# Geometry of Total Curvature and Tits Metric of Noncompact Riemannian Manifolds

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# Introduction

It is a well-known fact due to the celebrated theorem of Gauss-Bonnet that the total curvature of a compact Riemannian 2-manifold is actually a topological invariant. That is, on a compact surface the integration of the Gaussian curvature, which is primarily an invariant in Riemannian geometry, is completely determined by the Euler characteristic of the surface, which is an invariant in topology.

Contrary to this, for a complete noncompact Riemannian 2-manifold, the total curvature is no longer a topological invariant and reflects considerably the metric structure of the surface. In fact, it gives rise to a quantity measuring the sum of the expanding growth rate of each end of the surface.

In higher dimensional cases, we can define the so-called Tits metric for a Hadamard manifold, i.e. a complete connected simply connected Riemannian n-manifold of nonpositive sectional curvature, which gives rise to a quantity of the same kind.

In this thesis, we shall study the geometry of total curvature and Tits metric of complete noncompact Riemannian manifolds. Our object is to give some characterizations of these manifolds from a point of view of the geometry of total curvature and Tits metric.

Let $M$ be a connected, complete, noncompact and oriented Riemannian 2-

manifold. $M$ is said to be *finitely connected* if it is homeomorphic to a compact 2-manifold with finitely many points removed, and *infinitely connected* if otherwise. The total curvature $C(M)$ of $M$ is defined to be the improper integral

$$C(M) = \int_M G d_M$$

of the Gaussian curvature $G$ of $M$ over $M$, where $d_M$ denotes the volume element of $M$. It has been investigated by many authors to what extent the metric structure of $M$ has an influence on the total curvature of $M$. Amongst them, a pioneering work due to Cohn-Vossen [Co1] proved in 1935 that if a finitely connected $M$ admits total curvature $C(M)$, then it is dominated by $2\pi$ times the Euler characteristic $\chi(M)$ of $M$, that is $C(M) \leqq 2\pi\chi(M)$. On the other hand, when $M$ is infinitely connected and admits total curvature, we know by Huber's theorem [Hu] in 1957 that $C(M) = -\infty$.

Under some restrictions on the global geometry of $M$, there have been obtained several estimates for $C(M)$. For instance, Cohn-Vossen [Co2] also proved in 1936 that if a Riemannian plane $M$ admits total curvature and if there exists a straight line on $M$, then $C(M) \leqq 0$. Here by a Riemannian plane we mean a complete Riemannian manifold homeomorphic to $\mathbf{R}^2$, and by a straight line a distance preserving maximal geodesic. It is known that this is generalized to the case that $M$ is a connected, complete, noncompact, oriented and finitely connected Riemannian 2-manifold with only one end. In fact, if such an $M$ admits total curvature and contains a straight line, then $C(M) \leqq 2\pi(\chi(M) - 1)$.

It is then natural to ask the converse of this theorem. In Section 1 of Chapter 3, we shall study the existence of a straight line from this point of view and prove the following

THEOREM A. *Let $M$ be a connected, complete, non-compact, oriented and finitely connected Riemannian 2-manifold having one end. If the total curvature of $M$ is smaller than $2\pi(\chi(M) - 1)$, then $M$ contains a straight line.*

In the case where $C(M) = 2\pi(\chi(M) - 1)$, Theorem A does not remain true. In fact, we can construct a $C^2$-plane $M$ whose total curvature is equal to 0 and on which there are no straight lines. ( See §1, 1.2 of Chapter 3. ) Note that if $M$ is the standard Euclidean plane, then $C(M) = 0$ and $M$ contains a straight line. Furthermore it should be noted that if $M$ has more than one end, then it is easy to see that there is a straight line in $M$ which combines two distinct ends.

To prove Theorem A, we need the estimate for the measure of rays via total curvature. Here a ray is a distance preserving geodesic defined on $[0, \infty)$. This estimate is originally due to Maeda [Ma2], [Ma3] who studied in 1984 the measure of rays on a noncompact Riemannian 2-manifold of nonnegative curvature. Subsequently, through his idea, the geometric significance of total curvature has been clarified in a more precise form. There are several results for the measure of rays in more general situation ( cf. [Og], [Sg2] and [Sy2] ) as well as that for the isoperimetric problem ( cf. [Sh3], [Sh4], [Sh5] and [SST] ).

As other investigations to give a geometric significance of the total curvature, Shiohama [Sh1], [Sh2] proved the conditions for a Busemann function to be an exhaustion or to be a nonexhaustion, and Shioya [Sy5] then studied the self-intersection number of a maximal geodesic.

From a point of view of analytic methods, there are also many results for the estimate of total curvature. For example, we know the inequality of Osserman [Os] for a complete minimal surface in the Euclidean 3-space and the result of White [Wh] for a complete surface whose second fundamental form has finite $L^2$ norm.

In 1973, Eberlein and O'Neill [EO] introduced the concept of the ideal boundary $M(\infty)$ for a Hadamard manifold $M$, which marked a milestone in the study of the geometry of arbitrary dimensional noncompact manifolds. They define the *ideal boundary* $M(\infty)$ of a given Hadamard manifold $M$ by the set of *points at infinity* of $M$, which are defined to be the equivalence classes of the geodesics under the asymptotic relation, originally due to Busemann [Bu]. In general the asymptotic relation is not an equivalence relation. However, if the sectional curvature of $M$ is nonpositive, then it is not hard to see that the asymptotic relation becomes an equivalence relation and we can define the ideal boundary.

Note that since $M$ is simply connected and nonpositively curved, if $p$ is a point in $M$ and $z$ is a point in $M(\infty)$, then there is a unique ray $\gamma$ such that $\gamma(0) = p, \gamma \in z$. Hence for an arbitrary fixed point $p \in M$, there is defined a natural bijection $\psi : B_p(M) \to \overline{M} = M \cup M(\infty)$, where $B_p(M)$ denotes a closed disk on the tangent space of $M$ at $p$. Eberlein and O'Neill induced the so-called *cone topology* on $\overline{M}$ via this bijection $\psi$ so that $\overline{M}$ with the cone topology gives rise to a compactification of $M$. This fact played a crucial role in their study. For instance,this enabled them to classify visibility manifolds into three types: parabolic, axial and fuchsian.

Subsequently, in 1985 Gromov [BGS] defined the Tits metric on the ideal boundary $M(\infty)$ of a Hadamard manifold $M$ and obtained further information on $M(\infty)$. The Tits metric on $M(\infty)$ is closely related to the flatness or more precisely the asymptotic flatness of a Hadamard manifold. To illustrate this, we only remark here typical examples. For the explicit definition, see Section 2 of Chapter 2. If $M$ is Euclidean, then $(M(\infty), \text{Td})$ is isometric to a standard sphere and if $M$ is a hyperbolic space, that is, a complete connected and simply connected Riemannian manifold of constant negative sectional curvature, then $\text{Td}\,(z_1, z_2) = \infty$ for any two distinct points $z_1, z_2 \in M(\infty)$.

In 2-dimensional Hadamard manifolds, the total curvature and the Tits metric both measure the expanding growth rates at infinity. In fact, studying a relation between the total curvature and the Tits metric on a 2-dimensional Hadamard manifold, we obtain the following

THEOREM 1. *Let $M$ be a 2-dimensional Hadamard manifold and $\alpha$ a diameter of $M(\infty)$ with respect to the Tits metric. Then the total curvature $C(M)$ of $M$ equals to $2(\pi - \alpha)$. In particular, if $\alpha = \infty$ then $C(M) = -\infty$.*

This theorem will be proved in Section 3 of Chapter 2. We will also prove the following theorem concerning relationship among the total curvature, the Tits topology induced from the Tits metric, and the sphere topology which is the restriction of the cone topology on the ideal boundary.

THEOREM 2. *Let $M$ be a Hadamard manifold. Then the following three conditions are equivalent :*

(1) *$M(\infty)$ is compact in the Tits topology.*

(2) *The Tits topology is equivalent to the sphere topology on $M(\infty)$.*

(3) *For given $x \in M$ and $\varepsilon > 0$, there exists a positive number $\delta(x, \varepsilon)$ such that the total curvature $C(F)$ of $F = F(\gamma_u, \gamma_v)$ satisfies that*

$$C(F) > -\varepsilon \qquad \text{for every } u, v \in S_xM \text{ with } \angle(u, v) < \delta,$$

*where $F(\gamma_u, \gamma_v)$ denotes a component consisting of minimizing geodesic segments joining $\gamma_u(t)$ to $\gamma_v(t)$ for all $t \geqq 0$ and $S_xM$ is the unit tangent sphere at $x$.*

The ideal boundary can be also defined for other classes of Riemannian manifolds and their geometry has been investigated. For instance, according to Gromov's suggestion, Kasue [Ks] constructed a metric space $M(\infty)$ for a Riemannian manifold $M$ with asymptotically nonnegative curvature and defined a counterpart of the Tits metric. For a Riemann surface, Shioya [Sy1],[Sy3],[Sy4] defined an equivalence relation on rays on the surface and, using the total curvature of the domain bounded by two rays, defined a distance on the equivalence classes of rays.

For arbitrary dimensional Hadamard manifolds we shall also study the relation between the global geometry of them and the information related to the Tits metric on their ideal boundaries.

It is an interesting problem to study to what extent the structure of $(M(\infty), \mathrm{Td})$ determines the structure of $M$. In fact, for given two Hadamard manifolds $M$ and $M^*$, even if $(M(\infty), \mathrm{Td})$ is isometric to $(M^*(\infty), \mathrm{Td})$, $M$ is not necessarily isometric to $M^*$. However, it is known that if $M$ is a symmetric space of rank $\geqq 2$ and if the isometry $g : (M(\infty), \mathrm{Td}) \to (M^*(\infty), \mathrm{Td})$ is a homeomorphism in the sphere topology, then $M$ is isometric to $M^*$ up to a normalizing constant ( Appendix 4 of [BGS] ). In particular, for symmetric spaces of rank $\geqq 2$, the Tits metric is closely related to their Tits building and has been utilized by many authors to characterize symmetric spaces (e.g., [BGS], [BS2], [EH] and [Th]).

As one of other rigidity properties, we shall prove the following theorem in Section 2 of Chapter 3.

THEOREM B. *Let $M$ be a nontrivial product Hadamard manifold, i.e., $M = M_1 \times M_2$ and $M^*$ a Hadamard manifold with $\dim M = \dim M^*$ such that there exists a continuous, bijective and projective map $\Phi : M \to M^*$. Then the map $\Phi : M \to M^*$ is an isometry up to a normalizing constant if and only if $\widetilde{\Phi} : (M(\infty), \mathrm{Td}) \to (M^*(\infty), \mathrm{Td})$ induced by $\Phi$ is an isometry.*

Here a projective map is meant a geodesic preserving map.

On the other hand, by the structure of $(M(\infty), \mathrm{Td})$ we can also characterize the structure of $M$ itself. We shall here remark that, as a result from this point of view, we obtained in [AO] that the Euclidean factor of $M$ is characterized by certain class of points of $M$.

As another characterization, recently, Kubo [Ku] proved that given two connected complete oriented and noncompact Riemannian 2-manifolds with finite total curvature, if there is a Hausdorff approximation between them, then their ideal boundaries are isometric. ( For a definition of a Hausdorff approximation, see Section 3 in Chapter 3. ) This means that if ideal boundaries are not isometric, then there is no Hausdorff approximation between their underlying open surfaces.

The same rigidity property on ideal boundaries for Hadamard manifolds is valid and we prove the following theorem.

THEOREM C. *Let $M$ and $N$ be Hadamard manifolds with ideal boundaries $M(\infty)$ and $N(\infty)$ respectively, which are assumed to be compact with respect to the Tits-topology. If there exists a Hausdorff approximation from $M$ to $N$, then $(M(\infty), \mathrm{Td})$ is isometric to $(N(\infty), \mathrm{Td})$.*

In this thesis we study on Hadamard manifolds, but this property is valid also for manifolds of asymptotically nonnegative curvature and is able to prove in a similar fashion. So we refer to this case. Namely the following theorem holds:

THEOREM D. *Let $M$ and $N$ be manifolds of asymptotically nonnegative curvature with ideal boundaries $M(\infty)$ and $N(\infty)$ respectively. If there exists a Hausdorff approximation from $M$ to $N$, then $(M(\infty), \mathrm{Td}\ )$ is isometric to $(N(\infty), \mathrm{Td}\ )$.*

Finally we summarize the content of this thesis. This thesis is organized as follows.

Chapter 1 is devoted to fundamental definitions and properties of Riemannian manifolds. In Chapter 2, we first recall the definitions and some fundamental properties of total curvature of a noncompact Riemannian 2-manifold and Tits metric on a Hadamard manifold in Sections 1 and 2. In Section 3 of this chapter, we shall investigate some relations between total curvature and Tits metric and prove Theorems 1 and 2 stated above. Chapter 3 is the main content of this thesis, where the three characterization theorems on Riemann surfaces and Hadamard manifolds from a point of view of total curvature and Tits metric will be proved. In Section 1, we study the existence of a straight line and prove Theorem A. Section 2 is concerned with the rigidity of products and Theorem B is proved there. Last in Section 3, we study the rigidity of ideal boundaries in a term of a Hausdorff approximation and prove Theorem C. Then continuously, we refer to Theorem D.

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# Chapter 1. Preliminaries

## §1. Fundamental properties of Riemannian manifolds

We shall start with reviewing relevant fundamental definitions and properties of Riemannian manifolds. For background materials we refer mainly to [BGS].

Throughout this thesis, unless otherwise stated, Riemannian manifolds are always assumed to be connected and complete, and geodesics are understood to be parametrized by arc length.

A geodesic $\gamma$ of $M$ is said to be a *maximal geodesic*, a *geodesic ray* or a *geodesic segment* according as its domain is $\mathbf{R}$, $[0, \infty)$ or a compact interval, respectively. If $v$ is a unit vector, then $\gamma_v$ denotes a unique geodesic such that $\gamma_v'(0) = v$. A geodesic segment $\gamma : I \to M$ is said to be *minimizing* if for any $s, t \in I$, it holds that

$$(*) \qquad d(\gamma(s), \gamma(t)) = |s - t|.$$

We often call a geodesic ray simply a *ray*. Also a maximal geodesic is called a *straight line* if $(*)$ holds for all $s, t \in \mathbf{R}$.

The *exponential map* $\exp_p : T_pM \to M$ of $M$ at $p$ is defined by $\exp_p(v) = \gamma_v(1)$ for all $v \in T_pM$, where $T_pM$ is the tangent space of $M$ at $p$. In general, $\exp_p$ is defined on a neighborhood of the origin in $T_pM$. However, it follows from the following Hopf-Rinow theorem that $\exp_p$ is defined on the entire tangent space provided $M$ is complete.

THE HOPF-RINOW THEOREM. *The following statements are equivalent:*

(a) *$M$ is a complete metric space, where the distance from $p$ to $q$ in $M$, denoted by $d(p,q)$, is defined to be the infimum of the length of all piecewise smooth curves from $p$ to $q$.*

(b) *For some $p \in M$, $\exp_p$ is defined on the whole $T_pM$.*

(c) *For all $p \in M$, $\exp_p$ is defined on the whole $T_pM$.*

*Furthermore, each of these conditions implies that any two points $p, q$ of $M$ can be joined by a geodesic whose length is the distance from $p$ to $q$.*

The last statement is most important in the study of complete Riemannian manifolds. In particular, if $M$ has nonpositive curvature, then the exponential map $\exp_p$ is nonsingular everywhere and hence we obtain the following

THE HADAMARD-CARTAN THEOREM. *Let $M$ be an $n$-dimensional complete Riemannian manifold of nonpositive curvature. Then for any $p \in M$, $\exp_p : T_pM \to M$ is a covering map. Hence the universal covering space of $M$ is diffeomorphic to $\mathbf{R}^n$.*

A complete, connected and simply connected Riemannian manifold of nonpositive curvature is called a *Hadamard manifold.* Note that by the Hadamard-Cartan theorem, a Hadamard manifold is diffeomorphic to $\mathbf{R}^n$. Other simple examples of Riemannian manifolds are provided by the spaces of constant sectional curvature. We call a connected, simply connected and complete Riemannian manifold of constant sectional curvature a *standard sphere*, a *hyperbolic space* or a *Euclidean space* according as the sectional curvature is positive, negative or 0. In this thesis, we shall only deal with the spaces of constant sectional curvature $1, -1$ or $0$, which we denote by $S^n, H^n$ or $\mathbf{R}^n$, respectively.

Now, we shall review the "convexity" which is an important tool in the study of manifolds of nonpositive curvature. We recall here some elementary facts concerning the convexity of sets and functions.

A subset $W$ of a Riemannian manifold $M$ is said to be *convex* if for any $p, q \in W$ there is, up to parametrization, a unique minimizing geodesic segment of $M$ contained in $W$, which joins $p$ to $q$. Let $W \subset M$ be a convex set. For a subset $A \subset W$ we define the *convex hull* of $A$ as the smallest convex subset of $W$ which contains $A$.

A function $g : \mathbf{R} \to \mathbf{R}$ is said to be *convex* if we have the inequality

$$g(a + s(b - a)) \leqq g(a) + s(g(b) - g(a))$$

for $a < b$ and $s \in (0, 1)$. If the inequality is strict, then $g$ is said to be *strictly convex*.

A function $f$ on a Riemannian manifold $M$ is said to be (*strictly*) *convex* if for every nontrivial geodesic segment $c$ the function $f \circ c$ is (strictly) convex. Note that this definition is independent of the parametrization of geodesics. If $f$ is a convex function on a convex set $W$, then for any $a \in \mathbf{R}$ the sublevel set

$\{p \in W | \, f(p) \leqq a\}$ is convex.

Let $f$ be a differentiable convex function on a convex set $W$. If there exist critical points of $f$ in the interior $\mathrm{Int}(W)$ of $W$, then they attain the absolute minimum of $f$. In fact, let $p \in \mathrm{Int}(W)$ be a critical point of $f$ with $f(p) = a$. Suppose that there is a point $q \in W$ with $f(q) < a$ and let $c : [0,1] \to W$ be the geodesic segment ( which is not necessarily unit speed ) from $q$ to $p$. By the convexity we have $f(c(t)) < a$ for $t \in [0,1)$ and $(f \circ c)'(1) = <\mathrm{grad} f(p), c'(1)> > 0$, contradicting to $\mathrm{grad} f(p) = 0$. An analogous argument shows that for any convex function $f$ on $W$, if $p$ is a local extremum of $f$ in the interior of $W$, then $p$ is an absolute minimum.

