TOPOLOGICAL PROPERTIES OF FREE TOPOLOGICAL GROUPS

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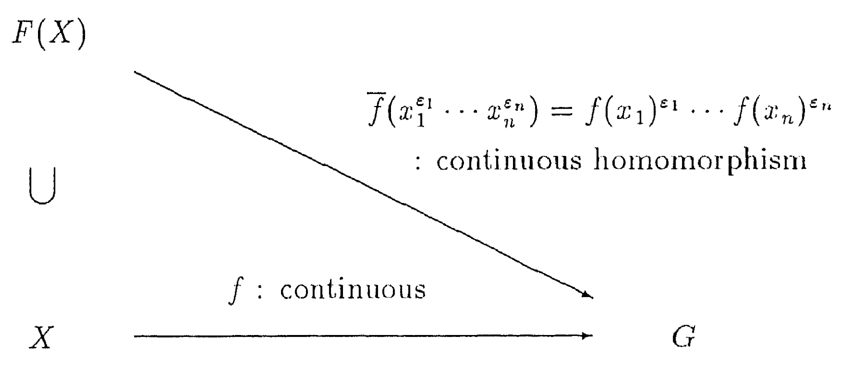
Introduction

In this paper we investigate topological properties of the free topological group $F(X)$ and the free Abelian topological group $A(X)$ over a Tychonoff space X, which were first introduced by A. A. Markov [16] in 1941, as follows.

> The *free topological group* $F(X)$ over a space X in the sense of A. A. Markov is the free algebraic group over the set X equipped with the group topology $\mathcal{T}$ having the following properties:
>
> (1) X is a subspace of $F(X)$,
>
> (2) each continuous mapping from X to an arbitrary topological group G extends to a continuous homomorphism from $F(X)$ to G.
>
> The *free Abelian topological group* $A(X)$ over a space X in the sense of A. A. Markov is the free algebraic Abelian group over the set X equipped with the group topology $\mathcal{T}$ having the properties (1) and (2) for an arbitrary Abelian topological group G.



He proved the existence and the uniqueness of $F(X)$ $(A(X))$ over a Tychonoff space X. The complete proof was appeared in A. A. Markov [17] in 1945, but his proof was very complicated. T. Nakayama [19] and S. Kakutani [14] gave the simple proofs. After a few years, M. I. Graev [9] extended the construction of $F(X)$ $(A(X))$ in the sense of A. A. Markov. And also, he gave another proof of the existence and the uniqueness of $F(X)$ $(A(X))$ in the sense of M. I. Graev. His main method of proving the existence of $F(X)$ $(A(X))$ is the construction of invariant "Graev" metric on $F(X)$ $(A(X))$ over a metric space X. The Graev metric was useful not only for the proof of the existence of $F(X)$ $(A(X))$ but also for investigations of topological properties of $F(X)$ $(A(X))$. On the other hand, it is well known that, in a similar argument, the Graev pseudometric can be constructed on $F(X)$ $(A(X))$ in the sense of A. A. Markov over a Tychonoff space X.

In Chapter 2, we introduce the construction, and discuss infinite dimensionality of the free topological group, as applications of the Graev pseudometric. In particular, we show that

> $F(X)$ $(A(X))$ *cannot contain the Hibert cube* $I^{\aleph_0}$ *for finite dimensional metric spaces* X *(for example, the unit n-dimensional cube* I^n *and Euclidean n-dimensional space* R^n*), while* $F(X)$ $(A(X))$ *is an infinite dimensional group.*

The structure of $F(X)$ $(A(X))$ is very simple from the algebraic point of view - it is exactly the free (Abelian) group over the set X. On the contrary, the topology of $F(X)$ $(A(X))$ is rather complicated even for very simple spaces X. Indeed, if the space $F(X)$ $(A(X))$ is first-countable then X is a discrete space (cf. §1.2). Furthermore, the definition of $F(X)$ $(A(X))$ says nothing about any constructive form of open sets, i.e. an open neighborhood base of the unit element. Therefore, it seems difficult to investigate the topological properties on $F(X)$ $(A(X))$. From the early 1980s, the Moscow group, V. G. Pestov [20], O. V. Sipacheva [22], M. G. Tkačenko [24],[25], and V. V. Uspenskiĭ [26], gave some open neighborgood bases of the unit element of $F(X)$ $(A(X))$, respectively. Unfortunately, their constructions are very complicated to study topological properties.

In Chapter 3, we construct an alternative form of an open neighborhood base of the unit element 0 in $A(X)$. It is sufficiently simple for our investigations. Furthermore, we give an open neighborhood base of 0 in $A_{2n}(X)$, where $F_n(X)$ $(A_n(X))$ is a subspace of $F(X)$ $(A(X))$ formed by reduced words whose lengths are less than or equal to the natural number n.

In our main part Chapter 4, applying the neighborhood base of 0, we discuss about the k-property on $A(X)$. Recently, A. V. Arhangel'skiĭ, O. G. Okunev and V. G. Pestov [4] showed the characterizations of a metrizable space X such that $F(X)$ and $A(X)$ is a k-space, respectively. In the proof, they used the following concrete spaces such that the free (Abelian) topological groups over them are not k-spaces:

the Fréchet-Urysohn fan $V(\aleph_1)$ of cardinality $\aleph_1$,

the hedgehog space $J(\aleph_0)$ of spininess $\aleph_0$ such that each spininess is a sequence which converges to the center point, and

$Y = C \oplus \{x_\alpha : \alpha < \omega_1\}$, where C is a convergent sequence with its limit and $\{x_\alpha : \alpha < \omega_1\}$ is a discrete collection.

In fact, they proved that neither $A_3(V(\aleph_1))$, $F_2(V(\aleph_1))$, $A(J(\aleph_0))$, $F(J(\aleph_0))$, nor $F_4(Y)$ is a k-space (cf. [23]). Since, by most of their ways, it was shown that $F_n(X)$ ($A_n(X)$), as a closed subspace of $F(X)$ ($A(X)$), is not a k-space for some natural number n, we naturally raise the following question:

if $F_n(X)$ ($A_n(X)$) is a k-space for each natural number n, then is $F(X)$ ($A(X)$) a k-space ?

However, we obtain the negative answer for $A(X)$. Namely,

$A_n(J(\kappa))$ is a k-space for each n and each cardinality κ, but $A(J(\kappa))$ is not a k-space if $\kappa \geq \aleph_0$.

Although the subspaces $A_n(X)$, $n = 1, 2, \ldots$, play an important role to investigate the topology of $A(X)$, the above result shows that there is a gap between the k-property of $A(X)$ and the one of $A_n(X)$, $n = 1, 2, \ldots$. In fact, we shall give characterizations of a metrizable space X such that every $A_n(X)$ is a k-space. As a consequence, we have a *stability theorem* of the property of $A_n(X)$, $n = 1, 2, \ldots$, as follows.

For a metrizable space X, the following are equivalent,

(a) *$A_4(X)$ is a k-space,*

(b) *$A_n(X)$ is a k-space, for each $n = 1, 2, \ldots$.*

Furthermore, we obtain characterizations of a metrizable space X such that $A_3(X)$ is a k-space and $A_2(X)$ is a k-space, respectively. As a result, we have the following.

(1) *There is a metrizable space X such that $A_2(X)$ is a k-space but $A_3(X)$ is not a k-space.*

(2) *There is a metrizable space X such that $A_3(X)$ is a k-space but $A_4(X)$ is not a k-space.*

From these results, we shall answer to the questions of T. H. Fay, E. T. Ordman and B. V. S. Thomas [8] for $A(X)$.

On the other hand, we obtain that

if each $F_n(X)$ $(A_n(X))$ is locally compact, then $F(X)$ $(A(X))$ is a k-space.

In addition, we obtain characterizations of a metrizable space X such that each $F_n(X)$ is locally compact.

Chapter 1

Definitions and preliminaries

All topological spaces are assumed to be Tychonoff and topological groups are assumed to be T_0. By N we denote the set of all natural numbers. Our terminology and notations of general topology follow [6], and we refer [11], [21] for elementary properties of topological groups. Especially, the readers are refered to the survey papers [3] for the theory of free topological groups.

1.1 Topological groups

A topological group is a set endowed with two structures: that of a group and that of a topological space. These structures are connected in such a way that algebraic properties of the group affect topological properties of the space, and *vice versa*. We begin with the definition.

Definition 1.1.1 Let G be a set that is a group and also a topological space. Suppose that:

(1) the mapping $(x, y) \longrightarrow xy$ of $G \times G$ onto G is a continuous mapping;

(2) the mapping $x \longrightarrow x^{-1}$ of G onto G is continuous.

Then G is called a *topological group* and the topology a *group topology* on G.

The following elementary properties of a group topology are often used in this paper without notice. In particular, the description of a group topology by properties of a neighborhood base of e is important.

Theorem 1.1.2 *Let G be a topological group. Then, the following properties hold.*

(1) *For each $g \in G$, the left and right translations by g are homeomorphisms on G. The inversation is also a homeomorphism on G.*

(2) *Let $\mathcal{U}$ be a neighborhood base of e. Then the families $\{gU : g \in G \text{ and } U \in \mathcal{U}\}$ and $\{Ug : g \in G \text{ and } U \in \mathcal{U}\}$ are open base for G.*

(3) *G has a neighborhood base of e consisting of neighborhood U such that $U = U^{-1}$* (sets having this property are called *symmetric*).

(4) *Let A and B be subsets of G. If A is open and B is arbitrary, then AB and BA are open. If A and B are compact, then AB is compact. If A is closed and B is compact, then AB and BA are closed.*

(5) *Let A and B be subsets of G, then*

  (a) $\overline{A}\,\overline{B} \subset \overline{AB}$,

  (b) $\overline{A}^{-1} = \overline{A^{-1}}$, *and*

  (c) $x\overline{A}y = \overline{xAy}$ *for all $x, y \in G$.*

(6) *Every open subgroup of G is closed.*

Theorem 1.1.3 *Let G be a topological group, and $\mathcal{U}$ a neighborhood base of e. Then*

(i) $\bigcap \mathcal{U} = \{e\}$,

(ii) *for every $U, V \in \mathcal{U}$, there is a $W \in \mathcal{U}$ such that $W \subset U \cap V$,*

(iii) *for every $U \in \mathcal{U}$, there is a $V \in \mathcal{U}$ such that $V^2 \subset U$,*

(iv) *for every $U \in \mathcal{U}$, there is a $V \in \mathcal{U}$ such that $V^{-1} \subset U$,*

(v) *for every $U \in \mathcal{U}$ and $x \in U$, there is a $V \in \mathcal{U}$ such that $xV \subset U$,*

(vi) *for every $U \in \mathcal{U}$ and $x \in G$, there is a $V \in \mathcal{U}$ such that $xVx^{-1} \subset U$.*

Conversely, let G be a group, and let $\mathcal{U}$ be a family of subsets of G with the finite intersection property for which (i) $\sim$ (vi) *hold. Then the family $\{xU : x \in G \text{ and } U \in \mathcal{U}\}$ and $\{Ux : x \in G \text{ and } U \in \mathcal{U}\}$ are open base for a group topology on G.*

We call the above properties (i) $\sim$ (vi) *the axioms of group topology* on G. The axiom (iii) implies the regularity for topological groups.

Corollary 1.1.4 *Let G be a topological group. For every neighborhood U of e, there is a neighborhood V of e such that $\overline{V} \subset U$. Thus, G is regular.*

The following result shows in particular that the axiom (vi) can be radically improved for compact groups. This fact will be useful in sevral places in the sequel.

Corollary 1.1.5 *Let G be a topological group, let U be any neighborhood of e, and let F be any compact subset of G. Then there is a neighborhood V of e such that $xVx^{-1} \subset U$ for all $x \in F$.*

The following result is the homomorphism theorem for topological groups.

Theorem 1.1.6 *Let G and $\tilde{G}$ be topological groups with unit elements e and $\tilde{e}$, respectively, and let f be an open, continuous homomorphism of G onto $\tilde{G}$. Then the topological quotient group $G/f^{-1}(\tilde{e})$ is topologically isomorphic to $\tilde{G}$.*

We conclude this section with the description of a group topology by means of a family of left invariant pseudometrics. This implies that every topological group is Tychonoff.

Definition 1.1.7 A metric or pseudometric d on a group G is said to be *left invariant* if $d(ax, ay) = d(x, y)$ for all $a, x, y \in G$. If $d(xa, ya) = d(x, y)$ for all $a, x, y \in G$, then d is said to be *right invariant.* If d is both left and right invariant, it is said to be *two-sided invariant.*

Theorem 1.1.8 *Let $\{U_k : k \in N\}$ be a sequence of symmetric neighborhoods of e in a topological group G such that $U_{k+1}^2 \subset U$ for each $k \in N$. Let $H = \bigcap_{k=1}^{\infty} U_k$. Then there is a left invariant pseudometric d on G such that:*

(1) d *is continuous,*

(2) $d(x, y) = 0$ *if and only if* $y^{-1}x \in H$,

(3) $d(x, y) \leq 2^{-k+2}$ *whenever* $y^{-1}x \in U_k$,

(4) $2^{-k} \leq d(x, y)$ *whenever* $y^{-1}x \notin U_k$.

If, in addition, $x\,U_k\,x^{-1} = U_k$ *for all* $x \in G$ *and* $k \in N$, *then* d *is also right invariant and*

(5) $d(x^{-1}, y^{-1}) = d(x, y)$ *for all* $x, y \in G$.

Theorem 1.1.9 *Let* G *be a topological group,* g *an element of* G, *and* F *a closed subset of* G *not containing* g. *Then there is a continuous real function* ψ *on* G *such that* $\psi(g) = 0$ *and* $\psi(x) = 1$ *for each* $x \in F$. *Thus* G *is Tychonoff.*

1.2 Free topological goups

In this section, we have a definition of the free (Abelian) topological groups and some preliminaries.

Definition 1.2.1 The *free topological group* $F(X)$ over a space X in the sense of Markov [17] is the free algebraic group over the set X equipped with the group topology $\mathcal{T}$ having the following properties:

(1) X is a subspace of $F(X)$, and

(2) each continuous mapping from X to an arbitrary topological group G extends to a continuous homomorphism from $F(X)$ to G.

