ELASTIC WAVE PROPAGATION PROBLEMS IN STRATIFIED MEDIA $\mathbf{R}^3$

SENJO SHIMIZU

THESIS

Submitted in partial fulfillment of the requirements for the degree of
Doctor of Philosophy (Science)
in Doctoral Program in Mathematics at the
University of Tsukuba

January, 1994

Preface

This paper deals with elastic wave propagation in stratified media $\mathbf{R}^3$.

The eigenfunction expansion theory for elastic wave propagation problems has been studied by several authors. J. R. Schulenberger [I-9], [I-10] gave eigenfunction expansions in the half-space $\mathbf{R}^n_+$ $(n = 2, 3)$, using the method developed by S. Wakabayashi [I-16]. Y. Dermenjian and J. C. Guillot [I-3] studied scattering theory for elastic wave propagation starting with the basic elastic operators (symmetric systems of second order). J. C. Guillot [I-5] proved the existence and uniqueness of a Rayleigh surface wave propagation along the free boundary of a transversely isotropic elastic half space. In Chapter I, we shall derive eigenfunction expansions associated with the stationary problems for elastic wave propagation in plane-stratified media $\mathbf{R}^3$ using the methods due to S. Wakabayashi, and also J. C. Guillot. The eigenfunction expansion is given in terms of a family of generalized eigenfunctions corresponding to incident, reflected, refracted and Stoneley waves.

Energy distribution of the solutions of various wave propagation problems has been studied by C. H. Wilcox ([II-10], [II-11], [II-12], [II-13]). Chapter II provides energy distribution of the solutions of elastic wave propagation problems in plane-stratified media $\mathbf{R}^3$ using methods due to Wilcox. We construct asymptotic wave functions by using spectral integral representations of the solutions and the method of stationary phase. The integral representations are based on an eigenfunction expansion theory derived in Chapter I. We calculate asymptotic energy of the solutions for large times of the interface problems for elastic waves and show that the energy of the Stoneley components of the solutions with finite energy is asymptotically concentrated along the interface.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

TABLE OF CONTENTS

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

CHAPTER I

EIGENFUNCTION EXPANSIONS FOR ELASTIC WAVE PROPAGATION PROBLEMS IN STRATIFIED MEDIA $\mathbf{R}^3$

Abstract

This paper provides eigenfunction expansions associated with the stationary problems for elastic wave propagation in stratified media $\mathbf{R}^3$. The eigenfunction expansion is given in terms of a family of generalized eigenfunctions corresponding to incident, reflected, refracted and Stoneley waves.

1991 Mathematical Subject Classification: Primary 35P10; Secondary 73C35.

Key words and phrases: Eigenfunction expansions, elastic wave propagation problems, interface conditions, stratified media, Stoneley wave.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

§ 0. Introduction

This paper provides eigenfunction expansions associated with the stationary problems for elastic wave propagation in *stratified* media $\mathbf{R}^3$. The eigenfunction expansion is given in terms of a family of generalized eigenfunctions corresponding to incident, reflected, refracted and Stoneley waves.

The eigenfunction expansion theory for wave propagation problems has been studied by several authors (for example, K. Mochizuki [8], J. R. Schulenberger and C. H. Wilcox [11], C. H. Wilcox [18]). S. Wakabayashi [16] provided eigenfunction expansions associated with the stationary problems in the half-space $\mathbf{R}^n_+$ for symmetric hyperbolic systems with constant coefficients. Such systems were first studied in $\mathbf{R}^n_+$ by M. Matsumura [7]. The eigenfunction expansion is given in terms of a family of generalized or improper eigenfunctions corresponding to incident, reflected and surface or boundary waves.

For elastic wave propagation, J. R. Schulenberger [9], [10] gave eigenfunction expansions in the half-space $\mathbf{R}^n_+$ ($n = 2, 3$), using the method developed by S. Wakabayashi. He transformed the 2×2 second order system of linear elasticity into a 5×5 first order system. But the defect of this approach is to introduce static solutions corresponding to a zero propagation speed which do not appear in the elastic wave propagation. The treatment (for example the definition domain) for the selfadjoint operator associated with non elliptic spatial part is somewhat complicated (see [16, Section 7]). Moreover the relations between the displacement vector solutions of the original system and solutions of the transformed system are complicated.

Y. Dermenjian and J. C. Guillot [3] studied scattering theory for elastic wave propagation starting with the basic elastic operators (symmetric systems of second order). J. C. Guillot [5] proved the existence and uniqueness of a Rayleigh surface wave propagation along the free boundary of a transversely isotropic elastic half space, by reducing the basic operator to a family of operators which is easier to study. Concerning stratified media, there is an interesting work by C. H. Wilcox [17] on eigenfunction expansions for the Pekeris differential operator in terms of free wave eigenfunctions and guided wave eigenfunctions.

In this paper we shall derive eigenfunction expansions associated with the stationary problems for elastic wave propagation in plane-stratified media $\mathbf{R}^3$ using the methods due to S. Wakabayashi [16], and also J. C. Guillot [5]. Schulenberger's works [9], [10] are useful references in our study.

We consider the plane stratified medium $\mathbf{R}^3 = \{x = (x_1, x_2, x_3); x_i \in \mathbf{R}\}$ with the planar interface $x_3 = 0$, which is defined by

$$(\lambda(x_3), \mu(x_3), \rho(x_3)) = \begin{cases} (\lambda_1, \mu_1, \rho_1), & x_3 < 0, \\ (\lambda_2, \mu_2, \rho_2), & x_3 > 0. \end{cases}$$

Here λ_1, λ_2, μ_1, μ_2 are certain quantities called the Lamé constants and $\rho_1, \rho_2 > 0$ are the densities.

