Blow-up Solutions of One Dimensional Quasilinear Degenerate Parabolic Equations

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THESIS

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Preface

In this paper we shall consider the one dimensional boundary value problem for euation

$$(1) \qquad \partial_t \beta(u) - u_{xx} + g(u)_x = f(u)$$

where $\beta(v)$, $g(v)$, $f(v)$ with $v \geq 0$ and the intial data $\varphi(x)$ are nonnegative continuous functions.

Equation (1) describes the combution process with convection in a stationary medium in which the thermal conductivity $\beta'(u)^{-1}$ and convection $g(u)$ are depending in a nonlinear way on the temperature $\beta(u) = \beta(u(x,t))$.

We assume the conditions which guarantee the uniqueness and local existence of the nonnegative continuous weak solution to above problem. Let T be the maximum existence time of the solution. If $u(x,t)$ does not exist globally in time, then $T < \infty$ and

$$(2) \qquad \limsup_{t \uparrow T} \sup_{x \in \mathbb{R}} u(x,t) = \infty.$$

In this case we say that u is a blow-up solution and T is a blow-up time.

The main purpose of the present paper is the study of blow-up solutions. Especially we are interested in the shape of the blow-up set which locates the "hot-spots" at the blow-up time. In addition, since our quasilinear equation (1) has a property of the finite propagation of an interface, there are some interesting subjects such as asymptotic behavior of the interface near the blow-up time.

Firstly we consider the Neumann and Dirichlet problem for (1) in a bounded domain. Furthermore we consider only the case $g(\xi) \equiv 0$ for $\xi \geq 0$. Especially in the semilinear case $\beta(\xi) \equiv \xi$, Chen-Matano [1] shows that the blow-up set of a solution is finite, if

(I) $f(\xi)$ grows more rappidly than ξ.

In Chapter I, we extend this result to more general $\beta(\xi)$. In our result we do not use the analyticity condition on the initial data $\varphi(x)$ and $f(\xi)$ which is required by Chen-Matano in the case of the Dirichlet problem and in case $f(0) > 0$. But we have to add some technical condition on $\varphi(x)$.

Next, we consider the Cauchy problem. We assume that the initial data $\varphi(x)$ has a compact support $[-a_1, a_1]$. If we add some assumptions on $\beta(\xi)$ and $f(\xi)$, we can obtain a finite propagation of the interface of the blow-up solution $u(x,t)$ in $t < T$. In the Cauchy problem, we do not consider only the condition (I) but also the following condition (II):

(II) $f(\xi)$ grows more slowly than ξ.

In Chapter I, we also treat this problem for the case $g(\xi) \equiv 0$. In this case, the shape of the blow-up set will be very different to each other under the conditions (I) and (II). Roughly speaking, under (I) we show that the blow-up set S is contained in $[-a_1, a_1]$ and the interface stays bounded as $t \uparrow T$. Furthermore, if the technical condition is added on $\varphi(x)$, then S becomes a finite set. On the other hand, under (II) we show that $S = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ and consequently the interface propagates to the infinity as $t \uparrow T$.

In Chapter **II**, we consider this problem for (1) with $g(\xi) \fallingdotseq 0$. Further we treat only the case (I) such as $f(t)$ grows more rappidly than ξ and $g(\xi)$. Then we have to assume that $\varphi(x)$ has the unique locally maximum point in $\mathbb{R}$. Because, the reflectional symmetry for equation which plays important role in the proof for the case $g(\xi) \equiv 0$ does not hold. Especially, in the semilinear case $\beta(\xi) \equiv \xi$, Friedman-Lacey showed the existence of single point blow-up solutions of (1) for Dirichlet problem. We extend this result to the Cauchy problem of a degenetrate quasilinear equation whose initial data $\varphi(x)$ has a compact support. Moreover we can show also that the left side interface stays bounded as t tends to the blow-up time T. Namely, we get $S = \{\eta_0\}$ for some $-\infty < \eta_0 \leq \infty$.

Finally, we consider the problem whether $\eta_0 < \infty$ or $\eta_0 = \infty$ holds. For this problem, we obtain $\eta_0 < \infty$, if we add another conditions on f and φ such that $f(\xi)$ grows rappidly than ξ and $g(\xi)$, and we show that the right side interface stays bounded as t tends to the blow-up time T.

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CONTENTS

Chapter I

On Blow-up Sets and Asymptotic Behavior of Interfaces of One Dimensional Quasilinear Degenerate Parabolic Equations

1. Introduction

In this paper we study the initial boundary value problem for a quasilinear degenerate parabolic equation of the form

(1.1) $$b(u)_t = u_{xx} + f(u) \qquad \text{in} \quad x \in \Omega,\ t > 0,$$

with one dimensional open interval $\Omega \subset \mathbb{R}$ under the initial condition

(1.2) $$u(x,0) = u_0(x) \qquad \text{in} \quad x \in \Omega,$$

together with one of the following three types of boundary conditions:

(a) the Dirichlet boundary conditions with $\Omega = (0,L)$

(1.3a) $$u(0,t) = u(L,t) = 0 \qquad \text{in} \quad t > 0;$$

(b) the Neumann boundary conditions with $\Omega = (0,L)$

(1.3b) $$u_x(0,t) = u_x(L,t) = 0 \qquad \text{in} \quad t > 0;$$

(c) the Cauchy problem, namely,

(1.3c) $\Omega = \mathbb{R}$.

We assume the following conditions on $b(u)$, $f(u)$ and $u_0(x)$, respectively:

(A1) $b(u) \in C([0,\infty)) \cap C^{\infty}((0,\infty))$, $b^{-1}(v) \in C^1([0,\infty))$, $\lim_{u\to\infty} b(u) = \infty$, $b(0)=0$, $b(u) \geq 0$ for $u \geq 0$ and $b'(u)>0$ $b''(u) \leq 0$ for $u>0$ where $u=b^{-1}(v)$ is the inverse function of $v=b(u)$;

(A2) $f(u) \in C([0,\infty)) \cap C^{\infty}((0,\infty))$, $f(b^{-1}(v)) \in C^1([0,\infty))$ and $f(u)>0$ for $u>0$;

(A3) $u_0(x) \in B(\bar{\Omega})$ and $u_0(x) \geq 0$ for $x \in \bar{\Omega}$

where $B(K)$ is the set of all bounded continuous functions on a closed subset K of $\mathbb{R}$.

In the case of the Dirichlet problem, we assume in addition the following compatibility condition:

(A3a) $u_0(0) = u_0(L) = 0.$

Furthermore, we assume the following condition so that weak solutions of (1.1) may blow up in a finite time:

(A4) $$\int_1^{\infty} \frac{b'(\xi)}{f(\xi)}\, d\xi < \infty.$$

The equation (1.1) is called a porous media type equation and it represents the process of thermal diffusion in a non-linear continuous medium with the emission of thermal energy. And $u(x,t)$ represents a temperature and $f(u)$ represents a heat source.

Remark 1.1. Assumptions (A1), (A2) and (A4) are satisfied if, for example, the equation (1.1) is

$$(1.4) \qquad (u^{1/m})_t = u_{xx} + u^{p/m} \qquad (p > 1,\ m \geq 1).$$

Let us put $Q_\tau = \Omega\times(0,\tau)$. We know that if $\tau > 0$ is small enough, there exists a unique non-negative weak solution $u(x,t)$ of (1.1)(1.2)(1.3abc) (see, e.q., [2], [3], [12] and [13]). The definition of "weak" solutions will be given below in Section 2.

Now let us put

$$(1.5) \qquad T = \sup\{\ \tau \mid u(x,t) \text{ exists in } Q_\tau = \Omega\times(0,\tau)\}.$$

If $u(x,t)$ does not exist globally in time, namely,

$$0 < T < \infty,$$

then we call this solution a *blow-up weak solution* and we call T a *blow-up time*. The local existence theorem implies

$$(1.6) \qquad \limsup_{t\uparrow T}\ \{\ u(x,t) \mid x \in \Omega\ \} = \infty.$$

For studies on blow-up or non-blow-up of solutions, see references [6], [8], [9] and [10].

By a *blow-up point* of a blow-up weak solution $u(x,t)$ we mean a point $x \in \bar{\Omega}\cup\{\infty\}\cup\{-\infty\}$ such that there is a sequence $\{x_n, t_n\} \subset \bar{\Omega}\times(0,T)$ satisfying

$$t_n \uparrow T,\ x_n \to x \quad \text{and} \quad u(x_n, t_n) \to \infty \quad \text{as} \quad n \to \infty.$$

Also we call the set of all blow-up points a *blow-up set*.

From the definition, we see that the blow-up set is a closed subset in $\mathbb{R}\cup\{\infty\}\cup\{-\infty\}$. We shall study the shape of the blow-up set of each blow-up solution to (1.1)(1.2)(1.3abc) and furthermore, in the case of the Cauchy problem (1.1)(1.2)(1.3c) we shall study the asymptotic behavior of an interface of each blow-up solution of this problem near the blow-up time $t=T$.

In the case the of the Dirichlet or Neumann problem, we shall show that if $f(u)$ grows more rapidly than u (see (A5)), then the blow-up set of a blow-up solution is finite (Theorem 5.1). In the semilinear case $b(\xi)=\xi$, this result has already been obtained by Chen-Matano [4] and our methods are based on theirs. Note that we have to add some technical conditions on the initial data $u_0(x)$ (see (A6) and (A7)). On the other hand, we do not use the analyticity condition on $u_0(x)$ and $f(u)$ which is required in the case of the Dirichlet problem and in case $f(0) > 0$.

In the case of the Cauchy problem, we assume that the initial data $u_0(x)$ has a compact support $[0,L]$ (see (A8)). If we add some assumptions on $b(u)$ and $f(u)$ (see (A9)(A10)(A11)), we can obtain a finite propagation of the interface of the blow-up solution $u(x,t)$ in $t<T$. We are interested in the behavior of the interface near the blow-up time $t=T$ as well as the shape of the blow-up set.

We consider the following two cases:

(I) $f(u)$ grows more rapidly than u (see (A5)).

(II) $f(u)$ grows more slowly than u (see (A12)).

If (I) holds, we obtain that the blow-up set $S(u_0)$ is contained in $[0,L]$ and the interface stays bounded as $t \uparrow T$. Furthermore, if the technical condition (A7) is added on $u_0(x)$, then $S(u_0)$ becomes a finite set (Theorem 6.2 (i)). On the other hand, if (II) holds, we obtain that the blow-up set $S(u_0)$ is equal to $\mathbb{R}$ and consequently the interface propagates to the infinity as $t \uparrow T$ (Theorem 6.2 (ii)).

To prove these results, we can use the finite propagation property of the interface for (I) and (II). We can also use the non-blow-up result for the Dirichlet problem due to Imai-Mochizuki [8] for (II) only. In order to prove results for (I) we can also use the methods developed Friedman-McLeod [5] and Chen-Matano [4] for the semilinear problem. Note here that [8] studied the initial-boundary value problem (1.1)(1.2)(1.3ab) and asserted that the above two conditions (I) and (II) on $f(u)$ bring on the completely different blow-up situations.

Similar results to Theorem 6.3 were already obtained in [6] and [7] for the special equation (1.4) with $m>1$. However, the proof in [6,7] strongly depends on the equation and it seems difficult to apply it directly to our general quasilinear equation.

The paper is structured as follows: In Section 2, we state the definition of weak solutions of (1.1)(1.2)(1.3abc) and we state some lemmas used throughout this paper. In Section

3, we also state some lemmas which will be directly used in our blow-up problems. In Section 4, we shall show that the blow-up set becomes finite under some special conditions on the blow-up solution $u(x,t)$. In Section 5 we consider the Dirichlet or Neumann problem using the results in Section 4 and in Section 6 we consider the Cauchy problem.

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2. Definitions and Preliminaries

In this section, we assume (A1)-(A4).

Definition 2.1. Let Ω be an open interval in $\mathbb{R}$ and let $T > 0$. A function $u(x,t)$ defined in $Q_T = \Omega \times (0,T)$ is called a *weak solution* of (1.1), if:

1) $u(x,t) \in B(\bar{\Omega} \times [0,T'])$ for each $T' \in (0,T)$, and $u(x,t) \geq 0$ for $(x,t) \in Q_T$;

2) For any $T' \in (0,T)$ and any bounded open interval $\Omega' = (\alpha,\beta)$ in Ω, the identity

$$\int_{\Omega'} b(u(x,T'))\varphi(x,T')\,dx - \int_{\Omega'} b(u(x,0)\varphi(x,0)\,dx$$

$$- \int_0^{T'}\int_{\Omega'} b(u(x,t))\varphi_t dxdt = \int_0^{T'}\int_{\Omega'} u\varphi_{xx}(x,t)dxdt - \int_0^{T'} u\frac{\partial\varphi}{\partial x}\, dt\Big|_{x=\beta}^{x=\alpha} + \int_0^{T'}\int_{\Omega'} f(u)\varphi(x,t)\, dxdt$$

holds for any test function $\varphi(x,t) \in C^{2.1}(\bar{\Omega}' \times [0,T))$ satisfying $\varphi(\alpha,t) = \varphi(\beta,t) = 0$ for $t \in (0,T)$ and $\varphi(x,t) \geq 0$ for $t \in (0,T)$.

A function $u(x,t)$ defined in $Q_T = \Omega \times (0,T)$ is called a *weak super-(sub-)solution* of (1.1), if $u(x,t)$ satisfies 1) and 2) with equality replaced by $\geq$ ($\leq$).

A function $u(x,t)$ defined in $(0,L) \times (0,T)$ is called a *weak solution of the Dirichlet problem* (1.1)(1.2)(1.3a), if $u(x,t)$ is a weak solution of (1.1) in $(0,L) \times (0,T)$ and, if $u(0,t) = u(L,t) = 0$ in $t \in (0,T)$ and $u(x,0) = u_0(x)$ in $x \in (0,L)$.

A function $u(x,t)$ defined in $(0,L) \times (0,T)$ is called a *weak solution of the Neumann problem* (1.1)(1.2)(1.3b), if $u(x,t)$ is a weak solution of (1.1) in $(0,L) \times (0,T)$ and, if $u(x,t)$ satisfies 1) and 2) with $\varphi(x,t)$ replaced by $\varphi(x,t) \in C^{2,1}(\bar{\Omega}' \times [0,T))$ satisfying $\varphi(x,t) = 0$ for $(x,t) \in (\{\alpha,\beta\}\setminus\{0,L\}) \times [0,T)$ and $\varphi_x(x,t) = 0$ for $(x,t) \in \{\alpha,\beta\}\cap\{0,L\}\times(0,T)$, and if $u(x,0) = u_0(x)$ for $x \in \Omega$.

A function $u(x,t)$ defined in $(0,L) \times (0,T)$ is called a *weak super-(sub)solution of the Neumann problem* (1.1)(1.3b), if $u(x,t)$ satisfies 1) and 2) with equality replaced by $\geq$ ($\leq$) and with $\varphi(x,t)$ replaced by $\varphi(x,t) \in C^{2,1}(\bar{\Omega}'\times[0,T))$ satisfying $\varphi(x,t) = 0$ for $(x,t) \in (\{\alpha,\beta\}\setminus$

$\{0,L\}) \times [0,T)$ and $\varphi_x(x,t) = 0$ for $(x,t) \in \{\alpha,\beta\} \cap \{0,L\} \times (0,T)$.

A function $u(x,t)$ defined in $\mathbb{R} \times (0,T)$ is called a *weak solution of the Cauchy problem* (1.1)(1.2)(1.3c), if $u(x,t)$ is a weak solution of (1.1) in $\mathbb{R} \times (0,T)$ and, if $u(x,0) = u_0(x)$ for $x \in \mathbb{R}$.

Lemma 2.2. *(the comparison theorem). Assume* (A1)-(A4). *Let* Ω *be an open interval in* $\mathbb{R}$. *Then, the two following results hold:*

(i) *Suppose that* $u(x,t)$ *is a super-solution of* (1.1) *in* $Q_T = \Omega \times (0,T)$ *and* $v(x,t)$ *is a sub-solution of* (1.1) *in* $Q_T = \Omega \times (0,T)$. *Then, if* $u(x,t) \geq v(x,t)$ *for* $(x,t) \in \partial\Omega \times (0,T)$ *and if* $u(x,0) \geq v(x,0)$ *for* $x \in \Omega$, $u(x,t) \geq v(x,t)$ *for all* $(x,t) \in \Omega \times (0,T)$.

(ii) *Suppose that* $u(x,t)$ *is a super-solution of the Neumann problem* (1.1)(1.3b) *in* $Q_T = \Omega \times (0,T)$ *and* $v(x,t)$ *is a sub-solution of the Neumann problem* (1.1)(1.3b). *Then, If* $u(x,0) \geq v(x,0)$ *for* $x \in (0,L)$, $u(x,t) \geq v(x,t)$ *for all* $(x,t) \in \Omega \times (0,T)$.

Proof. See Aronson-Grandall-Peletier [2] and Bertsch-Kersner-Peletier [3]. □

Finally we show the following lemma:

Lemma 2.3. *Assume* (A1)-(A4). *Let* $u(x,t)$ *be a weak solution of* (1.1) *in* Q_T. *Then, if there exists a point* $(x,t) \in Q_T$ *such that* $u(x_0,t_0) > 0$, $u(x,t)$ *is a* C^∞- *function in a* *neighborhood of* (x_0, t_0) *and*

$$u(x_0, t) \geq \eta(t) > 0 \qquad t \in (t_0, T) \tag{2.1}$$

where $\eta(t)$ *is a solution of an ordinary differential equation* $\eta' = -\lambda\eta/b'(\eta)$ *for some positive constant* λ.

