THE CAUCHY PROBLEM FOR THE SCHRÖDINGER TYPE EQUATIONS

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THESIS

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1. Introduction

We consider the following Cauchy problem in $(0,T) \times \mathbb{R}^n$

$$(1.1) \qquad \begin{cases} L[u(t,x)] = f(t,x), & (t,x) \in (0,T) \times \mathbb{R}^n \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n \end{cases}$$

where $L = \partial_t - \frac{\sqrt{-1}}{2}\Delta + \sum_{j=1}^n a_j(t,x)\partial/\partial x_j + b(t,x)$ and $\Delta = \sum_{j=1}^n \partial^2/\partial x_j^2$. When the coefficients $a_j(t,x)$ are complex valued functions, the equation (1.1) is called a Schrödinger type equation. It is well known that if $a_j(t,x)$ are real valued smooth functions, the Cauchy problem is well-posed in $L^2(\mathbb{R}^n)$. But if the imaginary parts of $\{a_j(t,x)\}$ do not vanish identically, the Cauchy problem (1.1) is not necessarily well posed in $L^2(\mathbb{R}^n)$ nor in $H^\infty = \bigcap_{s>0} H^s$, where H^s stands for a usual Sobolev space in $\mathbb{R}^n$.

Let $T > 0$ and $\kappa > 0$. We say that the Cauchy problem (1.1) is H_κ^∞ (respectively. H^∞)-well-posed in $[0,T]$, if for any $u_0 \in H_\kappa^\infty$ (respectively. H^∞) and $f \in C^0([0,T]; H_\kappa^\infty)$ (respectively. $C^0([0,T]; H^\infty)$ there is a unique solution $u \in C^1([0,T]; H_\kappa^\infty)$ (respectively. $C^1([0,T]; H^\infty)$ of (1.1). Furthermore we say that the Cauchy problem (1.1) is L^2-well-posed in $[0,T]$, if the Cauchy problem (1.1) is H^∞-well-posed in $[0,T]$ and the solution of (1.1) satisfies

$$\|u(t)\|_{L^2} \leq C(T)\Big\{\|u_0\|_{L^2} + \int_0^t \|f(s)\|_{L^2} ds\Big\}$$

for $t \in [0,T]$.

In [8], Mizohata gave a necessary condition for the Cauchy problem (1.1) to be well-posed in $L^2(\mathbb{R}^n)$ as following

$$(*) \qquad \sup_{\omega \in S^{n-1},(x,t)\in\mathbb{R}^n\times\mathbb{R}} \left| \sum_{j=1}^n \int_0^t \Im a_j(x + \omega\cdot\tau)\omega_j d\tau \right| < +\infty$$

where $\Im a_j$ is the imaginary part of a_j.

And in [5] Ichinose gave a necessary condition for the Cauchy problem (1.1) to be well-posed in H^∞ such as for any $\rho \geq 0$ there are some positive constants C and C' such that

$$(**) \qquad \sup_{x\in\mathbb{R}^n,\omega\in S^{n-1}} \left| \sum_{j=1}^{n} \int_0^\rho \Im a_j(x+\theta\omega)\omega_j d\theta \right| \leq C\log(1+\rho) + C'.$$

We can find in [8] a sufficient condition for the Cauchy problem (1.1) to be well-posed in $L^2(\mathbb{R}^n)$ that is under $(*)$ for any multi indices α ($|\alpha| \geq 1$)

$$\sup_{x\in\mathbb{R}^n,\omega\in S^{n-1}} \sum_{j=1}^{n} \int_0^\infty |\partial_x^\alpha a_j(x+\omega\tau)| d\tau < +\infty.$$

In [9], Takeuchi claimed the following conditions are sufficient ones (1.1) to be well-posed in $L^2(\mathbb{R}^n)$

$$\begin{cases} |\partial_x^\alpha \Im a(x)| \leq C_\alpha \langle x\rangle^{-1-\varepsilon_0-|\alpha|}, \\ |\partial_x^\alpha \Re a(x)| \leq C_\alpha \langle x\rangle^{-1}, \qquad (|\alpha| \geq 1) \end{cases}$$

where $\langle x\rangle = (1+|x|^2)^{1/2}$ and $\Re a_j$ is the real part of a_j.

In [2], we can find a sufficient condition for the Cauchy problem (1.1) to be well-posed in H^∞

$$\begin{cases} |\Im a_j(t,x)| \leq C\langle x\rangle^{-1}, \\ |\partial_x^\alpha a_j(t,x)| \leq C_\alpha \langle x\rangle^{-1} \end{cases}$$

for any α ($|\alpha| \geq 1$).

In [4], Ichinose gave a sufficient condition for the Cauchy problem (1.1) to be well-posed in H^∞ which under $(**)$ for any α ($|\alpha| \geq 1$)

$$\begin{cases} (i) \sup_{x\in\mathbb{R}^n,\omega\in S^{n-1}} \sum_{j=1}^{n} \int_0^{+\infty} |\partial_x^\alpha a_j(x+\theta\omega)| d\theta < +\infty, \\ (ii) \sup_{S\in\mathcal{T}} \left| \sum_{i<j} \iint_S \left(\partial_{x_i}\Im a_j - \partial_{x_j}\Im a_i\right) dx_i \wedge dx_j \right| \leq +\infty \end{cases}$$

where $\int\int_S(\cdots)dx_i \wedge dx_j$ denotes the integral of two form over S.

In [10], Takeuchi gave a sufficient condition for the Cauchy problem (1.1) to be well-posed in H^∞ which under Ichinose's condition (i)

$$\sup_{\rho\geq 0, x\in\mathbb{R}^n, \omega\in S^{n-1}} \left| \sum_{j=1}^n \int_0^\rho \Im a_j(x+s\omega)\omega_j ds - \kappa(\omega)\log\frac{G(x+\rho\omega)}{G(x)} \right| < +\infty.$$

In this paper we shall investigate the Cauchy problem (1.1) in Gevrey classes when $\{a_j(t,x)\}$ does not satisfy the necessary conditions derived in [5,8]. To do so, we introduce some function spaces. For a topological space X and an interval I in $\mathbb{R}$ we denote by $C^k(I;X)$ the set of functions which are k times continuously differentiable with respect to $t\in I$ in X. For $m\in\mathbb{R}$, $\rho>0$ and $\kappa>0$ we define a Hilbert space $H^m_{\kappa,\rho} = \{u(x)\in L^2(\mathbb{R}^n_x); \langle\xi\rangle^m e^{\rho\langle\xi\rangle^{1/\kappa}}\hat{u}(\xi)\in L^2(\mathbb{R}^n_\xi)\}$, where $\hat{u}(\xi)$ stands for the Fourier transform of u and $\langle\xi\rangle = \{1+\xi_1^2+\cdots+\xi_n^2\}^{1/2}$. For $\rho<0$ we define $H^m_{\kappa,\rho}$ as the dual space of $H^{-m}_{\kappa,-\rho}$. For $\rho=0$ define $H^m_{\kappa,0}=H^m$ the usual Sobolev space in $\mathbb{R}^n$. Then note that the dual space of $H^m_{\kappa,\rho}$ becomes $H^{-m}_{\kappa,-\rho}$ for any $\rho, m\in\mathbb{R}$. Denote $H^m_\kappa = \bigcup_{\rho>0} H^m_{\kappa,\rho}$ and $H^\infty_\kappa = \bigcap_{m\in\mathbb{R}} H^m_\kappa$. We say $a(x)$ belongs to the set of functions $\mathcal{B}_{\kappa,A}$ that there are constants $A>0$ and $\kappa>0$ such that

$$|D_x^\alpha a(x)| \leq CA^{|\alpha|}|\alpha|!^\kappa \tag{1.2}$$

for any $x\in\mathbb{R}^n$ and $\alpha\in\mathbb{N}^n$, where $D_{x_j} = -\sqrt{-1}\partial/\partial x_j$ and $D_x^\alpha = D_{x_1}^{\alpha_1}\cdots D_{x_n}^{\alpha_n}$. Define $\mathcal{B}_\kappa = \bigcup_{A>0}\mathcal{B}_{\kappa,A}$ and denote by $\mathcal{B}^\infty$ the set of functions whose all derivatives are bounded in $\mathbb{R}^n$. We note that $H^\infty_\kappa \subset \mathcal{B}_\kappa$, that is, H^∞_κ is a subspace of the Gevrey class of exponent κ and that H^∞_1 is the space of real analytic functions defined in $\mathbb{R}^n$.

Now we can state our theorems.

Theorem 1.1. *Assume that a_j $(j = 1, \cdots, n)$ and b are in $C^0([0,T]; \mathcal{B}^\infty)$ and there are positive constants C and $\sigma \geq 1$ such that*

$$\Im a_j(t,x) = \mathcal{O}\left(|x|^{-\sigma}\right) \quad (|x| \to \infty) \tag{1.3}$$

uniformly in $t \in [0,T]$. Then the Cauchy problem (1.1) is H^∞-well-posed in $[0,T]$ if $\sigma = 1$ and L^2-well-posed in $[0,T]$ if $\sigma > 1$.

Theorem 1.2. *Let $T > 0$. Assume that the coefficients $a_j(t,x)$ and $b(t,x)$ belong to $C^0([0,T]; \mathcal{B}_{s_0,A_0})$ and moreover that the imaginary parts of $a_j(t,x)$ $(j = 1, \cdots, n)$ satisfy*

$$\Im a_j(t,x) = o(|x|^{-\sigma}) \quad (|x| \to \infty) \tag{1.4}$$

uniformly in $t \in [0,T]$. Then if $0 < \sigma < 1$, $\kappa > 1$ and $s_0 \leq \kappa \leq (1-\sigma)^{-1}$ the Cauchy problem (1.1) is H^∞_κ-well-posed in $[0,T]$.

We note that if the condition (1.3) is verified, the imaginary parts of a_j satisfy $(*)$ and $(**)$. Theorem 1.1 is a generalization of the results which are obtained in [2,3], where they imposed some technical conditions on the real parts of a_j.

To prove Theorem 1.1 and Theorem 1.2, by use of the idea introduced in [2] and [3] we shall seek a function $\Lambda(t,x,\xi)$ defined in $[0,T] \times \mathbb{R}^n \times \mathbb{R}^n$ which satisfies the conditions

$$\Lambda_t(t,x,\xi) + \sum_{j=1}^n \xi_j \Lambda_{x_j}(t,x,\xi) + \sum_{j=1}^n \Im a_j(t,x)\xi_j \leq 0, \tag{1.5}$$

$$|\Lambda(t,x,\xi)| \leq \begin{cases} \rho_1 \langle\xi\rangle^{1/\kappa} & (\sigma < 1) \\ C \log(2 + |\xi|) & (\sigma = 1) \\ C & (\sigma > 1) \end{cases} \tag{1.6}$$

for x, $\xi \in \mathbb{R}^n$ and $t \in [0,T]$, where $\Lambda_t = \partial\Lambda/\partial t$ and $\Lambda_{x_j} = \partial\Lambda/\partial x_j$. Besides, we change the unknown function u as $v(t,x) = e^{\Lambda}(t,x,D)u(t,x)$, where $e^{\Lambda}(t,x,D)$ stands for the pseudo-differential operator with symbol $e^{\Lambda(t,x,\xi)}$. Then we can prove that in the case of $\sigma < 1$ $e^{\Lambda}(t,x,D)$ is continuous from $H^0_{\kappa,\rho}$ to $L^2(\mathbb{R}^n)$ if $\rho > \rho_1$, where ρ_1 is given in (1.6) and that in the case of $\sigma = 1$ (respectively. $\sigma > 1$) $e^{\Lambda}(t,x,D)$ is also continuous from H^∞ (respectively. L^2) to L^2 and has the inverse $(e^{\Lambda}(t,x,D))^{-1}$. Moreover we can write $L_\Lambda = e^{\Lambda}(t,x,D)L(e^{\Lambda}(t,x,D))^{-1} = \partial_t - \sqrt{-1}\Delta + A(t,x,D)$, where $A(t,x,D)$ is a pseudo-differential operator of first order, and besides if Λ satisfies (1.5) and (1.6), we can prove that $A(t,x,D) + A(t,x,D)^*$ is a negative operator in $L^2(\mathbb{R}^n)$, where A^* denotes the adjoint operator of A in $L^2(\mathbb{R}^n)$.

Then the new unknown function v satisfies

$$(1.7) \qquad \begin{cases} (\partial_t - \sqrt{-1}\Delta - A)v(t,x) = g(t,x) \left(= e^{\Lambda}(t,x,D)f\right), \\ v(0,x) = v_0 \left(= e^{\Lambda}(0,x,D)u_0\right). \end{cases}$$

The negativity of $A + A^*$ in $L^2(\mathbb{R}^n)$ evidently shows that the Cauchy problem (1.7) is L^2-well posed in $[0,T]$, and consequently we can see that the Cauchy problem (1.1) is H^∞_κ (respectively. H^∞ if $\sigma = 1$ and L^2 if $\sigma > 1$)-well-posed in $[0,T]$.