The convexity of the distance function on a Riemannian manifold of nonpositive curvature will be used frequently in the later discussion.

LEMMA 1.1.1 ( Theorem 1.3 in [BGS] ). *Let $M$ be a Riemannian manifold of nonpositive curvature. Then the distance function $d : W \times W \to \mathbf{R}$ is convex for every convex subset $W \subset M$.*

PROOF. It suffices to prove that for any geodesic segments $c_i : [0,1] \to W (i = 1,2)$ the function $t \to d(c_1(t), c_2(t))$ is convex. Let $\sigma_t$ be a unique geodesic segment from $c_1(t)$ to $c_2(t)$ and $L(t) := \mathrm{length}(\sigma_t) = d(c_1(t), c_2(t))$. If $c_1(t) \neq c_2(t)$ for all $t \in [0,1]$, then $L$ is differentiable and it follows from nonpositive curvedness and the second variation formula that $L''(t) \geqq 0$. If $c_1(t_0) = c_2(t_0)$, then $L(t_0) = 0$ is an absolute minimum, and $L$ is also convex in this case. ∎

Let $W$ be a convex subset in a Riemannian manifold $M$ of nonpositive curvature and $W_0 \subset W$ a closed convex subset. Lemma 1.1.1 implies that for each $p \in W$ there is a unique point $\tilde{p}$ in $W_0$ of the minimal distance to $p$. We define the *projection* $\pi_{W_0} : W \to W_0$ by $\pi_{W_0}(p) = \tilde{p}$. Then we get the following

LEMMA 1.1.2 ( cf. Section 1.6 in [BGS] ). *The distance function* $d(W_0, \cdot)$ *is convex on* $W$.

REMARK. The term cf. in the brackets means that the following statement can be seen there without proof.

PROOF. For any geodesic segment $c : [0,1] \to W$, let $p_1 = c(0)$, $p_2 = c(1)$, $q_1 = \pi_{W_0}(p_1)$, $q_2 = \pi_{W_0}(p_2)$. If $c_0 : [0,1] \to W$ is the geodesic segment from $q_1$ to $q_2$, then

$$\begin{aligned} d(c(t), W_0) &\leqq d(c(t), c_0(t)) \\ &\leqq (1-t)d(c(0), c_0(0)) + t \cdot d(c(1), c_0(1)) \\ &= (1-t)d(c(0), W_0) + t \cdot d(c(1), W_0), \end{aligned}$$

which means that $d(W_0, \cdot)$ is convex on $W$. ∎

We continue to prepare some definitions and properties concerning noncompact Riemannian manifolds.

DEFINITION 1.1.1. An *end* of a Hausdorff space $X$ is a function $\varepsilon$ that assigns to each compact subset $K$ of $X$ a connected component $\varepsilon(K)$ of $X \backslash K$

subject to the requirement that $\varepsilon(K) \supseteqq \varepsilon(L)$ whenever $K \subseteqq L$.

A subset $N \subseteqq X$ is a *neighborhood* of an end $\varepsilon$ if $N \supseteqq \varepsilon(K)$ for some compact set $K$ of $X$. A curve $c : [0, \infty) \to X$ is *divergent* if $c$ ultimately leaves any compact subset of $X$. For each divergent curve $c$, we say that $c$ *converges* to an end $\varepsilon$ if for any compact set $K$ there is a constant $t(K)$ such that $c(t) \in \varepsilon(K)$ for any $t \geqq t(K)$.

Let $M$ be a connected, complete and noncompact Riemannian manifold. By the completeness of $M$, for any end $\varepsilon$ of $M$ there is a ray emanating from any point $p \in M$ and converging to $\varepsilon$.

DEFINITION 1.1.2. A noncompact surface $M$ is said to be *finitely connected* if $M$ is homeomorphic to a compact 2-manifold with finitely many points removed. $M$ is said to be *infinitely connected* if otherwise.

When a noncompact surface $M$ is finitely connected, the ends of $M$ correspond precisely to the points removed.

The Busemann function for a geodesic $c$, which is very useful to the study of noncompact manifolds, is defined as follows.

Let $c$ be a ray on $M$. For $x \in M$ we consider the function $t \to d(x, c(t)) - t$. This function is bounded from below and monotone decreasing. In fact, by the triangle inequality we have

$$t \leqq d(x, c(0)) + d(x, c(t)) \qquad \text{for } t > 0$$

and

$$d(x, c(t)) \leqq d(x, c(s)) + t - s \qquad \text{for } s < t.$$

Hence the function is bounded from below by $-d(x, c(0))$ and monotone decreasing. Thus the function

$$f_c(x) = \lim_{t \to \infty} (d(x, c(t)) - t)$$

is well defined on $M$.

DEFINITION 1.1.3. The function $f_c$ is called the *Busemann function* for a ray $c$.

A Busemann function $f_c$ is Lipschitz continuous with Lipschitz constant 1, that is, $|f_c(p) - f_c(q)| \leqq d(p, q)$. In general, Busemann functions are not necessarily differentiable, but it is known by Eberlein that Busemann functions are $C^2$ on a Hadamard manifold. ( For the proof, see Prop. 3.1 in [HI] )

## §2. Toponogov's theorem and local rigidity

We recall here the Toponogov's comparison theorem in the case of nonpositive curvature and, applying the rigidity part of this theorem, we shall prove some rigidity results for later use.

THE TOPONOGOV'S THEOREM. *Let $M$ be a Riemannian manifold of nonpositive curvature and $W$ a convex subset of $M$. Let $c_i (i = 1, 2, 3)$ be geodesic segments forming a triangle in $W$ and $\widetilde{c}_i$ a comparison triangle in $\mathbf{R}^2$ with length $L(\widetilde{c}_i) = L(c_i)$ and $\alpha_i, \widetilde{\alpha}_i$ corresponding angles. Then $\alpha_i \leqq \widetilde{\alpha}_i$ and the equality for one $i \in \{1, 2, 3\}$ implies that the $c_i$ span a totally geodesic triangle isometric to the Euclidean one. In particular, $\alpha_1 + \alpha_2 + \alpha_3 \leqq \pi$ and the equality holds if and only if the triangle is Euclidean.*

For a triangle spanned by $c_i$ with $a_i = L(c_i)$ , it follows from this theorem together with the lows of cosine in the Euclidean plane that the following two inequalities hold :

(1) first low of cosine

$$a_3^2 \geqq a_1^2 + a_2^2 - 2a_1 a_2 \cos \alpha_3,$$

(2) second low of cosine

$$a_3 \leqq a_2 \cos \alpha_1 + a_1 \cos \alpha_2.$$

In this section, applying the rigidity part of the Toponogov's comparison theorem, we prepare some results concerning local rigidity.

LEMMA 1.2.1 ( cf. Section 1.4 in [BGS] ). *If there are geodesic segments* $c_i : [0, l_i] \to M (i = 1, 2, 3, 4)$ *with* $c_i(l_i) = c_{i+1}(0)$ *which determine a quadrilateral with angles* $\alpha_i$ *at* $c_i(0)$, *then* $\sum_{i=1}^{4} \alpha_i \leqq 2\pi$ *and the equality holds if and only if the* $c_i$ *span a totally geodesic flat Euclidean quadrilateral.*

PROOF. Let $\gamma$ be a geodesic segment from $c_1(0)$ to $c_3(0)$. Then we get two triangles $T_1$ and $T_2$ where $T_1$ is spanned by $c_1, c_2$ and $\gamma$ and $T_2$ is spanned by $c_3, c_4$ and $\gamma$. Applying the rigidity part of the Toponogov's comparison theorem to $T_1$ and $T_2$, we see the sum of three angles for $T_1$ ( resp. $T_2$ ) is less than or equal to $\pi$, and hence the sum of the angles of quadrilateral is less than or equal to $2\pi$. If the equality holds, then $T_1$ and $T_2$ are Euclidean. Since $\angle(c_1'(0), -c_4'(l_4)) = \angle(c_1'(0), \gamma'(0)) + \angle(\gamma'(0), -c_4'(l_4))$, $T_1$ and $T_2$ form the Euclidean quadrilateral. ∎

LEMMA 1.2.2 ( cf. Section 1.5 in [BGS] ). *Let* $c_1 : [0, a] \to M$ *be a geodesic segment and* $c_2, c_3 : [0, \infty) \to M$ *geodesic rays emanating from* $c_2(0) = c_1(0)$, $c_3(0) = c_1(a)$ *such that* $d(c_2(t), c_3(t))$ *is bounded from above for all* $t \geqq 0$. *Let* $\alpha_2 = \angle(c_2'(0), c_1'(0))$ *and* $\alpha_3 = \angle(c_3'(0), -c_1'(a))$. *Then* $\alpha_2 + \alpha_3 \leqq \pi$ *and equality implies that the geodesics span a totally geodesic flat Euclidean strip.*

PROOF. Let $\gamma_t : [0, l_t] \to M$ be a geodesic segment from $c_3(0)$ to $c_2(t)$ and $\theta_t := \angle(-c_1'(a), \gamma_t'(0))$, $\omega_t = \angle(-\gamma_t'(l_t), -c_2'(t))$. Then by the Toponogov's comparison theorem, $\alpha_2 + \theta_t + \omega_t \leqq \pi$. Since $\theta_t$ converges to $\alpha_3$ as $t \to \infty$, we have $\alpha_2 + \alpha_3 \leqq \pi$.

To show the rigidity part, let $\sigma_t$ be a geodesic segment joining $c_2(t)$ and $c_3(t)$ and let $F$ be a component spanned by geodesic segments $c_2, c_3$ and $\sigma_t (t \geqq 0)$. If $\alpha_2 + \alpha_3 = \pi$ then $\left.\frac{d(c_2(t), c_3(t))}{dt}\right|_{t=0} = 0$ by the first variation formula. Since $d(c_2(t), c_3(t))$ is bounded, it holds that $d(c_2(t), c_3(t)) \equiv L(c_1) = a$ for all $t \geqq 0$. Hence also by the first variation formula, we have $\angle(c_2'(t), \sigma_t'(0)) \equiv \alpha_2$ and $\angle(c_3'(t), -\sigma_t'(a)) \equiv \alpha_3$. By Lemma 1.2.1, the domain bounded by $c_1, c_2, c_3$ and $\sigma_t$ in $F$ is a Euclidean rectangle for any $t$. Therefore $c_1, c_2$ and $c_3$ span a totally geodesic flat Euclidean strip. ∎

For two subsets $X, Y \subset M$ we define the *Hausdorff distance* $\mathrm{Hd}(X, Y)$ by

$$\mathrm{Hd}(X, Y) := \inf\{r > 0 | X \subset B_r(Y), Y \subset B_r(X)\},$$

where $B_r(X) := \{x \in M | d(x, X) < r\}$ is the r-neighborhood of $X$.

DEFINITION 1.2.1. Two totally geodesic submanifolds $Y_1$ and $Y_2$ of $M$ are said to be *parallel* if the Hausdorff distance between them is finite.

Then the following is valid for a Hadamard manifold $M$.

LEMMA 1.2.3 ( Lemma 2.3 in [BGS] ). *Let $Y_1, Y_2$ be parallel complete totally geodesic submanifolds and $a := \mathrm{Hd}(Y_1, Y_2)$. Then there is an isometric and totally geodesic embedding $\varphi : Y_1 \times [0, a] \to M$ with $\varphi(Y_1 \times \{0\}) = Y_1$ and $\varphi(Y_1 \times \{a\}) = Y_2$.*

PROOF. If $c : \mathbf{R} \to Y_2$ is a geodesic, then $d(c(t), Y_1) \leqq a$. Hence $d(c(t), Y_1)$

is constant, for it is a bounded convex function. In consequence, it follows that $d(\cdot, Y_1)$ is constant on $Y_2$, and by the same argument $d(\cdot, Y_2)$ is constant on $Y_1$. Clearly, the constant equals to $a$ in both cases. Let $\pi_{Y_1} : M \to Y_1, \pi_{Y_2} : M \to Y_2$ be projections. Then $d(p, \pi_{Y_2}(p)) = a$ for $p \in Y_1$, and hence $\pi_{Y_1} \circ \pi_{Y_2}$ is the identity on $Y_1$. Since the projections are distance decreasing, $\pi_{Y_1}$ and $\pi_{Y_2}$ are isometries on $Y_1$ or $Y_2$.

Now define $\varphi : Y_1 \times [0, a] \to M$, $\varphi(p, t) = c_p(t)$, where $c_p : [0, a] \to M$ is the geodesic segment from $p$ to $\pi_{Y_2}(p)$. For $(p_1, t_1), (p_2, t_2) \in Y_1 \times [0, a]$ we consider the geodesic segments $c_{p_1}, c_{p_2}, c_1 : [0, 1] \to Y_1$ from $p_1$ to $p_2$ and $c_2 : [0, 1] \to Y_2$ from $\pi_{Y_2}(p_1)$ to $\pi_{Y_2}(p_2)$. By Lemma 1.2.1 the geodesics bounds a totally geodesic rectangle since each angle is equal to $\frac{\pi}{2}$. Therefore $d(\varphi(p_1, t_1), \varphi(p_2, t_2)) = \hat{d}((p_1, t_1), (p_2, t_2))$, where $\hat{d}$ is distance with respect to the product metric of $Y_1 \times [0, a]$. Note that the geodesic segment on $M$ from $\varphi(p_1, t_1)$ to $\varphi(p_2, t_2)$ is contained in the image of $\varphi$. This completes the proof of the lemma. ∎

Let $Y$ be a complete totally geodesic submanifold of $M$. We define the subset $P_Y \subset M$ to be the union of all totally geodesic submanifolds parallel to $Y$.

LEMMA 1.2.4 ( Lemma 2.4 in [BGS] ). *$P_Y$ is isometric to $Y \times N$, where $N$ is a closed convex subset of $M$.*

PROOF. Let $Y_s, s \in S$ be the set of all parallels to $Y$. Here $S$ denotes an index set. Lemma 1.2.3 implies that for $s_1, s_2 \in S$ the convex hull of $Y_{s_1} \cup Y_{s_2}$ consists of a family of parallels to $Y$. Hence $P_Y$ is a convex subset, which is clearly closed. The interior of $P_Y$ is a manifold which possesses a parallel foliation given by the parallels to $Y$. This parallel foliation defines canonically a parallel distribution $\nu$. Let $\nu^\perp$ be the orthogonal distribution. Because $\nu$ is parallel, $\nu^\perp$ is integrable. Let $H$ be a maximal integral manifold for the distribution $\nu^\perp$. Let $p \in H$. Then by Lemma 1.2.3, $\pi_{Y_s}(p) \in H$ for all projections $\pi_{Y_s}$. Furthermore it holds that

$$\pi_{Y_{s_1}} \circ \pi_{Y_{s_2}}(x) = \pi_{Y_{s_1}}(x)$$

for arbitrary $x$ and $Y_{s_1}, Y_{s_2}$ in $P_Y$. Now fix $x \in Y$ and let $N := \{\pi_{Y_s}(x) | s \in S\}$. Define

$$\Phi : Y \times N \longrightarrow P_Y : (y, \pi_{Y_s}(x)) \rightarrow \pi_{Y_s}(y).$$

Then for any $y_1, y_2 \in Y$

$$\begin{aligned}
&d^2(\pi_{Y_{s_1}}(y_1), \pi_{Y_{s_2}}(y_2)) \\
=&d^2(\pi_{Y_{s_1}}(y_1), \pi_{Y_{s_2}} \circ \pi_{Y_{s_1}}(y_1)) + d^2(\pi_{Y_{s_2}} \circ \pi_{Y_{s_1}}(y_1), \pi_{Y_{s_2}}(y_2)) \\
=&d^2(\pi_{Y_{s_1}}(x), \pi_{Y_{s_2}}(x)) + d^2(y_1, y_2) \\
=&\hat{d}^2((y_1, \pi_{Y_{s_1}}(x)), (y_2, \pi_{Y_{s_2}}(x))),
\end{aligned}$$

where $\hat{d}$ is distance with respect to the product metric on $Y \times N$. Hence $\Phi$ is an isometry. Because $P_Y$ is convex, $N$ is convex. ∎

# Chapter 2. Total curvature and Tits metric

## §1. Total curvature of a noncompact Riemannian manifold

In this section we will briefly review basic matters on total curvature and show some examples to illustrate the geometric meaning of it.

Let $M$ be a connected, complete, noncompact and oriented Riemannian 2-manifold and $G$ the Gaussian curvature of $M$. Define nonnegative functions $G_+(p)$ and $G_-(p)$ by

$$G_+(p) := max\{G(p), 0\} \qquad \text{and} \qquad G_-(p) := max\{-G(p), 0\}$$

for $p \in M$, respectively. Then the total positive curvature $C_+(M)$ ( resp. the total negative curvature $C_-(M)$ ) of $M$ is defined to be

$$C_\pm(M) := \int_M G_\pm d_M,$$

the improper integral of $G_+$ ( resp. $G_-$ ) over $M$ with respect to the volume element $d_M$ of $M$.

DEFINITION 2.1.1. When either $C_+(M)$ or $C_-(M)$ is finite, the improper integral of $G$ over $M$ with respect to $d_M$ is defined and is denoted by $C(M)$, that is,

$$C(M) := \int_M G d_M.$$

In this case, we call $C(M)$ the *total curvature* of $M$.

We say that $M$ *admits total curvature* if $C(M)$ is defined. Note that the total curvature $C(M)$ may attain the infinity values, namely, $-\infty \leqq C(M) \leqq \infty$. It also holds that

$$C(M) = C_+(M) - C_-(M).$$

To be more precise, $C(M)$ is obtained as follows. Let $\{K_i\}_{i=1}^{\infty}$ be a monotone increasing sequence of compact subsets of $M$ such that $\bigcup_{i=1}^{\infty} K_i = M$. Let $C(K_i) := \int_{K_i} G d_M$ be the total curvature of each $K_i$. If $M$ admits total curvature, then the limit of $C(K_i)$ as $i$ tends to $\infty$ exists independently of the choice of $\{K_i\}$, and we have

$$C(M) = \lim_{i\to\infty} C(K_i).$$

On the other hand, when $M$ does not admit total curvature, i.e., $C_\pm(M) = \infty$, then the following is known.