For simplicity, we shall denote the lower halfspace $\mathbf{R}^3_- = \{x \in \mathbf{R}^3; x_3 < 0\}$ by *medium I* and the upper halfspace $\mathbf{R}^3_+ = \{x \in \mathbf{R}^3; x_3 > 0\}$ by *medium II*, as in Figure 1.

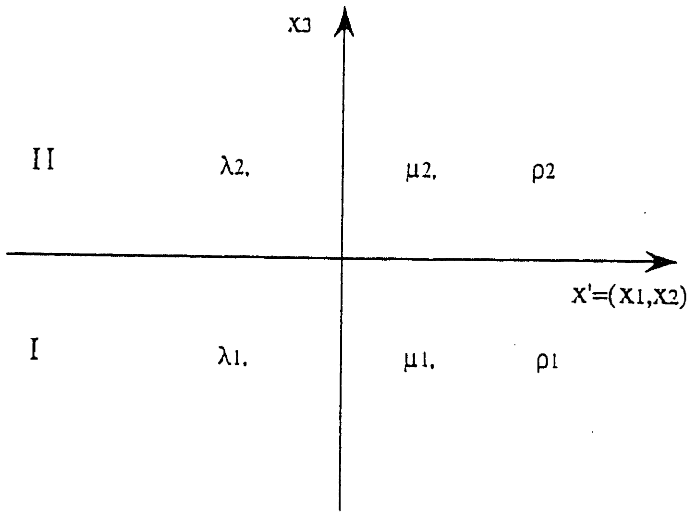

FIGURE 1 STRATIFIED MEDIA I AND II

The equations describing the propagation of elastic waves in the stratified medium are given by

$$(0.1)\qquad \frac{\partial^2 u_i}{\partial t^2} = \sum_{j=1}^{3} \frac{1}{\rho(x_3)} \frac{\partial \sigma_{ij}}{\partial x_j}(u), \quad i = 1, 2, 3,$$

where $u(x,t) = {}^t(u_1(x,t), u_2(x,t), u_3(x,t))$ is the displacement vector, and $\{\sigma_{ij}\}$ is the symmetric stress tensors defined by

$$\sigma_{ij}(u) = \lambda(x_3)(\nabla \cdot u)\delta_{ij} + 2\mu(x_3)\varepsilon_{ij}(u), \quad \varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Here ${}^tM$ denotes the transpose of a matrix M.

The $c^I_{ki\ell j}, c^{II}_{ki\ell j} (i, j, k, \ell = 1, 2, 3)$ are the stress-strain tensors given by

$$(0.2)\qquad \begin{aligned} c^I_{ki\ell j} &= \lambda_1 \delta_{ki}\delta_{\ell j} + \mu_1(\delta_{k\ell}\delta_{ij} + \delta_{kj}\delta_{i\ell}), \\ c^{II}_{ki\ell j} &= \lambda_2 \delta_{ki}\delta_{\ell j} + \mu_2(\delta_{k\ell}\delta_{ij} + \delta_{kj}\delta_{i\ell}) \end{aligned}$$

with the properties

$$\begin{aligned} c^I_{ki\ell j} &= c^I_{ik\ell j} = c^I_{kij\ell} = c^I_{\ell jki}, \\ c^{II}_{ki\ell j} &= c^{II}_{ik\ell j} = c^{II}_{kij\ell} = c^{II}_{\ell jki}, \end{aligned}$$

and δ_{ki} is the Kronecker delta. We assume that the constants $c^I_{ki\ell j}$, $c^{II}_{ki\ell j}$ satisfy the following stability conditions

$$(0.3)\qquad \begin{aligned} 3\lambda_1 + 2\mu_1 > 0, &\quad \mu_1 > 0, \\ 3\lambda_2 + 2\mu_2 > 0, &\quad \mu_2 > 0, \end{aligned}$$

which are equivalent to the conditions

$$(0.3')\qquad \begin{aligned} \sum_{k,i,\ell,j=1}^{3} c^I_{ki\ell j} s_{\ell j}\overline{s_{ki}} &\geq \exists \delta_1 \sum_{k,i=1}^{3} |s_{ki}|^2, \qquad \delta_1 > 0, \\ \sum_{k,i,\ell,j=1}^{3} c^{II}_{ki\ell j} s_{\ell j}\overline{s_{ki}} &\geq \exists \delta_2 \sum_{k,i=1}^{3} |s_{ki}|^2, \qquad \delta_2 > 0, \end{aligned}$$

for all complex symmetric 3×3 matrices (s_{ki}), $s_{ki} = s_{ik} \in \mathbf{C}$ (cf. [6]).

The wave equations (0.1) should be supplemented by interface conditions at the interface $x_3 = 0$ of the medium. We now impose on u the following conditions at the interface $x_3 = 0$.

$$u^I\big|_{x_3=0} = u^{II}\big|_{x_3=0}, \tag{0.4}$$

$$\sigma_{i3}(u^I)\big|_{x_3=0} = \sigma_{i3}(u^{II})\big|_{x_3=0}, \tag{0.5}$$

where $u = u^I$ for $x \in \mathbf{R}^3_-$, and $u = u^{II}$ for $x \in \mathbf{R}^3_+$.

