Renormalization of Gauge Theories

by

SHOICHI ICHINOSE

Dissertation submitted to the Faculty of the Graduate School
of the University of Tsukuba in partial fulfilment
of the requirements for the degree of
Doctor of Science
December 1980

Abstract:

Renormalization of gauge theories is performed in a manifestly covariant manner using the technique of the background-field method. A one-loop counter-term formula for bose systems with gravitational couplings is obtained and then applied to a system of gravity interacting with a scalar field. The formula contains 'tHooft and Veltman's as a special case and is more powerful than theirs in two respects: 1) It gives counter-terms without recourse to the doubling trick; 2) the generalization to higher-loop calculations is straightforward. Further, a two-loop counter-term formula for bose systems with (non gravitational) derivative couplings is obtained.

Contents

1. Introduction

In this century physics experienced two revolutions: quantum theory and relativistic theory. Both theories have provided mankind with deep understanding of nature covering a wide range of energy scale. The special theory of relativity has naturally been combined with quantum theory as a relativistic field theory. Today we thus have the well-established relativistic field theory as exemplified by quantum electrodynamics. It describes the dynamics of electrons and photons. For the general theory of relativity, the status is not very satisfactory yet from the viewpoint of quantum theory.

However the recent progress in field theories, especially in gauge theories, seems to indicate a new approach to the above-mentioned problem. Let us thus begin by briefly surveying the history of gauge thories. By the end of 1940's the renormalizability of quantum electrodynamics was proved by S. Tomonaga [1], J. Schwinger [2], R. P. Feynman [3] and F. J. Dyson [4]. In 1950 the general Hamiltonian formalism for gauge theories was presented by P. A. M. Dirac [5]. In 1953 C. N. Yang and R. L. Mills generalized for the first time the principle of local gauge invariance to non-Abelian gauge theories. In so doing they had to introduce a vector field, which later became known as the Yang-Mills field. R. Utiyama [7] further generalized their argument and dealt with gravity from the viewpoint of gauge theories (1955). In 1963 an explicit perturbative calculation of quantum gravity was given by R. P. Feynman [8]. He discovered that if one introduced a standard gauge and calculated diagrams with one closed loop in a way analogous to quantum electrodynamics, one would arrive at amplitudes violating unitarity. To avoid such a phenomenon, he introduced a fictitious vector field satisfying Fermi statistics. Later these peculiar features of non-Abelian gauge theories were intensively studied by B. S. DeWitt (1967)[9], L. D. Faddeev and V. N. Popov (1967) [10], S. Mandelstam (1968)[11] and E. S. Fradkin and I. V. Tyntin (1970)[12]. Non-Abelian gauge theories gradually became a central problem in physics. The unification of weak and electromagnetic interactions was achieved by S. Weinberg (1967)[13] and by A. Salam (1968)[14] using SU(2)×U(1) non-Abelian gauge theory with the Higgs mechanism. In 1971 G. 'tHooft [15] proved renormalizability of a massless Yang-Mills theory. He also proved renromalizability for the massive case but with spontaneous symmetry breakdown (1971)[16]. On the other hand, the discovery of asymptotic freedom of quantum chromodynamics in 1973 (H. D. Politzer [17], G. 'tHooft [18], D. Gross and F. Wilczek [19]) has given a great expectation such that this may be the right theory of the strong interaction.

When compared with the development of gauge theories in the flat space, the quantum theory of gravity still has a number of unsolved problems with regard to its renormalizability. The main reason for this is that quantum gravity has a coupling constant with dimension of length. Thus, it is not renormalizable in the usual sense [20]. However there still remains a possibility such that no infinity appears in on-shell quantities which are not contained in the original Lagrangian. It is known from the invariance argument that Einstein gravity is renormalizable up to one-loop level [21, 22, 23], and that supergravity is renormalizable up to two-loop level [24,25]. In both cases it should be possible to know by carrying out explicit calculations whether the theory is renormalizable or not at higher orders. For Einstein gravity one-loop counter-terms were explicitly obtained by G. 'tHooft and M. Veltman [21], but no one has ever calculated counter-terms for higher orders. A possible way to understand the mechanism of ultraviolet divergences of gravity is clearly to calculate them explicitly.

In this paper we present a way toward such a direction. First we will obtain a one-loop counter-term formula for bose systems. It is valid both for usual renormalizable theories and for theories with gravitational interactions. The formula is so powerful as to enable us to obtain one-loop counter-terms for Einstein gravity in a way simpler than the case of G. 'tHooft and M. Veltman. The generalization to two-loop calculations is straightforward. All types of two-loop counter-terms which will appear in the corresponding formula are obtained. For bose systems with non-gravitational interactions we will obtain a two-loop counter-term formula explicitly.

In sect.2 we will explain the background-field method on which our argument will be based. In sect.3 we will discuss some interesting transformation properties of one-loop Lagrangian, thereby finding a one-loop counter-term formula. The corresponding formula will be discussed further for the case of gravity in sect.4. In sect.5 the formula will be applied to the case of gravity interacting with a scalar field. A two-loop counter-term formula will be obtained in sect.6. Conclusion and discussion are given in sect.7. In Appendix A a method to calculate the divergent parts of two-loop integrals is described. The results for various two-loop integrals are given in Appendix B, and Feynman rules are given in Appendix C. For a system with derivative couplings up to second order, all types of two-loop counter-terms are listed in Appendix D. A proof of an relation (6.28) which will be used in the text is given in Appendix E. Renormalization of the ϕ^4-theory is performed up to two-loop order in Appendix F.

2. The background-field method

In analysing ultraviolet divergences of local gauge theories, the background-field method is very convenient [9,22,26~30,40]. Compared with the usual method of calculating indivisual Feynman graphs, the advantage of this method is the following. Firstly, the method manifestly maintains the symmetries even after a special gauge is fixed. This is due to the fact that a background-field functional is invariant under a certain transformation of background fields. Secondly, this method enables one to perform the renormalization procedure in a way more economical than with the usual method.

To begin with let us consider a field theory with real Bose fields φ_i, where i denotes any kind of indices, including a Lorentz index. The background-field functional is defined by

$$\exp i W[\varphi^{cl}] = \int d\phi \exp i \int d^4x \; \left(\mathcal{L}[\varphi^{cl}+\phi] - \mathcal{L}[\varphi^{cl}] - \mathcal{L}_{,i}[\varphi^{cl}]\phi_i \right) , \tag{2.1}$$

where φ^{cl}_i stand for classical background fields, and ϕ_i for quantum fields. $\mathcal{L}_{,i}[\varphi^{cl}]$ denotes the functional derivative of $\mathcal{L}[\varphi^{cl}]$ with respect to φ^{cl}_i. The functional generates all diagrams with loops. We expand $\mathcal{L}[\varphi^{cl} + \phi]$ in (2.1) around the classical fields φ^{cl}_i :

$$\begin{aligned} &\mathcal{L}[\varphi^{cl}+\phi] - \mathcal{L}[\varphi^{cl}] - \mathcal{L}_{,i}[\varphi^{cl}]\phi_i \\ &= \frac{1}{2!}\mathcal{L}_{,ij}[\varphi^{cl}]\phi_i\phi_j + \frac{1}{3!}\mathcal{L}_{,ijk}[\varphi^{cl}]\phi_i\phi_j\phi_k \\ &\quad + \frac{1}{4!}\mathcal{L}_{,ijkl}[\varphi^{cl}]\phi_i\phi_j\phi_k\phi_l + O(\phi^5) . \end{aligned} \tag{2.2}$$

The first term of (2.2) gives one-loop diagrams such as

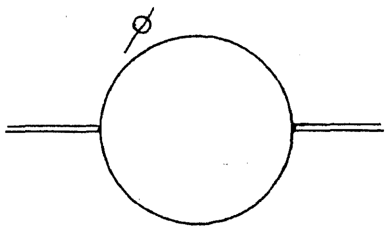


Fig.1.

where the external double lines represent 'vertices' containing φ^{cl} lines. The second and third terms give two-loop diagrams such as

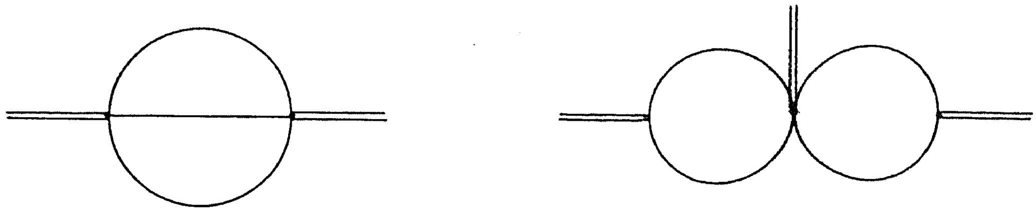


Fig.2.

When the S-matrix contains infinities, we have to introduce a counter Lagrangian $\Delta\mathcal{L}[\varphi]$ in the Lagrangian in order to subtract them. The counter Lagrangian $\Delta\mathcal{L}[\varphi^{cl}+\phi]$ may also be expanded as follows:

$$\Delta\mathcal{L}[\varphi^{cl}+\phi] - \Delta\mathcal{L}[\varphi^{cl}] - \Delta\mathcal{L}_{,i}[\varphi^{cl}]\phi_i$$
$$= \frac{1}{2!}\Delta\mathcal{L}_{,ij}[\varphi^{cl}]\phi_i\phi_j + O(\phi^3) . \tag{2.3}$$

These vertices give the diagrams which cancel the divergences of subintegrals. We now choose φ^{cl} in such a way that

$$\mathcal{L}_{,i}[\varphi^{cl}] + \Delta\mathcal{L}_{,i}[\varphi^{cl}] = 0 . \tag{2.4}$$

The counter Lagrangian $\Delta\mathcal{L}[\varphi^{cl}]$ can then be obtained by calculating divergent Feynman diagrams. In so doing we adopt the dimensional regularization procedure and the minimal subtraction scheme.

3. One-loop counter-term formula

Let us consider a one-loop Lagrangian $\mathcal{L}_2$:

$$\mathcal{L}_2 = \frac{1}{2!} \mathcal{L}_{,ij}[\varphi^{cl}]\phi_i\phi_j$$
$$= \frac{1}{2}\partial_\mu\phi_i W^{ij}\partial_\mu\phi_j + \phi_i N_\mu^{ij}\partial_\mu\phi_j + \frac{1}{2}\phi_i M^{ij}\phi_j , \tag{3.1}$$

where W, N_μ and M are functionals of classical fields φ_i^{cl} . For usual renormalizable field theories (including gauge theories with a Feynman-like gauge) we have

$$W^{ij} = -\delta^{ij} \tag{3.2}$$

This case was considered by G. 'tHooft [22]. For theories with gravitational interactions, the relation (3.2) is no longer valid. Here we consider (3.1) without the relation (3.2).

By adding total space-time derivatives to $\mathcal{L}_2$ (3.1) we can always so arrange the system that the following symmetry holds:

$$W^{ij} = W^{ji} , \qquad N_\mu^{ij} = -N_\mu^{ji} ,$$
$$M^{ij} = M^{ji} . \tag{3.3}$$

$\mathcal{L}_2$ is invariant under the following transformation:

$$\phi_i' = \phi_i + S_i{}^j(x)\,\phi_j ,$$
$$\Delta W^{ij} \equiv W^{ij\,\prime} - W^{ij} = -(WS)^{ij} - (WS)^{ji} ,$$
$$\Delta N_\mu^{ij} = -(N_\mu S)^{ij} + (N_\mu S)^{ji} + \frac{1}{2}\left\{(W\partial_\mu S)^{ij} - (W\partial_\mu S)^{ji}\right\} ,$$

$$\Delta M^{ij} = -(MS)^{ij} - (MS)^{ji} + \frac{1}{2}\partial_\mu\{(W\partial_\mu S)^{ij} + (W\partial_\mu S)^{ji}\}$$
$$- (N_\mu \partial_\mu S)^{ij} - (N_\mu \partial_\mu S)^{ji} , \tag{3.4}$$

where $S_i{}^j(x)$ is an arbitrary infinitesimal function of x. The local gauge symmetry (3.4) implies that of the original Lagrangian $\mathcal{L}[\varphi]$. $\Delta\mathcal{L}[\varphi^{cl}]$ must also be invariant under (3.4).

We now define covariant (contravariant) quantities as follows:

$$X^{ij} = M^{ij} + \frac{1}{2}[W(-\partial_\mu A_\mu + A_\mu A_\mu)]^{ij} + \frac{1}{2}[W(-\partial_\mu A_\mu + A_\mu A_\mu)]^{ji},$$
$$Y_{\lambda\sigma,i}{}^j = \partial_\lambda A_{\sigma,i}{}^j - \partial_\sigma A_{\lambda,i}{}^j - (A_\lambda A_\sigma - A_\sigma A_\lambda)_i{}^j , \tag{3.5}$$

where

$$A_{\lambda,i}{}^j = (Z N_\lambda)_i{}^j - \frac{1}{2}(Z\partial_\lambda W)_i{}^j ,$$
$$W^{ij} Z_{jk} = \delta^i{}_k . \tag{3.6}$$

The quantity A_λ transforms as

$$\Delta A_{\lambda,i}{}^j = (SA_\lambda - A_\lambda S + \partial_\lambda S)_i{}^j . \tag{3.7}$$

We see this A_λ plays the role of a guage field associated with the gauge transformation (3.4). The quantities X and $Y_{\lambda\sigma}$ transform as

$$\Delta X^{ij} = -(XS)^{ij} - (XS)^{ji} ,$$
$$\Delta Y_{\lambda\sigma,i}{}^{j} = (S Y_{\lambda\sigma})_i{}^j - (Y_{\lambda\sigma} S)_i{}^j . \tag{3.8}$$

The covariant derivatives are defined by

$$D_\lambda W^{ij} = \partial_\lambda W^{ij} + (WA_\lambda)^{ij} + (WA_\lambda)^{ji} = 0 ,$$
$$D_\lambda X^{ij} = \partial_\lambda X^{ij} + (XA_\lambda)^{ij} + (XA_\lambda)^{ji} . \tag{3.9}$$

Latin indices are raised and lowered by means of W^{ij} and Z_{ij}, respectively:

$$X_{ij} = Z_{ik} Z_{jl} X^{kl} ,$$
$$\Delta X_{ij} = (SX)_{ij} + (SX)_{ji} . \tag{3.10}$$

From the dimensional analysis we know that the one-loop counter-terms are written as

$$\Delta\mathcal{L}[\varphi^\alpha] = \frac{-1}{8\pi^2\varepsilon}\left(a X_{ij} X^{ji} + b Y_{\lambda\sigma,i}{}^{j} Y_{\lambda\sigma,j}{}^{i} \right), \tag{3.11}$$

where ε = 4-n. The result for the case $W^{ij} = -\delta^{ij}$ [22] is found to be

$$a = \frac{1}{4} \quad , \quad b = \frac{1}{24} \quad . \tag{3.12}$$

Similarly we can obtain invariant quantities for two-loop counter-terms (See cf. Appendices A, B).

4. Counter-term formula for gravity

We now consider the following one-loop Lagrangian:

$$\mathcal{L} = \sqrt{g}\,\Big(\frac{1}{2} W^{ij} \partial_\mu \varphi_i\, g^{\mu\nu} \partial_\nu \varphi_j + \varphi_i (N^\mu)^{ij} \partial_\mu \varphi_j + \frac{1}{2} \varphi_i M^{ij} \varphi_j \Big) , \tag{4.1}$$

where $g^{\mu\nu}$ is an external gravitational field and W, N^μ and M are functionals of the background fields. As before, φ_i denotes a quantum field. Under general coordinate transformations φ_i, W^{ij} and M^{ij} transform as scalars and N^μ as a contravariant vector, respectively, so that the Lagrangian (4.1) remains invariant. For the case of pure gravity or gravity interacting with a scalar field, the corresponding one-loop Lagrangian reduces to (4.1). The counter-term formula for (4.1) must reduce to the formula (3.11) for the flat metric $g_{\mu\nu} = \delta_{\mu\nu}$. The formula must be invariant under general coordinate transformations. We define the following quantities:

$$W^{ij} Z_{jk} = \delta^i{}_k \ ,$$
$$A^\lambda{}_{,i}{}^j = (ZN^\lambda - \frac{1}{2} Z \nabla^\lambda W)_i{}^j \ ,$$
$$X^{ij} = M^{ij} + \frac{1}{2}[W(-\nabla_\mu A^\mu + A_\mu A^\mu)]^{ij}$$
$$+ \frac{1}{2}[W(-\nabla_\mu A^\mu + A_\mu A^\mu)]^{ji} + aRW^{ij} \ ,$$
$$Y_{\lambda\alpha,i}{}^j = \nabla_\lambda A_{\alpha,i}{}^j - \nabla_\alpha A_{\lambda,i}{}^j - (A_\lambda A_\alpha - A_\alpha A_\lambda)_i{}^j \ ,$$

(4.2)

where a is a constant to be determined later and R is the Riemann scalar. The operation ∇_μ is the covariant derivative with respect to general coordinate transformations.

From the result of the previous section and the invariance argument we obtain

$$\Delta\mathcal{L} = \frac{-1}{8\pi^2\varepsilon} \sqrt{g} \left(\frac{1}{4} X_{ij} X^{ij} + \frac{1}{24} Y_{\lambda\alpha,i}{}^j Y^{\lambda\alpha}{}_{,j}{}^i + bR^2 + c R_{\mu\nu} R^{\mu\nu} \right) ,$$

(4.3)

where we have taken into account the identity [31]

$$\int d^4x \sqrt{g} \left(R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right) = 0 \qquad (4.4)$$

We can take R^2 and $R_{\mu\nu}R^{\mu\nu}$ to be independent invariants which are constructed from two Riemann tensors. One-loop calculation in the case of weak gravity ($g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$, $|h_{\mu\nu}|<<1$) gives three constants a, b and c as follows:

$$a = \frac{1}{6}, \quad b = -\frac{N}{360}, \quad c = \frac{N}{120} \tag{4.5}$$

where N is the number of independent scalar fields φ_i: $i = 1,2,\cdots,N$. In the case $W^{ij} = -\delta^{ij}$ our formula reduces to that of G. 'tHooft and M. Veltman [21].

5. Gravity interacting with a scalar field

We apply (4.3) to the case of gravity interacting with a scalr field:

$$\mathcal{L} = \sqrt{\bar{g}}\left(-\bar{R} - \frac{1}{2}\partial_\mu\bar{\varphi}\ \bar{g}^{\mu\nu}\partial_\nu\bar{\varphi}\right), \tag{5.1}$$

where we have placed a bar on $g_{\mu\nu}$ and φ to distinguish the full fields $\bar{g}_{\mu\nu}$ and $\bar{\varphi}$ from the classical fields, $\bar{R}$ is the Riemann scalar constructed from $\bar{g}_{\mu\nu}$. We use the same notations as those in ref.[21].

