ (i.e., T is $\aleph_0$-categorical).

Is there a theory T such that $I(\aleph_0, T) = n$, for every n with $1 < n < \omega$? For every n with $3 \leq n < \omega$, there is a theory T which satisfies $I(\aleph_0, T) = n$ (Ehrenfeucht). On the other hand it is known that there is no T such that $I(\aleph_0, T) = 2$ (Vaught). To prove the Vaught theorem, we need some notions and results of classical model theory.

0.3.2. Definition. Let T be a complete theory in L.
(1) A **complete n-type** of T ($n < \omega$) is a set of Σ of L-formulas in n free variables, say $x_1, x_2, ..., x_n$, which is maximal consistent with T. A **type** of T is an n-type for some $n < \omega$.
(2) The set of n-types of T is denoted by $S_n(T)$ and we put $S(T) = \bigcup_{n<\omega} S_n(T)$. Types are denoted by $p, q, r, ...$.
(3) If M is an L-structure, $A \subset M$, and $\bar{b}$ is a tuple from M, then $tp_M(\bar{b}/A) = \{\phi(\bar{x}) \in L(A)$ and $M \models \phi(\bar{b})\}$. Let $S_n(A)$ denote $S_n(Th(M, a)_{a\in A})$. Thus $tp_M(\bar{b}/A) \in S_n(A)$. Let $tp_M(\bar{b})$ denote $tp_M(\bar{b}/\emptyset)$.

0.3.3. Definition. Let $\Sigma(\bar{x})$ be a consistent set of L-formulas. Then Σ is said to be **isolated** if there is a consistent L-formula $\psi(\bar{x})$ such that $T \vdash (\forall \bar{x})(\psi(\bar{x}) \rightarrow \phi(\bar{x}))$ for every $\phi \in \Sigma$, and ψ is said to **isolate** Σ (relative to T of course).

0.3.4. Theorem (Omitting Types). *Let T be a countable theory in L For each $n < \omega$ let Σ_n be a consistent set of L-formula with free variables. Suppose that for each n, Σ_n is non-isolated. Then T has a countable model which omits each Σ_n.*

Proof. To simplify our proof we concentrate on omitting "one" non-isolated type $\Sigma(x)$. Let M be a countable model of T. By compactness we get next claim:

Claim: Let $A \subset M$ and let $\phi(x) \in L(A)$. Then if $\Sigma(x)$ is non-isolated over A, then there is a realization a of ϕ in M such that Σ is non-isolated over Aa.

Using the claim repeatedly, we can construct in M a (countable) model such that Σ is non-isolated over N. Clearly N omits Σ. $\square$

0.3.5. Theorem ($\aleph_0$-Categoricity). *Let T be a countable theory in L. Then T is $\aleph_0$-categorical iff for each $n < \omega$, all $p \in S_n(T)$ is isolated.*

Proof. (only if) If there is a non-isolated type p, then, by 0.3.4, we have a model which omits p. On the other hand there is a model realizing p. This is contradicts $\aleph_0$-categoricity of T.
(if) By the back-and-forth method. □

0.3.6. Definition. (1) A **prime model** of the theory T is a model M of T such that , for all $N \models T$, there is an elementary mapping of M into N.
(2) M is said to be an **atomic model** of T if $tp_M(\bar{a})$ is isolated for every $\bar{a} \in M$.

0.3.7. Remark. By 0.3.4 and (one half of) the back-and-forth method, we obtain that M is prime if and only if M is countable atomic.

By 0.3.7 , we have:

0.3.8. Theorem (Prime Models). *Let T be a complete theory in L.*
(1) If $S(T)$ is countable, then T has a prime model.
(1) Any two prime models of T are isomorphic.

0.3.9. Definition. (1) M is said to be **κ-saturated** if, for any $A \subset M$ with $|A| < \kappa$, for all $p(\bar{x}) \in S(A)$, p is realized in M.
(2) M is said to be **countably saturated** if it is countable and $\aleph_0$-saturated.

0.3.10. Theorem (Saturated Models). *Let T be a complete theory in L.*
(1) If $S(T)$ is countable, then T has a countably saturated model.
(2) Any two countably saturated models of T are isomorphic.

Proof. (1) Let M_0 be a countable model of T. Let $X = \bigcup\{S(F) : F \subset M_0, |F| < \aleph_0\}$. Then X is countable, since $S(T)$ is countable. By 0.2.8 we can get a countable elementary extension M_1 of M_0 which realizes every member of X. In the similar way we obtain a countable elementary chain $(M_i)_{i<\omega}$. Set $M = \bigcup M_i$. By 0.2.2 M is a model of T and moreover countably saturated. □

0.3.11. Theorem (Vaught). *For any complete T, $I(\aleph_0, T) \neq 2$.*

Proof. Suppose that $I(\aleph_0, T) = 2$. Then $S(T)$ is countable. By 0.3.8 and 0.3.10, there are a countably saturated model M and a countable atomic model N. Since T is not $\aleph_0$-categorical we can find, by 0.3.5, a non-isolated type $p \in S(T)$. Hence M and N is non-isomorphic. Let $\bar{a}$ realize p. Since $S(\bar{a})$ is also countable, there is a countable model N' atomic over $\bar{a}$. Then we obtain that N' is both non-saturated and non-atomic. It follows that $I(\aleph_0, T) \geq 3$. A contradiction. □

0.3.12. Example. (1) Let T be the theory of algebraically closed fields of some fixed characteristic. Any model of T is determined up to isomorphism by its transcendence dimension over the prime field. Namely, for any model M of T all maximal algebraically independent subsets of M have same cardinality, which is the transcendence dimension of M. Moreover two models with same transcendence dimension are isomorphic. Therefore we have $I(\aleph_0, T) = \aleph_0$.
(2) Let T be the theory which says that E_n is an equivalence relation for each $n < \omega$, such that E_0 has two infinite classes and E_{n+1} partition each En class into two infinite classes. Let E be the infinitary conjunction of all the E_n. Each model will have infinitely many E-classes, but the cardinality of each class can be chosen arbitrary. Thus we have $I(\aleph_0, T) = 2^{\aleph_0}$.

Is there a theory T such that $I(\aleph_0, T) = \kappa$, for some κ with $\aleph_0 < \kappa < 2^{\aleph_0}$? But the question is still open:

Vaught Conjecture. $I(\aleph_0, T) \geq \aleph_0$ *implies* $I(\aleph_0, T) = 2^{\aleph_0}$.

Shelah partially solved the conjecture, using machinery of so-called "stability theory".

The following table summarizes the discussion above. The columns indicate the number of countable models. An entry of "×" means there is no theory of this kind; an entry of "?" means the existence of such a theory is unknown.

The Number of Countable Models I

1	2	3	....	$\aleph_0$	....	$2^{\aleph_0}$
o	×	o	o	o	?	o

0.4 Stability Theory

0.4.1. Definition. (1) Let $\lambda \geq \aleph_0$. Then T is **λ-stable** if, for all A, $|A| \leq \lambda$ implies $|S_1(A)| \leq \lambda$.
(2) T is **stable** if T is λ-stable for some λ.
(3) T is **superstable** if T is λ-stable for all $\lambda \geq 2^{\aleph_0}$.

0.4.2. Remark. (1) T is said to have the order property if T has a model which contains a set totally ordered by a formula. Then it is well-known that T is unstable iff T has the order property.

(2) We can see that if T is ω-stable, then T is λ-stable for all λ. Hence we have "ω-stable $\Rightarrow$ superstable $\Rightarrow$ stable".

0.4.3. Definition. (1) Let $A \subset B$. Let $p \in S(A)$ and $q \in S(B)$ such that $p \subset q$. Then q is said to be a **forking extension** of p if there are $\phi(x, \bar{b}) \in L(B)$ and $M \supset A$ such that ϕ is not satisfied in M. And we say that q is a **non-forking extension** of p if q is not a forking extension of p.
(2) Let $\bar{a}$ be a tuple and let A and B sets. Then we say that $\bar{a}$ is **independent** from B over A (in symbols $\bar{a} \downarrow_A B$) if $tp(\bar{a}/B \cup A)$ is non-forking extension of $tp(\bar{a}/A)$.
(3) Let A, B and C be sets. Then we say that C is independent from B over A (in symbols $C \downarrow_A B$) if, for every $\bar{a} \in C$ is independent from B over A.

0.4.4. Remark. If T is stable, we have the symmetry property: $B \downarrow_A C \Rightarrow C \downarrow_A B$.

0.4.5. Theorem (Existence of non-forking extensions). *Let T be stable. Then for any $p \in S(A)$ and $B \supset A$ there is a non-forking extension of p over B.*

0.4.6. Definition. Let A be any set and let $p \in S(A)$. We say that p is **stationary** if it has exactly one non-forking extension over any set ($\supset A$).

0.4.7. Definition. (1) Let A be a set. Let p and q be stationary types over A. Then p and q are **almost orthogonal** (in symbols $p \perp^a q$) if whenever $a \models p$ and $b \models q$ then a and b are independent over A.
(2) Let A be a set. Let p and q be stationary types over A. Then p and q are **orthogonal** (in symbols $p \perp q$) if any nonforking extensions of p and q over any set are almost orthogonal.
(3) Let A and B be any sets. Let p be a stationary type over A and let q a stationary type over B. Then p and q are **orthogonal** if non-forking extensions of p and q over $A \cup B$ are orthogonal.
(4) Let A and B be any sets. Let $p \in S(A)$ be stationary. Then p is **orthogonal** to B (in symbols $p \perp B$) if any type over B is orthogonal to p.