The equations (0.1) may be written in the following form:

$$\frac{\partial^2 u}{\partial t^2} + Mu = 0, \tag{0.6}$$

(0.7)

$$\begin{aligned} Mu &= -\frac{\lambda+\mu}{\rho}\nabla(\nabla\cdot u) - \frac{\mu}{\rho}\Delta u \\ &= -\frac{1}{\rho}\begin{pmatrix} (\lambda+2\mu)\frac{\partial^2}{\partial x_1^2} + \mu(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}) & (\lambda+\mu)\frac{\partial^2}{\partial x_1 \partial x_2} & (\lambda+\mu)\frac{\partial^2}{\partial x_1 \partial x_3} \\ (\lambda+\mu)\frac{\partial^2}{\partial x_1 \partial x_2} & (\lambda+2\mu)\frac{\partial^2}{\partial x_2^2} + \mu(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_3^2}) & (\lambda+\mu)\frac{\partial^2}{\partial x_2 \partial x_3} \\ (\lambda+\mu)\frac{\partial^2}{\partial x_1 \partial x_3} & (\lambda+\mu)\frac{\partial^2}{\partial x_2 \partial x_3} & (\lambda+2\mu)\frac{\partial^2}{\partial x_3^2} + \mu(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}) \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \end{aligned}$$

where $\lambda = \lambda(x_3)$, $\mu = \mu(x_3)$, $\rho = \rho(x_3)$.

We interpret (0.1), (0.4), and (0.5) as an abstract wave equation

$$\frac{d^2 u}{dt^2} + Au = 0.$$

As we shall show later, A is a non-negative selfadjoint operator associated with (0.4), (0.5), (0.6) and (0.7) in the Hilbert space

$$\mathcal{H} = L^2(\mathbf{R}^3, \mathbf{C}^3, \rho(x_3)\,dx),$$

with inner product

$$(u, v) = \int_{\mathbf{R}^3} u \cdot v \rho(x_3)\,dx,$$

where $u \cdot v$ denotes the usual scalar product in $\mathbf{C}^3$: $u \cdot v = \sum_{i=1}^3 u_i \bar{v}_i$.

Let $\eta' = (\eta_1, \eta_2) \in \mathbf{R}^2$ be the dual variables of $x' = (x_1, x_2)$ and let $F_{x'}$ denote the partial Fourier transformation with respect to x';

$$\hat{u}(\eta', x_3) = (F_{x'}u)(\eta', x_3) = \underset{R\to\infty}{\text{l.i.m.}}\ \frac{1}{2\pi}\int_{|x'|\le R} e^{-i(x_1\eta_1 + x_2\eta_2)}u(x)\,dx'$$

for u in $\mathcal{H}$. Let

$$D(\hat{A}) = F_{x'}D(A) = \{\hat{u}; u \in D(A)\},$$
$$\hat{A}\hat{u} = F_{x'}AF_{\eta'}^{-1}\hat{u}, \qquad \hat{u} \in D(\hat{A}).$$

For every $\eta' \neq 0$, let

$$\mathrm{U} = \frac{1}{|\eta'|}\begin{pmatrix} \eta_1 & -\eta_2 & 0 \\ \eta_2 & \eta_1 & 0 \\ 0 & 0 & |\eta'| \end{pmatrix}, \quad \mathrm{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where U and C are unitary matrices and $|\eta'| = (\eta_1^2 + \eta_2^2)^{\frac{1}{2}}$. Then we have

$$Au = F_{\eta'}^{-1}\mathrm{UC}(A_1(\eta') \oplus A_2(\eta'))(\mathrm{UC})^{-1}F_{x'}u \quad \text{for } u \in D(A),$$

where $A_1(\eta')$ and $A_2(\eta')$ are non-negative selfadjoint operators (see Proposition 1.7).

We can get an explicit representation of the Green function $G_1(x_3, y_3, \eta'; \zeta)$ for the operator $A_1(\eta') - \zeta I$ ($\zeta \notin \mathbf{R}$) from the expression of the solution for the following problem:

$$(A_1(\eta', D) - \zeta)v(\eta', x_3) = f(\eta', x_3), \tag{0.8}$$
$$v(\eta', x_3)|_{x_3=-0} = v(\eta', x_3)|_{x_3=+0}, \tag{0.9}$$
$$B_1(\eta')v(\eta', x_3)|_{x_3=-0} = B_1(\eta')v(\eta', x_3)|_{x_3=+0}. \tag{0.10}$$

Here (0.9) and (0.10) are the interface conditions for $A_1(\eta', D)$ corresponding to (0.4) and (0.5). $A_1(\eta', D)$ ($D = \frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}x_3}$) is the differential operators corresponding to the selfadjoint operator $A_1(\eta')$. Since the solution v of (0.8) should satisfy the interface conditions (0.9) and (0.10), the denominator of v has the Lopatinski determinant $\Delta(\eta', \zeta)$ as follows:

$$\Delta(\eta', \zeta) = |\eta'|^6 D(z),$$

$$\begin{aligned} D(z) = &\left(2(\mu_1 - \mu_2) - \frac{\mu_1 z}{c_{s_1}^2} + \frac{\mu_2 z}{c_{s_2}^2}\right)^2 + 4(\mu_1 - \mu_2)^2 a_1 a_2 b_1 b_2 \\ &- a_1 b_1 \left(2(\mu_1 - \mu_2) + \frac{\mu_2 z}{c_{s_2}^2}\right)^2 - a_2 b_2 \left(2(\mu_1 - \mu_2) - \frac{\mu_1 z}{c_{s_1}^2}\right)^2 \\ &- \frac{\mu_1 \mu_2}{c_{s_1}^2 c_{s_2}^2}(a_1 b_2 + a_2 b_1) z^2, \end{aligned}$$

where

$$z = \frac{\zeta}{|\eta'|^2},$$

$$a_1 = \sqrt{1-\frac{z}{c_{p_1}^2}}, \quad a_2 = \sqrt{1-\frac{z}{c_{p_2}^2}}, \quad b_1 = \sqrt{1-\frac{z}{c_{s_1}^2}}, \quad b_2 = \sqrt{1-\frac{z}{c_{s_2}^2}}.$$

The squares of propagation speeds of shear(S) and pressure(P) waves are given by

$$c_{s_i}^2 = \frac{\mu_i}{\rho_i}, \quad c_{p_i}^2 = \frac{\lambda_i + 2\mu_i}{\rho_i}, \quad (i = 1, 2),$$

respectively. From the conditions (0.3), the minimum speed of $\{c_{s_1}, c_{p_1}, c_{s_2}, c_{p_2}\}$ is either c_{s_1} or c_{s_2}.