Following the procedure of the background-field method, we write

$$\bar{\varphi} = \tilde{\varphi} + \varphi, \quad \bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \tag{5.2}$$

We regard $\tilde{\varphi}$ and $g_{\mu\nu}$ as obeying the classical equation of motion. Note that so far as ultraviolet divergences up to one-loop level are concerned, the second term of (2.4) may be neglected. The one-loop Lagrangian for (5.1) is

$$\mathcal{L}_2 = \sqrt{g}\,\Big[-\tfrac{1}{2}\partial_\mu\varphi\,\partial_\nu\tilde{\varphi}\,(g^{\mu\nu}h^\alpha{}_\alpha - 2h^{\mu\nu}) - \tfrac{1}{2}\partial_\mu\tilde{\varphi}\,\partial_\nu\tilde{\varphi}\,(h^\mu{}_\alpha h^{\alpha\nu} - \tfrac{1}{2}h^\alpha{}_\alpha h^{\mu\nu})$$
$$-\tfrac{1}{2}\partial_\mu\varphi\, g^{\mu\nu}\partial_\nu\varphi - \big(\tfrac{1}{8}(h^\alpha{}_\alpha)^2 - \tfrac{1}{4}h^\alpha{}_\beta h^\beta{}_\alpha\big)\big(R + \tfrac{1}{2}\partial_\mu\tilde{\varphi}\, g^{\mu\nu}\partial_\nu\tilde{\varphi}\big)$$
$$-h^\nu{}_\beta h^\beta{}_\alpha R^\alpha{}_\nu + \tfrac{1}{2}h^\alpha{}_\alpha h^\nu{}_\beta R^\beta{}_\nu - \tfrac{1}{4}h^\beta{}_{\alpha,\nu}h^\alpha{}_\beta{}^{,\nu} + \tfrac{1}{4}h^\alpha{}_{\alpha,\mu}h^\beta{}_\beta{}^{,\mu}$$
$$-\tfrac{1}{2}h^\alpha{}_{\alpha,\beta}h^\beta{}_\mu{}^{,\mu} + \tfrac{1}{2}h^\nu{}_\beta{}^{,\alpha}h^\beta{}_{\alpha,\nu}\Big], \tag{5.3}$$

where the covariant or contravariant derivative contains the Christoffel symbol Γ made up from the classical field $g^{\mu\nu}$. Greek indices are raised and lowered by means of the classical field $g^{\mu\nu}$. The Lagrangian (5.3) is invariant under the gauge transformation:

$$\delta\varphi = \eta^\alpha(\tilde{\varphi} + \varphi)_{,\alpha},$$
$$\delta h_{\mu\nu} = (g_{\alpha\nu} + h_{\alpha\nu})\eta^\alpha{}_{,\mu} + (g_{\mu\alpha} + h_{\mu\alpha})\eta^\alpha{}_{,\nu} + \eta^\alpha h_{\mu\nu,\alpha}, \tag{5.4}$$

where η^α's are the parameters of infinitesimal general coordinate transformations. To fix the symmetry we adopt the following gauge-fixing:

$$\mathcal{L}_{gauge} = -\tfrac{1}{2}(C_\alpha)^2$$
$$C_\alpha = \sqrt[4]{g}\,(h^\nu{}_{\mu,\nu} - \tfrac{1}{2}h^\nu{}_{\nu,\mu} - \varphi\,\partial_\mu\tilde{\varphi})\,t^{\mu\alpha}, \tag{5.5}$$

where $t^{\mu\alpha}$ is the root of the tensor $g^{\mu\nu}$:

$$t^{\mu\alpha} t^{\alpha\nu} = g^{\mu\nu} . \qquad (5.6)$$

The Faddeev-Popov ghost Lagrangian is

$$\mathcal{L}_{ghost} = \sqrt{g}\, \eta^{\mu *} \{ \eta_{\mu}{}^{,\alpha}{}_{,\alpha} - (R_{\alpha\mu} + \partial_\alpha \tilde{\varphi} \partial_\mu \tilde{\varphi}) \eta^{\alpha} \} , \qquad (5.7)$$

Tensor and Scalar Part

The matrices W, N^{μ} and M in (3.1) take the following forms:

$$W^{I,J} = \begin{array}{c|c} & \begin{array}{cc} \mathrm{T} & \mathrm{S} \end{array} \\ \begin{array}{c} \mathrm{T} \\ \mathrm{S} \end{array} & \left(\begin{array}{c|c} W^{\alpha\beta,\mu\nu} & 0 \\ \hline 0 & -1 \end{array} \right) \end{array} , \qquad (N^{\lambda})^{I,J} = \begin{array}{c|c} & \begin{array}{cc} \mathrm{T} & \mathrm{S} \end{array} \\ \begin{array}{c} \mathrm{T} \\ \mathrm{S} \end{array} & \left(\begin{array}{c|c} (N^{\lambda})^{\alpha\beta,\mu\nu} & 0 \\ \hline 0 & 0 \end{array} \right) \end{array} ,$$

$$M^{I,J} = \begin{array}{c|c} & \begin{array}{cc} \mathrm{T} & \mathrm{S} \end{array} \\ \begin{array}{c} \mathrm{T} \\ \mathrm{S} \end{array} & \left(\begin{array}{c|c} M^{\alpha\beta,\mu\nu} & M^{\alpha\beta} \\ \hline M^{\alpha\beta} & -\partial_\mu \tilde{\varphi} \partial^\mu \tilde{\varphi} \end{array} \right) \end{array} , \qquad (5.8)$$

where

$$W^{\alpha\beta,\mu\nu} = \frac{1}{4} (g^{\alpha\beta} g^{\mu\nu} - g^{\alpha\mu} g^{\beta\nu} - g^{\alpha\nu} g^{\beta\mu}) ,$$

$$Z_{\alpha\beta,\mu\nu} = g_{\alpha\beta} g_{\mu\nu} - g_{\alpha\mu} g_{\beta\nu} - g_{\alpha\nu} g_{\beta\mu} ,$$

$$W^{\alpha\beta,\mu\nu} Z_{\mu\nu,\lambda\alpha} = \frac{1}{2} (\delta^{\alpha}_{\lambda} \delta^{\beta}_{\alpha} + \delta^{\alpha}_{\alpha} \delta^{\beta}_{\lambda}) ,$$

$$(N^{\lambda})^{\alpha\beta,\mu\nu} = -\frac{1}{2} g^{\lambda\alpha} \{ W^{\mu\nu,\alpha\omega} \Gamma^{\beta}_{\alpha\omega} + W^{\mu\nu,\beta\omega} \Gamma^{\alpha}_{\alpha\omega} - (\alpha\beta \leftrightarrow \mu\nu) \} ,$$

$$M^{\alpha\beta} = -\frac{1}{2}\left(\nabla^\alpha\nabla^\beta + \nabla^\beta\nabla^\alpha\right)\tilde{\varphi} + \frac{1}{2} g^{\alpha\beta}\,\nabla^\mu\nabla_\mu\tilde{\varphi} ,$$

$$\begin{aligned}
M^{\alpha\beta,\mu\nu} = \Big[& \frac{1}{16}\left(-g^{\alpha\beta}g^{\mu\nu} + 2g^{\alpha\mu}g^{\beta\nu}\right)R \\
& + \frac{1}{8}\left(g^{\alpha\beta}R^{\mu\nu} + g^{\mu\nu}R^{\alpha\beta} - 2g^{\alpha\mu}R^{\beta\nu}\right) + \frac{1}{4}R^{\alpha\mu,\beta\nu} \\
& + \frac{1}{2} g^{\lambda\sigma}\left(W^{\alpha\tau,\mu\omega}\Gamma^\beta_{\lambda\tau}\Gamma^\nu_{\sigma\omega} + W^{\mu\tau,\alpha\omega}\Gamma^\nu_{\lambda\tau}\Gamma^\beta_{\sigma\omega}\right) \\
& + \frac{1}{4}\Gamma^\tau_{\lambda\tau}\, g^{\lambda\sigma}\left(W^{\mu\nu,\alpha\omega}\Gamma^\beta_{\sigma\omega} + W^{\alpha\beta,\mu\omega}\Gamma^\nu_{\sigma\omega}\right) \\
& + \frac{1}{4}\partial_\lambda\left\{ g^{\lambda\sigma}\left(W^{\mu\nu,\alpha\omega}\Gamma^\beta_{\sigma\omega} + W^{\alpha\beta,\mu\omega}\Gamma^\nu_{\sigma\omega}\right)\right\} \\
& - \frac{1}{4} g^{\alpha\mu}\,\partial^\beta\tilde{\varphi}\,\partial^\nu\tilde{\varphi} - \frac{1}{8}W^{\alpha\beta,\mu\nu}\,\partial_\lambda\tilde{\varphi}\,\partial^\lambda\tilde{\varphi} \\
& + \frac{1}{16}\left(g^{\alpha\beta}\partial^\mu\tilde{\varphi}\,\partial^\nu\tilde{\varphi} + g^{\mu\nu}\partial^\alpha\tilde{\varphi}\,\partial^\beta\tilde{\varphi}\right) + \alpha\leftrightarrow\beta \ \Big] + \mu\leftrightarrow\nu .
\end{aligned} \tag{5.9}$$

From these quantities we can obtain invariant quantities:

$$\begin{aligned}
X^{IJ}X_{IJ} &= X^{T,T}X_{T,T} + 2X^{T,S}X_{T,S} + X^{S,S}X_{S,S} , \\
X^{T,T}X_{T,T} &= \frac{5}{18}R^2 + 6R_{\mu\nu}R^{\mu\nu} + 3\left(\partial^\alpha\tilde{\varphi}\,\partial_\alpha\tilde{\varphi}\right)^2 , \\
X^{T,S}X_{T,S} &= 2R_{\alpha\beta}\,\partial^\alpha\tilde{\varphi}\,\partial^\beta\tilde{\varphi} + \left(\nabla^2\tilde{\varphi}\right)^2 , \\
X^{S,S}X_{S,S} &= \left(\partial_\mu\tilde{\varphi}\,\partial^\mu\tilde{\varphi} + \frac{1}{6}R\right)^2 , \\
Y_{\lambda\sigma,I}{}^{J}\,Y^{\lambda\sigma}{}_{,J}{}^{I} &= 6R^2 - 24R_{\alpha\beta}R^{\alpha\beta} ,
\end{aligned} \tag{5.10}$$

where the symbols T and S mean the metric tensor and scalar, respectively.

Ghost Part

Applying formula (4.3) to the case of complex fields, we rewrite them in terms of real fields:

$$\eta_\mu = \frac{1}{\sqrt{2}}\left(\eta_{1\mu} + i\,\eta_{2\mu}\right) \tag{5.11}$$

The matrices W, N^μ and M are given by

$$W^{1\mu,1\nu} = W^{2\mu,2\nu} = -g^{\mu\nu},$$

$$(N^\alpha)^{1\mu,1\nu} = (N^\alpha)^{2\mu,2\nu} = \frac{1}{2}\left(g^{\nu\sigma} g^{\alpha\beta}\Gamma^{\mu}_{\sigma\beta} - \mu\leftrightarrow\nu\right),$$

$$M^{1\mu,1\nu} = M^{2\mu,2\nu} = -\frac{1}{2\sqrt{g}}\,\partial_\alpha\left\{\sqrt{g}\left(g^{\mu\lambda} g^{\alpha\beta}\Gamma^{\nu}_{\lambda\beta} + g^{\lambda\nu} g^{\alpha\beta}\Gamma^{\mu}_{\beta\lambda}\right)\right\}$$
$$- g^{\lambda\sigma} g^{\alpha\beta}\Gamma^{\mu}_{\alpha\lambda}\Gamma^{\nu}_{\sigma\beta} - R^{\mu\nu},$$

$$Z_{1\mu,1\nu} = Z_{2\mu,2\nu} = -g_{\mu\nu}\,. \tag{5.12}$$

All other components vanish.

The invariant quantities are found to be

$$X_{IJ}X^{IJ} = \frac{8}{9}R^2 + 2R_{\mu\nu}R^{\mu\nu}$$
$$+\left(\frac{2}{3}R g^{\mu\nu} + 4R^{\mu\nu}\right)\partial_\mu\tilde{\varphi}\,\partial_\nu\tilde{\varphi} + 2\left(\partial^\mu\tilde{\varphi}\,\partial_\mu\tilde{\varphi}\right)^2,$$

$$Y_{\lambda\alpha;I}{}^{J}\,Y^{\lambda\alpha,}{}_{J}{}^{I} = 2R^2 - 8R_{\mu\nu}R^{\mu\nu}. \qquad (5.13)$$

Noting the minus sign that appears in the ghost part, the total one-loop counter-terms are found to be

$$\Delta\mathcal{L} = \frac{-1}{8\pi^2\varepsilon}\sqrt{g}\left\{\frac{1}{80}R^2 + \frac{43}{120}R_{\alpha\beta}R^{\alpha\beta} + \frac{1}{2}(\partial_\mu\tilde{\varphi}\,\partial^\mu\tilde{\varphi})^2 - \frac{1}{12}R\,\partial_\mu\tilde{\varphi}\,\partial^\mu\tilde{\varphi} + 2(\nabla^2\tilde{\varphi})^2\right\}, \qquad (5.14)$$

the result being in agreement with that of ref.[21].

6. Two-loop counter-term formula

In order to obtain counter-terms which are to subtract divergences of two-loop diagrams, we must take into account the Lagrangian (2.2) up to $O(\phi^4)$:

$$\begin{aligned}
&\mathcal{L}[\varphi^{cl}+\phi] - \mathcal{L}[\varphi^{cl}] - \mathcal{L}_{,i}[\varphi^{cl}]\phi_i \\
&= \frac{1}{2}\partial_\mu\phi_i W^{ij}\partial_\mu\phi_j + \phi_i N_\mu^{ij}\partial_\mu\phi_j + \frac{1}{2}\phi_i M^{ij}\phi_j \\
&+ \Xi_{\mu\nu}^{ijk}\phi_i\partial_\mu\phi_j\partial_\nu\phi_k + \Omega_\mu^{ijk}\phi_i\phi_j\partial_\mu\phi_k + \Lambda^{ijk}\phi_i\phi_j\phi_k \\
&+ \Gamma_{\mu\nu}^{ijkl}\phi_i\phi_j\partial_\mu\phi_k\partial_\nu\phi_l + \Sigma_\mu^{ijkl}\phi_i\phi_j\phi_k\partial_\mu\phi_l + \Theta^{ijkl}\phi_i\phi_j\phi_k\phi_l \\
&+ O(\phi^5)
\end{aligned} \qquad (6.1)$$

where W, N_μ, M, $\Xi_{\mu\nu}$, Ω_μ, Λ, $\Gamma_{\mu\nu}$, Σ_μ and Θ are functionals of the background fields φ^{cl}_i . In (6.1) we assumed that the original Lagrangian $\mathcal{L}[\varphi]$ has derivative couplings up to second order. This is satisfied both for Einstein gravity and for usual renormalizable theories. The symmetries of (6.1) and all invariants are listed in Appendix D. Hereafter we consider the following case for simplicity:

$$W^{ij} = -\delta^{ij} \quad , \quad \Xi_{\mu\nu} = 0 \quad ,$$

$$\Gamma_{\mu\nu} = 0 \quad , \quad \Sigma_\mu = 0 \; . \tag{6.2}$$

The relations (6.2) hold for usual renormalizable theories, including gauge theories, with a Feynman-like gauge. Thus let us now deal with the following Lagrangian:

$$\begin{aligned} &\mathcal{L}[\varphi^{cl}+\phi] - \mathcal{L}[\varphi^{cl}] - \mathcal{L}_{,i}[\varphi^{cl}]\phi_i \\ &= -\frac{1}{2}\partial_\mu\phi_i\,\partial_\mu\phi_i + \phi_i N^\mu_{ij}\,\partial_\mu\phi_j + \frac{1}{2}\phi_i M_{ij}\phi_j \\ &\qquad + \Omega^\mu_{ijk}\phi_i\phi_j\partial_\mu\phi_k + \Lambda_{ijk}\phi_i\phi_j\phi_k \\ &\qquad + \Theta_{ijkl}\phi_i\phi_j\phi_k\phi_l \\ &\qquad + O(\phi^5) \; , \end{aligned} \tag{6.3}$$

where all Latin indices have been lowered. As in sect.3 we can always arrange the system in the following way:

$$N^{\mu}_{ij} = -N^{\mu}_{ji} \quad , \quad M_{ij} = M_{ji} \; ,$$

$$\Omega^{\mu}_{ijk} = \Omega^{\mu}_{jik} \quad , \quad \Omega^{\mu}_{ijk} + \Omega^{\mu}_{jki} + \Omega^{\mu}_{kij} = 0 \; ,$$

Λ_{ijk} totally symmetric with respect to (i, j, k) ,

Θ_{ijkl} totally symmetric with respect to (i, j, k, l).

(6.4)

The Lagrangian (6.3) is invariant under the following transformation:

$$\phi_i' = \phi_i + S_{ij}(x)\,\phi_j \quad ,$$

$$\Delta N^{\mu}_{ij} = -\partial_\mu S_{ij} + S_{ik} N^{\mu}_{kj} - N^{\mu}_{ik} S_{kj} \quad ,$$

$$\Delta M_{ij} = S_{ik} M_{kj} - M_{ik} S_{kj} - N^{\mu}_{ik}\,\partial_\mu S_{kj} - \partial_\mu S_{ik} N^{\mu}_{kj} \quad ,$$

$$\Delta \Omega^{\mu}_{ijk} = S_{il}\,\Omega^{\mu}_{ljk} + S_{jl}\,\Omega^{\mu}_{ilk} + S_{kl}\,\Omega^{\mu}_{ijl} \quad ,$$

$$\Delta \Lambda_{ijk} = S_{il}\,\Lambda_{ljk} + \frac{1}{3}\,\partial_\mu S_{il}\,\Omega^{\mu}_{jkl} + \text{cyclic}\,(i, j, k) \; ,$$

$$\Delta \Theta_{ijkl} = S_{im}\,\Theta_{mjkl} + \text{cyclic}\,(i, j, k, l) \quad ,$$

(6.5)

where cyclic (i, j, k) means those terms which are obtained by cyclically exchanging Latin indices i, j and k. $S_{ij}(x)$ is an arbitray infinitesimal parameters with antisymmetric property $S_{ij}(x) = -S_{ji}(x)$. Note that the antisymmetry property is due to the relation $W^{ij} = -\delta^{ij}$ in (6.2).

In view of the above transformation properties we can construct covariant quantities:

$$X_{ij} = M_{ij} - (N^\mu N^\mu)_{ij} \ ,$$

$$Y^{\mu\nu}_{ij} = \partial_\mu N^\nu_{ij} - \partial_\nu N^\mu_{ij} + N^\mu_{ik} N^\nu_{kj} - N^\nu_{ik} N^\mu_{kj} \ ,$$

$$\tilde{\Lambda}_{ijk} = \Lambda_{ijk} + \frac{1}{3}\left(N^\mu_{il} \Omega^\mu_{jkl} + N^\mu_{jl} \Omega^\mu_{ikl} + N^\mu_{kl} \Omega^\mu_{ijl} \right) . \tag{6.6}$$

The quantities X and $Y^{\mu\nu}$ in (6.6) are those given by G. 'tHooft [22], corresponding to a special case of (3.5). These quantities transform as

$$\Delta X_{ij} = S_{ik} X_{kj} - X_{ik} S_{kj} \ ,$$

$$\Delta Y^{\mu\nu}_{ij} = S_{ik} Y^{\mu\nu}_{kj} - Y^{\mu\nu}_{ik} S_{kj} \ ,$$

$$\Delta \tilde{\Lambda}_{ijk} = S_{il} \tilde{\Lambda}_{ljk} + S_{jl} \tilde{\Lambda}_{lik} + S_{kl} \tilde{\Lambda}_{lij} \ . \tag{6.7}$$

The transformation properties of X_{ij} and $Y^{\mu\nu}_{ij}$ correspond to a special case of (3.8). The covariant derivatives are defined by

$$D_\mu \Omega^\nu_{ijk} = \partial_\mu \Omega^\nu_{ijk} + N^\mu_{il}\Omega^\nu_{ljk} + N^\mu_{jl}\Omega^\nu_{ilk} + N^\mu_{kl}\Omega^\nu_{ijl} \,,$$

$$D_\mu \tilde{\Lambda}_{ijk} = \partial_\mu \tilde{\Lambda}_{ijk} + N^\mu_{il}\tilde{\Lambda}_{ljk} + N^\mu_{jl}\tilde{\Lambda}_{ilk} + N^\mu_{kl}\tilde{\Lambda}_{ijl} \,,$$

$$D_\lambda Y^{\mu\nu}_{ij} = \partial_\lambda Y^{\mu\nu}_{ij} + N^\lambda_{il} Y^{\mu\nu}_{lj} - N^\lambda_{jl} Y^{\mu\nu}_{li} \,. \tag{6.8}$$

One-loop counter-terms for (6.3) are obtained by putting $w^{ij} = -\delta^{ij}$ in (3.11):

$$\Delta \mathcal{L}^{one\text{-}loop}[\varphi^{cl}] = \frac{-1}{8\pi^2\varepsilon}\left(\frac{1}{4}X_{ij}X_{ji} + \frac{1}{24}Y^{\lambda\alpha}_{ij}Y^{\lambda\alpha}_{ji}\right), \tag{6.9}$$

where

$$\varepsilon = 4 - n \tag{6.10}$$

The formula (6.9) was previously obtained by G.'tHooft [22].