0.4.8. Definition. T be stable. Then U**-rank** is defined on (complete) types as follows:

(i) If $p \in S(A)$, then $U(p) \geq 0$;

(ii) If δ is limit, $p \in S(A)$ and $U(p) \geq \alpha$ for all $\alpha < \delta$, then $U(p) \geq \delta$;

(iii) If $\alpha = \beta + 1$, $p \in S(A)$, and p has a forking extension $q \in S(B)$ for some $B \supset A$ such that $U(q) \geq \beta$, then $U(p) \geq \alpha$;

(iv) We say that $U(p) = \alpha$ if $U(p) \geq \alpha$ and not $U(p) \geq \alpha + 1$.

(v) If $U(p) \geq \alpha$ for all α, we say $U(p) = \infty$, with the convention that $\alpha < \infty$ for all ordinals α.

0.4.9. Remark. (1) Let p, q be types such that $p \subset q$ and $U(q) < \infty$. Then $U(p) = U(q)$ if and only if q is a non-forking extension of p.
(2) We can see that T is superstable if and only if, for all $p \in S(A)$, $U(p) < \infty$.

0.4.10. Definition. Let A be a set and let p a stationary type over A. Then p is said to be **regular** if any forking extension of p is orthogonal to p.

0.4.11 Remark. If $p \in S(A)$ is regular and $M \supset A$, then all maximal independent sets for p in M have the same cardinality. So "dimension" exists for regular types.

0.4.12. Theorem (Existence of Regular Types). *Let T be ω-stable. Let $M \subset N$. Then there is an element $a \in N - M$ such that $tp(a/M)$ is regular.*

0.4.13. Theorem (Existence of Prime Models). *If T is ω-stable, then T has a unique prime model over any set.*

By the above existence theorems, many useful results can be proved. We summarize such results:

0.4.14. Theorem (Morley). *If for an uncountable cardinal λ a theory T is λ-categorical, then T is μ-categorical for every uncountable cardinal μ.*

0.4.15. Theorem (Baldwin-Lachlan). *If for an uncountable cardinal λ a theory T is λ-categorical, then $I(\aleph_0, T) = 1$ or $\aleph_0$.*

0.4.16. Theorem (Lachlan). If T is superstable and $\aleph_0$-categorical, then T is ω-stable.

0.4.17. Theorem (Lachlan). *If T is superstable then $I(\aleph_0, T) = 1$ or $\geq \aleph_0$.*

0.4.18. Theorem (Shelah). *If T is ω-stable then $I(\aleph_0, T) > \aleph_0$ implies $I(\aleph_0, T) = 2^{\aleph_0}$.*

0.4.19. Theorem (Hrushovski). There is an $\aleph_0$-categorical theory which is stable and unsuperstable.

The next problems are still open.

Open Problems. *(1) Is there a stable unsuperstable theory T such that $1 < I(\aleph_0, T) < \aleph_0$?*
(2) Is there a non-ω-stable theory T such that $\aleph_0 < I(\aleph_0, T) < 2^{\aleph_0}$?

The following table summarizes the above results. The rows describe a place in the stability classification. The columns indicate the number of countable models. An entry of "×" means there is no theory of this kind; an entry of "?" means the existence of such a theory is unknown.

The Number of Countable Models II

	1	....	$\aleph_0$	....	$2^{\aleph_0}$
$\aleph_1$-categorical	o	×	o	×	×
not $\aleph_1$-categorical, ω-stable	o	×	o	×	o
not ω-stable, superstable	×	×	o	?	o
not superstable, stable	o	?	o	?	o
unstable	o	o	o	?	o

0.5. Uncountable Models and DOP

We consider the number of uncountable models. Let T be a complete theory. We say that T has **many models** if $I(\lambda, T) = 2^\lambda$ for every $\lambda > |T|$. And we say that T is **classifiable** if T does not have many models. Shelah has solved the central problem on the number of uncountable models. His strategy is as follows:

(1) First he introduced a number of conditions of T (stable, superstable, NDOP, shallow and NOTOP).

(2) Next he proved that if T does not have these condition, then T has many models.

(3) On the other hand if T has these condition then he proved structure theorems (Existence of the independence relation, of regular types, of prime models and of presentations....).

Then Shelah's main theorem states that

0.5.1. Theorem(Shelah). *T is classifiable if and only if T is superstable, NDOP, shallow and NOTOP.*

The number of uncountable models

$$\text{a theory}\begin{cases}\text{unsuperstable} \Rightarrow \text{many models}\\ \text{superstable}\begin{cases}\text{DOP} \Rightarrow \text{many models}\\ \text{NDOP}\begin{cases}\text{deep} \Rightarrow \text{many models}\\ \text{shallow}\begin{cases}\text{OTOP} \Rightarrow \text{many models}\\ \text{NOTOP} \Rightarrow \text{classifiable}\end{cases}\end{cases}\end{cases}\end{cases}$$

We observe here one of the above conditions, the dimensional order property.

0.5.2. Definition. We say that a theory has the **dimensional order property** (for short **DOP**) if there exist tuples a, b, a finite set F and a non-algebraic type p such that $p \not\perp abF$, $p \perp aF$, $p \perp bF$ and $a \downarrow_F b$.

0.5.3. Example. Let $L = \{V, E, R\}$, where V(vertex) and E(edge) are unary predicates and R is a ternary predicate. Let T be the following theory in L: any model of T is the union of the two disjoint infinite sets V and E; $R \subset V \times V \times E$; for each pair $\langle u, v\rangle \in V \times V$ and $e \in E$, $R(u, v, e)$ holds iff $R(v, u, e)$ holds; for each pair $\langle u, v\rangle \in V \times V$ satisfying $u \neq v$ there are infinitely many elements $e \in E$ such that $R(u, v, e)$ holds; for each $e \in E$ there is a unique pair $\langle u, v\rangle \in V \times V$ satisfying $u \neq v$ such that $R(u, v, e)$ holds. Take any distinct elements a, b of V. Let $p(x) = R(a, b, x)$. Then a, b and p witness that T has the DOP. Each model of T above is constructed by first choosing its set of vertices, and then for each pair of vertices, by choosing the edges between them. In this case second-level choice is a function of a pair of vertices chosen at the first level. Then we can count the number of models: $I(\aleph_0, T) = 1$ and $I(\kappa, T) = 2^{\kappa}$. It follow that T has many models.

In chapter 3 we shall show that if T is an ω-categorical ω-stable theory with DOP, then essentially T can be regarded as one of three concrete examples.

0.6. Minimal Models

We restrict ourselves here to a countable complete theory T. M is said to be **minimal** if it has no proper elementary submodel (and hence it is countable). Our concern here is to count the number of minimal models. We observe first easy examples:

0.6.1. Example. Let T be the theory of an infinite set (see example 0.3.1). Then any countable models of T is prime. But each of them is not minimal. Hence T has no minimal models.

0.6.2. Example. Let T be the theory of an algebraically closed field with a given characteristic. Then the algebraic closure of the prime field is prime and minimal. Moreover it is the unique minimal model.

Each of the above theories has a prime model. In general if a theory with a prime model has a minimal model, it has the unique minimal one. So we consider a theory having a minimal non-prime model. In [6] Fuhrken gave an example of such a theory:

0.6.3. Example ([6]). For each $\nu < 2^{<\omega}$ we define a function $F_\nu : 2^\omega \to 2^\omega$ by $(F_\nu(\eta))(i) = \nu(i) + \eta(i) \bmod 2$ for $\eta \in 2^\omega$, $i < \omega$. And for $\eta \in 2^{<\omega}$, $P_\eta = \{\tau \in 2^\omega : \eta \prec \tau\}$. Let $M = (2^\omega, \{F_\nu\}_{\nu<2^{<\omega}}, \{P_\eta\}_{\eta<2^{<\omega}})$ and $T = Th(M)$. Then each model generated by only one element is minimal and non-prime. Then T has $2^{\aleph_0}$ minimal models.

A natural question will arise: Is there a theory which has κ minimal models for some κ with $1 < \kappa < 2^{\aleph_0}$? In [12] Marcus showed that if T is a theory of one unary function symbol and T has a minimal non-prime model then T has $2^{\aleph_0}$ minimal models. On the other hand Shelah [15] showed that for every κ with $1 \le \kappa \le \aleph_0$, there is a theory with exactly κ minimal non-prime models. By Morley's result, we can see that if T has more than $\aleph_1$ minimal models then it has $2^{\aleph_0}$ minimal ones. The following problem is still open:

Open Problem. *Is there a theory which has exactly $\aleph_1$ minimal models?*

In chapter 1 we prove a theorem on indiscernible sets in a minimal model. In chapter 2 we give a result on the number of minimal models. Our result is an expansion of Marcus's result [12].

The following table summarizes the discussion above. The columns indicate the number of minimal models. An entry of "×" means there is no theory of this kind; an entry of "?" means the existence of such a theory is unknown.