Proof. We only show (2.1). By the fact that $u(x_0, t_0) > 0$, there exist $\delta > 0$ and $\alpha > 0$ such that

$$u(x, t) \geq \alpha > 0 \quad \text{for} \quad |x - x_0| < \delta \quad \text{and} \quad |t - t_0| < \delta. \tag{2.2}$$

Set

$$v(x, t) = g(x)\eta(t) \tag{2.3}$$

where

$$g(x) = \sin((x - (x_0 - \delta))\pi/2\delta) \tag{2.4}$$

and $\eta(t)$ satisfies a differential equation

$$\eta' = -\lambda \frac{\eta}{b'(\eta)} \tag{2.5}$$

with $\eta(t_0) = \alpha$ and $\lambda = (\pi/2\delta)^2$.

We shall show

$$u(x, t) \geq v(x, t) \quad \text{for all} \quad (x, t) \in [x_0-\delta, x_0+\delta]\times[t_0, T). \tag{2.6}$$

First, η can be represented explicity by

$$\eta(t) = W^{-1}(W(\alpha)-\lambda(t-t_0))$$

where

$$(2.7) \qquad W(\eta) = \int_1^{\eta} \frac{b'(\xi)}{\xi}\, d\xi \qquad , \qquad \eta > 0$$

and W^{-1} is the inverse function of W. Noting that $W(\eta)$ is an increasing function and $W(\eta) \to -\infty$ as $\eta \downarrow 0$, we have that $\eta(t) > 0$ in $t \in (t_0,\infty)$ and $\eta(t) \downarrow 0$ as $t \to \infty$. Since $1/b'(\eta)$ is an increasing function, a simple calculation shows that

$$(2.8) \qquad b(v)_t \le v_{xx} + f(v) \quad \text{for} \quad (x,t) \in (x_0-\delta,x_0+\delta)\times(t_0,T).$$

Thus, we see that $v(x,t)$ is a sub-solution of (1.1) in $(x_0-\delta,x_0+\delta)\times(t_0,T)$ and that

$$(2.9) \qquad v(x_0\pm\delta,t) = 0 \le u(x_0\pm\delta,t) \quad \text{for} \quad t \in (t_0,T)$$

and

$$(2.10) \qquad v(x,t_0) = \alpha g(x) \le u(x,t_0) \quad \text{for} \quad x \in [x_0-\delta,x_0+\delta].$$

Applying the comparison theorem to $u(x,t)$ and $v(x,t)$, we obtain that $u(x,t) \ge v(x,t)$, $(x,t) \in [x_0-\delta,x_0+\delta]\times[t_0,T)$. This proof is complete. □

3. Fundamental Lemmas

In this section, we assume (A1)-(A4). We state some fundamental lemmas used after this section.

Lemma 3.1. *Assume* (A1)-(A4). *Let* $\Omega = (a,d)$ *be a bounded open interval and, let* $u(x,t)$ *be a weak solution of*

(1.1) *in* $Q_T = \Omega\times(0,T)$. *Then, the following two results hold:*

(i) *Suppose that* $u(a,t) > 0$ *for* $t \in [0,T)$, $u(a,0) > u(d,0) \geq 0$ *and* $\sigma_c u(x,t) \geq u(x,t)$ *for* $(x,t) \in [c,d]\times[0,T)$ *where* $\sigma_c u(x,t) = u(2c-x,t)$ *with* $c = (a+d)/2$. *Then, If there exists* $t_0 \in (0,T)$ *such that* $u(x,t) > 0$ *for* $x \in [a,c]$,

$$u_x(c,t_0) < 0. \tag{3.1}$$

(ii) *Suppose that* $u(d,t) > 0$ *for* $t \in [0,T)$, $u(d,0) > u(a,0) \geq 0$ *and* $\sigma_c u(x,t) \geq u(x,t)$ *for* $(x,t) \in [a,c]\times[0,T)$ *where* $\sigma_c u(x,t)$ *is as above. Then, if there exists* $t_0 \in (0,T)$ *such that* $u(x,t_0) > 0$ *for* $x \in [c,d]$,

$$u_x(c,t_0) > 0. \tag{3.2}$$

Proof. We shall only show (3.1).

First, we show that

$$\sigma_c u(x,t_0) > u(x,t_0) \qquad \text{for} \quad c < x \leq d. \tag{3.3}$$

Assume that $\sigma_c u(x_0,t_0) - u(x_0,t_0) = 0$ for some $x_0 \in (c,d)$. Set $w = v-u$ where $v = \sigma_c u$. Then we see that $w(x,t)$ satisfies a linear parabolic equation

$$w_t = \frac{1}{b'(u)}\{w_{xx} + (\bar{f}-\bar{b}'v_t)w\} \tag{3.4}$$

in (x,t) satisfying $w(x,t) > 0$, where $\bar{\varphi} = \bar{\varphi}(v,u) = \int_0^1 \bar{\varphi}'(\theta v+(1-\theta)u)\,d\theta$. Note that $(x_0,t_0) \in (c,d)\times(0,T)$ is a minimum point of $w(x,t)$ in $(c,d)\times[0,T)$. Applying the maximum principle to $w(x,t)$, we obtain that $w(x,t_0) = 0$ in $x \in [c,d]$, which implies $w(d,t) = 0$ in $t \in [0,t_0]$. Namely $w(d,0) = u(a,0) - u(d,0) = 0$. However this contradicts the assumption of Lemma 3.1.

Next we show (3.1). Note that $u(x,t) > 0$ in the neighborhood of $t=t_0$ and $x \in [a,c]$. Then, by the same methods by which we demonstrated to show (3.3) we have that $w(x,t)>0$ in $(x,t) \in (c,d]\times(t_0-\delta,t_0+\delta)$ for some $\delta>0$. Applying the maximum principle to $w(x,t)$ and using the fact that $w(c,t) = 0$, we have that $w_x(c,t) > 0$ for $t \in (t_0-\delta,t_0+\delta)$. Namely $u_x(c,t) < 0$ for $t \in (t_0-\delta,t_0+\delta)$. This is a proof of (3.1). □

Next, we further assume the following condition:

(A5) There exists a C^∞-function $F : [0,\infty) \to [0,\infty)$ such that

(i) $F(u) > 0$, $F'(u) \geq 0$, $F''(u) \geq 0$, for $u > 0$;

(ii) there exist $c > 0$ and $M_0 > 0$ such that,

$$f'F - fF' \geq c(F'F - \frac{b''}{b'} F^2) \qquad \text{for } u > M_0; \tag{3.5}$$

(iii)

$$\int_1^\infty \frac{du}{F(u)} < \infty. \tag{3.6}$$

Remark 3.2. If $p/m > 1$, equation (1.4) satisfies this condition.

Lemma 3.3 *(cf, Friedman-McLeod*[5], *Chen-Matano*[4]). *Assume* (A1)-(A5). *Let* $\Omega = (a,b)$ *be a bounded open interval* *and let* $u(x,t)$ *be a positive weak solution of* (1.1) *in* $Q_T = \Omega\times(0,T)$. *Furthermore suppose that*

(3.7) $\quad u_x(x,t)>0$ [or $u_x(x,t)<0$] in $(x,t)\in[c-\delta,c+\delta]\times(\tau,T)$,

for some $c\in(a,b)$ and $\delta>0$ *with* $(c-\delta,c+\delta)\subset(a,b)$ *and* some $\tau \in (0,T)$. *Then there are no blow-up points in* $(c-\delta,c+\delta)$.

Proof. We shall show this lemma in case

(3.8) $\quad u_x(x,t) > 0 \quad$ in $(x,t) \in (c-\delta,c+\delta)\times(t,T)$.

We give an indirect proof. Assume that $x_0 \in (c-\delta,c+\delta)$ is a blow-up point of $u(x,t)$. Then we see that

(3.9) $$\lim_{t\uparrow T} u(x,t) = \infty \qquad \text{for } x \in(x_0,c+\delta).$$

In fact, let $d \in(x_0,c+\delta)$ be fixed. Since x_0 is a blow-up point, there exist sequences $\{x_k\} \subset (c-\delta,d)$ and $\{t_k\} \subset (t,T)$ such that $x_k \to x_0$, $t_k \to T$ and $u(x_k,t_k) \uparrow \infty$ as $k \to \infty$. By (3.8), we obtain

(3.10) $$u(x,t_k) > u(x_k,t_k) \qquad \text{for } d \le x \le c+\delta.$$

Hence, by Lemma 2.3 we have

(3.11) $$u(x,t) \ge \eta_k(t)\sin\left(\frac{x-d}{c+\delta-d}\pi \right) \quad \text{for } x\in[d,c+\delta],\ T>t\ge t_k.$$

Here $\eta_k(t) = W^{-1}(W(\alpha) - \lambda(t-t_k))$ and $W(\eta) = \int_1^{\eta} \frac{b'(\xi)}{\xi}\, d\xi$ with $\alpha = u(x_k,t_k)$ and $\lambda = (\pi/(c+w-d))^2$. Since $W(\eta)$ is a monotone increasing function, we have

$$\eta_k(t) \geq W^{-1}(W(u(x_k,t_k)) - \lambda(T-t_k)) \text{ in } t \in [t_k,T).$$

Noting that $\lim_{k\to\infty} \{W(u(x_k,t_k) - \lambda(T-t_k)\} = \lim_{\eta\uparrow\infty} w(\eta)$, we obtain

$$\min_{t\in[t_k,T)} \eta_k(t) \geq W^{-1}(W(u(x_k,t_k)) - \lambda(T-t_k)) \to \infty \quad \text{as} \quad k \to \infty.$$

This and (3.11) show (3.9) since $d \in (x_0,c+\delta)$ is chosen arbitrarily.

Choose $d \in (x_0,c + d)$ again and set

(3.12) $$J = u_x - \varepsilon\rho(x)F(u(x,t)), \quad (x,t) \in Q = (d,c+\delta)\times(t_1,T)$$

and

(3.13) $$\rho(x) = \sin\frac{\pi(x-d)}{c+\delta-d}$$

where $\varepsilon > 0$ and $t_1 \in (\tau,T)$. Noting the assumptions (A1), (A4), (A5) and (3.7), and assuming that ε is sufficiently small and t_1 is sufficiently close $t = T$, we have

(3.14) $$(b'J)_t - J_{xx} \geq B(x,t)J + C(x,t)J \ , \quad (x,t) \in Q;$$

(3.15) $$J(d,t) > 0, \quad J(c+\delta,t) > 0, \qquad t_1 \leq t < T;$$

(3.16) $$J(x,t_1) > 0, \qquad d \leq x \leq c+\delta.$$

(cf, Chen-Matano [4] and Imai-Mochizuki [8]). Applying the maximum principle to (3.14) (3.15) and (3.16), we obtain $J(x,t) > 0$ for $(x,t) \in Q$, or

(3.17) $$\frac{u_x(x,t)}{F(u(x,t))} > \varepsilon\rho(x) \qquad \text{in} \quad (x,t) \in Q.$$

Integrating this inequality over $d \le x \le c+\delta$ yields

$$(3.18) \qquad \int_{u(d,t)}^{u(c+\delta,t)} \frac{du}{F(u)} > \varepsilon \int_d^{c+\delta} \rho(x)\,dx \quad \text{in} \quad t_1 < t < T.$$

The right-hand side of (3.18) is a positive constant, while the left-hand side tends to zero as $t \uparrow T$ by virtue of condition (A5)(iii) and (3.9). This contradiction shows that x_0 is a not blow-up point of $u(x,t)$. The proof is complete. □

Finally we show

Lemma 3.4. *Let* $u(x,t)$ *be as in Lemma 3.3. Then* $u(x,t)$ *can be extend to a* $C^{2,1}$*-function in* $(c-\delta,c+\delta)\times(0,T]$. *Moreover if we represent this* $C^{2,1}$*-function as* $u(x,t)$ *again, then*

$$(3.19) \qquad u_x(x,T) > 0 \ [\text{or } u_x(x,T) < 0] \quad \text{in} \quad x \in (c-\delta,c+\delta).$$

Proof. We sall show this lemma in case $u_x > 0$. By Lemma 2.3 and Lemma 3.3, for any $c-\delta < d_1 < d_2 < c+\delta$, there exists $M' = M'(d_1,d_2) > 0$ such that

$$\frac{1}{M'} \le u(x,t) \le M' \qquad \text{for} \quad (x,t) \in [d_1,d_2]\times[\tau,T).$$

Therefore we can easily extend $u(x,t)$ to $C^{2,1}$-function $u(x,t)$ in $(c-\delta,c+\delta)\times[\tau,T]$ by means of standard L^p and Schauder's estimates (cf, Chen-Matano [4]).

Choose $d \in (c-\delta,c+\delta)$ arbitrarily and choose $\ell > 0$ such that $[d-\ell,d+\ell] \subset (c-\delta,c+\delta)$. Let us consider the following initial boundary value problem:

$$
(3.20)\qquad \begin{cases} b(v)_t = v_{xx} + f(v) & \text{in} \quad x \in (d-\ell, d+\ell),\ t > \tau, \\ v(x,\tau) = u(x,\tau) & \text{in} \quad x \in (d-\ell, d+\ell), \\ v(d\pm\ell, t) = \eta_\pm(t) & \text{in} \quad t > \tau, \end{cases}
$$

where $\eta_\pm(t)$ are continuous functions on $[\tau,\infty)$ and satisfies that $\eta_\pm(t) = u(a\pm\ell, t)$ for $t \in [\tau,T)$ and $\eta_-(t) \le \eta_+(t)$ for $t \in [t,\infty)$. Then, there exists $T' > 0$ such that a solution $v(x,t)$ of (3.20) exists in $(d-\ell,d+\ell)\times(\tau,T+T')$ by the existence theorem, and the uniqueness theorem implies

$$
(3.21)\qquad v(x,t) = u(x,t) \quad \text{in} \quad (x,t) \in (d-\ell,d+\ell)\times[\tau,T].
$$

We compare $\sigma_d v(x,t)$ and $v(x,t)$ in $[d-\ell,d]\times[\tau,T+T')$. We can see easily that $\sigma_d v(x,\tau) \ge v(x,\tau)$ for $x \in [d-\ell,d]$, $\sigma_d v(d,t) = v(d,t)$ for $t \in [\tau,T+T')$ and $\sigma_d v(d-\ell,t) - v(d-\ell,t) = \eta_+(t) - \eta_-(t) \ge 0$ for $t \in [\tau,T+T')$. Noting $\sigma_d v$ and v are solutions of (1.1) and applying the comparison theorem, we obtain

$$
(3.22)\qquad \sigma_d v(x,t) \ge v(x,t) \quad \text{in} \quad (x,t) \in [d-\ell,d]\times[\tau,T+T').
$$

Then it follows from Lemma 3.1 that

$$
(3.23)\qquad u_x(d,t) > 0 \qquad \text{for} \quad t \in (\tau,T+T'),
$$

so $u_x(d,T) > 0$. Since $d \in (c-\delta,c+\delta)$ is chosen arbitrarily, we obtain (3.19). □

4. Key Lemma

In this section, we assume (A1)-(A5) and prove the following key lemma for the case of a bounded domain:

Lemma 4.1. *Assume* (A1)-(A5). *Let* $u(x,t)$ *be a positive weak solution of* (1.1) *in* $Q_T = \Omega \times (0,T) = (0,L) \times (0,T)$ *and let* $0 < a_1 < a_1 < L$. *Suppose that for any* $t_0 \in (0,T)$ *there exists* $\delta = \delta(t_0) > 0$ *such that*

$$(4.1) \quad u_x(x,t) > 0 \ [\mathit{or}\ u_x(x,t) < 0], \quad (x,t) \in [a_1-\delta, a_1+\delta] \times (t_0,T)$$

and

$$(4.2) \quad u_x(x,t) < 0 \ [\mathit{or}\ u_x(x,t) > 0], \quad (x,t) \in [a_2-\delta, a_2+\delta] \times (t_0,T)$$

where $[a_i-\delta, a_i+\delta] \subset \Omega$ $(i=1,2)$ *and* $[a_1-\delta, a_1+\delta] \cap [a_2-\delta, a_2+\delta] = \phi$. *Then, the blow-up set of* $u(x,t)$ *is finite in* $(a_1-\delta, a_2+\delta)$.

Remark 4.2. Consider the Dirichlet problem (1.1)(1.2) (1.3a) with $b(u)=u$. Let $u(x,t)$ be a blow-up solution of the problem. Then, we see that $u(x,t) > 0$ for $(x,t) \in (0,L)\times(0,T)$ and for any $\tau \in (0,T)$ there exists $\delta=\delta(\tau) > 0$ such that $u_x(x,t) > 0$, $(x,t) \in (0,\delta)\times(\tau,T)$ and $u_x(x,t) < 0$, $(x,t) \in (L-\delta,L)\times(\tau,T)$ (see Friedman-McLeod [5]). Using Lemma 4.1 and the fact that the blow-up set of $u(x,t)$ is contained in $(0,L)$ (Friedman-McLeod [5]), we obtain that the blow-up set of $u(x,t)$ is a finite set. Then, we do not use the analyticity condition on a initial data $u_0(x)$ and a heat source $f(u)$ which is required in Chen-Matano [4] in case $f(0) > 0$.