From the dimensional analysis we expect that two-loop counter-terms $\Delta\mathcal{L}^{\text{two-loop}}[\varphi^{cl}]$ can be written as a linear combination of the following terms:

$$D^2\tilde{\Lambda}^2, \quad \tilde{\Lambda}^2 X, \quad D\tilde{\Lambda}\Omega X, \quad D\tilde{\Lambda}\Omega Y, \quad D^4\Omega^2,$$

$$D^2\Omega^2 X, \quad \Omega^2 X^2, \quad \Omega^2 X Y, \quad D^2\Omega^2 Y, \quad \Omega^2 Y^2,$$

$$H X^2 \,. \tag{6.11}$$

The explicit expressions for the coefficients of these terms are obtained by calculating divergent Feynman diagrams. The Lagrangian

(6.3) contains the following vertices:

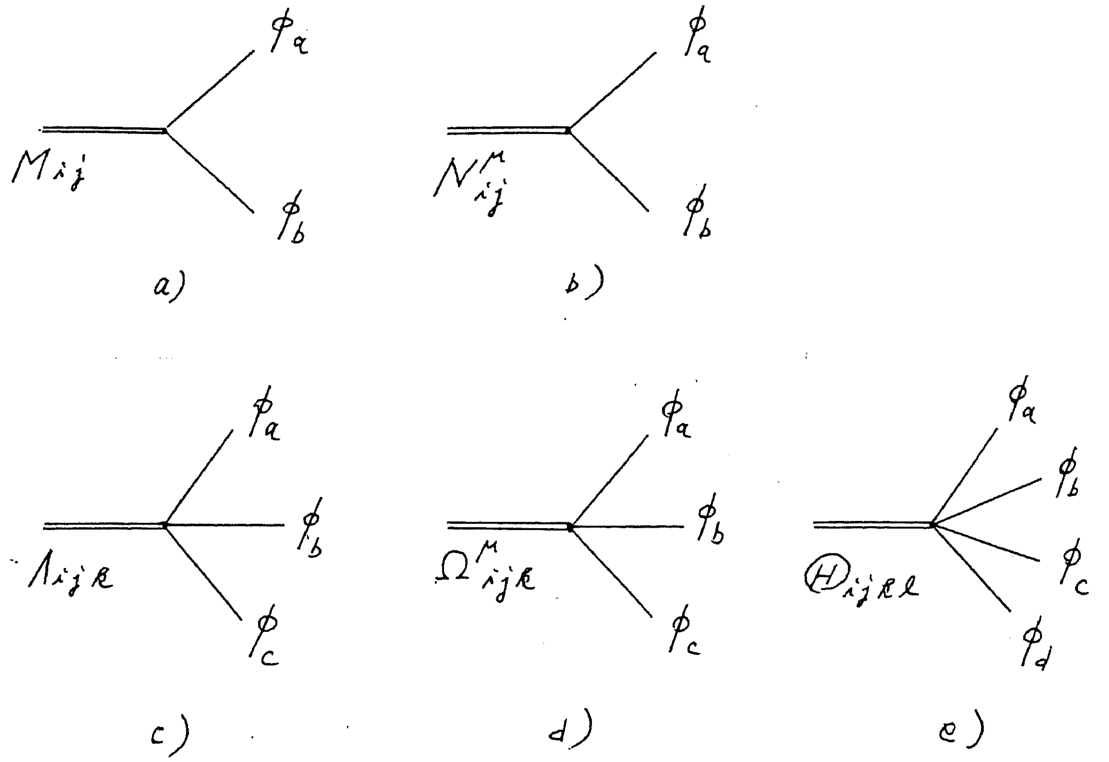
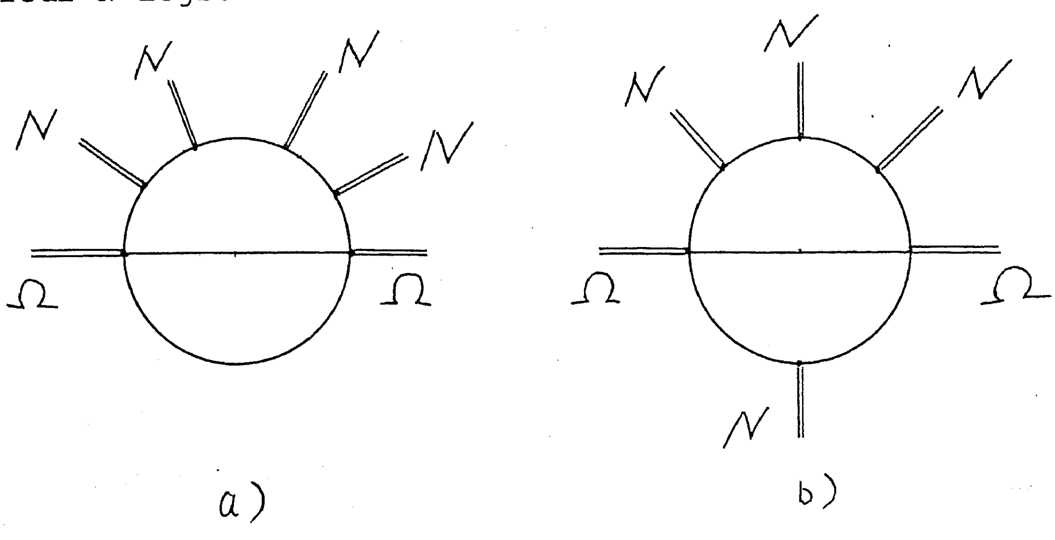


Fig.3.

The Feynman rules relevant to such vertices are given in Appendix C. An easy way to find all coefficients of (6.11) except ΘX^2 is to compute the logarithmically divergent two-loop diagrams with four N-legs:



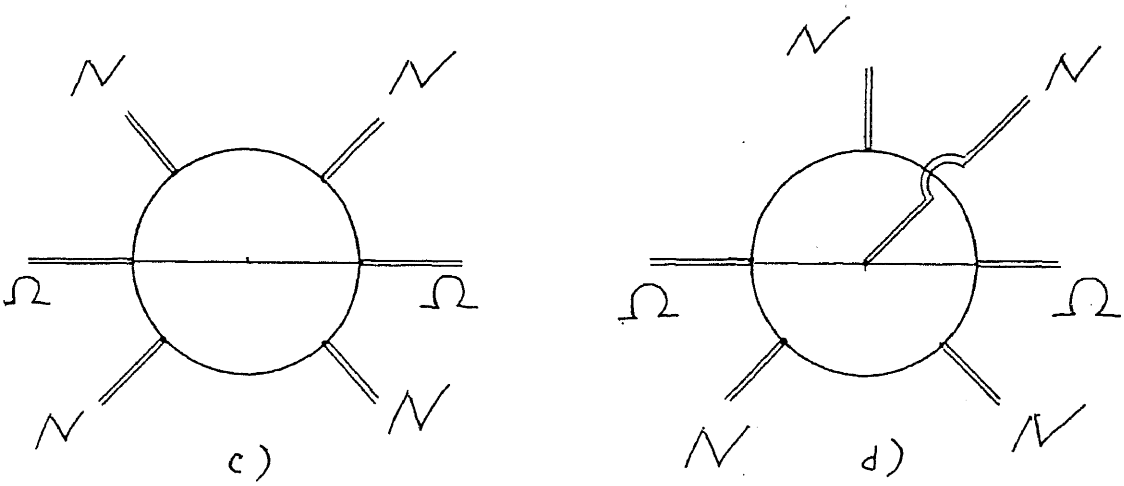


Fig.4.

We have checked that the diagrams a), b) and c), in fact, are sufficient to fix all the coefficients. In order to obtain the coefficient in front of $H x^2$, we can compute, for example, a diagram shown in Fig.5:

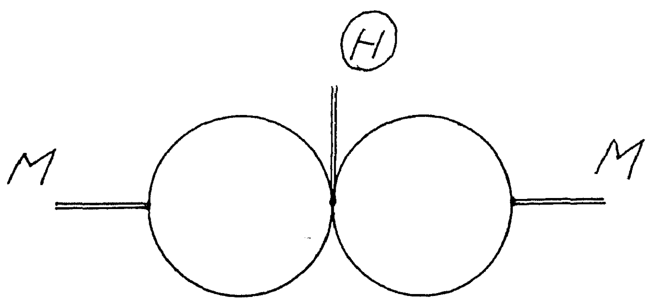


Fig.5.

Technical details of computing two-loop integrals are relegated to Appendix A.

Let us now consider the problem of subdivergences. As an example we consider the following diagram:

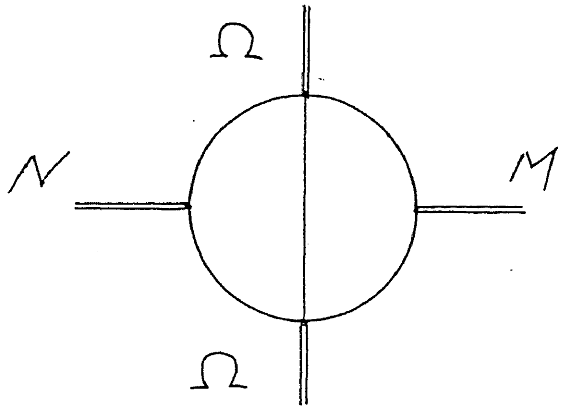


Fig.6.

The above two-loop diagram contains three one-loop subdiagrams,

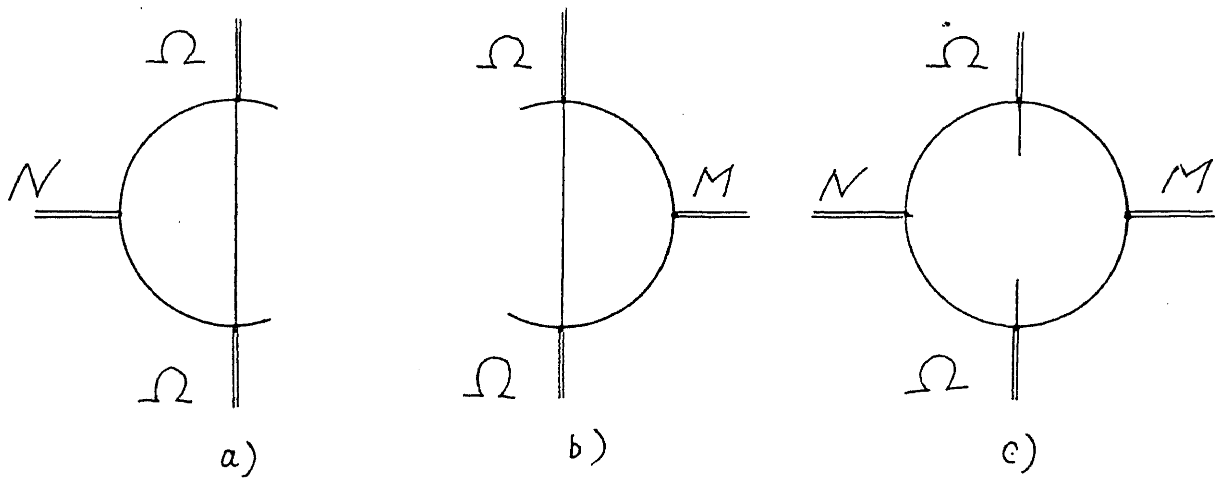
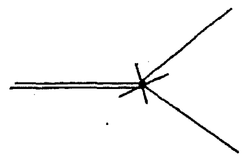


Fig.7.

and the power counting shows that the diagrams a) and b) diverge. These one-loop subdivergences are subtracted by the one-loop counter-terms. The vertices in(2.3) play the role:

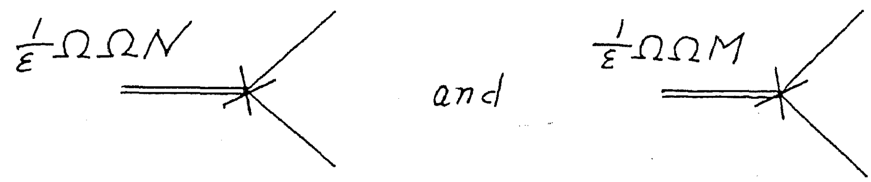
$$\mathcal{L}^{sub} = \frac{1}{2!}\,\Delta\mathcal{L}^{one-loop}{}_{,ij}[\varphi^{\alpha}]\,\phi_i\phi_j \quad , \tag{6.12}$$

where we have neglected the terms cubic and higher in ϕ's because they contribute only to subdivergences of higher-loop diagrams. Note further that $\Delta\mathcal{L}[\varphi^{cl}]$ has been replaced by $\Delta\mathcal{L}^{\text{one-loop}}[\varphi^{cl}]$ for the same reason. The vertices in (6.12) are represented graphically as



. Fig.8.

In Appendix E, it is shown that these vertices cancel the divergent part of one-loop subdiagrams. For the one-loop subdiagrams a) and b) in Fig.7, the vertices corresponding to (6.12) are represented graphically as



a) b)

Fig.9.

respectively. These vertices give subtraction diagrams for the two-loop diagram shown in Fig.6:

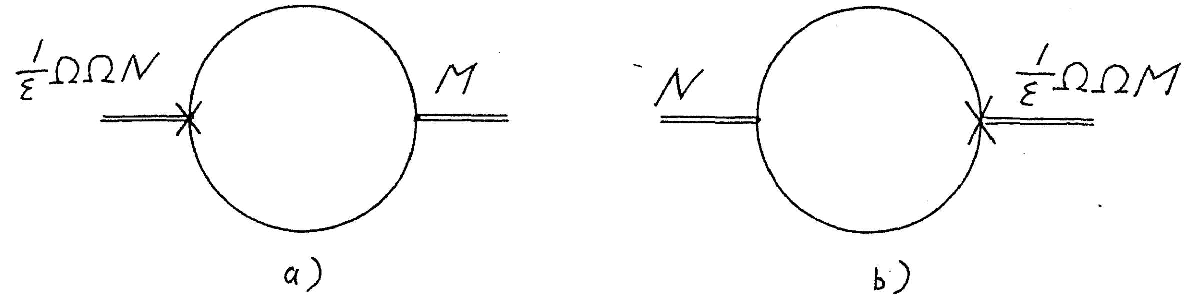


Fig.10.

In order to obtain $\Delta\mathcal{L}^{\text{two-loop}}[\varphi^{cl}]$ we must take into account both two-loop diagrams (such as a diagram in Fig.6) and subtraction diagrams (such as diagrams a) and b) in Fig.10). The subtraction diagrams cancel the one-loop subdivergences contained in the two-loop diagrams. Here, however, we consider the contribution to $\Delta\mathcal{L}^{\text{two-loop}}[\varphi^{cl}]$ from the two subtraction diagrams separately. The subtraction diagram a) in Fig.10, for example, contributes to two-loop counter-terms in the following way:

$$\left(\frac{a}{\varepsilon} + b\right)\left(\frac{1}{\varepsilon}\Omega\Omega N\right)M \,, \tag{6.13}$$

where the constants a and b are determined by explicitly calculating the diagram a) in Fig.10. We have suppressed all indices contained in $\Omega^{\mu}{}_{ijk}$, N^{μ}_{ij} and M_{ij} for simplicity. When the field indices i, j,$\cdots$ contain Lorentz indices μ, ν, $\cdots$ we must cautiously deal with contraction of Lorentz indices in (6.13). According to the minimal subtraction scheme the counter Lagrangian $\Delta\mathcal{L}[\varphi]$ is inserted in such a way that only divergent parts of diagrams are subtracted. Thus for the one-loop counter-terms (6.9), if $\delta_{\mu\mu}$ appears in $X_{ij}X_{ji}$ or $Y^{\lambda\sigma}_{ij}Y^{\lambda\sigma}_{ji}$, we must take

$$\delta_{\mu\mu} = 4 \quad . \tag{6.14}$$

Therefore, if $\delta_{\mu\mu}$ appears in $\Omega\Omega N$ of (6.13), which originally arises from $X_{ij}X_{ji}$ or $Y^{\lambda\sigma}_{ij}Y^{\lambda\sigma}_{ji}$ through (6.12), we must take

$$\delta_{\mu\mu} = 4 \quad . \tag{6.15}$$

For $\delta_{\mu\mu}$'s which are irrelevent to one-loop counter-terms we must, of course, take

$$\delta_{\mu\mu} = n \quad , \tag{6.16}$$

where n is the space-time dimension. Therefore we first consider the case when the field indices do not contain Lorentz idices, and then consider how to modify the formula for the case when the field indices do contain Lorentz indices.