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2. Preliminaries

Let $\kappa \geq 1$ be a real number and put

$$\Gamma_\kappa(r) = \begin{cases} \lambda_0 r!^\kappa / r^{\kappa+1}, & r > 0 \\ \lambda_0, & r = 0. \end{cases}$$

Lemma. 2.1. *With notations as above, there is a constant $\lambda_0 > 0$ such that for all $p = 0, 1, \cdots$ and $q = 1, 2, \cdots$,*

$$(i) \qquad \sum_{\alpha' \leq \alpha} \binom{\alpha}{\alpha'} \Gamma_\kappa(|\alpha - \alpha'| + p + 1)\, \Gamma_\kappa(|\alpha'| + q) \leq \Gamma_\kappa(|\alpha'| + p + q).$$

Furthermore, we have

$$(ii) \qquad \sum_{\alpha' \leq \alpha} \binom{\alpha}{\alpha'} \Gamma_\kappa(|\alpha - \alpha'| + p)\, \Gamma_\kappa(|\alpha'|) \leq \Gamma_\kappa(|\alpha'| + p).$$

Remark. In Lemma 2.1 we may put λ_0 such as

$$\lambda_0 \leq \left(2 \sum_{j=0}^{\infty} \frac{1}{(j+1)^2} \right)^{-1}.$$

For $x \in \mathbb{R}^n$, $x \in \mathbb{R}^m$, denoting by $\phi(x) = (\phi_1(x), \cdots, \phi_m(x))$, we set

$$X_j = \frac{\partial}{\partial x_j} + \sum_{j=1}^{m} \frac{\partial \phi(x)}{\partial x_j} \frac{\partial}{\partial y_j} \qquad for\ j = 1, \cdots, n$$

with $X = (X_1, \cdots, X_n)$.

Given any function $u(y) \in C^\infty(\mathbb{R}^m)$, we can interpret

$$D_x^\alpha u(\phi(x)) = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n} u(y)\Big|_{y=\phi(x)}.$$

From now on, we shall write $X_1^{\alpha_1}\cdots X_n^{\alpha_n}=X^\alpha$. We now consider a function $u(y)$ which satisfies

$$
(2.1)\qquad \left|D_y^\alpha u(y)\right| \le C_{uA_0}A_0^{|\alpha|}\Gamma_\kappa\left(|\alpha|\right)
$$

for any $\alpha\in\mathbb{N}^n$ and $y\in K_0$, and $\phi(x)$ such that

$$
(2.2)\qquad \left|D_x^\alpha \phi(x)\right| \le C_{\phi A}A^{|\alpha|}\Gamma_\kappa\left(|\alpha|\right), \quad x\in K
$$

for $|\alpha|\le r$ and $\kappa\ge 1$.

Lemma 2.2. *If $u(y)$ and $\phi(x)$ satisfy the conditions (2.1) and (2.2) respectively, then we have*

$$
(2.3)\qquad \begin{aligned} &\left|D_x^\alpha D_y^\beta X^\gamma u(y)\right| \\ &\le C_{uA_0}A^{|\alpha|+|\gamma|}A_0^{|\beta|}\sum_{j=1}^{|\gamma|}\binom{|\gamma|-1}{j-1}\left(nC_{\phi AA_0}\right)^j\Gamma_\kappa\left(|\alpha|+|\gamma|-j+1\right)\Gamma_\kappa\left(|\beta|+j\right)\end{aligned}
$$

for $|\alpha|+|\gamma|\le r$, $|\gamma|\ge 1$.

Proof of Lemma 2.2. We shall prove (2.3) inductively for $|\gamma|$. It is obviously true for $|\gamma|=1$ ($|\alpha|+|\gamma|\le r$). Hypothesize that (2.3) is true for $|\alpha|+|\gamma|\le r$, $|\gamma|=l<r$, $|\gamma|\ge 1$,

then for $|e| = 1$, in fact, to be such that $e = (1, \cdots, 0)$, we have

$$
\begin{aligned}
&\left|D_x^\alpha D_y^\beta X^{\gamma+e} u(y)\right| = \left|D_x^\alpha X^{\gamma+e} D_y^\beta u(y)\right| \\
&= \left|D_x^\alpha \left(\frac{\partial}{\partial y_1} + \sum_{j=1}^n \phi_{x_j} \frac{\partial}{\partial y_j}\right) \left(X^\gamma D_y^\beta u(y)\right)\right| \\
&= \left|D_x^{\alpha+e} X^\gamma D_y^\beta u(y) + \sum_{j=1}^n \sum_{\alpha' \le \alpha} \binom{\alpha}{\alpha'} D_x^{\alpha-\alpha'} \phi_{x_1} D_x^{\alpha'} X^\gamma \left(D_y^\beta D_{y_j} u(y)\right)\right| \\
&\le C_{uA_0} A^{|\alpha|+|\gamma|+1} A_0^{|\beta|} \sum_{j=1}^{|\gamma|} \binom{|\gamma|-1}{j-1} (C_{\phi A} A_0)^j \, \Gamma_\kappa(|\alpha| + 1 + |\gamma| - j + 1) \Gamma_\kappa(|\beta| + j) \\
&\quad + C_{uA_0} \sum_{\alpha' \le \alpha} \binom{\alpha}{\alpha'} n C_{\phi A} A^{|\alpha-\alpha'|+1} \Gamma_\kappa(|\alpha - \alpha'| + 1) A_0^{|\beta|+1} \sum_{j=1}^{|\gamma|-1} \Gamma_\kappa(|\alpha'| + |\gamma| - j + 1) \\
&\quad \times A^{|\alpha|+|\gamma|} \binom{|\gamma|-1}{j-1} (n C_{\phi A} A_0)^j \, \Gamma_\kappa(|\beta| + 1 + j) \\
&\le C_{uA_0} A^{|\alpha|+|\gamma|+1} A_0^{|\beta|} \sum_{j=1}^{|\gamma|} \binom{|\gamma|-1}{j-1} (C_{\phi A} A_0)^j \, \Gamma_\kappa(|\alpha| + |\gamma| + 1 - j + 1) \Gamma_\kappa(|\beta| + j) \\
&\quad + C_{uA_0} A^{|\alpha|+|\gamma|+1} A_0^{|\beta|} \sum_{j=1}^{|\gamma|-1} (n C_{\phi A})^j \, \Gamma_\kappa(|\alpha| + |\gamma| + 1 - j + 1) \Gamma_\kappa(|\beta| + j) \\
&\le C_{uA_0} A^{|\alpha|+|\gamma|+1} A_0^{|\beta|} \sum_{j=1}^{|\gamma|+1} \binom{|\gamma|}{j-1} (n C_{\phi A})^j \, \Gamma_\kappa(|\alpha| + |\gamma| + 1 - j + 1) \Gamma_\kappa(|\beta| + j)
\end{aligned}
$$

where we used the fact

$$
\binom{|\gamma|-1}{j-1} + \binom{|\gamma|-1}{j} = \binom{|\gamma|}{j-1}, \quad (j \ge 2).
$$

□

Corollary 2.3. *Under the hypotheses of (2.1), (2.2) and $nC_{\phi A_0} \le 1$, there are constants C_{uA_0} and $C_{\phi A}$ such that for any α which satisfies $0 \ne |\alpha| \le r$, $x \in K$,*

$$
(2.4) \qquad |D_x^\alpha u(\phi(x))| \le C_{uA_0} A^{|\alpha|} C_{\phi A} A_0 \Gamma_\kappa(|\alpha|).
$$

Proof. By (2.3) ($|\alpha| \neq 0$) and Lemma 2.1, we have

$$\begin{aligned}
&|D_x^\alpha u(\phi(x))| = \left| X^\alpha u(y)|_{y=\phi} \right| \\
&\leq C_{uA_0} A^{|\alpha|} \sum_{j=1}^{|\alpha|} \binom{|\alpha|-1}{j-1} (nC_{\phi A} A_0)^j \, \Gamma_\kappa(|\alpha| - j + 1) \, \Gamma_\kappa(j) \\
&\leq C_{uA_0} A^{|\alpha|} (nC_{\phi A} A_0) \sum_{j=1}^{|\alpha|-1} \binom{|\alpha|-1}{j} \Gamma_\kappa(|\alpha| - j) \, \Gamma_\kappa(j+1) \\
&\leq C_{uA_0} (nC_{\phi A} A_0) \, A^{|\alpha|} \Gamma_\kappa(|\alpha|) .
\end{aligned}$$

□

Remark. Similarly as the proof of Corollary 2.3 we can get

$$(2.4)' \qquad |D_x^\alpha u_y(\phi(x))| \leq C_{uA_0} A^{|\alpha|} n C_{\phi A} A_0^2 \Gamma_\kappa(|\alpha| + 1) .$$

We define $F_k(x, y) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, $(k = 1, \cdots, m)$ as

$$(2.5) \qquad \left| D_x^\alpha D_y^\beta F_k(x, y) \right| \leq C_{FA_0} A_0^{|\alpha|+|\beta|} \Gamma_\kappa(|\alpha| + |\beta|)$$

for any α, $\beta \in \mathbb{N}^n$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$.

And we assume $F(x, y) = (F_1(x, y), \cdots, F_m(x, y))$ satisfy

$$(2.6) \qquad \left| F_y(x, y)^{-1} \right| \leq C_1$$

for $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$.

Let $\Omega \subset \mathbb{R}^n$ be closed domain and $f(x)$ to be a solution of

$$(2.7) \qquad F(x, f(x)) = 0$$

for $x \in \Omega$.

Lemma 2.4. *If $F = (F_1, \cdots, F_m)(x, y)$ are functions which satisfy (2.5) and (2.6), and $f(x)$ is a solution of (2.7). Then there exist constants $C_{f,A}$ and $A > 0$ such that*

$$|D_x^\alpha f(x)| \le C_{f,A} A^{|\alpha|} \Gamma_\kappa(|\alpha|) \tag{2.8}$$

for any $\alpha \in \mathbb{N}^n$ and $x \in \Omega(\subset \mathbb{R}^n)$.

Proof. For $\hat{x} \in \mathbb{R}^n$ we put $K_\varepsilon(\hat{x}) = K_\varepsilon = \{x \in \Omega; |x - \hat{x}| \le \varepsilon\}$. It is enough to show that the following (2.9) to be true instead of (2.8)

$$|D_x^\alpha f(x)| \le C_{fA} A^{|\alpha|} \Gamma_\kappa(|\alpha|), \tag{2.9}$$

for any α and $x \in K_\varepsilon(\hat{x})$ where C_{fA} and A are independent of $\hat{x}$. We put $\varphi(x) = (x - \hat{x}, f(x) - f(\hat{x}))$ and $u(w) = F(x + \hat{x}, y + f(\hat{x}))$ for $w = (x, y) \in \mathbb{R}^{n+m}$.

By (2.7) we have

$$u(\varphi(x)) = 0 \tag{2.10}$$

for $x \in \Omega$, and the estimate for $\varphi(x)$ such that

$$\sup_{K_\varepsilon} |\varphi(x)| \le \sup_{K_\varepsilon} \{|x - \hat{x}| + c_0 |x - \hat{x}|\} \le (1 + c_0)\,\varepsilon \tag{2.11}$$

where

$$c_0 = \sup_{x \in \Omega} |f(x)| < \infty. \tag{2.12}$$

Differentiating (2.7), we get

$$F_x(x, f) + F_y(x, f) \cdot f_x(x) = 0, \tag{2.12$'$}$$

thus, $|f_x(x)| \le c_1 C_{F,A_0} A_0 \Gamma_\kappa(1)$,

$$\sup_{|\alpha|=1} \frac{|D_x^\alpha f(x)|}{\Gamma_\kappa(|\alpha|)A^{|\alpha|}} \le \frac{C_1 C_{FA_0} A_0}{A} \tag{2.13}$$

(A is sufficiently large) and

$$\sup_{|\alpha|=1} \frac{|D_x^\alpha (x-\hat{x})|}{\Gamma_\kappa(|\alpha|)A} \le \frac{1}{\Gamma(1)A}$$

provided A to be sufficiently large. We now put

$$\varphi_A = \max\left\{\frac{C_1 C_{FA_0}}{A}, (1+C_0)\varepsilon, \frac{1}{\Gamma_\kappa(1)A}\right\}. \tag{2.14}$$

We shall show that

$$|D_x^\alpha \varphi(x)| \le C_{\varphi A} A^{|\alpha|}\Gamma_\kappa(|\alpha|) \tag{2.15}$$

for any $x \in K_\varepsilon$ and $\alpha \in \mathbb{N}^n$.