The *free Abelian topological group* $A(X)$ over a space X in the sense of Markov is the free algebraic Abelian group over the set X equipped with the group topology $\mathcal{T}$ having the above properties (1) and (2) for an arbitrary Abelian topological group G.

Remark 1.2.2 M. I. Graev extended the construction of $F(X)$ ($A(X)$) in the sense of Markov as follows.

Let X be a space with a distinguished point e. Then, the *free topological group* $F_G(X)$ in the sense of Graev [9] is the free algebraic group over the set $X \setminus \{e\}$ with the unit element e and equipped with the group topology having the following properties:

(1) X is a subspace of $F_G(X)$, and

(2) for each continuous mapping f of X into an arbitrary topological group G such that $f(e)$ is the unit element of G, there is a unique continuous homomorphism $\overline{f}$ of $F_G(X)$ into G such that $\overline{f}|_X = f$.

Similarly, let X be a space with a distinguished point 0. Then, the *free Abelian topological group* $A(X)$ over a space X in the sense of Graev is the free algebraic Abelian group over the set $X \setminus \{0\}$ with the unit element 0 equipped with the group topology having the above properties (1) and (2) for an arbitrary Abelian topological group G.

In this paper, we use the definition of the free (Abelian) topological group $F(X)$ ($A(X)$) given by A. A. Markov.

Notations. We denote the *unit element* of $F(X)$ by e. Any $g \in F(X)$ except e has the unique reduced representation of the form $g = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}$, where $x_i \in X$ and $\varepsilon_i = \pm 1$ for $i = 1, \ldots, n$. We put

$$\ell_+(g) = |\{i \leq n : \varepsilon_i = 1\}|, \quad \ell_-(g) = |\{i \leq n : \varepsilon_i = -1\}|, \quad \text{and} \quad \ell(g) = \ell_+(g) + \ell_-(g).$$

We call the number $\ell(g)$ the *length* of g (by definition, $\ell(e) = 0$). For each $n \in N$, let $F_n(X) = \{g \in F(X) : \ell(g) \leq n\}$ (by definition $F_0(X) = \{e\}$) and define the mapping $i_n : \widetilde{X}^n \longrightarrow F_n(X)$ by $i_n((x_1, x_2, \ldots, x_n)) = x_1\, x_2 \cdots x_n$ for each $x_i \in \widetilde{X}^n$, where $\widetilde{X} = X \oplus \{e\} \oplus X^{-1}$.

Analogously, we denote the *unit element* of $A(X)$ by 0. Any $g \in A(X)$ except 0 has the unique reduced representation of the form $g = \varepsilon_1 x_1 + \varepsilon_2 x_2 + \cdots + \varepsilon_n x_n$, where $x_i \in X$ and $\varepsilon_i = \pm 1$ for $i = 1, \ldots, n$. We put

$$\ell_+(g) = |\{i \leq n : \varepsilon_i = 1\}|, \quad \ell_-(g) = |\{i \leq n : \varepsilon_i = -1\}|, \quad \text{and} \quad \ell(g) = \ell_+(g) + \ell_-(g).$$

We call the number $\ell(g)$ the *length* of g (by definition, $\ell(0) = 0$). For each $n \in N$, let $A_n(X) = \{g \in A(X) : \ell(g) \leq n\}$ (by definition $A_0(X) = \{0\}$) and define the mapping $i_n : \widetilde{X}^n \longrightarrow A_n(X)$ by $i_n((x_1, x_2, \ldots, x_n)) = x_1 + x_2 + \cdots + x_n$ for each $x_i \in \widetilde{X}^n$, where $\widetilde{X} = X \oplus \{0\} \oplus -X$.

In this paper, all results which hold for both $F(X)$ and $A(X)$ will be formulated for $F(X)$ and the corresponding results for $A(X)$ will be formulated in round brackets. If proofs in both cases are similar to each other, we will give only a proof for $F(X)$ omitting that for $A(X)$.

The following fundamental properties related to $F(X)$ $(A(X))$ are often used in this paper (see [2] and [9]).

Lemma 1.2.3 (1) *Let $\mathcal{T}_1$ be a group topology for $F(X)$ $(A(X))$ which induces the original topology for X, then $\mathcal{T}_1 \leq \mathcal{T}$.*

(2) *The mapping i_n is continuous for each $n \in N$, so that $F_n(X)$ $(A_n(X))$ is compact for each $n \in N$ if X is compact.*

(3) *X and $F_n(X)$ $(A_n(X))$, $n \in N$, are closed subspaces of $F(X)$ $(A(X))$.*

Proof. The statements (1) and (2) are easily seen by the definition of $F(X)$ $(A(X))$. Therefore, we shall show the statement (3). Let βX be the Stone-Čech compactification of X. Then, for the inclusion mapping $i : X \longrightarrow F(\beta X)$, there is a continuous homomorphism $\bar{i}$ of $F(X)$ into $F(\beta X)$ such that $\bar{i}|_X = i$. This follows that

$$\bar{i}^{-1}(F_n(\beta X)) = F_n(X) \text{ for each } n \in N.$$

By the statements (2), each $F_n(\beta X)$ is compact and hence closed in $F(\beta X)$. Therefore, $F_n(X)$ is closed in $F(X)$ for each $n \in N$. ∎

Now, we introduce an important subgroup F_0 (A_0) of $F(X)$ $(A(X))$, which is very useful for investigations of topological properties of $F(X)$ $(A(X))$. In fact, the following result for F_0 (A_0) has been often used. Here, we give a simple proof.

Lemma 1.2.4 *Let $F_0 = \{g \in F(X) : \ell_+(g) = \ell_-(g)\}$ and $A_0 = \{g \in A(X) : \ell_+(g) = \ell_-(g)\}$. Then F_0 (A_0) is a clopen subgroup of $F(X)$ ($A(X)$).*

Proof. Let f be the constant mapping of X into the additive discrete group Z of the integers such that $f(x) = 1$ for each $x \in X$. Then, there is a continuous homomorphism $\overline{f}$ of $F(X)$ onto Z such that $\overline{f}|_X = f$. Since $F_0 = \overline{f}^{-1}(0)$, this yields that F_0 is a clopen subgroup of $F(X)$. ■

For a compact space X, M. I. Graev [9] showed that the free group topology is determined by the collection $\{F_n(X) : n \in N\}$ ($\{A_n(X) : n \in N\}$), as follows.

Theorem 1.2.5 *Let X be a compact space. Then a subset U of $F(X)$ ($A(X)$) is open in $F(X)$ ($A(X)$) if and only if $U \cap F_n(X)$ is open in $F_n(X)$ ($U \cap A_n(X)$ is open in $A_n(X)$) for each $n \in N$.*

Using the above theorem, we shall show the following fact, which is also well known and has been often used.

Corollary 1.2.6 *Let X be a space and K a compact subset of $F(X)$ ($A(X)$). Then, K is contained in $F_n(X)$ ($A_n(X)$) for some $n \in N$.*

Proof. Let $\overline{i}$ be the continuous homomorphism of $F(X)$ into $F(\beta X)$ such that $\overline{i}|_X = i$, as in the proof of the statements (3) of Lemma 1.2.3. Since K is a compact subset of $F(X)$, $\overline{i}(K)$ is a compact subset of $F(\beta X)$. Now, we shall show that $\overline{i}(K)$ is contained in $F_n(\beta X)$ for some $n \in N$. Assume the contrary, then there is a sequence $\{g_n : n \in N\}$ in $\overline{i}(K)$ such that

$$g_n \in F_{k_n+1}(\beta X) \setminus F_{k_n}(\beta X),$$

where $\{k_n : n \in N\}$ is an infinite subsequence of N. By virtue of Theorem 1.2.5, $\{g_n : n \in N\}$ is a discrete closed subset of $F(\beta X)$. This is a contradiction, and hence there is an $n \in N$ such that

$$K = \overline{i}^{-1}\, \overline{i}(K) \subset \overline{i}^{-1}(F_n(\beta X)) = F_n(X). \qquad ■$$

We conclude this section with the following result obtained by M. I. Graev [9], which show that the topology of $F(X)$ $(A(X))$ is complicated even for simple spaces X.

Theorem 1.2.7 *Let X be a space. If $F(X)$ $(A(X))$ is first-countable, then the space X must be a discrete space.*

Chapter 2

The Graev pseudometric

In this chapter, we introduce the construction of the Graev pseudometric on $F(X)$ ($A(X)$). As an application, one of the most important results for investigations of toplogical properties of $F(X)$ ($A(X)$) is obtained, which was proved by A. V. Arhangel'skiĭ [2] and C. Joiner [13]. Finally, we discuss infinite-dimensionality of $F(X)$ ($A(X)$).

2.1 Construction and properties

We shall construct the Graev pseudometric on $F(X)$, since it can be constructed for Abelian case, analogously. Before the construction, we introduce some notions.

Let $S(X)$ be the semigroup of all (reduced and unreduced) words in the letter $\widetilde{X}$. Then $F(X)$ is the set of all reduced words from $S(X)$. The relation $g \equiv h$ for $g, h \in S(X)$ implies that the words g and h consist of the same number of letters and thier corresponding letters are identical. By $g = h$ we imply the equality of the reduced forms of these words. Let $g \equiv x_1 x_2 \cdots x_n \in S(X)$, where each $x_i \in \widetilde{X}$. The number n is called the *length* of g denoted $\ell(g)$. Notice that for each $g \in F(X)$, the definitions of length are coincided.

Let X be a space and d a continuous pseudometric on X. At first, we consider the following continuous pseudometric d' on $\widetilde{X}$:

$$
\begin{aligned}
&d'(x, e) = d'(x^{-1}, e) = 1 \quad \text{for each } x \in X, \\
&d'(x, y) = d'(x^{-1}, y^{-1}) = d(x, y) \quad \text{for each } x, y \in X, \\
&d'(x, y^{-1}) = d'(x^{-1}, y) = d'(x, e) + d'(y, e) \quad \text{for each } x, y \in X.
\end{aligned}
$$

Next we extend d' to all of $F(X)$. Let $g, h \in F(X)$. We define

$$\overline{d}(g,h) = \inf\{ \sum_{i=1}^{k} d'(x_i, y_i) \ : \ g' \equiv x_1x_2\cdots x_k, h' \equiv y_1y_2\cdots y_k \in S(X) \text{ such that } g = g' \text{ and } h = h'\}$$

For convenience, we put $\phi(g',h') = \sum_{i=1}^{k} d'(x_i, y_i)$. M. I. Graev showed the following important property of $\overline{d}$. Since we use the proof later, we give an outline of the proof here.

Proposition 2.1.1 *For every $g,h \in F(X)$, there are $g',h' \in S(X)$ such that*

$$g = g', h = h', \text{ and } \overline{d}(g,h) = \phi(g',h').$$

Outline of the proof. Let $g,h \in F(X)$ and $g \equiv x_1x_2\cdots x_m$, $h \equiv y_1y_2\cdots y_n$ reduced forms g and h, respectively, where each x_i, $y_j \in X \cup X^{-1}$. Take any $g',h' \in S(X)$ such that $g = g', h = h'$ and

$$(*) \begin{pmatrix} g' \equiv a_1a_2\cdots a_s \equiv A_0x_1A_1x_2\cdots A_{m-1}x_mA_m \\ h' \equiv b_1b_2\cdots b_s \equiv B_0y_1B_1y_2\cdots B_{n-1}y_nB_n, \end{pmatrix}$$

where each $a_i, b_j \in \widetilde{X}$, each $A_i, B_j \in S(X)$ such that $A_i = B_j = e$. In the above representation $(*)$, we call each $\begin{pmatrix} a_i \\ b_i \end{pmatrix}$ a *column*. Take an x_i in the top line of $(*)$. Under this x_i is an element u_1. If u_1 is an e or one of the y_j, then we stop. Otherwise u_1 is in some B_{j_1}, so there is a u_1^{-1} in B_{j_1}. Above u_1^{-1} is an element u_2 in the top line. If u_2 is either e or one of the $x_{i'}$, then we stop. We continue this process until it stops as it must. There are four possible cases, essentially.

$$(1) \begin{cases} x_i & u_2 & u_2^{-1} & \cdots & u_{2n} & u_{2n}^{-1} \\ u_1 & u_1^{-1} & u_3 & \cdots & u_{2n-1}^{-1} & e \end{cases}$$

$$(2) \begin{cases} x_i & u_2 & u_2^{-1} & \cdots & u_{2n-2}^{-1} & e \\ u_1 & u_1^{-1} & u_3 & \cdots & u_{2n-1} & u_{2n-1}^{-1} \end{cases}$$

$$(3) \begin{cases} x_i & u_2 & u_2^{-1} & \cdots & u_{2n} & u_{2n}^{-1} \\ u_1 & u_1^{-1} & u_3 & \cdots & u_{2n-1}^{-1} & y_j \end{cases}$$

$$(4) \begin{cases} x_i & u_2 & u_2^{-1} & \cdots & u_{2n-2}^{-1} & x_{i'} \\ u_1 & u_1^{-1} & u_3 & \cdots & u_{2n-1} & u_{2n-1}^{-1} \end{cases}$$

In the case (1) and (2), let replace all u_i by e. And, in the case (3), let replace all u_{2i-1} by y_j, and all u_{2i} by y_j^{-1}. Finally, in the last case (4), let replace all u_{2i-1} by $x_{i'}^{-1}$, and all u_{2i} by $x_{i'}$. We carry out the above process for the remaining x_k and next, for the remaining y_l (this means that for example, we don't carry out for $x_{i'}$ if the case of the process for some x_i is (4)). If after all these process we are left with a certain number of columns, we replace all letters in these columns by e. At last, we remove all columns $\begin{pmatrix} e \\ e \end{pmatrix}$ in the given words. Consequently, we obtain two new words g_1 and h_1 in $S(X)$. And, we can prove that g_1 and h_1 have the following properties:

(1) $g_1 = g,\ h_1 = h$ and $\ell(g_1) = \ell(h_1)$,

(2) each letters in the words g_1 and h_1 is either $x_i^{\varepsilon_i}$, $y_j^{\varepsilon_j}$, or e, where $\varepsilon_i = \pm 1 = \varepsilon_j$,

(3) under each x_i there stands either e, $x_{i'}^{-1}$ $(i \neq i')$, or y_j; on the top of each y_j there stands either e, $y_{j'}^{-1}$ $(j \neq j')$, or x_i,

(4) the number of columns in $\begin{pmatrix} g_1 \\ h_1 \end{pmatrix}$ which contribute to $\phi(g_1, h_1)$ is at most $m + n$,

(5) $\phi(g_1, h_1) \leq \phi(g', h')$.