[A] <u>The Case when the field indices do not contain Lorentz indices</u>

This is the case with scalar fields φ_i (ϕ^4-theory, σ-model, etc.). Evaluating the two-loop diagrams shown in Fig.4 and Fig.5, and subtracting the divergences of subdiagrams in a simple-minded manner we find the explicit form for (6.11) as follows:

$$D^2\tilde{\Lambda}^2 \; : \; -\frac{3}{2^9\pi^4\varepsilon}\,\tilde{\Lambda}_{ijk}\,D^2\tilde{\Lambda}_{ijk} \;, \tag{6.17}$$

$$\tilde{\Lambda}^2 X \; : \; +\frac{3^2}{2^8\pi^4\varepsilon^2}\,(2-\varepsilon)\,\tilde{\Lambda}_{ijk}\,\tilde{\Lambda}_{ij\ell}\,X_{k\ell} \;, \tag{6.18}$$

$$D\tilde{\Lambda}\Omega X \ : \ +\frac{3}{2^8\pi^4\varepsilon^2} X_{kl} \Big\{ \big(-1+\tfrac{5}{4}\varepsilon\big)\tilde{\Lambda}_{ijk} D_\mu \Omega^\mu_{ijl} + \big(-3+\tfrac{3}{4}\varepsilon\big) D_\mu \tilde{\Lambda}_{ijk}\cdot \Omega^\mu_{ijl} \Big\}, \tag{6.19}$$

$$D\tilde{\Lambda}\Omega Y \ : \ +\frac{3}{2^8\pi^4\varepsilon^2} \tilde{\Lambda}_{ijk} \Big\{ \big(1+\tfrac{5}{12}\varepsilon\big) D_\nu Y^{\nu\mu}_{ka}\cdot \Omega^\mu_{ija} + \varepsilon\, Y^{\nu\mu}_{ka} D_\nu \Omega^\mu_{ija} \Big\}, \tag{6.20}$$

$$D^4\Omega^2 \ : \ -\frac{1}{3\cdot 2^{13}\pi^4\varepsilon} \Omega^\mu_{ijk}\big(3D^2D^2\Omega^\mu_{ijk} - 4D^2 D_\nu D_\mu \Omega^\nu_{ijk}\big), \tag{6.21}$$

$$\begin{aligned} D^2\Omega^2 X \ : \ &+\frac{1}{3^2\cdot 2^8\pi^4\varepsilon^2} X_{kl} \\ &\times\Big[D^2\Omega^\mu_{kij}\Big\{\big(14-\tfrac{41}{6}\varepsilon\big)\Omega^\mu_{lij} + \big(13-\tfrac{107}{12}\varepsilon\big)\Omega^\mu_{lji}\Big\} \\ &+ D_\lambda\Omega^\mu_{kij}\Big\{\big(11+\tfrac{5}{12}\varepsilon\big)D_\lambda\Omega^\mu_{lij} + \big(16-\tfrac{8}{3}\varepsilon\big)D_\lambda\Omega^\mu_{lji}\Big\} \\ &+ D_\mu D_\nu\Omega^\mu_{kij}\Big\{-\big(2+\tfrac{49}{6}\varepsilon\big)\Omega^\nu_{lij} + \big(2-\tfrac{59}{6}\varepsilon\big)\Omega^\nu_{lji}\Big\} \\ &- D_\mu\Omega^\mu_{kij}\Big\{\big(4+\tfrac{35}{6}\varepsilon\big)D_\nu\Omega^\nu_{lij} + \big(5+\tfrac{65}{12}\varepsilon\big)D_\nu\Omega^\nu_{lji}\Big\} \\ &+ D_\mu\Omega^\nu_{kij}\Big\{\big(14-\tfrac{4}{3}\varepsilon\big)D_\nu\Omega^\mu_{lij} + \big(13-\tfrac{11}{12}\varepsilon\big)D_\nu\Omega^\mu_{lji}\Big\}\Big], \end{aligned} \tag{6.22}$$

$$\begin{aligned} D^2\Omega^2 Y \ : \ &-\frac{1}{3^2\cdot 2^8\pi^4\varepsilon^2} Y^{\lambda a}_{kl} \\ &\times\Big\{\big(1-\tfrac{5}{3}\varepsilon\big)D_\lambda\Omega^\mu_{kij}D_a\Omega^\mu_{lij} - \tfrac{7}{2}\varepsilon D_\nu\Omega^\lambda_{kij}D_a\Omega^\nu_{lij} + \big(10+\tfrac{7}{3}\varepsilon\big)D_a\Omega^\lambda_{kij}D_\nu\Omega^\nu_{lij} \end{aligned}$$

$$+(10-\frac{7}{6}\varepsilon)\Omega^{\lambda}_{kij}D_\nu D_\sigma\Omega^{\nu}_{lij}+(-14+\frac{61}{12}\varepsilon)D^2\Omega^{\lambda}_{kij}\cdot\Omega^{\sigma}_{lij}+(-14+\frac{4}{3}\varepsilon)D_\mu\Omega^{\lambda}_{kij}D_\mu\Omega^{\sigma}_{lij}$$

$$-(1+\frac{41}{24}\varepsilon)D_\lambda\Omega^{\mu}_{kij}D_\sigma\Omega^{\mu}_{lji}-\frac{13}{4}\varepsilon D_\nu\Omega^{\lambda}_{kij}D_\sigma\Omega^{\nu}_{lji}+(8+\frac{35}{12}\varepsilon)D_\sigma\Omega^{\lambda}_{kij}D_\nu\Omega^{\nu}_{lji}$$

$$+(8-\frac{1}{3}\varepsilon)\Omega^{\lambda}_{kij}D_\nu D_\sigma\Omega^{\nu}_{lji}+(-13+\frac{121}{24}\varepsilon)D^2\Omega^{\lambda}_{kij}\Omega^{\sigma}_{lji}+(-13+\frac{11}{12}\varepsilon)D_\mu\Omega^{\lambda}_{kij}D_\mu\Omega^{\sigma}_{lji}\Big\}. \tag{6.23}$$

When applying this formula to the pure Yang-Mills theory with the background gauge we find that the counter-terms (6.17)~(6.23) vanish. Then only the following four types of counter-terms remain:

$$\Omega^2X^2:\ +\frac{1}{2^8\pi^4\varepsilon^2}(X^2)_{kl}\Big\{(2-\frac{\varepsilon}{3})\Omega^{\mu}_{kij}\Omega^{\mu}_{lij}+(\frac{5}{2}-\frac{19}{24}\varepsilon)\Omega^{\mu}_{kij}\Omega^{\mu}_{lji}\Big\}$$

$$+\frac{1}{2^8\pi^4\varepsilon^2}X_{il}X_{jm}\Big\{(5-\frac{7}{4}\varepsilon)\Omega^{\mu}_{kij}\Omega^{\mu}_{klm}+(4-\frac{1}{2}\varepsilon)\Omega^{\mu}_{kij}\Omega^{\mu}_{kml}\Big\}, \tag{6.24}$$

$$\Omega^2XY:\ -\frac{1}{3\cdot 2^8\pi^4\varepsilon^2}(XY^{\mu\nu})_{kl}$$

$$\times\Omega^{\mu}_{kij}\Big\{(-\frac{26}{3}+\frac{65}{18}\varepsilon)\Omega^{\nu}_{lij}+(-\frac{28}{3}+\frac{35}{9}\varepsilon)\Omega^{\nu}_{lji}\Big\}$$

$$+\frac{1}{3^2\cdot 2^8\pi^4\varepsilon^2}X_{lb}Y^{\mu\nu}_{ka}$$

$$\times\Big[\Omega^{\mu}_{ilk}\Big\{(14+\frac{37}{6}\varepsilon)\Omega^{\nu}_{iab}+(14-\frac{22}{3}\varepsilon)\Omega^{\nu}_{iba}\Big\}$$

$$-(10+\frac{19}{3}\varepsilon)\Omega^{\mu}_{ikl}\Omega^{\nu}_{iab}\Big], \tag{6.25}$$

$$\Omega^2 Y^2 \ : \ -\frac{1}{2^{10}\cdot 3^2 \pi^4 \varepsilon^2}$$

$$\times \Big[\Omega^{\mu}_{kij} \Omega^{\nu}_{bij} \times \{ (44+\tfrac{\varepsilon}{6}) (Y^{\mu\tau} Y^{\nu\tau})_{kb}$$

$$+ (-28+\tfrac{37}{6}\varepsilon) (Y^{\nu\tau} Y^{\mu\tau})_{kb} + (-12+\tfrac{5}{4}\varepsilon) (Y^{\lambda\tau} Y^{\lambda\tau})_{kb} \, \delta_{\mu\nu} \}$$

$$+ \Omega^{\mu}_{kij} \Omega^{\nu}_{bji} \times \{ (46-\tfrac{11}{12}\varepsilon) (Y^{\mu\tau} Y^{\nu\tau})_{kb}$$

$$+ (-26+\tfrac{61}{12}\varepsilon) (Y^{\nu\tau} Y^{\mu\tau})_{kb} + (-15+\tfrac{35}{8}\varepsilon) (Y^{\lambda\tau} Y^{\lambda\tau})_{kb} \, \delta_{\mu\nu} \} \Big]$$

$$+ \frac{1}{2^{9}\cdot 3^2 \pi^4 \varepsilon^2}$$

$$\times \Big[\Omega^{\mu}_{bdc} \Omega^{\nu}_{bae} \{ (28-\tfrac{2}{3}\varepsilon) Y^{\mu\tau}_{ce} Y^{\nu\tau}_{da} + (-5+\tfrac{23}{6}\varepsilon) Y^{\lambda\tau}_{ce} Y^{\lambda\tau}_{da} \, \delta_{\mu\nu} \}$$

$$+ \Omega^{\mu}_{bac} \Omega^{\nu}_{bed} \{ (-4+\tfrac{131}{12}\varepsilon) Y^{\mu\tau}_{ce} Y^{\nu\tau}_{da} + (-4+\tfrac{23}{12}\varepsilon) Y^{\nu\tau}_{ce} Y^{\mu\tau}_{da}$$

$$+ (4-\tfrac{25}{24}\varepsilon) Y^{\lambda\tau}_{ce} Y^{\lambda\tau}_{da} \, \delta_{\mu\nu} \} \Big] , \tag{6.26}$$

$$\text{Ⓗ} X^2 \ : \ + \frac{3}{2^6 \pi^4 \varepsilon^2} X_{ab} X_{cd} \text{Ⓗ}_{abcd} \quad . \tag{6.27}$$

For the theory with non-derivative couplings ($N^{\mu} = \Omega^{\mu} = 0$), only (6.17), (6.18) and (6.27) remain. This case was considered by S. Tamura [32]. The ϕ^4-theory and the σ-model are well-known examples of this case.

[B] <u>The case when the field indices contain Lorentz indices</u>

This is the case with Bose fields φ_i with integer spin ≥ 1 (Yang-Mills theory, gravity, etc.). In this case we have only to consider how to modify the formula given for the previous case [A]. As explained previously, both two-loop diagrams and subtraction diagrams contribute to $\Delta\mathcal{L}^{\text{two-loop}}[\varphi^\alpha]$. For the counter-terms corresponding to subtraction diagrams, we must specify the part which comes from one-loop counter-terms (such as $\Omega\Omega N$ in (6.13)). This is necessary only for double pole parts and not for single pole parts, because the difference between $\delta_{\mu\mu} = 4$ and $\delta_{\mu\mu} = n$ gives finite terms for the latter case. We therefore redirive the double pole parts of two-loop counter-terms which come from the subtraction diagrams. Substituting the one-loop counter-term formula (6.9) in (6.12) we obtain (for derivation, see Appendix E)

$$\frac{1}{2!}\Delta\mathcal{L}^{\text{one-loop}}[\varphi^\alpha]_{,ij}\,\phi_i\phi_j$$
$$=\frac{1}{\varepsilon}\left(\frac{1}{2}W^{\mu\nu}_{ij}D_\mu\phi_i D_\nu\phi_j + N^{\mu}_{ij}\phi_i D_\mu\phi_j + \frac{1}{2}M_{ij}\phi_i\phi_j\right), \tag{6.28}$$

where

$$W^{\mu\nu}_{ij} = \frac{1}{24\pi^2}\Big[\delta_{\mu\nu}(\Omega^\alpha_{iab}-\Omega^\alpha_{iba})\Omega^\alpha_{jab}$$
$$-(14\Omega^\nu_{iab}+13\Omega^\nu_{iba})\Omega^\mu_{jab}-(14\Omega^\mu_{iab}+13\Omega^\mu_{iba})\Omega^\nu_{jab}\Big], \tag{6.29}$$

$$
\begin{aligned}
\mathcal{N}^{\mu}_{ij} = -\frac{1}{64\pi^2}\Big[& 6\,(6\tilde{\Lambda}_{abi} + D_\nu\Omega^{\nu}_{abi})\Omega^{\mu}_{abj} \\
& -\frac{4}{3}D_\alpha(\Omega^{\beta}_{iab}-\Omega^{\beta}_{iba})(\delta_{\mu\alpha}\Omega^{\beta}_{jab}-\delta_{\mu\beta}\Omega^{\alpha}_{jab}) \\
& +9(D_\nu\Omega^{\mu}_{abi}\cdot\Omega^{\nu}_{abj}+\Omega^{\mu}_{abi}D_\nu\Omega^{\nu}_{abj}) \\
& +\frac{2}{3}D_\nu\{(\Omega^{\nu}_{iab}-\Omega^{\nu}_{iba})\Omega^{\mu}_{jab}\} - i\leftrightarrow j \quad \Big],
\end{aligned}
\tag{6.30}
$$

$$
\begin{aligned}
\mathcal{M}_{ij} = -\frac{1}{32\pi^2}\Big[& \{24\,\Theta_{abij} - 2(\Omega^{\mu}_{iac}-\Omega^{\mu}_{ica})(\Omega^{\mu}_{jcb}-\Omega^{\mu}_{jbc})\}X_{ab} \\
& +(6\tilde{\Lambda}_{abi}+D_\mu\Omega^{\mu}_{abi})(6\tilde{\Lambda}_{abj}+D_\nu\Omega^{\nu}_{abj}) \\
& +\frac{2}{3}(\Omega^{\alpha}_{iac}-\Omega^{\alpha}_{ica})(\Omega^{\beta}_{jcb}-\Omega^{\beta}_{jbc})Y^{\alpha\beta}_{ba} \\
& -\frac{2}{3}\{D_\alpha(\Omega^{\beta}_{iab}-\Omega^{\beta}_{iba})-D_\beta(\Omega^{\alpha}_{iab}-\Omega^{\alpha}_{iba})\}D_\alpha\Omega^{\beta}_{jab} \\
& -\frac{1}{2}Y^{\mu\nu}_{kj}\{9\Omega^{\mu}_{abi}\Omega^{\nu}_{abk}+\frac{1}{3}(\Omega^{\nu}_{iab}-\Omega^{\nu}_{iba})(\Omega^{\mu}_{kab}-\Omega^{\mu}_{kba})\} \\
& -\frac{1}{2}D_\mu\{6(6\tilde{\Lambda}_{abi}+D_\nu\Omega^{\nu}_{abi})\Omega^{\mu}_{abj}-\frac{4}{3}D_\alpha(\Omega^{\beta}_{iab}-\Omega^{\beta}_{iba})(\delta_{\mu\alpha}\Omega^{\beta}_{jab}-\delta_{\mu\beta}\Omega^{\alpha}_{jab})\} \\
& + i\leftrightarrow j \quad \Big].
\end{aligned}
\tag{6.31}
$$

The quantities $\mathcal{W}^{\mu\nu}$, $\mathcal{N}^{\mu}$ and $\mathcal{M}$ defined in (6.29), (6.30) and (6.31) respectively, have the following symmetry:

$$
\begin{aligned}
& \mathcal{W}^{\mu\nu}_{ij} = \mathcal{W}^{\nu\mu}_{ij}\,, \qquad \mathcal{W}^{\mu\nu}_{ij} = \mathcal{W}^{\mu\nu}_{ji}\,, \\
& \mathcal{N}^{\mu}_{ij} = -\mathcal{N}^{\mu}_{ji}\,, \qquad \mathcal{M}_{ij} = \mathcal{M}_{ji}\,.
\end{aligned}
\tag{6.32}
$$

From the invariance argument we find that the double pole parts of two-loop counter-terms which come from the subtraction diagrams are written as

$$\Delta\mathcal{L}^{sub} = \frac{1}{\varepsilon^2} \mathcal{W}^{\mu\nu}_{ij} \{ \delta_{\mu\nu} (a X_{jk} X_{ki} + b Y^{\lambda\alpha}_{jk} Y^{\lambda\alpha}_{ki} + c D^2 X_{ji}) + d Y^{\mu\lambda}_{jk} Y^{\lambda\nu}_{ki} + e D_\mu D_\nu X_{ji} \}$$
$$+ \frac{1}{\varepsilon^2} f \mathcal{N}^{\mu}_{ij} D_\lambda Y^{\lambda\mu}_{ji} + \frac{1}{\varepsilon^2} g \mathcal{M}_{ij} X_{ji} \quad , \tag{6.33}$$

where a, b, c, d, e, f and g are constants to be determined later. For convenience let us rewrite (6.28) in the following way:

$$\frac{1}{2!} \Delta\mathcal{L}^{one-loop}[\varphi^\alpha]_{,ij} \phi_i \phi_j$$
$$= \frac{1}{\varepsilon} \left(\frac{1}{2} \bar{W}^{\mu\nu}_{ij} \partial_\mu \phi_i \partial_\nu \phi_j + \bar{N}^{\mu}_{ij} \phi_i \partial_\mu \phi_j + \frac{1}{2} \bar{M}_{ij} \phi_i \phi_j \right) , \tag{6.34}$$

where

$$\bar{W}^{\mu\nu}_{ij} = \mathcal{W}^{\mu\nu}_{ij} ,$$

$$\bar{N}^{\mu}_{ij} = -\frac{1}{2} (\mathcal{W}^{\mu\nu}_{im} N^{\nu}_{mj} - \mathcal{W}^{\mu\nu}_{jm} N^{\nu}_{mi}) + \mathcal{N}^{\mu}_{ij} ,$$

$$\bar{M}_{ij} = -\frac{1}{2} \partial_\mu (\mathcal{W}^{\mu\nu}_{im} N^{\nu}_{mj} + \mathcal{W}^{\mu\nu}_{jm} N^{\nu}_{mi})$$
$$+ \mathcal{W}^{\mu\nu}_{\ell m} N^{\mu}_{\ell i} N^{\nu}_{mj} + \mathcal{N}^{\mu}_{i\ell} N^{\mu}_{\ell j}$$
$$+ \mathcal{N}^{\mu}_{j\ell} N^{\mu}_{\ell i} + \mathcal{M}_{ij} . \tag{6.35}$$

The quantities in (6.35) possess the same symmetry as those given in (6.32).

The vertices in (6.34) are expressed graphically as in Fig.11:

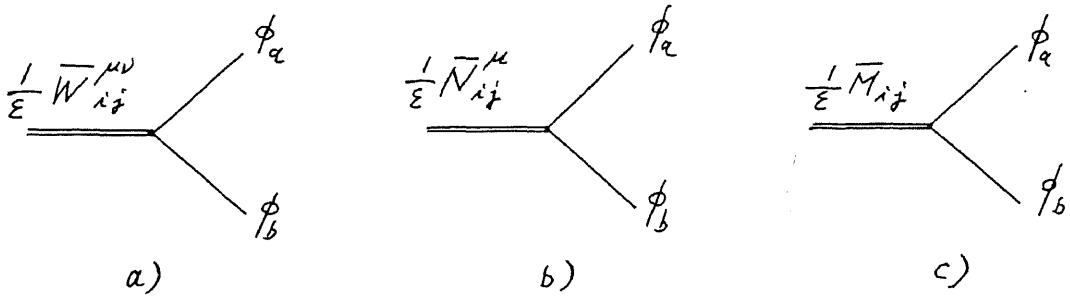


Fig.11.

The Feynman rule for the vertices in Fig.11 is given in Appendix C. An easy way to fix all constants in (6.33) is to calculate logarithmically divergent diagrams such as given in Fig.12:

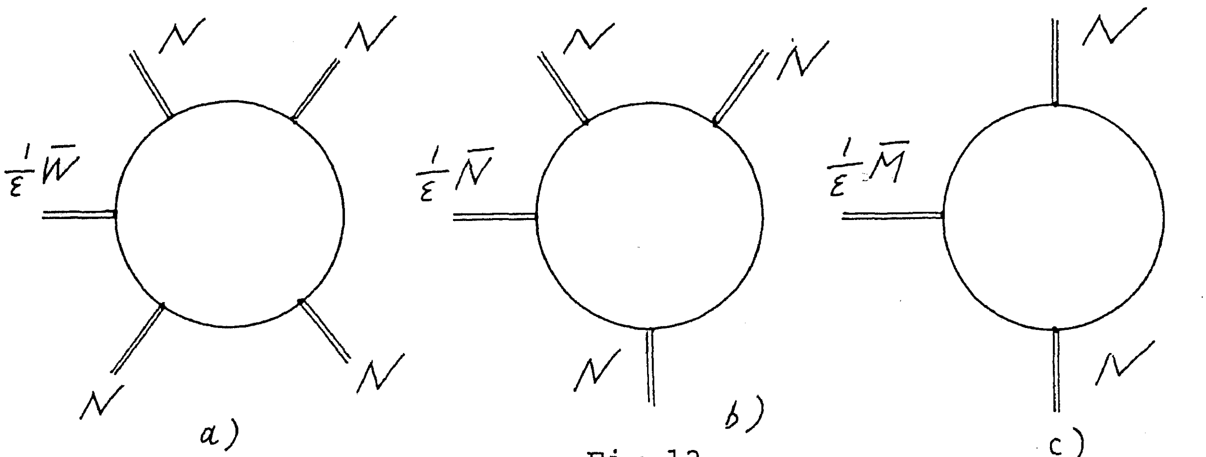


Fig.12.