We can see that $D(z)$ has the only one real zero when $D(z)$ has zeros. Denote by c_{St}^2 its real zero. Then the zero of $\Delta(\eta', \zeta)$ is $c_{St}^2|\eta'|^2$ and is the origin of the Stoneley wave propagating along the interface $x_3 = 0$ in the elastic space R^3, and c_{St} is its speed.

By virtue of principle of the argument, the conditions for the existence of zeros of the Lopatinski determinant $\Delta(\eta', \zeta) = |\eta'|^6 D(z)$ (the existence of the Stoneley waves) are given as follows:
If $c_{s_1} < c_{s_2}$, then

(i) $D(c_{s_1}^2) > 0 \implies$ The zero $\zeta = c_{St}^2|\eta'|^2$ of $\Delta(\eta', \zeta)$ in ζ exists in $[0,\ c_{s_1}^2|\eta'|^2)$ with order 1. More precisely, we shall prove in the proof of Theorem 6.5 that $c_{St} \neq 0$.

(ii) $D(c_{s_1}^2) = 0 \implies c_{St} = c_{s_1}$ and we shall consider this case under some restricted conditions (cf. Lemma 6.4).

(iii) $D(c_{s_1}^2) < 0 \implies \Delta(\eta', \zeta)$ has no zero.

If $c_{s_2} < c_{s_1}$, then we must replace $D(c_{s_1}^2)$ by $D(c_{s_2}^2)$.

We also obtain an explicit representation of the Green function $G_2(x_3, y_3, \eta'; \zeta)$ for the operator $A_2(\eta') - \zeta I$ $(\zeta \notin \mathbf{R})$ by the same method as $G_1(x_3, y_3, \eta'; \zeta)$. The Lopatinski determinant corresponding to the operator $A_2(\eta') - \zeta I$ $(\zeta \notin \mathbf{R})$ has no zero. By using the Green functions $G_1(x_3, y_3, \eta'; \zeta)$ and $G_2(x_3, y_3, \eta'; \zeta)$, we define

$$\psi_{1j}(x_3, \eta; \zeta) = F_{y_3}^{-1}[G_1(x_3, y_3, \eta'; \zeta)](\xi)(\lambda_j(\eta) - \zeta)P_j(\eta)\rho(x_3)^{-1}, \quad j \in M,$$
$$\psi_{1j}^{St}(x_3, \eta; \zeta) = \frac{\zeta - c_{St}^2|\eta'|^2}{\zeta - \lambda_j(\eta)}\psi_{1j}(x_3, \eta; \zeta), \quad j \in M,$$
$$\psi_{2k}(x_3, \eta; \zeta) = F_{y_3}^{-1}[G_2(x_3, y_3, \eta'; \zeta)](\xi)(\lambda_j(\eta) - \zeta)\rho(x_3)^{-1}, \quad k \in N.$$

Here $\eta = (\eta_1, \eta_2, \xi) = (\eta', \xi)$, $\lambda_j(\eta) = c_j^2|\eta|^2$ are the eigenvalues of $A_1(\eta')$, $P_j(\eta)$ are mutually orthogonal projections for $A_1(\eta')$, $\lambda_k(\eta) = c_k^2|\eta|^2$ are the eigenvalues of $A_2(\eta')$, $M = \{s_1, p_1, s_2, p_2\}$ and $N = \{s_1, s_2\}$. When $\zeta \to \lambda_j(\eta) \pm i0$, $\zeta \to c_{St}^2|\eta|^2$, and $\zeta \to \lambda_k(\eta) \pm i0$, the limits $\psi_{1j}^{\pm}(x_3, \eta)$, $\psi_{1j}^{St}(x_3, \eta)$, and $\psi_{2k}^{\pm}(x_3, \eta)$ exist and these limit functions are generalized eigenfunctions for $A_1(\eta')$, $A_2(\eta')$, respectively.

Using these generalized eigenfunctions for $A_1(\eta')$, $A_2(\eta')$, we define generalized eigenfunctions for A as follows:

$$\psi_{1j}^{\pm}(x, \eta) = \frac{1}{2\pi}e^{i(x_1\eta_1 + x_2\eta_2)}\mathrm{UC}(\psi_{1j}^{\pm}(x_3, \eta) \oplus O_{1\times 1}), \quad j \in M,$$

$$\psi_{1j}^{St}(x,\eta) = \frac{1}{2\pi} e^{i(x_1\eta_1+x_2\eta_2)} \mathrm{UC}(\psi_{1j}^{St}(x_3,\eta) \oplus O_{1\times 1}), \quad j \in M,$$
$$\psi_{2k}^{\pm}(x,\eta) = \frac{1}{2\pi} e^{i(x_1\eta_1+x_2\eta_2)} \mathrm{UC}(O_{2\times 2} \oplus \psi_{2k}^{\pm}(x_3,\eta)), \quad k \in N.$$

where $O_{n\times n}$ denotes the $n \times n$ zero matrix.