By virtue of the property (4), it is readily seen that for all such words having the above properties the function ϕ may take only a finite number of values. Let $g_2, h_2 \in S(X)$ which have the properties such that $\phi(g_2, h_2)$ takes the minimum value. Then we can show that these g_2 and h_2 are required words. ∎

Using Proposition 2.1.1, M. I. Graev proved that $\overline{d}$ is a pseudometric on $F(X)$ which induces the group topology on $F(X)$, and which is weaker than the free group topolory $\mathcal{T}$. Furthermore, A. V. Arhangel'skiĭ [2] and C. Joiner [13] obtained the following important result.

Theorem 2.1.2 *Let X be a space. Then, for each $n \in N$, the mapping*

$$\begin{aligned} f_n &= i_n|_{i_n^{-1}(F_n(X)\setminus F_{n-1}(X))} \\ &\quad : i_n^{-1}(F_n(X) \setminus F_{n-1}(X)) \longrightarrow F_n(X) \setminus F_{n-1}(X) \end{aligned}$$

is a homeomorphism.

Proof. It suffices to show that each f_n is an open mapping. That is, let $g \in F_n(X) \setminus F_{n-1}(X)$ and $g \equiv x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}$ be the reduced form. For each open neighborhood U of $(x_1^{\varepsilon_1}, x_2^{\varepsilon_2}, \ldots, x_n^{\varepsilon_n}) \in \widetilde{X}^n$, let d be a continuous pseudometric on X such that

$$U_{\delta'}(x_1^{\varepsilon_1}) \times U_{\delta'}(x_2^{\varepsilon_2}) \times \cdots \times U_{\delta'}(x_n^{\varepsilon_n}) \subset U$$

for some $\delta' > 0$, where each $U_{\delta'}(x_i^{\varepsilon_i})$ is a δ'-ball of $x_i^{\varepsilon_i}$ in $\widetilde{X}$. Then, for sufficiently small $\delta > 0$, it suffices to show that

$$B_\delta(g) \cap (F_n(X) \setminus F_{n-1}(X)) \subset f_n(U_\delta(x_1^{\varepsilon_1}) \times \cdots \times U_\delta(x_n^{\varepsilon_n})),$$

where $B_\delta(g)$ is a δ-ball of g in $(F(X), \overline{d}\,)$. In fact, let $\delta > 0$ such that

$$2\delta < \min(\{\, d'(x_i, x_j) : x_i \neq x_j\}, 1, \delta').$$

Thus, by virtue of the proof of Proposition 2.1.1, for each $h \equiv y_1^{\xi_1} y_2^{\xi_2} \cdots y_n^{\xi_n} \in B_\delta(g) \cap (F_n(X) \setminus F_{n-1}(X))$ let $g', h' \in S(X)$ such that $g' = g$, $h = h'$, and $\overline{d}(g,h) = \phi(g', h')$. Then it can be proved that under each $x_i^{\varepsilon_i}$ there stands $y_i^{\xi_i}$. This implies that

$$\overline{d}(g,h) = \phi(g', h') = \sum_{i=1}^{n} d'(x_i^{\varepsilon_i}, y_i^{\xi_i}) < \delta.$$

Consequently, each $y_i^{\xi_i} \in U_\delta(x_i^{\varepsilon_i})$, that is $h \in f_n(U_\delta(x_1^{\varepsilon_1}) \times \cdots \times U_\delta(x_n^{\varepsilon_n})$. ■

For $A(X)$, in a similar argument we obtain the following.

Theorem 2.1.3 *Let X be a space. Then, for each $n \in N$, the mapping*

$$\begin{aligned} f_n &= i_n|_{i_n^{-1}(A_n(X)\setminus A_{n-1}(X))} \\ &\quad : i_n^{-1}(A_n(X) \setminus A_{n-1}(X)) \longrightarrow A_n(X) \setminus A_{n-1}(X) \end{aligned}$$

is an open and closed $n!$ to 1 mapping.

Corollary 2.1.4 *Let X be a metrizable space and d a metric on X which is compatible with the topology on X. Then each $F_n(X)$ $(A_n(X))$ is a closed subset of $(F(X), \overline{d}\,)$ $((A(X), \overline{d}\,))$.*

Proof. In the above proof, it can be easily seen that $B_\delta(g) \cap F_{n-1}(X) = \emptyset$. ■

2.2 Infinite dimensionality

In this section, we discuss infinite dimensionality of $F(X)$ $(A(X))$ using the Graev pseudometric. At first, we introduce the following as a corollary of the results in §2.1, which was also proved by A. V. Arhangel'skiĭ [2].

Theorem 2.2.1 *Let X be a metrizable space. Then $F(X)$ $(A(X))$ is a hereditarily paracompact F_σ-metrizable group.*

Proof. Let X be a metrizable space and d a metric on X which is compatible with the topology on X. By Corollary 2.1.4, for each $n \in N$ $Y_n = F_n(X) \setminus F_{n-1}(X)$ is an open subset of the metric space $(F_n(X), \overline{d}|_{F_n(X)})$. Therefore, we have

$$Y_n = \bigcup_{i=1}^{\infty} Z_{(n,i)},$$

where each $Z_{(n,i)}$ is closed in $(F_n(X), \overline{d}|_{F_n(X)})$, and then in $(F(X), \overline{d})$. Since the topology induced by $\overline{d}$ on $F(X)$ is weaker than the free group topology on $F(X)$, $Z_{(n,i)}$ is a closed subset of $F(X)$ for each $n, i \in N$. On the other hand, by Theorem 2.1.2, it can be proved that each $Z_{(n,i)}$ is metrizable. Thus $F(X)$ is F_σ-metrizable.

Next, we shall show that $F(X)$ is paracompact. Let $\mathcal{U}$ be an arbitrary open cover of $F(X)$, and for each $n \in N$,

$$\mathcal{U}_n = \{\, U \cap Y_n : U \in \mathcal{U}\}.$$

Then we can take a σ-discrete closed refinement $\mathcal{H}_n = \bigcup_{i=1}^{\infty} \mathcal{H}_{(n,i)}$ in $(F_n(X), \overline{d}|_{F_n(X)})$, and in $(F(X), \overline{d}\,)$. Now, choose a set $U(H) \in \mathcal{U}$ for each $H \in \mathcal{H}_n, n \in N$ such that $H \subset U(H)$. Thus, for each $n, i \in N$, there is a discrete open collection $\mathcal{W}_{(n,i)} = \{\, W(H) : H \in \mathcal{H}_{(n,i)}\}$ in $(F(X), \overline{d}\,)$ such that

$H \subset W(H)$ for each $H \in \mathcal{H}_{(n,i)}$ and

$W(H) \cap Y_n \subset U(H)$ for each $H \in \mathcal{H}_{(n,i)}$.

Let

$$\mathcal{G}_{(n,i)} = \{\, W(H) \cap U(H) : H \in \mathcal{H}_{(n,i)}\} \text{ and } \mathcal{G} = \bigcup_{n,i=1}^{\infty} \mathcal{G}_{(n,i)}.$$

Then, it is easy to see that $\mathcal{G}$ is a σ-discrete open refinement of $\mathcal{U}$ in $F(X)$. Therefore, $F(X)$ is paracompact. Analogously, it can be proved that $F(X)$ is hereditarily paracompact. ■

Before the discussion, we introduce some concepts of infinite dimensional spaces.

Definition 2.2.2 ([12]) A space X is *(strongly) countable-dimensional* ((s.) c.d.) if X can be represented as the union of a sequence $X_1, X_2, \ldots$ of (closed) subpaces such that $\dim X_i < \infty$ for each $i \in N$.

Definition 2.2.3 ([1]) A space X is said to be a *C-space* (to *have property C*) if for every sequence $\{\mathcal{U}_n : n \in N\}$ of open covers of X there is a sequence $\{\mathcal{V}_n : n \in N\}$ of open collections in X such that

(1) each $\mathcal{V}_n$ is pairwise disjoint,

(2) each V in $\mathcal{V}_n$ is contained in some U in $\mathcal{U}_n$, and

(3) $\bigcup\{\mathcal{V}_n : n \in N\}$ is a cover of X.

By the product theorems for the above infinite dimensional spaces (cf. [7], [10])) and the proof of Theorem 2.2.1, the following result is obtained.

Theorem 2.2.4 *Let X be a metrizable space. Then the following statements hold.*

(1) *X is c.d. if and only if $F(X)$ $(A(X))$ is c.d.*

(2) *X is s.c.d. if and only if $F(X)$ $(A(X))$ is s.c.d.*

(3) *X is a compact C-space if and only if $F(X)$ $(A(X))$ is a C-space.*

Since every c.d. (s.c.d., C-) space cannot contain the Hilbert cube $I^{\aleph_0}$, we have the following.

Corollary 2.2.5 *For the unit n-dimensional cube I^n or Euclidean n-dimensional space R^n, the free (Abelian) topological groups over them cannot contain the Hilbert cube $I^{\aleph_0}$, while their dimensions are infinite.*

Chapter 3

Neighborhood bases of the unit element

In this chapter, we shall discuss neighborhood bases of the unit element in $F(X)$ $(A(X))$. In the first section, we introduce the neighborhood bases using pseudometrics on X, which were obtained by M. G. Tkačendo [25] and V. V. Uspenskiĭ [26]. And also, we introduce some important and useful results which are obtained applying the neighborhood bases. In the second section, we shall construct a neighborhood base of 0 in $A(X)$. Furthermore, we shall construct a neighborhood base of 0 in $A_{2n}(X)$, $n = 1, 2, \ldots$. These neighborhood bases were constructed in our paper [28].

3.1 Pseudometrics

Let X be a space and $\mathcal{D}$ the family conststing of all continuous pseudometrics on X. For each $d \in \mathcal{D}$, let $\overline{d}$ be the Graev pseudometric on $F(X)$ $(A(X))$, and $\mathcal{T}_{\overline{d}}$ the group topology on $F(X)$ $(A(X))$ induced by $\overline{d}$. Now we define the group topology $\mathcal{T}_1$ as follows,

$$\mathcal{T}_1 = \sup\{\mathcal{T}_{\overline{d}} : d \in \mathcal{D}\}.$$

Then, it is clear that $\mathcal{T}_1|_X$ is equal to the original topology on X. Therefore, by Lemma 1.2.3 (1),

$$\mathcal{T}_1 \leq \mathcal{T}.$$

Furthermore, M. G. Tkačenko [24] proved the following results.

Theorem 3.1.1 *Let $\mathcal{V} = \{B_{(1,\overline{d})}(0) : d \in \mathcal{D}\}$, where $B_{(1,\overline{d})}(0) = \{g \in A(X) : \overline{d}(g,0) < 1\}$. Then $\mathcal{V}$ is a neighborhood base of 0 in $A(X)$, and therefore, $\mathcal{T}_1 = \mathcal{T}$ for $A(X)$.*

After a few years, V. V. Uspenskiĭ [26] constructed a neighborhood base of e (0) in $F(X)$ ($A(X)$) using continuous pseudometrics on X, as follows.

Each $g \in F_0$ (cf. Lemma 1.2.4), admits a decomposition of the form

$$(*) \qquad g = g_1 x_1^{\varepsilon_1} y_1^{-\varepsilon_1} g_1^{-1} g_2 x_2^{\varepsilon_2} y_2^{-\varepsilon_2} g_2^{-1} \cdots g_n x_n^{\varepsilon_n} y_n^{-\varepsilon_n} g_n^{-1},$$

where $x_i, y_i \in X$, and $\varepsilon_i = \pm 1$, and $g_i \in F(X)$.

Theorem 3.1.2 *Let X be a space and $S = \{d_g : g \in F(X)\} \in \mathcal{D}^{F(X)}$ be a set of continuous pseudometrics on X with $F(X)$ as the index set. For each $g \in F_0$ let*

$$p_S(g) = \inf\{\sum_{i=1}^{n} d_{g_i}(x_i, y_i)\},$$

where the infimum is over all decompositions of the form $(*)$. Then:

(1) *Each p_S is a continuous seminorm on F_0.*

(2) *As S runs through $\mathcal{D}^{F(X)}$, the set $\{g : p_S(g) < 1\}$ forms a neighborhood base of e in F_0 (and then in $F(X)$).*

As a result, he proved the following result which will be often used in the next chapter.

Theorem 3.1.3 *Let X be a metrizable space, and Y a closed subspace of X. Then the free (Abelian) topological group $F(Y)$ ($A(Y)$) is canonically topological isomorphic to the subgroup of $F(X)$ ($A(X)$) generated by Y.*

Remark 3.1.4 Recently, O. V. Sipacheva [22] obtained the following result.

Let X be a space and Y a subspace of X. Then, the free (Abelian) topological group $F(Y)$ ($A(Y)$) is canonically topological isomorphic to the subgroup of $F(X)$ ($A(X)$) generated by Y if and only if any continuous bounded pseudometric defined on Y can be extended to a continuous pseudometric on X.

3.2 Constructions

In this section, we will construct a neighborhood base $\mathcal{W}$ of 0 in $A(X)$. The idea of our construction is due to M. G. Tkačenko [25]. In order to construct the neighborhood base Σ^* of the unit element of $F(X)$, he used the universal uniformity on $\widetilde{X}^n$ for each $n \in N$. On the other hand, we used only the universal uniformity $\mathcal{U}_X$ on X for $A(X)$. The neighborhood base $\mathcal{W}$ of 0 in $A(X)$ can be obtained from the neighborhood base of e in $F(X)$ by V. G. Pestov [20]. Nevertheless, since he gave it without any proof and the construction of $\mathcal{W}$ is important in this paper, we give its construction and proof here. Furthermore, we will construct a neighborhood base $\mathcal{W}_n$ of 0 in $A_{2n}(X)$ for each $n \in N$. The existence of the neighborhood base $\mathcal{W}_n$ is one of the important facts in this paper.