The result is

$$\Delta\mathcal{L}^{sub} = -\frac{1}{2^6\varepsilon^2\pi^2}\,\mathrm{Tr}\Big[\, w^{\mu\nu}\Big\{ \delta_{\mu\nu}\big(X^2 + \frac{1}{6}Y^{\lambda\alpha}Y^{\lambda\alpha} + \frac{1}{3}D^2X\big) + \frac{2}{3}Y^{\mu\lambda}Y^{\lambda\nu} + \frac{2}{3}D_\mu D_\nu X \Big\} - \frac{4}{3}n^{\mu}D_\lambda Y^{\lambda\mu} + 4mX \,\Big], \tag{6.36}$$

where the symbol Tr means trace. As noted in (6.13) and (6.14), we must take

$$\delta_{\mu\mu} = 4 \tag{6.37}$$

within $w^{\mu\nu}$, n^{μ} and m and

$$\delta_{\mu\mu} = n \tag{6.38}$$

otherwise. Rewriting $w^{\mu\nu}_{ij}$ as

$$w^{\mu\nu}_{ij} = \delta_{\mu\nu}V_{ij} + z^{\mu\nu}_{ij} , \tag{6.39}$$

where

$$V_{ij} = \frac{1}{24\pi^2}\left(\Omega^{\alpha}_{iab} - \Omega^{\alpha}_{iba}\right)\Omega^{\alpha}_{jab}$$

$$z^{\mu\nu}_{ij} = \frac{-1}{24\pi^2}\Big\{ (14\Omega^{\nu}_{iab} + 13\Omega^{\nu}_{iba})\Omega^{\mu}_{jab} + (14\Omega^{\mu}_{iab} + 13\Omega^{\mu}_{iba})\Omega^{\nu}_{jab}\Big\}, \tag{6.40}$$

we obtain

$$\Delta\mathcal{L}^{sub} = \frac{1}{2^6\varepsilon\pi^2}\,Tr\{\,V(X^2+\frac{1}{6}Y^{\lambda\alpha}Y^{\lambda\alpha}+\frac{1}{3}D^2X)\,\}$$

$$-\frac{1}{2^6\varepsilon^2\pi^2}\,Tr\{\,(4V+Z^{\mu\nu}\cdot\delta_{\mu\nu})(X^2+\frac{1}{6}Y^{\lambda\alpha}Y^{\lambda\alpha}+\frac{1}{3}D^2X)$$

$$+\frac{2}{3}(\delta_{\mu\nu}V+Z^{\mu\nu})(Y^{\mu\lambda}Y^{\lambda\nu}+D_\mu D_\nu X)$$

$$-\frac{4}{3}n^\mu D_\lambda Y^{\lambda\mu}+4mX\,\}\,. \tag{6.41}$$

We notice that the first part of (6.41) is already contained in the single pole parts of the counter-terms (6.17)~(6.27). Further we find that the double pole parts of (6.41) are just twice as large as those of counter-terms (6.17)~(6.27). Noting that the latter include contribution both from two-loop diagrams and from subtraction diagrams, we conclude that the double pole parts of counter-terms (6.17)~(6.27) must be replaced by the following:

$$\Delta\mathcal{L}^{two-loop}_{double}[\varphi^{cl}] = -\frac{1}{2}\Delta\mathcal{L}_1[\varphi^{cl}]+\Delta\mathcal{L}_2[\varphi^{cl}]\,, \tag{6.42}$$

where the first term comes from the two-loop diagrams, the second term from the subtraction diagrams, and $\Delta\mathcal{L}_1$ and $\Delta\mathcal{L}_2$ have a common expression such as

$$-\frac{1}{2^6\varepsilon^2\pi^2}\,Tr\,\{\,(4V+Z^{\mu\nu}\cdot\delta_{\mu\nu})(X^2+\frac{1}{6}Y^{\lambda\alpha}Y^{\lambda\alpha}+\frac{1}{3}D^2X)$$

$$+\frac{2}{3}(\delta_{\mu\nu}V+Z^{\mu\nu})(Y^{\mu\lambda}Y^{\lambda\nu}+D_\mu D_\nu X)$$

$$-\frac{4}{3} n^\mu D_\lambda Y^{\lambda\mu} + 4mX \} . \tag{6.43}$$

However the rules for contracting Lorentz indices are different for the two cases $\Delta\mathcal{L}_1$ and $\Delta\mathcal{L}_2$. In $\Delta\mathcal{L}_1$

$$\delta_{\mu\mu} = n \tag{6.44}$$

for all contractions in (6.43). In $\Delta\mathcal{L}_2$

$$\delta_{\mu\mu} = 4 \text{ (within } V,\ Z^{\mu\nu},\ n^\mu \text{ and } m \text{)},$$
$$\delta_{\mu\mu} = n \text{ (otherwise)} . \tag{6.45}$$

We summarize the results in this section.

Case [A] <u>The field indices do not contain Lorentz indices</u>

The complete two-loop counter-terms are given by (6.17)~(6.27). Especially for a theory with non-derivative couplings ($N^\mu = \Omega^\mu = 0$), they are given by (6.17), (6.18) and (6.27).

Case [B] <u>The field indices contain Lorentz indices</u>

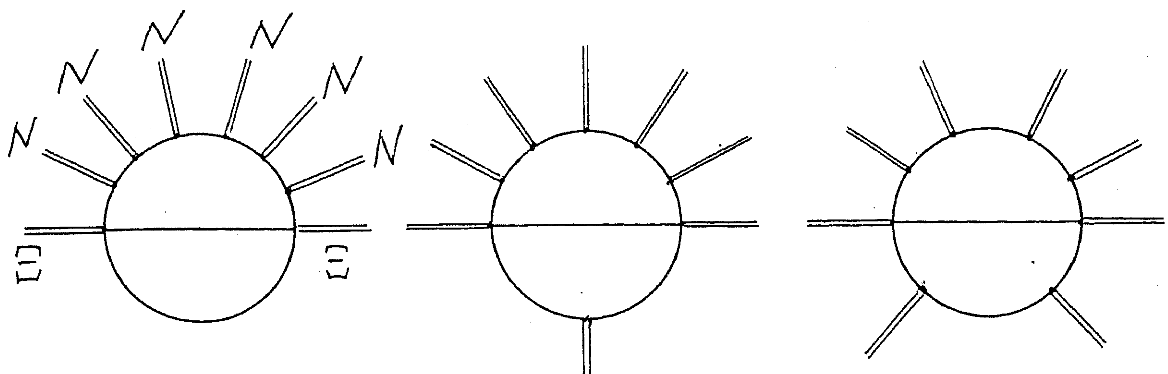
The double pole formula is given by (6.42), (6.43), (6.44) and (6.45). The single pole formula is given by the single pole parts of (6.17)~(6.27).

7. Conclusion and discussion

The formula (4.3) is valid for bose systems with gravitational interaction. It is more powerful than that of G. 'tHooft and M. Veltman in the following respects. Firstly ours gives counter-terms without recourse to the doubling trick [2]. Secondly it is easily generalized to the case of higher-order calculations.

In sect.6 we have obtained the two-loop counter-term formula for a system with (non-gravitational) derivative couplings. The formula can be applied to various types of field theories such as ϕ^4-theory, scalar electrodynamics, σ-model and Yang-Mills theory. When combined with 'tHooft's trick [22], the formula becomes applicable to the system including Fermi fields, such as quantum electrodynamics and quantum chromodynamics.

In order to calculate the two-loop counter-terms for the case of gravity, the formula given in sect.6 is insufficient. We must take into account all the terms of (6.1). In Appendix D all types of two-loop counter-terms for this case are listed. From the analogy to the case of sect.6 it is expected that some of the following logarithmically divergent diagrams must be calculated in order to determine the necessary coefficients.



(a) (b) (c)

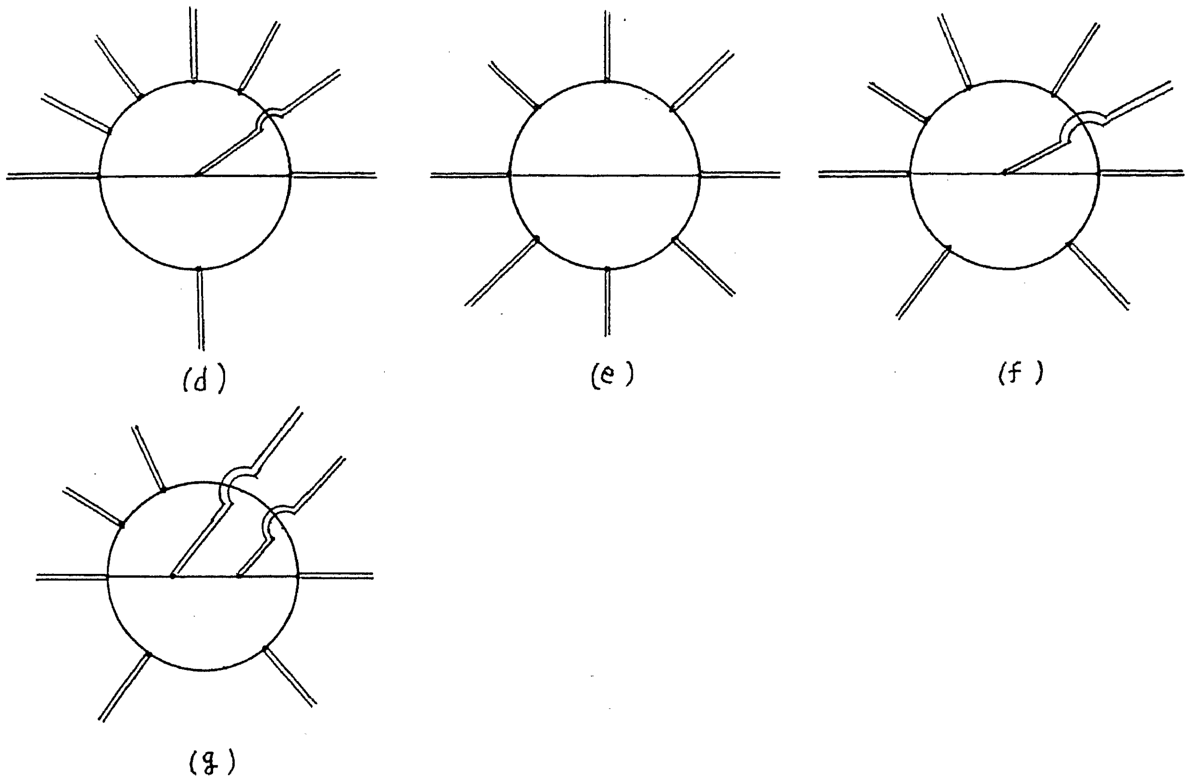


Fig.13.

It does not seem to be very difficult to calculate these diagrams when use is made of computers.

We now add some comments concerning the renormalizability of pure Einstein gravity. It is well known [21, 23, 27] that owing to the relation (4.4) we may take R^2 and $R_{\mu\nu}R^{\mu\nu}$ as independent one-loop counter-terms, both of which vanish on shell ($R_{\mu\nu} = 0$). At the lowest level, therefore, there appears no infinity in the S-matrix of pure Einstein gravity. G. 'tHooft and M. Veltman [21] pointed out that the one-loop counter-terms can be absorbed by the renormalization of an unusual type such as

$$g'_{\mu\nu} = g_{\mu\nu} + \frac{1}{\varepsilon}\left(a\, g_{\mu\nu} R + b\, R_{\mu\nu}\right) , \tag{7.1}$$

where a and b are certain constants. As for two-loop counter-terms only the following is nonvanishing on shell [23, 27]:

$$R_{\mu\nu\alpha\beta}\, R^{\alpha\beta\tau\omega}\, R_{\tau\omega}{}^{\mu\nu} . \tag{7.2}$$

Consequently two-loop quantum correction to pure Einstein gravity will be finite only if the coefficient of this invariant happens to vanish due to some miraculous mechanism of cancellation.

Finally let us touch on the meaning of quantum gravity. The gravitational force is one of the forces we experience in the daily life. However, due to smallness of the coupling constant κ, its quantum effects are hardly measurable by ordinary experiments. As is well known, the coupling constant κ gives the Planck length

$$L = \sqrt{\frac{\kappa\hbar}{c^3}} \approx 10^{-35}\ m , \tag{7.3}$$

and the Planck time

$$T = \sqrt{\frac{\kappa\hbar}{c^5}} \approx 10^{-43}\ sec , \tag{7.4}$$

At present, however, we have no satisfactory theory which properly describes a region of such a small space-time scale. The difficulties encountered in the application of quantum theory to gravity may thus provide us with a clue to new physics.

The calculations of diagrams shown in Fig.4 have been carried out by means of computers.

Acknowledgements

The author is grateful to Professors S. Kamefuchi and Y. Iwasaki for enlightening me on the problem of renormalizability and to Professor Y. Hara for encouragement during this work. A special acknowledgement is due to Doctor M. Omote for suggesting him the present problem and for discussions throughout all stages of the work. He would also like to thank Professor Y.Miyamoto for carefully reading the manuscript.

Appendix A. A method to compute the divergent parts of two-loop integrals

In this appendix we show how to evaluate the divergent parts of two-loop integrals. Some methods have already been given in the literature [34, 37]. We present a new method, which has the following features. Firstly our method is applicable to a wide range of two-loop integrals. Secondly actual calculations can easily be carried out by a computer. Let us explain the method by referring to an example:

$$I = \int d^n p_1 d^n p_2 \frac{1}{(p_1^2+m_1^2)(p_2^2+m_2^2)^2[(k+p_1-p_2)^2+m_3^2]} . \tag{A.1}$$

First we perform the P_1-integral

$$I_1 = \int d^n p_1 \frac{1}{(p_1^2+m_1^2)[(k+p_1-p_2)^2+m_3^2]}$$

$$= i\pi^{\frac{n}{2}}\Gamma(2-\tfrac{n}{2})\int_0^1 dx \frac{m^2(x)}{\{x(1-x)\}^{2-\frac{n}{2}}(m^2(x))^{3-\frac{n}{2}}} , \tag{A.2}$$

where

$$m^2(x) = (k-p_2)^2 + \frac{m_1^2}{x} + \frac{m_3^2}{1-x} , \tag{A.3}$$

Next we perform the P_2-integral:

$$I_2 = \int d^n p_2 \frac{m^2(x)}{(p_2^2+m_2^2)^2(m^2(x))^{3-\frac{n}{2}}} . \tag{A.4}$$

The Feynman parameter method gives us

$$I_2 = \frac{\Gamma(5-\frac{n}{2})}{\Gamma(3-\frac{n}{2})} \int_0^1 dy\,(1-y)\,y^{2-\frac{n}{2}} \int d^n p_2' \frac{1}{(p_2'^2 + m'(x,y)^2)^{5-\frac{n}{2}}}$$

$$\times \left\{ (p_2' - k(1-y))^2 + \frac{m_1^2}{x} + \frac{m_3^2}{1-x} \right\}, \tag{A.5}$$

where $\quad p_2' = p_2 - ky,$

$$m'^2(x,y) = m_2^2(1-y) + k^2 y(1-y) + \frac{y}{x} m_1^2 + \frac{y}{1-x} m_3^2. \tag{A.6}$$

We proceed further,

$$I_2 = \frac{\Gamma(5-\frac{n}{2})}{\Gamma(3-\frac{n}{2})} \int_0^1 dy\,(1-y)\,y^{2-\frac{n}{2}} \frac{i\pi^{\frac{n}{2}}}{(m'^2)^{5-n}} \frac{1}{\Gamma(5-\frac{n}{2})}$$

$$\times \left[\Gamma(4-n)\frac{m'^2}{2} n + \Gamma(5-n)\left\{ (1-y)\frac{dm'^2}{dy} + m'^2 \right\} \right], \tag{A.7}$$

where use is made of the relation:

$$k^2(1-y)^2 + \frac{m_1^2}{x} + \frac{m_3^2}{1-x} = (1-y)\frac{dm'^2}{dy} + m'^2 . \tag{A.8}$$

From (A.2) and (A.7) we obtain

$$I = -\frac{\pi^n}{2-\frac{n}{2}} \int_0^1 dx \int_0^1 dy \frac{(1-y)\,y^{2-\frac{n}{2}}}{(x(1-x))^{2-\frac{n}{2}}}$$

$$\times \left[\Gamma(4-n)\frac{(m'^2)^{-4+n}}{2} n + \Gamma(5-n)(1-y)\frac{1}{(m'^2)^{5-n}} \frac{dm'^2}{dy} \right.$$

$$+\Gamma(5-n)(m'^2)^{-4+n} \qquad] \,. \tag{A.9}$$

So far our calculation is exact. We are now interested only in the divergent part of (A.9). Let us evaluate the first term.

$$\begin{aligned} J_1 &= \int_0^1 dx \int_0^1 dy \frac{(1-y)\, y^{2-\frac{n}{2}}}{[x(1-x)]^{2-\frac{n}{2}}} (m'^2)^{-4+n} \\ &= \frac{4-n}{2}\int_0^1 dx \int_0^1 dy\,(1-y)\ln y + \int_0^1 dx \int_0^1 dy \frac{(1-y)(m'^2)^{-4+n}}{[x(1-x)]^{2-\frac{n}{2}}} \\ &\quad + O((4-n)^2) \,, \end{aligned} \tag{A.10}$$

where we have used an expansion

$$y^{2-\frac{n}{2}} = 1 + \frac{4-n}{2}\ln y + O((4-n)^2) \,. \tag{A.11}$$

The calculation of J_1 proceeds as follows:

$$\begin{aligned} J_1 &= -\frac{3}{8}(4-n) - \frac{1}{2}\int_0^1 dx \frac{1}{\{x(1-x)\}^{2-\frac{n}{2}}} \int_0^1 dy \left\{\frac{d(1-y)^2}{dy}\right\}(m'^2)^{-4+n} \\ &\qquad + O((4-n)^2) \\ &= -\frac{3}{8}(4-n) - \frac{1}{2}\int_0^1 dx \frac{1}{\{x(1-x)\}^{2-\frac{n}{2}}} \Big\{ \Big[(1-y)^2 (m'^2)^{-4+n}\Big]_0^1 \\ &\qquad - \int_0^1 dy\,(1-y)^2 \frac{d(m'^2)^{-4+n}}{dy} \Big\} + O((4-n)^2) \\ &= -\frac{3}{8}(4-n) - \frac{1}{2}\int_0^1 dx \frac{1}{\{x(1-x)\}^{2-\frac{n}{2}}} \Big\{ -(1-(4-n)\ln m_2^2) \end{aligned}$$

$$+\frac{1}{2}(4-n)\int_0^1 dy(1-y)^2\frac{1}{m'^2}\frac{dm'^2}{dy}\Big\} + O((4-n)^2).$$

(A.12)

From (A.12) the first term of (A.9) is

$$K_1 = -\frac{\pi^n}{(4-n)^2}\Big(4-(4-n)-4\gamma(4-n)\Big)$$
$$\times\Big\{-\frac{3}{8}(4-n)+\frac{1}{2}+\frac{1}{2}(4-n)-\frac{4-n}{2}\ln m_2^2$$
$$-\frac{1}{2}(4-n)\int_0^1 dx\int_0^1 dy(1-y)^2\frac{1}{m'^2}\frac{dm'^2}{dy}\Big\} + O(1). \quad \text{(A.13)}$$

The second and third terms of (A.9) are not particularly difficult to evaluate:

$$K_2 = -\frac{2\pi^n}{4-n}\int_0^1 dx\int_0^1 dy\,(1-y)^2\frac{1}{m'^2}\frac{dm'^2}{dy} + O(1),$$
$$K_3 = -\frac{\pi^n}{4-n} + O(1). \quad \text{(A.14)}$$

The final result is

$$I = \pi^n\Big[-2\Big\{\frac{1}{(4-n)^2}-\frac{1}{4-n}(\ln m_2^2+\gamma)\Big\}-\frac{1}{4-n}\Big]$$
$$+ O(1). \quad \text{(A.15)}$$

Note that K_2 cancels a term of the same type in (A.13). We have checked such cancellation for individual cases, and expect that they may occur generally. In the Appendix B we list the results for other examples.