We need some notations and preliminary lemmas (see Chen-Matano [4] and Angenent [1]).

Notation 4.3. Let $w(x)$ be a continuous real value function on K where K is $S^1 = \mathbb{R}/\mathbb{Z}$ or a bounded closed interval in $\mathbb{R}$. We define the *nodal number* of w by

$$\nu(w) = \text{the number of points } x \in K \text{ with } w(x) = 0.$$

This defines a functional $\nu : C(K) \to \mathbb{N}\cup\{0\}\cup\{\infty\}$.

Definition 4.4. We say that $w \in C^1(K)$ poses only simple zeroes if $w'(x) \neq 0$ for any $x \in K$ such that $w(x) = 0$. The set of all such functions is denoted by $\sum(K)$.

Lemma 4.5 *(Angenent). Let* $p(x,t)$, $q(x,t)$ *and* $r(x,t)$ *be locally bounded continuous functions on* $S^1\times(t_0,T)$ *with* p_{xx}, p_{xt}, p_{tt}, p_x, p_t, q_x, q_t, *all locally bounded continuous. Furthermore, let* $p(x,t) > 0$ *and let* $w(x,t)$ *be a classical solution of*

$$(4.3) \quad w_t = p(x,t)w_{xx} + q(x,t)w_x + r(x,t)w, \quad (x,t)\in S^1\times(t_0,T).$$

Assume that w *is not identically equal to zero. Then*

(i) $\nu(w(\cdot,t))$ *is finite for any* $t \in(t_0,T)$ *and is monotone nonincreasing in* t;

(ii) *there exists a strictly decreasing sequence of points* $\{t_k\} \subset (t_0,T)$ *such that* $\{t_k\} \downarrow t_0$ *and* $w(\cdot,t) \in$

$\sum(S^1)$ *for any* $t \in (t_0,T)\setminus\{t_k\}$.

Lemma 4.6 *(Angenent). The assertions of lemma 4.5 hold with* S^1 *replaced by a closed interval* $[a,b]$ *in* $\mathbb{R}$, *if we assume in addition that* $w(a,t) \neq 0$ *and* $w(b,t) \neq 0$ *for any* $t \in (t_0,T)$.

Remark 4.7. Lemma 4.5 and Lemma 4.6 follow immediately from the next lemma due to Angenent [1]:

Lemma 4.8 ([1]). *Under the assumption of Lemma 4.1 or of Lemma 4.2, we have*

(i) $\nu(w(\cdot,t))$ *is finite for* $t \in (t_0,T)$,

(ii) *If* (x_0,t_1) *is multiple zero of* w, then $\nu(w(\cdot,t_2)) > \nu(w(\cdot,t_3))$ *for all* $t_2 < t_1 < t_3 < T$.

With Lemma 4.6 we can now prove Lemma 4.1.

Proof of Lemma 4.1 (cf, Chen-Matano). We note that the point of $(a_1-\delta(t_0),a_1+\delta(t_0)) \cup (a_2-\delta(t_0),a_2+\delta(t_0))$ is not a blow-up point.

By differentiating equation (1.1) with respect to x, we see that $w = u_x(x,t)$ satisfies a parabolic equation of the form (4.3) in $[a_1,a_2] \times (t_0,T)$. Therefore noting (4.1) and (4.2) and using Lemma 4.6 (ii), we can see the existence of $\tau \in (t_0,T)$ such that

(4.4) $$u_x(\cdot,t) \in \sum([a_1,a_2]) \qquad \text{for all} \quad t \in [\tau,T).$$

Applying the implicit function theorem to u_x, we obtain

C^1-curves $\xi_1, \xi_2, \ldots, \xi_n : [\tau, T) \to (a_1, a_2)$ such that

(i) $\xi_1(t) < \xi_2(t) < \ldots < \xi_n(t)$ for $t \in [\tau, T)$,

and

(ii) $\{x \in [a_1, a_2] \mid u_x(x,t) = 0\} = \{\xi_1(t), \ldots, \xi_n(t)\}$

for each $t \in [\tau, T)$.

Let $S(u_0)$ be the blow-up set of $u(x,t)$. We shall show

$$\lim_{t \uparrow T} \xi_i(t) = \alpha_i \quad \text{exists for each } 1 \leq i \leq n \tag{4.5}$$

and

$$S(u_0) \cap (a_1, a_2) \subset \{\alpha_1, \alpha_2, \ldots, \alpha_n\}. \tag{4.6}$$

First, set $\alpha_i^- = \liminf_{t \uparrow T} \xi_i(t)$, $\alpha_i^+ = \limsup_{t \uparrow T} \xi_i(t)$ and $J_i = [\alpha_i^-, \alpha_i^+]$ for each $1 \leq i \leq n$. Then, by (4.1) and (4.2) we have

$$\bigcup_{i=1}^{n} J_i \subset (a_1, a_2). \tag{4.7}$$

Moreover we obtain

$$(a_1, a_2) \setminus \bigcup_{i=1}^{n} J_i \subset (a_1, a_2) \setminus S(u_0). \tag{4.8}$$

In fact, choose a closed interval $[c,d] \subset (a_1, a_2) \setminus \bigcup_{i=1}^{n} J_i$ $(c<d)$ arbitrarily. Then, there exists $t_1 \in (\tau, T)$ such that $u_x(x,t) \neq 0$ does not change its sign in the rectangular region $[c,d] \times [t_1, T)$. It follows from lemma 3.3 that (c,d)

$\subset (a_1,a_2)\setminus S(u_0)$. Since $[c,d] \subset (a_1,a_2)\setminus \bigcup_{i=1}^{n} J_i$ is chosen arbitrarily, (4.8) follows.

Next we define the family $\{W_i\}$ of sets inductively as follows: For closed intervals $A_1=[c_1,d_1]$ and $A_2=[c_2,d_2]$, set $(A_1,A_2) = ((c_1+c_2)/2,(d_1+d_2)/2)$ and define W_j $(j=0,1,..)$ inductively as

$$W_0 = \{(a_1-\delta,a_1+\delta),(a_2-\delta,a_2+\delta)\}$$

$$W_1 = \{A \mid A = (\bar{A}_1,\bar{A}_2), \quad A_1,\ A_2 \in W_0\}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$W_{j+1} = \{A \mid A = (\bar{A}_1,\bar{A}_2), \quad A_1,\ A_2 \in W_j\ \}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

Then, we see easily the following properties with $\tilde{W}_j = \bigcup_{A\in W_j} A$:

(4.9) $\qquad W_j \subset W_{j+1} \qquad$ for $j \geq 0$,

(4.10) $\qquad \tilde{W}_j \subset (a_1-\delta,a_2+\delta) \qquad$ for each j,

and there exists $m \geq 1$ such that

(4.11) $$(a_1,a_2) \subset \bigcup_{j=0}^{m} \tilde{W}_j .$$

Hence, if we show

(4.12) $$\left(\bigcup_{i=1}^{n} \mathring{J}_i\right) \cap \left(\bigcup_{j=0}^{m} \tilde{W}_j\right) = \phi$$

where $\mathring{K}$ is the set of all interior points of a subset K in

$\mathbb{R}$, then we have $\bigcup_{i=1}^{n} \mathring{J}_i = \phi$ by (4.7) ,and therefore we obtain (4.5) and (4.6) by (4.8).

Let us show (4.12). Suppose that (4.12) does not hold. Then, by (4.9) there exists j_0 $(1 \le j_0 \le m)$ such that

$$(4.13) \qquad \widetilde{W}_{j_0-1} \cap \left(\bigcup_{i=1}^{n} \mathring{J}_i \right) = \phi$$

$$(4.14) \qquad \widetilde{W}_{j_0} \cap \left(\bigcup_{i=1}^{n} \mathring{J}_i \right) \neq \phi .$$

Since A is represented as $A=(\bar{A}_1, \bar{A}_2)$ for some A_1, $A_2 \in W_{j_0-1}$, it follows from (4.8) and (4.13) that

$$(4.15) \qquad \{A_1 \cup A_2\} \setminus \{ \alpha \mid \alpha = \alpha_i^- = \alpha_i^+ \} \subset (a_1, a_2) \setminus S(u_0).$$

On the other hand, by (4.14), there exists i_0 $(1 \le i_0 \le n)$ such that

$$(4.16) \qquad A \cap \mathring{J}_{i_0} \neq \phi .$$

Noting that $A=(\bar{A}_1, \bar{A}_2)$, there exist $a \in A_1 \setminus \{ \alpha \mid \alpha = \alpha_i^- = \alpha_i^+ \}$ and a non-empty open interval $D \subset A_2 \setminus \{ \alpha \mid \alpha = \alpha_i^- = \alpha_i^+ \}$ such that

$$(4.17) \qquad (a, \bar{D}) = \{ \frac{a+x}{2} \mid x \in D \} \subset A \cap \mathring{J}_{i_0} .$$

Hence, by (4.15) we have

$$(4.18) \qquad D \subset (a_1, a_2) \setminus S(u_0).$$

Noting that $D \subset A_2 \setminus \{\alpha \mid \alpha = \alpha_i^- = \alpha_i^+ \}$, there exist $t_2 \in (t_1, T)$ and a non-empty closed interval $K \subset D$ such that $u_x \neq 0$ does not change its sign in $K\times[t_2,T)$. Therefore, by Lemma 3.3, we can extend $u(x,t)$ to a $C^{2,1}$-function in $\mathring{K}\times[t_2,T]$ uniquely and we may see that $u_x(x,T) > 0$ for all $x\in\mathring{K}$. Since the $\lim_{t\uparrow T} u(a,t) = u(a,T)$ also exists by Lemma 3.3, we can see the existence of $x \in \mathring{K}$ such that $u(x_0,T) \neq u(a,T)$. Hence there exists $t_2'\in [t_2,T)$ such that

$$(4.19) \qquad u(x_0,t) \neq u(a,t) \quad \text{for all} \quad t\in[t_2',T).$$

Set $w(x,t) = \sigma_b u(x,t)-u(x,t)$ where $\sigma_b u(x,t)=u(2b-x,t)$ with $b=(x_0+a)/2$. Assume that $x_0<a$ for convenience. Then we can see that $w(x,t)$ is a solution of a linear parabolic equation of the form (4.3) in $[x_0,a]\times(t_2',T)$. Moreover, it follows from (4.19) that

$$(4.20) \quad \begin{cases} w(x_0,t)=\sigma_b u(x_0,t)-u(x_0,t)=u(a,t)-u(x_0,t)\neq 0, & t\in(t_2',T), \\ w(a,t)=u(x_0,t)-\sigma_b u(x_0.t)\neq 0, & t\in(t_2',T). \end{cases}$$

Applying Lemma 4.6 to $w(x,t)$ in $[x_0,a]\times(t_2',T)$, we see that there exists $t_3 \in (t_2',T)$ such that $w(\cdot,t) \in \sum([x_0,a])$ for all $t \in [t_3,T)$. Hence, noting $w(b,t) = u(b,t)-u(b,t) = 0$, we have $w_x(b,t)\neq 0$ for $t \in [t_3,T)$, namely,

$$(4.21) \qquad u_x(b,t) \neq 0 \qquad \text{for all} \quad t \in [t_3,T).$$

On the other hand, since $b \in \mathring{J}_{i_0}=(\alpha_{i_0}^-,\alpha_{i_0}^+)$, there exists a sequence $\{t_n\} \subset (0,T)$ such that $t_n \uparrow T$ and $b=\xi_{i_0}(t_n)$, that is, $u_x(b,t_n) = 0$ $n\geq 1$. This contradicts (4.21). So we have (4.12), namely (4.5) and (4.6). Thus the proof is complete. □

5. The Case of a Bounded Domain

Throughout this section, we assume (A1)-(A5) and consider the Dirichlet or Neumann problem (1.1)(1.2)(1.3ab). We assume further the following technical conditions on an initial data $u_0(x)$:

(A6) $u_0(0) = u_0(L) = 0$ and $u_0(x) > 0$ for $x \in \Omega = (0,L)$;

(A7) there exist a_1 and a_2 in $\Omega = (0,L)$ such that $2a_1 \leq a_2$, $a_1 \leq 2a_2 - L$, $\sigma_{a_1} u_0(x) \geq u_0(x)$ on $x \in [0,a_1]$ and $\sigma_{a_2} u_0(x) \geq u_0(x)$ on $x \in [a_2, L]$ where $\sigma_a u(x) = u(2a-x)$.

The main result in this section is as follows:

Theorem 5.1. *Assume* (A1)-(A7). *Let* $u(x,t)$ *be a blow-up weak solution of problem* (1.1)(1.2)(1.3a) *or* (1.1)(1.2)(1.3b) *with blow-up time* $t=T$ *and, let* $S(u_0)$ *be the blow-up set of* $u(x,t)$. *Then*

(5.1) $\qquad S(u_0)$ *is a finite set,*

and furthermore in the case of the Dirichlet problem,

(5.2) $\qquad S(u_0) \subset (a_1, a_2)$.

First, we consider a weak solution $u(x,t)$ of the Dirichlet problem (1.1)(1.2)(1.3a). We need some lemmas.

Lemma 5.2. *Assume* (A1)-(A7). *Let* $u(x,t)$ *be a blow-up weak solution of the Dirichlet problem* (1.1)(1.2)(1.3a). *Then*

$$\sigma_{a_1} u(x,t) \geq u(x,t) \quad \textit{for all} \quad (x,t)\in[0,a_1]\times[0,T); \tag{5.3}$$

$$\sigma_{a_2} u(x,t) \geq u(x,t) \quad \textit{for all} \quad (x,t)\in[a_2,L]\times[0,T). \tag{5.4}$$

Proof. We show only (5.3). Set $v(x,t)=\sigma_{a_1} u(x,t)$. Then we see that $v(x,t)$ is also a weak solution of (1.1) in $(0,a_1)\times(0,T)$. By the assumption (A7), we obtain that $v(x,0)=\sigma_{a_1} u_0(x)\geq u_0(x)=u(x,0)$. For boundary values of $u(x,t)$ in $(0,a_1)\times(0,T)$, we obtain that $v(0,t)=\sigma_{a_1} u(0,t)\geq 0=u(0,t)$ and $v(a_1,t)=u(a_1,t)$ for $t \in [0,T)$. Applying the comparison theorem to $v(x,t)$ and $u(x,t)$, we have that $v(x,t) \geq u(x,t)$ for $(x,t) \in (0,a_1)\times(0,T)$. This is a proof of (5.3). □

This and Lemma 3.1 imply the following

Lemma 5.3. *Let* $u(x,t)$ *be as in Lemma 5.2. Then*

$$u_x(a_1,t) > 0 \quad \textit{for all} \quad t \in (0,T); \tag{5.5}$$

$$u_x(a_2,t) < 0 \quad \textit{for all} \quad t \in (0,T). \tag{5.6}$$

Lemma 5.4. *Let* $u(x,t)$ *be as in Lemma 5.3. Then for any* $t_0 \in (0,T)$, *there exists* $\delta=\delta(t_0) > 0$ *such that for any* $a \in [a_1-\delta,a_1+\delta]$

$$\sigma_a u(x,t) \geq u(x,t) \quad \textit{for all} \quad (x,t)\in[0,a]\times[t_0,T) \tag{5.7}$$

and for any $a \in [a_2-\delta, a_2+\delta]$

(5.8) $\sigma_a u(x,t) \geq u(x,t)$ *for all* $(x,t) \in [a,L] \times [t_0,T)$.

Proof. We only show (5.7). Noting the proof of Lemma 5.2, we see that it is enough to show the following: For any $t_0 \in (0,T)$, there exists $\delta=\delta(t_0) > 0$ such that for any $a \in [a_1-\delta, a_1+\delta]$,

(5.9) $\sigma_a u(x,t_0) - u(x,t_0) > 0$ for all $x \in [0,a)$.

Therefore we show (5.9).

By (5.5), for any $t_0 \in (0,T)$ there exists $\delta=\delta(t_0)>0$ such that

(5.10) $u_x(x,t_0) > 0$ for $|x-a_1| < \delta$.

Set $h(a) = \min_{0 \leq x \leq a-\delta/2} \{\sigma_a u(x,t_0) - u(x,t_0)\}$ for each $a \in [0,L/2]$. Then, we see clearly that $h(a)$ is a continuous function in $[0,L/2]$. Noting the proof of (3.3), we obtain

(5.11) $h(a_1) > 0$.

Hence there exists $\delta'>0$ $(0<\delta' \leq \delta/2)$ such that $h(a) > 0$ for all $|a-a_1| \leq \delta'$, namely

(5.12) $\sigma_a u(x,t_0)-u(x,t_0) > 0$ for $0 \leq x \leq a-\delta/2$, $|a-a_1| \leq \delta'$.

On the other hand, since $a-\delta/2 \leq x \leq a$ and $|a-a_1| \leq \delta/2$ imply $-\delta \leq x-a_1 \leq \delta/2$, by (5.10) we have

(5.13) $\quad \sigma_a u(x,t_0)-u(x,t_0) > 0 \quad \text{for} \quad a-\delta/2\le x\le a, \ |a-a_1|\le\delta'.$

Combining (5.12) and (5.13), we obtain (5.9). The proof is complete. □

This and Lemma 3.1 imply the following

Lemma 5.5. *Let* $u(x,t)$ *be as in Lemma 5.4. Then,*

(5.14) $\quad u_x(x,t) > 0 \quad$ *for all* $\quad (x,t) \in [a_1-\delta, a_1+\delta]\times(t_0,T);$

(5.15) $\quad u_x(x,t) < 0 \quad$ *for all* $\quad (x,t) \in [a_2-\delta, a_2+\delta]\times(t_0,T).$

We are now ready to prove Theorem 5.1 in the case of the Dirichlet problem.