The Number of Minimal Models

0	1	2	....	$\aleph_0$	$\aleph_1$	....	$2^{\aleph_0}$
o	o	o	o	o	?	×	o

Chapter 1. Indiscernible Sets in Minimal Models

A model M is said to be *minimal* if there is no proper elementary submodel of M. We consider the size of indiscernible sets in a minimal model. Shelah showed that if a theory T is totally transcendental then there is no infinite indiscernible set in a minimal model of T (see [15, IV, Theorem 4.21]). On the other hand, in [11] Marcus constructed a minimal (and prime) structure with an infinite indiscernible set. His structure is stable and non-superstable. Our aim is therefore to extend Shelah's result to a superstable theory.

Theorem A. *Let T be superstable and let A be any set. Then there is no minimal model over A which has an infinite set of indiscernibles over A.*

To illustrate our proof, we observe Shelah's proof: Let M be a model having an infinite indiscernible set I. Pick any $a \in I$ and let $J = I - \{a\}$. Since T is totally transcendental, there is $N \prec M$ which is atomic over J. By indiscernibility of I, we have $a \notin N$. Hence M is not minimal.

However, if T is not totally transcendental , then we do not necessarily have an atomic model inside M. So, instead of N above, we take in M a maximal set E which includes J but is independent from a. We call such E *a $tp(a)$-envelope of J in M* (see Definition 1.2.1 for the exact definition). First we show that if T is superstable, E is an elementary submodel of M (Lemma 1.2.3). It follows that M is not minimal, and hence we can obtain our theorem. In the end of the paper, we give a stable structure having an infinite indiscernible set (Example 1.3.2). The way of the construction is essentially the same as Marcus's one [11].

1.1. Notation.

We fix a (possibly uncountable) stable theory T. We usually work in a big model $\mathbb{C}$ of T. Our notations are fairly standard. $A, B, ...$ are used to denote small subsets of $\mathbb{C}$. $\bar{a}, \bar{b}, ...$ are used to denote finite sequences of elements in $\mathbb{C}$. $\varphi, \psi, ...$ are used to denote formulas (with parameter). $p, q, ...$ are used to denote types (with parameter). The nonforking extension of a stationary types p to the domain A is denoted by $p|A$. The type of a over A is denoted by $tp(a/A)$.

$R^\infty(p)$ (resp. $R^\infty(\varphi)$) is the infinity rank of a type p (resp. a formula φ). We simply write $R^\infty(a/A)$ instead of $R^\infty(tp(a/A))$. The set of realizations of a type p (resp. a formula φ) in a model M is denoted by p^M (resp. φ^M).

1.2. Lemma.

1.2.1. Definition. Let M be a model and $A \subset B \subset M$. Let $p \in S(A)$ be stationary. Then a *p-envelope* of B in M is a maximal set E such that $B \subset E \subset M$ and any element of $(p|B)^M$ is independent from E over A.

1.2.2. Remark. The notion of "envelopes" was introduced in [4], and was defined in the context of totally categorical theories. Our definition is a generalization of that in [4].

1.2.3. Lemma. *Let T be superstable. Let M be a model and $A \subset M$. Let $p \in S(A)$ be stationary. Suppose that M contains some infinite Morley sequence I of p. Then a p-envelope of IA in M is an elementary submodel of M.*

Proof. For the simplicity of the notation, we may assume that $A = \emptyset$. Take any p-envelope E of I in M. If $(p|I)^M = \emptyset$, then $E = M$. So we assume that $(p|I)^M \neq \emptyset$. Assume by way of contradiction that E is not an elementary submodel of M. Then, by the Tarski criterion, there is a consistent formula $\varphi(x, \bar{e_0}) \in L(E)$ such that $\varphi^M \cap E = \emptyset$. By superstability, pick an element b of φ^M such that $R^\infty(b/E)$ is minimal.

Claim. *Any $a \in (p|I)^M$ is independent from b over E.*

Proof. Assume otherwise. Then there is an element a of $(p|I)^M$ such that $tp(a/Eb)$ forks over E. Take a formula $\theta(x, \bar{e_1}) \in tp(b/E)$ such that $R^\infty(b/E) = R^\infty(\theta)$. Now $tp(a/Eb)$ forks over $\emptyset$, so there is $\bar{e} \in E$ such that $tp(a/\bar{e}b)$ forks over $\emptyset$. Then we may assume that $\bar{e_0}, \bar{e_1} \subset \bar{e}$. Note that $tp(a/\bar{e})$ does not fork over $\emptyset$ (because $\bar{e} \in E$). It follows that $tp(b/\bar{e}a)$ forks over $\bar{e}$. So we can get a formula $\psi(x, \bar{e}, a) \in tp(b/\bar{e}a)$ such that, if $\models \psi(b', \bar{e}, a)$ then $tp(b'/\bar{e}a)$ forks over $\bar{e}$. Let $\Gamma(a, \bar{e})$ denote $(\exists x)(\varphi(x, \bar{e_0}) \wedge \psi(x, \bar{e}, a) \wedge \theta(x, \bar{e_1}))$. Now the weight of $\bar{e}$ is finite since $R^\infty(\bar{e}) < \infty$. Therefore we can pick $a' \in I$ such that $tp(a'/\bar{e})$ does not fork over $\emptyset$. Remember that $tp(a/\bar{e})$ does not forks over $\emptyset$. It follows that $tp(a/\bar{e}) = tp(a'/\bar{e})$. Hence $\Gamma(a', \bar{e})$ holds. Therefore there is an element $b' \in \varphi^M$ such that $R^\infty(b'/\bar{e}) \leq R^\infty(b/E)$ and $tp(b'/\bar{e}a')$ forks over $\bar{e}$. Thus $R^\infty(b/E) \geq R^\infty(b'/\bar{e}) > R^\infty(b'/\bar{e}a') \geq R^\infty(b'/E)$. Moreover $R^\infty(b'/E) \neq 0$

because b' satisfies φ. But this contradicts the minimality of $R^\infty(b/E)$. Hence the claim holds.

Thus any $a \in (p|I)^M$ is independent from bE over I. But this contradicts that E is an envelope. Hence E is an elementary submodel. This completes the proof of the lemma. □

1.2.4. Example. Let $Per(\omega)$ denote the set of permutations of ω which move only a finite number of elements. For each $i < \omega$, define a function $\pi_i : Per(\omega) \to \omega$ such that $\pi_i(\sigma) = \sigma(i)$. Let $A = \omega \cup Per(\omega)$. Consider the structure $M = (A; \omega, Per(\omega), \{\pi_i\}_{i<\omega})$. Then ω is a Morley sequence of $tp(0)$. Note that for any $\sigma \in Per(\omega)$, $\omega \subset dcl(\sigma)$(=the definable closure of σ). Therefore $\omega - \{0\}$ is the $tp(0)$-envelope of $\omega - \{0\}$ in M. However $\omega - \{0\}$ is not a model. Moreover $T = Th(M)$ is not superstable (since the weight of σ is infinite). This example shows that we need, in lemma 1.2.3, the assumption that T is superstable.

1.3. Theorem and Example.

1.3.1 Theorem A. *Let T be superstable and let A be any set. Then there is no minimal model over A which has an infinite set of indiscernibles over A.*

Proof. Suppose that M has an infinite set I of indiscernibles over some set A. We can assume that I is already an infinite Morley sequence of some $p \in S(A)$ because $\kappa(T)$ is countable. Pick any $a \in I$. By lemma 1.2.3, a p-envelope E of $(I - \{a\}) \cup A$ in M is an elementary submodel of M. It is clear that $a \notin E$. Hence M is not minimal. A contradiction. □

1.3.2. Example (see [11]). Theorem 1.3.1 can not be extended to a stable theory. We construct a minimal structure with an infinite indiscernible set. Recall the structure $M = (A; \omega, Per(\omega), \{\pi_i\}_{i<\omega})$ (see Example 1.2.4). Note that this structure is not minimal. But by modifying the construction, we can obtain a minimal one: For each $n < \omega$, we define inductively P_n and $\{\pi_a^n : a \in P_n\}$ which satisfy the following properties:
(i) $P_0 = \omega$, and $\pi_a^0 = \pi_a$ $(a \in P_0)$;
(ii) $P_{n+1} = Per(P_n)$ $(n < \omega)$;
(iii) $\pi_a^{n+1} : P_{n+1} \to P_n$ is a function such that $\pi_a^{n+1}(\sigma) = \sigma(a)$ $(a \in P_n,\ n < \omega)$.
Let $A^* = \bigcup\{P_n : n < \omega\}$. Consider the structure $M^* = (A^*; \{P_n : n < \omega\}, \{\pi_a^n : a \in P_n, n < \omega\})$. Then for each $n < \omega$, if $\sigma \in P_{n+1}$ then we have $P_n \subset dcl(\sigma)$.

Hence M^* is a minimal model (Proof: Take any $N \prec M^*$ and $a \in M^*$. Then there is some n such that $a \in P_n$. Now $P_{n+1} \cap N \neq \emptyset$, so we can pick some $\sigma \in P_{n+1} \cap N$. Therefore $a \in dcl(\sigma) \subset N$, so $a \in N$. It follows that $N = M^*$). It is easy to see that $P_0 = \omega$ is an infinite indiscernible set. Moreover M^* is not superstable, since M is interpreted in M^* (Recall that M is not superstable).