Now we define the Fourier transform of $f \in \mathcal{H}$ with respect to these generalized eigenfunctions: $f \mapsto (\hat{f}_{1j}^{\pm}, \hat{f}_{1j}^{St}, \hat{f}_{2k}^{\pm})$,

$$\hat{f}_{1j}^{\pm}(\eta) = \operatorname*{l.i.m.}_{R\to\infty} \int_{|x|\le \mathbf{R}} \psi_{1j}^{\pm}(x,\eta)^* f(x)\rho(x_3)\, dx, \quad j \in M,$$
$$\hat{f}_{1j}^{St}(\eta) = \operatorname*{l.i.m.}_{R\to\infty} \int_{|x|\le \mathbf{R}} \psi_{1j}^{St}(x,\eta)^* f(x)\rho(x_3)\, dx, \quad j \in M,$$
$$\hat{f}_{2k}^{\pm}(\eta) = \operatorname*{l.i.m.}_{R\to\infty} \int_{|x|\le \mathbf{R}} \psi_{2k}^{\pm}(x,\eta)^* f(x)\rho(x_3)\, dx, \quad k \in N.$$

Our main results are the following three theorems. Theorem 0.1 corresponds to the Parseval and Plancherel formulas.

Theorem 0.1. *We assume that $D(c_{s_1}^2) > 0$ if $c_{s_1} < c_{s_2}$ and that $D(c_{s_2}^2) > 0$ if $c_{s_2} < c_{s_1}$. Let f, $g \in \mathcal{H}$ and $0 < a < b < \infty$. Then we have*

$$(f,g) = \sum_{j\in M} \left(\int_{\mathbf{R}^3} \hat{f}_{1j}^{\pm}(\eta) \cdot \hat{g}_{1j}^{\pm}(\eta)\, d\eta + \int_{\mathbf{R}^3} \hat{f}_{1j}^{St}(\eta) \cdot \hat{g}_{1j}^{St}(\eta)\, d\eta \right)$$
$$+ \sum_{k\in N} \int_{\mathbf{R}^3} \hat{f}_{2k}^{\pm}(\eta) \cdot \hat{g}_{2k}^{\pm}(\eta)\, d\eta.$$

The first half of Theorem 0.2 expresses the Fourier inversion formula with respect to generalized eigenfunctions. The latter half gives the canonical form for A.

Theorem 0.2. *We assume the same assumption as Theorem 0.1.*

(1) For $f \in \mathcal{H}$,

$$f(x) = \sum_{j\in M} \operatorname*{l.i.m.}_{R\to\infty} \int_{|\eta|\le R} \left(\psi_{1j}^{\pm}(x,\eta)\hat{f}_{1j}^{\pm}(\eta) + \psi_{1j}^{St}(x,\eta)\hat{f}_{1j}^{St}(\eta) \right) d\eta$$
$$+ \sum_{k\in N} \operatorname*{l.i.m.}_{R\to\infty} \int_{|\eta|\le R} \psi_{2k}^{\pm}(x,\eta)\hat{f}_{2k}^{\pm}(\eta)\, d\eta.$$

(2) For $f \in D(A)$,

$$Af(x) = \sum_{j\in M} \operatorname*{l.i.m.}_{R\to\infty} \int_{|\eta|\le R} \left(\lambda_j(\eta)\psi_{1j}^{\pm}(x,\eta)\hat{f}_{1j}^{\pm}(\eta) + c_{St}^2|\eta'|^2\psi_{1j}^{St}(x,\eta)\hat{f}_{1j}^{St}(\eta) \right) d\eta$$
$$+ \sum_{k\in N} \operatorname*{l.i.m.}_{R\to\infty} \int_{|\eta|\le R} \lambda_k(\eta)\psi_{2k}^{\pm}(x,\eta)\hat{f}_{2k}^{\pm}(\eta)\, d\eta,$$

and

$$(\widehat{Af})_{1j}^{\pm}(\eta) = \lambda_j(\eta)\hat{f}_{1j}^{\pm}(\eta), \quad j \in M,$$
$$(\widehat{Af})_{1j}^{St}(\eta) = c_{St}^2|\eta'|^2\hat{f}_{1j}^{St}(\eta), \quad j \in M,$$
$$(\widehat{Af})_{2k}^{\pm}(\eta) = \lambda_k(\eta)\hat{f}_{2k}^{\pm}(\eta), \quad k \in N.$$

Theorem 0.3 gives an explicit expression of the ranges $R(\Phi^{\pm})$.

Theorem 0.3. *Assume the same assumption as Theorem 0.1. We define the mappings by*

$$\Phi_{1j}^{\pm}: \mathcal{H} \ni f \to \hat{f}_{1j}^{\pm}(\eta) \in (P_j(\eta) \oplus O_{1\times 1})L^2(\mathbf{R}_{\pm}^3, \mathbf{C}^3),\ j \in M$$
$$\Phi_{1j}^{St}: \mathcal{H} \ni f \to \hat{f}_{1j}^{St}(\eta) \in (P_j(\eta) \oplus O_{1\times 1})L^2(\mathbf{R}^3, \mathbf{C}^3),\ j \in M$$
$$\Phi_{2k}^{\pm}: \mathcal{H} \ni f \to \hat{f}_{2k}^{\pm}(\eta) \in (O_{2\times 2} \oplus 1)L^2(\mathbf{R}_{\pm}^3, \mathbf{C}^3),\ k \in N,$$

and put

$$\Phi^{\pm} = \sum_{j\in M} \Phi_{1j}^{\pm} \oplus \sum_{j\in M} \Phi_{1j}^{St} \oplus \sum_{k\in N} \Phi_{2k}^{\pm}.$$