Proof of Theorem 5.1 (the case of the Dirichlet problem). By Lemma 5.5 and Lemma 4.1, we see that $S(u_0)\cap(a_1-\delta,a_2+\delta)$ is a finite set. Hence, if we show that $[0,a_1-\delta]\cup[a_2+\delta,L] \subset [0,L]\setminus S(u_0)$, the proof is complete. We show only

(5.16) $\quad [0,a_1-\delta] \subset [0,L]\setminus S(u_0)$

Now we assume that $x_0 \in[0,a_1-\delta]$ is a point of $S(u_0)$. Then, by (A7) we have that $2a_1-x \in[a_1,a_2]$. It follows from Lemma 5.2 that $2a_1-x_0\in S(u_0)$. Since $S(u_0)\cap(a_1-\delta,a_2+\delta)$ is a finite set, there exists $a_3\in(a_1-\delta,a_1+\delta)$ such that $2a_3-x_0\notin S(u_0)$ and $0\le x_0<a_3$. By the fact that $\sigma_{a_3}u(x,t) \ge u(x,t)$ for $(x,t)\in[0,a_3]\times[t_0,T)$, we obtain that $x_0 \notin S(u_0)$. This contradicts the assumption that $x_0 \in S(u_0)$, and thus

we prove (5.16). □

Next we consider a blow-up weak solution $u(x,t)$ of the Neumann problem (1.1)(1.2)(1.3b). We can consider the following three cases:

(I) $\quad u(0,t) = u(L,t) = 0 \quad$ for all $t \in [0,T)$.

(II) (i) $u(0,t) = 0$ for all $t \in [0,T)$ and there exists $t_1 \in (0,T)$ such that $u(L,t_1) > 0$,

(ii) $u(L,t) = 0$ for all $t \in [0,T)$ and there exists $t_1 \in (0,T)$ such that $u(0,t_1) > 0$.

(III) there exist t_1, $t_2 \in (0,T)$ such that $u(0,t_1) > 0$ and $u(L,t_2) > 0$.

Proof of Theorem 5.1 (the case of the Neumann problem).

<u>Case (I)</u>. Since $u(x,t)$ can be regarded as a solution of the Dirichlet problem, this case comes to the case of the Dirichlet problem.

<u>Case (II)</u>. We only prove in case (ii). Extend a weak solution $u(x,t)$ as

$$(5.17) \qquad \tilde{u} = \begin{cases} u(x,t) & 0 \le x \le L \\ u(-x,t) & -L \le x \le 0 . \end{cases}$$

Then, we see that $\tilde{u}(x,t)$ is a weak solution of the Neumann problem and that $\tilde{u}(\pm L,t)=0$ for $t \ge t_1$ and $\tilde{u}(x,t_1) > 0$ for $x \in (-L,L)$. Hence this case comes to the case (I).

<u>Case (III)</u>. By extending $u(x,t)$ as (5.17) and

appropriate rescaling of the variables, this problem can be converted into the form (1.1)(1.2) with $\Omega = S^1 = \mathbb{R}/\mathbb{Z}$. Set $t_0 = \max\{t_1, t_2\}$. Then, it follows from the assumption (III) and Lemma 2.3 that $u(x,t)>0$ for all $(x,t) \in S^1 \times (t_0, T)$ and $u(x,t)$ is a classical solution of (1.1) on $S^1 \times (t_0, T)$. Noting Lemma 4.5 and Lemma 3.2 and using the same methods as Chen-Matano [4], we can prove the assertion in this case. The proof is complete. □

Remark 5.6. Considering the proof in this section, we can also get the similar result as Theorem 5.1, even if we extend conditions (A6) and (A7) on $u_0(x)$ to the following condition:

(*) there exists a finite family $\{\Omega_i\}_{i=1}^{i=\ell}$ of open intervals such that $\bigcup_{i=1}^{\ell} \Omega_i \subset (0,L)$, $\Omega_i \cap \Omega_j = \phi$ $(i \neq j)$ and $u_0(x) = 0$ in $(0,L) \setminus \bigcup_{i=1}^{\ell} \Omega_i$, and such that (A6) and (A7) hold with Ω replaced by Ω_i.

6. Asymptotic Behavior of an Interface

In this section, we consider the Cauchy problem (1.1)(1.2)(1.3c). We assume (A1)-(A4) and assume the following conditions on an initial data $u_0(x)$:

(A8) $u_0(x) > 0$, $x \in (0,L)$ and $u_0(x) = 0$, $x \in \mathbb{R} \setminus (0,L)$.

Furthermore, we assume the following conditions for the finite propagation of the interface of a weak solution $u(x,t)$ to (1.1)(1.2)(1.3c) in $t \in (0,T)$:

(A9) $$\int_0^1 \frac{d\eta}{b(\eta)} < \infty ;$$

(A10) $$f(0) = 0;$$

(A11) there exists a C^1-function $G(u)$ on $[0,\infty)$ such that

(6.1) $$G(u) \geq f(u) \quad \text{and} \quad G'b - Gb' \geq 0 \qquad \text{for all} \quad u \geq 0.$$

Now we can show a finite propagation of an interface of a weak solution.

Theorem 6.1 *(a finite propagation of an interface). Assume* (A1)-(A4) *and* (A8)-(A11). *Let* $u(x,t)$ *be a blow-up weak solution of* (1.1)(1.2)(1.3c) *with blow-up time* $t=T$. *Then, there exist continuous functions* $\xi_i(t) : [0,T) \to \mathbb{R}$ $(i=1,2)$ *such that*

(6.2) $$(\xi_1(t),\xi_2(t)) = \{ x \in \mathbb{R} \mid u(x,t) > 0 \} \quad \textit{for each} \quad t \in [0,T),$$

$\xi_1(t)$ *is a monotone decreasing function and* $\xi_2(t)$ *is a monotone increasing function, and furthermore,*

(6.3) $$-\infty < \xi_1(t) < \xi_2(t) < \infty \qquad \textit{for each} \quad t \in [0,T).$$

Next we state behavior of the interface of $u(x,t)$ near the blow-up time $t=T$. For this aim, we further assume (A5) or the condition

(A12) $$\lim_{u\to\infty} \frac{H(u)}{u^2} = 0$$

where $H(u) = \int_0^u f(\xi)\, d\xi$.

Theorem 6.2. *Let* $S(u_0)$ *be the blow-up set of* $u(x,t)$ *where* $u(x,t)$ *is as in Theorem 6.1 and let* $\xi_i(t)$ $(i=1,2)$ *is the interface of* $u(x,t)$. *Then the following two results hold*

(i) *if* (A5) *holds, then*

(6.4) $$S(u_0) \subset [0,L]$$

and

(6.5) $$-\infty < \lim_{t\uparrow T} \xi_1(t) < \lim_{t\uparrow T} \xi_2(t) < \infty,$$

and furthermore if (A7) *holds, then*

(6.6) $$S(u_0) \subset (a_1,a_2) \quad \textit{and} \quad S(u_0) \textit{ is a finite set.}$$

(ii) *if* (A12) *holds,*

(6.7) $$S(u_0) = \mathbb{R}\cup\{\infty\}\cup\{-\infty\}$$

and

(6.8) $$\lim_{t\uparrow T} \xi_1(t) = -\infty \quad \textit{and} \quad \lim_{t\uparrow T} \xi_2(t) = \infty.$$

Remark 6.3. If $p>1$ and $m>1$, equation (1.4) satisfies these conditions (A9)-(A11) with $G(u)=f(u)=u^{p/m}$. If $p/m < 1$, equation (1.4) satisfies condition (A12).

First we prove Theorem 6.1. We need the following lemma:

Lemma 6.4. *Assume* (A1)-(A4) *and* (A9)-(A11). *Let* $u(x,t)$ *be a weak solution of* (1.1)(1.2)(1.3c). *Suppose that there exist* $(a,t_1) \in \mathbb{R}\times[0,T)$ *and* $M > 0$ *such that*

$$u(x,t_1) = 0 \quad \textit{for} \quad x \geq a; \tag{6.9}$$

$$u(a,t) \leq M \quad \textit{for} \quad t \in [t_1,T). \tag{6.10}$$

Then, there exist $\ell > 0$ *and* $h > 0$ *depending on only* M *such that*

$$u(x,t) = 0 \quad \textit{for} \quad (x,t) \in [a+\ell,\infty)\times[t_1,t_1+h]\cap[t_1,T). \tag{6.11}$$

Proof. Consider the function $\psi : [0,\infty) \to [0,\infty)$ such that

$$\psi(u) = \int_0^u \frac{d\eta}{b(\eta)} \text{ (see Knerr [11]).} \tag{6.12}$$

Then, by (A1) and (A9) we see that $\psi(u)$ is well defined and is an onto and one to one mapping. Put

$$v(x,t) = \psi(u(x,t)). \tag{6.13}$$

Since $u(x,t)$ satisfies the equation (1.1) in (x,t)-set where $u(x,t) > 0$, a simple computation gives

$$v_t = \frac{1}{b'(\psi^{-1}(v))} v_{xx} + (v_x)^2 + \frac{f(\psi^{-1}(v))}{b(\psi^{-1}(v))b'(\psi^{-1}(v))}. \tag{6.14}$$

where $\psi^{-1}(v)$ is the inverse function of $v=\psi(u)$.

Let us consider the following function $v(x,t)$:

$$v(x,t) = (\eta(t) - C(x-a))^{+} \quad \text{for} \quad x \geq a,\ t \geq t_1 \tag{6.15}$$

where $\eta(t)$ satisfies the problem

$$\begin{cases} \eta(t_1) = \eta_1 = \psi(M) > 0 \\ \eta_t = C^2 + \dfrac{G(\psi^{-1}(\eta))}{b(\psi^{-1}(\eta))b'(\psi^{-1}(\eta))} \end{cases} \tag{6.16}$$

and C is a positive constant. Then $\eta(t)$ is represented as the following:

$$\eta(t) = \bar{G}^{-1}(\bar{G}(\eta_1) - (t-t_1)) \tag{6.17}$$

where

$$\bar{G}(\eta) = \int_{\eta}^{\infty} \frac{1}{C^2 + \dfrac{G(\psi^{-1}(\eta))}{b(\psi^{-1}(\eta))b'(\psi^{-1}(\eta))}}\, d\eta. \tag{6.18}$$

Here we note that

$$\bar{G} \leq \int_{\eta}^{\infty} \frac{b(\psi^{-1}(\eta))b'(\psi^{-1}(\eta))}{G(\psi^{-1}(\eta))}\, d\eta = \int_{\psi^{-1}(\eta)}^{\infty} \frac{b'(\xi)}{f(\xi)}\, d\xi < \infty \quad (\psi' = \tfrac{1}{b})$$

and $\xi = \bar{G}^{-1}(\eta)$ is the inverse function of $\eta = \bar{G}(\xi)$. And we have

$$\eta(t) < \infty, \quad t \in [t_1, t_1 + \bar{h}) \quad \text{and} \quad \eta(t) \uparrow \infty \quad \text{as} \quad t \uparrow t_1 + \bar{h}, \tag{6.19}$$

where $\bar{h} = \bar{G}(\eta_1)$. Since $v(x,t) = \eta(t) - C(x-a)$ where $\eta(t) > C(x-a)$, we obtain

$$v_t = \frac{1}{b'(\psi^{-1}(v))} v_{xx} + (v_x)^2 + \frac{G(\psi^{-1}(\eta))}{b(\psi^{-1}(\eta))b'(\psi^{-1}(\eta))}.$$

Considering that $\frac{G(\xi)}{b(\xi)b'(\xi)}$ is a monotone increasing function by (A1) and (A11), and that $v(x,t) = (\eta(t)-C(x-a))^+ \le \eta(t)$ for $x \ge a$, namely $\psi^{-1}(v) \le \psi^{-1}(\eta)$ for $x \ge a$, we have

$$(6.20) \qquad v_t \ge \frac{1}{b'(\psi^{-1}(v))} v_{xx} + (v_x)^2 + \frac{G(\psi^{-1}(v))}{b(\psi^{-1}(v))b'(\psi^{-1}(v))}.$$

Hence, if we put $w(x,t) = \psi^{-1}(v(x,t))$, we obtain

$$(6.21) \qquad b(w)_t \ge w_{xx}+f(w), \qquad t\in[t_1,t_1+\bar{h}),\ \eta(t)>C(x-a),\ x\ge a.$$

On the other hand, since $v(x,t)=0$ namely $w(x,t)=0$ in (x,t)-set where $x\ge a$, $t\in[t_1,t_1+\bar{h})$ and $\eta(t)<C(x-a)$, and since $f(0)=0$, we have

$$(6.22) \qquad b(w)_t = w_{xx}+f(w), \qquad t\in[t_1,t_1+\bar{h}),\ \eta(t)<C(x-a),\ x\ge a.$$

Since $w=0$ on $\eta(t)=C(x-a)$ and $b(0)=0$, we have

$$(6.23) \qquad w_x = v_x b(w)|_{\eta=C(x-a)} = 0.$$

Hence, combining (6.21), (6.22) and (6.23), we see that $w(x,t)$ is a super-solution of (1.1) in $[a,\infty)\times[t_1,t_1+\bar{h})$

On the other hand, by (6.9) and (6.10) we have

$$(6.24) \qquad w(a,t)=\psi^{-1}(v(a,t))\ge\psi^{-1}(\eta(t))\ge\psi^{-1}(\psi(M))=M\ge u(a,t)$$

for $t \in [t_1,t_1+\bar{h})\cap[t_1,T)$ and

$$(6.25) \qquad w(x,t_1) \ge 0 = u(x,t_1) \qquad \text{for } x \ge a .$$

Applying the comparison theorem to $w(x,t)$ and $u(x,t)$ on $[a,\infty)\times[t_1,t_0)$ where $t_0 = \min\{t_1+\bar{h},T\}$, we obtain

(6.26) $$w(x,t) \ge u(x,t) \qquad \text{for } (x,t) \in [a,\infty)\times[t_1,t_0).$$

Namely,

(6.27) $$u(x,t) = 0 \qquad \text{for } x\ge a,\ \eta(t)<C(x-a),\ t\in[t_1,t_0).$$

If we choose $h = \bar{h}/2$ and $\ell = \eta(t_1+h)/C$, we can show the assertions of Lemma 6.3. □

Proof of Theorem 6.1. Set $\xi_1(t) = \inf\{ x \mid u(x,t)>0 \}$ and $\xi_2(t) = \sup\{ x \mid u(x,t)>0 \}$. Then, by Lemma 2.2 and Lemma 2.3 it is obvious that $\xi_1(t)$ is a monotone decreasing function and $\xi_2(t)$ is a monotone increasing and that (6.2) holds (cf, Knerr [11]).

Next, we show (6.3) and here we only prove

(6.28) $$\xi_2(t) < \infty \qquad \text{for each } t \in (0,T).$$

Assume that (6.28) does not hold. Then, it follows that

(6.29) $$\xi_2(t) = \infty \qquad \text{for all } t \in (0,T)$$

or there exists $t_0 \in (0,T)$ such that

(6.30) $$\xi_2(t) < \infty \quad \text{for } t \in (0,t_0) \text{ and } \xi_2(t_0) = \infty.$$

We only drive a contradiction in case of (6.30). In case of (6.29), we can also drive it similarly.

Set

$$M = \sup\{ u(x,t) \mid (x,t) \in \mathbb{R}\times[0,t_0+\delta] \}$$

where $0 < \delta < T-t_0$. Then, by Lemma 6.3 there exist $\ell > 0$ and $h > 0$ such that (6.11) holds with T replaced by $t_0+\delta$. Choose $a > 0$ and $t_1 \in(0,t_0)$ such that $t_0 < t_1+h$ and $a = \xi_2(t_1)$. Then, we obtain

$$u(x,t) = 0 \quad \text{for} \quad (x,t) \in [a+\ell,\infty) \times [t_1,t_1+h]\cap[t_1,t_0+\delta),$$

that is,

$$\xi_2(t_0) \le a + \ell. \tag{6.31}$$

This contradicts (6.30) and thus (6.28) is shown.

Finally we can show the continuity of $\xi_i(t)$ by similar methods as that show (6.28), and we omit this proof. □

We need some lemmas to show Theorem 6.2.

Lemma 6.5. *Assume* (A1)-(A4) *and* (A8). *Let* $u(x,t)$ *be a weak solution of* (1.1)(1.2)(1.3c) *and let* $a \in \mathbb{R}\setminus(0,L)$. *Then*

$$\sigma_a u(x,t) \ge u(x,t), \quad \begin{matrix} \textit{for all} & x\in[a,\infty) & \textit{if} & a\ge L \\ \textit{for all} & x\in(-\infty,a] & \textit{if} & a\le 0 \end{matrix}, \quad t\in(0,T). \tag{6.32}$$

Therefore. $u(x,t)$ *is a monotone decreasing function on* $x\ge L$ *and a monotone increasing function on* $x\le 0$ *for each* $t\in(0,T)$.

Proof. This proof is similar to it of Lemma 5.2. We omit it. □

Lemma 6.6. *Let* $\xi_i(t)$ $(i=1,2)$ *be as in Theorem 6.1. Then*

$$\lim_{t \uparrow T} \xi_1(t) = -\infty \quad \textit{if and only if} \quad \lim_{t \uparrow T} \xi_2(t) = \infty.$$

Proof. This proof is obvious by Lemma 6.5. □

Proof of Theorem 6.2 (i). First, we show that $S(u_0) \subset [0,L]$.