By (2.11) and (2.13), (2.15) is obviously true for $|\alpha| = 0, 1$. Taking into account that $\varphi(x) = (x - \hat{x}, f(x) - f(\hat{x}))$, it suffices to show that

$$|D_x^\alpha f(x)| \le C_{\varepsilon A} A^{|\alpha|}\Gamma_\kappa(|\alpha|) \tag{2.16}$$

for $|\alpha| \ge 2$.

Assuming that (2.16) holds for $|\alpha| \le r$, we can get from (2.12)' that

$$u_x(\varphi(x)) + u_y(\varphi(x)) f_x(x) = 0. \tag{2.17}$$

Therefore, we have

$$|D_x^\alpha f(x)| \le C_{\varphi A} A^{|\alpha|}\Gamma_\kappa(|\alpha|)$$

for $|\alpha| = r+1$ and $x \in K_\varepsilon(\hat{x})$, where $C_{\varepsilon A}$ and A are not depend on $\hat{x}$. $\square$

We avail oneself for $(2.4)'$, we have

$$|D_x^\alpha u_y(\varphi(x))| + |D_x^\alpha u_x(\phi(x))| \le C_{FA_0} C_{\varphi A} A_0^2 A^{|\alpha|}\Gamma_\kappa(|\alpha|+1)$$

for $|\alpha| \le r$, thus

$$\begin{aligned}
&|D_x^\alpha f_x(x)| \\
&= \left| -F_y(x,f)^{-1}\left\{ D_x^\alpha u_x(\phi(x)) - \sum_{\alpha'<\alpha}\binom{\alpha}{\alpha'} D^{\alpha-\alpha'} u_y(\varphi)\, D^{\alpha'} f_x(x)\right\}\right| \\
&\le C_1 \Big\{ C_{FA_0} C_{\varphi A} A_0^2 A^{|\alpha|}\Gamma_\kappa(|\alpha|+1) \\
&+ \sum_{\alpha'<\alpha}\binom{\alpha}{\alpha'} C_{FA_0} C_{\varphi A} A_0^2 A^{|\alpha-\alpha'|}\Gamma_\kappa(|\alpha-\alpha'|+1) C_{\varphi A} A^{|\alpha'|+1}\Gamma_\kappa(|\alpha'|+1)\Big\} \\
&\le C_{\varphi A}\left\{ \frac{C_1 C_{FA_0} A_0^2}{A} + C_1 C_{FA_0} A_0^2 C_{\varphi A}\right\} A^{|\alpha|+1}\Gamma_\kappa(|\alpha|+1) \\
&\le C_{\varphi A} A^{|\alpha|+1}\Gamma_\kappa(|\alpha|+1),
\end{aligned}$$

if $C_1 C_{FA_0} A_0^2 \le 1/2$ and $C_1 C_{FA_0} A_0^2 C_{\varphi A} \le 1/2$.

We set

$$F((x,y;\xi,\eta);\Xi) = \Xi + \sqrt{-1}\int_0^1 \nabla_x \Lambda(x+\theta y, \Xi)\, d\theta - (\xi + \sqrt{-1}\eta)$$

where $\Xi(x,y,\xi,\eta)$ is a solution of the following equation

$$F(x,y,\xi,\eta;\Xi(x,y,\xi,\eta)) = 0. \tag{2.18}$$

Next, we shall prove that

$$\begin{aligned}
&\left|D_x^\beta D_\xi^\alpha D_y^\delta D_\eta^\lambda \left\{\Xi(x,y,\xi,\eta) - \xi - \sqrt{-1}\eta\right\}\right| \\
&\le C_{\Xi A} A^{|\alpha+\beta+\delta+\lambda|} |\alpha+\beta+\delta+\lambda|!^\kappa \langle\xi\rangle_h^{1/\kappa - |\alpha+\lambda|}.
\end{aligned} \tag{2.19}$$

We define as

$$f(x,y,\xi,\eta)=\Xi\left(x,y,\langle\xi\rangle_h^{-1/\kappa}\xi,\langle\xi\rangle_h^{-1/\kappa}\eta\right)-\langle\xi\rangle_h^{-1}(\xi+\sqrt{-1}\eta)$$

where $f(x,y,\xi,\eta)$ satisfy

$$f(x,y,\xi,\eta)+\sqrt{-1}\int_0^1\nabla_x\Lambda\left(x+\theta y,f+\langle\xi\rangle_h^{-1/\kappa}(\xi+\sqrt{-1}\eta)\right)d\theta=0$$

for $x,y,\xi,\ \eta\in\mathbb{R}^n$, $|\eta|\le M\langle\xi\rangle_h^{1/\kappa}$, and

$$F\left(x,y,\xi,\eta;\zeta\right)=\zeta+\sqrt{-1}\int_0^1\nabla_x\Lambda\left(x+\theta y,\xi+\langle\xi\rangle^{-1/\kappa}(\xi+\sqrt{-1}\eta)\right)d\theta. \tag{2.20}$$

We derive by changing variables (x,y,ξ,η) and ζ to x and y respectively that $F(x,y)$ implies

$$\left|D_x^\alpha D_y^\beta F(x,y)\right|\le C_{FA_0}A_0^{|\alpha|+|\beta|}\Gamma_\kappa(|\alpha|+|\beta|)$$

for $x\in\mathbb{R}^{4n}$ and $y\in\mathbb{R}^m$. Putting $\Omega=\left\{x\in\mathbb{R}^{4n};x=(x^1,x^2,x^3,x^4),\quad|x^4|\le M\right\}$, it is obtained that $f(y)$ to be a solution of $F\left(x,f(y)\right)=0$ for $x\in\Omega$. By virtue of (2.20), we have

$$|F_y(x,y)|\ge 1-Ch^{-1+1/\kappa}\ge 1/2,$$

provided h to be sufficiently large. Then $F(x,y)$ implies (2.6), therefore by Lemma 2.4 we have

$$|D_x^\alpha f(x)|\le C_{fA}A^{|\alpha|}\Gamma_\kappa(|\alpha|)$$

for any $x\in\Omega$ and $\alpha\in\mathbb{N}^n$, that is, using the variable (x,y,ξ,η) which was changed to x,

$$\left|D_x^\alpha D_y^\beta D_\xi^\lambda D_\eta^\delta f(x,y,\xi,\eta)\right|\le C_{fA}A^{|\alpha|+|\beta|+|\lambda|+|\delta|}\Gamma_\kappa(|\alpha|+|\beta|+|\lambda|+|\delta|)$$

for $(x,y,\xi,\eta)\in\Omega$. This implies (2.19).

Similarly as Corollary 2.3 we can get the following;

Lemma 2.5. *(Lemma 5.3 of [6]) Let* $K_1 \subset \mathbb{R}^{n_1}$ *and* $K_2 \subset \mathbb{R}^{n_2}$ *be closed sets,* $F(x,y)$ *in* $C^k(K_1 \times K_2)$ *satisfying*

$$|D_x^\alpha D_y^\beta F(x,y)| \le C_{FB} B^{|\alpha|+|\beta|}(|\alpha|+|\beta|)!^\kappa,$$

for $x \in K_1$, $y \in K_2$ *and* $|\alpha|+|\beta| \le k$, *and* $\varphi(x)$ *a mapping of* K_1 *to* K_2 *satisfying*

$$|D_x^\alpha \varphi(x)| \le C_{\varphi A} A^{|\alpha|} |\alpha|!^\kappa,$$

for $x \in K_1$, *and* $|\alpha| \le k$. *Then the composition* $F(x,\varphi(x))$ *satisfies*

$$\left| D_x^\alpha \left\{ F_{(\delta)}^{(\beta)}(x,\varphi(x)) \right\} \right| \le C_{FB} B^{|\beta|+|\delta|} (MA)^{|\alpha|} (|\alpha|+|\beta|+|\delta|)!^\kappa,$$

for $x \in K_1$ *and* $|\alpha|+|\beta|+|\delta| \le k$, *where* $F_{(\delta)}^{(\beta)}(x,y) = D_x^\delta D_y^\beta F(x,y)$ *and*

$$M = \max\left\{2^\kappa(1+2BC_{\varphi A}), 2B\right\}.$$

3. Symbol of Gevrey class

In this section we shall construct a function $\Lambda(t,x,\xi)$ which satisfies (1.5) and (1.6) following the idea in [1,2].

Let $g(x,\xi)$ be a function in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n\backslash 0)$ and consider the following equation

$$\sum_{j=1}^{n} \xi_j \partial_{x_j} \lambda(x,\xi) = |\xi| g(x,\xi) \tag{3.1}$$

for $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n\backslash 0$. Then we can find easily a solution of (3.1) has a following form

$$\lambda(x,\xi) = \lambda_0(\xi) + \int_0^{x\cdot\omega} g(x-\tau\omega,\xi)d\tau, \tag{3.2}$$

where $\omega = \xi/|\xi|$, $x\cdot\omega = \sum_{j=1}^n x_j\omega_j$ and $\lambda_0(\xi)$ is an arbitrary function of ξ.

Let $\kappa > 1$ and $\chi(t)$ be a function in $C_0^\infty(\mathbb{R})$ such that $\chi(t) = 1$ for $|t| \le 1/2$, $\chi(t) = 0$ for $|t| \ge 1$, $t\chi'(t) \le 0$ and $0 \le \chi(t) \le 1$, and moreover $\chi(t)$ satisfies

$$|\partial_t^m \chi(t)| \le A_0^{m+1} m!^\kappa \tag{3.3}$$

for $t \in \mathbb{R}$ and $m = 0, 1, 2, \cdots$. For $\varepsilon > 0$ we define

$$\begin{cases} \chi_1(x,\xi) = \chi\left(\dfrac{\langle x\rangle}{\varepsilon|\xi|}\right) \\ \chi_2(x,\xi) = \chi\left(\dfrac{2x\cdot\omega}{\langle x\rangle}\right) \quad (\omega = \xi/|\xi|), \end{cases} \tag{3.4}$$

then similarly in the proof of Corollary 2.3 we have

$$\begin{cases} |D_x^\alpha \langle x\rangle^m| \le C_0 {A_0}^{|\alpha|}|\alpha|!\langle x\rangle^{m-|\alpha|}, \\ |D_\xi^\alpha |\xi|^m| \le C_0 {A_0}^{|\alpha|}|\alpha|!|\xi|^{m-|\alpha|} \end{cases} \tag{3.5}$$

for x, $\xi \in \mathbb{R}^n$ with $|\xi| \ge 1$ and α, $\beta \in \mathbb{N}^n$, where $m \in \mathbb{R}^n$.

We have by virtue of (3.3), (3.5) and Lemma 2.5

$$|\chi_{k}{}^{(\alpha)}_{(\beta)}(x,\xi)| \le C_1 {A_1}^{|\alpha+\beta|}|\alpha+\beta|!^{\kappa}\langle x\rangle^{-|\beta|}|\xi|^{-|\alpha|} \tag{3.6}$$

for $x,\ \xi \in \mathbb{R}^n$ with $|\xi| \ge 1$, $\alpha,\beta \in \mathbb{N}^n$, $0 < \varepsilon \le 1$, and $k = 1,2$, where $\chi_{k}{}^{(\alpha)}_{(\beta)} = (\partial/\partial\xi)^{\alpha}\left(-\sqrt{-1}\partial/\partial x\right)^{\beta}\chi_k$, and C_1 and A_1 are independent of ε.