LEMMA 2.1.1. *If $M$ does not admit total curvature, then for any number $\alpha$ such that $-\infty \leqq \alpha \leqq \infty$, there exists a sequence $\{K_i\}$ as above satisfying*

$$\lim_{i\to\infty} C(K_i) = \alpha.$$

PROOF. We can construct such $\{K_i\}$ as follows. Let

$$M^+ = \{p \in M | G(p) \geqq 0\},$$

$$M^- = \{p \in M | G(p) \leqq 0\}.$$

Then $M^\pm$ are closed sets of $M$ and $M = M^+ \cup M^-$. Fix a point $p \in M$ and set

$$B^\pm(a) = B(a) \cap M^\pm,$$

where $B(a)$ denotes a closed ball with center $p$ and radius $a$. Then the total curvature $C(B^+(a))$ ( resp. $C(B^-(a))$ ) is a monotone increasing ( resp. decreasing ) divergent function on $[0, \infty)$. Hence for $\infty > \alpha > 0$ given, we may choose two divergent sequences $\{a_i\}$ and $\{b_i\}$ such that $C(B^+(a_i)) = i \cdot \alpha$ and $C(B^-(a_i)) = -i \cdot \alpha$.

Now set

$$K_i = B^+(a_{i+1}) \cup B^-(a_i).$$

Then $\{K_i\}$ is a monotone increasing sequence of compact sets, and $C(K_i) \equiv \alpha$.

In the case $\alpha \leqq 0$ or $\alpha = \pm\infty$, we can construct $\{K_i\}$ in a similar fashion. Hence we obtain the lemma. ∎

Note that when the total curvature of an open manifold exists, it is obtained as the limit of $C(K_i)$ as above. The following Gauss-Bonnet theorem for a compact domain with piecewise smooth boundary then plays an essential role in investigations of total curvature in noncompact case. We recall it here for later use.

THE GAUSS-BONNET THEOREM. *Let $D$ be a compact domain of an oriented surface $M$, whose boundary consists of closed, simple and piecewise smooth curves $C_1, \cdots, C_n$. Suppose that each $C_i$ is positively oriented, parametrized by arc length $s$ and let $\theta^i_1, \cdots, \theta^i_{m_i}$ be the external angles of $C_i$. Then*

$$\sum_{i=1}^{n} \int_{C_i} \kappa_g(s) d_s + \int_D G d_M + \sum_{i=1}^{n} \sum_{j=1}^{m_i} \theta^i_j = 2\pi\chi(D),$$

*where $\kappa_g(s)$ is the geodesic curvature of $C_i$ and $\chi(D)$ is the Euler characteristic of $D$.*

A pioneering work due to Cohn-Vossen [Co1] on the total curvature of an open surface states that if a finitely connected $M$ admits total curvature, then

$$C(M) \leqq 2\pi\chi(M).$$

On the other hand, by Huber's theorem [Hu], if an infinitely connected $M$ admits total curvature, then

$$C(M) = -\infty.$$

Hence if $M$ admits total curvature, then $C(M)$ is bounded from above, which yields that the total positive curvature $C_+(M)$ of $M$ is finite. Therefore $M$ admits total curvature if and only if $C_+(M)$ is finite.

Next we will show some examples to illustrate the geometric meaning of total curvature.

EXAMPLE 2.1.1. The total curvature of a Euclidean plane is 0, while that of a hyperbolic plane is $-\infty$.

To explain the next example we give the following definition.

DEFINITION 2.1.2. We say that $M$ is *conical* if $M$ is a Riemannian surface which is flat outside some compact set.

It is easy to see that each conical $M$ is finitely connected and admits finite total curvature. If $M$ has $k$-ends, then outside some compact set there are $k$ flat tubes $U_1, \cdots, U_k$. Each flat tube $U_i$ is isometrically embedded in a flat cylinder or in an object obtained by identifying edges of a sector with vertex angle $\theta_i$, $0 < \theta_i < \infty$. Since this object is a cone if $0 < \theta_i < 2\pi$, we call this also a *cone.*

EXAMPLE 2.1.2. Let $M$ be conical with $k$-ends and $\theta_i (i = 1, \cdots, k)$ the vertex angle of a cone containing a flat tube $U_i$ which is a neighborhood of each end, where $\theta_i$ is understood to be 0 if $U_i$ is isometrically embedded in a flat cylinder. Then the total curvature $C(M)$ of $M$ is given by

$$C(M) = 2\pi\chi(M) - \sum_{i=1}^{k} \theta_i.$$

In fact, if $U_i$ is embedded in a flat cylinder, there is a closed geodesic $\gamma_i$ in $U_i$. If $U_i$ is in a cone, there is a broken geodesic $\gamma_i$ homotopic to $\partial U_i$ with $n_i$ broken points, such that the angle at each broken point is $\pi - \frac{\theta_i}{n_i}$, $n_i$ being a positive integer satisfying $(n_i - 1)\pi \leqq \theta_i < n_i \pi$. Let $K$ be a compact domain on $M$ whose boundary consists of ( broken ) geodesics $\gamma_1, \cdots, \gamma_k$. Since the Gaussian curvature of $M$ is 0 outside $K$, it follows by applying the Gauss-Bonnet theorem on the compact set $K$ that

$$C(M) = C(K) = 2\pi\chi(M) - \sum_{i=1}^{k} \theta_i.$$

EXAMPLE 2.1.3. Let $S \subset \mathbf{R}^3$ be a surface of revolution around $z$-axis parametrized by

$$S(s, t) = (a(t)\cos s, a(t)\sin s, b(t)),$$

where $a(t)$ is a nonnegative function. Assume that the generating curve $c(t) = (a(t), b(t))$ is parametrized by arc length. Then $S$ admits total curvature if and only if the right and left limits of $a'(t)$ exist. Furthermore if $S$ is homeomorphic to a plane, then

$$C(S) = 2\pi(1 - a'(\infty)).$$

While if it is homeomorphic to a cylinder, then

$$C(S) = 2\pi(a'(-\infty) - a'(\infty)).$$

In fact, from a simple calculation, we see that the Gaussian curvature $G(s,t)$ at $S(s,t)$ equals to $-\frac{a''(t)}{a(t)}$, and the volume form of $S$ is $a(t)dsdt$.

When $S$ is homeomorphic to a plane, the generating curve $c$ is defined on $[0,\infty)$ and $a'(0) = 1, b'(0) = 0$. Then the total curvature $C(S)$ of $S$ exists if and only if there exists the limit of $a'(t)$ as $t \to \infty$, denoted by $a'(\infty)$. Therefore we have

$$\begin{aligned} C(S) &= \int_0^\infty \int_0^{2\pi} -\frac{a''(t)}{a(t)} \cdot a(t) ds dt \\ &= \int_0^{2\pi} ds \int_0^\infty -a''(t) dt \\ &= 2\pi(1 - a'(\infty)). \end{aligned}$$

Similarly, when $S$ is homeomorphic to a cylinder, if $a'(\infty)$ and $a'(-\infty)$ exist, then

$$C(S) = 2\pi(a'(-\infty) - a'(\infty)).$$

A typical example of the former case is given by a connected component of the hyperboloid of two sheets $S = \{(x, y, z) \in \mathbf{R}^3 | (x\sin\theta)^2 - (y^2 + z^2)\cos^2\theta = 1\}$. Then $C(S) = 2\pi(1 - \sin\theta)$. A typical example of the latter case is given by a hyperboloid of one sheet $S = \{(x, y, z) \in \mathbf{R}^3 | (x\sin\theta)^2 - (y^2 + z^2)\cos^2\theta = -1\}$. Then $C(S) = -4\pi\sin\theta$.

These examples show that the total curvature of an open surface is no longer a topological invariant and depends deeply on its metric. More precisely speaking, the total curvature of the surface depends on the sum of the expanding growth rate of each end.

## §2. Tits metric on a Hadamard manifold

In this section we will recall definitions and some properties about Tits metric. We will also give some examples to illustrate the geometric meaning of it.

DEFINITION 2.2.1. For two rays $c_1$ and $c_2$ on a Riemannian manifold $M$, we say $c_1$ is *asymptotic* to $c_2$ if there exist a convergent sequence $\{p_n\}$ of points, $p_n \to c_1(0)$, and a divergent sequence $\{t_n\}$, $t_n \to \infty$, such that $\gamma_n'(0)$ converges to the vector $c_1'(0)$, where $\gamma_n$ is the minimizing geodesic segment on $M$ joining $p_n$ and $c_2(t_n)$.

Note that being asymptotic is not an equivalence relation in general.

Now we define the ideal boundary of a Hadamard manifold using the asymptotic relation. Throughout the rest of this section, we assume that $M$ is a Hadamard manifold. On a Hadamard manifold, the asymptotic relation has the following more tractable and explicit expression.

LEMMA 2.2.1 ( Proposition 1.2 in [EO] ). *In a Hadamard manifold, a ray $c_1$ is asymptotic to a ray $c_2$ if and only if there is a constant $a \in \mathbf{R}$ such that*

$$d(c_1(t), c_2(t)) \leqq a \qquad \textit{for all } \ t \geqq 0.$$

PROOF. Let $c_1$ be asymptotic to $c_2$. Then there exist, by the definition, a convergent sequence $\{p_n\}$ of points and a divergent sequence $\{t_n\}$ such that

$\gamma_n'(0)$ converges to the vector $c_1'(0)$, where $\gamma_n$ is the geodesic from $p_n$ to $c_2(t_n)$. Let $a$ be a constant such that $d(p_n, c_2(0)) \leqq a$ for all $n$. Let $s_n$ be the number such that $\gamma_n(s_n) = c_2(t_n)$. By the triangle inequality it holds that

$$|s_n - t_n| \leqq d(p_n, c_2(0)) \leqq a.$$

Now fix $s \geqq 0$. For a large $n$ such that $s \leqq s_n$, we have

$$d(\gamma_n(s), c_2(s)) \leqq \max\{d(\gamma_n(0), c_2(0)), d(\gamma_n(s_n), c_2(s_n))\}$$

from the convexity of the function $t \to d(\gamma_n(t), c_2(t))$. But

$$d(\gamma_n(s_n), c_2(s_n)) = d(c_2(t_n), c_2(s_n)) = |s_n - t_n| \leqq a$$

and

$$d(\gamma_n(0), c_2(0)) = d(p_n, c_2(0)) \leqq a.$$

Then

$$d(\gamma_n(s), c_2(s)) \leqq a \qquad \text{for all } s \geqq 0.$$

Hence, by continuity of the exponential map, we have

$$d(c_1(s), c_2(s)) \leqq a \qquad \text{for all } s \geqq 0.$$

Conversely, we suppose that two rays $c_1(t)$ and $c_2(t)$ satisfying $d(c_1(t), c_2(t)) \leqq a$ for all $t \geqq 0$ are given. For a divergent sequence $\{t_n\}$, let $\gamma_n$ be a geodesic segments from $c_1(0)$ to $c_2(t_n)$. Then $\gamma_n'(0)$ converges to $c_1'(0)$, that is, $c_1$ is asymptotic to $c_2$, because of the uniqueness of a ray $c$ emanating from some fixed point such that $d(c(t), c_2(t)) \leqq a'$ for all $t \geqq 0$ and for some constant $a'$.

The uniqueness of such a ray is obtained from the convexity of the function $t \to d(\sigma_1(t), \sigma_2(t))$ for any two geodesics $\sigma_1, \sigma_2$. In fact, if there are two rays $c, \widetilde{c}$ emanating from same point $x$ such that $d(c(t), c_2(t))$ and $d(\widetilde{c}(t), c_2(t))$ are bounded from above, then there is a constant $\widetilde{a}$ such that $d(c(t), \widetilde{c}(t)) \leqq \widetilde{a}$ for all $t$. From the convexity of the distance function between two rays and $d(c(0), \widetilde{c}(0)) = 0$, we have $c(t) = \widetilde{c}(t)$. ∎

This lemma means that the asymptotic relation is an equivalence relation on a Hadamard manifold. Then the ideal boundary of $M$ is defined as follows.

DEFINITION 2.2.2. The equivalence classes of all rays on a Hadamard manifold $M$ with respect to the asymptotic relation are called *points at infinity* of $M$ and the set of these classes is called the *ideal boundary* of $M$, denoted by $M(\infty)$.

For a geodesic $c : \mathbf{R} \to M$, let $c(\infty) \in M(\infty)$ be the corresponding class of the ray $c|[0, \infty)$, and $c(-\infty)$ the class of the reversed geodesic $t \to c(-t)$. We will note that for a point $x \in M$ and a point $z \in M(\infty)$ given, there is a unique ray $c : [0, \infty) \to M$ with $c(0) = x$ and $c(\infty) = z$. In fact, take a ray $\gamma$ such that $\gamma(\infty) = z$. Then we can obtain such a ray $c$ as a limit of the sequence $\{\gamma_i\}$ of geodesic segments from $x$ to $\gamma(t_i)$ for a divergent sequence $\{t_i\}$. Therefore, for any point $p \in M$, there is a bijective map between the ideal boundary $M(\infty)$ of $M$ and the unit tangent sphere $S_pM$ at $p$ of $M$. This map is explicitly expressed by

$$\Psi : S_pM \to M(\infty) : \Psi(u) := \gamma_u(\infty),$$

where $\gamma_u$ is a ray with initial vector $u$. The topology on $M(\infty)$ induced from $S_pM$ by $\Psi$ is called the *sphere topology.* The sphere topology is independent of the choice of a point $p \in M$.

For $z_1,\ z_2 \in \overline{M} = M \cup M(\infty)$ and $x \in M$ we define

$$\angle_x(z_1, z_2) := \angle(c_1'(0),\ c_2'(0)),$$

where $c_i$ is a unique ray from $x$ to $z_i$ $(i = 1, 2)$. For $x \in M$, $z \in M(\infty)$, $\varepsilon > 0$ let $C_x(z, \varepsilon)$ be the cone $\{y \in \overline{M} | \, y \neq x \text{ and } \angle_x(z, y) < \varepsilon\}$. The *cone topology* on $\overline{M}$ is the topology generated by the open sets in $M$ and these cones. In other words, this is the topology induced from $B_pM$ by $\psi : B_pM \to \overline{M}$ such that

$$\psi(v) = \begin{cases} \exp_p\left(\dfrac{v}{1 - \|v\|}\right) & \text{for } \|v\| < 1, \\ \gamma_v(\infty) & \text{for } \|v\| = 1, \end{cases}$$

where $B_pM$ is the unit closed disk of $T_pM$ and hence the relative topology on $M(\infty)$ coincides with the sphere topology.

Next, we will define a metric on $M(\infty)$. Note that we allow that points have the infinite distance. The angle $\angle$ on $M(\infty)$ is defined as follows.

DEFINITION 2.2.3. The *angle* $\angle(z_1, z_2)$ between $z_1$ and $z_2$ in $M(\infty)$ is defined by

$$\angle(z_1, z_2) := \sup_{x \in M} \angle_x(z_1, z_2).$$

Clearly $\angle$ is a distance function on $M(\infty)$. Furthermore $(M(\infty), \angle)$ is a complete metric space. In fact, let $z_i$ be a Cauchy sequence in $(M(\infty), \angle)$. Then for all $x \in M$, $z_i$ is a Cauchy sequence in the metric $\angle_x(\cdot, \cdot)$. Hence $z_i$ converges to a point $z \in M(\infty)$ with respect to the metric $\angle_x(\cdot, \cdot)$. For $x, y \in M$, $\angle_x(\cdot, \cdot)$ and $\angle_y(\cdot, \cdot)$ define the same topology in $M(\infty)$. Hence $z$ does not depend on $x$. To prove that $z_i \to z$ in $\angle$, let $\varepsilon > 0$ be arbitrary. Then there exists $i_0 \in \mathbf{N}$ such that

$$\angle(z_i, z_j) < \frac{\varepsilon}{2} \qquad \text{for } i, j \geqq i_0.$$

For a fixed $k \geqq i_0$ there is a point $x \in M$ such that

$$\angle(z, z_k) - \angle_x(z, z_k) < \frac{\varepsilon}{2}.$$

Now $z_i \to z$ in $\angle_x(\cdot, \cdot)$ and $\angle_x(z_i, z_k) \leqq \angle(z_i, z_k) < \frac{\varepsilon}{2}$ for all $i \geqq i_0$. This implies $\angle_x(z, z_k) < \frac{\varepsilon}{2}$. Thus $\angle(z, z_k) < \varepsilon$ for all $k \geqq i_0$.

We will here introduce the interior metric of a metric space. Let $(X, d)$ be a metric space. For a continuous curve $c : [0,1] \to X$, we denote the length of $c$ with respect to the metric $d$ by $L(c)$. Then we define a new metric $d_i$ on $X$ by

$$d_i(x, y) := \begin{cases} \inf L(c) & \text{if there is a continuous curve } c \text{ from } x \text{ to } y, \\ \infty & \text{otherwise}, \end{cases}$$

where inf is attained over all continuous curves from $x$ to $y$. This metric $d_i$ is called the *interior metric* of $(X, d)$. A metric space $(X, d)$ is called a *length space* if $d = d_i$.

DEFINITION 2.2.4. The *Tits metric* Td on $M(\infty)$ is defined as the interior metric of the angle $\angle$, that is,

$$\mathrm{Td}(z, w) := \angle_i(z, w).$$

By the definition, it holds that for any $z, w \in M(\infty)$

$$\mathrm{Td}(z, w) \geqq \angle(z, w).$$

Furthermore the following is known.

LEMMA 2.2.2 ( Lemma 4.7 in [BGS] ). *For $z, w \in M(\infty)$ if there is no geodesic $c : \mathbf{R} \to M$ with $c(-\infty) = z$ and $c(\infty) = w$, then*

$$\mathrm{Td}(z, w) = \angle(z, w).$$

PROOF. To estimate the Tits distance from above, we have to construct a curve in $M(\infty)$ from $z$ to $w$ with length $\angle(z, w)$.

First we prove that there is a point $m \in M(\infty)$ with $\angle(z, m) = \angle(m, w) = \frac{1}{2}\angle(z, w)$. Fix a point $x$ arbitrary and let $c_z, c_w$ be rays from $x$ to $z, w$. For $j \in \mathbf{N}$ let $p_j$ be the unique point on the geodesic segment from $c_z(j)$ to $c_w(j)$ which has the minimal distance to $x$. Since there are no geodesics in $M$ joining $z$ and $w$, the geodesic segment from $c_z(j)$ to $c_w(j)$ have no accumulation geodesics and hence the sequence $\{p_j\}$ has no accumulation points in $M$. Therefore there is an accumulation point $m \in M(\infty)$ of $\{p_j\}$ with respect to the cone topology on $\overline{M}$. By a careful estimation we can see that $\angle(z, m) = \angle(m, w) = \frac{1}{2}\angle(z, w)$.