Then we have

$$R(\Phi^{\pm}) = \sum_{j\in M} \oplus(P_j(\eta) \oplus O_{1\times 1})L^2(\mathbf{R}_{\pm}^3, \mathbf{C}^3) \oplus \sum_{j\in M} \oplus(P_j(\eta) \oplus O_{1\times 1})L^2(\mathbf{R}^3, \mathbf{C}^3)$$
$$\oplus \sum_{k\in N} \oplus(O_{2\times 2} \oplus 1)L^2(\mathbf{R}_{\pm}^3, \mathbf{C}^3).$$

This implies that $\Phi^{\pm}$ are unitary operators in $\mathcal{H}$, and that the system of generalized eigenfunctions $\{\psi_{1j}^{\pm}, \psi_{1j}^{St}, \psi_{2k}^{\pm}\}$ is complete.

The remainder of this paper consists of seven sections. In Section 1, we prove the selfadjointness of the operator A governing the wave propagation of the elastic waves in plane-stratified media $\mathbf{R}^3$. In Section 2, we give a construction and an explicit representation of the Green function $G_1(x_3, y_3, \eta'; \zeta)$ for the operator $A_1(\eta') - \zeta I$ ($\zeta \notin \mathbf{R}$). In Section 3, the number and nature of the zeros of the Lopatinski determinant of $A_1(\eta')$ are studied by using Cagniard's method. In Section 4, we define a family of generalized eigenfunctions for $A_1(\eta')$ by using the Green function $G_1(x_3, y_3, \eta'; \zeta)$. In Section 5, we give an explicit representation of the Green function $G_2(x_3, y_3, \eta'; \zeta)$ for the operator $A_2(\eta') - \zeta I$ ($\zeta \notin \mathbf{R}$) and a family of generalized eigenfunctions for $A_2(\eta')$. In Section 6, we construct the spectral family of A by means of the generalized eigenfunctions of $A_1(\eta')$ and $A_2(\eta')$. We also prove the Parseval formula (Theorem 0.1). Finally in Section 7, we prove the eigenfunction expansion theorems (Theorem 0.2 and 0.3).

§ 1. The Selfadjoint Operator A

In this section, we shall prove the selfadjointness of the operator A along standard results in the theory of linear operators in Hilbert space.

Let us describe the operator A more carefully. We have

$$Mu = \frac{1}{\rho(x_3)} \sum_{i,j=1}^{3} M_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j},$$

with

$$M_{11} = -\begin{pmatrix} \lambda(x_3) + 2\mu(x_3) & 0 & 0 \\ 0 & \mu(x_3) & 0 \\ 0 & 0 & \mu(x_3) \end{pmatrix}, M_{12} = -\begin{pmatrix} 0 & \lambda(x_3) & 0 \\ \mu(x_3) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$M_{13} = -\begin{pmatrix} 0 & 0 & \lambda(x_3) \\ 0 & 0 & 0 \\ \mu(x_3) & 0 & 0 \end{pmatrix}, M_{22} = -\begin{pmatrix} \mu(x_3) & 0 & 0 \\ 0 & \lambda(x_3)+2\mu(x_3) & 0 \\ 0 & 0 & \mu(x_3) \end{pmatrix},$$

$$M_{23} = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \lambda(x_3) \\ 0 & \mu(x_3) & 0 \end{pmatrix}, M_{33} = -\begin{pmatrix} \mu(x_3) & 0 & 0 \\ 0 & \mu(x_3) & 0 \\ 0 & 0 & \lambda(x_3)+2\mu(x_3) \end{pmatrix},$$

$$M_{21} = {}^t M_{12}, \quad M_{31} = {}^t M_{13}, \quad M_{32} = {}^t M_{23}.$$

We represent M and $M_{ij}(1 \leq i, j \leq 3)$ as follows:

$$M = \begin{cases} M^I, & x_3 < 0, \\ M^{II}, & x_3 > 0, \end{cases} \qquad M_{ij} = \begin{cases} M^I_{ij}, & x_3 < 0, \\ M^{II}_{ij}, & x_3 > 0. \end{cases}$$

The interface condition (0.5) can be written as follows:

$$\text{(1.1)} \qquad \left. \sum_{j=1}^{3} M^I_{3j} \frac{\partial u^I}{\partial x_j} \right|_{x_3=0} = \left. \sum_{j=1}^{3} M^{II}_{3j} \frac{\partial u^{II}}{\partial x_j} \right|_{x_3=0}.$$

The Sobolev spaces for an open subset Ω of $\mathbf{R}^3$ are defined by

$$H^m(\Omega, \mathbf{C}^3) = \{u \in \mathbf{C}^3; D^\alpha u \in L^2(\Omega, \mathbf{C}^3), \text{ for } |\alpha| \leq m\}.$$

Here m is a non-negative integer and the multi-index notation is used for derivatives. Thus $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ where each α_j is a non-negative integer, $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3}$, $D_j = \frac{\partial}{\partial x_j}$ $(j = 1, 2, 3)$ and $|\alpha| = \alpha_1 + \alpha_2 + \alpha_2$. $H^m(\Omega, \mathbf{C}^3)$ is a Hilbert space with inner product

$$\text{(1.2)} \qquad (u, v)_m = \int_\Omega \sum_{|\alpha| \leq m} D^\alpha u(x) \cdot D^\alpha v(x)\, dx.$$

Definition 1.1. $u \in H^1(\mathbf{R}^3, \mathbf{C}^3) \cap \{Mu \in \mathcal{H}\}$ is said to satisfy the *generalized free interface condition* on $x_3 = 0$ if one has

$$\text{(1.3)} \qquad \int_{\mathbf{R}^3} Mu \cdot v\rho(x_3)\, dx + \sum_{i,j=1}^{3} \int_{\mathbf{R}^3} M_{ij} \frac{\partial u}{\partial x_j} \cdot \frac{\partial v}{\partial x_i}\, dx = 0$$

for all $v \in H^1(\mathbf{R}^3, \mathbf{C}^3)$.