For $M > 1$ we put

$$\tag{3.7} \begin{aligned} g_{1\sigma}(x,\xi) &= \begin{cases} M\langle x\rangle^{-\sigma}\chi_1(x,\xi) & (\sigma<1) \\ M\langle x\rangle^{-\sigma}\chi(\langle x\rangle/|\xi|) & (\sigma \ge 1), \end{cases} \\ g_{2\sigma}(x,\xi) &= \begin{cases} M\left((x\cdot\omega)^2+1\right)^{-\sigma/2}\chi_1(x,\xi) & (\sigma<1), \\ M\left((x\cdot\omega)^2+1\right)^{-\sigma/2}\chi(\langle x\rangle/|\xi|) & (\sigma \ge 1), \end{cases} \end{aligned}$$

where $\omega = \xi/|\xi|$. Taking account of (3.5) we can see from (3.6) and Lemma 2.5

$$\tag{3.8} |g_{k\sigma}{}^{(\alpha)}_{(\beta)}(x,\xi)| \le \begin{cases} C_2 M {A_2}^{|\alpha+\beta|}|\alpha+\beta|!^{\kappa}\langle x\rangle^{-\sigma-|\beta|}|\xi|^{-|\alpha|} & (k=1) \\ C_2 M {A_2}^{|\alpha+\beta|}|\alpha+\beta|!^{\kappa}\left((x\cdot\omega)^2+1\right)^{-\sigma/2-|\alpha+\beta|/2} & (k=2) \end{cases}$$

for $x,\xi \in \mathbb{R}^n$ with $|\xi| \ge 1$, $\alpha,\beta \in \mathbb{N}^n$ and $0 < \varepsilon \le 1$, where C_2 and A_2 are independent of ε and M. Define for $k = 1,2$

$$\lambda_{k\sigma}(x,\xi) = \int_0^{x\cdot\omega} g_{k\sigma}(x-\tau\omega,\xi)d\tau, \tag{3.9}$$

which satisfies (2.1) with $g = g_{k\sigma}$. Since $\left(1+(x\cdot\omega)^2\right)^{1/2} \le \langle x\rangle$ implies

$$g_{1\sigma}(x,\xi) - g_{2\sigma}(x,\xi) \le 0, \tag{3.10}$$

we can see

$$(x\cdot\omega)\left(\lambda_{1\sigma}(x,\xi) - \lambda_{2\sigma}(x,\xi)\right) \le 0. \tag{3.11}$$

We define

$$\lambda_\sigma(x,\xi) = -\lambda_{1\sigma}(x,\xi)\chi_2(x,\xi) - \lambda_{2\sigma}(x,\xi)\left(1-\chi_2(x,\xi)\right). \tag{3.12}$$

Recalling that $\lambda_{k\sigma}$ is a solution of (3.1) with $g = g_{k\sigma}$ and noting that $(x\cdot\omega)\chi'\left(2x\cdot\omega/\langle x\rangle\right) \le 0$ we obtain from (3.11)

$$\begin{aligned}\sum_{j=1}^{n}\xi_j\partial_{x_j}\lambda_\sigma(x,\xi) &= -|\xi|g_{1\sigma}(x,\xi)\chi_2(x,\xi) - |\xi|g_{2\sigma}(x,\xi)\left(1-\chi_2(x,\xi)\right)\\ &\quad - \left(\lambda_{1\sigma}(x,\xi)-\lambda_{2\sigma}(x,\xi)\right)\chi'\left(\frac{2x\cdot\omega}{\langle x\rangle}\right)\frac{2|\xi|}{\langle x\rangle}\left(1-\frac{(x\cdot\omega)^2}{\langle x\rangle^2}\right)\\ &\le -|\xi|g_{1\sigma}(x,\xi).\end{aligned} \tag{3.13}$$

Moreover noting that $\langle x-\tau\omega\rangle \ge \langle x\rangle/2$ on suppχ_2 for $|\tau| \le |x\cdot\omega|$ and $|x\cdot\omega| \ge \langle x\rangle/2$ on supp$(1-\chi_2)$ are valid, we obtain by virtue of (3.6) and (3.8)

$$\begin{cases} |\lambda_\sigma{}^{(\alpha)}(x,\xi)| \le \begin{cases} C_3 M\varepsilon^{1-\sigma}A_3{}^{|\alpha|}|\alpha|!^\kappa|\xi|^{1-\sigma-|\alpha|} & (\sigma<1)\\ C_\alpha M\log(2+|\xi|)|\xi|^{-|\alpha|} & (\sigma=1)\\ C_\alpha M|\xi|^{-|\alpha|} & (\sigma>1)\end{cases}\\ |\lambda_\sigma{}^{(\alpha)}_{(\beta)}(x,\xi)| \le \begin{cases} C_3 M A_3^{|\alpha+\beta|}|\alpha+\beta|!^\kappa|\xi|^{-|\alpha|} & (\sigma<1)\\ C_{\alpha\beta}M|\xi|^{-|\alpha|} & (\sigma\ge 1)\end{cases}\end{cases} \tag{3.14}$$

for $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$ with $|\xi| \ge 1$, $\alpha,\beta \in \mathbb{N}^n$ $(|\beta| \ne 0)$ and $0 < \varepsilon \le 1$, where C_3 and A_3 are independent of ε.

Now we define

$$\Lambda_\sigma(t,x,\xi) = \begin{cases} \rho(t)\langle\xi\rangle_h^{1/\kappa} + \lambda_\sigma(x,\xi)\left(1-\chi(h^{-1}|\xi|)\right) & (\sigma<1)\\ \rho(t)\log(1+\langle\xi\rangle_h) + \lambda_{1\sigma}(x,\xi)\left(1-\chi(h^{-1}|\xi|)\right) & (\sigma=1)\\ \lambda_\sigma(x,\xi)\left(1-\chi(h^{-1}|\xi|)\right) & (\sigma>1),\end{cases} \tag{3.15}$$

where $\rho(t)$ is a real valued function in $C^1([0,T])$ with $\rho'(t) < 0$ in $[0,T]$ and $\langle\xi\rangle_h = \left(h^2+|\xi|^2\right)^{1/2}$ (h a large parameter).

Lemma 3.2. *Assume that (1.3) or (1.4) is valid. Let $\rho > 0$ and $N > 0$. Then there are a real valued function $\rho(t)$ in $C^1([0,T])$ with $\rho'(t) < 0$ in $[0,T]$ and positive constants ε and M such that $\Lambda_\sigma(t,x,\xi)$ defined in (3.15) satisfies the following properties*

$$\left(\partial_t + \sum_{j=1}^n \xi_j \partial_{x_j}\right) \Lambda_\sigma(t,x,\xi) + \sum_{j=1}^n \Im a_j(t,x)\xi_j \tag{3.16}$$
$$\leq \begin{cases} C(h) - N|\rho(t)|\langle\xi\rangle_h^{1/\kappa} & (\sigma < 1) \\ C(h) - N\left(|\rho(t)|+1\right)\log(1+\langle\xi\rangle_h) & (\sigma = 1) \\ C(h) & (\sigma > 1) \end{cases}$$

and

$$\begin{cases} |\Lambda_\sigma{}^{(\alpha)}(t,x,\xi)| \leq \begin{cases} \rho A^{|\alpha|}|\alpha|!^\kappa \langle\xi\rangle_h^{1/\kappa-|\alpha|} & (\sigma < 1) \\ C_\alpha M \log(1+\langle\xi\rangle_h)\langle\xi\rangle_h^{-|\alpha|} & (\sigma = 1) \\ C_\alpha M\langle\xi\rangle_h^{-|\alpha|} & (\sigma > 1) \end{cases} \\ |\Lambda_{\sigma}{}^{(\alpha)}_{(\beta)}(t,x,\xi) \leq \begin{cases} CMA^{|\alpha+\beta|}|\alpha+\beta|!^\kappa\langle\xi\rangle_h^{-|\alpha|} & (\sigma < 1) \\ C_{\alpha\beta}M\langle\xi\rangle_h^{-|\alpha|} & (\sigma \geq 1) \end{cases} \end{cases} \tag{3.17}$$

for $x,\xi \in \mathbb{R}^n$, $\alpha,\beta \in \mathbb{N}^n$ ($|\beta| \neq 0$) and $h \geq 1$, where A, C_α and $C_{\alpha\beta}$ are independent of h.

Proof. If $0 < \sigma < 1$, (1.4) implies that for any $\delta > 0$ there is $R_\delta > 0$ such that

$$\sum_{j=1}^n |\Im\, a_j(t,x)| \leq \frac{\delta}{\langle x\rangle^\sigma}$$

for $\langle x\rangle \geq R_\delta$. Hence we have from the above estimate and from (1.3)

$$|\sum_{j=1}^n \Im a_j(t,x)\xi_j| \leq \begin{cases} \tilde{C}\langle x\rangle^{-\sigma}|\xi| & (\langle x\rangle \leq R_\delta,\ \sigma < 1) \\ \delta\langle x\rangle^{-\sigma}|\xi| & (\langle x\rangle \geq R_\delta,\ \sigma < 1) \\ \tilde{C}\langle x\rangle^{-\sigma}|\xi| & (\sigma \geq 1). \end{cases} \tag{3.18}$$

On the other hand from (3.13) and from the definition of $\Lambda_\sigma(t,x,\xi)$ it follows that

$$\left(\partial_t + \sum_{j=1}^{n} \xi_j \partial_{x_j}\right) \Lambda_\sigma(t,x,\xi)$$

$$\tag{3.19} \leq \begin{cases} \rho'(t)\langle\xi\rangle_h^{1/\kappa} - |\xi| g_{1\sigma}(x,\xi)\left(1-\chi(h^{-1}|\xi|)\right) & (\sigma<1) \\ \rho'(t)\log(1+\langle\xi\rangle_h) - |\xi| g_{1\sigma}(x,\xi)\left(1-\chi(h^{-1}|\xi|)\right) & (\sigma=1) \\ -|\xi| g_{1\sigma}(x,\xi)\left(1-\chi(h^{-1}|\xi|)\right) & (\sigma>1). \end{cases}$$

Recalling the definition (3.7) of $g_{1\sigma}$ we obtain from (3.18)

$$|\xi| g_{1\sigma}(x,\xi)\left(1-\chi(h^{-1}|\xi|)\right) = \begin{cases} M|\xi|\langle x\rangle^{-\sigma}\chi(\langle x\rangle/\varepsilon|\xi|) & (\sigma<1) \\ M|\xi|\langle x\rangle^{-\sigma}\chi(\langle x\rangle/|\xi|) & (\sigma\geq 1) \end{cases}$$

$$\tag{3.20} \geq \begin{cases} |\sum_{j=1}^{n} \Im a_j(t,x)\xi_j| - \delta\varepsilon^{-\sigma}|\xi|^{1-\sigma} & (\sigma<1) \\ |\sum_{j=1}^{n} \Im a_j\xi_j| - 2M & (\sigma\geq 1) \end{cases}$$

for $M \geq \tilde{C}$ ($\tilde{C}$ is a constant of (3.18)), $|\xi| \geq h$ and $\varepsilon|\xi| \geq \varepsilon h \geq R_\delta$ ($\sigma < 1$). Therefore in the case of $\sigma > 1$ (3.19) and (3.20) implies (3.16) directly. Besides, (3.19) yields (3.16) in the case of $\sigma \leq 1$, if $\rho(t)$ satisfies

$$\rho'(t) + N\rho(t) + \delta\varepsilon^{-\sigma} = 0 \quad (\sigma<1)$$

$$\rho'(t) + N(\rho(t)+1) + 2M = 0 \quad (\sigma=1)$$

for $t \in [0,T]$, which is solved as follows

$$\tag{3.21} \rho(t) = \begin{cases} e^{-Nt}\rho(0) - \dfrac{1-e^{-Nt}}{N}\delta\varepsilon^{-\sigma} & (\sigma<1) \\ e^{-Nt}\rho(0) - \dfrac{1-e^{-Nt}}{N}(2M+N) & (\sigma=1). \end{cases}$$

Next we shall prove that $\Lambda_\sigma(t,x,\xi)$ satisfies (3.17). By (3.3) and (3.5) there exist positive constants C_1, A_1 such that

$$\tag{3.22} |\partial_\xi^\alpha \chi(h^{-1}|\xi|)| \leq C_1 A_1^{|\alpha|}|\alpha|!^\kappa |\xi|^{-|\alpha|}$$

for $\xi \in \mathbb{R}^n$, $|\xi| \geq 1$ and $\alpha \in \mathbb{N}^n$. Then using Leibniz' rule and the inequality $|\xi| \leq \langle \xi \rangle_h \leq 3|\xi|$ on $\mathrm{supp}(1-\chi(h^{-1}|\xi|))$ we obtain from (3.14) and (3.22)

$$
(3.23) \qquad \begin{cases} |\Lambda_\sigma{}^{(\alpha)}(t,x,\xi)| \leq \begin{cases} \left(\rho(t) + CM\varepsilon^{1-\sigma}\right) A^{|\alpha|}|\alpha|!^\kappa \langle\xi\rangle_h^{1/\kappa-|\alpha|} & (\sigma < 1) \\ C_\alpha M \log(1+\langle\xi\rangle_h)\langle\xi\rangle_h^{-|\alpha|} & (\sigma = 1) \\ C_\alpha M \langle\xi\rangle_h^{-|\alpha|} & (\sigma > 1) \end{cases} \\ |\Lambda_\sigma{}_{(\beta)}^{(\alpha)}(t,x,\xi)| \leq \begin{cases} CMA^{|\alpha+\beta|}|\alpha+\beta|!^\kappa \langle\xi\rangle_h^{-|\alpha|} & (\sigma < 1) \\ C_{\alpha\beta} M \langle\xi\rangle_h^{-|\alpha|} & (\sigma \geq 1) \end{cases} \end{cases}
$$

for $x, \xi \in \mathbb{R}^n$, $\alpha, \beta \in \mathbb{N}^n$ and $h \geq 1$, where C, A and $C_{\alpha\beta}$ are independent of h. In the case of $\sigma \geq 1$ (3.23) implies (3.17) immediately. In the case of $\sigma < 1$ we have to choose $\rho(t)$, M, ε and δ such that

$$
(3.24) \qquad \left(\rho(t) + CM\varepsilon^{1-\sigma}\right) \leq \rho
$$

for $t \in [0, T]$. We can find $\rho(t)$ satisfying (3.21) and (3.24) if we take M, ε and δ suitably.