Thus we can construct the point $m$. Clearly $\angle(z,m) = \angle(m,w) \leqq \frac{\pi}{2}$ and hence $m$ and $z$ ( resp. $w$ ) cannot be joined by a geodesic on $M$. Therefore we can use this construction inductively and construct a map $h$ from $\left\{ \frac{k}{2^n} \middle| n, k \in \mathbf{N} \cup \{0\}, k \leqq 2^n \right\} \subset [0,1]$ into $M(\infty)$ with $\angle \left( h\left(\frac{k}{2^n}\right), h\left(\frac{k+1}{2^n}\right) \right) = \frac{1}{2^n}\angle(z,w)$. By the completeness of $(M(\infty), \angle)$ this map extends to a continuous map $h : [0,1] \to M(\infty)$ with length $L(h) = \angle(z,w)$. Hence $\mathrm{Td}(z,w) \leqq \angle(z,w)$. ∎

This lemma implies that if $\mathrm{Td}(z,w) > \pi$, then there is a geodesic $c : \mathbf{R} \to M$ with $c(-\infty) = z$ and $c(\infty) = w$. If there exists a ray joining $z$ and $w$, then $\mathrm{Td}(z,w) \geqq \angle(z,w) = \pi$. Therefore this also implies that $\angle(z,w) = \min(\mathrm{Td}(z,w), \pi)$ and hence Td induces the same topology on $M(\infty)$ as $\angle$. It should be noted that Tits metric Td inherits the completeness of angle $\angle$.

To illustrate the geometric meaning of Tits metric and to use later, we show the following lemmas.

LEMMA 2.2.3 ( Lemma 4.2 in [BGS] ). *For $z, w \in M(\infty)$, let $c : [0,\infty) \to M$ be a ray with $c(\infty) = z$. Set $\varphi(t) := \angle_{c(t)}(z,w)$. Then*

$$\lim_{t\to\infty} \varphi(t) = \angle(z,w).$$

PROOF. By Lemma 1.2.2 it holds that $\varphi(t) + (\pi - \varphi(s)) \leqq \pi$ for $s > t$. Hence $\varphi(t)$ is monotone increasing and therefore the limit exists. Clearly $\varphi(t) \leqq \angle(z,w)$.

To prove the opposite inequality let $x \in M$ be arbitrary and let $\alpha = \angle_x(z,w)$. We have to show that $\lim_{t\to\infty} \varphi(t) \geqq \alpha$.

For $t \geqq 0$ let $c_t : \mathbf{R} \to M$ be the geodesic with $c_t(0) = x, c_t(1) = c(t)$. Let

$$\begin{cases} \alpha_t := \angle_x(c_t(\infty), w), \\ \beta_t := \angle_{c(t)}(c_t(\infty), w), \\ \gamma_t := \angle_{c(t)}(z, c_t(\infty)). \end{cases}$$

Because $c_t(\infty)$ converges to $z$ as $t \to \infty$, $\alpha_t$ converges to $\alpha$. The argument of monotonicity of $\varphi(t)$ applied for the geodesic $c_t$ implies $\beta_t \geqq \alpha_t$. Applying Toponogov's theorem for the triangle $x, c(0), c(t)$, we see that $\gamma_t$ converges to 0. Thus $\varphi(t) + \gamma_t \geqq \beta_t \geqq \alpha_t \to \alpha$ and $\gamma_t \to 0$ implies $\lim_{t\to\infty} \varphi(t) \geqq \alpha$. ∎

LEMMA 2.2.4 (Lemma 4.3 in [BGS]). *Let $z, w \in M(\infty)$, $x \in M$ and let $c_i : [0,\infty) \to M$ $(i = 1, 2)$ be rays with $c_i(0) = x$, $c_1(\infty) = z$ and $c_2(\infty) = w$. Set $\alpha_t := \angle_{c_1(t)}(x, c_2(t))$, $\beta_t := \angle_{c_2(t)}(x, c_1(t))$. Then*

$$\angle(z,w) = \lim_{t\to\infty} (\pi - \alpha_t - \beta_t).$$

PROOF. For $s > t$ we consider the quadrilateral with vertices $c_1(t), c_1(s)$, $c_2(t), c_2(s)$ and angles $\pi - \alpha_t, \pi - \beta_t, \alpha_s, \beta_s$. Because the sum of the angles is not greater than $2\pi$, we have $\alpha_s + \beta_s \leqq \alpha_t + \beta_t$. It follows that $\pi - \alpha_t - \beta_t$ is monotone increasing and the limit exists.

Let $\varphi(t) := \angle_{c_1(t)}(z, w)$. Then $\varphi(t)$ converges to $\angle(z,w)$ as $t \to \infty$ by Lemma 2.2.3. Consider the triangle $c_1(t), c_2(t), w$. Since $\angle_{c_2(t)}(c_1(t), w) = \pi - \beta_t$, we have $\angle_{c_1(t)}(c_2(t), w) \leqq \beta_t$. Furthermore as $\varphi(t) + \angle_{c_1(t)}(c_2(t), w) + \alpha_t \geqq \pi$, we have

$\varphi(t) \geqq \pi - \alpha_t - \beta_t$. Therefore

$$\lim_{t\to\infty} (\pi - \alpha_t - \beta_t) \leqq \angle(z, w).$$

On the other hand, let $t_0 \in [0, \infty)$ and $\varepsilon > 0$ be arbitrary. For $t > t_0$ large enough, we have

$$|\angle_{c_1(t_0)}(z, c_2(t)) - \varphi(t_0)| < \frac{\varepsilon}{2}$$

and

$$|\angle_{c_2(t)}(c_1(t_0), c_1(t)) - \beta_t| < \frac{\varepsilon}{2}.$$

Because of

$$\angle_{c_1(t_0)}(z, c_2(t)) + \angle_{c_2(t)}(c_1(t_0), c_1(t)) + \alpha_t \leqq \pi,$$

we have

$$\begin{aligned}
\varphi(t_0) &< \angle_{c_1(t_0)}(z, c_2(t)) + \frac{\varepsilon}{2} \\
&\leqq \pi - \alpha_t - \angle_{c_2(t)}(c_1(t_0), c_1(t)) + \frac{\varepsilon}{2} \\
&< \pi - \alpha_t - \beta_t + \varepsilon.
\end{aligned}$$

Because $t_0$ and $\varepsilon$ are arbitrary,

$$\angle(z, w) \leqq \lim_{t\to\infty} (\pi - \alpha_t - \beta_t). \quad \blacksquare$$

Now we introduce other approaches to define Tits metric here.

For $z, w \in M(\infty)$ and $x \in M$ let $c_1, c_2$ be rays from $x$ to $z, w$. The function $t \to \dfrac{d(c_1(t), c_2(t))}{t}$ is monotone increasing and bounded by 2. Therefore we can define the metric $l$ on $M(\infty)$ by

$$l(z, w) := \lim_{t \to \infty} \frac{d(c_1(t), c_2(t))}{t}.$$

It is easy to check that the definition of $l$ is independent of the choice of $x$ and that $l$ is indeed a metric on $M(\infty)$. Then we have the following relation between $l$ and $\angle$.

LEMMA 2.2.5 ( Lemma 4.4 in [BGS] ). *For $z, w \in M(\infty)$ it holds that*

$$l(z, w) = 2 \sin \left( \frac{\angle(z, w)}{2} \right).$$

PROOF. For $x \in M$ let $c_1, c_2$ be the rays from $x$ to $z, w$ and let $f(t) := \dfrac{d(c_1(t), c_2(t))}{t}$. Because the differential of $\exp_x : T_x M \to M$ is an isometry at the origin in $T_x M$, we have $\lim_{t \to 0} f(t) = 2 \sin \left( \dfrac{\angle_x(z, w)}{2} \right)$. Now $f(t)$ is increasing with $\lim_{t \to \infty} f(t) = l(z, w)$. Therefore $l(z, w) \geqq 2 \sin \left( \dfrac{\angle_x(z, w)}{2} \right)$ and since $x$ is arbitrary

$$l(z, w) \geqq 2 \sin \left( \frac{\angle(z, w)}{2} \right).$$

On the other hand, let $\alpha_t := \angle_{c_1(t)}(x, c_2(t))$ and $\beta_t := \angle_{c_2(t)}(x, c_1(t))$. By the second law of cosine we have $d(c_1(t), c_2(t)) \leqq t \cos \alpha_t + t \cos \beta_t$ which implies

$$f(t) \leqq \cos \alpha_t + \cos \beta_t.$$

Since $\alpha_t, \beta_t \geqq 0$ and $\alpha_t + \beta_t \leqq \pi$, we have

$$\cos\alpha_t + \cos\beta_t = 2\cos\left(\frac{\alpha_t+\beta_t}{2}\right)\cos\left(\frac{\alpha_t-\beta_t}{2}\right) \leqq 2\cos\left(\frac{\alpha_t+\beta_t}{2}\right).$$

Thus

$$f(t) \leqq 2\cos\left(\frac{\alpha_t+\beta_t}{2}\right) = 2\sin\left(\frac{\pi-\alpha_t-\beta_t}{2}\right).$$

Because $\pi - \alpha_t - \beta_t$ converges to $\angle(z,w)$ by Lemma 2.2.4 we have

$$l(z,w) \leqq 2\sin\left(\frac{\angle(z,w)}{2}\right). \blacksquare$$

REMARK. Similarly to the proof of this lemma the following formula holds. For $z, w \in M(\infty)$ and $x \in M$ let $c_1, c_2$ be as above. Then for positive constants $a, b$

$$\lim_{t\to\infty} \frac{d(c_1(at), c_2(bt))}{t} = (a^2 + b^2 - 2ab\cos\angle(z,w))^{\frac{1}{2}}.$$

This lemma implies that $l(z,w) \leqq \angle(z,w)$ for all $z, w \in M(\infty)$ and hence for the interior metric $l_i$ of $l$ we also have $l_i(z,w) \leqq \angle_i(z,w) = \mathrm{Td}(z,w)$. Furthermore the opposite inequality also holds. In fact, For any point $x \in M$ and any curve $c : [0,1] \to (M(\infty), \mathrm{Td})$, $c$ induces a curve in the unit sphere in $T_xM$. It is easy to see that the $l$-length of $c$ is greater than the length of the induced curve in the tangent sphere, thus the $l$-length of $c$ is greater than $\angle_x(c(0), c(1))$ and hence $\angle(c(0), c(1))$. Therefore $l_i(z,w) \geqq \angle_i(z,w) = \mathrm{Td}(z,w)$. Hence we have $l_i = \mathrm{Td}$

The metrics $\angle$ and $l$ lead to the same length space $(M(\infty), \mathrm{Td})$. There is a third definition of Tits metric. For $x \in M$, we consider the spherical distance $d_t^s$ on the distance sphere $S_t(x) := \{y \in M | d(x,y) = t\}$, which is defined as the infimum of the lengths of the curves in $S_t(x)$ joining given two points. For points $z, w \in M(\infty)$, let $c_1, c_2$ be the rays from $x$ to $z, w$. Then the function $\widetilde{d}(z,w) := \lim_{t \to \infty} \dfrac{d_t^s(c_1(t), c_2(t))}{t}$ is well-defined and coincides with Td.

We will note that the topology of $(M(\infty), \mathrm{Td})$ does not necessarily coincide with the sphere topology on $M(\infty)$ as shown in the following examples. To distinguish these topologies we call the topology on $M(\infty)$ defined by the Tits metric the *Tits topology*. But there are some relations between these topologies. It is clear that if a sequence $\{z_i\} \subset M(\infty)$ converges to $z$ in the Tits topology, then it also converges to $z$ in the sphere topology. Conversely for $z \in M(\infty)$, the function $\mathrm{Td}(z, \cdot)$ on $M(\infty)$ is not necessarily continuous in the sphere topology. However this is semicontinuous in this topology. In fact using the third definition of the Tits metric, it holds that for $\{w_i\}$ converging to $w$ in the sphere topology

$$\liminf_{i \to \infty} \mathrm{Td}(z, w_i) \geqq \liminf_{i \to \infty} \frac{d_{t_0}^s(z, w_i)}{t_0} = \frac{d_{t_0}^s(z, w)}{t_0},$$

where $t_0$ is arbitrary. Thus $\liminf_{i \to \infty} \mathrm{Td}(z, w_i) \geqq \mathrm{Td}(z, w)$.

In the next section we will give the condition for these topologies to coincide each other.

Finally, we will give some examples.

EXAMPLE 2.2.1. If $M$ is an $n$-dimensional Euclidean space, then $(M(\infty),$

Td) is isometric to an $(n-1)$-dimensional standard sphere.

EXAMPLE 2.2.2. If $M$ is an $n$-dimensional hyperbolic space, then

$$\mathrm{Td}(z_1, z_2) = \begin{cases} \infty, & \text{for } z_1 \neq z_2 \in M(\infty), \\ 0, & \text{for } z_1 = z_2 \in M(\infty). \end{cases}$$

Therefore the Tits topology on a hyperbolic space is a discrete topology.

EXAMPLE 2.2.3. Let $M^2$ be conical. Hence there is a compact domain $K$ such that $M \backslash K$ is isometrically embedded in a cone of vertex angle $\theta$, $2\pi \leqq \theta$. Then $(M(\infty), \mathrm{Td})$ is isometric to a circle of girth $\theta$.

EXAMPLE 2.2.4. Let $M$ be a product of the hyperbolic plane $\mathbf{H}^2$ and the real line $\mathbf{R}$, i.e., $M = \mathbf{H}^2 \times \mathbf{R}$. We denote by $\mathbf{R}(\infty)$ a pair of two asymptotic classes $\{N, S\}$, where $N$ is the asymptotic class of a ray $\gamma$ defined by $\gamma(t) = (p, t) \in \mathbf{H}^2 \times \mathbf{R}$ for $p \in \mathbf{H}^2$ and $S$ is that of a ray $\gamma$ with $\gamma(t) = (p, -t)$. We also denote by $\mathbf{H}^2(\infty)$ the set of asymptotic classes of rays on $\mathbf{H}^2$-factor. Then we consider $M(\infty)$ as the unit sphere, where the north pole is $N$ and the south pole is $S$ and the equator is $\mathbf{H}^2(\infty)$.

Now we consider the Tits length from $N$ to $S$. For a ray $\gamma$ in $\mathbf{H}^2$, let $H$ be the flat half plane $\{\gamma\} \times \mathbf{R}$ of $M$. Then the set of asymptotic classes of rays on $H$, denoted by $H(\infty)$, is a half circle from $N$ to $S$. Hence

$$\mathrm{Td}(N, S) = \pi.$$

Next we consider two points $z, w$ on the equator. There is a geodesic $\alpha$ in $\mathbf{H}^2$ joining these points and $F = \{\alpha\} \times \mathbf{R}$ is a flat plane in M. Then $F(\infty)$ is a circle in $M(\infty)$ containing $z, w, N$ and $S$. We can choose a curve in $F(\infty)$ from $z$ to $N$ ( or $S$ ) and from this pole back to $w$ as a minimal curve in $M(\infty)$ joining $z$ and $w$. Therefore we have

$$\mathrm{Td}(z, w) = \pi.$$

For any points $z_1, z_2 \in M(\infty)$, a minimal curve in $M(\infty)$ joining $z_1$ and $z_2$ is also a curve from $z_1$ to $N$ ( or $S$ ) and from this pole back to $z_2$ and it holds

$$\mathrm{Td}(z_1, z_2) \leqq \pi.$$

In this example, we should note that a closed distance ball $B_\varepsilon(N)$ at the north pole is not compact, and hence Tits topology is not necessarily locally compact.

As understood from Lemma 2.2.3, the second definition or these examples, Tits metric expresses the expanding growth rate of the equivalence class of rays.

## §3. Relations between total curvature and Tits metric

In the previous sections we saw that both quantities of total curvature and Tits metric are concerned with the growth rate of ends. Motivated this fact, we investigate some relations between total curvature and Tits metric and prove two theorems.

Let $M$ be a Hadamard manifold. First we will prove the following theorem.

THEOREM 1. *Let $M$ be a 2-dimensional Hadamard manifold and $\alpha$ the diameter of $M(\infty)$ with respect to the Tits metric. Then the total curvature $C(M)$ of $M$ equals to $2(\pi - \alpha)$, where in the case that $\alpha = \infty$ this means $C(M) = -\infty$.*

First applying Lemma 2.2.4 we shall show a relation between total curvature and Tits metric.

LEMMA 2.3.1. *Let $M$ be an n-dimensional Hadamard manifold and $\alpha$, $\beta$ rays emanating from $p \in M$ satisfying $\mathrm{Td}(\alpha(\infty), \beta(\infty)) < \pi$. Let $F$ be a component consisting of geodesic segments joining $\alpha(t)$ and $\beta(t)$ for all $t \geqq 0$. Then*

$$C(F) = \angle(\alpha'(0),\ \beta'(0)) - \mathrm{Td}(\alpha(\infty),\ \beta(\infty)),$$

*where $C(F)$ denotes the total curvature of $F$.*

PROOF. From Lemma 2.2.4, we have

$$\angle(\alpha(\infty),\ \beta(\infty)) = \lim_{t \to \infty} (\pi - \alpha_t - \beta_t),$$

where $\alpha_t = \angle_{\alpha(t)}(p, \beta(t))$ and $\beta_t = \angle_{\beta(t)}(\alpha(t), p)$. Now from the assumption that $\mathrm{Td}(\alpha(\infty), \beta(\infty)) < \pi$, we have $\mathrm{Td}(\alpha(\infty), \beta(\infty)) = \angle(\alpha(\infty), \beta(\infty))$. Hence we have

$$\lim_{t\to\infty}(\pi - \alpha_t - \beta_t) = \mathrm{Td}(\alpha(\infty), \beta(\infty)).$$

Then, applying the Gauss-Bonnet theorem to the triangle $p, \alpha(t), \beta(t)$, we have

$$\begin{aligned} C(F) &= \lim_{t\to\infty}\{\angle(\alpha'(0), \beta'(0)) + \alpha_t + \beta_t - \pi\} \\ &= \angle(\alpha'(0), \beta'(0)) - \mathrm{Td}(\alpha(\infty), \beta(\infty)), \end{aligned}$$

as required. ∎

In a 2-dimensional Hadamard manifold, this lemma implies the following proposition.