Assume that $x_0 \in S(u_0)$ is not a point in $[0,L]$. Without loss of generality, we may assume that $x_0 > L$. By Lemma 6.5, we have that $[L,x_0) \subset S(u_0)$. It follows from Lemma 3.1 and. Lemma 6.5 that for any $\delta>0$ small enough there exists $t_0=t_0(\delta) \in (0,T)$ such that $u(x,t) > 0$ for $(x,t) \in [L,x_0-\delta] \times [t_0,T)$ and $u_x(x,t) > 0$ in $(x,t) \in [L,x_0-\delta] \cup [t_0,T)$. Usig Lemma 3.3, we have that $(L,x_0-\delta) \subset S(u_0)^c$. This contradicts that $[L,x_0) \subset S(u_0)$ and thus we show that $S(u_0) \subset [0,L]$.

Hence, noting that $u(x,t) \le M(\delta)$ for $(x,t) \in (-\infty,-\delta] \cup [L+\delta.\infty) \times [0,T)$ $(\delta>0)$ and using the similar method as in the proof of Theorem 6.1, we see (6.5).

Next we further assume (A7) and prove (6.6). Consider $u(x,t)$ and $\sigma_{a_2} u(x,t)$ in $(a_2,\infty) \times (0,T)$. By the comparison theorem on a half space, we have that $\sigma_{a_2} u(x,t) \ge u(x,t)$ in $(a_2,\infty) \times (0,T)$ (cf, proof of Lemma 5.2). Using Lemma 3.1, we obtain that $u_x(a_2,t) < 0$ for $t \in (0,T)$. Hence, by the similar method as in the proof of Lemma 5.4 and by the continuity of $\xi_2(t)$, we have that, if $t_1 \in (0,T)$ is sufficiently close to $t=0$, then $\sigma_a u(x,t_1) \ge u(x,t_1)$ in $[a,\infty) \times [t_1,T)$ where $|a-a_2| \le \delta$ for some $\delta>0$. Using Lemma 3.1 again, we obtain that $u_x(a,t) < 0$ for $|a-a_2| < \delta$, $t_1 < t < T$.

Also similarly we obtain that $u_x(a,t) > 0$ for $|a-a_1| < \delta'$, $t_1 < t < T$ for some $\delta'>0$. Therefore as we prove Theorem 5.1, we obtain (6.6). The proof is complete. □

Next we prove Theorem 6.2 (ii). We need the following lemma due to Imai-Mochizuki [8].

Lemma 6.7. *Assume* (A1)-(A4) *and* (A12). *Let* $u(x,t)$ *be a weak solution of the Dirichlet problem* (1.1)(1.2)(1.3a). *Then,* $u(x,t)$ *exists globally in time and stays bounded (in* $L^\infty(0,L)$) *as* $t \uparrow \infty$.

Proof of Theorem 6.2 (ii). First, we show

(6.33) $$(-\infty,0]\cup(L,\infty) \subset S(u_0).$$

Here, we only show that $[L,\infty) \subset S(u_0)$.

Assume that $x_0 \in [L,\infty)$ is not a point in $S(u_0)$. Then, by Lemma 6.5 we have that for some $M_1>0$

(6.34) $$u(x,t) \le M_1 \quad \text{for} \quad (x,t) \in [x_0,\infty)\times[0,T),$$

and by (A8) we see

(6.35) $$u(x,0) = u_0(x) = 0 \qquad \text{for} \quad x \ge x_0.$$

As we prove Theorem 6.1, we obtain

(6.36) $$\lim_{t\uparrow T} \xi_2(t) < \infty.$$

Using Lemma 6.6, we have

(6.37) $$\lim_{t\uparrow T} \xi_1(t) > -\infty.$$

Choosing $x_1 < \lim_{t\uparrow T} \xi_1(t)$ and $\lim_{t\uparrow T} \xi_2(t) < x_2$, we see that $u(x_1,t) = 0$ and $u(x_2,t) = 0$ for $t \in (0,T)$. Since we can look $u(x,t)$ as a solution of the Dirichlet problem with $\Omega = (x_1,x_2)$, we see that $u(x,t)$ does not blow up at $t=T$ by Lemma 6.7 . This is a contradiction since $t=T$ is assumed to be the blow-up time of $u(x,t)$ and thus we prove (6.33). Noting Lemma 6.5, we obtain that $(0,L) \subset S(u_0)$. Hence we see that $S(u_0)=\mathbb{R}\cup\{\infty\}\cup\{-\infty\}$ and that $\lim_{t\uparrow T} \xi_2(t) = \infty$ and $\lim_{t\uparrow T} \xi_1(t) = -\infty$. The proof is complete. □

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Chapter II

Blow-up Solutions
of Quasilinear Degenerate Parabolic Equations
with Convection

0. Introduction

In this paper we shall consider the Cauchy problem in $\mathbb{R}$:

$$\partial_t \beta(u) - u_{xx} + g(u)_x = f(u) \quad (x,t)\in\mathbb{R}\times(0,T) \tag{0.1}$$

$$u(x,0) = \varphi(x) \qquad x \in \mathbb{R} \tag{0.2}$$

where $\beta(v)$, $g(v)$, $f(v)$ with $v\geq 0$ and $\varphi(x)$ are nonnegative continuous functions.

Equation (0.1) describes the combution process with convection in a stationary medium in which the thermal conductivity $\beta'(u)^{-1}$ and the volume heat source $f(u)$ and convection $g(u)$ are depending in a nonlinear way on the temperature $\beta(u) = \beta(u(x,t))$ of the medium.

Throughout this paper we assume

(A1) $\beta(v)$, $f(v)$, $g(v)$ $\in$ $C^\infty(\mathbb{R}_+)\cap C(\bar{\mathbb{R}}_+)$, $\beta(v)>0$, $\beta'(v)>0$, $\beta''(v)\leq 0$ and $f(v)$, $g(v)$, $g'(v)>0$ for $v>0$. $\lim_{v\to\infty} \beta'(v) = \infty$. $f\circ\beta^{-1}$ and $g\circ\beta^{-1}$ are locally Lipschitz continuous in $[\beta(0),\infty)$.

(A2) $(g\circ\beta^{-1})'(u) \leq C\sqrt{(\beta^{-1})'(u)}$ in the neighborhood of $u=0$

for some positive constant C.

(A3) $\varphi(x) \geq 0$, $\not\equiv 0$ and $\in \mathbb{B}(\mathbb{R})$ (bounded continuous in $\mathbb{R}$).

With these conditions above Cauchy problem has a unique local solution $u(x,t)$ (in time) which satisfies (0,1) in $\mathbb{R} \times (0,T)$ in the following weak sense, where $T > 0$ is assumed suffisiently small.(see e.g. Oleinik et all [9, 12, 16])

Definition 0.1. Let G be an open interval in $\mathbb{R}$. By a weak solution of equation (0.1) in $G \times (0,T)$ we mean a function $u(x,t)$ such that

1) $u(x,t) \geq 0$ in $\bar{G} \times [0,T)$ and $\in C(\bar{G} \times [0,\tau])$ for each $0 < \tau < T$.

2) For any bounded open interval $\Omega = (x_1,x_2) \subset G$, $0 < \tau < T$ and $\varphi(x,t) \in C^2(\bar{\Omega} \times [0,T))$ which vanishes on $x = x_1$, x_2, the following identity holds:

$$\int_\Omega \beta(u(x,\tau))\varphi(x,\tau)dx - \int_\Omega \beta(u(x,0))\varphi(x,0)\, dx$$

$$= \int_0^\tau\int_\Omega \{\beta(u)\varphi_t + u\varphi_{xx} + g(u)\varphi_x + f(u)\varphi\}dxdt - \int_0^\tau u\varphi_x dt\Big|_{x=x_1}^{x=x_2}. \quad (0.3)$$

If $u(x,t)$ does not exist globally in time, it's existence time $T < \infty$ defined by

$$T = \sup\{ \tau > 0 \ ; u(x,t) \text{ is bounded in } \mathbb{R} \times [0,\tau]\} \quad (0.4)$$

and we see that

$$\limsup_{t\uparrow T}\ \sup_{x\in\mathbb{R}} u(x,t) = \infty. \tag{0.5}$$

In this case we say that u is a blow-up solution and T is a blow-up time.

The main purpose of the present paper is the study of blow-up solutions. Especially we are interested in the shape of the blow-up set which locates the "hot-spots" at the blow-up time. In addition, since our quasilinear equation (0.1) has a property of the finite propagation of an interface, there are some interesting subjects such as asymptotic behavior of the interface near the blow-up time. These problems have been studied by the authors [6, 7, 8, 17] and Mochizuki-Suzuki [15] for equation (0.1) without convection term.

Firstly we consider the finite propagation of interfaces of solutions of (0.1)(0.2). To deal with this we require the additional conditions:

(A4) $\varphi(x) > 0$ for $x \in (-a_1.a_1)$ and $= 0$ for $x \notin (-a_1,a_1)$.

(A5) $\lim_{v\to 0} \frac{g'}{\beta'}(v) = 0$ and $\frac{g'}{\beta'}$ is increasing in $v\geq 0$.

(A6) $\beta(0) = f(0) = 0$, $\displaystyle\int_0^1 \frac{dv}{\beta(v)} < \infty$,

Put

$$\Omega(t) = \{\ x \in \mathbb{R}\ ;\ u(x,t) > 0\ \},\ \Gamma(t) = \partial\Omega(t) \tag{0.6}$$

for each $t \in (0,T)$. Then the interface Γ is given by

$$\Gamma = \bigcup_{0\le t\le T} \Gamma(t) \times \{t\}, \qquad (0.7)$$

and under these assumptions (A1)-(A6) we can show that $\Omega(t)$ is bounded and nonincreasing in $t \in [0,T)$ (Theorem 1.7). Moreover $\Omega(t)$ is represented by continuous functions $\xi_i(t) : [0,T) \longrightarrow \mathbb{R}$ $(i=1,2)$ like $\Omega(t) = \{ x \mid x \in (\xi_1(t),\xi_2(t))\}$. In the case without convection term in (0.1), these results have been shown by knerr [12], R. Suzuki [15] and Mochizuki-Suzuki [17].

Next, we restrict ourselves to the blow-up solution of (0.1)(0.2) and shall study the shape of blow-up set and the behavior of the interface of u near the blow-up time. The existence and non-existence of blow-up solution (0.1)(0.2) is discussed in Friedman-Lacey [5], Imai-Mochizuki [10] and Imai-Mochizuki-Suzuki [11]. We assume the following condition is given in [10] as a "necessary" condition to raise a blow-up:

(A7) $$\int_1^\infty \frac{\beta'(v)}{f(v)}\, dv < \infty.$$

Furthermore we assume that $f(u)$ grows rappidly than $g(u)$ and u (see (A10)) and assume for the initial data $\varphi(x)$ that

(A8) $\varphi'' - \{g(\varphi)\}' + f(\varphi) \ge 0$ in $\mathscr{D}'$,

(A9)) the lap-number of $\varphi(x)$ in $[a_1,a_1]$ is two.

Here, we denote the lap-number of $\varphi(x)$ in the following :

Definition 0.2. Let $\bar{I} = [a,b]$ be a closed interval and $w =$

$w(x)$ be a real-valued function on $[a,b]$. We say w is piecewise monotone if $\bar{I}$ can be divided into a finite number of non-overlapping sub-intervals J_1, J_2, $\cdots J_m (\bigcup_{i=1}^{m} J_i = \bar{I})$ on each of which monotone. Then there is the least value of the numbers m for which we can fined a division $\{J_i\}$ as above. This valued is called the lap-number of w on $[a,b]$ and denoted by $\ell(w)$.

Then, in the semilinear case $\beta(u) = u$, Friedmann-Lacey [5] showed the existence of single point blow-up solutions of (0.1) for Dirichlet problem . In this paper we extend this result to the Cauchy problem of a degenerate quasilinear equation. Moreover we can get also that the left side interfaces stay bounded as t tends to the blow-up time T. Namely, if we denote

Definition 0.3. The blow-up set of u is defined as

$S = \{ x \in \mathbb{R}$; there is a sequence $(x_i, t_i) \in \mathbb{R} \times (0,T)$ such that $x_i \longrightarrow x$, $t_i \longrightarrow T$ and $u(x_i, t_i) \longrightarrow \infty$ as $i \longrightarrow \infty \}$

and each $x \in S$ is called a blow-up point of u,

then we obtain that $S = \{\eta_0\}$ for some $-\infty < \eta_0 \leq \infty$ and $-\infty < \lim_{t \uparrow T} \xi_1(t)$ (see Theorem 3.3).

Finally, the rest problem is that whether $\eta_0 < \infty$ or $\eta_0 = \infty$ holds. About this problem, we anser that $\eta_0 < \infty$ (Theorem 4.2) if we add another conditions on f and φ such that f grows rappidly than $g(u)$ and u (see (A11)).

Remark 0.4. If $p+m > 2q$ $(2m-1 \le q)$ or $p+1 > q+m$ $(m/2 + 1/2 \le q \le 2m-1)$, equation

$$(u^{1/m})_t = u_{xx} - (u^{q/m})_x + u^{p/m} \quad (m, p, q > 1) \tag{0.8}$$

satisfies (A1)(A2)(A5)-(A7)(A10)(A11).

Remark 0.5. Condition (A2) is needed to show uniqueness of weak solution of (0.1). If uniqueness of weak solution to (0.1) holds, above theorems are given without (A2).

The methods to prove these results are essentially the same to thoes of Friedman-Lacey [5] and R. Suzuki [17]. We use smoothness, comparison principle and property of the zero set of $u_x(x,t)$.

The paper is structured as follows. In §1 we summarize above two principles and show the finite propagation of the interfaces of solutions (Theorem 1.7). In §2 we study the property of the zeroes set of $u_x(x,t)$ where the lap-number of the initial data of the solution $u(x,t)$ is two. Using this property we prove the existence of single point blow-up solutions in §3. Finally if we add some assumptions we show that the blow-up point is bounded in §4.

1. A Comparison Principole and Finite Propagation of an Interface.

In this section we begin with two proposition which will be fundamental tools in our study of interfaces and the blow-up sets.

Proposition 1.1 (Smoothness principle). *Assume* (A1)-(A3). *Let* G *be an open interval and let* u *be a solution of* (0.1) *in* $G \times (0,T)$ *in the sense of Definition* 0.1. *If* $u(\bar{x},\bar{t}) > 0$ *for some* $(\bar{x},\bar{t}) \in G\times(0,T)$, *then* u *is a classical solution in a neighborhood* W *of* $(\bar{x},\bar{t})$ *and hence* $u \in C^{\infty}(W)$.

Proof. Note that $\beta(v)$, $f(v)$, $g(v) \in C^{\infty}(\mathbb{R}_+)$ and $\beta'(v) > 0$ for $v > 0$. Then the above proposition follows from the usual parabolic regularization method (see e.g. Ladyzenskaja [13] et all).

Definition 1.2. For each open interval $G \subset \mathbb{R}$, a super-solution (or sub-solution) of (0.1) in $G \times (0,T)$ is defined by 1), 2) of Definition 0.1 with equality (0.3) replacd by $\geq$ (or $\leq$).

Proposition 1.3 (Comparison principle). *Assume* (A1)-(A3). *Let* u *(or* v*) be a super-solution (or sub-solution) of* (0.1) *in* $G \times$ (0.T). *If* $u \geq v$ *on the parabolic boundary of* $G \times (0,T)$, *then we have* $u \geq v$ *in the whole* $\bar{G} \times (0,T)$.

Proof. See e.g. Gilding [9]. □

Remark 1.4. Condition (A2) is required in the proof of Proposotion 1.3. But this condition could have replaced by weaker condition if we add some regularity conditions on $\varphi(x)$ (see Diaz-Kersner [4]).

In the rest of this section, based on these principle, we shall show finite propagation of the interface in $t < T$. Firstly we prove sevral lemmas.

Lemma 1.5 (positivity). *Assume* (A1)-(A3)(A5). *Let* u *be a weak solution of* (0.1)(0.2). *If* $u(\bar{x},\bar{t}) > 0$ *for some* $(\bar{x},\bar{t}) \in \mathbb{R}\times(0,T)$, *then*

$$u(\bar{x},\bar{t}) > 0 \quad \textit{for } t \geq \bar{t}. \tag{1.1}$$

Proof. (c.f. R.Suzuki [17], Friedman-Lacey [15]). Without loss of generality we can assume $\bar{x} = 0$. Since u is continuous in $\mathbb{R} \times [0,T)$, there exist $a_0 > 0$ and $\delta > 0$ such that

$$u(x,t) \geq a_0 \qquad \text{in } [-2\delta,2\delta] \times [\bar{t},\bar{t}+2\delta).$$

Let $\rho(t)$ be the solution to

$$\rho'(t) = -\frac{\lambda\rho}{\beta'(\rho)} \qquad \text{in } (\bar{t},\infty) \text{ with } \rho(\bar{t}) = a \tag{1.2}$$

where $\lambda = (\frac{\pi}{2\delta})^2$ and $0 < a < a_0$. Integrating this, we have

$$\rho(t) = W^{-1}(W(a) - \lambda(t-\bar{t})), \text{ where } W(s) = \int_1^s \frac{\beta'(v)}{v}\, dv \tag{1.3}$$

Note that $\beta'(v) > 0$ and $\beta''(v) \leq 0$ in $v > 0$. Then as is easily seen, $W(s)$ is increasing in $s > 0$ and $W(s) \longrightarrow -\infty$ as $s \downarrow 0$. Thus $\rho(t) > 0$ for each $t > \bar{t}$.