□

4. Pseudo-differential operators in Gevrey classes

Let $\kappa > 1$ and $\Lambda(x,\xi)$ be a real valued function defined in $\mathbb{R}^{2n}$ satisfying

$$|\Lambda^{(\alpha)}_{(\beta)}(x,\xi)| \le \lambda(\Lambda) A_0^{|\alpha+\beta|} |\alpha+\beta|!^{\kappa} \langle\xi\rangle_h^{1/\kappa - |\alpha|} \tag{4.1}$$

for $x, \xi \in \mathbb{R}^n$ $\alpha, \beta \in \mathbb{N}^n$ and $h \ge 1$, where $\lambda(\Lambda) > 0$ and $A_0 > 0$ are independent of the parameter h. Following the Part I in [6] we introduce pseudo-differential operators

$$e^{\Lambda}(x,D)u(x) = \int e^{\sqrt{-1}x\cdot\xi + \Lambda(x,\xi)} \hat{u}(\xi) \tilde{d}\xi$$

and the reversed operator ${}^{R}e^{\Lambda}(x,D)$ of $e^{\Lambda}(x,D)$ as follows,

$${}^{R}e^{\Lambda}(x,D)u(x) = \int \Big[\int e^{\sqrt{-1}(x+y)\cdot\xi + \Lambda(y,\xi)} u(y) dy \Big] \tilde{d}\xi,$$

for $u \in H^0_{\kappa,\rho}$, where $x\cdot\xi = x_1\xi_1 + \cdots + x_n\xi_n$ and $\tilde{d}\xi = (2\pi)^{-n} d\xi$.

Lemma 4.1. *(Proposition 6.7 of [6]) Let $\Lambda(x,\xi)$ be a symbol satisfying (4.1). Then there is $\delta > 0$ such that $e^{\Lambda}(x,D)$ maps continuously $L^2_{\kappa,\rho}$ to $L^2_{\kappa,\rho-\tau}$ for $|\rho - \tau| < \delta A^{-1/\kappa}$ and $\tau > \lambda(\Lambda)$ and $\{e^{\Lambda}(x,D)\}^R$ from $L^2_{\kappa,\rho}$ to $L^2_{\kappa,\rho-\tau}$ for $|\rho| < \delta A^{-1/\kappa}$ and $\tau > \lambda(\Lambda)$.*

Then it follows from Lemma 4.1 that $e^{\Lambda}(x,D)$ and ${}^{R}e^{\Lambda}(x,D)$ map continuously $H^0_{\kappa,\rho}$ into $H^0_{\kappa,\rho-\tau}$ for $|\rho-\tau| < \delta A^{-1/\kappa}$ and $\tau > \lambda(\Lambda)$ and $H^0_{\kappa,\rho}$ into $H^0_{\kappa,\rho-\tau}$ for $|\rho| < \delta A^{-1/\kappa}$ and $\tau > \lambda(\Lambda)$ respectively, where δ is a positive constant and ρ_1 is that of (1.6).

We put

$$\varphi(x,\xi) = x\xi - \sqrt{-1}\Lambda(x,\xi)$$

and define

$$I_{\varphi}(x,D)u(x) = \int e^{\sqrt{-1}\varphi(x,\xi)} \hat{u}(\xi) \tilde{d}\xi,$$

$$\mathcal{F}\left(I_{\varphi}^{R}(x,D)u\right)(\xi) = \int e^{\sqrt{-1}\varphi(y,\xi)}\hat{u}(y)\tilde{d}y.$$

Let $\Xi(x,y,\xi)$ and $Y(z,\xi,\eta)$ to be solutions of the equation

$$\Xi - \sqrt{-1}\int_0^1 \nabla_x\Lambda\left(x+\theta(x-y),\Xi\right)d\theta = \xi,$$

$$Y - \sqrt{-1}\int_0^1 \nabla_\xi\Lambda(Y,\xi+\theta\eta)d\theta = z,$$

respectively, and

$$J(x,y,\xi) = \det\left\{\frac{\partial\Xi}{\partial\xi}(x,y,\xi)\right\},$$

$$\hat{J}(z,\xi,\eta) = \det\left\{\frac{\partial Y}{\partial x}(z,\xi,\eta)\right\}.$$

Lemma 4.2. *(Corollary 6.13 in [6]) Let $\Lambda(x,\xi)$ be a symbol satisfying (4.1) and $\varphi = x\xi - i\Lambda(x,\xi)$. Then*

$$I_{\varphi}(x,D)I_{-\varphi}^{R}(x,D) = J(x,D) + r(x,D),$$

$$I_{-\varphi}^{R}(x,D)I_{\varphi}(x,D) = \hat{J}(x,D) + \hat{r}(x,D),$$

where

$$J(x,\xi) = \sum_{|\alpha|<N}\frac{1}{\alpha!}D_y^{\alpha}\partial_\zeta^{\alpha}J(x,x+y,\xi+\zeta)\Big|_{y=\zeta=0} + J_N(x,\xi),$$

$$\hat{J}(x,\xi) = \sum_{|\alpha|<N}\frac{1}{\alpha!}D_y^{\alpha}\partial_\eta^{\alpha}\hat{J}(x+z,\xi,\eta)\Big|_{z=\eta=0} + \hat{J}_N(x,\xi),$$

and for $C>0$ and $\delta>0$, J_N, $\hat{J}_N$ and r, $\hat{r}$ satisfy

$$\left|J_{N}{}^{(\alpha)}_{(\beta)}(x,\xi)\right|, \left(\left|\hat{J}_{N}{}^{(\alpha)}_{(\beta)}(x,\xi)\right|\right) \le C_{NA}(CA)^{|\alpha+\beta|}|\alpha+\beta|!^{\kappa}\langle\xi\rangle_h^{-1-|\alpha|-N(1-1/\kappa)},$$

for any non-negative integer N,

$$|r^{(\alpha)}_{(\beta)}(x,\xi)|, \left(|\hat{r}^{(\alpha)}_{(\beta)}(x,\xi)\right) \le C_{NA}(CA)^{|\beta|}|\beta|!^{\kappa}\exp\left\{-\delta A^{-1/\kappa}\langle\xi\rangle_h^{1/\kappa}\right\}$$

respectively.

By virtue of Lemma 4.2 we have

$$(4.2)\qquad \begin{cases} e^{\Lambda}(x,D)\circ {}^{R}e^{-\Lambda}(x,D)=I+r_1(x,D) \\ {}^{R}e^{-\Lambda}(x,D)\circ e^{\Lambda}(x,D)=I+r_2(x,D) \end{cases}$$

where r_k $(k=1,2)$ satisfy

$$|r_{k}{}^{(\alpha)}_{(\beta)}(x,\xi)|\le C_{\alpha\beta}\langle\xi\rangle_h^{1/\kappa-1-|\alpha|}\le C_{\alpha\beta}h^{1/\kappa-1}\langle\xi\rangle_h^{-|\alpha|}$$

for $x,\xi\in\mathbb{R}^n$, $\alpha,\beta\in\mathbb{N}^n$, $h\ge 1$ and $C_{\alpha\beta}$ is independent of h. In (4.2) $A\circ B$ means the operator product of A and B. Therefore Neumann series assures the existence of the inverse of $I+r_k(x,D)$ if h is taken sufficiently large. Hence we have also the inverse of $e^{\Lambda}(x,D)$ from (4.2). Let $p(x,\xi)$ be a symbol satisfying

$$(4.3)\qquad |p^{(\alpha)}_{(\beta)}(x,\xi)|\le CA^{|\alpha+\beta|}|\alpha+\beta|!^{\kappa}\langle\xi\rangle_h^{m-|\alpha|}$$

for $x,\xi\in\mathbb{R}^n$, $\alpha,\beta\in\mathbb{N}^n$ and $h\ge 1$.

Lemma 4.3. *(Theorem 6.14 in [6]) Let $\Lambda(x,\xi)$ and $p(x,\xi)$ be symbols satisfying (4.1) and (4.3) respectively, and $\varphi=x\xi-\sqrt{-1}\Lambda(x,\xi)$. Then we have*

$$I_\varphi(x,D)p(x,D)=p(\varphi;x,D)I_\varphi(x,D)+r(\varphi;x,D),$$

where $p(\varphi;x,D)$ maps continuously $H^{m+s}_{\kappa,\rho}$ to $H^{s}_{\kappa,\rho}$ if $|\rho|<\delta A^{-1/\kappa}$ and $s\in\mathbb{R}$, and $r(\varphi;x,D)$ maps $L^2_{\kappa,\rho}$ to $L^2_{\kappa,\rho+\delta A^{-1/\kappa}-\lambda(\Lambda)}$ if $|\rho|<\delta A^{-1/\kappa}$ and $\rho+\delta A^{-1/\kappa}<\lambda(\Lambda)$. Moreover $p(\varphi;x,D)$ satisfies for any integer $N\ge 0$

$$p(\varphi;x,D)=p_N(x,D)+R_N(x,D),$$

$$p_N(x,\xi) = p\left(x - \sqrt{-1}\Lambda_\xi(x,\Xi), \xi + \sqrt{-1}\Lambda_x(x,\Xi)\right)$$
$$+ \sum_{0<|\alpha+\beta|<N} p_{(\beta)}^{(\alpha)}\left(x - \sqrt{-1}\Lambda_\xi(x,\Xi), \xi + \sqrt{-1}\Lambda_x(x,\Xi)\right)\omega_\alpha^\beta(\varphi; x, \xi),$$

where $R_N(x,D)$ maps continuously $H_{\kappa,\rho}^{m+s}$ to $H_{\kappa,\rho}^{s-N(1-1/\kappa)-1}$ for $|\rho| < \delta A^{-1/\kappa}$, and $\omega_\alpha^\beta(\varphi; x, \xi)$ is a solution of the equation

$$\Xi + \sqrt{-1}\nabla_x\Lambda(x,\Xi) = \xi.$$

Then it follows from Lemma 4.3 that there are $\delta > 0$ and $h_0 > 0$ such that if $\lambda(\Lambda) < \delta A^{-1/\kappa}$

$$e^\Lambda(x,D) \circ p(x,D) \circ \left(e^\Lambda(x,D)\right)^{-1} = p(x,D) + q(x,D) + r(x,D) \tag{4.4}$$

where $q(x,\xi)$ and $r(x,\xi)$ respectively satisfy

$$q(x,\xi) = \sum_{|\alpha+\beta|=1} p_{(\beta)}^{(\alpha)}(x,\xi)\left(\partial_\xi^\beta\Lambda(x,\xi)\right)\left((\sqrt{-1}\partial_x)^\alpha\Lambda(x,\xi)\right) \tag{4.5}$$

$$|r_{(\beta)}^{(\alpha)}(x,\xi)| \le C_{\alpha\beta}\langle\xi\rangle_h^{m-2(1-1/\kappa)-|\alpha|} \tag{4.6}$$

for $x, \xi \in \mathbb{R}^n$, $\alpha, \beta \in \mathbb{N}^n$ and $h \ge h_0$. For $\Lambda(x,\xi)$ and $p(x,\xi)$ satisfying (4.1) and (4.3) respectively we define a pseudo-differential operator $(e^\Lambda p)(x,D)$ as follows

$$(e^\Lambda p)(x,D)u(x) = \int e^{\sqrt{-1}x\cdot\xi + \Lambda(x,\xi)}p(x,\xi)\hat{u}(\xi)\tilde{d}\xi \tag{4.7}$$

for $u \in H_{\kappa,\rho}^0$.