Constructing the neighborhood bases, we introduce some definitions and notations. Let $(X, \mathcal{U})$ be a uniform space. The *inverse relation* of $U \in \mathcal{U}$ will be denoted be U^{-1}, and the *composition* of U and V in $\mathcal{U}$ will be denoted by $U \circ V$; thus we have

$$U^{-1} = \{(x, y) \in X \times X : (y, x) \in U\} \text{ and}$$

$$U \circ V = \{(x, z) \in X \times X : \text{ there is a } y \in X \text{ such that } (x, y) \in U \text{ and } (y, z) \in V\}$$

The *diagonal* of $X \times X$ is the set $\Delta_X = \{(x, x) : x \in X\}$. A set $U \in \mathcal{U}$ is called *symmetric* if $U = U^{-1}$.

Lemma 3.2.1 *Let $k \in N \cup \{0\}$, $p, k_1, \ldots, k_p \in N$ such that $\sum_{i=1}^{p} 2^{-k_i} < 2^{-k}$.*

(1) *Let $(X, \mathcal{U})$ be a uniform space and $\{U_n : n \in N \cup \{0\}\}$ a countable subcollection of $\mathcal{U}$ such that $U_{n+1} \circ U_{n+1} \circ U_{n+1} \subset U_n$ for each $n \in N \cup \{0\}$, then $U_{k_1} \circ U_{k_2} \circ \cdots \circ U_{k_p} \subset U_k$.*

(2) *Let G be a group with the unit element e and $\{V_n : n \in N \cup \{0\}\}$ a countable collection of subsets of G such that $e \in V_n$ and $V_{n+1} \cdot V_{n+1} \cdot V_{n+1} \subset V_n$ for each $n \in N \cup \{0\}$, then $V_{k_1} \cdot V_{k_2} \cdot \cdots \cdot V_{k_p} \subset V_k$* (cf. [25]).

Proof. Since the proof is straightforward, we only give an outline of the proof of (1). We prove by induction with respect to p. Assume that for each $n \leq p$, the condition (1) is obtained. If there is a $j \in \{1, 2, \ldots, p+1\}$ such that $k_j = k+1$, by inductive assumption,

$$U_{k_1} \circ U_{k_2} \circ \cdots \circ U_{k_{p+1}} \subset U_{k+1} \circ U_{k+1} \circ U_{k+1} \subset U_k.$$

Thus, let $k_j < k+1$ for each $j \in \{1, 2, \ldots, p+1\}$. In this case, if $\sum_{i=1}^{p+1} 2^{-k_i} < 2^{-(k+1)}$, we can show that

$$U_{k_1} \circ U_{k_2} \circ \cdots \circ U_{k_p} \subset U_{k+1},$$

therefore

$$U_{k_1} \circ U_{k_2} \circ \cdots \circ U_{k_{p+1}} \subset U_{k+1} \circ U_{k+1} \subset U_k.$$

Otherwise, i.e. $2^{-(k+1)} \leq \sum_{i=1}^{p} 2^{-k_i} < 2^{-k}$, then there is a $j \in \{2, \ldots, p\}$ such that

$$\sum_{i=1}^{j} 2^{-k_i} < 2^{-(k+1)} \text{ and } \sum_{i=1}^{j+1} 2^{-k_i} \geq 2^{-(k+1)}.$$

It follows that

$$U_{k_1} \circ U_{k_2} \circ \cdots \circ U_{k_{p+1}} \subset U_{k+1} \circ U_{k_{j+1}} \circ U_{k+1} \subset U_{k+1} \circ U_{k+1} \circ U_{k+1} \subset U_k.$$

Consequently, the condition (1) is obtained. ■

Let $\mathcal{U}_X$ be the universal uniformity on a space X and put $\mathcal{P} = \{P \subset \mathcal{U}_X : P$ is countable $\}$. For each $P = \{U_1, U_2, \ldots\} \in \mathcal{P}$, let

$$\begin{aligned} W(P) &= \{x_1 - y_1 + x_2 - y_2 + \cdots + x_k - y_k \\ &\quad : (x_i, y_i) \in U_i \text{ for } i = 1, 2, \ldots, k, k \in N\}, \text{ and} \\ \mathcal{W} &= \{W(P) : P \in \mathcal{P}\}. \end{aligned}$$

Furthermore, fix any $n \in N$. Let

$$\begin{aligned} \mathcal{Q}_n(P) &= \{Q \subset P : |Q| = n\}, \\ W_n(P) &= \{x_1 - y_1 + x_2 - y_2 + \cdots + x_n - y_n \\ &\quad : (x_j, y_j) \in U_{i_j} \text{ for } j = 1, 2, \ldots, n, \{U_{i_1}, U_{i_2}, \ldots, U_{i_n}\} \in \mathcal{Q}_n(P)\}, \text{ and} \\ \mathcal{W}_n &= \{W_n(P) : P \in \mathcal{P}\}. \end{aligned}$$

Remark 3.2.2 In the above definition, for $P \in \mathcal{P}$, there may be the same elements in P. In particular, for each $U \in \mathcal{U}_X$, the countable collection $\{U, U, \ldots\}$ is also in $\mathcal{P}$.

The reader should remark that the representations of elements of $W(P)$ and $W_n(P)$ need not be a reduced representation.

For the definition of $W_n(P)$, let

$$\mathcal{R}_n(P) = \{Q \subset P : |Q| \leq n\}.$$

Since, Δ_X is contained in each $U \in \mathcal{U}_X$, it is easy to see that

$$\begin{aligned} W_n(P) &= \{x_1 - y_1 + x_2 - y_2 + \cdots + x_k - y_k \\ &\quad : (x_j, y_j) \in U_{i_j} \text{ for } j = 1, 2, \ldots, k, \{U_{i_1}, U_{i_2}, \ldots, U_{i_k}\} \in \mathcal{R}_n(P)\}. \end{aligned}$$

Theorem 3.2.3 *$\mathcal{W}$ is a neighborhood base of* 0 *in* $A(X)$.

Proof. First, we shall show that $\mathcal{W}$ satisfies the axioms for open sets in an Abelian topological group $A(X)$ (cf. Theorem 1.1.3), i.e. $\mathcal{W}$ satisfies the the following properties:

(i) for every $V \in \mathcal{W}$, there is a $W \in \mathcal{W}$ such that $W + W \subset V$;

(ii) for every $V \in \mathcal{W}$, there is a $W \in \mathcal{W}$ such that $-W \subset V$;

(iii) for every $V \in \mathcal{W}$ and every $g \in V$, there is a $W \in \mathcal{W}$ such that $g + W \subset V$;

(iv) for every $U,\ V \in \mathcal{W}$, there is a $W \in \mathcal{W}$ such that $W \subset U \cap V$;

(v) $\{0\} = \bigcap \mathcal{W}$.

Let $P = \{U_1, U_2, \ldots\} \in \mathcal{P}$ and $g \in W(P)$. Assume that $g = x_1 - y_1 + x_2 - y_2 + \cdots + x_n - y_n$ such that $(x_i, y_i) \in U_i$ for $i = 1, 2, \ldots, n$, for some $n \in N$. Take $P_1 = \{A_1, A_2, \ldots\}$, $P_2 = \{B_1, B_2, \ldots\}$ and $P_3 = \{C_1, C_2, \ldots\}$ such that

(1) $P_1,\ P_2,\ P_3 \in \mathcal{P}$,

(2) $A_i \subset U_{2i-1} \cap U_{2i}$ for each $i \in N$,

(3) $B_i \subset U_i$ and B_i is symmetric for each $i \in N$,

(4) $C_i \subset U_{i+n}$ for each $i \in N$.

Then it can be shown that $W(P_1) + W(P_1) \subset W(P)$, $-W(P_2) \subset W(P)$, and $g + W(P_4) \subset W(P)$. These imply that the conditions (i), (ii), and (iii) hold. The conditions (iv) and (v) are easily seen, so that we omit the proof.

Thus, by Theorem 1.1.3, let $\mathcal{T}_1$ be a group topology for $A(X)$ generated by $\mathcal{W}$. Take $P = \{U_1, U_2, \ldots\} \in \mathcal{P}$ and $x \in X$, and put $W(x) = \{y \in X : (y, x) \in U_1\}$. Then, since $\mathcal{U}_X$ is compatible with the original topology for X, $W(x)$ is open in X. Also, we can show that $x \in W(x) \subset (W(P) + x) \cap X$, and this means that $\mathcal{T}_1|_X$ is weaker than the original topology for X.

Claim. *$\mathcal{T}_1$ is stronger than the topology of $A(X)$.*

Proof of Claim. Let V be an open neighborhood of 0 in $A(X)$. Put $V_0 = V$ and take a sequence $\{V_n : n \in N\}$ of neighborhoods of 0 in $A(X)$ such that $V_n + V_n + V_n \subset V_{n-1}$ for each $n \in N$. Let $U_n = \{(x, y) \in X \times X : x - y \in V_n\}$ for each $n \in N$, and $P = \{U_1, U_2, \ldots\}$. Since $U_n \in \mathcal{U}_X$ for each $n \in N$, $P \in \mathcal{P}$. Take any point $g \in W(P)$, then there is an $n \in N$ such that

$$g = x_1 - y_1 + \cdots + x_n - y_n$$

for some $(x_i, y_i) \in U_n$ for $i = 1, 2, \ldots, n$. Thus, by Lemma 3.2.1 (2),

$$g \in V_1 + V_2 + \cdots + V_n \subset V_0 = V.$$

It follows that $W(P) \subset V$.

By Claim, $\mathcal{T}_1|_X$ coincides with the original topology for X. Thus, by Lemma 1.1 (1), $\mathcal{T}_1$ is weaker than the topology for $A(X)$. Consequently, $\mathcal{T}_1$ coincides with the topology for $A(X)$, and $\mathcal{W}$ is a neighborhood base of 0 in $A(X)$. ■

Theorem 3.2.4 *$\mathcal{W}_n$ is a neighborhood base of 0 in $A_{2n}(X)$ for each $n \in N$.*

Proof. Fix any $n \in N$. By Theorem 3.2.3, $\mathcal{W}|_{A_{2n}(X)} = \{W(P) \cap A_{2n}(X) : P \in \mathcal{P}\}$ is a neighborhood base of 0 in $A_{2n}(X)$. For each $P = \{U_1, U_2, \ldots\} \in \mathcal{P}$, it is clear that $W_n(P) \subset W(P) \cap A_{2n}(X)$. Thus, it suffices to show the following claim.

Claim. *For each $P \in \mathcal{P}$, there is a $P_1 \in \mathcal{P}$ such that $W(P_1) \cap A_{2n}(X) \subset W_n(P)$.*

Proof of Claim. Let $P = \{U_1, U_2, \ldots\} \in \mathcal{P}$. Put $V_0 = U_1$ and inductively take a collection $\{V_m : m \in N\} \subset \mathcal{U}_X$ such that

$$V_m \circ V_m \circ V_m \subset V_{m-1} \cap U_{m+1} \text{ for each } m \in N.$$

For the collection $P_1 = \{V_m : m \in N\}$, we shall show that

$$W(P_1) \cap A_{2n}(X) \subset W_n(P).$$

Take any $g \in W(P_1) \cap A_{2n}(X)$, and let

$$(1) \qquad g = x_1 - y_1 + x_2 - y_2 + \cdots + x_k - y_k,$$

where $(x_i, y_i) \in V_i$ for $i = 1, 2, \ldots, k$ and $k \in N$. Now, we put

$$\begin{aligned} A(g) &= \{x_i : x_i \text{ is not reduced in the representation (1) of } g, i = 1, 2, \ldots, k\}, \\ B(g) &= \{y_i : y_i \text{ is not reduced in the representation (1) of } g, i = 1, 2, \ldots, k\}. \end{aligned}$$

If there are $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$ such that $x_i = x_j$ ($y_i = y_j$), we regard that these elements x_i and x_j (y_i and y_j) are different elements in $A(g)$ ($B(g)$), respectively. Since $g \in A_{2n}(X)$, we can see that $|A(g)| = |B(g)|$ and $|A(g)| + |B(g)| \leq 2n$. Thus, we put $A(g) = \{a_1, a_2, \ldots, a_\ell\}$ for some $\ell \leq n$, and take $a_i \in A(g)$. Then there is a $k(i,1) \in \{1, 2, \ldots, k\}$ such that $a_i = x_{k(i,1)}$. If $y_{k(i,1)} \in B(g)$, we put $b_{\varphi(i)} = y_{k(i,1)}$. Otherwise, $y_{k(i,1)}$ is reduced in the representation (1) of g, so that there is a $k(i,2) \in \{1, \ldots, k\}$ such that $y_{k(i,1)} = x_{k(i,2)}$, and clearly, $k(i,2) \neq k(i,1)$. If $y_{k(i,2)} \in B(g)$, we put $b_{\varphi(i)} = y_{k(i,2)}$. Otherwise, in the same way, take a $k(i,3) \in \{1, 2, \ldots, k\}$ such that $y_{k(i,2)} = x_{k(i,3)}$, and $k(i,3) \notin \{k(i,1),\ k(i,2)\}$.

We continue this process till an element of $B(g)$ appears, and denote the element of $B(g)$ by $b_{\varphi(i)}$. Clearly, the element $b_{\varphi(i)}$ must be appeared. Furthermore, we carry out this work for every element of $A(g)$. Thus, we get a permutation φ on $\{1, 2, \ldots, \ell\}$ and sequences $\{k(i,1), k(i,2), \ldots, k(i, j(i))\} \subset \{1, 2, \ldots, k\}$, $i = 1, 2, \ldots, \ell$, such that

$$(2) \qquad a_i = x_{(i,1)} \text{ and } b_{\varphi(i)} = y_{k(i,j(i))}, \text{ for } i = 1, 2, \ldots, \ell,$$

$$(3) \qquad y_{k(i,j)} = x_{k(i,j+1)} \text{ for } j = 1, 2, \ldots, j(i) - 1, i = 1, 2, \ldots, \ell, \text{ and}$$

(4) $\quad \{k(i,j) : j = 1, 2, \ldots, j(i), i = 1, 2 \ldots, \ell\}$ consists of distinct numbers.