Chapter 2. The Number of Minimal Models

The algebraic closure $\bar{Q}$ of the rationals Q in the complex number field C is small in the following two senses: (i) There is no proper elementary subfield K of $\bar{Q}$, and (ii) every field which is elementarily equivalent to $\bar{Q}$ has a copy of $\bar{Q}$ in it. In algebraic model theory, it is often the case that one of the two properties stated above implies the other. But in general model theory we have to distinguish these two notions. The notion expressing the first property is called 'minimal', and the other for the the second 'prime' (see Definition 2.1.1). The following is an example of a theory having a minimal non-prime model:

Example (see [6]). The following theory T satisfies our assumption: For each $\nu < 2^{<\omega}$ we define a function $F_\nu : 2^\omega \to 2^\omega$ by $(F_\nu(\eta))(i) = \nu(i) + \eta(i)$ mod 2 for $\eta \in 2^\omega$, $i < \omega$. And for $\eta \in 2^{<\omega}$, $P_\eta = \{\tau \in 2^\omega : \eta \prec \tau\}$. Let $M = (2^\omega, \{F_\nu\}_{\nu<2^{<\omega}}, \{P_\eta\}_{\eta<2^{<\omega}})$ and $T = Th(M)$. Then each model generated by only one element is minimal and non-prime.

Our concern is the number of minimal models of a theory with no prime model (In fact if a theory has a prime model then it has at most one minimal model). In [12] Marcus showed that if T is a theory of one unary function symbol and T has a minimal non-prime model then T has continuously many such models. On the other hand, Shelah proved that for every κ, $1 \leq \kappa \leq \aleph_0$, there is a theory with exactly κ minimal non-prime models(see [15]).

Here we extend Marcus' result. Theories of one unary function symbol may have the Lascar rank greater than 1 ($U(T) > 1$), however if such a theory T has a minimal model then any element a of the model has the minimum Lascar rank (i.e. $U(a) \leq 1$). Moreover a theory of one unary function symbol has some 'naive' property called trivial. In this paper we show the following theorem:

Theorem B. *Let T be stable and trivial. Suppose that T has a model M such that (i) M is minimal and non-prime, and (ii) $U(a) \leq 1$, for all $a \in M$. Then T has $2^{\aleph_0}$ many minimal models.*

2.1. Definitions and Preliminary results

Our notations and conventions are standard. We fix a complete theory T formulated in a countable language L. We work in a big model $\mathbf{C}$ of T. $A, B, \ldots$ are used to denote small subsets of $\mathbf{C}$. $\bar{a}, \bar{b}, \ldots$ are used to denote finite sequences of elements in $\mathbf{C}$. $\varphi, \psi, \ldots$ are used to denote formulas (with parameter). $p, q, \ldots$ are used to denote types (with parameter). The types of a over A is denoted by $tp(a/A)$. φ^M denotes the set of realizations of φ in M. The Lascar rank of p is denoted by $U(p)$. We simply write $U(a/A)$ instead of $U(tp(a/A))$. $U(a)$ means $U(a/\emptyset)$.

2.1.1. Definition. Let M be a model of the theory T.

(1) M is said to be *minimal* if there is no proper elementary submodel oaf M.

(2) M is said to be *prime* if M can be elementarily embedded in any model of T.

2.1.2. Definition. (1) Let A be a set. Then an $L(A)$-type $\Gamma(x)$ (not necessarily complete) is said to be *principal over* A if it is generated by one $L(A)$-formula $\varphi(x)$ (φ need not be a formula in Γ).

(2) A formula $\varphi(x) \in L$ is said to be *atomless* if there is no formula $\psi(x)$ with the following properties:

(i) $T \vdash \forall x(\psi(x) \to \varphi(x))$;

(ii) $\psi(x)$ is complete i.e. $\psi(x)$ determines a complete type $p(x)$.

If $S(\emptyset)$ is countable, then there is a prime (and atomic) model. On the other hand, if $S(\emptyset)$ is uncountable then there is an atomless formula.

We prove a version of Lemma 1.3 of [12].

2.1.3. Lemma. *Let $\Gamma(\bar{x})$ be a non-principal (possibly incomplete) type over a countable set A. Suppose that there is an atomless formula $\psi(y)$ over $\emptyset$ such that if d is a realization of ψ then d and A are independent. Then there are continuously many countable models ($\supset A$) omitting Γ.*

Proof. First we show the following claim:

Claim 1. *Let $\theta(\bar{x}, y)$ and $\varphi(y)$ be $L(A)$-formulas. If $\theta(\bar{x}, y) \wedge \varphi(y)$ is consistent then there is an $L(A)$-formula $\varphi^*(y)$ with $\varphi^{*\mathbf{C}} \subset \varphi^{\mathbf{C}}$ such that $\theta(\bar{x}, d)$ does not generate Γ for any realization d of φ^*.*

Proof. Since Γ is non-principal over A there is a realization d of φ such that $\theta(\bar{x}, d)$ does not generate Γ. So we can pick $\gamma \in \Gamma$ such that $\theta(\bar{x}, d) \wedge \neg\gamma(\bar{x})$ is consistent. Define $\varphi^*(y) = (\exists \bar{x})(\varphi(y) \wedge \theta(\bar{x}, y) \wedge \neg\gamma(\bar{x}))$. Then φ^* is a consistent $L(A)$-formula. It is clear that Γ is not generated by $\theta(x, d)$ for any $d \in \varphi^{*\mathbf{C}}$.

Let $\Gamma(\bar{x})$ have k-variables. Let $\theta_n(\bar{x}, y)$ ($n < \omega$) be an enumeration of all $L(A)$-formula with $(k+1)$-variables.

Claim 2. *We can define inductively $L(A)$-formulas $\psi_\eta(y)$ and L-formula $\alpha_\eta(y)$ ($\eta < 2^{<\omega}$) satisfying the following conditions: for each $\eta < 2^{<\omega}$,*

(1) $\psi_{<>}(y) = \psi(y)$;

(2) $\models (\forall y)(\psi_{\eta^\frown i}(y) \to \psi_\eta(y))$ ($i = 0, 1$);

(3) there is an L-formula $\alpha_\eta(y)$ such that $\models (\forall y)(\psi_{\eta^\frown 0}(y) \to \alpha_\eta(y))$ and $\models (\forall y)(\psi_{\eta^\frown 1}(y) \to \neg\alpha_\eta(y))$;

(4) If $\psi_\eta(y) \wedge \theta_n(\bar{x}, y)$ is consistent then $\theta_n(\bar{x}, a)$ does not generate Γ for any realization a of ψ_η (the length of η is $n+1$).

Proof. Suppose that ψ_η's (the length of η is $\leq n+1$) have been defined. Fix any η with length $n+1$. First we see that there is an L-formula $\alpha(y)$ such that both $\alpha(y) \wedge \psi_\eta(y)$ and $\neg\alpha(y) \wedge \psi_\eta(y)$ are consistent. If not, ψ_η generates some complete L-type q. Since ψ is atomless q is non-principal. On the other hand, by the assumption, ψ_η does not fork over $\emptyset$. So ψ_η is realized by every model. This means that q is principal, which is a contradiction. Therefore we get such an $\alpha(y)$. Put $\alpha_\eta(y) = \alpha(y)$. Let $\psi_0(y) = \alpha_\eta(y) \wedge \psi_\eta(y)$ and $\psi_1(y) = \neg\alpha_\eta(y) \wedge \psi_\eta(y)$. Suppose that $\psi_0(y) \wedge \theta_{n+1}(x, y)$ is consistent. By claim 1 we obtain an $L(A)$-formula $\psi_0^*(y)$ with $\psi_0^{*\mathbf{C}} \subset \psi_0{}^{\mathbf{C}}$ such that $\theta_{n+1}(\bar{x}, d)$ does not generate $\Gamma(\bar{x})$ for any realization d of ψ_0^*. Put $\psi_{\eta^\frown 0} = \psi_0^*$. Similarly we can get $\psi_{\eta^\frown 1}$. Then they satisfy our requirement. This completes our construction.

For $\tau < 2^\omega$, define $\Sigma_\tau(y) = \{\psi_{\tau|n}(y) : n < \omega\}$. It is easy to see that Σ_τ's are $L(A)$-types which satisfy that i) $\tau \neq \lambda$ implies $tp(d_\tau) \neq tp(d_\lambda)$ for any realization d_τ of Σ_τ and d_λ of Σ_λ, and ii) if d_τ is a realization of Σ_τ then Γ is non-principal over $A \cup d_\tau$. By ii), for every $\tau < 2^\omega$ there is a countable model M_τ ($\supset A \cup d_\tau$) omitting Γ. By i), for any M_τ there are at most countably many M_λ's isomorphic to M_τ. Thus there is an $X \subset 2^\omega$ with $|X| = 2^{\aleph_0}$ such that M_τ ($\tau \in X$) are pairwise non-isomorphic. Hence we obtain continuously many countable models omitting Γ. This completes the proof of the lemma. □

2.1.4. Definition. T is said to be *trivial* if it has the following property: for any three elements $a, b, c \in \mathbf{C}$ and any set $A \subset \mathbf{C}$, if a, b and c are pairwise independent over A then they are independent over A.