Lemma 4.4. *(Theorem 6.10 in [6]) $p(x,\xi)$ and $\lambda(x,\xi)$ be symbols satisfying (4.3) and (4.1) respectively and $\varphi = x\xi - \sqrt{-1}\Lambda(x,\xi)$. Then there are $C > 0$, $\delta > 0$ and $h_0 > 0$ independent of A such that the product of $p_\varphi(x,D)$ and $I_{-\varphi}^R(x,D)$ is given as follows:*

$$p_\varphi(x,D)I_{-\varphi}^R(x,D) = \hat{p}(x,D) + \hat{r}(x,D),$$

where

$$\hat{p}(x,\xi) = \sum_{|\alpha|<N} \frac{1}{\alpha!} D_y^\alpha \partial_\eta^\alpha \left\{ J(x, x+y, \xi+\eta) p(x, \Xi(x, x+y, \xi+\eta)) \right\} \Big|_{y=\eta=0} + \hat{p}_N(x,\xi),$$

$$|\hat{p}_N{}_{(\beta)}^{(\alpha)}(x,\xi)| \le C_{NA}(CA)^{|\beta|}|\beta|!^\kappa \langle\xi\rangle_h^{m-1-|\alpha|-N(1-1/\kappa)},$$

for any non-negative integer N,

$$|\hat{r}_{(\beta)}^{(\alpha)}(x,\xi)| \le C_{A\alpha}(CA)^{|\beta|} \exp\left\{ -\delta A^{-1/\kappa} \langle\xi\rangle_h^{1/\kappa} \right\},$$

for x, $\xi \in \mathbb{R}^n$, α, $\beta \in \mathbb{N}^n$, $h \ge h_0$ and $\Xi(x,y,\xi)$ is a solution of the following equation

$$\Xi - \sqrt{-1} \int_0^1 \Lambda_x \left(x + \theta(x-y), \Xi \right) d\theta = \xi,$$

and $J(x,y,\xi) = \det\{\partial\Xi/\partial\xi(x,y,\xi)\}$.

Then quoting Lemma 4.4, if $\lambda(\Lambda) < \delta A^{-1/\kappa}$ we have

$$\left(e^\Lambda p \right)(x,D) \circ \left(e^\Lambda(x,D) \right)^{-1} = p(x,D) + q(x,D) + \tilde{r}(X,D), \tag{4.8}$$

where $q(x,\xi)$ and $\tilde{r}(x,\xi)$ satisfy (4.5) and (4.6) respectively.

We try to apply the above results to $\Lambda_\sigma(t,x,\xi)$ defined by (3.15) in the case of $\sigma < 1$. First of all we note that the estimates (3.17) yields

$$|\Lambda_\sigma{}_{(\beta)}^{(\alpha)}(t,x,\xi)| \le \left(\rho + CMh^{-1/\kappa} \right) A^{|\alpha+\beta|} |\alpha+\beta|!^\kappa \langle\xi\rangle_h^{1/\kappa-|\alpha|} \tag{4.9}$$

for $x, \xi \in \mathbb{R}^n$, $t \in [0,T]$, $\alpha, \beta \in \mathbb{N}^n$ and $h \ge 1$. In Lemma 2.1 we choose $\rho > 0$ and $h > 0$ such that

$$\lambda(\Lambda_\sigma) = \left(\rho + CMh^{-1/\kappa} \right) \le \delta A^{-1/\kappa} \tag{4.10}$$

for $t \in [0,T]$. If $a_j(t,x)$ $(j=1,\cdots,n)$ belong to $C^0([0,T];\mathcal{B}_{s_0,A_0})$, $s_0 \le \kappa$, $\kappa > 1$ and $\lambda(\Lambda_\sigma) < \delta A_0^{-1/\kappa}$ are valid, by virtue of (4.4) and (4.5) with $p(x,\xi) = \sum_{j=1}^n a_j(t,x)\xi_j$, we have

$$
\begin{aligned}
&e^{\Lambda_\sigma}(t,x,D) \circ \left(-\sqrt{-1}\sum_{j=1}^n a_j(t,x)\partial_{x_j}\right) \circ \left(e^{\Lambda_\sigma}(t,x,D)\right)^{-1} \\
&= -\sqrt{-1}\sum_{j=1}^n a_j(t,x)\partial_{x_j} + q(t,x,D) + r(t,x,D)
\end{aligned} \tag{4.11}
$$

where q and r respectively satisfy

$$
q(t,x,\xi) = \sqrt{-1}\sum_{j=1}^n a_j(t,x)\Lambda_{\sigma x_j} - \sqrt{-1}\sum_{j,l=1}^n a_{j\,x_l}(t,x)\xi_j\Lambda_{\sigma\xi_l}, \tag{4.12}
$$

$$
|r^{(\alpha)}_{(\beta)}(t,x,\xi)| \le C_{\alpha\beta}\langle\xi\rangle_h^{1-2(1-1/\kappa)-|\alpha|} \tag{4.13}
$$

for $x,\xi \in \mathbb{R}^n$, $t \in [0,T]$, $\alpha,\beta \in \mathbb{N}^n$ and $h \ge h_0$.

Next we shall calculate $e^{\Lambda_\sigma}(t,x,D) \circ \left(\partial_t - \sqrt{-1}\Delta\right) \circ \left(e^{\Lambda_\sigma}(t,x,D)\right)^{-1}$. We have

$$
\begin{aligned}
&\left(\partial_t - \sqrt{-1}\Delta\right) u \\
&= \left(\partial_t - \sqrt{-1}\Delta\right) \circ \left\{e^{\Lambda_\sigma}(t,x,D) \circ \left(e^{\Lambda_\sigma}(t,x,D)\right)^{-1} u\right\} \\
&= \left\{\left(e^{\Lambda_\sigma}\Lambda'_\sigma\right)(t,x,D) + e^{\Lambda_\sigma}(t,x,D) \circ \partial_t\right\} \circ \left(e^{\Lambda_\sigma}(t,x,D)\right)^{-1} u \\
&\quad - \sqrt{-1}\sum_{j=1}^n \left\{\left(e^{\Lambda_\sigma}\Lambda_{\sigma x_j}^2\right)(t,x,D) + 2\left(e^{\Lambda_\sigma}\Lambda_{\sigma x_j}\right)(t,x,D) \circ \partial_{x_j} + e^{\Lambda_\sigma}(t,x,D) \circ \partial_{x_j}^2\right\} \\
&\quad \circ \left(e^{\Lambda_\sigma}(t,x,D)\right)^{-1} u
\end{aligned}
$$

where we write $\Lambda'_\sigma = \partial\Lambda_\sigma/\partial t$ and $\Lambda_{\sigma x_j} = \partial\Lambda_\sigma/\partial x_j$. Hence we obtain

$$
\begin{aligned}
&e^{\Lambda_\sigma}(t,x,D) \circ \left(\partial_t - \sqrt{-1}\Delta\right) \circ \left(e^{\Lambda_\sigma}(t,x,D)\right)^{-1} u \\
&= \left(\partial_t - \sqrt{-1}\Delta\right) u - \left(e^{\Lambda_\sigma}\Lambda'_\sigma\right)(t,x,D) \circ \left(e^{\Lambda_\sigma}(t,x,D)\right)^{-1} u \\
&\quad + \sqrt{-1}\sum_{j=1}^n \left\{\left(e^{\Lambda_\sigma}\Lambda_{\sigma x_j}^2\right)(t,x,D) + \left(e^{\Lambda_\sigma}\Lambda_{\sigma x_j}\right)(t,x,D) \circ \partial_{x_j}\right\} \\
&\quad \circ \left(e^{\Lambda_\sigma}(t,x,D)\right)^{-1} u.
\end{aligned}
$$

Noting that from (3.15) and (3.17) it follows that $\partial_t \Lambda_\sigma(t,x,\xi) = \rho'(t)\langle\xi\rangle_h^{1/\kappa}$, and $\partial_{x_j}\Lambda_\sigma(t,x,\xi) = \lambda_{\sigma x_j}(x,\xi)\left(1-\chi(h^{-1}|\xi|)\right)$ is a symbol of order zero, we obtain using (4.4) and (4.8)

$$
\begin{aligned}
&e^{\Lambda_\sigma}(t,x,D)\circ\left(\partial_t - \sqrt{-1}\Delta\right)\circ\left(e^{\Lambda_\sigma}(t,x,D)\right)^{-1}u \\
&= \left(\partial_t - \sqrt{-1}\Delta\right)u - \rho'(t)\langle D\rangle_h^{1/\kappa}u + \sqrt{-1}\sum_{j=1}^{n}\Lambda_{\sigma x_j}(x,D)\circ\partial_{x_j}u + c_\sigma(t,x,D)u
\end{aligned} \tag{4.14}
$$

where $c_\sigma(t,x,\xi)$ satisfies

$$
|c_{\sigma}{}^{(\alpha)}_{(\beta)}(t,x,\xi)| \le C_{\alpha\beta}\langle\xi\rangle_h^{1/\kappa-(1-1/\kappa)-|\alpha|} \tag{4.15}
$$

for $x,\xi\in\mathbb{R}^n$, $\alpha,\beta\in\mathbb{N}^n$ and $h\ge h_0$. Thus summing up, if $0<\sigma<1$ and $s_0\le\kappa\le(1-\sigma)^{-1}$, we can transform L by $e^{\Lambda_\sigma}(t,x,D)$ to the following form

$$
\begin{aligned}
&e^{\Lambda_\sigma}(t,x,D)\circ L\circ\left(e^{\Lambda_\sigma}(t,x,D)\right)^{-1} \\
&= L - \rho'(t)\langle D\rangle_h^{1/\kappa} + \sqrt{-1}\sum_{j=1}^{n}\Lambda_{\sigma x_j}(t,x,D)\circ\partial_{x_j} - q(t,x,D) - \tilde{c}_\sigma(t,x,D)
\end{aligned} \tag{4.16}
$$

where $q(t,x,\xi)$ is given by (4.12) and $\tilde{c}_\sigma$ satisfies (4.15). In the case of $\sigma\ge1$ applying the asymptotic expansion of products of pseudo-differential operators of (1,0)-type, we can obtain the formulas (4.4) and (4.8) replacing (4.6) by the following estimates

$$
|r^{(\alpha)}_{(\beta)}(t,x,\xi)| \le \begin{cases} C_{\alpha\beta}\left\{(\log(1+\langle\xi\rangle_h))\langle\xi\rangle_h^{-1}\right\}^2\langle\xi\rangle_h^{m-|\alpha|} & (\sigma=1) \\ C_{\alpha\beta}\langle\xi\rangle_h^{m-2-|\alpha|} & (\sigma>1) \end{cases} \tag{4.17}
$$

for $x,\xi\in\mathbb{R}^n$, $\alpha,\beta\in\mathbb{N}^n$ and $h\ge h_0$. Therefore in this case $r(t,x,\xi)$ in the formula (4.11) satisfies (4.17) with $m=1$ instead of (4.13). Thus we can see that the formula (4.16) in the case of $\sigma\ge1$ also holds, replacing (4.15) by (4.17) with $m=1$.

5. Proof of Theorems

In this section we shall prove Theorem 1.1 and 1.2. Let $\Lambda_\sigma(t,x,\xi)$ be a symbol defined by (3.15). Put $v(t,x) = e^{\Lambda_\sigma}(t,x,D)u(t,x)$ and $L_\sigma = e^{\Lambda_\sigma}(t,x,D) \circ L \circ \left(e^{\Lambda_\sigma}(t,x,D)\right)^{-1}$. Then if u is a solution of (1.1) and satisfies that $e^{\Lambda_\sigma}(t,x,D)u \in C^1([0,T];H^\infty)$, $v(t,x)$ belongs to $C^1([0,T];H^\infty)$ and satisfies

$$(5.1) \qquad \begin{cases} L_\sigma v(t,x) = g(t,x), & (t,x) \in (0,T)\times\mathbb{R}^n \\ v(0,x) = v_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where $g = e^{\Lambda_\sigma}(t,x,D)f(t,x)$ and $v_0 = e^{\Lambda_\sigma}(0,x,D)u_0$. Conversely if $v(t,x) \in C^1([0,T];$ $H^\infty)$ satisfies (5.1), then $u(t,x) = \left(e^{\Lambda_\sigma}(t,x,D)\right)^{-1} v(t,x)$ is a solution of (1.1) and satisfies that $e^{\Lambda_\sigma}(t,x,D)u(t,x)$ belongs to $C^1([0,T];H^\infty)$. Therefore it suffice to show that for any $v_0 \in H^l$ and for any $g(t,x) \in C^0([0,T];H^l)$ there is a solution $v(t,x)$ of (5.1) which belongs to $C^0([0,T];H^l) \cap C^1([0,T];H^{l+2})$, where l is a real number.

Proposition 5.1. *Assume that the conditions of Theorem 1.1 or of Theorem 1.2 are valid. Let l be a real number and $v(t,x)$ be in $C^1([0,T];H^\infty)$. If we take suitably $\rho(t)$, M, ε and h in the definition of Λ_σ, there is a positive constant $C = C(l,T)$ such that*

$$(5.2) \qquad \|v(t)\|_{H^l} \le e^{Ct}\|v(0)\|_{H^l} + \int_0^t e^{C(t-\tau)}\|L_\sigma v(\tau)\|_{H^l}\,d\tau$$

for $t \in [0,T]$.