Now we put

$$v(x,t) = \rho(t)\sin\frac{\pi}{2\delta}(x-\delta). \tag{1.4}$$

Then since $\beta'(\rho) \le \beta'(v)$, we see

$$\beta(v)_t \le v_{xx} \qquad (x,t) \in (-\delta,\delta)\times(\bar{t},T). \tag{1.5}$$

Next we put

$$r(t) = \int_0^t \frac{g'}{\beta'}(\rho(t))\, dt. \tag{1.6}$$

Cosidering condition (A5), we can see that

$$r(t) < \infty \quad \text{for each } t \in [\bar{t},T).$$

Set

$$R_1 = \{\ 0 < x < \delta,\ 0 < t < T\ \},$$

$$R_2 = \{r(t) - r(T) < x < 0,\ 0 < t < T\},$$

and

$$R_3 = \{\ r(t) - r(T) - \delta < x < r(t) - r(T),\ 0 < t < T\ \}$$

Then we define a functon $w(x,t)$ in the following:

$$w(x,t) = v(x,t) \quad \text{in } R_1,$$

$$w(x,t) = \rho(t) \quad \text{in } R_2$$

$$w(x,t) = v(x-r(t)+r(T),t) \quad \text{in } R_3$$

$$w(x,t) = 0 \quad \text{in the else case.}$$

Note

$$r(T) \le T \max_{0\le\xi\le a} \frac{g'}{\beta'}(\xi). \tag{1.7}$$

Then, in virtue of (A5), if $a>0$ is small enough, we have

$$0 < r(T) < \delta.$$

Hence we obtain

$$w(x,\bar{t}) = 0 \qquad \text{in} \quad x \le -2\delta,\ x \ge 2\delta,$$

$$w(x,\bar{t}) \le a \qquad \text{in} \quad -2\delta \le x \le 2\delta.$$

we get

$$w(x,\bar{t}) \le u(x,\bar{t}) \quad x \in \mathbb{R}. \tag{1.8}$$

On the other hand, we computate that

$$\beta(w)_t - w_{xx} + g(w)_x$$

$$= \beta(v)_t - v_{xx} + g'(v)v_x \le g'(v)v_x \le 0 \qquad \text{in } R_1,$$

$$\text{(since } v_x \le 0 \text{ in } R_1\text{)}$$

$$\beta(w)_t - w_{xx} + g(w)_x = \beta'(\rho(t))\rho'(t) \le 0 \quad \text{in } R_2,$$

$$\text{(since } \rho'(t) \le 0\text{)}$$

and

$$\beta(w)_t - w_{xx} + g(w)_x$$

$$= \beta' w_t - w_{xx} + g'(w)w_x$$

$$= \beta' v_t - \beta' r' v_x - v_{xx} + g'(v)v_x$$

$\leq (g'(v) - r'\beta')v_x$

$=(g'(v) - \frac{g'}{\beta'}(\rho(t))\beta'(v))v_x$

$=\beta'(v)(\frac{g'}{\beta'}(v) - \frac{g'}{\beta'}(\rho(t))v_x$

≤ 0 in R_3 (since $v \leq \rho$, $v_x \geq 0$ in R_3 and (A5)).

It follows from this computation and (1.8) that we can see that $w(x,t)$ is a sub-solution of (0.1) in $\mathbb{R} \times [0,T)$. Applying the comparison theorem to w and u, we get

$$u(x,t) \geq w(x,t) \quad (x,t) \in \mathbb{R} \times [\bar{t},T) \tag{1.9}$$

namely,

$$u(0,t) \geq \rho(t) > 0 \quad t \in [\bar{t},T). \quad \square \tag{1.10}$$

This lemma and the comparison theorem implies the existence of the interface: That is, if we put

$$\xi_1(t) = \inf \{ x \mid u(x,t) > 0 \} \tag{1.11}$$

$$\xi_2(t) = \sup \{ x \mid u(x,t) > 0 \} \tag{1.12}$$

and assume (A4), then we have

$$\{ x \mid u(x,t) > 0 \} = (\xi_1(t),\xi_2(t)) \text{ for each } t \in [0,T). \tag{1.13}$$

Furthermore assume (A6), Then we can obtain the finite propagation of the interface in t < T by the following lemma:

Lemma 1.6. *Assume* (A1)-(A6). *Let* $u(x,t)$ *be a weak* *solution of* (0.1)(0.2). *Suppose that there exist* $(a,t) \in \mathbb{R} \times [0,T)$ *and* $M > 0$ *such that*

$$u(x,t_1) = 0 \quad for\ x \geq a \tag{1.14}$$

$$u(a,t) \leq M \quad for\ t \in [t_1,T). \tag{1.15}$$

Then, there exist $\ell > 0$ *and* $h > 0$ *depending on only* M *such that*

$$u(x,t) = 0\ for \quad (x,t) \in [a+\ell,\infty) \times [t_1,t_1+h]\cap[t_1,T). \tag{1.16}$$

Furthermore, if $M > 0$ *is small enough, we can take* $\ell > 0$ *small enough.*

Proof. (C.F., Lemma 2.2 and Lemma 2.3 of Mochizuki- Suzuki [15] and Knerr [12]) We construct a super-solution $w(x,t)$ of (0.1) in the following form:

$$w(x,t) = \psi^{-1}([\rho(t) - (x-a)]^+) \tag{1.17}$$

where $[g]^+ = \max\ \{g,0\}$, $\psi(u) = \int_0^u \frac{dv}{\beta(v)}$, $\rho(t) = C(M)(t-t_1)+\psi(M)$ and $C(M) = 1 + \sup\limits_{0\leq v\leq 2M} \{\frac{g'}{\beta'} + \frac{f}{\beta\beta'}\}$ (these functions are well defined since we have assumed (A5)(A6)).

In fact, in the domain $\{\{x\geq a\}\times[t_1,t_1+k]\}\cap\{\rho(t)\geq x-a\}$ where $k = C(M)^{-1}(\psi(2M)-\psi(M))$, we have $w = \psi^{-1}(\rho(t)-x+a) \leq \psi^{-1}(\rho(t_1+k)) = 2M$ and hence

$$\frac{1}{\beta'(w)}\ \psi(w)_{xx} + \left|\psi(w)_x\right|^2 - \frac{g'(w)}{\beta'(w)}\ \psi(w)_x + \frac{f(w)}{\beta(w)\beta'(w)}$$

$$\leq 1 + \frac{g'(w)}{\beta'(w)} + \frac{f(w)}{\beta(w)\beta'(w)} \leq C(M) = \partial_t \psi(w). \qquad (1.18)$$

Therefore this w is extended by 0 to the whole $\{x \geq a\} \times [t_1, t_1+k]$ as a super-solution of (0.1). For this aim we have only to note that $f(0) = 0$ and $\partial_x w(\rho(t)+a, t) = (\psi^{-1})'(0) = \beta(0) = 0$.

Moreover, we have

$$w(x,t_1) \geq 0 = u(x,t_1) \quad \text{on} \quad x \geq a,$$

$$w(x,t) \geq \psi^{-1}(\mathrm{p}(t)) \geq \psi^{-1}(\rho(t)) \geq \psi^{-1}(\psi(M)) = M \geq u(x,t) \quad \text{on } x=a, \ t \geq t_1.$$

Thus, Proposition 1.3 implies that

$$w(x,t) \geq u(x,t) \quad \text{in } \{ x \geq a \} \times [t_1, t_1 + k') \qquad (1.19)$$

where $k' = \min \{ k, T - t_1 \}$

By the property of $w(\mathrm{x,t})$, choosing $h = k'$ and $\ell = \rho(t_1+h)$, we conclude the assertion of (1.16). Since $\psi(M)$ goes to 0 as M goes to 0, we can choose k small if $M > 0$ is small enough. Hence we can choose $\ell > 0$ small enough also.

Theorem 1.7. *Assume* (A1)-(A6). *Let $u(x,t)$ be any weak solution to* (0.1)(0.2). *Then $\Omega(t)$ formes a bounded set in $\mathbb{R}$ and is nonincreasing in t:*

$$\Omega(t_1) \subset \Omega(t_2) \qquad \text{if } t_1 < t_2,$$

and there exist continuous functions $\xi_i(t) : [0,T) \longrightarrow \mathbb{R}$ $(i=1.2)$ such that

$$\Omega(t) = \{ x \mid x \in (\xi_1(t),\xi_2(t)) \}.$$

Proof. Proposition 1.3, Lemma 1.5 and 1.6 reduce to Theorem 1.7 soon. □

2. The Property of Zeroe Set of $u_x(x,t)$

Throughout this section, assume (A1)-(A6). Furthermore we assume that the lap-number of the initial data $\varphi(x)$ is two (see (A9)). In this section we prove the next proposition:

Proposition 2.1 (see Chen-Matano-Mimura [3] Proposition 2.4). *Let $u(x,t)$ and $\xi_i(t)$ be as in Theorem* 1.7. *If we assume* (A9), *then there exists a* C^1*-function* $\eta(t) : (0,T) \longrightarrow \mathbb{R}$ *such that*

$$\{ x \in (\xi_1(t),\xi_2(t) \; ; \; u_x(x,t) = 0 \} = \{\eta(t)\} \qquad (2.1)$$

for each $t \in (0,T)$ *and for some* $\delta > 0$

$$- a_1 + \delta \le \eta(t) \qquad \text{for all } t \in (0,T). \qquad (2.2)$$

First we give the following lemma :

Lemma 2.2. *Let* $\varphi_n(x)$ *be a* C^∞*-function such that* $\varphi_n(x) \ge 1/n$, $\varphi_n(\pm n) = 1/n$ *and* $\varphi_n(x)$ *converges to* $\varphi(x)$ *as* n *goes to* ∞ *locally uniformly with respect to* x. *Furthermore assume that the lap-number of* $\varphi_n(x)$ *is two and* $\sigma_a\varphi_n(x) \ge \varphi_n(x)$ *in* $x \le a$ *if* $a \le -a_1+\delta$, *in* $x \ge a$ *if* $a \ge a_1$ *for some* $\delta > 0$. *Here we note that*

$(\sigma_a x+x)/2 = a$ *and* $\sigma_a u(x) = u(\sigma_a x)$. *Let* $u_n(x,t)$ *be a classical solution of the initial boundary value problem*

$$\begin{cases} \partial_t \beta(u) - u_{xx} + g(u - 1/n)_x = f(u - 1/n) & (x,t)\in\mathbb{R}\times(0,T) \\ u(x,0) = \varphi_n(x) & x \in \mathbb{R} \times (0,T) \\ u(\pm n,t) = 1/n & t > 0 \end{cases} \tag{2.3}$$

Then $u_n(x,t) \geq 1/n$ for $(x,t) \in \mathbb{R} \times (0,T)$ *and* $u_n(x,t) \longrightarrow u(x,t)$ *as* $n \longrightarrow \infty$ *locally uniformly in* $\mathbb{R} \times [0,T)$.

Proof. see Gillding [9].

Remark 2.3. The existence of above $\varphi_n(x)$ is guaranteed by the assumption (A4)(A9).

Lemma 2.4. *Let* $u_n(x,t)$ *be as in Lemma* 2.2. *Then for each* $T' \in (0,T)$ *there exists a* C^1*-function* $\eta_n(t) : (0,T') \longrightarrow \mathbb{R}$ *for large enough* n *such that*

$$\{x \in (-n,n) \ ; \ u_{n,x}(x,t) = 0\} = \{\eta_n(t)\} \quad \text{for each } t\in(0,T'). \tag{2.4}$$

Furthermore

$$- a_1 + \delta \leq \eta_n(t) \qquad \textit{for } t \in (0,T'). \tag{2.5}$$

where $\delta > 0$ *is appeared in Lemma* 2.2.

Before we show this lemma, we need some notations and definitions (c.f. Chen-Matano [2] and R.Suzuki [17]).

Notation 2.5. Let $w(x)$ be a continuous real value function on K where K is a bounded closed interval in $\mathbb{R}$. We define the nodal number of w by

$$\nu_K(w) = \text{the number of points } x \in K \text{ with } w(x) = 0.$$

Definition 2.6. we say that $w \in C^1(K)$ poses only simple zeroes if $w'(x) = 0$. The set of all such functions is denoted by $\sum(K)$.

Lemma 2.7 *(see Angenent [1] and note R. Suzuki [17])). Let $p(x,t)$, $q(x,t)$ and $r(x,t)$ be locally bounded continuous functions on $[a,b] \times (t_0,T)$ with p_{xx}, p_{xt}, p_{tt}, p_x, p_t, q_x, q_t all locally bounded continuous. Furthermore, let $p(x,t) > 0$ and let $w(x,t)$ be a classical solution of*

$$w_t = p(x,t)w_{xx} + q(x,t)w_x + r(x,t)w \quad (x,t)\in[a,b]\times(t_0,T). \quad (2.6)$$

Assume that $w(a,t) \neq 0$ and $w(b,t) \neq 0$ for any $t \in (t_0,T)$. Then

(i) *$\nu(w(\cdot,t))$ is finite for any $t \in (t_0,T)$ and is monotone nonincreasing in t;*

(ii) *there exists a strictly decreasing sequence of points $\{t_k\}$ such that $\{t_k\} \downarrow t_0$ and $w(x,t) \in \sum([a,b])$ for any $t \in (t_0)\setminus\{t_k\}$.*

On the other hand there is the following lemma about lap-number (see Matano [14]).

Lemma 2.8 *(Matano). Let $u(x,t)$ be a solution of the following Dirichlet problem :*

$$u_t = a(x,t)u_{xx} + b(x,t)u_x + f(t,u) \quad in \quad [a,b]\times(0,T)$$

$$u(x,0) = u_0(x) \quad in \quad [a,b]$$

$$u(a,t) = u(b,t) = 0 \quad in \ (0,T).$$

where $u_0(x) \in C([a,b]\times[0,T))$, $a \in C^1([a,b]\times[0,T))$, $b \in C^\alpha([a,b]\times[0,T))$ *for some* $0<\alpha<1$, $f \in C^1([0,T)\times\mathbb{R})$, $a(x,t) \geq \delta$ *in* $[a,b]\times[0,T)$ *for some* $\delta > 0$ *and* $f(t,0) = 0$. *Then if we assume that* $u(x,t) \geq 0$ *in* $[a,b]\times[0,T)$, *the lap-number* $\ell(u(\cdot,t))$ *is decreasing in* $t \in [0,T)$.

Proof of lemma 2.4. Applying the maximum principle to $u_n(x,t)$, we obtain

$$u_n(x,t) \geq 1/n \quad \text{in} \quad x \in [-n,n],\ t > 0 \tag{2.7}$$

and

$$\pm u_x(\pm n,t) < 0 \quad \text{for} \quad t > 0. \tag{2.8}$$

Note the lap-number $\ell(u_n(\cdot,t))$ for $t>0$ equals to two by Lemma 2.8. Hence since the nodal number $\nu_{[-n,n]}(u(\cdot,t)) = 1$, it follows from lemma 2.7 that there exist C^1-function $\eta_n(t)$ such that

$$\{x \in (-n,n)\ ;\ u_{n,x}(x,t) = 0\} = \{\eta_n(t)\} \quad \text{for } t>0 \tag{2.9}$$

Next we show (2.5) (see Friedman-Lacy [5]). Choose $a \in [-n,-a_1+\delta]$ and set

$$w = u(x,t) - v(x,t) \quad \text{in} \quad [-n,a]$$

where $u = u_n$ and $v = \sigma_a u_n$. Then w satisfies equation

$$\beta'(u)w_t - w_{xx} + Cw = -\{g'(u-1/n)+g'(v-1/n)\}u_x + g'(v-1/n)w_x \qquad (2.10)$$

where $C = C(x,t) = -\dfrac{f(u-1/n) - f(v-1/n)}{u-v} + \dfrac{\beta'(u) - \beta'(v)}{u-v}\, v_t$

Furthermore if we set $h(x,t) = e^{-\gamma t}w$ where γ is choesn later, then $h(x,t)$ satisfies the following equation

$$\beta'(v)h_t - h_{xx}\ \{\gamma\beta' + C\}h$$

$$= -\{g'(u-1/n) + g'(v-1/n)\}e^{-\gamma t}u_x + g'(v-1/n)h_x \qquad (2.11)$$

Since $\beta'(v) > 0$ and $C < \infty$ for each $t \in [0,T']$ and $x \in [-n,n]$, if γ is large enough then

$$\gamma\beta'(v) + C > 0$$

Further we note

$$h(a,t) = 0$$

$$h(-n,t) = e^{-\gamma t}\{u(-n,t) - v\} = e^{-\gamma t}\{1/n - v\} \le 0 \quad \text{for } t > 0$$

$$h(x,0) = \varphi_n - \sigma_a\varphi_n \le 0 \quad \text{for } x \in [-n,a] \qquad (2.12)$$

We shall claim that $h \le 0$ in $[-n,a] \times [0,T']$. Indeed otherwise we take a positive maximum at some point $(\bar{x},\bar{t})$ in $(-n.a) \times (0,T']$. Then we have

$$h(\bar{x},\bar{t}) > 0,\ h_x(\bar{x},\bar{t}) = 0,\ h_t(\bar{x},\bar{t}) \geq 0 \text{ and } h_{xx}(x,t) \leq 0. \tag{2.13}$$

and for $u(x,t)$ also we have

$$u(\bar{x},\bar{t}) > v(\bar{x},\bar{t}) \tag{2.14}$$

$$u_x(\bar{x},\bar{t}) = v_x(\bar{x},\bar{t}). \tag{2.15}$$

Suppose $\bar{x} \leq \eta_n(\bar{t})$. Then $u_x(\bar{x},\bar{t}) \geq 0$. Noting this and (2.13), we see

$$\beta'(v)h_t - h_{xx} + \{\gamma\beta'(v) + C\}h$$

$$> -\{g'(u - 1/n) + g'(v - 1/n)\}e^{-\gamma t}u_x + g'(v - 1/n)h_x$$

at $(x,t) = (\bar{x},\bar{t})$. This contradicts to (2.11).