Thus, from (2) and (3), we have for each $i = 1, 2, \ldots, \ell$,

$$(a_i, b_{\varphi(i)}) \in V_{k(i,1)} \circ V_{k(i,2)} \circ \cdots \circ V_{k(i,j(i))}.$$

Now, let $k(i) = \min\{k(i,1), k(i,2), \ldots, k(i,j(i))\}$, then by (4), the sequence $\{k(1), k(2), \ldots, k(\ell)\}$ is a subsequence of $\{1, 2, \ldots, n\}$ consisting of distinct elements. Thus, by Lemma 3.2.1 (1) and the definition of P_1,

$$(a_i, b_{\varphi(i)}) \in V_{k(i)-1} \subset U_{k(i)} \text{ for each } i = 1, 2, \ldots, \ell.$$

On the other hand, since φ is a permutation on $\{1, 2, \ldots, \ell\}$,

$$g = a_1 - b_{\varphi(1)} + a_2 - b_{\varphi(2)} + \cdots + a_\ell - b_{\varphi(\ell)}.$$

Consequently, since $\{U_{k(1)}, U_{k(2)}, \ldots, U_{k(\ell)}\} \in \mathcal{R}_n(P)$, by Remark 3.2.2, we have $g \in W_n(P)$, so that $W(P_1) \cap A_{2n}(X) \subset W_n(P)$. ■

For a space X and each $n \in N$, we define a mapping j_n from X^{2n} $(= X^n \times X^n)$ to $A_{2n}(X)$ as follows

$$j_n((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = x_1 + x_2 + \cdots + x_n - (y_1 + y_2 + \cdots + y_n)$$

for each $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n) \in X^n$. Theorem 3.2.4 gives the following important result, which is often used in the next chapter.

Corollary 3.2.5 *Let X be a space, $n \in N$ and E be a subset of $A_{2n}(X)$. Then, $0 \in \overline{E}$ if and only if $j_n^{-1}(E) \cap U^n \neq \emptyset$ for each $U \in \mathcal{U}_X$, where $U^n = \{((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) \in X^{2n} : (x_i, y_i) \in U, \ i = 1, 2, \ldots, n\}$.*

Proof. Necessity. Let $U \in \mathcal{U}_X$ and put $P = \{U_1, U_2, \ldots\} \in \mathcal{P}$ such that $U_i = U$ for each $i \in N$. Since $W_n(P)$ is a neighborhood of 0 in $A_{2n}(X)$, we can take a $g \in W_n(P) \cap E$. Then, we have

$$g = x_1 - y_1 + x_2 - y_2 + \cdots + x_n - y_n,$$

where $(x_i, y_i) \in U_i$ for $i = 1, 2, \ldots, n$.

Thus, for $x = ((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n))$, it is clear that $x \in j_n^{-1}(E) \cap U^n$.

Sufficiency. Let $P = \{U_1, U_2, \ldots\} \in \mathcal{P}$, and take $U \in \mathcal{U}$ such that $U \subset U_1 \cap U_2 \cap \cdots \cap U_n$. By the assumption, we can take $x = ((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) \in j_n^{-1}(E) \cap U^n$. Since, $(x_i, y_i) \in U \subset U_i$ for $i = 1, 2, \ldots, n$, it follows that $j_n(x) \in W_n(P) \cap E$. Therefore, by Theorem 3.2.4, $0 \in \overline{E}$. ∎

Corollary 3.2.6 *Let X be a paracompact space and E a subset of $A_2(X)$. Then, $0 \in \overline{E}$ if and only if $\overline{j_1^{-1}(E)} \cap \Delta_X \neq \emptyset$.*

Proof. Since X is paracompact, every open neighborhood U of Δ_X in X^2 is contained in $\mathcal{U}_X$. Thus, from Corollary 3.2.5 we obtain the result. ∎

Chapter 4

The k-property of $A_n(X)$

In this chapter, we shall discuss about the k-property of each $A_n(X)$ and the local compactness of each $F_n(X)$ $(A_n(X))$. In fact, we obtain the characterizations of a metrizable space X such that every $A_n(X)$ is a k-space, $A_3(X)$ is a k-space, and $A_2(X)$ is a k-space, respectively. And also, we obtain the characterizations of a metrizable space X such that every $F_n(X)$ is locally compact. The main results in this chapter were obtained in our papers, [27] and [28].

4.1 Preliminaries

We shall begin this section with some definitions.

Definition 4.1.1 Let $\mathcal{K}$ be a cover of a space X, then $\mathcal{K}$ is *generating* if a subset E of X is closed in X iff for each $K \in \mathcal{K}$, $E \cap K$ is closed in K. And, if the cover $\mathcal{K} = \{K \subset X : K \text{ is compact}\}$ is generating, we call such a space X a *k-space*. The spaces with a countable generating cover consisting of compact subsets are called *k_ω-spaces*.

The following facts concerning generating cover are often used in this chapter.

Lemma 4.1.2 (i) *If $\mathcal{K}$ is a generating cover of a space X and E is a closed subset of X, then $\{K \cap E : K \in \mathcal{K}\}$ is a generating cover of E.*

(ii) *If a cover $\mathcal{K}$ of a space X is refined by a generating cover of X, then $\mathcal{K}$ is generating.*

(iii) *If $\mathcal{H}$ is a generating cover of a space X, and $\mathcal{K}$ is a cover of X such that $\mathcal{K}_H = \{K \cap H : K \in \mathcal{K}\}$ is a generating cover of H for each $H \in \mathcal{H}$, then $\mathcal{K}$ is a generating cover of X.*

Definition 4.1.3 For each subset E in $F(X)$ $(A(X))$, we set

$$\mathrm{car}E = \min\{B \subset X : E \subset F(B,\ X)\ (A(B,\ X))\},$$

where $F(B,\ X)$ $(A(B,\ X))$ is the subgroup of $F(X)$ $(A(X))$ generated by elements of B, and it is called the *carrier* of E in X.

Recall that a subset E of a space Z is *bounded* (in Z) iff every real-valued continuous function on Z is bounded on E. The following theorem was proved in [4].

Theorem 4.1.4 *If E is a bounded set in $F(X)$ $(A(X))$, then* car$E$ *is bounded in X.*

One of the techniques of studying the topological structure of $F(X)$ $(A(X))$ is the canonical representation of the space $F(X)$ $(A(X))$ as the union of its closed subspaces $F_n(X)$ $(A_n(X))$. And, the technique is often used. For example, Theorem 1.2.5 is an important fact for our investigation. The theorem implies that if X is a compact space, then $F(X)$ $(A(X))$ is a k_ω-space. Later, the result was generalized: if X is a k_ω-space, then $F(X)$ is also a k_ω-space. In fact, J. Mack, S. A. Morris and E. T. Ordman [18] showed that if X is a k_ω-space, then $F_n(X)$ $(A_n(X))$ is a k_ω-space for each $n \in N$ and also E is a closed subset of $F(X)$ $(A(X))$ if and only if $E \cap F_n(X)$ is a closed subset of $F_n(X)$ ($E \cap A_n(X)$ is a closed subset of $A_n(X)$) for each $n \in N$. The class of k_ω-spaces includes not only compact spaces but all locally compact Lindelöf spaces, and is narrower than the one of σ-compact spaces.

However $F(X)$ $(A(X))$ need not be a k-space even if X is a k-space. That is, T. H. Fay, E. T. Ordman and B. V. S. Thomas [8] showed that the free topological group of the space of rationals Q is not a k-space, in fact, they showed $F_3(Q)$ is not a k-space. Recently, A. V. Arhangel'skiĭ, O. G. Okunev and V. G. Pestov [4] showed the following characterization of a metrizable space X such that $F(X)$ ($A(X)$) is a k-space.

Theorem 4.1.5 *If X is metrizable and X' is the set of all nonisolated points in X, then the following conditions are equivalent:*

(a) *$A(X)$ is a k-space,*

(b) *$A(X)$ is homeomorphic to a product of a k_ω-space with a discrete space,*

(c) *X is locally compact and X' is separable.*

Theorem 4.1.6 *If X is metrizable, then the following conditions are equivalent:*

(a) *$F(X)$ is a k-space,*

(b) *$F(X)$ is a k_ω-space or discrete,*

(c) *X is locally compact separable or discrete.*

In the proof of these theorems, they used some concrete spaces such that the free (Abelian) topological groups over them are not k-spaces; for example, the Fréchet-Urysohn fan $V(\aleph_1)$ of cardinality $\aleph_1$, the hedgehog space $J(\aleph_0)$ of spininess $\aleph_0$ such that each spininess is a sequence which converges to the center point, and $Y = C \oplus \{x_\alpha : \alpha < \omega_1\}$, where C is a convergent sequence with its limit and $\{x_\alpha : \alpha < \omega_1\}$ is a discrete collection. In fact, they proved that neither $A_3(V(\aleph_1))$, $F_2(V(\aleph_1))$, $A(J(\aleph_0))$, $F(J(\aleph_0))$, nor $F_4(Y)$ is a k-space (cf. [23]). Since, by most of their ways, it was shown that $F_n(X)$ ($A_n(X)$), as a closed subspace of $F(X)$ ($A(X)$), is not a k-space for some $n \in N$, we naturally raise the following question:

> *if $F_n(X)$ ($A_n(X)$) is a k-space (locally compact space) for each $n \in N$, then is $F(X)$ ($A(X)$) a k-space ?*

It is one of the aims of this chapter to discuss about the above question.

4.2 Test spaces

In this section, we introduce three spaces M_1, M_2 and M_3 which have the following properties:

(1) $A_n(M_1)$ is a k-space for each $n \in N$,

(2) $A_3(M_2)$ is not a k-space, and

(3) $A_4(M_3)$ is not a k-space.

The construction of these spaces are simple, but they play essential roles in the proof of the main results in the next section.

Constructions. Let M_1 be a metrizable space such that $M_1 = X_0 \cup \bigcup_{i=1}^{\infty} X_i$ such that

(1) X_i is an infinite discrete open subspace of M_1 for each $i \in N$, and

(2) X_0 is a compact subspace of M_1 and $\{V_k = X_0 \cup \bigcup_{i=k}^{\infty} X_i : k \in N\}$ is a neighborhood base of X_0 in X, i.e. for each open set U in X which contains X_0, there is a $k \in N$ such that $X_0 \subset V_k \subset U$.

In the above definition, if each X_i consists of countably many elements and X_0 is a one point set, we denote the space by M_1'. We put $C = \{\frac{1}{n} : n \in N\} \cup \{0\}$ with the subspace topology of I. Let $M_2 = \oplus\{C_i : i \in N\} \oplus M_1'$, where C_i is a copy of C for each $i \in N$. Let $M_3 = \oplus\{C_\alpha : \alpha < \omega_1\}$, where C_α is a copy of C for each $\alpha < \omega_1$.

Remark 4.2.1 The concrete example of M_1 is the hedgehog space $J(\kappa)$ of spininess κ such that each spininess is a sequence which convreges to the center point. In [27] we proved that $A_n(J(\kappa))$ is a k-space for each $n \in N$. On the other hand, it was proved in [4] that $A(J(\kappa))$ is not a k-space if $\kappa \geq \aleph_0$. Furthermore, by [4], we know that $A_4(M_2)$ is not a k-space.

Theorem 4.2.2 *$A_n(M_1)$ is a k-space for each $n \in N$.*

Proof. In order to prove the theorem, it suffices to show that

(*) for each $n \in N$ and $E \subset A_n(M_1)$ such that $E \cap K$ is closed in K for each compact subset K of $A_n(M_1)$, if $0 \in \overline{E}$ then $0 \in E$.

For, if $A_n(M_1)$ is not a k-space for some $n \in N$, then there is a subset H of $A_n(M_1)$ such that $H \cap K$ is closed in K for each compact subset K of $A_n(M_1)$ and $\overline{H} \setminus H \neq \emptyset$. Take a point $g \in \overline{H} \setminus H$, and let $E = H - g$. Then, it can be seen that E is a subset of $A_{2n}(M_1)$ such that $E \cap K$ is closed in K for each compact subset K of $A_{2n}(M_1)$ and $0 \in \overline{E} \setminus E$.

Now, let prove the property (*). Take an arbitrary $n \in N$ (we can assume that $n \geq 2$). Let E be a subset of $A_n(M_1)$ such that $E \cap K$ is closed in K for each compact subset K of $A_n(M_1)$, and assume that $0 \in \overline{E}$. Since A_0 (see Lemma 1.2.4) is an open neighborhood of 0 in $A(M_1)$, $0 \in \overline{E \cap A_0}$. Furthermore, note that

$$E \cap A_0 = \bigcup\{E \cap (A_{2m}(M_1) \setminus A_{2m-1}(M_1)) : m \leq \frac{n}{2}, m \in N\},$$

then there is an $m \in N$ with $m \leq \dfrac{n}{2}$ such that

$$0 \in \overline{E \cap (A_{2m}(M_1) \setminus A_{2m-1}(M_1))}.$$

And, we put $D = E \cap (A_{2m}(M_1) \setminus A_{2m-1}(M_1))$.