Proof. Denote $\|v\|_l = \|v\|_{H^l}$ and $(v,v)_l = (v,v)_{H^l}$ for short. Differentiating $\frac{1}{2}\|v(t)\|_l^2$

with respect to t we have from the formula (4.16) for L_σ

(5.3)

$$\|v(t)\|_l \frac{d}{dt}\|v(t)\|_l = \Re\left(v'(t), v(t)\right)_l$$
$$= \Re\left(\sqrt{-1}\Delta v + \sum_{j=1}^{n} a_j \partial_{x_j} v + g, v\right)_l$$
$$+ \Re\left(\left(\rho'(t)\langle D\rangle_h^{1/\kappa} - \sqrt{-1}\sum_{j=1}^{n} \Lambda_{\sigma x_j}(x, D) \circ \partial_{x_j} + q(t, x, D) + \tilde{c}_\sigma(t, x, D)\right) v, v\right)_l .$$

Moreover noting that the expression (4.12) and the estimate (3.17) yield

$$|q(t,x,\xi)| \le \begin{cases} C\left(\rho\langle\xi\rangle_h^{1/\kappa} + Ch^{-1/\kappa}\langle\xi\rangle_h^{1/\kappa}\right) & (\sigma < 1) \\ C\left\{\log(1 + \langle\xi\rangle_h) + |\rho(t)|\right\} & (\sigma = 1) \\ C & (\sigma > 1). \end{cases}$$

we have by virtue of (3.16) and (4.15) ($\sigma < 1$) (or (4.17) ($\sigma \ge 1$))

$$\rho'(t)\langle\xi\rangle_h^{1/\kappa} + \sum_{j=1}^{n} \xi_j \Lambda_{\sigma x_j}(x,\xi) + \sum_{j=1}^{n} \Im a_j \xi_j + \Re\left(q(t,x,\xi) + \tilde{c}_\sigma(t,x,\xi)\right) \le C$$

if we choose ρ, N and h in Lemma 3.2 suitably. Hence we obtain from (5.3) and Theorem 4.4 in Chapter.3 of [7]

$$\frac{d}{dt}\|v(t)\|_l \le C\|v(t)\|_l + \|g(t)\|_l,$$

which implies (5.2) directly. □

We can obtain the following proposition similarly as Theorem 4.1 of [1].

Proposition 5.2. *Assume that the conditions of Theorem 1.1 or of Theorem 1.2 are valid. Let l be a real number and take suitably $\rho(t)$, M, ε and h in the definition of Λ_σ, then for* *any* $v_0 \in H^l$ *and for any* $g(t,x) \in C^0([0,T];H^l)$ *there is an unique solution* $v(t,x)$ *of (5.1) which belongs to* $C^0([0,T];H^l) \cap C^1([0,T];H^{l+2})$.

Following the idea of Kumano-go [7], we shall prove the above theorem. We need several lemmas. First, we define $\{\zeta_\nu(\xi)\}_{\nu=1}^{\infty}$ as

$$\zeta_\nu(\xi) = \left(\nu \sin\frac{\xi_1}{\nu}, \cdots, \nu \sin\frac{\xi_n}{\nu}\right) \tag{5.4}$$

and $P_\nu(t) = p_\nu(t,x,D_x)$ as

$$p_\nu(t,x,\xi) = p_\nu(t,x,\zeta_\nu(\xi)). \tag{5.5}$$

We now consider the following Cauchy problem

$$\begin{cases} L_\nu v_\nu = \partial_t v_\nu - \sqrt{-1}P_\nu(t)v_\nu = g(t), \quad t \in (0,T), \\ v_\nu|_{t=0} = v_0. \end{cases} \tag{5.6}$$

We define the series of weight function $\{\lambda_\nu(\xi)\}_{\nu=1}^{\infty}$ as

$$\lambda_\nu(\xi) = \langle\zeta_\nu(\xi)\rangle = \left\{1 + \sum_{j=1}^{n}\left(\nu \sin\frac{\xi_j}{\nu}\right)^2\right\}, \tag{5.7}$$

then we have

$$\begin{cases} i)\ 1 \le \lambda_\nu(\xi) \le \min\left(\langle\xi\rangle, \sqrt{1+n\nu^2}\right), \\ ii)\ |\partial_\xi^\alpha \lambda_\nu(\xi)| \le A_\alpha \lambda_n u(\xi)^{1-|\alpha|}, \\ iii)\ \lambda_\nu(\xi) \to \langle\xi\rangle \quad (\nu \to \infty) \text{ on } \mathbb{R}^n_\xi, \\ \qquad (uniform\ convergence\ in\ a\ compact\ set). \end{cases} \tag{5.8}$$

In fact $\zeta_\nu(\xi)$ satisfies

$$
(5.9)\qquad \begin{cases} i)\ |\zeta_\nu(\xi)| \le \min\left(|\xi|, \sqrt{n}\nu\right), \\ ii)\ |\partial_\xi^\alpha \zeta_\nu(\xi)| \le A'_\alpha \lambda_n u(\xi)^{1-|\alpha|}, \\ iii)\ \zeta_\nu(\xi) \to \xi, \\ \qquad (uniform\ convergence\ in\ a\ compact\ set). \end{cases}
$$

Denote by $S^m_{\lambda_\nu}$ the set of symbols $p(x,\xi) \in C^\infty(\mathbb{R}^{2n})$ satisfying

$$
|p^{(\alpha)}_{(\beta)}(x,\xi)| \le C_{\alpha\beta}\lambda_\nu(\xi)^{m-|\alpha|}
$$

for any multi-index α, β. Then we get the following lemma.

Lemma 5.1. *For $p(x,\xi) \in S^m$ put $p_\nu(x,\xi) = p(x,\zeta_\nu(\xi))$. Then $p_\nu(x,\xi) \in S^m_{\lambda_\nu}$,* and α, β *there is constant $A_{\alpha,\beta}$ which is independent of ν and p, we have*

$$
(5.10)\qquad \begin{cases} |p_\nu{}^{(\alpha)}_{(\beta)}(x,\xi)| \le \left(A_{\alpha,\beta}|p|^{(m)}_{|\alpha+\beta}\right)\lambda_\nu(\xi)^{m-|\alpha|}, \\ p_\nu(x,\xi) \to p(x,\xi) \quad (uniformly)\ (\nu \to \infty)\ at\ \mathbb{R}^n_x \times K_\xi. \end{cases}
$$

where K_ξ is an arbitrary compact set of $\mathbb{R}^n_\xi$.

Denoting $H_{\lambda_\nu,s} = \left\{u \in \mathcal{S}'; \lambda_\nu(\xi)^s \hat{u} \in L^2\right\}$, we have the following;

Lemma 5.2. *$P = p(x,D_x) \in S^m_{\lambda_\nu}$ is continuous mapping from the Sobolev spaces $H_{\lambda_\nu,s+m}$ to $H_{\lambda_\nu,s}$ and for constants $C_{s,m}$ and $l = l(s,m)$ we have*

$$
(5.11)\qquad \|Pv\|_{\lambda_\nu,s} \le \left(C_{s,m}|p|^{(m)}_l\right)\|v\|_{\lambda_\nu,s+m}.
$$

Especially for $m = 0$ and $s = 0$ we have

$$
(5.12)\qquad \|Pv\|_{L^2} \le \left(C|p|^{(0)}_l\right)\|v\|_{L^2}, \quad v \in L^2(\mathbb{R}^n).
$$

Lemma 5.3. *For any $v_0 \in H^l$ and any $g(t) \in C_t^0\left([0,T];H^l\right)$ there exists a solution $v_\nu(t) \in C_t^0\left([0,T];H^l\right)$ of (5.6) which satisfies the energy inequalities*

$$\|v_\nu(t)\|_l \le e^{\gamma t}\|v_0\|_l + \int_0^t e^{\gamma(t-\tau)}\|g(\tau)\|_l d\tau \quad t \in [0,T] \tag{5.13}$$

$$\|\Lambda_\nu^j v_\nu(t)\|_l \le e^{\gamma_1 t}\|\Lambda_\nu^j v_0\|_l + \int_0^t e^{\gamma(t-\tau)}\|\Lambda_\nu^j g(\tau)\|_l d\tau \quad t \in [0,T];\ j = 1,2,\cdots \tag{5.14}$$

$$\|\frac{d}{dt}\Lambda_\nu^j v_\nu(t)\|_l \le C_T\left\{\|\Lambda_\nu^{j+2}v_0\|_l + \max_{[0,T]}\|\Lambda_\nu^{j+2}g(\tau)\|_l d\tau\right\} \quad t \in [0,T];\ j = 1,2,\cdots \tag{5.15}$$

$$\begin{aligned}&\|\Lambda_\nu^j\left(v_\nu(t) - v_\nu(t')\right)\|_l \le C_T'|t-t'|\left\{\|\Lambda_\nu^{j+2}v_0\|_l + \max_{[0,T]}\|\Lambda_\nu^{j+2}g(\tau)\|_l d\tau\right\}\\ &t,t' \in [0,T];\ j = 1,2,\cdots\end{aligned} \tag{5.16}$$

where γ, γ_1, C_T, C_T' are constants which are independent of ν, and $\Lambda_\nu = \lambda_\nu(D_x)$.

Proof of Lemma 5.3. I) If we fix ν arbitrarily, we have $p_\nu(t,x,\xi) \in C_t^k\left([0,T];\mathcal{B}^\infty(\mathbb{R}^{2n}_{x,\xi})\right)$ $(k = 0,1)$. Then $P_\nu(t)$ is an H^l-bounded operator uniformly with respect to t. Therefore $v_\nu(t)$ to be a unique solution of the integral equation

$$v_\nu(t) = v_0 + \sqrt{-1}\int_0^t P_\nu(\tau)u_\nu(\tau)d\tau + \int_0^t g(\tau)d\tau, \tag{5.17}$$

which can be solved as follows;

$$\begin{aligned}v_\nu(t) = v_0 + \int_0^t g(\tau)d\tau + \sum_{k=1}^\infty \sqrt{-1}^k\int_0^t\int_0^{\tau_1}\cdots\int_0^{\tau_{k-1}} P_\nu(\tau_1)\cdots P_\nu(\tau_k)v_0 d\tau_k\cdots\tau_1\\ \sum_{k=2}^\infty\sqrt{-1}^k\int_0^t\int_0^{\tau_1}\cdots\int_0^{\tau_{k-1}}P_\nu(\tau_1)\cdots P_\nu(\tau_k)g(\tau_k)d\tau_k\cdots\tau_1 \quad \tau_0 = t.\end{aligned}$$

Then we have $v_\nu(t) \in C_t^1\left([0,T];H^l\right)$.

II) From (5.6) we have

$$\frac{d}{dt}\|v_\nu\|_l^2$$
$$= 2\Re\left(\frac{d}{dt}v_\nu, v_\nu\right)_l$$
$$= 2\Re\left(\sqrt{-1}|\zeta_\nu(D)|^2 v + \sum_{j=1}^n a_j\zeta_{\nu j}v + g, v\right)_l$$
$$+ 2\Re\Big(\Big(\rho'(t)\langle\zeta_\nu(D)\rangle_h^{1/\kappa} - \sqrt{-1}\sum_{j=1}^n \Lambda_{\sigma x_j}(x,\zeta_\nu(D))\circ\zeta_{\nu j} + q(t,x,\zeta_\nu(D))$$
$$+ \tilde{c}_\sigma(t,x,\zeta_\nu(D))\Big)v, v\Big)_l.$$

On the other hand, we have

(5.18)

$$\rho'(t)\langle\zeta_\nu(\xi)\rangle_h^{1/\kappa} + \sum_{j=1}^n \zeta_{\nu j}\Lambda_{\sigma x_j}(x,\zeta_\nu(\xi)) + \sum_{j=1}^n \Im a_j\zeta_{\nu j} + \Re\left(q(t,x,\zeta_\nu(\xi)) + \tilde{c}_\sigma(t,x,\zeta_\nu(\xi))\right) \le C$$

if we choose ρ, N and h in Lemma 3.2 suitably. Thus by Lemma 5.2 and the Sharp Gårding inequality (for instant Theorem 4.4 of Chapter 3 in [7]), for a constant $\gamma > 0$ which independent of ν, t we have

$$\frac{d}{dt}\|v_\nu(t)\|_l^2 \le 2\gamma\|v_\nu(t)\|_l^2 + 2\|g(t)\|_l\|v_\nu(t)\|_l \quad t\in[0,T];\nu=1,2,\cdots$$

and

$$\frac{d}{dt}\|v_\nu(t)\|_l \le \gamma\|v_\nu(t)\|_l + \|g(t)\|\|v_\nu(t)\|_l.$$