Appendix B. The divergent parts of two-loop integrals

We list the divergent parts of various two-loop integrals. They are grouped according to the types of corresponding graphs. Here we define some quantities which will be used below.

$$\varepsilon = 4 - n \,,$$

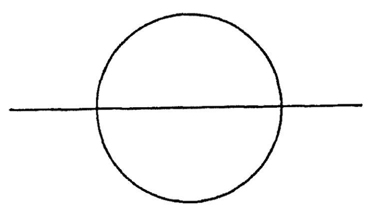
$$X = \frac{1}{\varepsilon}\left(\gamma + \ln m_2^2 + \ln \pi\right) - \frac{1}{\varepsilon^2} \,,$$

$$[\mu\nu\lambda\sigma] = \delta_{\mu\nu}\delta_{\lambda\sigma} + \delta_{\mu\lambda}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\lambda} \,,$$

$$\begin{aligned}[\mu\nu\lambda\sigma\tau\omega] = {} & \delta_{\mu\nu}[\lambda\sigma\tau\omega] + \delta_{\mu\lambda}[\nu\sigma\tau\omega] \\ & + \delta_{\mu\sigma}[\nu\lambda\tau\omega] + \delta_{\mu\tau}[\nu\lambda\sigma\omega] + \delta_{\mu\omega}[\nu\lambda\sigma\tau] \,,\end{aligned} \tag{B.1}$$

where m_2^2 is a parameter to appear below.

A-type



$$A = \int d^n p_1 d^n p_2 \frac{1}{p_1^2 (p_2^2 + m_2^2)(p_1 - p_2 + k)^2}$$

$$= \pi^4 \left[\left(\frac{k^2}{2} + 3m_2^2 \right) \frac{1}{\varepsilon} - 2X \right] + O(1) , \quad \text{(B.2)}$$

$$A_1^\mu = \int d^n p_1 d^n p_2 \frac{p_1^\mu}{(p_1^2 + m_1^2)(p_2^2 + m_2^2)(p_1 - p_2 + k)^2}$$

$$= \pi^4 \left[-k^\mu \left(\frac{5}{4} m_2^2 + \frac{1}{2} m_1^2 + \frac{1}{6} k^2 \right) \frac{1}{\varepsilon} + k^\mu m_2^2 X \right] + O(1) , \quad \text{(B.3)}$$

$$A_{11}^{\mu\nu} = \int d^n p_1 d^n p_2 \frac{p_1^\mu p_1^\nu}{p_1^2 (p_2^2 + m_2^2)(p_1 - p_2 + k)^2}$$

$$= \pi^4 \left[k^\mu k^\nu \left\{ \left(\frac{7}{9} m_2^2 + \frac{1}{12} k^2 \right) \frac{1}{\varepsilon} - \frac{2}{3} m_2^2 X \right\} \right.$$

$$\left. + \delta_{\mu\nu} \left\{ - \left(\frac{k^4}{48} + \frac{17}{12 \times 6} k^2 m_2^2 \right) \frac{1}{\varepsilon} + \frac{1}{6} k^2 m_2^2 X \right\} \right]$$

$$+ O(1) , \quad \text{(B.4)}$$

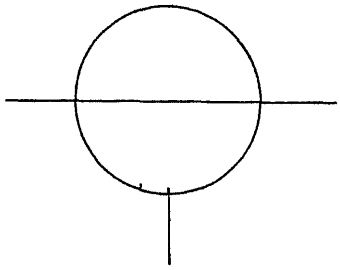
$$A_{12}^{\mu\nu} = \int d^n p_1 d^n p_2 \frac{p_1^\mu p_2^\nu}{p_1^2 (p_2^2 + m_2^2)(p_1 - p_2 + k)^2}$$

$$= \pi^4 \left[-k^\mu k^\nu \left(\frac{1}{6} m_2^2 + \frac{1}{24} k^2 \right) \frac{1}{\varepsilon} \right.$$

$$\left. + \delta_{\mu\nu} \left\{ - \left(\frac{7}{16} m_2^4 + \frac{1}{12} k^2 m_2^2 + \frac{1}{96} k^4 \right) \frac{1}{\varepsilon} + \frac{1}{4} m_2^4 X \right\} \right]$$

$$+ O(1) , \quad \text{(B.5)}$$

B-type



$$B = \int d^n p_1 d^n p_2 \frac{1}{(p_1^2+m_1^2)(p_2^2+m_2^2)^2((p_1-p_2+k)^2+m_3^2)}$$

$$= \pi^4 \left[-\frac{1}{\varepsilon} + 2X \right] + O(1) , \tag{B.6}$$

$$B_1^{\mu} = \int d^n p_1 d^n p_2 \frac{p_1^{\mu}}{(p_1^2+m_1^2)(p_2^2+m_2^2)^2((p_1-p_2+k)^2+m_3^2)}$$

$$= \pi^4 \left[\frac{1}{4} k^{\mu} \frac{1}{\varepsilon} - k^{\mu} X \right] + O(1) , \tag{B.7}$$

$$B_{11}^{\mu\nu} = \int d^n p_1 d^n p_2 \frac{p_1^{\mu} p_1^{\nu}}{(p_1^2+m_1^2)(p_2^2+m_2^2)^2(p_1-p_2+k)^2}$$

$$= \pi^4 \Big[k^{\mu} k^{\nu} \left(-\frac{1}{9}\frac{1}{\varepsilon} + \frac{2}{3} X \right)$$

$$+ \delta_{\mu\nu} \left\{ \left(\frac{3}{8} m_1^2 + \frac{5}{72} k^2 \right) \frac{1}{\varepsilon} - \left(\frac{1}{2} m_1^2 + \frac{1}{6} k^2 \right) X \right\} \Big]$$

$$+ O(1) , \tag{B.8}$$

$$B_{12}^{\mu\nu} = \int d^n p_1 d^n p_2 \frac{p_1^{\mu} p_2^{\nu}}{(p_1^2+m_1^2)(p_2^2+m_2^2)^2(p_1-p_2+k)^2}$$

$$= \pi^4 \Big[\frac{1}{6} k^\mu k^\nu \frac{1}{\varepsilon} + \delta_{\mu\nu} \Big\{ \Big(\frac{5}{8} m_2^2 + \frac{1}{4} m_1^2 + \frac{1}{12} k^2 \Big) \frac{1}{\varepsilon} - \frac{1}{2} m_2^2 X \Big\} \Big] + O(1), \tag{B.9}$$

$$B_{111}^{\mu\nu\lambda} = \int d^n p_1 d^n p_2 \frac{p_1^\mu p_1^\nu p_1^\lambda}{(p_1^2+m_1^2)(p_2^2+m_2^2)^2(p_1-p_2+k)^2}$$

$$= \pi^4 \Big[k^\mu k^\nu k^\lambda \Big(\frac{1}{16} \frac{1}{\varepsilon} - \frac{1}{2} X \Big) + [\mu\nu\lambda\,\omega] k^\omega m_1^2 \Big(\frac{7}{72} \frac{1}{\varepsilon} + \frac{1}{6} X \Big) + [\mu\nu\lambda\omega] k^\omega k^2 \Big(-\frac{7}{72\times 4} \frac{1}{\varepsilon} + \frac{1}{12} X \Big) \Big] + O(1), \tag{B.10}$$

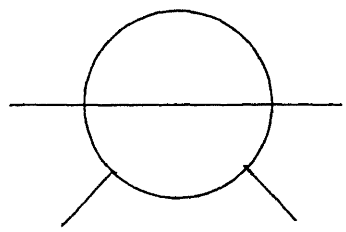
$$B_{112}^{\mu\nu\lambda} = \int d^n p_1 d^n p_2 \frac{p_1^\mu p_1^\nu p_2^\lambda}{(p_1^2+m_1^2)(p_2^2+m_2^2)^2(p_1-p_2+k)^2}$$

$$= \pi^4 \Big[-\frac{1}{12} k^\mu k^\nu k^\lambda \frac{1}{\varepsilon} + \delta_{\mu\nu} k^\lambda \Big\{ \Big(\frac{17}{3\times 24} m_2^2 + \frac{1}{6} m_1^2 + \frac{1}{24} k^2 \Big) \frac{1}{\varepsilon} - \frac{1}{6} m_2^2 X \Big\}$$

$$+(\delta_{\lambda\mu}k^{\nu}+\delta_{\lambda\nu}k^{\mu})\left\{-\left(\frac{7}{18}m_2^2+\frac{1}{12}m_1^2+\frac{1}{24}k^2\right)\frac{1}{\varepsilon}+\frac{1}{3}m_2^2 X\right\}\Bigg]$$

$$+\ O(1)\quad , \tag{B.11}$$

C-type



$$C=\int d^n p_1 d^n p_2\ \frac{1}{(p_1^2+m_1^2)(p_2^2+m_2^2)^3[(p_1-p_2+k)^2+m_3^2]}$$

$$=-\pi^4\frac{1}{m_2^2}\frac{1}{\varepsilon}+O(1)\quad , \tag{B.12}$$

$$C_1^{\mu}=\int d^n p_1 d^n p_2\ \frac{p_1^{\mu}}{(p_1^2+m_1^2)(p_2^2+m_2^2)^3[(p_1-p_2+k)^2+m_3^2]}$$

$$=\pi^4\frac{1}{2}\frac{k^{\mu}}{m_2^2}\frac{1}{\varepsilon}+O(1)\quad ,$$

(B.13)

$$C_2^{\mu} = \int d^n p_1 d^n p_2 \frac{p_2^{\mu}}{(p_1^2+m_1^2)(p_2^2+m_2^2)^3[(p_1-p_2+k)^2+m_3^2]}$$

$$= O(1) ,$$

(B.14)

$$C_{11}^{\mu\nu} = \int d^n p_1 d^n p_2 \frac{p_1^{\mu} p_1^{\nu}}{(p_1^2+m_1^2)(p_2^2+m_2^2)^3[(p_1-p_2+k)^2+m_3^2]}$$

$$= \pi^4\left[-\frac{1}{3}\frac{k^{\mu}k^{\nu}}{m_2^2}\frac{1}{\varepsilon} + \frac{1}{12}\delta_{\mu\nu}\frac{1}{m_2^2}(3m_3^2+3m_1^2+k^2)\frac{1}{\varepsilon}\right]$$

$$+ O(1) ,$$

(B.15)

$$C_{12}^{\mu\nu} = \int d^n p_1 d^n p_2 \frac{p_1^{\mu} p_2^{\nu}}{(p_1^2+m_1^2)(p_2^2+m_2^2)^3[(p_1-p_2+k)^2+m_3^2]}$$

$$= \pi^4 \delta_{\mu\nu}\left(-\frac{1}{16}\frac{1}{\varepsilon} + \frac{1}{4}X\right) + O(1) ,$$

(B.16)

$$C_{22}^{\mu\nu} = \int d^n p_1 d^n p_2 \frac{p_2^{\mu} p_2^{\nu}}{(p_1^2+m_1^2)(p_2^2+m_2^2)^3[(p_1-p_2+k)^2+m_3^2]}$$

$$= \pi^4 \delta_{\mu\nu}\left(-\frac{1}{8}\frac{1}{\varepsilon} + \frac{1}{2}X\right) + O(1) ,$$

(B.17)

$$C^{\mu\nu\lambda}_{111} = \int d^n p_1 d^n p_2 \frac{p_1^\mu p_1^\nu p_1^\lambda}{(p_1^2+m_1^2)(p_2^2+m_2^2)^3[(p_1-p_2+k)^2+m_3^2]}$$

$$= \pi^4 \Big[\frac{1}{4} \frac{k^\mu k^\nu k^\lambda}{m_2^2} \frac{1}{\varepsilon} - \frac{1}{24}[\mu\nu\lambda\omega] \frac{k^\omega}{m_2^2}(4m_3^2 +$$

$$2m_1^2 + k^2) \frac{1}{\varepsilon} \Big] + O(1) \quad , \tag{B.18}$$

$$C^{\mu\nu\lambda}_{112} = \int d^n p_1 d^n p_2 \frac{p_1^\mu p_1^\nu p_2^\lambda}{(p_1^2+m_1^2)(p_2^2+m_2^2)^3[(p_1-p_2+k)^2+m_3^2]}$$

$$= \pi^4 \Big[\delta_{\mu\nu} k^\lambda \Big(-\frac{5}{48\times 3} \frac{1}{\varepsilon} + \frac{1}{12} X \Big)$$

$$+ (\delta_{\lambda\mu} k^\nu + \delta_{\lambda\nu} k^\mu) \Big(\frac{1}{36} \frac{1}{\varepsilon} - \frac{1}{6} X \Big) \Big] + O(1) \quad , \tag{B.19}$$

$$C^{\mu\nu\lambda}_{122} = \int d^n p_1 d^n p_2 \frac{p_1^\mu p_2^\nu p_2^\lambda}{(p_1^2+m_1^2)(p_2^2+m_2^2)^3[(p_1-p_2+k)^2+m_3^2]}$$

$$= \pi^4 \Big[-\frac{1}{24} (\delta_{\mu\nu} k^\lambda + \delta_{\mu\lambda} k^\nu) \frac{1}{\varepsilon}$$

$$+ \delta_{\nu\lambda} k^\mu \Big(\frac{1}{48} \frac{1}{\varepsilon} - \frac{1}{4} X \Big) \Big] + O(1) \quad , \tag{B.20}$$

$$C^{\mu\nu\lambda}_{222} = \int d^n p_1 d^n p_2 \frac{p_2^\mu p_2^\nu p_2^\lambda}{(p_1^2+m_1^2)(p_2^2+m_2^2)^3 [(p_1-p_2+k)^2+m_3^2]}$$

$$= -\frac{1}{12}\pi^4 [\mu\nu\lambda\omega] k^\omega \frac{1}{\varepsilon} + O(1) , \tag{B.21}$$

$$C^{\mu\nu\lambda\alpha}_{1111} = \int d^n p_1 d^n p_2 \frac{p_1^\mu p_1^\nu p_1^\lambda p_1^\alpha}{(p_1^2+m_1^2)(p_2^2+m_2^2)^3 (p_1-p_2+k)^2}$$

$$= \pi^4 \Big[-\frac{1}{5} \frac{k_\mu k_\nu k_\lambda k_\alpha}{m_2^2} \frac{1}{\varepsilon}$$

$$-[\mu\nu\lambda\alpha] \frac{1}{m_2^2} \Big\{ \frac{1}{24}(m_1^2)^2 + \frac{1}{24} k^2 m_1^2 + \frac{1}{60}(k^2)^2 \Big\} \frac{1}{\varepsilon}$$

$$+[\mu\nu\lambda\alpha\alpha\beta] k^\alpha k^\beta \frac{1}{m_2^2}\Big(\frac{1}{48} m_1^2 + \frac{1}{80} k^2\Big)\frac{1}{\varepsilon} \Big] + O(1) , \tag{B.22}$$

$$C^{\mu\nu\lambda\alpha}_{1112} = \int d^n p_1 d^n p_2 \frac{p_1^\mu p_1^\nu p_1^\lambda p_2^\alpha}{(p_1^2+m_1^2)(p_2^2+m_2^2)^3 (p_1-p_2+k)^2}$$

$$= \pi^4 \Big[[\mu\nu\lambda\omega] k^\omega k^\alpha \Big(\frac{7}{48\times 12} \frac{1}{\varepsilon} - \frac{1}{24} X \Big)$$

$$+[\mu\nu\lambda\alpha] \Big\{ \Big(\frac{5}{24\times 12} m_1^2 + \frac{7}{48\times 24} k^2 \Big) \frac{1}{\varepsilon}$$

$$- \Big(\frac{1}{24} m_1^2 + \frac{1}{48} k^2 \Big) X \Big\}$$

$$+(\delta_{\alpha\mu}k^\nu k^\lambda+\delta_{\alpha\nu}k^\mu k^\lambda+\delta_{\alpha\lambda}k^\mu k^\nu)\left(-\frac{1}{64}\frac{1}{\varepsilon}+\frac{1}{8}X\right)\Big]+O(1),$$

(B.23)

$$C^{\mu\nu\lambda\alpha}_{1122}:=\int d^np_1d^np_2\,\frac{p_1^\mu p_1^\nu p_2^\lambda p_2^\alpha}{(p_1^2+m_1^2)(p_2^2+m_2^2)^3(p_1-p_2+k)^2}$$

$$=\pi^4\Big[-\frac{1}{24}\delta_{\mu\nu}k^\lambda k^\alpha\frac{1}{\varepsilon}$$

$$+[\mu\nu\lambda\alpha]\left\{\left(\frac{7}{72}m_2^2+\frac{1}{48}m_1^2\right)\frac{1}{\varepsilon}-\frac{1}{12}m_2^2X\right\}$$

$$+\delta_{\lambda\alpha}k^\mu k^\nu\left(-\frac{1}{36}\frac{1}{\varepsilon}+\frac{1}{6}X\right)$$

$$+\frac{1}{96}[\mu\nu\lambda\alpha\alpha\beta]k^\alpha k^\beta\frac{1}{\varepsilon}$$

$$+\delta_{\mu\nu}\delta_{\lambda\alpha}\Big\{\left(-\frac{5}{32}m_2^2+\frac{1}{32}m_1^2-\frac{1}{72\times4}k^2\right)\frac{1}{\varepsilon}$$

$$+\left(\frac{1}{8}m_2^2-\frac{1}{8}m_1^2-\frac{1}{24}k^2\right)X\Big\}\Big]+O(1),$$

(B.24)

$$C_{1222}^{\mu\nu\lambda\alpha} = \int d^n p_1 d^n p_2 \frac{p_1^\mu p_2^\nu p_2^\lambda p_2^\alpha}{(p_1^2+m_1^2)(p_2^2+m_2^2)^3(p_1-p_2+k)^2}$$

$$= \pi^4 \Big[-\frac{1}{96}(\delta_{\mu\nu} k^\lambda k^\alpha + \delta_{\mu\lambda} k^\nu k^\alpha + \delta_{\mu\alpha} k^\nu k^\lambda)\frac{1}{\varepsilon}$$

$$+ \frac{1}{32}[\nu\lambda\alpha\omega] k^\omega k^\mu \frac{1}{\varepsilon}$$

$$+[\mu\nu\lambda\alpha]\Big\{\Big(\frac{13}{96}m_2^2 + \frac{1}{24}m_1^2 + \frac{1}{64}k^2\Big)\frac{1}{\varepsilon} - \frac{1}{8}m_2^2 X\Big\}\Big] + O(1),$$

(B.25)

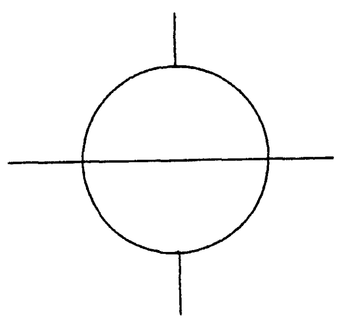
$$C_{2222}^{\mu\nu\lambda\alpha} = \int d^n p_1 d^n p_2 \frac{p_2^\mu p_2^\nu p_2^\lambda p_2^\alpha}{p_1^2(p_2^2+m_2^2)^3(p_1-p_2+k)^2}$$

$$= \pi^4 \Big[-\frac{1}{96}[\mu\nu\lambda\alpha\alpha\beta] k^\alpha k^\beta \frac{1}{\varepsilon}$$

$$+[\mu\nu\lambda\alpha]\Big\{\Big(\frac{13}{48}m_2^2 + \frac{1}{24}k^2\Big)\frac{1}{\varepsilon} - \frac{1}{4}m_2^2 X\Big\}\Big] + O(1),$$

(B.26)

D-type



$$D = \int d^n p_1 d^n p_2 \frac{1}{(p_1^2+m_1^2)^2(p_2^2+m_2^2)^2[(p_1-p_2+k)^2+m_3^2]} = O(1) \, ,$$

(B.27)

$$D_1^\mu = \int d^n p_1 d^n p_2 \frac{p_1^\mu}{(p_1^2+m_1^2)^2(p_2^2+m_2^2)^2[(p_1-p_2+k)^2+m_3^2]} = O(1) \, ,$$

(B.28)

$$D_{11}^{\mu\nu} = \int d^n p_1 d^n p_2 \frac{p_1^\mu p_1^\nu}{(p_1^2+m_1^2)^2(p_2^2+m_2^2)^2[(p_1-p_2+k)^2+m_3^2]}$$

$$= \pi^4 \delta_{\mu\nu} \left(-\frac{3}{8}\frac{1}{\varepsilon} + \frac{1}{2} X \right) + O(1) \, ,$$