PROPOSITION 2.3.2. *Let $M$ be a 2-dimensional Hadamard manifold and $\alpha$, $\beta$ be rays emanating from a same point $p \in M$. If $\mathrm{Td}(\alpha(\infty), \beta(\infty)) < \infty$, then, for the component $F$ bounded by two rays $\alpha$ and $\beta$ which contains the ray $\gamma$ such that $\mathrm{Td}(\alpha(\infty), \gamma(\infty)) = \mathrm{Td}(\gamma(\infty), \beta(\infty)) = \frac{1}{2}\mathrm{Td}(\alpha(\infty), \beta(\infty))$, we get*

$$C(F) = \angle(\alpha'(0), \beta'(0)) - \mathrm{Td}(\alpha(\infty), \beta(\infty)),$$

*where $\angle$ is measured with respect to $F$. If $\mathrm{Td}(\alpha(\infty), \beta(\infty)) = \infty$, then $C(F) = -\infty$ for each component $F$ bounded by $\alpha$ and $\beta$.*

PROOF. In the case that $\mathrm{Td}(\alpha(\infty), \beta(\infty)) < \infty$, there exists a continuous curve $c : [0, 1] \to M(\infty)$ from $\alpha(\infty)$ to $\beta(\infty)$ such that the length of $c$ is equal to $\mathrm{Td}(\alpha(\infty),\ \beta(\infty))$ on $M(\infty)$, since $M(\infty)$ is homeomorphic to $S^1$. Choose the subdivision $t_0 = 0 < t_1 < t_2 < \cdots < t_m = 1$ of the interval $[0, 1]$ satisfying that the length of $c|\,[t_{i-1},\ t_i]$ is smaller than $\pi$ for any $i \in \{1, 2, \cdots, m\}$ . Let $F_i$ be the component bounded by two rays $c_{i-1}$ and $c_i$ which are emanating from $p$ to $c(t_{i-1})$ and $c(t_i)$, respectively. Then from the above lemma, we have

$$C(F_i) = \angle(c'_{i-1}(0),\ c'_i(0)) - L(c|\,[t_{i-1},\ t_i]),$$

where $L(c|\,[t_{i-1},\ t_i])$ is the length of $c|\,[t_{i-1},\ t_i]$ . Hence

$$\begin{aligned} C(F) &= \sum_{i=1}^{m} C(F_i) \\ &= \angle(\alpha'(0),\ \beta'(0)) - L(c) \\ &= \angle(\alpha'(0),\ \beta'(0)) - \mathrm{Td}(\alpha(\infty), \beta(\infty)). \end{aligned}$$

Next we shall consider the case that $\mathrm{Td}(\alpha(\infty), \beta(\infty)) = \infty$. Let $r > 0$ be an arbitrary large number. Fix a positive integer $\tilde{r} > \{r + \angle(\alpha'(0), \beta'(0))\}/\pi$ and choose rays $\gamma_i (i = 0, 1, \cdots, \tilde{r})$ on $F$ satisfying the following conditions:

$$\begin{cases} \gamma_0 = \alpha, \qquad \gamma_{\tilde{r}} = \beta, & \\ \mathrm{Td}(\gamma_{i-1}(\infty),\ \gamma_i(\infty)) > \pi & \text{for } 1 \leqq i \leqq \tilde{r}, \\ \sum_{i=1}^{\tilde{r}} \angle(\gamma'_{i-1}(0), \gamma'_i(0)) = \angle(\alpha'(0), \beta'(0)). & \end{cases}$$

Let $F_i \subset F$ be a component bounded by $\gamma_{i-1}$ and $\gamma_i$ and $\widetilde{F}_i \subset F_i$ be a component bounded by the straight line $\sigma$ such that $\sigma(\infty) = \gamma_{i-1}(\infty)$ and $\sigma(-\infty) = \gamma_i(\infty)$. Note that $C(\widetilde{F}_i) \leqq 0$, because the sectional curvature is nonpositive. Then, applying the Gauss-Bonnet theorem, we have

$$C(F_i) = \angle(\gamma'_{i-1}(0),\ \gamma'_i(0)) - \pi + C(\widetilde{F}_i)$$
$$\leqq \angle(\gamma'_{i-1}(0),\ \gamma'_i(0)) - \pi.$$

Hence we obtain

$$C(F) = \sum_{i=1}^{\tilde{r}} C(F_i)$$
$$\leqq \angle(\alpha'(0),\ \beta'(0)) - \widetilde{r} \cdot \pi$$
$$< -r.$$

Because $r$ is arbitrary, $C(F) = -\infty$ . ▮

This proposition means Theorem 1 as follows.

PROOF OF THEOREM 1. In the case that $\alpha = \infty$, the above proposition implies that the total negative curvature of $M$ is infinity, hence $C(M) = -\infty$.

For a finite diameter $\alpha$ of $M(\infty)$, we can choose a pair $\{z, w\}$ of points of $M(\infty)$ such that $\mathrm{Td}(z, w) = \alpha$ because $M(\infty)$ is homeomorphic to $S^1$. Then there exists a geodesic $\sigma$ such that $\sigma(\infty) = z$ and $\sigma(-\infty) = w$. Let $F_1, F_2$ be two components of $M$ bounded by $\sigma$. Applying the above proposition to $F_1$ and $F_2$ respectively, we have

$$C(F_1) = C(F_2) = \pi - \alpha.$$

Therefore

$$C(M) = C(F_1) + C(F_2) = 2(\pi - \alpha).\ ▮$$

Next, as announced in the previous section, we show the equivalent conditions for the Tits topology to coincide with the sphere topology on $M(\infty)$. We recall the map $\Psi : S_p M \to M(\infty)$ defined by

$$\Psi(u) := \gamma_u(\infty),$$

where $\gamma_u$ is a ray with an initial vector $u$. For two rays $\gamma$ and $\sigma$ emanating from the same point, let $F(\gamma, \sigma)$ be a component consisting of geodesic segments joining $\gamma(t)$ to $\sigma(t)$ for all $t \geqq 0$. Then we have the following

THEOREM 2. *Following three conditions are equivalent:*

(1) *$M(\infty)$ is compact in the Tits topology.*

(2) *$\Psi$ is homeomorphic, that is, the Tits topology is equivalent to the sphere topology.*

(3) *For given $x \in M$ and $\varepsilon > 0$, there exists a positive number $\delta(x, \varepsilon)$ such that for every $u, v \in S_x M$ with $\angle(u, v) < \delta$, the total curvature $C(F(\gamma_u, \gamma_v))$ is greater than $-\varepsilon$.*

PROOF.

(1) $\Rightarrow$ (2) : It is clear that $\Psi$ is bijective. And $\Psi^{-1}$ is continuous since it holds always that for every $z, w \in M(\infty)$,

$$\mathrm{Td}(z, w) \geqq \angle(z, w) \geqq \angle_x(z, w).$$

Furthermore supposing (1), from the property that $(S_x M, \angle)$ is Hausdorff, we can conclude that $\Psi$ is homeomorphic.

(2) $\Rightarrow$ (1) : The compactness of $M(\infty)$ is induced from the compactness of $S_xM$ by $\Psi$.

(1),(2) $\Rightarrow$ (3) : Fix $x \in M$ and $\varepsilon > 0$ arbitrarily. We may suppose $\varepsilon < \pi$ without loss of generality. From the compactness of $M(\infty)$ we can choose a sequence $\{z_i | \, i = 1, 2, \cdots, m\}$ of finitely many points such that $\{U_i := B_{\frac{\varepsilon}{2}}(z_i) | \, i = 1, 2, \cdots, m\}$ is an open covering of $M(\infty)$, where $B_{\frac{\varepsilon}{2}}(z_i)$ denotes an open ball centered at $z_i$ with radius $\dfrac{\varepsilon}{2}$. Then $\{\Psi^{-1}(U_i)\}$ is also an open covering of a compact set $S_xM$. Now let $\delta$ be a Lebesgue number of $\{\Psi^{-1}(U_i)\}$. Then for $u, v \in S_xM$ with $\angle(u, v) < \delta$, there is at least one number $i_0 \in \{1, \cdots, m\}$ such that $\{u, v\} \subset \Psi^{-1}(U_{i_0})$. Hence $a := \mathrm{Td}(\gamma_u(\infty), \gamma_v(\infty))$ is smaller than $\varepsilon$. Therefore we have

$$C(F(\gamma_u, \gamma_v)) = \angle(u, v) - a \geqq -a > -\varepsilon,$$

where the equality follows from Lemma 2.3.1.

(3) $\Rightarrow$ (2) : In order to show this assertion, it is sufficient to prove that $\Psi$ is continuous. Fix $x \in M$ and $\varepsilon > 0$ arbitrarily. We may suppose $\varepsilon < \dfrac{\pi}{2}$ without loss of generality. Then we choose $\delta(x, \varepsilon) < \varepsilon$ in the condition (3). Now for $u, v \in S_xM$ with $\angle(u, v) < \delta$ we have

$$\angle(u, v) - C(F(\gamma_u, \gamma_v)) < \delta + \varepsilon < 2\varepsilon < \pi.$$

Therefore by Lemma 2.3.1 it holds that

$$\mathrm{Td}(\gamma_u(\infty), \gamma_v(\infty)) = \angle(u, v) - C(F(\gamma_u, \gamma_v)) < 2\varepsilon,$$

which means that $\Psi$ is continuous. ∎

REMARK. Note that $M(\infty)$ is not necessarily compact in the Tits topology even if the diameter of $M(\infty)$ determined by the Tits metric is bounded, as understood from Example 2.2.4.

# Chapter 3. Geometry of total curvature and Tits metric

## §1. Existence of a straight line

In this section, let $M$ be a connected complete noncompact and oriented Riemannian 2-manifold. The aim of this section is to investigate an existence of a straight line via total curvature. As stated in the introduction, it is known that if a connected, complete, non-compact, oriented and finitely connected Riemannian 2-manifold $M$ having one end admits total curvature and if $M$ contains a straight line, then the total curvature of $M$ is not greater than $2\pi(\chi(M)-1)$. We consider the opposite of this and prove the following theorem as the first characterization of noncompact manifolds.

THEOREM A. *Let $M$ be a connected, complete, noncompact, oriented and finitely connected Riemannian 2-manifold having one end. If $M$ admits total curvature which is smaller than $2\pi(\chi(M) - 1)$, then $M$ contains a straight line.*

We will also comment on a straight line in $M$ whose total curvature is greater than or equal to $2\pi(\chi(M) - 1)$ or which has more than one end.

## 1.1 Measure of rays

First we study about a measure of rays for the proof of Theorem A.

For any point $p \in M$, let $A_p$ be the set of all initial vectors of rays emanating from $p$, that is,

$$A_p = \{v \in S_p M \mid \gamma_v \text{ is a ray }\},$$

where $\gamma_v$ denotes the geodesic with initial vector $v$. Since $M$ is complete and noncompact it holds that $A_p \neq \emptyset$ for all $p \in M$.

To measure $A_p$, we consider the Lebesgue measure $\mathfrak{M}$ on $S_p M$, which is induced from the Riemannian metric on $M$ and hence satisfies $\mathfrak{M}(S_p M) = 2\pi$. Since rays converge to also a ray, $A_p$ is a closed subset of $S_p M$ and for a sequence $\{p_j\}$ of points on $M$ converging to $p$, it holds in the unit circle bundle over $M$ that

$$\limsup_{j \to \infty} A_{p_j} \subset A_p.$$

Therefore the function $p \to \mathfrak{M}(A_p)$ is upper semicontinuous and hence locally integrable in the sense of Lebesgue.

DEFINITION 3.1.1. We call $\mathfrak{M}(A_p)$ the *measure of rays* at $p$.

Through the following examples we consider the measure of rays.

EXAMPLE 3.1.1. If $M$ be a Hadamard 2-manifold, then for each point $p$, $A_p = S_p M$ and hence

$$\mathfrak{M}(A_p) = 2\pi.$$

EXAMPLE 3.1.2. Let $M$ be a conical Riemannian plane. Then a flat tube $U = M \backslash K$ is isometrically embedded in a cone with vertex angle $\theta, 0 \leqq \theta < \infty$, for some compact set $K$, where $\theta = 0$ means that $U$ is in a flat cylinder. If $0 \leqq \theta < 2\pi$, then for $p \in U$ far enough from $\partial U$

$$\mathfrak{M}(A_p) = \theta$$

and if $\theta \geqq 2\pi$, then

$$\sup_{p \in M} \mathfrak{M}(A_p) = 2\pi.$$

In fact, if $U$ is in a flat cylinder, then $A_p, p \in U$, consists of only one vector and hence $\mathfrak{M}(A_p) = 0$.

If $U$ is in a cone with vertex angle $0 < \theta < 2\pi$, then we can choose a pair of rays $\alpha$ and $\beta$ in $U$ such that $\beta$ is contained in a cut locus of any point on $\alpha$. Roughly speaking, $\alpha$ and $\beta$ correspond to the meridians, one of which is antipodal to another, on the corresponding flat cone. Reparametrizing $\alpha$, if necessary, we may assume that any minimizing geodesic segment joining $\alpha(0)$ and $\beta(t)$ does not intersect the boundary of $U$ for all $t \geqq 0$. Since there are exactly two minimizing geodesic segments joining $\alpha(0)$ and $\beta(t)$ in $U$, we can obtain two distinct rays $\beta_1$ and $\beta_2$ emanating from $\alpha(0)$ such that they are asymptotic to $\beta$. Let $D$ be the component in $U$ bounded by $\beta_1 \cup \beta_2$ and containing $\alpha$. Then

$$A_{\alpha(0)} = \{v \in S_pM | \gamma_v \subset D\}.$$

Since the inner angle of $D$ at $\alpha(0)$ is $\theta$, we have

$$\mathfrak{M}(A_{\alpha(0)}) = \theta.$$

Last we consider the case that $\theta \geqq 2\pi$. Let

$$K_p = \{v \in S_pM \mid \gamma_v \cap \operatorname{Int}K \neq \emptyset\},$$

where Int$K$ denotes the interior of $K$. For any $\varepsilon > 0$, we can choose a constant $R$ such that $\mathfrak{M}(K_p) < \varepsilon$ for a point $p \in U$ with $d(p, K) > R$. Since $A_p \supset S_pM \backslash K_p$, we have for such a point $p$

$$\mathfrak{M}(A_p) \geqq \mathfrak{M}(S_pM \backslash K_p) > 2\pi - \varepsilon.$$

EXAMPLE 3.1.3. Let $M$ be a paraboloid of revolution and $p$ the vertex of $M$. Then $A_p = S_pM$, while for any point $q \neq p$ in $M$, $A_q$ consists of only one vector tangent to the meridian. Therefore

$$\begin{cases} \mathfrak{M}(A_p) = 2\pi, & \\ \mathfrak{M}(A_q) = 0 & \text{for } q \neq p. \end{cases}$$

Example 3.1.3 implies that the measure $\mathfrak{M}(A_p)$ of rays at $p$ of a surface depends on the choice of $p$. Nevertheless some estimations for the measure of rays were given. The first one, which is due to Maeda [Ma2] [Ma3], states that if $M$ is a Riemannian plane with nonnegative Gaussian curvature, then

$$\inf_{p \in M} \mathfrak{M}(A_p) = 2\pi - C(M).$$

The extended version given by Shiga [Sg2], where the sign of the Gaussian curvature changes, is the following. If $M$ is a Riemannian plane admitting total curvature, then

$$2\pi - C_+(M) \leqq \inf_{p \in M} \mathfrak{M}(A_p) \leqq 2\pi - C(M).$$

Essentially these results come from the fact that we can estimate the measure of the complement of $A_p$ in $S_pM$ by the total curvature of the subset in a Riemannian plane $M$ which does not intersect the rays emanating from $p$. Maeda and Shiga treated only the case of Riemannian planes. However the proof of the estimation of the complement of rays is essentially independent of the topology of $M$. The following Lemma is the estimation of it and used to prove our theorem.

LEMMA 3.1.1 ( cf. [Sh5] ). *Assume that $M$ admits total curvature. Let $p \in M$ have the property that $M \backslash \{\gamma_v(t) | v \in A_p, t \geqq 0\} \neq \emptyset$ and let $D$ be a component of this set. If $M \backslash D$ is homeomorphic to a closed half plane, then*

$$C(D) = 2\pi(\chi(M) - 1) + \angle(u, v),$$

*where $u, v \in A_p$ are tangent to the rays consisting the boundary of $D$ and $\angle(u, v)$ is the angle measured with respect to $D$.*

PROOF. We divide the whole proof into a number of small steps. In the first two steps we shall show the existence of points and minimizing geodesic segments emanating from these points satisfying some angle estimation and in the last step prove this lemma using these points and geodesics.

Step 1 : Let $c$ be a ray and $x$ a point in $M$. Then for any $\varepsilon > 0$ there exists a divergent sequence $\{t_j\}$ such that

$$\angle(c'(t_j), \gamma_j'(l_j)) < \varepsilon$$

for any minimizing geodesic segment $\gamma : [0, l_j] \to M$ from $x$ to $c(t_j)$.

In fact, the function $f(t) := d(x, c(t))$ is Lipschitz continuous with Lipschitz constant 1 and hence it is differentiable almost everywhere. In particular, $f$ is differentiable at $t$ if and only if every minimizing geodesic from $x$ to $c(t)$ makes a constant angle $\theta(t)$ with $c$ and then $f'(t) = \cos\theta(t)$. Hence $t - f(t) = \int_0^t [1 - \cos\theta(u)] du - f(0)$. By the triangle inequality we have $t - f(t) \leqq f(0) < \infty$ for any $t \geqq 0$. Therefore $1 - \cos\theta(u)$ converges to 0 as $u \to \infty$. This implies that there exists a divergent sequence $\{t_j\}$ as required.

Step 2 : From the assumption of the lemma, it follows that $M$ has only one end. Then for any $t > 0$ there exists a curve $b_t : [0, 1] \to \overline{D}$ from $\gamma_u(t)$ to $\gamma_v(t)$ such that $d(p, b_t(s)) \geqq t$ for all $s \in [0, 1]$, where $\overline{D}$ denotes the closure of $D$. Then for any $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that for any $t > t_\varepsilon$ there are a point $x_t \in b_t((0, 1))$ and minimizing geodesic segments $\alpha_t, \beta_t : [0, l] \to \overline{D}$ from $p$ to $x_t$ satisfying

$$\angle(\alpha_t'(0), u) < \frac{\varepsilon}{2} \qquad \text{and} \qquad \angle(\beta_t'(0), v) < \frac{\varepsilon}{2},$$

where $l$ is the distance between $p$ and $x_t$.