2.2. Theorem and Proof

2.2.1 Theorem B. *Let T be stable and trivial. Suppose that T has a model M such that*

(1) M is minimal and non-prime;

(2) $U(a) \leq 1$, for all $a \in M$.

Then T has continuously many minimal models.

Proof. First we show the following claim:

Claim 1. *There are an element a of M and a finite subset F of M such that $tp(a/F)$ is non-principal.*

Proof. M is a non-prime model. So it is not atomic, hence there is a minimal finite subset E of M such that $tp(E)$ is non-principal. Pick any element a of E. Let $F = E - \{a\}$. By the minimality of E $tp(F)$ is principal, so $tp(a/F)$ is non-principal.

By a *minimal component* we mean a set whose any two elements are interalgebraic. Let $C = acl(a) - acl(\emptyset)$ and $A = M - C$. C is a minimal component since $U(a) = 1$.

Claim 2. *There are a finite subset F' of A and an atomless formula $\psi(y)$ over F' such that any realization d of ψ is independent from A over F'.*

Proof. Since M is a minimal model, by the Tarski-Vaught test, we can easily find an $L(A)$-formula $\psi(y, \bar{a})$ such that $\psi^M \subset C$. Let $F' = F \cup \bar{a}$. We notice that under the assumption (2), in M the general notion of independence coincides algebraic independence. So C and A are independent by using the triviality of T. First we will show that ψ is atomless over F'. If not, there is a complete formula $\psi'(y)$ over F' such that $\psi'^{\mathbf{C}} \subset \psi^{\mathbf{C}}$. Then ψ' is realized by some element e of C. On the other hand, by claim 1, $tp(e/F)$ is non-principal. Thus using the Open Map Theorem we obtain that $tp(e/F')$ is non-principal, which contradicts that ψ' is complete. Hence ψ is atomless over F'. Next we show that any realization d of ψ is independent from A over F'. Take any formula $\theta(y) \in tp(d/A)$. Then $\psi(y) \wedge \theta(y)$ is consistent. Notice that $\psi^M \subset C$. So we can pick a realization d' of θ in C. Now $tp(d'/A)$ does not fork over F' since C and A are independent. Hence θ does not fork over F'. This shows that $tp(d/A)$ does not fork over F'.

Define $\Gamma(x, y) = \{x \text{ and } y \text{ are not interalgebraic}\} \cup \{x \neq c : c \in A\} \cup \{y \neq c : c \in A\}$. Γ is non-principal over F' because our model M ($\supset F'$) omits it. From claim 2 it follows that Γ and ψ satisfy the assumptions of the lemma. So we get the following claim (Note that F' is finite):

Claim 3. *There are pairwise non-isomorphic countable models M_τ ($\tau < 2^{\aleph_0}$) omitting Γ.*

Claim 4. *Each M_τ is a minimal model.*

Proof. Since M_τ omits Γ there is a minimal component D such that $M_\tau = D \cup A$. Suppose that M_τ is not minimal. Then there is a proper subset B of A such that $D \cup B$ is an elementary submodel of M_τ. So we can pick a minimal component $E \subset A - B$. By the minimality of M there is a formula $\psi(x, \bar{b})$ over $M - E$ such th.at ψ^M is contained in E. By the triviality of T E and $\bar{b}$ are independent. Therefore ψ does not fork over $\emptyset$, so $\psi^{M_\tau} \cap D \neq \emptyset$. On the other hand, by the minimality of M, there is a formula $\varphi(x, \bar{a})$ over A such that φ^M is contained in C. By the triviality C and $\bar{a}$ are independent. Therefore φ does not fork over $\emptyset$, so $\varphi^{M_\tau} \cap D \neq \emptyset$. Hence we can assume that $\varphi^{M_\tau} \cap \psi^{M_\tau} \neq \emptyset$ (because any two elements of D are interalgebraic). So we have $M \models (\exists x)(\varphi(x, \bar{a}) \wedge \psi(x, \bar{b}))$. This contradicts that C and E are disjoint. Hence M_τ is minimal.

By claim 3, 4 we obtain $2^{\aleph_0}$ minimal models. This completes the proof of the theorem. □

2.2.2. Remarks. (1)It is known that a theory of one unary function symbol f is stable and trivial (see e.g. [18]). Moreover a minimal model of such a theory has minimum Lascar rank. This can be shown as follows: Pick any element a of a minimal model of the theory. Let $tp(a/B)$ be any forking extension of $tp(a)$.

Then by Lemma 1 in [12], there is an element b of B which is contained in the *connected component* $C(a)$ of a, where $b \in C(a)$ if and only if there are $n, m < \omega$ such that $f^n(a) = f^m(b)$. On the other hand we see that each connected component in a minimal model is a minimal component in our language (see Lemma 3.1 in [12]). Therefore $C(a)$ is a minimal component, so a and b are interalgebraic. Thus $tp(a/B)$ is algebraic. Hence $U(a) \leq 1$.) So our theorem is a generalization of Marcus' one.

(2) The following theory T satisfies our assumption: For each $\nu < 2^{<\omega}$ we define a function $F_\nu : 2^\omega \to 2^\omega$ by $(F_\nu(\eta))(i) = \nu(i) + \eta(i)$ mod 2 for $\eta \in 2^\omega$, $i < \omega$. And for $\eta \in 2^{<\omega}$, $P_\eta = \{\tau \in 2^\omega : \eta \prec \tau\}$. Let $M = (2^\omega, \{F_\nu\}_{\nu < 2^{<\omega}}, \{P_\eta\}_{\eta < 2^{<\omega}})$ and $T = Th(M)$. Then it is easy to see that T is stable and trivial. Each model generated by only one element is minimal and non-prime. And the Lascar rank of the model is minimum.

(3) In [15] Shelah has shown that any κ with $1 \leq \kappa \leq \aleph_0$ there is a complete theory, with no prime model, and exactly κ minimal models . Theories he gave are stable, trivial and have a minimal non-prime model. But all minimal models of them have the Lascar rank 2. This shows that the condition (2) of our theorem is essential.

Chapter 3. ω-Categorical ω-Stable theories with DOP

In [16] Shelah classified the sufficiently saturated models of a superstable theory. In his proof he introduced a dividing line, *the Dimensional Order Property* (DOP), and showed that if a superstable theory has the DOP then it has 2^λ models for uncountable λ.

We say that a theory has the DOP if there exist tuples a, b, a finite set F and a non-algebraic type p such that $p \not\perp abF$, $p \perp aF$, $p \perp bF$ and $a \downarrow_F b$. Shelah's proof distinguishes three types of superstable theories with the DOP, i.e.,

Case 1. $tp(a/F)$ and $tp(b/F)$ are trivial; $tp(a/F) \perp tp(b/F)$;

Case 2. $tp(a/F)$ and $tp(b/F)$ are trivial; $tp(a/F) = tp(b/F)$;

Case 3. Either $tp(a/F)$ or $tp(b/F)$ is nontrivial.

Examples 1, 2 and 3 below have the DOP, and fall into cases 1, 2 and 3 respectively.

Example 1. Let S_1 be the theory which says that E_1 and E_2 are equivalence relations with infinitely many infinite classes such that the intersection of each E_1-class and E_2-class is infinite (see, e.g., [1]). Let c be any element. Take a and b such that $E_1(a,c) \wedge \neg E_2(a,c)$ and $E_2(b,c) \wedge \neg E_1(b,c)$. Then $tp(a/c)$ and $tp(b/c)$ are trivial and orthogonal. Let $p(x) = E_1(x,b) \wedge E_2(x,a)$. Then a, b, c and p witness that S_1 has the DOP. Hence S_1 falls into case 1.

Example 2. Let $L = \{V, E, R\}$, where V(vertex) and E(edge) are unary predicates and R is a ternary predicate. Let S_2 be the following theory in L: any model of S_2 is the union of the two disjoint infinite sets V and E; $R \subset V \times V \times E$; for each pair $\langle u, v\rangle \in V \times V$ and $e \in E$, $R(u,v,e)$ holds iff $R(v,u,e)$ holds; for each pair $\langle u, v\rangle \in V \times V$ satisfying $u \neq v$ there are infinitely many elements $e \in E$ such that $R(u,v,e)$ holds; for each $e \in E$ there is a unique pair $\langle u, v\rangle \in V \times V$ satisfying $u \neq v$ such that $R(u,v,e)$ holds. Take any distinct elements a, b of V. Clearly $tp(a) = tp(b)$ is trivial. Let $p(x) = R(a,b,x)$. Then a, b and p witness that S_2 has the DOP. Hence S_2 falls into case 2.

Example 3. Let $q = p^n$ (p is a prime number, $n < \omega$). Let S_3^q be the theory of the structure $(D \cup G, \pi, \mathcal{G})$ defined as follows: $\mathcal{G} = (G, cl)$ is a projective geometry of infinite dimension over a finite field F_q, where cl is a closure operator on G; $D = \omega \times G$; π is a function from D onto G such that $\pi : (m, g) \mapsto g$. Take any distinct elements a, b of G. Then we can get $c \in cl(a, b) - \{a, b\}$. Define the type $p(x)$ by $\pi(x) = c$. Then a, b and p witness that S_3^q has the DOP: Now $p \perp \emptyset$. Thus we have $p \perp a$ and $p \perp b$ since $c \downarrow a$ and $c \downarrow b$. On the other hand we have $p \not\perp ab$ since $c \in acl(ab)$. Hence S_3^q has the DOP. Clearly $\mathcal{G}$ is non-trivial. It follows that S_3^q falls into case 3.