(B.29)

$$D_{12}^{\mu\nu} = \int d^n p_1 d^n p_2 \frac{p_1^\mu p_2^\nu}{(p_1^2+m_1^2)^2(p_2^2+m_2^2)^2[(p_1-p_2+k)^2+m_3^2]}$$

$$= -\frac{\pi^4}{4} \delta_{\mu\nu} \frac{1}{\varepsilon} + O(1) \, ,$$

(B.30)

$$D_{111}^{\mu\nu\lambda} = \int d^n p_1 d^n p_2 \frac{p_1^\mu p_1^\nu p_1^\lambda}{(p_1^2+m_1^2)^2(p_2^2+m_2^2)^2[(p_1-p_2+k)^2+m_3^2]}$$

$$= \pi^4 [\mu\nu\lambda\omega] k^\omega \left(\frac{5}{72}\frac{1}{\varepsilon} - \frac{1}{6} X \right) + O(1) \, ,$$

(B.31)

$$D_{112}^{\mu\nu\lambda} = \int d^n p_1 d^n p_2 \frac{p_1^\mu p_1^\nu p_2^\lambda}{(p_1^2+m_1^2)^2(p_2^2+m_2^2)^2[(p_1-p_2+k)^2+m_3^2]}$$

$$= \pi^4\left\{-\frac{1}{6}\delta_{\mu\nu}k^\lambda + \frac{1}{12}(\delta_{\lambda\mu}k_\nu + \delta_{\lambda\nu}k_\mu)\right\}\frac{1}{\varepsilon} + O(1) ,$$

(B.32)

$$D_{1111}^{\mu\nu\lambda\alpha} = \int d^n p_1 d^n p_2 \frac{p_1^\mu p_1^\nu p_1^\lambda p_1^\alpha}{(p_1^2+m_1^2)^2(p_2^2+m_2^2)^2(p_1-p_2+k)^2}$$

$$= \pi^4\Big[\,[\mu\nu\lambda\alpha\alpha\beta]k^\alpha k^\beta\left(-\frac{7}{32\times 18}\frac{1}{\varepsilon} + \frac{1}{24}X\right)$$

$$+[\mu\nu\lambda\alpha]\left\{\left(\frac{11}{72}m_1^2 + \frac{5}{24\times 6}k^2\right)\frac{1}{\varepsilon} - \left(\frac{1}{6}m_1^2 + \frac{1}{12}k^2\right)X\right\}\Big]$$

$$+ O(1) ,$$

(B.33)

$$D_{1112}^{\mu\nu\lambda\alpha} = \int d^n p_1 d^n p_2 \frac{p_1^\mu p_1^\nu p_1^\lambda p_2^\alpha}{(p_1^2+m_1^2)^2(p_2^2+m_2^2)^2(p_1-p_2+k)^2}$$

$$= \pi^4\Big[\,\frac{1}{24}[\mu\nu\lambda\omega]k^\omega k^\alpha\frac{1}{\varepsilon}$$

$$+[\mu\nu\lambda\alpha]\left\{\left(\frac{17}{24\times 6}m_2^2 + \frac{1}{12}m_1^2 + \frac{1}{48}k^2\right)\frac{1}{\varepsilon} - \frac{1}{12}m_2^2 X\right\}$$

$$-\frac{1}{24}(\delta_{\mu\alpha}k_\nu k_\lambda + \delta_{\nu\alpha}k_\mu k_\lambda + \delta_{\lambda\alpha}k_\mu k_\nu)\frac{1}{\varepsilon}\ \Big] + O(1) \,, \tag{B.34}$$

$$D^{\mu\nu\lambda\alpha}_{1122} = \int d^n p_1 d^n p_2 \frac{p_1^\mu p_1^\nu p_2^\lambda p_2^\alpha}{(p_1^2+m_1^2)^2(p_2^2+m_2^2)^3[(p_1-p_2+k)^2+m_3^2]}$$

$$= \pi^4 \Big[\ \delta_{\mu\nu}\delta_{\lambda\alpha}\left(-\frac{1}{32}\frac{1}{\varepsilon} + \frac{1}{8}X\right)$$

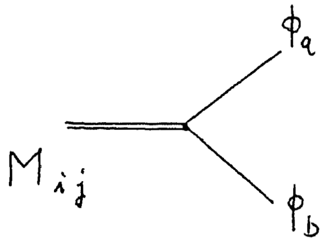
$$-\frac{1}{48}[\mu\nu\lambda\alpha]\frac{1}{\varepsilon}\ \Big] + O(1) \,. \tag{B.35}$$

Appendix C. Feynman Rules

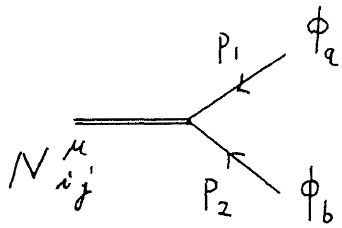
Feynman rules for the Lagrangian:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi_i\partial_\mu\phi_i + \phi_i N^\mu_{ij}\partial_\mu\phi_j + \frac{1}{2}\phi_i M_{ij}\phi_j$$
$$+ \Omega^\mu_{ijk}\phi_i\phi_j\partial_\mu\phi_k + \Lambda_{ijk}\phi_i\phi_j\phi_k$$
$$+ \Theta_{ijkl}\phi_i\phi_j\phi_k\phi_l \,. \tag{C.1}$$

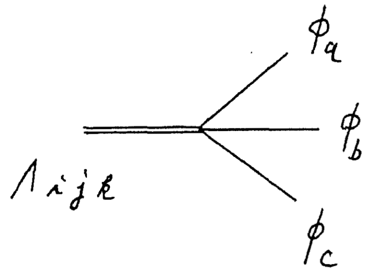
$$\phi : \quad {}_i\overset{k}{\overline{\qquad\qquad}}{}_j \qquad \frac{1}{ik^2}\delta_{ij} \,, \tag{C.2}$$



$$\frac{i}{2}\left(\delta_{ia}\delta_{jb}+\delta_{ib}\delta_{ja}\right) , \qquad \text{(C.3)}$$



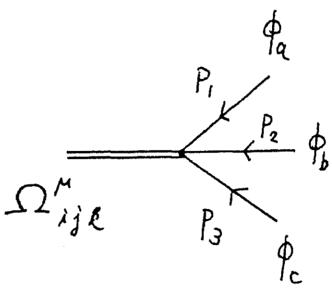
$$\frac{1}{2}\left(\delta_{ia}\delta_{jb}-\delta_{ib}\delta_{ja}\right)\left(P_1-P_2\right)^{\mu} , \qquad \text{(C.4)}$$



$$i\,\{ijk\,;\,abc\} , \qquad \text{(C.5)}$$

where

$$\{ijk\,;\,abc\} = \delta_{ia}\delta_{jb}\delta_{kc} + perm(a,b,c) , \qquad \text{(C.6)}$$

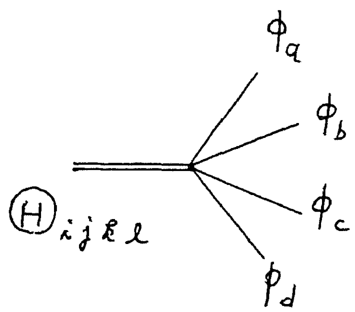


$$-\frac{1}{3}\Big\{(ijk\,;\,abc)P_1^{\mu}+(ijk\,;\,bca)P_2^{\mu} + (ijk\,;\,cab)P_3^{\mu}\Big\} , \qquad \text{(C.7)}$$

where

$$(ijk\,;abc) = 2\delta_{ak}(\delta_{bi}\delta_{cj}+\delta_{ci}\delta_{bj}) - \delta_{ai}(\delta_{bj}\delta_{ck}+\delta_{cj}\delta_{bk}) - \delta_{aj}(\delta_{bi}\delta_{ck}+\delta_{ci}\delta_{bk}), \tag{C.8}$$

$$\Omega^{\mu}{}_{ijk}(ijk\,;abc) = 6\,\Omega^{\mu}{}_{bca}, \tag{C.9}$$



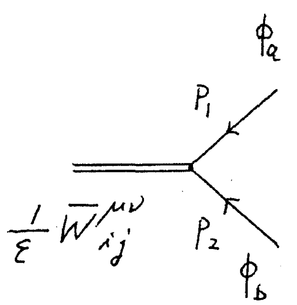
$$i\{ijkl\,;abcd\}, \tag{C.10}$$

where

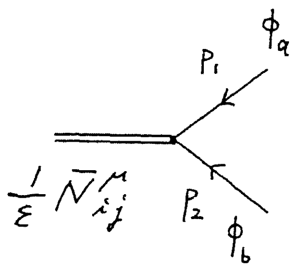
$$\{ijkl\,;abcd\} = \delta_{ia}\delta_{jb}\delta_{kc}\delta_{ld} + \mathrm{perm}(a,b,c,d). \tag{C.11}$$

Feynman rules for the interaction Lagrangian:

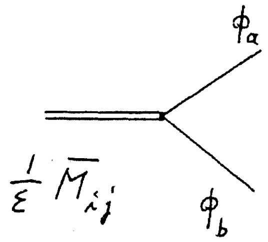
$$\mathcal{L}_{int} = \frac{1}{\varepsilon}\left\{\frac{1}{2}\overline{W}^{\mu\nu}_{ij}\,\partial_\mu\phi_i\,\partial_\nu\phi_j + \overline{N}^{\mu}_{ij}\,\phi_i\,\partial_\mu\phi_j + \frac{1}{2}\overline{M}_{ij}\,\phi_i\phi_j\right\}. \tag{C.12}$$



$$\frac{1}{4i\varepsilon}(\delta_{ai}\delta_{bj}+\delta_{aj}\delta_{bi})(P_{1\mu}P_{2\nu}+P_{1\nu}P_{2\mu}), \tag{C.13}$$



$$\frac{1}{2\varepsilon}(\delta_{ia}\delta_{jb}-\delta_{ib}\delta_{ja})(p_1-p_2)^{\mu} \tag{C.14}$$



$$\frac{1}{2\varepsilon}\, i\,(\delta_{ia}\delta_{jb}+\delta_{ib}\delta_{ja}) . \tag{C.15}$$

Appendix D. All types of two-loop counter-terms:

[1] <u>Invariants for the case of $\mathcal{L}_3 = \frac{1}{3!}\mathcal{L}_{ijk}[\varphi^{cl}]\phi_i\phi_j\phi_k$</u>

$$\mathcal{L}_3 = \Xi^{ijk}_{\mu\nu}\phi_i\,\partial_\mu\phi_j\,\partial_\nu\phi_k + \Omega^{ijk}_{\mu}\phi_i\phi_j\,\partial_\mu\phi_k + \Lambda^{ijk}\phi_i\phi_j\phi_k . \tag{D.1}$$

Symmetry:

$$\Lambda^{ijk} \quad \text{totally symmetric,}$$

$$\Omega^{ijk}_{\mu} = \Omega^{jik}_{\mu} ,$$

$$\Xi^{ijk}_{\mu\nu} = \Xi^{ikj}_{\nu\mu} ,$$

$$\Omega^{ijk}_{\mu} + \Omega^{jki}_{\mu} + \Omega^{kij}_{\mu} = 0 ,$$

$$\Xi^{ijk}_{\mu\nu} + \Xi^{jik}_{\mu\nu} = \Xi^{ikj}_{\mu\nu} + \Xi^{kij}_{\mu\nu} = \Xi^{kji}_{\mu\nu} + \Xi^{jki}_{\mu\nu} . \tag{D.2}$$

Transformation properties:

$$\Delta \Xi_{\mu\nu}^{ijk} = -\Xi_{\mu\nu}^{ljk} S_l{}^i - \Xi_{\mu\nu}^{ilk} S_l{}^j - \Xi_{\mu\nu}^{ijl} S_l{}^k ,$$

$$\Delta \Omega_\mu^{ijk} = -\Omega_\mu^{ljk} S_l{}^i - \Omega_\mu^{ilk} S_l{}^j - \Omega_\mu^{ijl} S_l{}^k$$
$$-\frac{1}{3}\{ (2\Xi_{\mu\nu}^{ikl} - \Xi_{\mu\nu}^{kil}) \partial_\nu S_l{}^j + (2\Xi_{\mu\nu}^{jkl} - \Xi_{\mu\nu}^{kjl}) \partial_\nu S_l{}^i$$
$$-(\Xi_{\mu\nu}^{jil} + \Xi_{\mu\nu}^{ijl}) \partial_\nu S_l{}^k \} ,$$

$$\Delta \Lambda^{ijk} = -\Lambda^{ljk} S_l{}^i - \Lambda^{ilk} S_l{}^j - \Lambda^{ijl} S_l{}^k$$
$$-\frac{1}{3}(\Omega_\mu^{ijl} \partial_\mu S_l{}^k + \Omega_\mu^{jkl} \partial_\mu S_l{}^i + \Omega_\mu^{kil} \partial_\mu S_l{}^j)$$
$$+\frac{1}{9}\partial_\mu \{ (\Xi_{\mu\nu}^{jkl} + \Xi_{\mu\nu}^{kjl}) \partial_\nu S_l{}^i + (\Xi_{\mu\nu}^{ikl} + \Xi_{\mu\nu}^{kil}) \partial_\nu S_l{}^j$$
$$+(\Xi_{\mu\nu}^{jil} + \Xi_{\mu\nu}^{ijl}) \partial_\nu S_l{}^k \} .$$

(D.3)

Covariant quantities:

$$\tilde{\Omega}_\mu^{ijk} = \Omega_\mu^{ijk} + \frac{1}{3}\{ (2\Xi_{\mu\nu}^{jkl} - \Xi_{\mu\nu}^{kjl}) A_{\nu,l}{}^i$$
$$+(2\Xi_{\mu\nu}^{ikl} - \Xi_{\mu\nu}^{kil}) A_{\nu,l}{}^j - (\Xi_{\mu\nu}^{jil} + \Xi_{\mu\nu}^{ijl}) A_{\nu,l}{}^k \} ,$$

$$\tilde{\Lambda}^{ijk} = \Lambda^{ijk} + \{ \frac{1}{3}(\tilde{\Omega}_\mu^{jkl} - \frac{1}{3} D_\nu (\Xi_{\nu\mu}^{jkl} + \Xi_{\nu\mu}^{kjl})) \cdot A_{\mu,l}{}^i$$
$$-\frac{1}{18}(\Xi_{\mu\nu}^{jkl} + \Xi_{\mu\nu}^{kjl})(\partial_\mu A_\nu + \partial_\nu A_\mu - A_\mu A_\nu - A_\nu A_\mu)_l{}^i$$

$$
-\frac{1}{9}\left(\Xi_{\mu\nu}^{klm}-\Xi_{\mu\nu}^{lkm}-\Xi_{\mu\nu}^{mlk}\right)A_{\mu,l}{}^{i}A_{\nu,m}{}^{j}
$$
$$
+\text{cyclic }(i,j,k)\qquad\Big\}\,. \tag{D.4}
$$

Covariant derivatives:

$$
D_\lambda \Xi_{\mu\nu}^{ijk}=\partial_\lambda \Xi_{\mu\nu}^{ijk}+\Xi_{\mu\nu}^{ljk}A_{\lambda,l}{}^{i}+\Xi_{\mu\nu}^{ilk}A_{\lambda,l}{}^{j}+\Xi_{\mu\nu}^{ijl}A_{\lambda,l}{}^{k}\,. \tag{D.5}
$$

Invariants:

$$
\begin{array}{lllll}
D^2\hat{\Lambda}^2\,, & \hat{\Lambda}^2X\,, & DX\tilde{\Lambda}\tilde{\Omega}\,, & DY\tilde{\Lambda}\tilde{\Omega}\,, & D^4\tilde{\Omega}^2\,,\\
D^2\tilde{\Omega}^2X\,, & D^2\tilde{\Omega}^2Y\,, & \tilde{\Omega}^2X^2\,, & \tilde{\Omega}^2Y^2\,, & \tilde{\Omega}^2XY\,,\\
D^4\tilde{\Lambda}\Xi\,, & D^2\tilde{\Lambda}\Xi X\,, & D^2\tilde{\Lambda}\Xi Y\,, & \tilde{\Lambda}\Xi X^2\,, & \tilde{\Lambda}\Xi XY\,,\\
\tilde{\Lambda}\Xi Y^2\,, & D^5\tilde{\Omega}\Xi\,, & D^3\tilde{\Omega}\Xi X\,, & D^3\tilde{\Omega}\Xi Y\,, & D\tilde{\Omega}\Xi X^2\,,\\
D\tilde{\Omega}\Xi XY\,, & D\tilde{\Omega}\Xi Y^2\,, & D^6\Xi^2\,, & D^4\Xi^2X\,, & D^4\Xi^2Y\,,\\
D^2\Xi^2X^2\,, & D^2\Xi^2XY\,, & D^2\Xi^2Y^2\,, & \Xi^2X^3\,, & \Xi^2X^2Y\,,\\
\Xi^2XY^2\,, & \Xi^2Y^3\,. & & &
\end{array}
$$

(D.6)

[2] Invariants for the case of $\mathcal{L}_4 = \frac{1}{4!}\mathcal{L}'_{ijk\ell}[\varphi^{\alpha}]\phi_i\phi_j\phi_k\phi_\ell$

$$\mathcal{L}_4 = \Gamma^{ijk\ell}_{\mu\nu}\phi_i\phi_j\partial_\mu\phi_k\partial_\nu\phi_\ell + \Sigma^{ijk\ell}_{\mu}\phi_i\phi_j\phi_k\partial_\mu\phi_\ell + \Theta^{ijk\ell}\phi_i\phi_j\phi_k\phi_\ell \tag{D.7}$$

Symmetry:

$\Theta^{ijk\ell}$ totally symmetric,

$\Sigma^{ijk\ell}_{\mu}$ totally symmetric with respect to i, j and k,

$$\Gamma^{ijk\ell}_{\mu\nu} = \Gamma^{jik\ell}_{\mu\nu} \quad ,$$

$$\Gamma^{ijk\ell}_{\mu\nu} = \Gamma^{ij\ell k}_{\nu\mu} \quad ,$$

$$\Sigma^{ijk\ell}_{\mu} + \Sigma^{\ell jki}_{\mu} + \Sigma^{i\ell kj}_{\mu} + \Sigma^{ij\ell k}_{\mu} = 0 \quad ,$$

$$\Gamma^{ikj\ell}_{\mu\nu} + \Gamma^{i\ell jk}_{\mu\nu} + \Gamma^{k\ell ji}_{\mu\nu} = \Gamma^{ik\ell j}_{\mu\nu} + \Gamma^{i\ell kj}_{\mu\nu} + \Gamma^{k\ell ij}_{\mu\nu} \quad ,$$

$$\Gamma^{ijk\ell}_{\mu\nu} + \Gamma^{ikj\ell}_{\mu\nu} + \Gamma^{\ell jki}_{\mu\nu} + \Gamma^{\ell kji}_{\mu\nu} = \Gamma^{ji\ell k}_{\mu\nu} + \Gamma^{j\ell ik}_{\mu\nu} + \Gamma^{ki\ell j}_{\mu\nu} + \Gamma^{k\ell ij}_{\mu\nu} \quad . \tag{D.8}$$