Next suppose $\eta_n(\bar{t}) < \bar{x}$. Then $u_x(\bar{x},\bar{t}) < 0$ and $v_x(\bar{x},\bar{t}) > 0$. These results contradict to (2.15).

Hence we obtain $h \leq 0$ in $[-n.a] \times [0,T')$, that is,

$$u_n(x,t) \leq \sigma_a u_n(x,t) \quad \text{for} \quad (x,t) \in [-n,a] \times (0,T']. \tag{2.16}$$

Therefore

$$u_{n,x}(a,t) \geq 0 \quad \text{for } t \in [0,T'] \tag{2.17}$$

Since $a \in [-n, a_1+\delta]$ is chosen arbitrarily, we get

$$\eta_n(t) \geq -a_1 + \delta.$$

This shows (2.5). The proof is complete. □

Proof of Proposition 2.1. Using lemma 2.4, lemma 2.5 and the limit proceduer of appromixmate solution u_n, we can prove Proposition 2.1. By Theorem 1.7, there exist continuous functions $\xi_i(t) : [0,T) \longrightarrow \mathbb{R}$ such that

$$\xi_1(0) = -a_1 \qquad \xi_2(0) = a_1 \quad \text{for } t \in [0,T) \tag{2.18}$$

and

$$\{ x ; u(x,t) > 0 \} = (\xi_1(t), \xi_2(t)) \quad \text{for each } t \in (0,T). \tag{2.19}$$

Hence for each $t_1 \in (0,T)$, there exist sequence $(x_j^{\pm})$ and $\delta_j > 0$ such that

$$x_j^- \longrightarrow \xi_1(t_1) \text{ and } x_j^+ \longrightarrow \xi_2(t_2) \text{ as } j \longrightarrow \infty, \tag{2.20}$$

$$\xi_1(t) < x_j^- < x_j^+ < \xi_2(t) \quad \text{for } t \in (t_1-\delta_j, t_1+\delta_j), \tag{2.21}$$

$$\pm u_x(x_j^{\pm}, t) < 0 \quad \text{for each } t \in (t_1-\delta_j, t_1+\delta_j). \tag{2.22}$$

Now we shall show that the nodal number of $u_x(\cdot, t_1)$ on $[x_j^-, x_j^+]$ is one, namely,

$$\nu_{[x_j^-, x_j^+]}(u_x(\cdot, t_1)) = 1. \tag{2.23}$$

Applying Lemma 2.7 to $u_x(\cdot, t_1)$ in $[x_j^-, x_j^+]$, we can see that $\nu_{[x_j^-, x_j^+]}(u_x(\cdot, t_1))$ is finite for each $t \in (t_1-\delta_j t_1+\delta_j)$ and that is decreasing in $t \in (t_1-\delta_j, t_1+\delta_j)$, and see that $u_x(\cdot, t_2) \in \Sigma([x_j^-, x_j^+)]$ for some $t_2 \in (t_1-\delta_j, t_1)$. Then, we get that if n is large enough,

$$\{\eta_n(t)\} \subset (x_j^-, x_j^+) \quad \text{for } t \in (t_1-\delta_j, t_1+\delta_j). \qquad (2.24)$$

In fact, assume that there exists a subsequence $\{\eta_{n_k}(t)\} \subset \{\eta_n(t)\}$ such that $x_j^+ \leq \eta_{n_k}(t)$. Then, by Lemma 2.4 we obtain that $u_{n_k}(x,t)$ is increasing in $x \in [x_j^-, x_j^+]$. Therefore, since $u_{n_k}(x,t)$ converges to $u(x,t)$ as $n_k \longrightarrow \infty$ by Lemma 2.2, we can see that $u(x,t)$ has same property as $u_{n_k}(x,t)$ and $u_x(x_j^+, t) \geq 0$. This contradicts to (2.22). In the other hand assume that there exists a subsequence $\{\eta_{n_k}(t)\} \subset \{\eta_n(t)\}$ such that $\eta_{n_k}(t) \leq x_j^-$, we can also show same contradiction.

For each $t \in (t_1-\delta_j, t_1+\delta_j)$ let $\eta_0(t)$ be accumulating point of $\{\eta_n(t)\}$. Then, since $u(\cdot, t)$ is decreasing in $x \in [x_j, \eta_0(t)]$ and $u(\cdot, t)$ is decreasing in $x \in [\eta_0(t), x_j^+]$ namely $u_x(x,t) \geq 0$ in $x \in [x_j^-, \eta_0(t)]$ and $u_x(x,t) \leq 0$ in $x \in [\eta_0(t), x_j^+]$, we obtain that $u_x(\eta_0(t), t)) = 0$ and $\eta_0(t) \in (x_j^-, x_j^+)$.

Take $t = t_2$ and assume that there is a point $x_1 \in (x_j^-, x_j)$ beside η_0 such that $u_x(x_1, t_2) = 0$. It follows from Lemma 2.7 that x_1 is a single zeroe point of $u_x(x, t_2)$ in $[x_j^-, x_j^+]$ namely, $u_{xx}(x_1, t_2) \neq 0$. This contradicts to the fact that $u_x(x, t_2) \geq 0$ [or ≤ 0] in the neighborhoods of x_1. Hence the zeroe points of $u_x(x, t_2)$ in $[x_j^-, x_j^+]$ coincides with $\eta_0(t_2)$, that is

$\nu_{[x_j^-, x_j^+]}(u_x(\cdot, t_1)) = 1.$

Since the nodal number of $u_x(\cdot, t)$ in $[x_j^-, x_j^+]$ is nonincreasing in t by Lemma 2.7, we get (2.23). Furthermore noting the cumulating point is only $\eta_0(t_1)$, we see that $\eta_n(t_1) \longrightarrow \eta_0(t_1)$ as $n \longrightarrow \infty$.

Therefore if $j \longrightarrow \infty$ in (2.23). we get

$$\nu_{(\xi_1(t_1), \xi_2(t_2))}(u_x(\cdot, t_1)) = 1.$$

and

$$\{x \in (\xi_1(t_1), \xi_2(t_1) \ ; \ u_x(x, t) = 0\} = \{\eta_0(t_1)\}.$$

Noting $t_1 \in (0, T)$ is chosen arbitrarily and $u_x(\cdot, t_1) \in \Sigma(\xi_1(t_1), \xi_2(t_1))$ by Lemma 2.7 and setting $\eta(t) = \eta_0(t)$, we have that $\{x \in (\xi_1(t), \xi_2(t)) \ ; \ u_x(x, t) = 0\} = \{\eta(t)\}$ for each $t \in (0, T)$ and $\eta(t)$ is C^1-function and that

$$\eta_n(t) \longrightarrow \eta(t) \ (n \longrightarrow \infty) \text{ for each } t \in (0, T). \qquad (2.25)$$

(2.2) is reduced by (2.5) soon. The proof is complete. □

3. Single Point Blow-up

In this section we show the existence of single point blow-up solutions of (0.1)(0.2) and study the asmyptotic behavior of an interface of the blow-up solutions. For this aim we need (A7)(A8) and the next assumptions beside the conditions (A1)-(A6)(A9).

(A10) There exists a C^2-function $F(v)$ such that

(i) $F(v),\ F'(v),\ F''(v) \geq 0$ for $v \geq 0$.

(ii) $\int_1^\infty \frac{d\xi}{F(\xi)} < \infty$

(iii) there are constants $c > 0$ and $v_0 > 0$ such that

$$f'F - F'f - \frac{1}{2}(g')^2F \geq c(F^2g'' + F'F) \quad v \geq v_0.$$

Condition (A10) shows that f grows more rappidly than g and u. Condition (A8) is required to ensure that $u(x,t)$ is increasing in t for each $x \in \mathbb{R}$. Namely,

Lemma 3.1. *Assume* (A1)-(A3) (A8). *Let $u(x,t)$ be a weak solution of* (0.1)(0.2) *in $\mathbb{R} \times (0,T)$. Then $u(x,t)$ is nondecreasing in t. If $u(x_0,t_0) > 0$ for some $(x_0,t_0) \in \mathbb{R} \times (0,T)$ then $\partial_t u(x,t) \geq 0$ in the neighborhood of (x_0,t_0).*

If we add (A7)(A10), we can get :

Lemma 3.2 *(Chen-Matano* [2], *R.Suzuki* [17]). *Assume* (A1)-(A3)(A7)(A10). *Let $\Omega = (a,b)$ be a bounded open interval and let $u(x,t)$ be a positive weak solution of* (0.1) *in $Q_T = \Omega \times (0,T)$. Furthermore suppose that*

$$u_x(x,t) > 0 \ [\text{or } u_x(x,t) < 0] \text{ in } (x,t) \in [c-\delta, c+\delta] \times (\tau,T) \quad (3.1)$$

for csome $c \in (a,b)$ and $\delta > 0$ with $(c-\delta, c+\delta) \subset (a,b)$ and some $\tau \in (0,T)$. Then there are no blow-up points in $(c-\delta, c+\delta)$.

Proof. We shall show this lemma in case

$$u_x(x,t) > 0 \quad \text{in} \quad (x,t) \in (c-\delta, c+\delta) \times (\tau, T). \tag{3.2}$$

Assume $x_0 \in (c-\delta, c+\delta)$ is a blow-up point of $u(x,t)$. Then, by (3.2) and Lemma 3.1, we soon see that

$$\lim_{t \uparrow T} u(x,t) = \infty \quad \text{for} \quad x \in (x_0, c+\delta). \tag{3.3}$$

Choose $d \in (x_0, c+\delta)$ and set

$$J = u_x - \varepsilon\rho(x)F(u(x,t) \quad (x,t) \in Q = (d, c+\delta) \times (\tau, T) \tag{3.4}$$

and

$$\rho(x) = \left[\sin \frac{\pi(x-d)}{c+\delta-d} \right]^2 \tag{3.5}$$

where $\varepsilon > 0$ and $t_1 \in (\tau, T)$ is chosen later. We computate

$$(\beta' J)_t - J_{xx}$$

$$= \varepsilon\rho A(x,t) + B(x,t)J - g'J_x - \varepsilon\beta' F u_t + \varepsilon\rho F''(u_x)^2 \tag{3.6}$$

where

$$A(x,t) = f'F - F'f + \{ -\frac{\rho'}{\rho}g' + \frac{\rho''}{\rho} \}F - \varepsilon\{\rho F^2 g'' + 2\rho' F' F\}$$

and

$$B(x,t) = f' + \varepsilon\rho F'g' + 2\varepsilon\rho' F' - \varepsilon g'\rho F' - g''J - 2\varepsilon g''\rho F.$$

Here we used the relation that

$$u_{xx} = J_x + \varepsilon\rho' F + \varepsilon\rho F' J + \varepsilon^2\rho^2 F' F$$

and

$$(u_x)^2 = J^2 + 2\varepsilon\rho F J + \varepsilon^2\rho^2 F^2.$$

If we note that

$$\rho' = 2\lambda \sin\lambda(x-d)\cdot\cos\lambda(x-d)$$

and

$$\rho'' = 2\lambda^2(1-2\rho) \quad \text{where } \lambda = \frac{\pi}{c+\delta-d},$$

we get that

$$-\frac{\rho'}{\rho}g' + \frac{\rho''}{\rho} = \frac{-2g'\sin\lambda(x-d)\cdot\cos\lambda(x-d) + 2\lambda^2(1-2\rho)}{\{\sin\lambda(x-d)\}^2}$$

$$= -4\lambda^2 + \frac{2\lambda\{\lambda - \cos\lambda(x-d)\cdot\sin\lambda(x-d)\cdot g'\}}{\{\sin\lambda(x-d)\}^2}$$

$$\geq -4\lambda^2 + \frac{2\lambda\{\lambda - \sin\lambda(x-d)\cdot|g'|\}}{\{\sin\lambda(x-d)\}^2}$$

Hence putting $\theta = \sin\lambda(x-d)$, we can write above inequality in the following :

$$-\frac{\rho'}{\rho}g' + \frac{\rho''}{\rho} \geq -4\lambda^2 + \frac{2\lambda(\lambda-\theta|g'|)}{\theta^2}$$

where $0 \leq \theta \leq 1$.

Set $h(\theta) = -4\lambda + \dfrac{2\lambda(\lambda-\theta|g'|)}{\theta^2}$ and assume that θ is indpendent of g'. Then, we see that h(θ) takes a minimum value $h(\frac{2\lambda}{|g'|}) = -4\lambda^2 - \frac{1}{2}(g')^2$ at $\theta = \frac{2\lambda}{|g'|}$ since $h'(\theta) = -4\lambda^2\theta^{-3} + 2\lambda\theta^{-2}g'$ and $h(\theta) = 0$ reduces to $\theta = \frac{2\lambda}{|g'|}$. Therefore we have

$$-\frac{\rho'}{\rho}g' + \frac{\rho''}{\rho} \geq -4\lambda^2 - \frac{1}{2}(g')^2 \tag{3.7}$$

Thus we get lower bounded estimates of $A(x,t)$:

$$A(x,t) \geq f'F - F'f + (4\lambda^2 + \tfrac{1}{2}(g')^2)F - \varepsilon\{\rho F^2 g'' + 2|\rho'|F'F\}. \tag{3.8}$$

Considering (A10), Lemma 3.1 and the fact that $F'(u)$ goes to infinity as $u \longrightarrow \infty$ if t_1 is closed enough to T, we have

$$(\beta' J)_t - J_{xx} \geq B(x,t)J - g'J_x. \tag{3.9}$$

On the other hand

$$J(d,t) = u_x(d,t) > 0, \quad J(c+\delta,t) = u_x(c+\delta,t) > 0 \tag{3.10}$$

and

$$J(x,t_1) > 0 \quad \text{(by (3.2)) for small enough } \varepsilon > 0. \tag{3.11}$$

Hence applying the maximum principle to $J(x,t)$, we obtain

$$J(x,t) > 0 \quad \text{in} \quad (t_1,T) \times (d,c+\delta),$$

namely,

$$\frac{u_x}{F} > \varepsilon\rho \quad \text{in } (t_1,T) \times (d,c+\delta). \tag{3.12}$$

Integrating this enequality over $d \le x \le c+\delta$ yieds

$$\int_{u(d,t)}^{u(c+\delta,t)} \frac{du}{F(u)} > \varepsilon\int_{d}^{c+\delta} \rho(x)\, dx \quad \text{in } t_1 < t < T. \tag{3.13}$$

The right-hand side of (3.13) is a positive constant, while the left-hand side tends to zero as $t \uparrow T$ by virtue of condition (A10)(ii) and (3.3). This contradiction shows that x_0 is a not blow-up point of $u(x,t)$. The proof is complete. □

Theorem 3.3. *Let* $u(x,t)$ *and* $\xi_i(t)$ *be as in Teorem* 1.7 *and* S *be a blow-up set of* $u(x,t)$. *Furthermore assume* (A7)-(A10). *Then*

$$S = \{ \eta_0 \} \tag{3.14}$$

for some $\eta_0 \in [-a_1 + \delta,\infty]$ *with a small* $\delta > 0$ *and*

$$- \infty < \lim_{t\uparrow T} \xi_1(t). \text{ (3.15)} \tag{3.15}$$

Proof (see Friedman-Lacey [17]). By Proposition 2.1. there exists a C^1-function $\eta(t) : (0,T) \longrightarrow \mathbb{R}$ such that

$$\{ x \in (\xi_1(t),\xi_2(t) \; ; \; u_x(x,t) = 0 \} = \{ \eta(t) \} \tag{3.16}$$

for each $t \in (0,T)$ and

$$-a_1 + \delta \le \eta(t) \quad \text{for all } t \in (0,T) \tag{3.17}$$

where $\xi_i(t)$ is defined by (1.11)(1.12). Therefore we see

$$u_x(x,t) > 0 \quad \text{for} \quad \xi_1(t) < x < -a_1+\delta, \; 0 < t < T \tag{3.18}$$

and it follows from Lemma 3.1 and Lemma 3.2 that

$$\{ x \; ; \; x < -a_1+\delta \} \subset S^c \tag{3.19}$$

where S is a blow-up set of $u(x,t)$.

Here if show

$$\lim_{t\uparrow T} \eta(t) = \eta_0 \quad \text{exists} \tag{3.20}$$

then by Lemma 3.1 and Lemma 3.2 we can obtain the results of Theorem 3.3. Hence we shall show (3.20).