Clearly these examples are ω-categorical and ω-stable. In this note we prove the following theorem.

Theorem C. *Let a theory T be ω-categorical and ω-stable. Then T has the DOP if and only if S_1, S_2 or S_3^q (for some $q < \omega$) can be fully interpreted in T^{eq}.*

3.1. Preliminaries.

We fix a countable stable theory T. We usually work in the big model $\mathbf{C}^{eq}$ of T^{eq}. Our notations are fairly standard. Types are complete types with parameters, and they are denoted by $p, q, \ldots$. The nonforking extension of a stationary type p to the domain A is denoted by $p|A$. The type of a over A is denoted by $tp(a/A)$. And the strong type of a over A is denoted by $stp(a/A)$. If p and q are stationary types over A then $p \otimes q$ denotes the type $tp(ab/A)$, where a realizes p and b realizes $q|Aa$. When a type p is orthogonal to the set A we write $p \perp A$. The canonical base of a strong type $stp(a/A)$ is denoted by $Cb(a/A)$. $RM(p)$ is a Morley rank of a type p. We simply write $RM(a/A)$ instead of $RM(tp(a/A))$. The set of realizations of a type p (resp. a formula φ) in a model M is denoted by p^M (resp. φ^M).

The following definition of the DOP is different from original one ([1], [7], [16]). But it is easy to see that they are equivalent (if T is superstable).

3.1.1. Definition. We say that T has *the Dimensional Order Property* (DOP for short), if there are tuples a, b, a finite set F and a non-algebraic type p such that $p \not\perp abF, p \perp aF, p \perp bF$ and $a \downarrow_F b$.

3.1.2. Definition. We say that a stationary type $p \in S(A)$ is *trivial* if it has the following property: for any set B containing A and any realizations a, b and c of $p|B$, if a, b and c are pairwise independent over B then they are independent over B.

The following definition is due to Baudisch [2].

3.1.3. Definition. Let a theory S be interpretable in T and φ a formula in T which defines the universe of an S-model. Then we say that S is *fully interpreted in T* if for any S-model M there is a T-model N such that $\varphi^N \cong M$.

In general the DOP is not always preserved under interpretations. But Baudisch showed that the situation changes if the interpretation is full, i.e.,

3.1.4. Fact ([2]). *Let T be superstable and assume that S is fully interpreted in T. If S has the DOP then T has the DOP.*

Finally we state some facts on ω-categorical ω-stable theories:

3.1.5. Fact ([4]). *A strongly minimal set is locally modular.*

3.1.6. Fact (The Coordinatization Theorem [4]). *For any element a and set A with $a \notin acl(A)$ there is $e \in acl(aA)$ with $RM(e/A) = 1$.*

3.1.6. Fact ([4]). *T is 1-based.*

3.1.7. Fact ([3]). *Suppose that p is a type with Morley rank 1 which is non-orthogonal to a set A. Then there is $r \in S(A)$ with Morley rank 1 such that $p \not\perp r$.*

3.2. Lemmas.

From now on T is ω-categorical and ω-stable.

3.2.1. Lemma. *Suppose that T has the DOP. Then there are tuples a, b, d and a finite set F which satisfy the following conditions:*
(i) $tp(d/abF) \perp aF, tp(d/abF) \perp bF, a \downarrow_F b$;
(ii) $tp(d/abF)$ has Morley rank 1;
(iii) $tp(a/F)$ and $tp(b/F)$ are strictly minimal;
(iv) $tp(a/F) = tp(b/F)$ or $tp(a/F) \perp tp(b/F)$.

Proof. First we prove the following claim:

Claim. *There are tuples a, b, a model M and a nonalgebraic type p satisfying the following:*
(a) $p \not\perp abM$, $p \perp aM$, $p \perp bM$ and $a \downarrow_M b$;
(b) p has Morley rank 1;

(c) $tp(a/M)$ and $tp(b/M)$ are strongly minimal;
(d) $tp(a/M) = tp(b/M)$ or $tp(a/M) \perp tp(b/M)$.

Proof. By the definition of the DOP, we get a, b, M, and p satisfying condition (a). In addition we can assume that p is regular. By fcat 3.1.5 there is a type p' with Morley rank 1 which is non-orthogonal to p. By the regularity of p, we have $p' \not\perp abM$. Replacing p by p', we may assume moreover that p satisfies (b). Now we show that there are a^* and M^* with $RM(a^*/M^*) = 1$ and $RM(b/M^*) = RM(b/M)$ such that a^*, b, M^* and p satisfy (a) and (b). This can be shown as follows. By fcat 3.1.5, there is $a' \in acl(aM)$ with $RM(a'/M) = 1$. If $p \perp a'bM$, then we finish (by renaming a' by a^* and M by M^*). So assume that $p \not\perp a'bM$. Take a prime model M' over Ma' such that $abC \downarrow_{Ma'} M'$, where C is the domain of p. Then we have $p \perp aM'$ and $p \perp bM'$. Thus a, b, M' and p satisfy (a) and (b). Moreover $RM(b/M) = RM(b/Ma') = RM(b/M')$ and $RM(a/M) > RM(a/Ma') = RM(a/M')$. So, by iterating this process we can get a^* and M^* as required.
Again, by using the similar argument above, we can assume that a, b, M, and p satisfy (a),(b), and (c). Finally we arrange that $tp(a/M) = tp(b/M)$ and $tp(a/M) \perp tp(b/M)$. Suppose that $tp(a/M)$ and $tp(b/M)$ are not orthogonal. By ω-stability they are not almost orthogonal. So we have a realization b' of $tp(b/M)$ such that a and b' are interalgebraic over M. Let a' be a realization of $tp(a/M)$ such that $tp(ab'/M) = tp(a'b/M)$. Replace a by ab' and b by $a'b$. Then they still satisfy (a), (b) and (c). Hence we assume that $tp(a/M) = tp(b/M)$. It follows that a, b, M and p satisfy (a), (b), (c), and (d).

Now we can assume that C(the domain of p) is finite. Thus we can replace the model M by some finite set F (by taking F satisfying $abC \downarrow_F M$). Then, by ω-categoricity and (a), we can assume that $tp(a/F)$ and $tp(b/F)$ are strictly minimal. Using (a), (b) and fact 5, we can get $q \in S(abF)$ with Morley rank 1 which is non-orthogonal to p. Let d be a realization of q. So the quadruple $\langle a, b, d, F \rangle$ satisfies conditions (i), (ii), (iii) and (iv). □

By a *DOP-quadruple* we mean a quadruple which satisfies conditions (i), (ii), (iii) and (iv) of lemma 3.2.1. By lemma 3.2.1 (iv), a DOP-quadruple $\langle a, b, d, F \rangle$ always satisfies one of the following conditions:

Case 1. $tp(a/F)$ and $tp(b/F)$ are trivial, $tp(a/F) \perp tp(b/F)$;

Case 2. $tp(a/F)$ and $tp(b/F)$ are trivial, $tp(a/F) = tp(b/F)$;

Case 3. $tp(a/F)$ or $tp(b/F)$ is nontrivial.

3.2.2. Lemma. *Let $\langle a, b, d, F \rangle$ be a DOP-quadruple. If case 1 or case 2 occurs, then*
(i) $acl(dF) \cap (tp(a/F)^{\mathbf{C}} \cup tp(b/F)^{\mathbf{C}}) = \{a, b\}$;
(ii) Any subset of $tp(a/F)^{\mathbf{C}} \cup tp(b/F)^{\mathbf{C}}$ which is pairwise independent over F is independent over F.

Proof. To simplify the notation we assume that $F = \emptyset$.
(i) First we show that $a, b \in acl(d)$. Suppose not. Without loss of generality, we assume that $a \notin acl(d)$, so $a \downarrow d$. Then we have $b \notin acl(d)$ (If not, we have $a \downarrow_b d$. Thus we get $tp(d/ab) \perp tp(d/ab)$ since $tp(d/ab) \perp b$. A contradiction). So we have $b \downarrow d$. Let $c = Cb(d/ab)$. Note that T is 1-based(see fact 3.1.6). Thus the set $\{a, b, c\}$ is pairwise independent, but each of a, b and c is algebraic over the other two. Let $b'c'$ be a realization of $tp(bc)$ such that $abc \downarrow b'c'$. Let a' and a'' be realizations of $tp(a)$ such that $tp(abc) = tp(a'b'c) = tp(a''bc')$. Then a, a' and a'' are realizations of $tp(a/b'c')$ which are pairwise independent but not independent. This contradicts that $tp(a)$ is trivial. It follows that $a, b \in acl(d)$. Next we show that $\{a, b\} = acl(d) \cap (tp(a)^{\mathbf{C}} \cup tp(b)^{\mathbf{C}})$. If not, there is $c \in acl(d) \cap (tp(a)^{\mathbf{C}} \cup tp(b)^{\mathbf{C}}) - \{a, b\}$. By strict minimality and triviality, $\{a, b, c\}$ is independent. So $c \downarrow ab$. Then we have $tp(d/ab) \perp tp(c/ab)$, which contradicts that $c \in acl(d)$. Hence (i) holds.
(ii) Clearly case 2 satisfies (ii). So we assume that case 1 occurs. Let X be a pairwise independent subset of $tp(a)^{\mathbf{C}} \cup tp(b)^{\mathbf{C}}$. Let X_a denote $X \cap tp(a)^{\mathbf{C}}$ and X_b denote $X \cap tp(b)^{\mathbf{C}}$. By the triviality and the strict minimality, both X_a and X_b are independent sets. Since $tp(a)$ and $tp(b)$ are orthogonal, X_a is independent from X_b. Hence $X (= X_a \cup X_b)$ is independent. □

3.2.3. Lemma. *Let $\langle a, b, d, F \rangle$ be a DOP-quadruple. If case 1 occurs, then*
(i) $a, b \in dcl(dF)$;
(ii) The copies of $tp(d/abF)$ under F-automorphisms are pairwise orthogonal.