Let $D(A)$ denote the set of functions $u \in H^1(\mathbf{R}^3, \mathbf{C}^3) \cap \{Mu \in \mathcal{H}\}$ which satisfy the generalized free interface condition (1.3). We then have the following theorem:

Theorem 1.2. *The following operator A with domain $D(A)$:*

$$Au = Mu, \quad u \in D(A),$$

is a non-negative selfadjoint operator in the Hilbert space $\mathcal{H} = L^2(\mathbf{R}^3, \mathbf{C}^3, \rho(x_3)\, dx)$.

And u belongs to $D(A)$ if and only if u belongs to $H^2(\mathbf{R}^3_-, \mathbf{C}^3) \oplus H^2(\mathbf{R}^3_+, \mathbf{C}^3)$ and satisfies the interface conditions (0.4) and (0.5) in the sense of trace on $x_3 = 0$.

In order to prove Theorem 1.2, we prepare some Lemmas.

Lemma 1.3. *The operator A is symmetric; that is,*

$$A \subset A^*. \tag{1.4}$$

Proof. To prove (1.4), note that the set

$$\mathcal{D}_0(\mathbf{R}^3, \mathbf{C}^3) = \mathcal{D}(\mathbf{R}^3, \mathbf{C}^3) \cap \{u; u(x) = 0 \text{ in a neighborhood of } x_3 = 0\}$$

is a subset of $D(A)$. And $\mathcal{D}_0(\mathbf{R}^3, \mathbf{C}^3)$ is dense in $\mathcal{H}$. Hence $D(A)$ is dense in $\mathcal{H}$, so the adjoint operator A^* is uniquely defined. If u and v are both in $D(A)$, then we have by using interface condition (1.3)

(1.5)

$$\begin{aligned}
(Au, v) &= (Mu, v) \\
&= \int_{\mathbf{R}^3} Mu \cdot v\rho(x_3)\, dx \\
&= \int_{\mathbf{R}^3_-} M^I u^I \cdot v^I \rho_1\, dx + \int_{\mathbf{R}^3_+} M^{II} u^{II} \cdot v^{II} \rho_2\, dx \\
&= + \sum_{i,j=1}^{3} \int_{\mathbf{R}^3_-} M^I_{ij} \frac{\partial^2}{\partial x_i \partial x_j} u^I \cdot v^I\, dx + \sum_{i,j=1}^{3} \int_{\mathbf{R}^3_+} M^{II}_{ij} \frac{\partial^2}{\partial x_i \partial x_j} u^{II} \cdot v^{II}\, dx \\
&= - \sum_{i,j=1}^{3} \int_{\mathbf{R}^3_-} M^I_{ij} \frac{\partial u^I}{\partial x_j} \cdot \frac{\partial v^I}{\partial x_i}\, dx + \sum_{j=1}^{3} \int_{\partial \mathbf{R}^3_-} M^I_{3j} \frac{\partial u^I}{\partial x_j} \cdot v^I\, dx' \\
&\quad - \sum_{i,j=1}^{3} \int_{\mathbf{R}^3_+} M^{II}_{ij} \frac{\partial u^{II}}{\partial x_j} \cdot \frac{\partial v^{II}}{\partial x_i}\, dx - \sum_{j=1}^{3} \int_{\partial \mathbf{R}^3_+} M^{II}_{3j} \frac{\partial u^{II}}{\partial x_j} \cdot v^{II}\, dx' \\
&= \int_{\mathbf{R}^3_-} u^I \cdot M^I v^I \rho_1\, dx + \int_{\mathbf{R}^3_+} u^{II} \cdot M^{II} v^{II} \rho_2\, dx \\
&= \int_{\mathbf{R}^3} u \cdot Mv\rho(x_3)\, dx \\
&= (u, Av),
\end{aligned}$$

which is equivalent to (1.4). □

Lemma 1.4. *The symmetric operator A is non-negative; that is*

$$A \geq 0. \tag{1.6}$$

Proof. Putting $v = u \in D(A)$ in the first half of the formula (1.5), we have

$$\begin{aligned}
(Au, u) &= \int_{\mathbf{R}^3_-} M^I u^I \cdot u^I \rho_1\, dx + \int_{\mathbf{R}^3_+} M^{II} u^{II} \cdot u^{II} \rho_2\, dx \\
&= - \sum_{i,j=1}^{3} \int_{\mathbf{R}^3_-} M^I_{ij} \frac{\partial u^I}{\partial x_j} \cdot \frac{\partial u^I}{\partial x_i}\, dx - \sum_{i,j=1}^{3} \int_{\mathbf{R}^3_+} M^{II}_{ij} \frac{\partial u^{II}}{\partial x_j} \cdot \frac{\partial u^{II}}{\partial x_i}\, dx.
\end{aligned}$$