On the other hand, by the properties (1) and (2) of the definition of M_1, we can find a countable uniform base $\mathcal{U}$ of the universal uniformity $\mathcal{U}_{M_1}$ of M_1 such that $\mathcal{U} = \{U_k = G_k \cup \Delta_{M_1} : k \in N\}$, where for each $k \in N$, G_k is an open neighborhood of Δ_{X_0} in $M_1 \times M_1$ such that $G_k \subset V_k \times V_k$. Now, apply Corollary 3.2.5, then we have

$$j_m^{-1}(D) \cap (U_k)^m \neq \emptyset \text{ for each } k \in N.$$

Let take a point $x_k \in j_m^{-1}(D) \cap (U_k)^m$ for each $k \in N$. Since $g_k = j_m(x_k) \in A_{2m}(M_1) \setminus A_{2m-1}(M_1)$, $x_k \in (G_k)^m$, and $\operatorname{car} g_k \subset V_k$. It follows that $K = \bigcup\{\operatorname{car} g_k : k \in N\} \cup X_0$ is a compact subset of M_1, and by Lemma 1.2.3 (3) and Theorem 3.1.3, $A_n(K)$ is a compact subset of $A_n(M_1)$. Hence we have $E \cap A_n(K)$ is closed in $A_n(K)$. Since $\{x_i : i \in N\} \subset j_m^{-1}(D)$,

$$\{g_i : i \in N\} \subset D \cap A_n(K) \subset E \cap A_n(K).$$

On the other hand,

$$\{x_i : i \in N\} \cap (U_k)^m \neq \emptyset \text{ for each } k \in N.$$

Hence, by Corollary 3.2.5, $0 \in \overline{\{g_i : i \in N\}}$. Thus we have

$$0 \in \overline{E \cap A_n(K)} = E \cap A_n(K) \subset E.$$

Consequently, $A_n(M_1)$ is a k-space for each $n \in N$. ∎

Corollary 4.2.3 *$A_n(J(\kappa))$ is a k-space for each $n \in N$.*

Theorem 4.2.4 *$A_3(M_2)$ is not a k-space.*

Proof. For each $n \in N$, we put $X_n = \{x_{n,i} : i \in N\}$, $C_n = \{c_{n,i} : i \in N\} \cup \{c_n\}$, and $X_0 = \{x\}$. For each $n, j \in N$, let

$$g_{n,j} = c_n - c_{n,j} + x_{n,j}, \text{ and } E = \{g_{n,j} : n, j \in N\}.$$

We shall prove that

(1) $E \cap K$ is closed in K for each compact subset K in $A_3(M_2)$, and

(2) $x \in \overline{E} \setminus E$.

Let K be a compact subset of $A_3(M_2)$, in fact of $A(M_2)$. Then, by Theorem 4.1.4, car$K$ is bounded in M_2. Hence, there are finite subsets F_1 and $F_{2(n)}$ of N, $n \in N$, such that

$$\text{car}K \subset \bigcup\{C_n : n \in F_1\} \cup \bigcup\{x_{n,i} : i \in F_{2(n)}, n \in N\} \cup \{x\}.$$

Thus, by the definition of E,

$$\text{car}(K \cap E) \subset \bigcup\{\text{car}g_{n,i} : n \in F_1 \text{ and } i \in F_{2(n)}\}.$$

Hence, $K \cap E$ is finite and thereby it is closed in K.

Next, we shall prove (2). Since $x \notin E$, we shall show that $x \in \overline{E}$. Since $A_3(M_2)$ is closed in $A(M_2)$, it suffices to show that $x \in \overline{E}^{A(M_2)}$. Let U be an open neighborhood of x in $A(M_2)$. Then we can choose an open neighborhood W of 0 in $A(M_2)$ such that $W + W + x \subset U$. Since $W + x$ is an open neighborhood of x in $A(M_2)$, there is an $N \in N$ such that $\{x_{n,i} : n \geq N, i \in N\} \subset W + x$. And, for each $n \geq N$, there is an $i_n \in N$ such that $c_n - c_{n,i_n} \in W$. It follows that

$$\{g_{n,i_n} : n \geq N\} \subset W + W + x \subset U.$$

Thus $x \in \overline{E}$. ∎

To prove that $A_4(M_3)$ is not a k-space, we need the following technical lemma which was obtained in [15].

Lemma 4.2.5 *There is a collection $\mathcal{E} = \{E_\alpha : \alpha < \omega_1\}$ of infinite subsets of N such that*

(1) *$E_\alpha \cap E_\beta$ is finite for each $\alpha, \beta < \omega_1$ with $\alpha \neq \beta$,*

(2) *for each $\alpha < \omega_1$, $\max(E_\alpha \cap E_\beta) \neq \max(E_\alpha \cap E_\gamma)$ for each $\beta, \gamma < \alpha$ with $\beta \neq \gamma$.*

Theorem 4.2.6 *$A_4(M_3)$ is not a k-space.*

Proof. Let $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ and $\mathcal{B} = \{B_\beta : \beta < \omega_1\}$ are uncountable subcollections of $\mathcal{E}$ defined in Lemma 3.5 such that $\mathcal{A} \cup \mathcal{B} = \mathcal{E}$ and $\mathcal{A} \cap \mathcal{B} = \emptyset$. Then it can be proved that

(3) there are no infinite subsets A of N such that $A_\alpha \setminus A$ is finite and $B_\alpha \cap A$ is finite for each $\alpha < \omega_1$.

We put $C_\alpha = \{x_{\alpha,n} : n \in N\} \cup \{x_\alpha\}$, where x_α is the limit point of $\{x_{\alpha,n} : n \in N\}$, for each $\alpha < \omega_1$. Let, for each $\alpha, \beta < \omega_1$,

$$\begin{aligned} A_{\alpha,\beta} &= \{x_{\alpha,n} - x_\alpha + x_{\beta,n} - x_\beta : n \in A_\alpha \cap B_\beta\}, \text{ and} \\ E &= \bigcup \{A_{\alpha,\beta} : \alpha, \beta < \omega_1\}. \end{aligned}$$

We shall show that

(4) $E \cap K$ is closed in K for each compact subset K of $A_4(M_3)$, and

(5) $0 \in \overline{E} \setminus E$.

Let K be an compact subset of $A_4(M_3)$, then car$K$ is bounded in M_3 by Theorem 4.1.4. We can take a finite subset $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of ω_1 such that car$K \subset \bigcup_{i=1}^{n} C_{\alpha_i}$, so that $E \cap K \subset \bigcup\{A_{\alpha_i,\alpha_j} : i, j = 1, 2, \ldots, n\}$. On the other hand, by the property (1) of Lemma 4.2.5, each $A_{\alpha,\beta}$ is finite. It follows that $E \cap K$ is finite, and (4) is proved.

Before we show (5), we shall define a uniform base $\mathcal{U}$ of the universal uniformity on M_3, as follows. For each $\alpha < \omega_1$ and $k \in N$, let $V_{\alpha,k} = \{x_{\alpha,m} : m \geq k\} \cup \{x_\alpha\}$, and we put

$$\mathcal{U}_\alpha = \{U_{\alpha,k} = (V_{\alpha,k} \times V_{\alpha,k}) \cup \Delta_\alpha : k \in N\},$$

where Δ_α is the diagonal of $C_\alpha \times C_\alpha$. For each $M = \{n_\alpha \in N : \alpha < \omega_1\} \in N^{\omega_1}$, let

$$U(M) = \bigcup\{U_{\alpha,n_\alpha} : \alpha < \omega_1\}, \text{ and } \mathcal{U} = \{U(M) : M \in N^{\omega_1}\}.$$

Then, $\mathcal{U}$ is a uniform base of the universal uniformity on M_3. To show that $0 \in \overline{E}$, by Corollary 3.2.5, it suffices show that

$$j_2^{-1}(E) \cap U^2 \neq \emptyset \text{ for each } U \in \mathcal{U}.$$

Take an $M = \{n_\alpha : \alpha < \omega_1\} \in N^{\omega_1}$, and for each $\alpha < \omega_1$, let

$$A'_\alpha = \{n \in A_\alpha : n \geq n_\alpha\} \text{ and } B'_\alpha = \{n \in B_\alpha : n \geq n_\alpha\}.$$

Clearly each A'_α is an infinite subset of N. Assume that $A'_\alpha \cap B'_\beta = \emptyset$ for each $\alpha, \beta < \omega_1$. Then $\bigcup_{\alpha<\omega_1} A'_\alpha \cap \bigcup_{\beta<\omega_1} B_\beta = \emptyset$. For the infinite set $A = \bigcup_{\alpha<\omega_1} A_\alpha$ in N, we can prove that

$$A \cap B_\beta \subset \{n \in N : n < n_\beta\} \text{ for each } \beta < \omega_1.$$

Hence $A \cap B_\beta$ is finite for each $\beta < \omega_1$, and also

$$A_\alpha \setminus A \subset A_\alpha \setminus A'_\alpha = \{n \in N : n < n_\alpha\}.$$

Thereby $A_\alpha \setminus A$ is finite for each $\alpha < \omega_1$. This contradicts the property (3). Thus, there are $\alpha, \beta < \omega_1$ such that $A'_\alpha \cap B'_\beta \neq \emptyset$. Take an $n \in A'_\alpha \cap B'_\beta$, then

$$\begin{aligned}(x_{\alpha,n}, x_\alpha) &\in V_{\alpha,n_\alpha} \times V_{\alpha,n_\alpha} \subset U(M), \text{ and}\\ (x_{\beta,n}, x_\beta) &\in V_{\beta,n_\beta} \times V_{\beta,n_\beta} \subset U(M).\end{aligned}$$

It follows that

$$x = ((x_{\alpha,n}, x_{\beta,n}), (x_\alpha, x_\beta)) \subset U(M) \times U(M).$$

Since $j_2(\boldsymbol{x}) = x_{\alpha,n} - x_\alpha + x_{\beta,n} - x_\beta \in E$,

$$\boldsymbol{x} \in j_2^{-1}(E) \cap U(M)^2 \neq \emptyset.$$

Consequently, $A_4(M_3)$ is not a k-space. ■

Remark 4.2.7 The essential idea of the proof of (5) is used by Malyhin to prove that the tightness of the product $V(\aleph_1) \times V(\aleph_1)$ is ω_1. The proof was cited with his permission in [4]. Therefore, $V(\aleph_1) \times V(\aleph_1)$ is not a k-space. Since X^2 is closed embedded in $A_3(X)$ for each space X (cf. [23]), $A_3(V(\aleph_1))$ is not a k-space. In fact, in the next section, we shall show that $A_2(V(\aleph_1))$ is not a k-space.

4.3 The k-property of $A_n(X)$

In this section, we shall give characterizations of a metrizable space X such that every $A_n(X)$ is a k-space for $n \geq 4$, $A_3(X)$ is a k-space, and $A_2(X)$ is a k-space, respectively. First, we introduce the result for the mapping i_n, which was pointed out in [3]. Recall that a space X is *Dieudonné complete* if there is a complete uniformity on the space X ([6]). Every paracompact space is Dieudonné complete, and the closure of every bounded subset of a Dieudonné complete space is compact.

Proposition 4.3.1 *Let X be a Dieudonné complete space. Then, for each $n \in N$, the mapping i_n is quotient if $A_n(X)$ is a k-space.*

Its proof can be easily obtained from the following well-known facts, and therefore we omit the proof.

(1) *For each compact subset K of $A_n(X)$, there is a compact subset C of $(X \oplus -X \oplus \{0\})^n$ such that $i_n(C) = K$ if X is a Diedonné complete space* ([4]).

(2) *A continuous mapping $f : X \longrightarrow Y$ of a topological space to a k-space Y is quotient if and only if for every compact subset $Z \subset Y$ the restriction $f|_{f^{-1}(Z)} : f^{-1}(Z) \longrightarrow Z$ is quotient* ([6], Theorem 3.3.22).

Theorem 4.3.2 *If X is a metrizable space, then the following statements are equivalent:*

(a) *$A_n(X)$ is a k-space for each $n \in N$,*

(b) *$A_4(X)$ is a k-space,*

(c) *i_n is a quotient mapping for each $n \in N$,*

(d) *i_4 is a quotient mapping,*

(e) *either X is locally compact and the set X' of all nonisolated points of X is separable, or X' is compact.*

Proof. The implications (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d) are clear, and by Proposition 4.3.1, (a) and (c), (b) and (d) are equivalent. Thus, it suffices to show that the implications (b) $\Rightarrow$ (e) and (e) $\Rightarrow$ (a).

Proof of (b) $\Rightarrow$ (e). Assuming the contrary, we can consider the following two cases.

Case 1. *X is not locally compact and X' is not compact.*

Case 2. *X' is not separable.*

In Case 1, we can take an infinite discrete sequence $\{c_n \in X' : n \in N\}$ in X. For each $n \in N$, since $c_n \in X'$, there is a convergent sequence $\{c_{n,i} : i \in N\}$ in X which converges to c_n. Furthermore, since X is not locally compact, we can find a point $x \in X$ and $\{B_n : n \in N\}$ be a countable neighborhood base of x in X such that for each $n \in N$, $B_n \supset B_{n+1}$ and $B_n \setminus B_{n+1}$ contains an infinite discrete (in X) sequence $X_n = \{x_{n,i} :\in N\}$. Without loss of generality, we may assume that the collection $\{B_1\} \cup \{C_n : n \in N\}$ is discrete in X, where $C_n = \{c_{n,i} : i \in N\} \cup \{c_n\}$. It follows that $Y = \bigcup\{C_n : n \in N\} \cup \bigcup\{X_n : n \in N\} \cup \{x\}$ is a closed subset of X. Thus Y is homeomorphic to M_2, and therefore, by Theorem 4.2.4, $A_3(M_2)$ is not a k-space. Hence $A_3(Y)$ is not a k-space. Since, by Theorem 3.1.3, $A_3(Y)$ is embedded into $A_3(X)$ as a closed subspace, $A_3(X)$ is not a k-space. Thus, $A_4(X)$ is not a k-space.

In Case 2, since X' has an uncountable discrete collection in X, we can take a closed subspace Y of X which is homeomorphic to M_3. Thus, by Theorem 4.2.6 and the same way as Case 1, we can prove that $A_4(X)$ is not a k-space.

Proof of (e) $\Rightarrow$ (a). By Theorem 4.1.5, if X is locally compact and X' is separable, $A(X)$ is a k-space. Since each $A_n(X)$ is closed in $A(X)$, $A_n(X)$ is a k-space for each $n \in N$. Next, suppose that X' is compact and X is not compact because $A(Y)$ of a compact space Y is a k-space. Then the space X is a space of type M_1. It follows, by Theorem 4.2.2, that $A_n(X)$ is a k-space for each $n \in N$.