Therefore we get (5.13). Moreover from (5.6) we have

$$\frac{d}{dt}\Lambda_\nu^j v_\nu = \sqrt{-1}\left(P_\nu + [\Lambda_\nu^j, P_\nu]\Lambda_\nu^{-j}\right)\Lambda_\nu^j v_\nu + \Lambda_\nu^j g$$

and $P_{\nu,j} = P_\nu + [\Lambda_\nu^j, P_\nu]\Lambda_\nu^{-j}$. Here, $[\Lambda_\nu^j, P_\nu]\Lambda_\nu^{-j} \in C_t^0([0,T]; S^0_{\lambda_\nu})$ and $P_{\nu,j}$ satisfies (5.18). Hence we get (4.14) similarly to (5.13). On the other hand, noting

$$\frac{d}{dt}\Lambda_\nu^j v_\nu = \sqrt{-1}\left(\Lambda_\nu^j P_\nu\Lambda_\nu^{-j-2}\right)\Lambda_\nu^{j+2}v_\nu + \Lambda_\nu^j g$$

where $\sigma\left(\Lambda_\nu^j P_\nu \Lambda_\nu^{-j-2}\right) \in C_t^0([0,T]; S^0_{\lambda_\nu})$ (uniformly on ν), we have

$$\|\frac{d}{dt}\Lambda_\nu^j v_\nu(t)\|_l \leq C_1\|\Lambda_\nu^{j+2} v_\nu\|_l + \|\Lambda_\nu^j g\|_l.$$

Then from this and (5.14), we get (5.15). Put $v_\nu(t) - v_\nu(t') = \int_{t'}^t \frac{d}{d\tau} v_\nu(\tau) d\tau$. Then we get (5.16) from (5.15). This completes the proof of Lemma 5.3. □

Proof of Proposition 5.2. I) First we assume that $v_0 \in H^{l+4}$, $g(t) \in C_t^0([0,T]; H^{l+4})$. Then it follows from Lemma 5.3 that there is a solution $v_\nu(t)$ of (5.6).Then, from (5.9) we have

$$\|\Lambda_\nu^j v_0\|_l \leq \|\Lambda^j v_0\|_l, \qquad \|\Lambda_\nu^j g\|_l \leq \|\Lambda^j g\|_l$$

for $j \leq 4$, where $\Lambda = \langle D_x \rangle$. Therefore by (5.13), (5.14), (5.16) we have

$$\|\Lambda_\nu^j v_\nu(t)\|_l \leq C_1\left\{\|\Lambda_\nu^j v_\nu\|_l + \max_{[0,T]}\|\Lambda_\nu^j g\|_l\right\}, \tag{5.20}$$

$$\|\Lambda_\nu^j (v_\nu(t) - v_\nu(t'))\|_l \leq C_2|t-t'|\left\{\|\Lambda_\nu^{j+2} v_\nu\|_l + \max_{[0,T]}\|\Lambda_\nu^{j+2} g\|_l\right\}, \tag{5.21}$$

for $j \leq 4$.

Therefore if we fix $t_0 \in [0,T]$ arbitrarily, then from (5.20) with $j = 0$, $\{v_\nu(t_0)\}_{\nu=1}^\infty$ is a bounded sequence in H^l. Hence there exists $v(t_0) \in H^l$ and a sequence such that $\{v_{\nu_m}\}_{m=1}^\infty$ of H^l we have

$$v_{\nu_m} \to v(t_0) \ (weakly) \ in \ H^l \ (m \to \infty).$$

Let $\{t_k\}_{k=1}^\infty$ be a dense set in $[0,T]$. Then by the diagonal method there exists $v(t_k) \in H^l$ and we have

$$v_{\nu_m}(t_k) \to v(t_k) \ (weakly) \ in \ H^l \ (m \to \infty). \tag{5.22}$$

Then from (5.21) with $j = 0$, for any $t, t' \in \{t_k\}_{k=1}^{\infty}$ we have

$$\|v(t) - v(t')\|_l \le C_2|t - t'| \left\{ \|\Lambda^2 v_0\|_l + \max_{[0,T]} \|\Lambda^2 g\|_l \right\}. \tag{5.23}$$

For any $t_0 \in [0, T]$ we choose subsequence $\{t_{k_s}\}_{s=1}^{\infty}$ of $\{t_k\}_{k=1}^{\infty}$ such that $t_{k_s} \to t_0$. Then by (5.23), $\{t_{k_s}\}_{s=1}^{\infty}$ becomes a Cauchy sequence in $H^l(\mathbb{R}^n)$. Thus there exists $v(t_0) \in H^l$ and we have $v(t_{k_s}) \to v(t_0)$ in H^l $(s \to \infty)$. Moreover for all $v(t)$ we obtain (4.23) and

$$v_{\nu_m}(t) \to v(t) \ (weakly) \ in \ H^l \ (m \to \infty).$$

Thus for $\varphi \in \mathcal{S}$ we have

$$\left(\Lambda_{\nu_m}^k v_{\nu_m}(t), \varphi\right)_l = \left(v_{\nu_m}(t), \Lambda_{\nu_m}^k \varphi\right)_l \to \left(v(t), \Lambda^k \varphi\right)_l \ (m \to \infty). \tag{5.24}$$

Therefore noting $|\left(\Lambda_{\nu_m}^k v_{\nu_m}(t), \varphi\right)_l| \le \|\Lambda_{\nu_m}^k v_{\nu_m}(t)\|_l \|\varphi\|_l$, we get from (5.20) and (5.24)

$$|\left(v(t), \Lambda^k \varphi\right)_l| \le C_1 \left\{\|\Lambda^k v_0\|_l + \|\Lambda^k g\|_l\right\} \|\varphi\|_l \ (\varphi \in \mathcal{S}) \tag{5.25}$$

for $k \le 4$. We define $v_{R,k}(t, x) \in H^l$ for any $t \in [0, T]$ and $R > 0$ such that $\hat{v}_{R,k}(t, \xi) = \langle\xi\rangle^k \hat{v}(t, \xi)$, and $\hat{v}_{R,k} = 0$ $(|\xi| > R)$ and take the sequence $\{\varphi_j\}_{j=1}^{\infty}$ of $\mathcal{S}$ as supp $\hat{\varphi} \subset \{|\xi| \le R\}$, $\hat{\varphi}_j \to \hat{v}_{R,k}(t, \xi)$ in H^l $(j \to \infty)$. Then we have

$$\left(u(t), \Lambda^k \varphi_j\right)_l = \int \hat{u}(t, \xi)\langle\xi\rangle^{2l+k}\overline{\hat{\varphi}_j(\xi)}d\xi \to \int_{|\xi| \le R} \langle\xi\rangle^{2(l+k)} |\hat{u}(t, \xi)|^2 d\xi$$
$$(j \to \infty)$$

Therefore from (5.25) we get

$$\|v_{R,k}(t)\|_l \le C_1 \left\{ \|\Lambda^k v_0\|_l + \max_{[0,T]} \|\Lambda^k g\|_l \right\} \quad (k \le 4).$$

Hence taking $R \to \infty$, we have $\Lambda^k v \in H^l$ and

$$(5.26) \qquad \|\Lambda^k v\|_l \le C_1 \left\{ \|\Lambda^k v_0\|_l + \max_{[0,T]} \|\Lambda^k g\|_l \right\} \quad (k \le 4).$$

From (5.21) we have similarly to (5.26)

$$(5.27) \qquad \|\Lambda^k (v(t) - v(t'))\|_l \le C_2' |t - t'| \left\{ \|\Lambda^{k+2} v_0\|_l + \max_{[0,T]} \|\Lambda^{k+2} g\|_l \right\} \quad (k \le 2).$$

Thus we obtain $u(t)$, $\Lambda u(t)$ and $\Lambda^2 u(t)$ are contained in $C_t^0([0,T]; H^l)$.

We take $\varphi(t,x) = \varphi_1(t)\varphi_2(x) \in C_0^\infty((0,T) \times \mathbb{R}^n)$ arbitrarily. Noting

$$L_{\nu_m}^* \varphi \to L^* \varphi \ in \ C_t^0([0,T]; H^l),$$

we have

$$\begin{aligned} \iint_{\Omega_T} Lv \cdot \bar{\varphi} dx dt &= \iint_{\Omega_T} v \cdot \overline{L^* \varphi} dx dt \\ &= \lim_{m \to \infty} \iint_{\Omega_T} v_{\nu_m} \cdot \overline{L_{\nu_m}^* \varphi} dx dt = \lim_{m \to \infty} \iint_{\Omega_T} Lv_{\nu_m} \cdot \overline{\varphi} dx dt \\ &= \iint_{\Omega_T} g \cdot \bar{\varphi} dx dt. \end{aligned}$$

Thus $Lv = g$ $(t \in [0,T])$. Therefore u is a solution of (5.2). Also noting $v(t)$, $\Lambda v(t)$ and $\Lambda^2 v(t)$ are contained in $C_t^0([0,T]; H^l)$ and $\partial_t v = \sqrt{-1} P(t) v + g$, we get $v(t) \in C_t^1([0,T]; H^l) \cap C_t^0([0,T]; H^{l+2})$.

II) If $v(t) \in C_t^1([0,T]; H^l) \cap C_t^0([0,T]; H^{l+2})$ is a solution of (5.2), we obtain the energy inequality (5.3) similarly to the proof of Lemma 5.3. Assume $v_0 \in H^{l+2}$ and $g(t) \in C_t^0([0,T]; H^{l+2})$. Put $v_{0,\varepsilon} = \chi_\varepsilon(D_x) v_0$ and $g_\varepsilon(t) = \chi_\varepsilon(D_x) g(t)$, where $\chi(\xi) \in \mathcal{S}_\xi$ satisfies $\chi(0) = 1$ and $\chi_\varepsilon(\xi) = \chi(\varepsilon \xi)$. Then $v_{0,\varepsilon} \in H^{l+4}$, $g_\varepsilon(t) \in C_t^0([0,T]; H^{l+2})$. Therefore from I), we get the solution $v_\varepsilon(t)$ of (5.2) for $v_{0,\varepsilon}$ and g_ε. Then by (5.3) we have

$$\|v_\varepsilon(t) - v_{\varepsilon'}(t')\|_{l+2} \le e^{\gamma_1 T} \left\{ \|v_{0,\varepsilon} - v_{0,\varepsilon'}\|_{l+2} + T \max_{[0,T]} \|g_\varepsilon(\tau) - g_{\varepsilon'}(\tau)\|_{l+2} \right\}.$$

On the other hand when ε, $\varepsilon' \to 0$ we have

$$\begin{cases} \|v_{0,\varepsilon} - v_{0,\varepsilon'}\|_{l+2} = \|(\chi_\varepsilon(D_x) - \chi_{\varepsilon'}(D_x))v_0\|_{l+2} \to 0, \\ \max_{[0,T]} \|g_\varepsilon(\tau) - g_{\varepsilon'}(\tau)\|_{l+2} = \max_{[0,T]} \|(\chi_\varepsilon(D_x) - \chi_{\varepsilon'}(D_x))g(\tau)\|_{l+2} \to 0. \end{cases}$$

Therefore, noting $\partial_t v_\varepsilon = \sqrt{-1}Pv_\varepsilon + g_\varepsilon$, we can see $\{v_\varepsilon\}_{0<\varepsilon<1}$ becomes a Cauchy sequence in $C_t^1([0,T];H^l) \cap C_t^0([0,T];H^{l+2})$ and the limit of this series $v(t)$ is a solution of (5.2). This completes the Proposition 5.2. □

Proof of Theorem 1.1 and Theorem 1.2. By Propositions 5.1 and 5.2 we see that there exists an unique solution $v(t,x) \in C_t^0([0,T];H^l) \cap C_t^1([0,T];H^{l+2})$ of the Cauchy problem (5.1) with initial data $v_0(x)$ and we get the energy inequality

$$\|v(t,\cdot)\|_l \leq C(T)\left(\|v_0\|_l + \int_0^t \|e^\Lambda f(\tau,\cdot)\|_l d\tau\right)$$

for $t \in [0,T]$. Hence, we obtain the unique solution $u(t,x) = \left(e^\Lambda\right)^{-1} v(t,x)$ of the equation (1.1) with initial data by using the Sharp Gårding inequality and the energy inequality

$$\|u(t,\cdot)\|_l \leq C'(T)\left(\|u_0\|_l + \int_0^t \|f(\tau,\cdot)\|_l d\tau\right)$$

for $t \in [0,T]$, $(0 < \sigma < 1$ or $1 < \sigma)$ and

$$\|u(t,\cdot)\|_l \leq C'(T)\left(\|u_0\|_{l'} + \int_0^t \|f(\tau,\cdot)\|_{l'} d\tau\right)$$

for $t \in [0,T]$ and $l' \in \mathbb{R}$ $(\sigma = 1)$. Thus we complete the proof of our Theorems. □

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