In fact, since there are no rays emanating from $p$ on $D$, there is a constant $t_\varepsilon$

such that it holds that either $\angle(u, \tau'(0)) < \dfrac{\varepsilon}{2}$ or $\angle(v, \tau'(0)) < \dfrac{\varepsilon}{2}$ for any $t > t_\varepsilon$, any $s \in (0,1)$ and any minimizing geodesic segment $\tau : [0, l] \to \overline{D}$ from $p$ to $b_t(s)$.

Fix $t > t_\varepsilon$ arbitrary. We may suppose $\varepsilon < \angle(u, v)$. Let $I_0 = \{s \in [0,1] | \angle(u, \tau'(0)) < \dfrac{\varepsilon}{2}$ for any minimizing geodesic segment $\tau$ from $p$ to $b_t(s)\}$ and $I_1 = \{s \in [0,1] | \angle(v, \tau'(0)) < \dfrac{\varepsilon}{2}$ for any minimizing geodesic segment $\tau$ from $p$ to $b_t(s)\}$. Then $I_0$ and $I_1$ are nonempty subsets containing 0 and 1, respectively. If there is an $s_0 \in [0,1] \backslash (I_0 \cup I_1)$, then the point $x_t = b_t(s_0)$ has the desired property. In the case that $[0,1] \backslash (I_0 \cup I_1) = \emptyset$, let $s_0 = \sup_{I_0} s$. Then we can see that the point $x_t = b_t(s_0)$ also has the desired property. Therefore $[0,1] \backslash (I_0 \cup I_1)$ is not empty.

Step 3 : Let $\{\varepsilon_j\}$ be a monotone decreasing sequence converging to 0 and $\{t_j\}$ a divergent sequence satisfying $t_j > t_{\varepsilon_j}$. By Step 2, for any $j$ there exist a point $x_j = b_{t_j}(s_0)$ and two minimizing geodesic segments $\alpha_j, \beta_j : [0, l_j] \to \overline{D}$ from $p$ to $x_j$ such that $\angle(\alpha_t'(0), u) < \dfrac{\varepsilon_j}{2}$ and $\angle(\beta_t'(0), v) < \dfrac{\varepsilon_j}{2}$. Let $D_j$ be a bounded domain whose boundary consists of $\alpha_j$ and $\beta_j$. Taking a subsequence, if necessary, we may suppose that $\{D_j\}$ is a monotone increasing sequence such that $\bigcup_j D_j = D$. Then we have

$$C(D) \geqq \lim_{j \to \infty} C(D_j) = 2\pi(\chi(M) - 1) + \angle(u, v).$$

In fact, let $\theta_j$ be $\angle(\alpha_j'(l_j), \beta_j'(l_j))$ measured with respect to $D_j$. Then, applying the Gauss-Bonnet theorem, we have

$$C(D_j) = 2\pi(\chi(D_j) - 1) + \angle(\alpha_j'(0), \beta_j'(0)) + \theta_j .$$

Note that from the assumption of this lemma it holds that

$$\lim_{j\to\infty} \chi(D_i) = \chi(M).$$

Since $C(D)$ exists, the sequence $\{\theta_j\}$ has its limit and hence

$$\begin{aligned} C(D) &= 2\pi(\chi(M) - 1) + \angle(u, v) + \lim_{j\to\infty} \theta_j \\ &\geqq 2\pi(\chi(M) - 1) + \angle(u, v). \end{aligned}$$

To prove the inverse inequality, we construct another monotone increasing sequence covering over $D$. For any $j$, according to Step 1, we choose numbers $m_j, n_j, \hat{m}_j$ and $\hat{n}_j$ large enough such that it holds that

$$\angle(\lambda_j'(m_j), \gamma_u'(\hat{m}_j)) < \frac{\varepsilon_j}{2} \qquad \text{and} \qquad \angle(\mu_j'(n_j), \gamma_v'(\hat{n}_j)) < \frac{\varepsilon_j}{2}$$

for two minimizing geodesic segments $\lambda_j : [0, m_j] \to \overline{D}$ from $\lambda_j(0) = x_j$ to $\lambda_j(m_j) = \gamma_u(\hat{m}_j)$ and $\mu_j : [0, n_j] \to \overline{D}$ from $\mu_j(0) = x_j$ to $\mu_j(n_j) = \gamma_v(\hat{n}_j)$.

Let $E_j \subset \overline{D}$ be a bounded domain whose boundary consists of $\gamma_u([0, \hat{m}_j])$, $\lambda_j([0, m_j]), \mu_j([0, n_j])$ and $\gamma_v([0, \hat{n}_j])$. It is clear that $D_j \subset E_j$. Taking a subsequence, if necessary, we may suppose that $\{E_j\}$ is a monotone increasing sequence such that $\bigcup_j E_j = D$. Let $\omega_j = \angle(\lambda_j'(0), \mu_j'(0)) \in [0, 2\pi)$ measured with respect to $E_j$. Then applying the Gauss-Bonnet theorem we have

$$C(E_j) \leqq 2\pi\chi(E_j) - 4\pi + \angle(u,v) + \omega_j + \varepsilon_j$$

$$\leqq 2\pi(\chi(E_j) - 1) + \angle(u,v) + \varepsilon_j.$$

Hence

$$C(D) = \lim_{j\to\infty} C(E_j) \leqq 2\pi(\chi(M) - 1) + \angle(u,v). \blacksquare$$

## 1.2 Proof of Theorem A

Now we are in the situation to prove Theorem A.

PROOF OF THEOREM A. First we consider the case that $\int_M G_- d_M < \infty$. We put $\varepsilon = \frac{1}{2}\{2\pi(\chi(M)-1) - C(M)\} > 0$. Then there exists a compact set $K \subset M$ such that

$$\begin{cases} \int_K G_- d_M > \int_M G_- d_M - \varepsilon \\ M \backslash K \text{ is homeomorphic to } S^1 \times [0,\infty), \end{cases}$$

where $S^1$ denotes a unit circle. For an arbitrary point $p$ on $M \backslash K$, we shall show that there exists a ray emanating from $p$ which intersects with the interior of $K$.

Now, we suppose that such a ray does not exist. Let $\Omega$ denote the set of all elements $(u,v) \in A_p \times A_p$. Note that $\Omega$ is nonempty and closed on $S_pM \times S_pM$ since $A_p$ is nonempty and closed. Then there exists the element $(u,v)$ in $\Omega$ satisfying

$$\angle(u,v) \leqq \angle(u',v') \qquad \text{for all } (u',v') \in \Omega,$$

where the angle is measured with respect to the domain containing $K$. It should be noted that if $u = v$, then the angle is understood as $\angle(u,v) = 2\pi$. Let $E$ be a component containing $K$ and bounded by $\gamma_u([0,\infty))$ and $\gamma_v([0,\infty))$. From Lemma 3.1.1, we have

$$C(E) = 2\pi(\chi(M)-1) + \angle(u,v) > 2\pi(\chi(M)-1).$$

On the other hand, we have

$$\int_K G_+ d_M \leqq \int_E G_+ d_M \leqq \int_M G_+ d_M$$

and

$$\int_E G_- d_M \geqq \int_K G_- d_M > \int_M G_- d_M - \varepsilon,$$

where the last inequality holds under the construction of $K$. Hence

$$C(E) < C(M) + \varepsilon < 2\pi(\chi(M) - 1).$$

This is a contradiction. Therefore there exists a ray emanating from $p$ which intersects with the interior of $K$.

Let $\{p_j\}$ be the sequence of points on $M \backslash K$ such that $\{d(p_j, K)\}$ is a monotone divergent sequence. As is shown above, for each $j$ there exists a ray $\gamma_j$ emanating from $p_j$ which intersects with the interior of $K$. Since $K$ is compact there exists a subsequence $\{\gamma_k\}$ of $\{\gamma_j\}$ such that $\gamma_k$ converges to a straight line as $k$ tends to infinity.

Next we consider the case that $\int_M G_- d_M = \infty$. Since $M$ admits total curvature, it holds that $\int_M G_+ d_M < \infty$. We can choose the positive number $\varepsilon$ satisfying $\varepsilon > \int_M G_+ d_M$. Then there exists a compact set $K \subset M$ such that

$$\begin{cases} \int_K G_- d_M > -2\pi(\chi(M) - 1) + \varepsilon \\ M \backslash K \text{ is homeomorphic to } S^1 \times [0, \infty). \end{cases}$$

Similarly to the previous case we can prove the existence of a straight line passing through $K$. Thus the proof of Theorem A is complete. ∎

In the sequel we will give some comments on straight lines in the different situations from the above.

First we note the case where $C(M) > 2\pi(\chi(M)-1)$. Then $M$ has no straight lines. This is understood from the generalized Cohn-Vossen theorem stated that if a connected, complete, non-compact, oriented and finitely connected Riemannian 2-manifold $M$ with only one end admits total curvature and if $M$ contains a straight line, then $C(M) \leqq 2\pi(\chi(M) - 1)$. ( Confer Section 4 in [Sh5]. )

The idea of the proof of this generalized theorem is summarized as follows. Construct a monotone increasing sequence $\{K_j\}$ of compact suitable domains such that $\bigcup K_j = M$, $\chi(K_j) = \chi(M)$ and each component of $\partial K_j$ is a simply closed curve consisting of a broken geodesic. Then, by applying the Gauss-Bonnet theorem to $K_j$ and by estimating the angles of edges of $\partial K_j$, the estimate of $C(M) = \lim_{j\to\infty} C(K_j)$ is obtained.

Similarly as above, it holds that for two components $M_1, M_2$ bounded by a straight line, $C(M_i) \leqq 2\pi(\chi(M_i) - 1)$ for $i = 1, 2$. Note that $\chi(M) = \chi(M_1) + \chi(M_2) - 1$.

When $C(M) = 2\pi(\chi(M) - 1)$, there are both cases that $M$ contains a straight line or not.

The typical example of a surface $M$ whose total curvature equals to $2\pi(\chi(M) - 1)$ and on which straight lines exist is a Euclidean plane.

As the counter example, we construct a $C^2$-surface $M$ homeomorphic to $\mathbf{R}^2$ in $\mathbf{R}^3$ whose total curvature is equal to 0 and on which there are no straight lines. The construction is carried out as follows. Consider the $C^2$-function $f : (-\infty, 1] \to [0, \infty)$ defined by

$$f(x) = \begin{cases} x^4 - \dfrac{x^2}{2} + 1 & \text{for } x \leqq 0 , \\ (1 - x^2)^{\frac{1}{2}} & \text{for } 0 \leqq x \leqq 1. \end{cases}$$

Then $M$ is defined as a surface of revolution around the $x$-axis whose generating line is the graph of $f$ in the (xz)-plane.

In fact, applying Example 2.1.3, it is easy to see that $C(M) = 0$. Next we will see that there are no straight lines on $M$. Let $K = \{(x, y, z) \in M \mid x \geqq -\frac{1}{2}\}$. Since the boundary of $K$ is a closed geodesic, it is obviously that there are no straight lines passing through any point on $K$. Furthermore there are no straight lines on $M \backslash K$ as follows. Suppose that there exists a straight line $\alpha$ on $M \backslash K$. Then $\alpha$ divides $M$ into two components $M_1 \supset K$ and $M_2$. Now from the comment stated above it follows that $C(M_1) \leqq 0$ and $C(M_2) \leqq 0$. In particular, $C(M_2) < 0$ because the Gaussian curvature is negative on $M \backslash K$. Hence $C(M) = C(M_1) + C(M_2) < 0$. This is a contradiction.

Now we note that if $M$ has more than one ends, then there is a straight line in $M$ combining two distinct ends. In fact, let $\varepsilon_1$ and $\varepsilon_2$ be distinct ends. Then there is a compact set $K$ such that $\varepsilon_1(K) \cap \varepsilon_2(K) = \emptyset$. Let $\{x_i^1\}$ ( resp. $\{x_i^2\}$ ) be a divergent sequence in $\varepsilon_1(K)$ ( resp. $\varepsilon_2(K)$ ) and $c_i$ a minimizing geodesic segment from $x_i^1$ to $x_i^2$. Since $c_i$ intersects $K$, there is an accumulation geodesic which is a straight line joining $\varepsilon_1$ and $\varepsilon_2$.

Finally we will note the following fact. As pointed out in Section 2 of

Chapter 2, for a Hadamard manifold $M$, if $\mathrm{Td}(z,w) > \pi$ for $z, w \in M(\infty)$ then there is a geodesic $c$ joining $z$ and $w$.

## §2. Rigidity of products

As the second characterization of noncompact manifolds, we will give a sufficient condition for a projective map to be isometric in terms concerning Tits metric.

Let $M, M^*$ be Hadamard manifolds with $\dim M = \dim M^*$ such that there exists a continuous, bijective and projective map $\Phi : M \to M^*$. By definition, a geodesic preserving map $\Phi$ is called a *projective map*. We will show that $\Phi$ preserves the asymptotic relation on rays and hence induces the map $\widetilde{\Phi}$ between their ideal boundaries, namely, $\widetilde{\Phi} : M(\infty) \to M^*(\infty)$. Then we have the following rigidity theorem.

THEOREM B. *Let $M$ be a nontrivial product manifold, i.e., $M = M_1 \times M_2$. Then a continuous, bijective and projective map $\Phi : M \longrightarrow M^*$ is an isometry up to a normalizing constant if and only if $\widetilde{\Phi} : (M(\infty), \mathrm{Td}) \to (M^*(\infty), \mathrm{Td})$ induced by $\Phi$ is an isometry.*

## 2.1 Isometry between ideal boundaries

Tits metric determines the whole structure of products. That is, the isometry on the ideal boundary of a nontrivial product induces the product structure on the Hadamard manifold whose ideal boundary is the target of this isometry. Here we shall see this fact which is the result of Schroeder in [BGS].

First we will investigate how structure exists on the ideal boundary of a nontrivial product $M = M_1 \times M_2$.

Let $A_i = M_i(\infty)$. Then $M_i(\infty)$ is canonically embedded in $M(\infty)$ and the following holds similarly to Example 2.2.4 :

(i) If $z_i \in A_i, i = 1, 2$, then $\mathrm{Td}(z_1, z_2) = \dfrac{\pi}{2}$.

(ii) If $z \in M(\infty)$, then there are $z_i \in A_i, i = 1, 2$ such that $z$ is contained in the minimal geodesic in $(M(\infty), \mathrm{Td})$ from $z_1$ to $z_2$.

In fact, for any $z_i \in A_i, i = 1, 2$ and $x \in M$, let $\gamma_i$ be maximal geodesics emanating from $x$ with $\gamma_i(\infty) = z_i$, respectively. Then $\gamma_1$ and $\gamma_2$ span a totally geodesic Euclidean plane on $M$ and $\gamma_1'(0)$ and $\gamma_2'(0)$ are orthogonal at $x$. Hence $\mathrm{Td}(z_1, z_2) = \dfrac{\pi}{2}$. For $z \in M(\infty)$, fix a ray $\gamma$ with $\gamma(\infty) = z$. Let $\gamma_i$ be the projection of $\gamma$ on $M_i$-factor. Then we can choose the asymptotic class of the ray $\gamma_i$ as $z_i \in A_i$ satisfying the condition (ii).

Conversely these conditions characterize the product of a manifold.

LEMMA 3.2.1 ( Lemma C in Appendix 4 in [BGS] ). *Let $M$ be a Hadamard* *manifold and let* $A_1, A_2 \subset M(\infty)$ *be subsets which satisfy (i),(ii). Then* $M$ *is a product* $M_1 \times M_2$ *with* $M_i(\infty) = A_i$.

Before the proof of this lemma, we prepare the following lemmas.

LEMMA 3.2.2 ( cf. Section 4.2 in [BGS] ). *If there is a point* $x \in M$ *such that* $\angle(z, w) = \angle_x(z, w) < \pi$ *for some* $z, w \in M(\infty)$, *then the geodesics* $c_z, c_w$ *from* $x$ *to* $z, w$ *bound a totally geodesic Euclidean sector.*

PROOF. Let $F$ be a component consisting of the rays $\gamma_t$ asymptotic to $c_z$ emanating from $c_w(t)$ for all $t \geqq 0$. By Lemma 1.2.2, $c_z, c_w, \gamma_t$ bound a totally geodesic Euclidean strip $F_t$ for all $t > 0$. Since $F_t$ is totally geodesic, $F_t$ is in $F$ and hence this lemma is obtained. ∎

LEMMA 3.2.3. *Let* $c$ *be a ray and* $f_c$ *the Busemann function for* $c$. *Then* $-gradf_c$ *is the initial vector of the ray asymptotic to* $c$.

PROOF. First we see the fact in [Eb2] that if $\alpha$ is asymptotic to $c$, then it holds that

$$f_c - f_\alpha \equiv f_c(\alpha(0)).$$

We put $p = c(0), q = \alpha(0) \in M$ and $x = c(\infty) = \alpha(\infty) \in M(\infty)$. Let $\{x_n\}$ be a sequence on $M$ converging to $x$ with respect to the cone topology of $\overline{M}$. Then for any point $r \in M$, it holds that $f_c(r) = \lim_{n\to\infty}\{d(r, x_n) - d(p, x_n)\}$ and $f_\alpha(r) = \lim_{n\to\infty}\{d(r, x_n) - d(q, x_n)\}$. Hence

$$f_c(r) - f_\alpha(r) = \lim_{n\to\infty}\{d(q, x_n) - d(p, x_n)\} = f_c(p).$$

For any $p \in M$ there is the unique ray $\alpha$ emanating from $p$ and asymptotic to $c$. Since $f_\alpha(\alpha(t)) = -t$, we have $f_c(\alpha(t)) = -t + f_c(\alpha(0))$. Hence $(f_c \circ \alpha)'(0) = \, < (\text{grad} f_c)_p, \alpha'(0) > = -1$. Because Busemann functions are Lipschitz continuous with constant 1, $\|\text{grad} f_c\| \leqq 1$. Therefore $-\text{grad} f_c = \alpha'(0)$. ∎

REMARK. Let $V$ be the vector field consisting of the initial vectors of rays asymptotic to a ray $c$. We proved this lemma simplicity assuming the differentiability of $f_c$, but the method of the proof of differentiability of the Busemann function $f_c$ in [EO]( to be $C^1$ ) and [HI], is to show that $(f_c \circ \gamma)'(0) = < V, \gamma'(0) >$ for any geodesic $\gamma$ by careful estimation.