Assume that $\lim_{t\uparrow T} \eta(t)$ doesn't exist. Then if we set $\eta_- = \liminf_{t\uparrow T} \eta(t)$ and $\eta_+ = \limsup_{t\uparrow T} \eta(t)$,

$$-a_1 + \delta \le \eta_- < \eta_+ \le \infty. \tag{3.21}$$

Choose $-a_1 + \delta < s_1 < \eta_-$ and $\eta_- < s_2 < \eta_+$ such that

$$\alpha = (s_1 + s_2)/2 \in (\eta_-,\eta_+). \tag{3.22}$$

Then, since $\lim_{t\uparrow T} u(\eta(t),t) = \infty$, by Lemma 3.1 we get $\lim_{t\uparrow T} u(x,t) =$

∞ for each $x \in (\eta_-,\eta_+)$. Hence, if T_0 is chosen close enough to T, we obtain

$$u_x(x,T_0) > 0 \ , \text{ for } s_1 < x < s_2 \tag{3.23}$$

and

$$u(s_1,t) < u(s_2,t) \quad \text{for} \quad t \in (T_0,T). \tag{3.24}$$

Set $w = u - v$ where $v(x,t) = \sigma_\alpha u = u(2\alpha - x,t)$ and consider $w(x,t)$ in the rectangle reigion $R = \{ s_1 < x < \alpha, T_0 < t < T \}$. Then we see

$$w(x,T_0) = u(x,T_0) - \sigma_\alpha u(x,T_0) \le 0 \text{ in } [s_1,\alpha] \tag{3.25}$$

and

$$w(x,s_1) = u(s_1,t) - u(s_2,t) \le 0 \text{ in } t \in [T_0,T). \tag{3.26}$$

By the same methods as it to show (2.16) we obtain

$$w(x,t) \le 0 \quad \text{for} \quad (x,t) \in [s_1,\alpha]\times[T \ ,T_0). \tag{3.27}$$

Since $w(\alpha,t) = 0$, we get

$$\frac{1}{2}w_x(a,t) = u_x(\alpha,t) \ge 0 \quad \text{for} \quad t \in [T_0,T). \tag{3.28}$$

This is contradiction to $\alpha \in (\eta_-,\eta_+)$. Therefore we obtain (3.20) and $S = \{\eta_0\}$. The proof is complete. □

4. The Upper Bound Estimates and Bounded Point Blow-up

In this section we show $\eta_0 < \infty$ where η_0 is appeared in Theorem 3.3. In order to show this, we need the upper bound estimates of the blow-up solution of (0.1)(0.2). For this aim, we further assume the following another assumptions for $f(u)$ and the initial data φ such that $f(u)$ grows more rappidly than u and $g(u)$:

(A11) there exists a C^2-function $\Phi(v)$ such that

(i) $\Phi, \Phi', \Phi'' > 0$ for $v > 0$ and $\Phi(0) = 0$

(ii) $\displaystyle\int_1^\infty \frac{d\xi}{\Phi(\xi)} < \infty$

(iii) there are constants $C > 0$ and $v_1 > 0$ such that

$$4\Phi''(f'\Phi - \Phi' f) \geq (g'')^2 \Phi \qquad \text{for } v \geq v_1$$

and

$$\frac{(g'')^2\Phi}{4\Phi''\beta'} + \frac{f\Phi'}{\Phi\beta'} < C \qquad \text{for } 0 \leq v \leq v_1$$

(iv) $$\int_0^1 \sup_{0 \leq v \leq H^{-1}(t) + 1} \frac{g'(v)}{\beta'(v)} dt < \infty$$

where $H(\xi) = \displaystyle\int_\xi^\infty \frac{d\eta}{\Phi(\eta)}$

(v) $\varphi'' - g(\varphi)' + f(\varphi) \geq \Phi(\varphi)$ in $\mathcal{D}'$.

Lemma 4.1. *Assume* (A1)-(A11). *Let* $u_n(x,t)$ *be a solution of the regularized problem* (2.3) *with blow-up time* T_n. *Let's add the following condition to* $\varphi_n(x)$ *appeared in Lemma* 2.2

$$\varphi_n'' - \{g(\varphi_n - 1/n)\}' + f(\varphi_n - 1/n) \geq \Phi(\varphi_n - 1/n) \quad \text{in } \mathcal{D}'. \qquad (4.1)$$

(the existence of above $\varphi_n(x)$ *is guaranteed by the assumption* (A11)(v)). *Then, for some* $c_1 > 0$.

$$u_n(x,t) \leq H^{-1}(c_1(T_n' - t)) + 1 \quad \text{for } (x,t) \in \mathbb{R} \times [0,T_n') \qquad (4.2)$$

where $H(\xi) = \int_\xi^\infty \frac{d\eta}{\Phi(\eta)}$ *and* $T_n' = \text{nin } \{T_n, T\}$

Proof (see Friedman-Lacey). Set

$$J = u_t - c(t)\Phi(u - 1/n) \qquad (4.3)$$

where $u = u_n$, $c(t) = e^{-Mt}$ and M is positive constant chosen later. We computate $(\beta' J)_t - J_{xx}$ in the following :

$$(\beta' J)_t - J_{xx} = B(x,t)J - g' J_x + c(t)A(x,t)$$

where

$$A(x,t) = f'\Phi - \Phi' f - c\beta''\Phi^2 + M\beta'\Phi - g''\Phi u_x + \Phi''(u_x)^2$$

and

$$B(x,t) = -g'' u_x - c\beta''\Phi + f'.$$

Here we used the next relation :

$$u_{xt} = J_x + c\Phi' u_x.$$

We further computate that

$$A(x,t) = \Phi''\{(u_x)^2 - \frac{g''}{\Phi''}\Phi u_x\} + f'\Phi - \Phi' f - c\beta''\Phi^2 + M\beta'\Phi$$

$$= (u_x - \frac{g''}{2\Phi''}\Phi)^2 - \frac{(g'')^2\Phi^2}{4\Phi''} + f'\Phi - \Phi' f$$

$$\geq \frac{4\Phi''[f'\Phi - \Phi' f - c\beta''\Phi^2 + M\beta'\Phi] - (g'')^2\Phi^2}{4\Phi''}$$

Choosing $M = C$ and noting $\beta'' \leq 0$, we get

$$4M\Phi''\Phi\beta' - 4\Phi''\Phi' f - (g'')^2\Phi^2$$

$$= 4\beta'\Phi''\Phi\{C - \frac{\Phi' f}{\Phi\beta'} - \frac{(g'')^2\Phi}{4\Phi''\beta'}\} \geq 0 \quad \text{for } 0 \leq v \leq v_1.$$

Hence considering condition (A11), we see that

$$A(x,t) \geq 0. \tag{4.4}$$

Thus we have

$$(\beta' J)_t - J_{xx} \geq B(x,t)J - g' J_x. \tag{4.5}$$

On the other hand

$$J(\pm n,t) = u_t(\pm n,t) - c(t)\Phi(u_n(\pm n,t) - 1/n)$$

$$= u_{n,t}(\pm n, t) - c(t)\Phi(0) = 0 \qquad (4.6)$$

and

$$J(x,0) = u_{n,t}(x,0) - \Phi(u_n(x,0) - 1/n)$$

$$= \varphi'' - \{g(\varphi_n - 1/n)\}' + f(\varphi_n - 1/n) - \Phi(\varphi_n - 1/n)$$

$$\geq 0 \qquad (\text{by } (4.1)). \qquad (4.7)$$

Applyiong the maximum principle to J, we obtain

$$J(x,t) \geq 0 \quad \text{in} \quad [-n,n] \times [0,T_n'), \qquad (4.8)$$

that is,

$$u_{n,t} \geq e^{-Mt}\Phi(u_n - 1/n) \geq c_1\Phi(u_n - 1/n) \qquad (4.9)$$

where $c_1 = e^{-Mt}$

Let $T' \in (0,T)$ fixed and choose n large enough such that $T' < T_n$. Furthermore, integrate (4.9) over $[t,T')$ for each $x \in [-n,n]$. Then

$$\int_t^{T'} \frac{u_{n,t}}{c_1\Phi(u_n - 1/n)}\, dt \geq T' - t. \qquad (4.10)$$

Setting $H(\xi) = \int_\xi^\infty \frac{d\eta}{\Phi(\eta)}$, we have

$$-\frac{1}{c_1}[\, H(u_n - 1/n)\,]_t^{T'} \geq T' - t,$$

that is,

$$-\frac{1}{c_1}\{ H(u_n(x,T') - 1/n) - H(u_n(x,t) - 1/n)\} \geq T' - t$$

Therefore

$$H(u_n(x,t) - 1/n) \geq c_1(T' - t). \tag{4.11}$$

Since $H(\xi)$ is a decreasing function in ξ, we obtain

$$u_n(x,t) - 1/n \leq H^{-1}(c_1(T'-t))$$

namely,

$$u_n(x,t) \leq H^{-1}(c_1(T' - t)) + \frac{1}{n} \leq H^{-1}(c_1(T' - t)) + 1.$$

The proof is complete. □

Theorem 4.2. *In Theorem* 3.3, *if we further assume* (A11), *we get*

$$n_0 < \infty \tag{4.12}$$

and

$$\lim_{t \uparrow T} \xi_2(t) < \infty \tag{4.13}$$

Proof. Let's $T' \in (0,T)$ be fixed and set $h(t) = H^{-1}(c_1(T'-t))+1$. Then using Lemma 4.2, if n is large enough, $T' < T'_n$ and

$$(4.14) \qquad u_n(x,t) \leq H^{-1}(c_1(T'_n - t)) + 1 \leq h(t). \tag{4.14}$$

Put

$$v(x,t) = u_n(x + k(t),t) \tag{4.15}$$

where $k(t) = \int_0^t \ell(h(s))ds$ and $\ell(\xi) = \sup_{0\le v\le \xi} \frac{g'(v)}{\beta'(v)}$.

Then $v(x,t)$ satisfies the following equation :

$$\beta'(v)v_t - v_{xx} = \beta'(\ k' - \frac{g'(v - 1/n)}{\beta'(v)}\ )v_x + f(v - 1/n).$$

Since $g'(v)$ is a decreasing function in v, we get

$$\frac{g'(v(x,t) - 1/n)}{\beta'(v(x,t))} \le \frac{g'(v)}{\beta'(v)} = \ell(v) \le \ell(h(t)) = k'(t).$$

Hence we obtain

$$k'(t)- \frac{g'(v-1/n)}{\beta'(v)} \ge 0 \text{ for } (x,t) \in [-k(t)-n,-k(t)+n]\times[0,T'). \tag{4.16}$$

On the other hand, noting condition (A11)(iv) we see that

$$\begin{aligned} k(t) \le k(T') &= \int_0^{T'} \ell(H^{-1}(c_1(T' - s)) + 1)\ ds \\ &= \frac{1}{c_1} \int_0^{c_1T'} \ell(H^{-1}(t) + 1)\ dt \quad (\text{put } t=c_1(T'-s)) \\ &\le k(T) = \frac{1}{c_1}\int_0^{c_1T} \ell(H^{-1}(t) + 1)\ dt \\ &< \infty \qquad \text{for t} \in (0,T'). \end{aligned} \tag{4.17}$$

Thus for each $a \geq a_1$, there exists N such that

$$a \leq n - k(T) \leq n - k(t) \text{ for all } n \geq N \text{ and each } t \in [0,T'] \quad (4.18)$$

Put $\tilde{v}(x,t) = \sigma_a v = v(2a - x,t)$. Then we can onsider $w = v(x,t) - \tilde{v}(x,t)$ in $\bigcup_{0 \leq t \leq T'} [a,n - k(t)] \times \{t\}$ since $n - k(T) \leq n - k(t)$ for $t \in [0,T']$. As $\tilde{v}$ satisfies equation

$$\beta'(v)w_t - w_{xx} + c(x,t)w$$

$$= \{ \beta'(\tilde{v})(k' - \frac{g'(\tilde{v} - 1/n)}{\beta'(\tilde{v})}) + \beta'(v)(k' - \frac{g'(v-1/n)}{\beta'(v)} \}v_x$$

$$- \{ \beta'(\tilde{v})(k' - \frac{g'(\tilde{v} - 1/n)}{\beta'(\tilde{v})}) \}w_x \quad (4.19)$$

where $c(x,t) = - \frac{f(v - 1/n) - f(\tilde{v} - 1/n)}{v - \tilde{v}} + \frac{\beta'(v) - \beta'(\tilde{v})}{v - \tilde{v}}\tilde{v}_t$. Furthermore, if we set $h(x,t) = e^{-\gamma t}w$ where γ is chosen later, then we obtain the following equation with respect to h :

$$\beta'(v)h_t - h_{xx} + \{\gamma\beta'(v) + c\}h$$

$$= \{ \beta'(\tilde{v})(k' - \frac{g'(\tilde{v} - 1/n)}{\beta'(\tilde{v})}) + \beta'(v)(k' - \frac{g'(v-1/n)}{\beta'(v)})\}e^{-\gamma t}v_x$$

$$- \{ \beta'(\tilde{v})(k' - \frac{g(v - 1/n)}{\beta'(\tilde{v})})\}h_x \quad (4.20)$$

Since $\beta'(v) > 0$ and $c < \infty$, choosing γ large enough we get

$$\gamma\beta'(v) + c > 0. \quad (4.21)$$

On the other hand, as $u(x,t) \geq \frac{1}{n}$ for $(x,t) \in [-n,n] \times [0,T')$ and

$v(n - k(t),t) = u(n,t) = \frac{1}{n}$, we obtain

$$h(n-k(t),t) = e^{-\gamma t}\{v(n-k(t),t) - \sigma_a v(n-k(t),t)\}$$

$$= e^{-\gamma t}\{ \frac{1}{n} - \sigma_a v(n - k(t),t)\} \le 0 \qquad (4.22)$$

and

$$h(a,t) = e^{-\gamma t}\{v(a,t) - v(a,t)\} = 0 \qquad (4.23)$$

Noting condition for φ_n in Lemma 2.2, we also obtain

$$h(x,0) = u_n(x + k(0),0) - \sigma_a u_n(x + k(0),0)$$

$$= \varphi_n(x) - \sigma_a \varphi_n(x) \le 0 \quad \text{for each } x \in [a,n]. \qquad (4.24)$$

We claim that $h \le 0$ in $\bigcup_{0 \le t \le T'} [a,n - k(t)] \times \{t\}$. Suppose $(\bar{x},\bar{t}) \in \bigcup_{0 < t \le T'} (a,n - k(t)) \times \{t\}$ is a maximum point and $h(\bar{x},\bar{t}) > 0$. Then

$$h_t(\bar{x},\bar{t}) \ge 0,\ h_{xx}(\bar{x},\bar{t}) \le 0,\ h_x(\bar{x},\bar{t}) = 0, \qquad (4.25)$$

namely

$$v(\bar{x},\bar{t}) > \tilde{v}(\bar{x},\bar{t}), \qquad (4.26)$$

$$v_x(\bar{x},\bar{t}) = \tilde{v}_x(\bar{x},\bar{t}). \qquad (4.27)$$

Assume $\bar{x} \ge \eta_n(\bar{t}) - k(\bar{t})$. Then

$$v_x(\bar{x},\bar{t}) \le 0.$$

Noting this , (4.16) and (4.25). We see that

$$\beta'(v)h_t - h_{xx} + \{\gamma\beta'(v) + c\}h$$

$$> \{ \beta'(\tilde{v})(k' - \frac{g'(\tilde{v} - 1/n)}{\beta'(\tilde{v})}) + \beta'(v)(k' - \frac{g'(v-1/n)}{\beta'(v)})\}e^{-\gamma t}v_x$$

$$- \{ \beta'(\tilde{v})(k' - \frac{g(v - 1/n)}{\beta'(\tilde{v})})\}h_x$$

This contradicts to ((4.20).

On the other hand, assume $\bar{x} < \eta_n(\bar{t}) - k(\bar{t})$. Then $v_x(\bar{x},\bar{t}) >$ 0 and $v_x(\bar{x},\bar{t}) < 0$. These results contradicts to (4.27). Hence we obtain $h \le 0$, that is,

$$w(x,t) \le 0 \quad (x,t) \in [a,n-k(t)) \times[0,T']. \qquad (4.28)$$

or,

$$u_n(x+k(t),t) \le u_n(2a-x+k(t),t) \quad (x,t)\in[a,n-k(t)]\times[0,T'] \quad (4.29)$$

Here, if n goes to ∞, then

$$u(x+k(t),t) \le u(2a-x+k(t),t) \quad \text{for} \quad (x,t) \in [a,\infty)\times[0,T'].(4.30)$$

Putting $x' = x + k(t)$, $a' = a + k(t)$ and noting $k(t) \le k(T)$, we obtain for each $a' \in [a_1 + k(T),\infty)$

$$u(x',t) \le \sigma_{a'}\mathrm{u}(x',t) \quad \text{for } (x',t) \in [0,T'].$$

Since this shows that $u(x,\mathrm{t})$ is decreasing function in $x \in$

$[a_1+k(T),\infty)$ for $t \in [0,T']$, we get

$$\eta(t) \le a_1 + k(T).$$

As $T' \in (0,T)$ is chosen arbitrarily, we conclude

$$\eta(t) \le a_1 + k(T) \quad \text{for } t \in [0,T). \tag{4.31}$$

Hence we get

$$\eta_0 = \lim_{t\uparrow T} \eta(t) \le a_1 + k(T)$$

and noting Thorem 3.3 we obtain

$$S = \{\eta_0\}.$$

Therefore by virtue of Lemma 1.6. We also obtain (4.13). The proof is complete. □

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