Consequently, we have Theorem 4.3.2. ■

Corollary 4.3.3 (Stability theorem) *For a metrizable space X, every $A_n(X)$, $n \in N$, is a k-space if and only if $A_4(X)$ is a k-space.*

Case 1 of the proof of the implication (b) $\Rightarrow$ (e) in Theorem 4.3.2 yields the following (cf. [8]).

Corollary 4.3.4 *Let Q be the space of rationals and P the space of irrationals. Then, neither $A_3(Q)$ nor $A_3(P)$ is a k-space.*

The following theorem can be proved from the result in [20]. Here, we shall give another proof.

Theorem 4.3.5 *For a paracompact space X, $A_2(X)$ is a k-space if and only if X^2 is a k-space.*

Proof. First, we show the "only if" part. Since X^2 is a clopen subset of $(X \oplus -X \oplus \{0\})^2$ and $X^2 \subset i_2^{-1}(A_2(X) \setminus A_1(X))$, by theorem 2.1.3, $i_2(X^2)$ is a clopen subset of $A_2(X) \setminus A_1(X)$. In particular, $i_2(X^2)$ is open in $A_2(X)$. Since $A_2(X)$ is a k-space, $i_2(X^2)$ is a k-space. On the other hand, since $i_2^{-1}(i_2(X^2)) = X^2$, by Theorem 2.1.3, the mapping $i_2|_{X^2} : X^2 \longrightarrow i_2(X^2)$ is a perfect mapping. Thus, X^2 is a k-space.

Next, we show the "if" part. Let E be a subset of $A_2(X)$ such that $E \cap K$ is closed in K for each compact subset K of $A_2(X)$, and take a word $g \in \overline{E}$ We shall show that $g \in E$. The proof is in three cases.

Case 1. $g \in A_2(X) \setminus A_1(X)$.

Since $A_2(X) \setminus A_1(X)$ is open in $A_2(X)$, there is an open neighborhood U of g in $A_2(X)$ such that $\overline{U} \subset A_2(X) \setminus A_1(X)$. We put $H = E \cap \overline{U}$ then we have that $g \in \overline{H}$ and $H \cap K$ is closed in K for each compact subset of $\overline{U}$. On the other hand, by Theorem 2.1.3, $i_2|_{i_2^{-1}(U)} : i_2^{-1}(\overline{U}) \longrightarrow \overline{U}$ is an open mapping, and $i_2^{-1}(\overline{U})$ is a closed subset of the k-space X^2. It follows that $\overline{U}$ is a k-space. Therefore, H is closed in $\overline{U}$, and hence in $A_2(X)$. Consequently, $g \in H \subset E$.

Case 2. $g \in X \oplus -X$.

Recall that A_0 is a clopen subgroup of $A(X)$, then $g+A_0$ is a clopen neighborhood of g in $A(X)$. Note that $(g+A_0) \cap \big((A_2(X) \setminus A_1(X)) \cup \{0\}\big) = \emptyset$. We put $H = (g+A_0) \cap E$, then it is easy to see that $g \in \overline{H} \subset X \oplus -X$ and $H \cap K$ is closed in K for each compact subset K of $X \oplus -X$. Since $X \oplus -X$ is a k-space, H is closed in $X \oplus -X$. Hence $g \in H \subset E$.

Case 3. $g = 0$.

Assume that $0 \in \overline{E} \setminus E$. Let $H = E \cap A_0$, then we can see that $0 \in \overline{H} \setminus H$ and $H \subset A_2(X) \setminus A_1(X)$. Moreover $j_1^{-1}(H) \subset X^2 \setminus \Delta_X$ and by Corollary 3.2.6, $\overline{j_1^{-1}(H)} \cap \Delta_X \neq \emptyset$. Take a point $\boldsymbol{x} = (x, x) \in \overline{j_1^{-1}(H)} \cap \Delta_X$. Let C be an arbitrary compact subset of X^2. Then $j_1(C)$ is a compact subset of $A_2(X)$. By the assumption, $H \cap j_1(C)$ is closed in $j_1(C)$, so that in $A_2(X)$. Thus, $j_1^{-1}(H \cap j_1(C))$ is closed in X^2. Since

$$\begin{aligned} j_1\big|_{j_1^{-1}((A_2(X)\setminus A_1(X))\cap A_0)} : X^2 \setminus \Delta_X &= j_1^{-1}((A_2(X) \setminus A_1(X)) \cap A_0) \\ &\longrightarrow (A_2(X) \setminus A_1(X)) \cap A_0 \end{aligned}$$

is one to one and onto, and $H \cap j_1(C) \subset (A_2(X) \setminus A_1(X)) \cap A_0$, we can see that $j_1^{-1}(H \cap j_1(C)) = j_1^{-1}(H) \cap C$. Thus $j_1^{-1}(H) \cap C$ is closed in X^2, so that in C. It follows that $j_1^{-1}(H)$ is closed in X^2 because X^2 is a k-space. This contradicts that $\boldsymbol{x} \in \overline{j_1^{-1}(H)} \setminus j_1^{-1}(H)$. Consequently, we have $0 \in E$. ∎

Remark 4.2.7 and Theorem 4.3.5 yield

Corollary 4.3.6 *$A_2(V(\aleph_1))$ is not a k-space.*

Corollary 4.3.7 *For a metrizable space X, $A_2(X)$ is a k-space, and the mapping i_2 is quotient.*

Theorem 4.3.8 *If X is a metrizable space, then the following statements are equivalent:*

(a) $A_3(X)$ *is a* k*-space,*

(b) *the mapping* i_3 *is quotient,*

(c) X *is locally compact or the set* X' *of all nonisolated points in* X *is compact.*

Proof. By Proposition 4.3.1, (a) and (b) are equivalent. In the proof of the implication (b) $\Rightarrow$ (e) in Theorem 4.3.2, it was already shown the implication (a) $\Rightarrow$ (c). Furthermore, by the proof of the implication (e) $\Rightarrow$ (a) in Theorem 4.3.2, $A_3(X)$ is a k-space if X' is compact. Thus, to complete the proof, it suffices to show that $A_3(X)$ is a k-space if X is locally compact.

Let X be a locally compact metrizable space. Then X can be represented as the sum of locally compact separable spaces, i.e. $X = \bigoplus\{X_\alpha : \alpha \in A\}$, where each X_α is locally compact separable. For each $\alpha \in A$, let $\mathcal{U}_\alpha$ be the universal uniformity on X_α and Δ_α the diagonal of $X_\alpha \times X_\alpha$. Then, $\mathcal{U} = \{\bigoplus_{\alpha \in A} U_\alpha : U_\alpha \in \mathcal{U}_\alpha\}$ is the universal uniformity on X. To prove that $A_3(X)$ is a k-space, take a subset E of $A_3(X)$ such that $E \cap K$ is closed in K for each compact subset K of $A_3(X)$, and an arbitrary point $g \in \overline{E}$. We shall show that $g \in E$. If $g \in (A_3(X) \setminus A_2(X)) \cup (A_2(X) \setminus A_1(X)) \cup \{0\}$, it can be shown that $g \in E$ in the similar way to the proof of Theorem 4.3.5. Thus we may assume that $g \in X \oplus -X$, in particular, $g \in X$ (if $g \in -X$, we can show similarly).

We put $H = (E - g) \cap A_0$, then H is a subset of $A_4(X)$ such that $H \cap K$ is closed in K for each compact subset K of $A_4(X)$ and $0 \in \overline{H}$. If $0 \in \overline{H \cap A_2(X)}$, then it can be shown that $0 \in H \cap A_2(X)$ and $g \in H$ from the proof of Theorem 4.3.5. Thus, we may assume that $H \subset (A_4(X) \setminus A_3(X)) \cup \{0\}$ because $H \cap (A_3(X) \setminus A_2(X)) = \emptyset$. By the definition of H, we put

$$H = \{h_\lambda = x_\lambda - y_\lambda + z_\lambda - g : \lambda \in \Lambda\},$$

where $x_\lambda, y_\lambda, z_\lambda \in X$ for each $\lambda \in \Lambda$. Since $g \in X$, there is an $\alpha_0 \in A$ such that $g \in X_{\alpha_0}$. Let

$$H_1 = \{h_\lambda \in H : x_\lambda \in X_{\alpha_0} \text{ or } z_\lambda \in X_{\alpha_0}\}.$$

Hence it is easy to see that $0 \notin \overline{H \setminus H_1}$ because $j_2^{-1}(H \setminus H_1) \cap U^2 = \emptyset$ for each $U \in \mathcal{U}$. Thus we have $0 \in \overline{H_1}$. Now, we assume that $0 \in \overline{H_2}$, where

$$H_2 = \{h_\lambda \in H_1 : x_\lambda, z_\lambda \in X_{\alpha_0}\}.$$

Then $j_2^{-1}(H_2) \cap U^2 \neq \emptyset$ for each $U \in \mathcal{U}$. By the definition of $\mathcal{U}$, $j_2^{-1}(H_3) \cap U^2 \neq \emptyset$ for each $U \in \mathcal{U}$, where

$$H_3 = \{h_\lambda \in H_2 : y_\lambda \in X_{\alpha_0}\}.$$

It follows that $0 \in \overline{H_3}$. On the other hand, H_3 is a subset of $A_4(X_{\alpha_0})$ and $A_4(X_{\alpha_0})$ can be considered as a closed subset of $A_4(X)$ by Theorem 3.1.3. Since X_{α_0} is locally compact separable, by Theorem 4.3.2, $A_4(X_{\alpha_0})$ is a k-space. Furthermore, $H_3 \subset H \cap A_4(X_{\alpha_0})$ and $H \cap A_4(X_{\alpha_0}) \cap K$ is closed in K for each compact subset K of $A_4(X_{\alpha_0})$. We can easily see that $0 \in H \cap A_4(X_{\alpha_0}) \subset H$. Therefore it suffices to show that $0 \in H$ if $0 \in \overline{H_1 \setminus H_2}$. Let

$$H_4 = \{h_\lambda : x_\lambda \notin X_{\alpha_0} \text{ and } z_\lambda \in X_{\alpha_0}\},$$

and we may assume that $0 \in \overline{H_4}$. Now, we put

$$\begin{aligned} \Lambda' &= \{\lambda \in \Lambda : h_\lambda \in H_4\}, \\ L &= \{(z_\lambda, g) : \lambda \in \Lambda'\}, \text{ and} \\ M &= \{(x_\lambda, y_\lambda) : \lambda \in \Lambda'\}. \end{aligned}$$

Since $j_2^{-1}(H_4) \cap U^2 \neq \emptyset$ for each $U \in \mathcal{U}$,

(1) $L \cap U \neq \emptyset$ for each $U \in \mathcal{U}$, and

(2) $M \cap U \neq \emptyset$ for each $U \in \mathcal{U}$.

Since $L \subset X_{\alpha_0}^2$, by (1), $\overline{L} \cap \Delta_{\alpha_0} \neq \emptyset$ and, in particular, we can see that $(g, g) \in \overline{L}$. Therefore $g \in \overline{\{z_\lambda : \lambda \in \Lambda'\}}$. Now, let $\mathcal{B}_g = \{B_n : n \in N\}$ be a countable neighborhood base of g in X_{α_0} such that $B_{n+1} \subset B_n$ and $B_1 = X_{\alpha_0}$. For each $n \in N$, let

$$M_n = \{(x_\lambda, y_\lambda) : z_\lambda \in B_n \setminus B_{n+1}\}.$$

Then $M = \bigcup_{n=1}^{\infty} M_n$. On the other hand, by (2) $\overline{M} \cap \Delta_X \neq \emptyset$. We consider the following two cases.

Case 1. *There is a subsequence $\{k_m : m \in N\}$ of N such that $\overline{M_{k_m}} \cap \Delta_X \neq \emptyset$ for each $m \in N$.*

For each $m \in N$, we can take an $\alpha_m \in A$ such that $\overline{M_{k_m}} \cap \Delta_{\alpha_m} \neq \emptyset$. We put $N_{\alpha_m} = M_{k_m} \cap X_{\alpha_m}^2$, then $\overline{N_{\alpha_m}} \cap \Delta_{\alpha_m} \neq \emptyset$. Thus, $\overline{j_2^{-1}(H_5)} \cap \Delta_Y \neq \emptyset$, so that $0 \in \overline{H_5}$, where

$$
\begin{aligned}
H_5 &= \{h_\lambda \in H_4 : z_\lambda \in B_{k_1} \text{ and } (x_\lambda, y_\lambda) \in \bigcup_{m=1}^{\infty} N_{\alpha_m}\} \text{ and} \\
Y &= \bigoplus_{m=0}^{\infty} X_{\alpha_m}.
\end{aligned}
$$

Since Y is a locally compact separable metrizable closed subspace of X, $A_4(Y)$ is closed in $A_4(X)$ and, by Theorem 4.3.2, $A_4(Y)$ is a k-space. Furthermore, $H_5 \subset H \cap A_4(Y)$ and $H \cap A_4(Y) \cap K$ is closed in K for each compact subset K of $A_4(Y)$. It follows that $0 \in H \cap A_4(Y) \subset H$.

Case 2. *There is an $n \in N$ such that $\overline{M_m} \cap \Delta_X = \emptyset$ for each $m \geq n$.*

Let

$$
\begin{aligned}
H_6 &= \{h_\lambda : z_\lambda \in B_n\} \text{ and} \\
M(n) &= \bigcup_{m \geq n} M_m.
\end{aligned}
$$

Then it is easy to see that $0 \in \overline{H_6}$, so that $\overline{M(n)} \cap \Delta_X \neq \emptyset$. Thus, we can take an $\alpha \in A$ such that $\overline{M(n)} \cap \Delta_\alpha \neq \emptyset$. Now, we can assume that $N_m = M_m \cap X_\alpha \neq \emptyset$ for each $m \geq n$. Hence, it can be seen that for each $U \in \mathcal{U}_\alpha$, $\{m \geq n : N_m \cap U \neq \emptyset\}$ is an infinite set because $\overline{N_m} \cap \Delta_\alpha = \emptyset$ for each $m \geq n$ and $\overline{\bigcup_{m \geq n} N_m} \cap \Delta_\alpha \neq \emptyset$. It follows that $0 \in \overline{H_7}$, where

$$
H_7 = \{h_\lambda \in H_6 : (x_\lambda, y_\lambda) \in \bigcup_{m \geq n} N_m\}.
$$

On the other hand, $H_7 \subset A_4(X_{\alpha_0} \oplus X_\alpha)$, $A_4(X_{\alpha_0} \oplus X_\alpha)$ is closed in $A_4(X)$ and $A_4(X_{\alpha_0} \oplus X_\alpha)$ is a k-space. Therefore, in the same argument as in Case 1, we can see that $0 \in H \cap A_4(X_{\alpha_0} \oplus X_\alpha) \subset H$.