PROOF OF LEMMA 3.2.1. We define $A_i(x) := \{v \in S_x(M) | c_v(\infty) \in A_i\}$, where $c_v$ is the geodesic with initial vector $v$. We note that the set $A_i(x)$ are invariant under reflection. In fact, if $c_v(\infty) \in A_i$ then $\text{Td}(c_v(-\infty), c_v(\infty)) \geqq \pi$. Now the conditions (i),(ii) imply that the points $z$ in $M(\infty)$ with $\text{Td}(z, c_v(\infty)) \geqq \pi$ are points in $A_i$. Therefore $c_v(-\infty) = c_{-v}(\infty) \in A_i$ and hence $-v \in A_i(x)$.

First we show that if $v_i \in A_i(x), i = 1, 2$, then $\angle(v_1, v_2) = \dfrac{\pi}{2}$ and the geodesics $c_{v_1}, c_{v_2}$ span a totally geodesic Euclidean plane. Because $\angle(v_1, v_2) \leqq \text{Td}(c_{v_1}(\infty), c_{v_2}(\infty)) = \dfrac{\pi}{2}$ and $\angle(-v_1, v_2) \leqq \dfrac{\pi}{2}$ we conclude $\angle(v_1, v_2) = \dfrac{\pi}{2}$ and Lemma 3.2.2 implies that $c_{v_1}([0, \infty))$ and $c_{v_2}([0, \infty))$ span a flat sector. By the same argument we see that three other sectors spanned by $c_{v_1}, c_{v_2}$ are flat and together span a flat plane.

Using the condition (ii) one easily sees that $A_1(x)$ and $A_2(x)$ are orthogonal great spheres in $S_xM$ with $\dim A_1(x) + \dim A_2(x) = n-2$. Let $D_1(x)$ and $D_2(x)$ be the subspaces of $T_xM$ spanned by $A_1(x)$ and $A_2(x)$, respectively. Then $D_1(x)$ and $D_2(x)$ are orthogonal subspaces which together span $T_xM$.

Next we show that $D_1$ and $D_2$ define integrable distributions and the integral manifolds are totally geodesic. For $x \in M$ let $v_1, \cdots, v_s$ be an orthonormal basis of $D_1(x)$ and $u_1, \cdots, u_k$ an orthonormal basis of $D_2(x)$. Let $z_i := c_{v_i}(\infty), w_\alpha := c_{u_\alpha}(\infty)$ and $b_i = f_{z_i}, g_\alpha = f_{w_\alpha}$ be the corresponding Busemann functions. Then by Lemma 3.2.3, $v_i = -\mathrm{grad} b_i(x)$ and $u_\alpha = -\mathrm{grad} g_\alpha(x)$. Let $V_i = -\mathrm{grad} b_i$ and $W_\alpha = -\mathrm{grad} g_\alpha$. Then $V_i$ and $W_\alpha$ are differentiable and $V_i(y) \in D_1(y)$, $W_\alpha(y) \in D_2(y)$ for any $y \in M$. Furthermore $V_1, \cdots, V_s$ span $D_1$ and $W_1, \cdots, W_k$ span $D_2$ for any point in $M$. Note that $V_j \perp W_\alpha$ by the former assertion and hence $< \nabla_{V_i} V_j, W_\alpha >= - < V_j, \nabla_{V_i} W_\alpha >$. Because at every point $y$, $V_i(y), W_\alpha(y)$ are tangent to a totally geodesic plane $P$ by the former assertion and in this plane $V_i$ and $W_\alpha$ are parallel vector fields, we have $\nabla_{V_i} W_\alpha = 0$. Therefore $\nabla_{V_i} V_j$ is normal to $D_2$ and hence in $D_1$. By the same argument $\nabla_{W_\alpha} W_\beta \in D_2$. This implies that $D_1$ and $D_2$ are integrable and the integral manifolds are totally geodesic.

Therefore for $x \in M$, the sets $M_i(x) = \exp_x(D_i(x))$ are complete totally geodesic submanifolds of $M$, where exp is the exponential map of $M$.

Because $M_i(x)$ and $M_i(y)$ are convex subsets of $M$, $f = d(M_i(x), \cdot)$ is a convex function on $M_i(y)$. One easily seen that $f$ is constant on $M_i(y)$, because $M_i(x)(\infty) = M_i(y)(\infty)$. Similarly $d(M_i(y), \cdot)$ is constant on $M_i(x)$. Hence the

Hausdorff distance is finite.

Using the argument in the proof of Lemma 1.2.4, it is easy to see that $M$ is the product $M_1 \times M_2$ where $M_i$ is isometric to $M_i(x)$ for any $x \in M$. ∎

Note that if $A_1$ consists only two points, then this lemma implies that $M$ splits into $\mathbf{R} \times M'$. Here $M'$ may have the nontrivial Euclidean factor. Then we have the following.

COROLLARY 3.2.4 ( cf. Appendix 4 in [BGS] ). *If* $(M(\infty), \mathrm{Td})$ *is isometric to the standard sphere* $S^{n-1}$, *then* $M$ *is isometric to* $\mathbf{R}^n$.

PROOF. Let $A_1 = \{N, S\}$ be the set of points in $M(\infty)$ corresponding to the north pole and the south pole in the standard sphere and $A_2$ the points corresponding to the equator. Then, applying Lemma 3.2.1, $M$ splits into $\mathbf{R} \times M'$. Note here that $M'(\infty)$ is isometric to $S^{n-2}$. Hence we can use this splitting inductively and obtain the result. ∎

## 2.2 Proof of Theorem B

Before the proof of Theorem B we see that $\Phi : M \to M^*$ induces $\widetilde{\Phi} : M(\infty) \to M^*(\infty)$

LEMMA 3.2.5. *The function $\Phi$ preserves the asymptotic relation.*

PROOF. If $\gamma_1$ is asymptotic to $\gamma_2$, then for a divergent sequence $\{t_i\}(i \in \mathrm{N})$ the sequence of geodesic segments $\{\mu_i : [0, a_i] \to M\}$ joining $\mu_i(0) = \gamma_1(0)$ and $\mu_i(a_i) = \gamma_2(t_i)$, where $a_i = d(\gamma_1(0), \gamma_2(t_i))$, converges to $\gamma_1$. By continuity of $\Phi$, the sequence of geodesic segments $\{\Phi(\mu_i)\}$ joining $\Phi(\gamma_1(0))$ and $\Phi(\gamma_2(t_i))$ converges to the ray $\Phi(\gamma_1)$, that is, $\Phi(\gamma_1)$ is asymptotic to $\Phi(\gamma_2)$. ∎

Therefore $\Phi$ induces $\widetilde{\Phi} : M(\infty) \to M^*(\infty)$ defined by $\widetilde{\Phi}(\alpha) = \Phi(\gamma)(\infty)$ for every $\alpha \in M(\infty)$, where $\gamma$ is a ray contained in $\alpha$.

Now we prove Theorem B which states that if $\widetilde{\Phi}$ is an isometry, then $\Phi$ is an isometry up to a normalizing constant.

PROOF OF THEOREM B. Since $\widetilde{\Phi}$ is an isometry, applying Lemma 3.2.1 we can obtain that $M^*$ is also a product manifold, precisely $M^*$ can be denoted $M^* = M_1^* \times M_2^*$ such that $M_i^*(\infty) = \widetilde{\Phi}(M_i(\infty))$ for $i = 1, 2$. For $x \in M$ and $\alpha_i \in M_i(\infty)$ $(i = 1, 2)$, let $E = E(x, \alpha_1, \alpha_2)$ ( resp. $E^* = E^*(x^*, \alpha_1^*, \alpha_2^*)$ ) be the totally geodesic Euclidean plane spanned by two unit speed geodesics $\gamma_i$ on $M$ ( resp. $\gamma_i^*$ on $M^*$ ) such that $\gamma_i(0) = x, \gamma_i(\infty) = \alpha_i$ ( resp. $\gamma_i^*(0) = \Phi(x) = x^*, \gamma_i^*(\infty) = \widetilde{\Phi}(\alpha_i) = \alpha_i^*$ ).

Now we divide the whole proof into a number of small steps.

Step 1 : If $y \in E$, then $\Phi(y) \in E^*$.

For any $y \in E$, let $\gamma$ be a geodesic asymptotic to $\gamma_1$ through $y$. The intersection of $\gamma$ and $\gamma_2$ is denoted by $y'$. Then $\Phi(\gamma)$ is a geodesic asymptotic to $\gamma_1^* = \Phi(\gamma_1)$ through $\Phi(y') \in E^*$. Hence $\Phi(\gamma)$ lies on $E^*$, namely, $\Phi(y) \in E^*$.

From Step 1, $\Phi|_E : E \to E^*$ is well-defined and it is clear that $\Phi|_E$ is bijective.

Step 2 : $\Phi|_E$ is an isometry up to a normalizing constant.

We denote a point $y = (\gamma_1(t_1), \gamma_2(t_2)) \in M_1 \times M_2$ on $E$ by $(t_1, t_2)$ in brief. For example, $x \in E$ is denoted by $(0,0)$. We also denote $y^* = (\gamma_1^*(t_1), \gamma_2^*(t_2)) \in M_1^* \times M_2^*$ on $E^*$ by $(t_1, t_2)$. Note that for any two points $y = (s_1, s_2), z = (t_1, t_2)$ on $E$ ( or $E^*$ ), $d(y,z) = \{(s_1 - t_1)^2 + (s_2 - t_2)^2\}^{\frac{1}{2}}$.

Now, since $\Phi(\gamma_1) = \gamma_1^*$, there is a positive number $r$ such that $\Phi(1,0) = (r,0)$. Then we have $\Phi(t,0) = (rt,0)$ and $\Phi(0,t) = (0,rt)$ for every $t \in \mathbf{R}$. In fact, let $\sigma_1$ be a geodesic on $E$ defined by $\sigma_1(t) = (t,t)$ and for any $t \in \mathbf{R}$ given, let $\sigma_2, \sigma_3, \sigma_4$ and $\sigma_5$ be geodesics asymptotic to $\sigma_1, \gamma_1, \gamma_2$ and $\gamma_2$ through $x_1 = (t,0), x_3 = (0,t), x_1$ and $x_2 = (2t,0)$, respectively. Furthermore $y_1$ and $y_2$ denote the intersections $\sigma_1 \cap \sigma_3 \cap \sigma_4$ and $\sigma_2 \cap \sigma_3 \cap \sigma_5$ respectively, namely, $y_1 = (t,t)$ and $y_2 = (2t,t)$. Then it is immediately seen that

$$d(\Phi(x), \Phi(x_1)) = d(\Phi(y_1), \Phi(y_2)) = d(\Phi(x_1), \Phi(x_2)),$$

since $\Phi$ preserves the asymptotic relation. Therefore we have $\Phi(x_2) = 2\Phi(x_1)$.

By the repetition of the same argument we see

(i) $$\Phi(nt,0) = n\Phi(t,0) \qquad \text{for every integer } n\ .$$

In particular, we have for $t = 1$

(ii) $$\Phi(n,0) = n\Phi(1,0) = (nr,0) \qquad \text{for every integer } n\ .$$

Hence for any rational number $a = \dfrac{q}{p}$ ( $p, q$ : integers ), we have

$$\begin{aligned}\Phi(a,0) &= \Phi\left(\frac{q}{p},0\right) = \frac{1}{p}\Phi(q,0)\\ &= \frac{1}{p}(qr,0) = \left(\frac{q}{p}r,0\right) = (ar,0),\end{aligned}$$

where the second equality is obtained from (i) in case that $n = p$ and $t = \dfrac{q}{p}$ and the third equality from (ii). Thus the first equation of our claim is clear by continuity. On the other hand, $\sigma_1^* := \Phi(\sigma_1)$ is a geodesic on $E^*$ which can be parametrized by $\sigma_1^*(t) = (t,t) \in E^*$, since $\widetilde{\Phi}$ is isometry. Furthermore $\sigma_4^* := \Phi(\sigma_4)$ is a geodesic asymptotic to $\gamma_2^*$ through $(r,0)$. Therefore the intersection $\Phi(y_1)$ of $\sigma_1^*$ and $\sigma_4^*$ is $(r,r)$. Hence the intersection $\Phi(x_3)$ of $\gamma_2^*$ and $\sigma_3^*$ is $(0,r)$, where $\sigma_3^* := \Phi(\sigma_3)$ is a geodesic asymptotic to $\gamma_1^*$ through $\Phi(y_1)$. Then similarly as we obtained the first equation, we have the second equation.

Using these two equations, we ultimately obtain $\Phi(y) = (t_1 r, t_2 r)$ for every point $y = (t_1,t_2) \in E$. In fact, let $\sigma_1$ and $\sigma_2$ be geodesics asymptotic to $\gamma_1$ and $\gamma_2$ through $(0,t_2)$ and $(t_1,0)$, respectively. Then $y$ is the intersection of $\sigma_1$ and $\sigma_2$. Hence $\Phi(y)$ is the intersection of $\sigma_1^*$ and $\sigma_2^*$, where $\sigma_i^* := \Phi(\sigma_i)$ $(i = 1,2)$.

Since $\sigma_1^*$ and $\sigma_2^*$ are asymptotic to $\gamma_1^*$ and $\gamma_2^*$ through $(0, t_2 r)$ and $(t_1 r, 0)$, we have $\Phi(y) = (t_1 r, t_2 r)$. After all, for any two points $y, z \in E$, it holds that $d(\Phi(y), \Phi(z)) = rd(y, z)$.

Since this normalizing constant $r$ may depend on the choice of a flat plane $E$, we denote this constant on $E$ by $r(E)$.

Step 3 : $r(E)$ does not depend on the choice of a flat plane $E$.

Fix a flat $E' = E(x', \alpha_1', \alpha_2')$ arbitrarily. Let $\gamma$ be a ray on $M$ emanating from $x$ through $x'$. Then there are two points $\alpha_i'' \in M_i(\infty)$ $(i = 1, 2)$ such that $\mathrm{Td}(\alpha_1'', \gamma(\infty)) + \mathrm{Td}(\gamma(\infty), \alpha_2'') = \frac{\pi}{2}$. We now consider the sequence of flats ; $E_1 := E(x, \alpha_1, \alpha_2) = E$, $E_2 := E(x, \alpha_1, \alpha_2'')$, $E_3 := E(x, \alpha_1'', \alpha_2'') = E(x', \alpha_1'', \alpha_2'')$, $E_4 := E(x', \alpha_1', \alpha_2'')$, $E_5 := E(x', \alpha_1', \alpha_2') = E'$. Since $\dim(E_i \cap E_{i+1}) \geqq 1$ for $i = 1, 2, 3, 4$, we have $r(E_i) = r(E_{i+1})$ for $i = 1, 2, 3, 4$. Hence it holds that $r = r(E) = r(E')$.

Finally we can conclude the proof as follows. For any two points $x_1, x_2 \in M$, let $\gamma$ be a ray emanating from $x_1$ through $x_2$. Then there are two points $\alpha_i \in M_i(\infty)$ $(i = 1, 2)$ such that $\mathrm{Td}(\alpha_1, \gamma(\infty)) + \mathrm{Td}(\gamma(\infty), \alpha_2) = \frac{\pi}{2}$. Since $\gamma$ lies on a flat $E(x_1, \alpha_1, \alpha_2)$, the argument in the above three steps shows $d(\Phi(x_1), \Phi(x_2)) = rd(x_1, x_2)$, that is, $\Phi$ is an isometry up to a normalizing constant $r$. ∎

REMARK. It is essential for $M$ to be a product manifold in our proof, but the author does not know any example that a projective map $\Phi$ on a Hadamard manifold which induces an isometry on the ideal boundary is not isometric.

## §3. Rigidity of compact ideal boundaries

The concept of ideal boundary is defined not only for Hadamard manifolds, but also for manifolds of asymptotically nonnegatively curvature or for open 2-manifolds admitting total curvature, as stated in the Introduction.

Recently, Kubo [Ku] proved that given two connected complete oriented and noncompact Riemannian 2-manifolds with finite total curvature, if there is a Hausdorff approximation between them, then their ideal boundaries are isometric. This means that if ideal boundaries are not isometric, then there is no Hausdorff approximation between their underlying open surfaces.

In this section, as the third characterization, we shall study the same rigidity problem on ideal boundaries for Hadamard manifolds using Hausdorff convergence, and prove the following theorem. We refer to the precise definition of a Hausdorff approximation and to the concept of Hausdorff convergence in the next small section.

THEOREM C. *Let $M$ and $N$ be Hadamard manifolds with ideal boundaries $M(\infty)$ and $N(\infty)$ respectively, which are assumed to be compact with respect to the Tits-topology. If there exists a Hausdorff approximation from $M$ to $N$, then $(M(\infty), \mathrm{Td})$ is isometric to $(N(\infty), \mathrm{Td})$.*

In this thesis we study on Hadamard manifolds, but this property is valid also for manifolds of asymptotically nonnegative curvature and is able to prove in a similar fashion. So we refer to this case. Namely the following theorem holds:

THEOREM D. *Let $M$ and $N$ be manifolds of asymptotically nonnegative curvature with ideal boundaries $M(\infty)$ and $N(\infty)$ respectively. If there exists a Hausdorff approximation from $M$ to $N$, then $(M(\infty), \mathrm{Td})$ is isometric to $(N(\infty), \mathrm{Td})$.*

It should be noted that the ideal boundaries treated in both Kubo's theorem and our Theorems C,D are compact with respect to the Tits-topology. The result seems to remain true even in the case when ideal boundaries are noncompact, but we shall need another approach to prove it.

### 3.1 Hausdorff convergence

First we shall introduce Hausdorff convergence following [Fu]. The definition in [Fu] is slightly different from the original one introduced by Gromov in [GLP]. However it is more tractable in our discussion.

Let $\mathfrak{MET}$ denote the set of all isometry classes of metric spaces. For any isometry class $X \in \mathfrak{MET}$, we denote a representative metric space of $X$ also by the same symbol $X$. For $X, Y \in \mathfrak{MET}$, a ( not necessary continuous ) map $\phi : X \longrightarrow Y$ is said to be a $\Delta$-*Hausdorff approximation* if $\phi$ satisfies the following conditions:

(1) The $\Delta$-neighborhood $B_\Delta(\phi(X)) = \{x \in Y |\ d(x, \phi(X)) < \Delta\}$ of $\phi(X)$ in $Y$ is equal to $Y$.

(2) For any points $x, y \in X$, we have

$$|d_X(x,y) - d_Y(\phi(x), \phi(y))| < \Delta.$$

The *Hausdorff distance* $d_H(X,Y)$ between $X$ and $Y$ is defined to be the infinimum of the positive numbers $\Delta$ such that there exist $\Delta$-Hausdorff approximations from $X$ to $Y$ and from $Y$ to $X$.