Transformation properties:

$$\Delta\Gamma^{ijk\ell}_{\mu\nu} = -\Gamma^{ajk\ell}_{\mu\nu}\delta_a{}^i - \Gamma^{iak\ell}_{\mu\nu}\delta_a{}^j - \Gamma^{ija\ell}_{\mu\nu}\delta_a{}^k - \Gamma^{ijka}_{\mu\nu}\delta_a{}^\ell \quad ,$$

$$\Delta\Sigma^{ijk\ell}_{\mu} = -\Sigma^{ajk\ell}_{\mu}\delta_a{}^i - \Sigma^{iak\ell}_{\mu}\delta_a{}^j - \Sigma^{ija\ell}_{\mu}\delta_a{}^k - \Sigma^{ijka}_{\mu}\delta_a{}^\ell$$

$$+\frac{1}{6}\left(-3\Gamma_{\mu\nu}^{jkla}+\Gamma_{\mu\nu}^{klja}+\Gamma_{\mu\nu}^{jlka}\right)\partial_\nu S_a{}^{i}+\frac{1}{6}\left(-3\Gamma_{\mu\nu}^{kila}+\Gamma_{\mu\nu}^{klia}+\Gamma_{\mu\nu}^{lika}\right)\partial_\nu S_a{}^{j}$$

$$+\frac{1}{6}\left(-3\Gamma_{\mu\nu}^{ijla}+\Gamma_{\mu\nu}^{ljia}+\Gamma_{\mu\nu}^{lija}\right)\partial_\nu S_a{}^{k}+\frac{1}{6}\left(\Gamma_{\mu\nu}^{jkia}+\Gamma_{\mu\nu}^{ikja}+\Gamma_{\mu\nu}^{ijka}\right)\partial_\nu S_a{}^{l},$$

$$\Delta H^{ijkl}=-H^{ajkl}S_a{}^{i}-\frac{1}{4}\Sigma_\mu^{ijka}\,\partial_\mu S_a{}^{l}$$

$$+\frac{1}{24}\partial_\mu\left\{\left(\Gamma_{\mu\nu}^{jkla}+\Gamma_{\mu\nu}^{klja}+\Gamma_{\mu\nu}^{jlka}\right)\partial_\nu S_a{}^{i}\right\}$$

$$+\text{cyclic}\,(i,j,k,l)\quad .$$

(D.9)

Covariant quantities:

$$\tilde{\Sigma}_\mu^{ijkl}=\Sigma_\mu^{ijkl}-\frac{1}{6}\left(-3\Gamma_{\mu\nu}^{jkla}+\Gamma_{\mu\nu}^{klja}+\Gamma_{\mu\nu}^{jlka}\right)A_{\nu,a}{}^{i}$$

$$-\frac{1}{6}\left(-3\Gamma_{\mu\nu}^{kila}+\Gamma_{\mu\nu}^{klia}+\Gamma_{\mu\nu}^{lika}\right)A_{\nu,a}{}^{j}$$

$$-\frac{1}{6}\left(-3\Gamma_{\mu\nu}^{ijla}+\Gamma_{\mu\nu}^{ljia}+\Gamma_{\mu\nu}^{lija}\right)A_{\nu,a}{}^{k}$$

$$-\frac{1}{6}\left(\Gamma_{\mu\nu}^{jkib}+\Gamma_{\mu\nu}^{ikjb}+\Gamma_{\mu\nu}^{ijka}\right)A_{\nu,a}{}^{l}\quad,$$

$$\tilde{H}^{ijkl}=H^{ijkl}+\left\{\frac{1}{4}\tilde{\Sigma}_\mu^{ijka}A_{\mu,a}{}^{l}-\frac{1}{24}D_\mu\sigma_{\mu\nu}^{jkl,a}A_{\nu,a}{}^{i}\right.$$

$$+\frac{1}{48}\sigma_{\mu\nu}^{jkl,a}\left(-\partial_\mu A_\nu-\partial_\nu A_\mu+A_\mu A_\nu+A_\nu A_\mu\right)_a{}^{i}$$

$$+\frac{1}{48}\left(\gamma_{\mu\nu}^{ij,ab}A_{\nu,b}{}^{k}+\gamma_{\mu\nu}^{jk,ab}A_{\nu,b}{}^{i}+\gamma_{\mu\nu}^{ki,ab}A_{\nu,b}{}^{j}\right)A_{\mu,a}{}^{l}$$

$$\left.+\text{cyclic}\,(i,j,k,l)\right\}\quad,$$

(D.10)

where

$$\gamma_{\mu\nu}^{ij,ab} = -2\Gamma_{\mu\nu}^{ijab} + \Gamma_{\mu\nu}^{aijb} + \Gamma_{\mu\nu}^{ajib} + \Gamma_{\mu\nu}^{ibaj} + \Gamma_{\mu\nu}^{jbai} ,$$

$$\sigma_{\mu\nu}^{jk\ell,a} = \Gamma_{\mu\nu}^{jk\ell a} + \Gamma_{\mu\nu}^{k\ell ja} + \Gamma_{\mu\nu}^{j\ell ka} . \tag{D.11}$$

Covariant derivatives:

$$D_\lambda \Gamma_{\mu\nu}^{ijk\ell} = \partial_\lambda \Gamma_{\mu\nu}^{ijk\ell} + \Gamma_{\mu\nu}^{ajk\ell} A_{\lambda,a}{}^{i} + \Gamma_{\mu\nu}^{iak\ell} A_{\lambda,a}{}^{j} + \Gamma_{\mu\nu}^{ija\ell} A_{\lambda,a}{}^{k} + \Gamma_{\mu\nu}^{ijka} A_{\lambda,a}{}^{\ell} . \tag{D.12}$$

Invariants:

$$X^2\tilde{\Theta},\quad Y^2\tilde{\Theta},\quad DX^2\tilde{\Sigma},\quad DXY\tilde{\Sigma},\quad DY^2\tilde{\Sigma},$$
$$D^2X^2\Gamma,\quad D^2XY\Gamma,\quad D^2Y^2\Gamma,\quad X^3\Gamma,\quad X^2Y\Gamma,$$
$$XY^2\Gamma,\quad Y^3\Gamma . \tag{D.13}$$

Appendix E. Proof of (6.28)

Here we show the derivation of (6.28). First we express the coefficient functionals N^μ, M, Ω^μ, Λ and Θ introduced in (6.3) as functional derivatives of $\mathcal{L}\,[\,\varphi^{c\alpha}\,]$:

$$-\delta_{\mu\nu}\delta_{ij} = \frac{\delta^2 \mathcal{L}}{\delta(\partial_\mu \varphi_i^{cl})\,\delta(\partial_\nu \varphi_j^{cl})} \quad ,$$

$$N^{\mu}_{ij} = \frac{1}{2}\left(\frac{\delta^2 \mathcal{L}}{\delta\varphi_i^{cl}\,\delta(\partial_\mu \varphi_j^{cl})} - \frac{\delta^2 \mathcal{L}}{\delta\varphi_j^{cl}\,\delta(\partial_\mu \varphi_i^{cl})}\right) ,$$

$$M_{ij} = \frac{\delta^2 \mathcal{L}}{\delta\varphi_i^{cl}\,\delta\varphi_j^{cl}} - \frac{1}{2}\partial_\mu\left(\frac{\delta^2 \mathcal{L}}{\delta\varphi_i^{cl}\,\delta(\partial_\mu \varphi_j^{cl})} + \frac{\delta^2 \mathcal{L}}{\delta\varphi_j^{cl}\,\delta(\partial_\mu \varphi_i^{cl})}\right) ,$$

$$\Omega^{\mu}_{ijk} = -\frac{1}{6}\left\{\frac{\delta^3 \mathcal{L}}{\delta\varphi_k^{cl}\,\delta\varphi_i^{cl}\,\delta(\partial_\mu \varphi_j^{cl})} + \frac{\delta^3 \mathcal{L}}{\delta\varphi_k^{cl}\,\delta\varphi_j^{cl}\,\delta(\partial_\mu \varphi_i^{cl})} - 2\frac{\delta^3 \mathcal{L}}{\delta\varphi_i^{cl}\,\delta\varphi_j^{cl}\,\delta(\partial_\mu \varphi_k^{cl})}\right\} ,$$

$$\Lambda_{ijk} = \frac{1}{6}\frac{\delta^3 \mathcal{L}}{\delta\varphi_i^{cl}\,\delta\varphi_j^{cl}\,\delta\varphi_k^{cl}} - \frac{1}{18}\partial_\mu\left\{\frac{\delta^3 \mathcal{L}}{\delta\varphi_i^{cl}\,\delta\varphi_j^{cl}\,\delta(\partial_\mu \varphi_k^{cl})} + \frac{\delta^3 \mathcal{L}}{\delta\varphi_i^{cl}\,\delta\varphi_k^{cl}\,\delta(\partial_\mu \varphi_j^{cl})} + \frac{\delta^3 \mathcal{L}}{\delta\varphi_k^{cl}\,\delta\varphi_j^{cl}\,\delta(\partial_\mu \varphi_i^{cl})}\right\} ,$$

$$\Theta_{ijkl} = \frac{1}{4!}\frac{\delta^4 \mathcal{L}}{\delta\varphi_i^{cl}\,\delta\varphi_j^{cl}\,\delta\varphi_k^{cl}\,\delta\varphi_l^{cl}} \quad .$$

(E.1)

Making use of relations (E.1), we find

$$\frac{\delta N^{\mu}_{jk}}{\delta \varphi_i^{ce}} = \Omega^{\mu}_{ijk} - \Omega^{\mu}_{ikj} \quad , \quad \frac{\delta M_{ij}}{\delta \varphi_k^{ce}} = 6\Lambda_{ijk} + \partial_\mu \Omega^{\mu}_{ijk} \, ,$$

$$\frac{\delta M_{ij}}{\delta(\partial_\mu \varphi_k^{ce})} = 3\Omega^{\mu}_{ijk} \, , \quad \frac{\delta(\partial_\nu N^{\mu}_{jk})}{\delta \varphi_i^{ce}} = \partial_\nu(\Omega^{\mu}_{ijk} - \Omega^{\mu}_{ikj}) \, ,$$

$$\frac{\delta(\partial_\nu N^{\mu}_{jk})}{\delta(\partial_\lambda \varphi_i^{ce})} = \delta_{\nu\lambda}(\Omega^{\mu}_{ijk} - \Omega^{\mu}_{ikj}) \, ,$$

$$\frac{\delta^2 M_{ij}}{\delta\varphi_l^{ce}\delta\varphi_k^{ce}} = 24\, \Theta_{ijkl} \, . \tag{E.2}$$

Substituting the one-loop counter-term formula (6.9) into (6.12)

$$\frac{1}{2!} \Delta\mathcal{L}^{\text{one-loop}}{}_{,ij}[\varphi^{ce}]\phi_i\phi_j$$

$$= -\frac{1}{2^5\pi^2\varepsilon}\Big[\!\Big[\phi_i\phi_j \Big[2\{12\Theta_{abij} - (\Omega^{\mu}_{iac} - \Omega^{\mu}_{ica})(\Omega^{\mu}_{jcb} - \Omega^{\mu}_{jbc})\} X_{ab}$$

$$+ (6\tilde{\Lambda}_{abi} + D_\mu\Omega^{\mu}_{abi})(6\tilde{\Lambda}_{abj} + D_\nu\Omega^{\nu}_{abj})$$

$$-\frac{2}{3}(\Omega^{\alpha}_{iac} - \Omega^{\alpha}_{ica})(\Omega^{\beta}_{jcb} - \Omega^{\beta}_{jbc}) Y^{\alpha\beta}_{ab}$$

$$-\frac{2}{3}\{D_\alpha(\Omega^{\beta}_{iab} - \Omega^{\beta}_{iba}) - D_\beta(\Omega^{\alpha}_{iab} - \Omega^{\alpha}_{iba})\} D_\alpha\Omega^{\beta}_{jab} \Big]$$

$$+ \phi_i D_\mu\phi_j \Big[6(6\tilde{\Lambda}_{abi} + D_\nu\Omega^{\nu}_{abi})\Omega^{\mu}_{abj} - \frac{4}{3}D_\alpha(\Omega^{\beta}_{iab} - \Omega^{\beta}_{iba})$$

$$\times(\delta_{\mu\alpha}\Omega^{\beta}_{jab} - \delta_{\mu\beta}\Omega^{\alpha}_{jab}) \Big]$$

$$+D_\mu\phi_i D_\nu\phi_j \Big[9\Omega^\mu_{abi}\Omega^\nu_{abj} - \frac{2}{3}\{ \delta_{\mu\nu}(\Omega^\beta_{iab} - \Omega^\beta_{iba}) - \delta_{\mu\beta}(\Omega^\nu_{iab} - \Omega^\nu_{iba})\}\Omega^\beta_{jab} \Big] \Big]\Big] , \tag{E.3}$$

where

$$D_\mu\phi_i = \partial_\mu\phi_i + N^\mu_{ij}\phi_j \quad . \tag{E.4}$$

Calculating one-loop subdiagrams such as Fig.7, we can check that the vertices in (E.3) cancel the divergent part of those diagrams. Adding total space-time derivatives we can write the equation (E.3) in the form (6.28).

Appendix F. Renormalization of the ϕ^4-theory with N components

In this appendix we apply the one-loop and two-loop counter-term formulas to the ϕ^4-theory with N components which possesses a global O(N)-symmetry. The Lagrangian is of the form:

$$\mathcal{L}[\varphi] = \sum_{i=1}^{N}\left[-\frac{1}{2}(\partial_\mu\varphi_i)^2 - \frac{1}{2}m^2\varphi_i^2\right] - \frac{\lambda}{8}\Big(\sum_i \varphi_i^2\Big)^2 . \tag{F.1}$$

The coefficient functionals in (6.3) are given by

$$N^\mu_{ij} = 0 , \quad M_{ij} = -m^2\delta_{ij} - \lambda\varphi_i\varphi_j - \frac{\lambda}{2}\varphi_k^2\delta_{ij} ,$$

$$\Omega^\mu_{ijk} = 0 , \quad \Lambda_{ijk} = -\frac{\lambda}{6}(\varphi_i\delta_{jk} + \varphi_j\delta_{ik} + \varphi_k\delta_{ij}) ,$$

$$\Theta_{ijkl} = -\frac{\lambda}{24}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \tag{F.2}$$

The covariant quantities (6.6) are

$$X_{ij} = M_{ij} \ , \qquad Y^{\mu\nu}_{ij} = 0 \ ,$$

$$\tilde{\Lambda}_{ijk} = \Lambda_{ijk} \ . \tag{F.3}$$

From the one-loop counter-term formula (6.9),

$$\Delta\mathcal{L}^{one-loop} = -\frac{1}{8\pi^2\varepsilon}\left\{\lambda\left(1+\frac{N}{2}\right)\frac{m^2}{2}\varphi^2 + \left(4+\frac{N}{2}\right)\frac{\lambda^2}{8}(\varphi^2)^2\right\} . \tag{F.4}$$

This result was previously obtained by G. 'tHooft [22]. From the two-loop counter-terms (6.17), (6.18) and (6.27), we find

$$\Delta\mathcal{L}^{two-loop} = -\frac{1}{2^{11}\pi^4}\Big[\frac{1}{\varepsilon}(N+2)\lambda^2\varphi_i\partial^2\varphi_i$$
$$+2\left\{\frac{2}{\varepsilon^2}(N+2)(N+5)-\frac{3}{\varepsilon}(N+2)\right\}\lambda^2 m^2\varphi^2$$
$$+\left\{\frac{1}{\varepsilon^2}(N+8)^2-\frac{1}{\varepsilon}(5N+22)\right\}\lambda^3(\varphi^2)^2\Big] . \tag{F.5}$$

These results enable us to obtain various renormalization constants up to two-loop order:

$$Z_3 = 1 - \frac{1}{2^{10}\pi^4\varepsilon}(N+2)\lambda^2 + O(\lambda^3) \ ,$$

$$m_B^2 = m^2\Big[1 + \frac{1}{2^4\pi^2\varepsilon}\lambda(2+N) + \frac{\lambda^2}{2^8\pi^4\varepsilon^2}\Big\{(N+2)(N+5)$$
$$-\frac{5}{4}\varepsilon(N+2)\Big\}\Big] \ ,$$

$$\lambda_B = \lambda \left[1 + \frac{1}{2^4 \pi^2 \varepsilon} \lambda (8+N) + \frac{\lambda^2}{2^8 \pi^4 \varepsilon^2} \left\{ (N+8)^2 - \frac{3}{2} \varepsilon (3N+14) \right\} \right], \tag{F.6}$$

where m_B and λ_B are bare parameters. The renormalization-group function $\beta(\lambda)$ is given as

$$\beta(\lambda) = \frac{\lambda^2}{2^4 \pi^2} (8+N) - \frac{3}{2^8 \pi^4} \lambda^3 (3N+14) , \tag{F.7}$$

The results (F.6) and (F.7) are in agreement with those given in [37, 38, 39].

References

[1] S. Tomonaga, Progr. Theor. Phys. 1, 27 (1946).

[2] J. Schwinger, Phys. Rev. 74, 1439 (1948).

[3] R. P. Feynman, Phys. Rev. 76, 769 (1949).

[4] F. J. Dyson, Phys. Rev. 75, 1736 (1949).

[5] P. A. M. Dirac, Can. J. Math. 2, 129 (1950).

[6] C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1954).

[7] R. Utiyama, Phys. Rev. 101, 1597 (1956).

[8] R. P. Feynman, Acta Phys. Polonica 24, 697 (1963).

[9] B. S. DeWitt, Phys. Rev. 162, 1195, 1239 (1967).

[10] L. D. Faddeev and V. N. Popov, Phys. Letters 25B, 29 (1967).

[11] S. Mandelstam, Phys. Rev. 175, 1580, 1604 (1968).

[12] E. S. Fradkin and I. V. Tyntin, Phys. Rev. D2, 2841 (1970).

[13] S. Weinberg, Phys. Rev. Letters 19, 1264 (1967).

[14] A. Salam, Elementary Particle Theory, ed. N. Svartholm (Almquist and Forlag, Stockholm, 1968) p.367.

[15] G. 'tHooft, Nucl. Phys. B33, 173 (1971).

[16] G. 'tHooft, Nucl. Phys. B35, 167 (1971).

[17] H. D. Politzer, Phys. Rev. Letters 30, 1346 (1973).

[18] G. 'tHooft (unpublished).

[19] D. Gross and F. Wilczek, Phys. Rev. Letters 30, 1343 (1973).

[20] S. Sakata, H. Umezawa, and S. Kamefuchi, Progr. Theor. Phys. 7, 327 (1952).

[21] G. 'tHooft and M. Veltman, Ann. Inst. Henri Poincaré 20, 69 (1974).

[22] G. 'tHooft, Nucl. Phys. B62, 444 (1973).

[23] P. van Nieuwenhuizen and C. C. Wu, J. Math. Phys. 18, 182 (1977).

[24] M. T. Grisaru, Phys. Letters B66, 75 (1977).

[25] S. Deser, J. Kay, and K. Stelle, Phys. Rev. Letters 38, 527 (1977).

[26] J. Honerkamp, Nucl. Phys. B48, 269 (1972).

[27] R. Kallosh, Nucl. Phys. B78, 293 (1974).

[28] H. Kluberg-Stern and J. B. Zuber, Phys. Rev. D12, 482 (1975).

[29] H. Kluberg-Stern and J. B. Zuber, Phys. Rev. D12, 467 (1975).

[30] I. Ya. Aref'eva, A. A. Slavnov, and L. D. Faddeev, Theor. Mat. Fiz, 21, 311 (1974).

[31] B. S. DeWitt, Phys. Rev. Letters 12, 742 (1964); Dynamical theory of groups and fields, (Gordon and Breach, 1965).

[32] S. Tamura, Lett. Nuovo Cimento 13, 639 (1975).

[33] G. 'tHooft, Nucl. Phys. B61, 455 (1973).

[34] D. R. T. Jones, Nucl. Phys. B75, 531 (1974).

[35] A. A. Vladimirov and O. V. Tarasov, Sov. J. Nucl. Phys. 25, 585 (1977).

[36] W. E. Caswell, Phys. Rev. Letters 33, 244 (1974).

[37] J. C. Collins, Phys. Rev. D10, 1213 (1974).

[38] A. A. Vladimirov, Dubna preprint E2-11096 (1977).

[39] F. M. Dittes, Yu. A. Kubyshin, and O. V. Tarasov, Dubna preprint E2-11100 (1977).

[40] S. Ichinose and M. Omote, Nucl. Phys. B142, 477 (1978).

[41] G. 'tHooft, Nucl. Phys. B44, 189 (1972).