Proof. For the simplicity of the notation, we assume that $F = \emptyset$. In this case we have $tp(a/d) \neq tp(b/d)$. Hence (i) follows from lemma 3.2.2 (i). To show (ii), it is enough to see that if $tp(d'/a'b')$ is a copy pf $tp(d/ab)$, then $tp(d/ab) \perp tp(d'/a'b')$. If $a \neq a'$ and $b \neq b'$, then $\{a, a', b, b'\}$ is pairwise independent. By lemma 3.2.2 (ii), we have $ab \downarrow a'b'$. Hence $tp(d/ab) \perp tp(d'/a'b')$, by lemma 3.2.1 (i). If $a \neq a'$ and $b = b'$, by lemma 3.2.2 (ii) again, we have $ab \downarrow_b a'b'$. Hence $tp(d/ab) \perp tp(d'/a'b')$. If $a = a'$ and $b \neq b'$, then we have $tp(d/ab) \perp tp(d'/a'b')$ as above. □

Let e be a tuple and A a set. Here we call the tuple e *strictly coordinatizable over A* if there are tuples $e_1, e_2, ..., e_n$ with $RM(e_i/A) = 1$ $(i = 1, 2, ..., n)$ such that e and $e_1, e_2, ..., e_n$ are interalgebraic over A.

3.2.4. Lemma. *Let $\langle a, b, d, F \rangle$ be a DOP-quadruple. If case 2 occurs then there is an element e which is strictly coordinatizable over abF such that*
(i) $tp(a/eF)$ is algebraic with multiplicity 2 and $tp(ab/eF) = tp(ba/eF)$;
(ii) The copies of $tp(e/abF)$ under F-automorphisms are pairwise orthogonal.

Proof. For the simplicity of the notation, we assume that $F = \emptyset$. Since a and b are independent, we have $tp(ab) = tp(ba)$. So there is an automorphism f such that $f(ab) = ba$. For $n < \omega$, define $D_n = \{f^i(d)\}_{i \leq n}$. Our proof separates into two cases:

Case A. D_n is independent over ab, for any $n < \omega$.

There is $n < \omega$ such that $stp(f^{n+1}(d)/ab) = stp(d/ab)$ since the multiplicity of $tp(d/ab)$ is finite. So we have $tp(f``D_n/ab) = tp(D_n/ab)$ since both d and $f^{n+1}(d)$ are independent from D_n over ab. Let e be an imaginary element for the finite set D_n. Clearly e is strictly coordinatizable over ab. By lemma 3.2.2 (i), we obtain moreover that $tp(ab/e) = tp(ba/e)$, and that $tp(a/e)$ is algebraic with multiplicity 2, by lemma 3.2.2 (i). Hence (i) holds. On the other hand, by lemma 3.2.1 (i), we have $tp(f^i(d)/ab) \perp a$ for any $i < \omega$. So we get $tp(e/ab) \perp a$. Similarly we have $tp(e/ab) \perp b$. So (ii) holds, by a similar argument as in the proof of lemma 3 (ii).

Case B. D_n is not independent over ab, for some $n < \omega$.

Take maximal $n < \omega$ such that D_n is independent over ab. Then, for every i, $f^i(d)$ is algebraic over D_n (Recall that $RM(d/ab) = 1$). So the size of $\bigcup_{i<\omega} D_i$ is finite, by ω-categoricity. Let e be an imaginary element for $\bigcup_{i<\omega} D_i$. Then e is strictly coordinatizable over ab. On the other hand, by the maximality of D_n, d and $f^{n+1}(d)$ are interalgebraic over $D_n - \{d\}$. So we have $f(eab) = eba$. Thus (i) holds, by using lemma 3.2.2 (i). By the similar argument of case A, (ii) also holds. □

3.2.5. Lemma. *Let $\langle a, b, d, F \rangle$ be a DOP-quadruple. If case 3 occurs then there are a tuple g and a finite set F' such that*
(i) $tp(g/F')$ is nontrivial strictly minimal modular, $RM(d/gF') = 1$ and $g \in dcl(dF')$;
(ii) The copies of $tp(d/gF')$ under F'-automorphisms are pairwise orthogonal.

Proof. Without loss of generality, we assume that $tp(a/F)$ is nontrivial. Then $tp(a/bF)$ is nontrivial and strongly minimal. In particular $tp(a/bF)$ is locally modular, by fact 3.1.5. By adding a suitable realization of $tp(a/bF)$ to the domain, we can assume that $tp(a/bF)$ is modular. Let $F' = Fb$. And let g be the set of realizations of $tp(a/F')$ which are algebraic over aF'. It is clear that $g \in dcl(aF')$ and $a \in acl(gF')$. Hence $tp(g/F')$ is nontrivial, strictly minimal and modular. Now we have $d \not\downarrow_{bF} a$, so $g \in acl(dF')$. We obtain therefore that $g \in dcl(dF')$ (Assume otherwise. Then we can pick g' such that $tp(g/dF') = tp(g'/dF')$ and $g \neq g'$. Recall that $tp(d/abF) \perp bF$, so $tp(d/gF') \perp F'$. Since $g \downarrow_{F'} g'$ we have $tp(d/gF') \perp tp(g'/gF')$. This contradicts that $g' \in acl(dF')$). Also it is clear that $RM(d/gF') = 1$. Hence (i) holds. Moreover (ii) holds, because $tp(d/gF') \perp F'$ and $tp(g/F')$ is strictly minimal. □

3.2.6. Lemma. *Let c be a tuple which is strictly coordinatizable over $\emptyset$. Let $p_c \in S(c)$ whose realization is strictly coordinatizable over c. Let I be an infinite*

Morley sequence of $tp(c)$. *Let* N_1 *be a prime model over* I. *For each* $c' \in tp(c)^{N_1}$ *let* $I_{c'}$ *be an infinite Morley sequence of some stationarization* $p^*_{c'}$ *of* $p_{c'}$ *over* N_1. *Let* N *be a prime model over* $N_1 \cup \bigcup_{c' \in tp(c)^{N_1}} I_{c'}$. *Suppose that the copies of* p_c *are pairwise orthogonal. Then*
(i) $tp(c)^{N_1} = tp(c)^N$;
(ii) If $p_{c'}$ *is a copy of* p_c *then* $|p^N_{c'}| = |I_{c'}|$, *for each* $c' \in N_1$.

Proof. (i) Assume otherwise. Then there is $a \in acl(tp(c)^N) - N_1$ with Morley rank 1 since c is strictly coordinatizable. In particular $a \downarrow N_1$. On the other hand we have $\otimes p^*_{c'} \perp \emptyset$ since the copies of p_c are pairwise orthogonal. So $\otimes p^*_{c'} \perp tp(a/N_1)$. Hence $a \notin N_1(\bigcup I_{c'}) = N$, which is a contradiction.
(ii) We show that $|p^N_{c'}| \leq |I_{c'}|$. Note that the multiplicity is finite. So it is enough to show that if q is a stationarization of $p_{c'}$ over $acl(c')$, then $|q^N| \leq |I_{c'}|$. Let d be a realization of q. Then there are $d_0, d_1, ..., d_n$ such that $acl(dc') = acl(d_0 d_1 ... d_n c')$ and $RM(d_i/acl(c')) = 1$ $(i = 0, 1, ..., n)$ since d is strictly coordinatizable over c'. Let $q_i = tp(d_i/acl(c'))$. Fix an arbitrary i.

Claim 1. $q_i^N = q_i^{N_1(I_{c'})}$.

Proof. Assume otherwise. Then we can pick $e \in q_i^N - N_1(I_{c'})$. Since $RM(q_i) = 1$, we have $e \not\downarrow_{N_1(I_{c'})} N$. So $e \not\downarrow_{N_1(I_{c'})} \bigcup_{c'' \in tp(c)^{N_1}} I_{c''}$ (because N is prime over $N_1 \cup \bigcup_{c'' \in tp(c)^{N_1}} I_{c''}$). Note that copies of p_c are pairwise orthogonal. Therefore there is $c''(\neq c')$ such that $e \not\downarrow_{N_1} I_{c''}$. Hence we have $q_i \not\perp p_{c''}$, so $p_{c'} \not\perp p_{c''}$. This is a contradiction.

Claim 2. $c' \in acl(I)$.