We should note that it is neither a metric nor a pseudometric. To be more precise, it holds that $d_H(X,X) = 0$, but $d_H(X,Y) = 0$ does not imply in general that $X$ is isometric to $Y$. For example, let $X = [0,1]$ and $Y = \mathbf{Q} \cap [0,1]$. Then $d_H(X,Y) = 0$, but $X$ is not isometric to $Y$. Furthermore, though $d_H$ satisfies a symmetric law, it does not satisfy the triangle inequality. Nevertheless it holds that

$$d_H(X, Z) \leqq 2\{d_H(X, Y) + d_H(Y, Z)\}$$

for $X, Y, Z \in \mathfrak{MET}$, and hence $d_H$ defines a metrizable uniform structure on $\mathfrak{MET}$. In fact, if $\phi_1 : X \to Y$ is a $\Delta_1$-Hausdorff approximation and if $\phi_2 : Y \to Z$ is a $\Delta_2$-Hausdorff approximation, then we have

$$|d_Z(\phi_2 \circ \phi_1(x), \phi_2 \circ \phi_1(x')) - d_X(x, x')| < \Delta_1 + \Delta_2 \qquad \text{for any } x, x' \in X,$$

$$Z = B_{\Delta_2}(\phi_2(Y)) = B_{\Delta_2}(\phi_2(B_{\Delta_1}(\phi_1(X)))) = B_{2\Delta_2+\Delta_1}(\phi_2 \circ \phi_1(X)),$$

and hence $\phi_2 \circ \phi_1 : X \to Z$ is a $(\Delta_1 + 2\Delta_2)$-Hausdorff approximation. Therefore $d_H(X, Z) \leqq \max\{d_H(X, Y) + 2d_H(Y, Z), 2d_H(X, Y) + d_H(Y, Z)\} \leqq 2\{d_H(X, Y) + d_H(Y, Z)\}$.

Now, let $\mathfrak{CMET}$ denote the set of all isometric classes of "compact" metric spaces. Then the following theorem holds.

THEOREM 3.3.1. ( Theorem 1.5 in [Fu] or Proposition 3.6 in [GLP] ) *$\mathfrak{CMET}$ is Hausdorff and complete.*

This means that if $d_H(X, Y) = 0$ for $X, Y \in \mathfrak{CMET}$, then $X$ is isometric to $Y$. Hence, noting the metrizable uniform structure on $\mathfrak{MET}$, we may treat $d_H$ as if it is a distance function. Also, Theorem 3.3.1 implies that for a Cauchy sequence $\{X_n\}$ in $\mathfrak{CMET}$, there exists the limit $X \in \mathfrak{CMET}$ such that $\lim_{n\to\infty} d_H(X_n, X) = 0$, which is denoted by $\lim_{n\to\infty} X_n$.

Finally we shall consider the noncompact case. In this case, we need to study in the category of pointed and locally compact metric spaces.

We denote by $\mathfrak{MET}_o$ the set of all isometry classes of pointed metric spaces $(X, p)$ with a base point $p$ such that the closure $\overline{B}_R(p, X)$ of $R$-neighborhood of $p$ in $X$ is compact for every $R > 0$. Let $(X, p), (Y, q) \in \mathfrak{MET}_o$ and $\phi : X \to Y$ be a pointed map, namely $\phi(p) = q$. We say that $\phi$ is a $\Delta$-*pointed Hausdorff approximation* if $\phi(\overline{B}_{\frac{1}{\Delta}}(p, X)) \subset \overline{B}_{\frac{1}{\Delta}}(q, Y)$ and if the restriction of $\phi$ on $\overline{B}_{\frac{1}{\Delta}}(p, X)$ into $\overline{B}_{\frac{1}{\Delta}}(q, Y)$ is a $\Delta$-Hausdorff approximation. Then the *pointed Hausdorff distance* $d_{p,H}((X, p), (Y, q))$ is also defined to be the infimum of the numbers $\Delta$ such that there exist $\Delta$-pointed Hausdorff approximations from $(X, p)$ to $(Y, q)$ and from $(Y, q)$ to $(X, p)$.

The counterpart of Theorem 3.3.1 also holds and we write $(X,p) = \lim_{n\to\infty}(X_n, p_n)$ if it satisfies that $\lim_{n\to\infty} d_{p,H}((X_n, p_n), (X, p)) = 0$. We remark that the limit space depends on the choice of base points.

### 3.2 Proof of Theorem C

On a Hadamard manifold $M$, if its ideal boundary $(M(\infty), \mathrm{Td})$ is compact , then there exists the tangent cone of $M$ at infinity, that is, the pointed Hausdorff limit of pointed spaces $((M, \frac{1}{t}d_M), p)$ for $t \to \infty$, and it is isometric to the cone of $M(\infty)$. We shall prove this and make use of it in the proof of Theorem C.

We here recall the definition of the cone $(\mathfrak{C}(M(\infty)), o)$ of $M(\infty)$ with vertex $o$. For a pair of points $(s, w), (t, z) \in [0, \infty) \times M(\infty)$, we set

$$\delta((s, w), (t, z)) := \sqrt{s^2 + t^2 - 2st\cos(\widetilde{\mathrm{Td}}(w, z))},$$

where $\widetilde{\mathrm{Td}}(w, z) := \min\{\pi, \mathrm{Td}(w, z)\}$. Using this function, we define an equivalence relation as follows :

$$(s, w) \sim (t, z) \Longleftrightarrow \delta((s, w), (t, z)) = 0.$$

Then it is clear that $\delta$ is a distance function on the quotient space $\{[0, \infty) \times M(\infty)\}/\sim$. This metric space $\left(\{[0, \infty) \times M(\infty)\}/\sim,\ \delta\ \right)$ is called the cone of $M(\infty)$ and denoted by $\mathfrak{C}(M(\infty))$, We call the equivalence class $[(0, z)](z \in M(\infty))$ the vertex of $\mathfrak{C}(M(\infty))$.

Then the following holds:

PROPOSITION 3.3.2. *If the ideal boundary is compact ( with respect to the Tits-topology ), then the sequence of the pointed metric spaces $((M, d_t^M), p)$ for* *any fixed point p on M, converges to the cone* $(\mathfrak{C}(M(\infty)), o)$ *of* $M(\infty)$ *with vertex* $o$ *in the sense of pointed Hausdorff distance, where* $d_t^M = \frac{1}{t} d_M$ ;

$$\lim_{t \to \infty} ((M, d_t^M), p) = (\mathfrak{C}(M(\infty)), o).$$

PROOF. Let $R$ be an arbitrary large number. In a natural way, we can identify the closed geodesic ball $\overline{B}_R^t(p)$ around $p$ with radius $R$ in $M_t := (M, d_t^M)$, with the closed disk $\overline{B}_R = \{v \in T_pM |\ \|v\| \leqq R\}$ and also the closed ball $\overline{B}_R(o)$ in $\mathfrak{C}(M(\infty))$ with it ;

$$T_pM \supset \overline{B}_R \ni v \longleftrightarrow \gamma_v(t) \in \overline{B}_R^t(p) \subset M_t$$

$$T_pM \supset \overline{B}_R \ni v \longleftrightarrow [(\|v\|, \gamma_{\frac{v}{\|v\|}}(\infty))] \in \overline{B}_R(o) \subset \mathfrak{C}(M(\infty)).$$

The induced metric on $\overline{B}_R$ from $(\overline{B}_R^t(p), d_t^M)$ or $(\overline{B}_R(o), \delta)$ through this identification is denoted by the same symbol $d_t^M$ or $\delta$, respectively.

Now we prove that a sequence $\{d_t^M\}$ of functions on $\overline{B}_R \times \overline{B}_R$ converges uniformly to the function $\delta$ on $\overline{B}_R \times \overline{B}_R$, where $\overline{B}_R$ is equipped a standard metric. Note that it is known in [BGS] that a sequence $\{d_t^M\}$ converges to the function $\delta$.

Since $\overline{B}_R$ is homeomorphic to $(\overline{B}_R^t(p), d_t^M)$, $d_t^M$ is a continuous function on $\overline{B}_R \times \overline{B}_R$. On the other hand, it is proved in Theorem 2 in the previous Chapter that $(M(\infty), \mathrm{Td})$ is compact if and only if the unit tangent sphere is homeomorphic to $(M(\infty), \mathrm{Td})$. Therefore $\overline{B}_R$ is homeomorphic to $\overline{B}_R(o)$. Hence $\delta$ is also continuous on $\overline{B}_R \times \overline{B}_R$. Since the sequence $\{d_t^M\}$ of monotone non-decreasing continuous functions converges to a continuous function $\delta$ on the compact set $\overline{B}_R \times \overline{B}_R$, the convergence is uniform.

This means that

$$\varepsilon_R(t) := \max_{(u,v)\in\overline{B}_R\times\overline{B}_R} \left|\delta(u,v) - d_t^M(u,v)\right|$$

converges to 0 as t tends to $\infty$.

Now for any $\varepsilon > 0$, let $R = \dfrac{1}{\varepsilon}$. Then there is a number $t_\varepsilon$ such that $\varepsilon_R(t) < \varepsilon$ for all $t > t_\varepsilon$. Since the map

$$\mathbf{\Phi}_t : (M, d_t^M) \to (\mathfrak{C}(M(\infty)), \delta) : \mathbf{\Phi}_t(x) = [(d_t^M(p,x), \gamma_{px}(\infty))],$$

where $\gamma_{px}$ denotes the geodesic emanating from $p$ through $x$, is an $\varepsilon$-Hausdorff approximation for $t > t_\varepsilon$, this completes the proof. ∎

REMARK. We can also see from the proof that if $(M(\infty), \mathrm{Td})$ is noncompact, then a sequence $\{((M, d_t^M), p)\}$ of pointed metric spaces does not converge in the sense of pointed Hausdorff distance.

In fact, if the sequence converges in this sense, then the sequence $\{d_t^M\}$ of the continuous functions on $\overline{B}_R \times \overline{B}_R$ converges uniformly to $\delta$. Hence $\delta$ is also continuous on $\overline{B}_R \times \overline{B}_R$. This means that $(M(\infty), \mathrm{Td})$ is homeomorphic to a standard sphere, and hence compact.

PROOF OF THEOREM C. Now we shall recall the assumption of Theorem C. We assume that a $\Delta$-Hausdorff approximation $\phi : M \to N$ is given. Let $p$ be any fixed point of $M$ and $q := \phi(p) \in N$.

From the above proposition, there is a sequence of $\varepsilon(t)$-pointed Hausdorff approximation $\Phi_t : ((N, d_t^N), q) \to (\mathfrak{C}(N(\infty)), o)$ such that $\varepsilon(t) \to 0$ as $t \to \infty$. If we regard $\phi$ as a map from $(M, d_t^M)$ to $(N, d_t^N)$, then $\phi$ is a $\frac{\Delta}{t}$-Hausdorff approximation and the composite $\Psi_t := \Phi_t \circ \phi : ((M, d_t^M), p) \to (\mathfrak{C}(N(\infty)), o)$ is a $\left(\frac{\Delta}{t} + 2\varepsilon(t)\right)$-pointed Hausdorff approximation. Since $\left(\frac{\Delta}{t} + 2\varepsilon(t)\right) \to 0$ as $t \to \infty$, it holds that

$$\lim_{t \to \infty} ((M, d_t^M), p) = (\mathfrak{C}(N(\infty)), o).$$

On the other hand, the left side of this equality coincides with $(\mathfrak{C}(M(\infty)), o)$, and hence $(\mathfrak{C}(M(\infty), o)$ is isometric to $(\mathfrak{C}(N(\infty)), o)$. Since $(M(\infty), \mathrm{Td})$ is isometric to the metric sphere in $\mathfrak{C}(M(\infty))$ around a vertex $o$ of radius 1 with the interior distance induced from the restriction of $\delta$, we can conclude that $(M(\infty), \mathrm{Td})$ is isometric to $(N(\infty), \mathrm{Td})$. ∎

To conclude this section, we shall give an example of Hadamard manifolds such that their ideal boundaries are isometric but there is no Hausdorff approximation between them.

EXAMPLE. Let $M$ be a Hadamard 2-manifold equipped with a metric given as $ds^2 = dr^2 + f(r)^2 d\theta^2$, where $(r, \theta)$ is a polar coordinate of $M$ with origin $o_M$ and $f : [0, \infty) \to [0, \infty)$ is a smooth function satisfying

$$\begin{cases} f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0 \\ f''(0) \geqq 0 \quad (\text{ for any } t \geqq 0 \text{ }) \\ f'(t) \equiv 2 \quad (\text{ for } t \geqq 2 \text{ }). \end{cases}$$

Let $N$ be a Hadamard 2-manifold with a metric $ds^2 = dr^2 + g(r)^2 d\theta^2$, where $g$ satisfies

$$\begin{cases} g(0) = 0, \quad g'(0) = 1, \quad g''(0) = 0 \\ g''(0) \geqq 0 \quad (\text{ for any } t \geqq 0 \text{ }) \\ g'(t) = 2 - \dfrac{1}{t} \quad (\text{ for } t \geqq 2 \text{ }). \end{cases}$$

Then the difference of the girths of the geodesic spheres of $M$ and $N$ with center $o_M, o_N$, respectively, and radius $t$, equals to $2\pi(f(t) - g(t))$. This is $\log(\frac{t}{2}) + f(2) - g(2)$ for $t \geqq 2$, and goes to infinity as $t \to \infty$. Hence there is no Hausdorff approximation between them.

### 3.3 The case of manifolds of asymptotically nonnegative curvature

Now we shall introduce the definitions and basic properties concerning the manifolds we shall study, that is, manifolds of asymptotically nonnegative curvature. For details, we refer to [Ks].

We call $M$ a manifold of asymptotically nonnegative curvature, if the sectional curvature $K_M$ of $M$ satisfies

$$K_M \geqq -k \circ r_o,$$

where $r_o$ is the distance function from a fixed point $o \in M$, called the base point of $M$, and $k(t)$ is a nonnegative monotone nonincreasing function on $[0, \infty)$ such that the integral $\int_0^\infty t \cdot k(t) dt$ is finite.

Let $p$ be an arbitrarily fixed point of $M$. For sufficiently large $t$, the metric sphere $S_t(p)$ around $p$ of radius $t$ is a Lipschitz hypersurface of $M$ consisting of $k$ connected components, where $k$ is the number of the ends of $M$. On $S_t(p)$, we introduce the interior distance, denoted by $d_{p,t}$, induced from the distance $d_M$ restricted on $S_t(p)$.

Now, similarly to the case of Hadamard manifolds, we can define an equivalence relation $\sim$ on the set $\mathfrak{R}_M$ of all rays of $M$ and the ideal boundary as the set of equivalence classes. Furthermore we can define the Tits metric Td on the ideal boundary.

In fact, two rays $\sigma, \gamma \in \mathfrak{R}_M$ are called *equivalent* and denoted by $\sigma \sim \gamma$ if $\lim_{t\to\infty} \dfrac{d_M(\sigma(t), \gamma(t))}{t} = 0$. We write $\sigma(\infty)$ for the equivalence class of $\sigma$ and $M(\infty)$ for the quotient space $\mathfrak{R}_M/\sim$. We call $M(\infty)$ the *ideal boundary* of $M$. Moreover we define the *Tits metric* Td on $M(\infty)$ by

$$(*) \qquad \mathrm{Td}(\sigma(\infty), \gamma(\infty)) := \lim_{t\to\infty} \frac{d_{p,t}(\sigma \cap S_t(p), \gamma \cap S_t(p))}{t},$$

where $p$ is any fixed point of $M$. Then Td is well-defined on $M(\infty)$. Actually, the following properties are valid for any fixed point $p \in M$ and any pair $\sigma, \gamma \in \mathfrak{R}_M$ ( cf. Proposition 2.1 in [Ks] );

(1) $\sigma \sim \gamma \Longleftrightarrow \lim_{t\to\infty} \dfrac{d_{p,t}(\sigma \cap S_t(p), \gamma \cap S_t(p))}{t} = 0$,

(2) there exists the limit : $\lim_{t\to\infty} \dfrac{d_{p,t}(\sigma \cap S_t(p), \gamma \cap S_t(p))}{t}$, which is independent of the choice of $p$.

Remark. Here we make some comments on the relation between the definitions made for these two classes of Riemannian manifolds, that is, the class of Hadamard manifolds and that of manifolds of asymptotically nonnegatively curvature.

In general, the asymptotic relation due to Busemann [Bu] is not an equivalence relation. But on Hadamard manifolds, this relation coincides with the boundedness of the distance between two points on rays stated previously, and hence it is an equivalence relation. Hence this can be used for defining points at infinity. On the other hand, on asymptotically nonnegatively curved manifolds, though the asymptotic relation is not an equivalence relation, this gives rise to the equivalence relation $\sim$ defined above. That is, if $\sigma$ is asymptotic to $\gamma$, then $\sigma \sim \gamma$. As inferred from the definition of the metric $l$, the equivalence relation $\sim$ is a natural extension of the asymptotic relation. We can also see that the equivalence relation $\sim$ coincides with the asymptotic relation on Hadamard manifolds.

The equation (*) defining the metric Td for asymptotically nennegatively curved manifolds is similar to the third definition of Td for Hadamard manifolds.

For an asymptotically nonnegatively curved manifold, its ideal boundary is always compact and there is a counterpart of Proposition 3.3.2, which is seen in the proof of Proposition 2.4 of [Ks] due to Kasue.

PROPOSITION 3.3.3. *Let $M$ be a manifold of asymptotically nonnegative curvature and $p$ be a base point of $M$. Then the sequence of the pointed metric spaces $((M, d_t^M), p)$ converges to the cone $(\mathfrak{C}(M(\infty)), o)$ of $M(\infty)$ with vertex $o$ in the sense of pointed Hausdorff distance, where $d_t^M = \frac{1}{t} d_M$ ;*

$$\lim_{t \to \infty} ((M, d_t^M), p) = (\mathfrak{C}(M(\infty)), o).$$

It should be noted that the Hausdorff limit in the proposition is independent of the choice of base points $p \in M$.

Theorem D can be proved in a quite similar fashion to the case of Hadamard manifolds by applying the above proposition.

Finally we note that for two asymptotically nonnegatively curved manifolds, there exists no Hausdorff approximation between them in general, even if their ideal boundaries are isometric. Indeed, the example of the previous section also gives a counter example in this case.

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