Consequently, in any case, we can prove that $0 \in H$, so that $g \in E$. Thus, E is a closed subset of $A_3(X)$. It follows that $A_3(X)$ is a k-space. ∎

Corollary 4.3.9 *Let M_3 be the space constructed in §4.3, then $A_3(M_3)$ is a k-space but $A_4(M_3)$ is not a k-space.*

Finally, we discuss about the questions which are asked by T. H. Fay, E. Ordman and B. V. S. Thomas in [8]. Since they discussed about the free topological group $F_G(X)$ over a space in the sense of Graev, we introduce some notations. For each $n \in N$, we denote the subspace of $F_G(X)$ consisting of words of length not exceeding n by $F_G(X)_n$. Since the unit element e of $F_G(X)$ is a point of X, the mapping i_n is defined on $X \cup X^{-1}$. Analogously, let $A_G(X)$ be the free Abelian topological group over a space X in the sense of Graev, and the subspace $A_G(X)_n$ and the mapping i_n can be defined similarly to these for $F_G(X)$. T. H. Fay, E. Ordman and B. V. S. Thomas [8] asked the following questions.

Question 4.3.10 *Is i_n always a quotient map if X is locally compact ?*

Question 4.3.11 *Is $i_2 : (Q \cup Q^{-1})^2 \longrightarrow F_G(Q)_2$ a quotient map ?*
Is i_2 always a quotient map ?

V. G. Pestov [20] answered these questions as follows.

Theorem 4.3.12 (1) *The mapping $i_2 : (X \oplus X^{-1} \oplus \{e\})^n \longrightarrow F_n(X)$ $\Big((X \cup X^{-1})^n \longrightarrow F_G(X)_n\Big)$ is quotient in either sense if and only if each neighborhood of Δ_X in X^2 is in $\mathcal{U}_X$. In particular, if X is paracompact, i_2 is quotient. Hence, the first part of Question 4.3.11 is yes.*

rm (2) Let X be a locally compact space which is not paracompact, then i_2 is not quotient by 1. Hence, Question 4.3.10 is no.

On the other hand, About the space M_3 described in §4.3, we note the following fact. That implies the negative answer of the abelian version of Question 4.3.10 even if a space X is locally compact and metrizable.

Theorem 4.3.13 *The space M_3 is a locally compact metrizable space such that $A_4(M_3)$ is not a k-space and $A_n(M_3)$ is homeomorphic to $A_G(M_3)_n$ for each $n \in N$. Thus, the mapping i_n is not quotient in either sense.*

Proof. It suffices to show that $A_n(M_3)$ is homeomorphic to $A_G(M_3)_n$ for each $n \in N$. Since M_3 has infinite many isolated points, for each isolated point x in X, M_3 is homeomorphic to $M_3 \setminus \{x\} = M_3'$. On the other hand, from the argument in [9], $F_G(M_3)$ $(A_G(M_3))$ is topologically isomorphic to $F(M_3')$ $(A(M_3'))$, respectively. In fact, the topological isomorphism is the continuous homomorphic extension of the identity on M_3, so that we can prove that $F_G(M_3)_n$ $(A_G(M_3)_n)$ is homeomorphic to $F_n(M_3')$ $(A_n(M_3'))$ for each $n \in N$. Thus, it follows that $A_n(M_3)$ is homeomorphic to $A_G(M_3)_n$ for each $n \in N$. ∎

4.4 Local compactness of $F_n(X)$

In this section we discuss the property that each $F_n(X)$ $(A_n(X))$ is locally compact. As a consequence we will characterize a metrizable space X having the above property. Now, we shall prove the following result which is a generalization of Theorem 1.2.5. The essential idea of the proof is due to M. I. Graev [9].

Theorem 4.4.1 *If $F_n(X)$ $(A_n(X))$ is locally compact for each $n \in N$, then a subset V is open in $F(X)$ $(A(X))$ if and only if $V_n = V \cap F_n(X)$ $(V \cap A_n(X))$ is open in $F_n(X)$ $(A_n(X))$ for each $n \in N$.*

Proof. The necessity is obvious and therefore we shall show only the sufficiency. Let V be an arbitrary open set in $F(X)$, then, each V_n is open in $F_n(X)$. Therefore, we shall show that the converse is also true.

Let $\mathcal{V} = \{V \subset F(X) : V_n \text{ is open in } F_n(X) \text{ for each } n \in N\}$. At first, we shall show that $\mathcal{V}$ induces a group topology on $F(X)$. Indeed, it suffices to show that $\mathcal{V}$ satisfies the following;

> for each $V \in \mathcal{V}$ and $a, b \in F(X)$ such that $ab^{-1} \in V$, there are $U(a), U(b) \in \mathcal{V}$ such that $a \in U(a)$, $b \in U(b)$, and $U(a)\,U(b)^{-1} \subset V$.

Let $V \in \mathcal{V}$ and $a, b \in F(X)$ such that $ab^{-1} \in V$, and assume that $\ell(a), \ell(b) \leq k$ for some $k \in N$. Then, by induction on $i \geq k$, we shall construct the sets $U_i(a)$ and $U_i(b)$ such that

(1) $a \in U_i(a)$ and $b \in U_i(b)$,

(2) $U_i(a)$ and $U_i(b)$ are open in $F_i(X)$,

(3) $U_j(a) \subset U_i(a)$ and $U_j(b) \subset U_i(b)$ for $j \leq i$,

(4) $\overline{U_i(a)}\,\overline{U_i(b)}^{-1} \subset V_{2i}$,

(5) $\overline{U_i(a)}$ and $\overline{U_i(b)}$ are compact.

Since V_{2k} is open in $F_{2k}(X)$, there is an open set V' in $F(X)$ such that $V_{2k} = V' \cap F_{2k}(X)$. Thus, we can take open sets U_a, U_b in $F(X)$ such that

$$\overline{U_a}\,\overline{U_b}^{-1} \subset V', \text{ and } \overline{U_a \cap F_k(X)} \text{ and } \overline{U_b \cap F_k(X)} \text{ are compact,}$$

because $F_k(X)$ is a locally compact closed subset of $F(X)$. Now put

$$U_k(a) = U_a \cap F_k(X) \text{ and } U_k(b) = U_b \cap F_k(X).$$

Then, it is clear that $U_k(a)$ and $U_k(a)$ satisfy the above propertis for k.

Suppose that the sets $U_i(a)$ and $U_i(b)$ have been constructed for $i = k, k+1, \ldots, n$, we shall construct the sets $U_{n+1}(a)$ and $U_{n+1}(b)$, as follows. Put

$$E = \overline{U_n(a)}^{-1}(F_{2n+2}(X) \setminus V_{2n+2})\,\overline{U_n(b)}.$$

Then E is a closed subset of $F(X)$ by the property (5), and also we have $e \notin E$. For, if $e = u_n^{-1} v w_n$ for some $u_n \in \overline{U_n(a)}$, $v \in F_{2n+2}(X) \setminus V_{2n+2}$, and $w_n \in \overline{U_n(b)}$, then it follows that

$$v = u_n w_n^{-1} \in \overline{U_n(a)}\,\overline{U_n(b)}^{-1} \subset V_{2n} \subset V_{2n+2},$$

and this is a contradiction. Therefore we can take a neighborhood U_e of e such that

$$\overline{U_e}\,\overline{U_e}^{-1} \subset F(X) \setminus E, \text{ and } \overline{U_e \cap F_{2n+1}(X)} \text{ is compact.}$$

Now we define the sets

$$U_{n+1}(a) = (U_n(a)\,U_e) \cap F_{n+1} \text{ and } U_{n+1}(b) = (U_n(b)\,U_e) \cap F_{n+1}.$$

Clearly, these sets satisfy the properties (1), (2) and (3). We shall show that they satisfy the properties (4) and (5). Let $u = vw \in F_{n+1}(X)$, where $v \in U_n(a)$ and

$w \in U_e$. Since $\ell(v) \leq n$ and $\ell(u) \leq n+1$, $\ell(w)$ must be less than or equal to $n+n+1 = 2n+1$. Hence this means that

$$U_{n+1}(a) \subset U_n(a)\,(U_e \cap F_{2n+1}(X)).$$

Thus we have

$$\begin{aligned}
\overline{U_{n+1}(a)} &\subset \overline{U_n(a)\,(U_e \cap F_{2n+1}(X))} \\
&\subset \overline{\overline{U_n(a)}\;\overline{(U_e \cap F_{2n+1}(X))}} \\
&\subset \overline{\overline{U_n(a)}\,(\overline{U_e} \cap F_{2n+1}(X))}.
\end{aligned}$$

Since $\overline{U_n(a)}\ \overline{(U_e \cap F_{2n+1}(X))}$ is compact, we have that $\overline{U_{n+1}(a)}$ is compact. Analogously, it can be seen that $\overline{U_{n+1}(b)}$ is compact, i.e. the property (5) is satisfied. Furthermore, since $\overline{U_n(a)}\,(\overline{U_e} \cap F_{2n+1}(X))$ is closed in $F(X)$,

$$\overline{U_{n+1}(a)} \subset \overline{U_n(a)}\,(\overline{U_e} \cap F_{2n+1}(X)) \subset \overline{U_n(a)}\;\overline{U_e}.$$

Similarly

$$\overline{U_{n+1}(b)} \subset \overline{U_n(b)}\,(\overline{U_e} \cap F_{2n+1}(X)) \subset \overline{U_n(b)}\;\overline{U_e}.$$

Therefore

$$\begin{aligned}
\overline{U_{n+1}(a)}\;\overline{U_{n+1}(b)} &\subset \overline{U_n(a)}\;\overline{U_e}\;\overline{U_e}^{-1}\;\overline{U_n(b)}^{-1} \\
&\subset \overline{U_n(a)}\,(F(X) \setminus E)\,\overline{U_n(b)}^{-1}.
\end{aligned}$$

On the other hand

$$\overline{U_{n+1}(a)}\;\overline{U_{n+1}(b)} \subset F_{2n+2}(X),$$

Now assume that there is a word

$$x \in \overline{U_{n+1}(a)}\;\overline{U_{n+1}(b)} \cap (F_{2n+2}(X) \setminus V_{2n+2}).$$

By the above result, put $x = u_n y v_n^{-1}$, where $u_n \in \overline{U_n(a)}$, $y \in F(X) \setminus E$ and $v_n \in U_n(b)$. Then we have

$$y = u_n^{-1} x v_n \in \overline{U_n(a)}^{-1} (F_{2n+2}(X) \setminus V_{2n+1})\,\overline{U_n(b)} = E,$$

but it is impossible. Consequently we can show that

$$\overline{U_{n+1}(a)}\;\overline{U_{n+1}(b)}^{-1} \subset V_{2n+2}.$$

We now define the sets

$$U(a) = \bigcup_{i=k}^{\infty} U_i(a) \text{ and } U(b) = \bigcup_{i=k}^{\infty} U_i(b).$$

Thus, by the properties (1) $\sim$ (4), it is easily seen that

$$U(a), U(b) \in \mathcal{V}, \text{ and } U(a)U(b)^{-1} \subset V.$$

From the above argument, $\mathcal{V}$ induces a certain group topology $\mathcal{T}_1$. Clearly, $\mathcal{T}_1|_X$ is equal to the original topology on X. This means that $\mathcal{T}_1$ is weaker than the free group topology on $F(X)$. Consequently each $V \in \mathcal{V}$ is open in $F(X)$, and the theorem is proved. ■

Corollary 4.4.2 *For a space X, if $F_n(X)$ $(A_n(X))$ is locally compact for each $n \in N$, then $F(X)$ $(A(X))$ is a k-space.*

Now, we introduce some characterizations of a metrizable space X such that each $F_n(X)$ is locally compact.

Proposition 4.4.3 *Let X be a metrizable space. $F_n(X)$ is locally compact for each $n \in N$ if and only if X is compact or discrete.*

Proof. Assume that there is a metrizable space X such that each $F_n(X)$ is locally compact, but X is noncompact and nondiscrete. Then we can find a discrete sequence $\{x_n : n = 0, 1, \ldots\}$ in X such that x_0 is nonisolated. Let $\{y_n : n \in N\}$ be a sequence in X which converges to x_0 such that $\{y_n : n \in N\} \cap \{x_n : n = 0, 1, \ldots\} = \emptyset$. Now, we put $C = \{y_n : n \in N\} \cup \{x_0\}$ and $Z = C \cup \{x_n : n \in N\}$, then Z is a closed subset of X. For each $n \in N$, let

$$C_n = \{x_n^{-1} x_0 y_i^{-1} x_n : i \in N\} \cup \{e\}.$$

By Corollary 4.4.2, $F(X)$ is a k-space, and hence it can be proved that $E = \bigcup_{n \in N} C_n$ is a closed subset of $F_4(X)$. Moreover, it is homeomorphic to the Fréchet-Urysohn fan of cardinality $\aleph_0$ (see the proof of Proposition 3.2 in [4]). By the assumption, $F_4(X)$ is locally compact and, in particular, so is E. But this is a contradiction. ■

Theorem 4.4.4 *For a nondiscrete metrizable space X, the following are equivalent:*

(a) $F_n(X)$ *is compact for each* $n \in N$,

(b) $F_n(X)$ *is locally compact for each* $n \in N$,

(c) X *is compact.*

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