Proof. Assume otherwise. Then we can pick $a \in acl(c') - acl(I)$ with Morley rank 1 (because c is strictly coordinatizable). So $a \downarrow I$. Then $tp(a/I)$ is isolated over some finite $I_0 (\subset I)$ since N_1 is atomic over I. On the other hand we can pick $a' \in acl(I) - acl(I_0)$ such that $stp(a) = stp(a')$. Thus $a' \downarrow I_0$. So we have $tp(a/I_0) = tp(a'/I_0)$. But this contradicts that $tp(a/I)$ is isolated over I_0.

Let $N_0 (\subset N_1)$ be a prime model over $acl(c')$. Pick a maximal subset X of N_1 such that $X \downarrow N_0$.

Claim 3. *We can assume that* $N_1(I_{c'}) \subset N_0(X)(I_{c'})$.

Proof. First we obtain $I \subset N_0(X)$ (If not, then there is $a \in acl(I) - N_0(X)$ with Morley rank 1. Then we have $a \downarrow N_0 X$, so $aX \downarrow N_0$. This contradicts the maximality of X). Since $I_{c'}$ is a Morley sequence of $p^*_{c'}$, we have $I_{c'} \downarrow_{c'} N_1$. So, by claim 2, we have $I_{c'} \downarrow_{acl(I)} N_1$. On the other hand we have $I_{c'} \downarrow_{N_0} X$ since $p_{c'}$ is orthogonal to $\emptyset$. So, by claim 2 again, we have $I_{c'} \downarrow_{acl(I)} N_0(X)$ (because $I \subset N_0(X)$). Thus there is an automorphism f fixing $II_{c'}$ such that $N_1 \subset f``N_0(X)$ since N_1 is prime over I. Hence we can assume that $N_1(I_{c'}) \subset N_0(X)(I_{c'})$.

Claim 4. $q_i^{N_0(I_{c'})(X)} = q_i^{N_0(I_{c'})}$.

Proof. Recall that $X \downarrow_{N_0} I_{c'}$. So $tp(X/N_0(I_{c'}))$ does not fork over $\emptyset$. Thus $q_i \perp tp(X/N_0(I_{c'}))$ (since q_i is orthogonal to $\emptyset$). Hence the claim holds.

By claim 1, 3 and 4, we have $|q_i^{N_1(I_{c'})}| \leq |q_i^{N_0(X)(I_{c'})}| = |q_i^{N_0(I_{c'})(X)}| = |q_i^{N_0(I_{c'})}| \leq |I_{c'}|$. Hence we obtain $|q^N| \leq |q_0^N| \times |q_1^N| \times \cdots \times |q_n^N| \leq |I_{c'}|^n = |I_{c'}|$. This completes the proof of the lemma. □

3.3. Theorem and Proof.

3.3.1. Theorem C. *Let a theory T be ω-categorical and ω-stable. Then T has the DOP if and only if S_1, S_2 or S_3^q (for some $q < \omega$) can be fully interpreted in T_F^{eq} for some finite set F.*

Proof. ($\rightarrow$) Assume that T has the DOP. By lemma 3.2.1 there is a DOP-quadruple $\langle a, b, d, F\rangle$. The proof separates into three cases:

Case 1. *$tp(a/F)$ and $tp(b/F)$ are trivial; $tp(a/F) \perp tp(b/F)$.*

For the simplicity of the notation we assume that $F = \emptyset$. By lemma 3.2.3 (i), there are $\emptyset$-definable functions f, g such that $f(d) = a$ and $g(d) = b$. Let $\varphi(x) = tp(d)$. We define $E_1(x, y)$ by "$f(x) = f(y)$" $\wedge\, \varphi(x) \wedge \varphi(y)$ and $E_2(x, y)$ by "$g(x) = g(y)$" $\wedge\, \varphi(x) \wedge \varphi(y)$. We prove that for any S_1-model M there is a T^{eq}-model N such that $M \cong (\varphi^N, E_1^N, E_2^N)$.
Take an arbitrary S_1-model M. Let $d_i/E_1 (i < \kappa_1)$ be all the E_1-classes appearing in M and $e_i/E_2 (i < \kappa_2)$ all the E_2-classes in M. Let $\lambda_{ij} = |E_1(x, d_i)^M \cap E_2(x, e_j)^M|$ for each $i < \kappa_1$, $j < \kappa_2$. Note that λ_{ij}'s are infinite. Now we construct a T^{eq}-model N satisfying our requirement. Let I_1 be a Morley sequence of length κ_1 of $tp(a)$ and I_2 a Morley sequence of length κ_2 of $tp(b)$. Let $I = I_1 \times I_2$. Then I is a Morley sequence of $tp(ab)$. Let N_1 be prime over I. Then $|tp(a)^{N_1}| = \kappa_1$ and $|tp(b)^{N_1}| = \kappa_2$ (since $tp(a) \perp tp(b)$). Let $\{a_i\}_{i<\kappa_1}$ be an enumeration of $tp(a)^{N_1}$ and $\{b_j\}_{j<\kappa_2}$ an enumeration of $tp(b)^{N_1}$. Then $\{a_i b_j\}_{i<\kappa_1, j<\kappa_2}$ is also an enumeration of $tp(ab)^{N_1}$. Let $p_{ab} = tp(d/ab)$. Let p_{ij} be a copy of p_{ab} over $a_i b_j$. Let I_{ij} be a Morley sequence of length λ_{ij} of some stationarization of p_{ij} over N_1. Let N be a prime model over $N_1 \cup \bigcup_{i<\kappa_1, j<\kappa_2} I_{ij}$. Remember that $tp(a)$, $tp(b)$ and p_{ab} have Morley rank 1. Thus $tp(ab)$ and p_{ab} are strictly coordinatizable. Moreover p_{ab} satisfies the assumption of lemma 6 (by lemma 3.2.3 (ii)). So, by lemma 6, $\{a_i\}_{i<\kappa_1}$ and $\{b_j\}_{j<\kappa_2}$ are enumerations of $tp(a)^N$ and $tp(b)^N$ respectively such that $|p_{ij}^N| = \lambda_{ij}$ for each $i < \kappa_1, j < \kappa_2$. Hence $(\varphi^N, E_1^N, E_2^N) \cong M$.

Case 2. *$tp(a/F)$ and $tp(b/F)$ are trivial; $tp(a/F) = tp(b/F)$.*

To simplify the notation we assume $F = \emptyset$. We can choose such e as in lemma 3.2.4. Define $V(x) = tp(a)$, $E(x) = tp(e)$and $\varphi(x) = V(x) \vee E(x)$. And define a relation $R(x, y, z)$ by $tp(abe)$. We prove that for any S_2-model M there is a T^{eq}-model N such that $M \cong (\varphi^N, V^N, E^N, R^N)$.
Take an arbitrary S_2-model M. Let $\{v_i\}_{i<\kappa}$ be an enumeration of V^M. Let $\lambda_{ij} = |R(v_i, v_j, z)^M|$ for each $i, j < \kappa$, $i \neq j$. Now we construct a T^{eq}-model N satisfying our requirement. Let I be a Morley sequence of length κ of $tp(ab)$. Let N_1 be prime over I. Let $\{a_i\}_{i<\kappa}$ be an enumeration of $tp(a)^{N_1}$. Then $\{a_i a_j\}_{i,j<\kappa, i\neq j}$ is an enumeration of $tp(ab)^{N_1}$. Let $p_{ab} = tp(d/ab)$. Let p_{ij} be a copy of p_{ab} over $a_i b_j$. Note that $p_{ij} = p_{ji}$. Let I_{ij} be a Morley sequence of length λ_{ij} of some stationarization of p_{ij} over N_1. Let N be a prime model over $N_1 \cup \bigcup_{i,j<\kappa, i\neq j} I_{ij}$. From lemma 3.2.4 it follows that $tp(ab)$ and p_{ab} are strictly coordinatizable, and that the copies of p_{ab} are pairwise orthogonal. Thus the assumption of lemma 6 is satisfied. So $\{a_i\}_{i<\kappa}$is an enumeration of V^N such that $|R(a_i, a_j, z)^N| = \lambda_{ij}$ for each $i, j < \kappa, i \neq j$. Hence $(\varphi^N, V^N, E^N, R^N) \cong M$.

Case 3. *$tp(a/F)$ or $tp(b/F)$ is nontrivial.*

Take d, g and F' as in lemma 3.2.5. To simplify the notation we assume that $F' = \emptyset$. Define $D(x) = tp(d)$, $G(x) = tp(g)$ and $\varphi(x) = D \cup G$. Since $g \in dcl(d)$ there is a definable function $\pi : D^{\mathbf{C}^{eq}} \to G^{\mathbf{C}^{eq}}$ such that $\pi(d) = g$. $G^{\mathbf{C}^{eq}}$ is a nontrivial modular strictly minimal set. So $G^{\mathbf{C}^{eq}}$ associates the projective geometry $\mathcal{G}$ of infinite dimension over a finite field F_q. Then we can prove that S_3^q is fully interpreted in $T_{F'}^{eq}$, by the similar argument as in the proof of case 1 and 2.

($\leftarrow$) Suppose that S_1, S_2 or S_3^q is fully interpreted in T_F^{eq}. Each of S_1, S_2 and S_3^q has the DOP (see introduction). So, by fact 3.1.4 we obtain that T_F^{eq} has the DOP. Hence T has the DOP. $\square$

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