Furthermore, we have

$$M_{ij}^I \frac{\partial u^I}{\partial x_j} \cdot \frac{\partial u^I}{\partial x_i} = -\begin{pmatrix} c_{1i1j}^I & \cdot & c_{1i3j}^I \\ \cdot & \cdot & \cdot \\ c_{3i1j}^I & \cdot & c_{3i3j}^I \end{pmatrix} \begin{pmatrix} \frac{\partial u_1^I}{\partial x_j} \\ \cdot \\ \frac{\partial u_3^I}{\partial x_j} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial u_1^I}{\partial x_i} \\ \cdot \\ \frac{\partial u_3^I}{\partial x_i} \end{pmatrix}$$
$$= -\sum_{k,\ell=1}^{3} c_{ki\ell j}^I \frac{\partial u_\ell^I}{\partial x_j} \overline{\frac{\partial u_k^I}{\partial x_i}}$$
$$= -\sum_{k,\ell=1}^{3} \frac{1}{2}\left(c_{ki\ell j}^I \frac{\partial u_\ell^I}{\partial x_j} \overline{\frac{\partial u_k^I}{\partial x_i}} + c_{ik\ell j}^I \frac{\partial u_\ell^I}{\partial x_j} \overline{\frac{\partial u_i^I}{\partial x_k}} \right)$$
$$= -\sum_{k,\ell=1}^{3} c_{ki\ell j}^I \frac{\partial u_\ell^I}{\partial x_j} \overline{\varepsilon_{ki}^I}$$
$$= -\sum_{k,\ell=1}^{3} \frac{1}{2}\left(c_{ki\ell j}^I \frac{\partial u_\ell^I}{\partial x_j} \overline{\varepsilon_{ki}^I} + c_{kij\ell}^I \frac{\partial u_j^I}{\partial x_\ell} \overline{\varepsilon_{ki}^I} \right)$$
$$= -\sum_{k,\ell=1}^{3} c_{ki\ell j}^I \varepsilon_{\ell j}^I \overline{\varepsilon_{ki}^I},$$

and also

$$M_{ij}^{II} \frac{\partial u^{II}}{\partial x_j} \cdot \frac{\partial v^{II}}{\partial x_i} = -\sum_{k,\ell=1}^{3} c_{ki\ell j}^{II} \varepsilon_{\ell j}^{II} \overline{\varepsilon_{ki}^{II}}.$$

From the condition of (0.3) and Korn's inequality (cf. [6], [12])

$$\|\nabla u^I\|_{L^2(\mathbf{R}_-^3)}^2 \le c \sum_{k,i=1}^{3} \int_{\mathbf{R}_-^3} |\varepsilon_{ki}^I|^2\,dx, \quad \|\nabla u^{II}\|_{L^2(\mathbf{R}_+^3)}^2 \le c \sum_{k,i=1}^{3} \int_{\mathbf{R}_+^3} |\varepsilon_{ki}^{II}|^2\,dx,$$

we have

$$(Au,u) = -\sum_{i,j=1}^{3} \int_{\mathbf{R}_-^3} M_{ij}^I \frac{\partial u^I}{\partial x_j} \cdot \frac{\partial u^I}{\partial x_i}\,dx - \sum_{i,j=1}^{3} \int_{\mathbf{R}_+^3} M_{ij}^{II} \frac{\partial u^{II}}{\partial x_j} \cdot \frac{\partial u^{II}}{\partial x_i}\,dx$$
$$= \sum_{i,j=1}^{3} \Big(\int_{\mathbf{R}_-^3} c_{ki\ell j}^I \varepsilon_{\ell j}^I \overline{\varepsilon_{ki}^I}\,dx + \int_{\mathbf{R}_+^3} c_{ki\ell j}^{II} \varepsilon_{\ell j}^{II} \overline{\varepsilon_{ki}^{II}}\,dx \Big)$$
$$\ge \int_{\mathbf{R}_-^3} \delta_1 \sum_{k,i=1}^{3} |\varepsilon_{ki}^I|^2\,dx + \int_{\mathbf{R}_+^3} \delta_2 \sum_{k,i=1}^{3} |\varepsilon_{ki}^{II}|^2\,dx$$
$$\ge c\delta \|\nabla u\|_{L^2(\mathbf{R}^3)}^2,$$

which implies (1.6). □

Lemma 1.5. *The range of $I + A$ is $\mathcal{H}$:*

$$\mathrm{R}(I + A) = \mathcal{H}. \tag{1.7}$$

Proof. If $f \in \mathrm{R}(I + A)$, there exists an element $u \in D(A)$ such that $u + Au = f$. Then we have for any $v \in H^1(\mathbf{R}^3, \mathbf{C}^3)$

$$\begin{aligned}(f, v) &= (Au, v) + (u, v) \\ &= -\int_{\mathbf{R}^3} \sum_{i,j=1}^{3} M_{ij} \frac{\partial u}{\partial x_j} \cdot \frac{\partial v}{\partial x_i}\, dx + \int_{\mathbf{R}^3} u \cdot v \rho(x_3)\, dx.\end{aligned} \tag{1.8}$$

Now we can define by using the right-hand side of (1.8) an inner product on $H^1(\mathbf{R}^3, \mathbf{C}^3)$

$$\{u, v\} = -\int_{\mathbf{R}^3} \sum_{i,j=1}^{3} M_{ij} \frac{\partial u}{\partial x_j} \cdot \frac{\partial v}{\partial x_i}\, dx + \int_{\mathbf{R}^3} u \cdot v \rho(x_3)\, dx$$

$$\text{for } \forall u, v \in H^1(\mathbf{R}^3, \mathbf{C}^3).$$

Then it follows from Korn's inequality as in the proof of Lemma 1.4 that

$$\{u, u\} \geq c\delta \, \|\nabla u\|^2_{L^2(\mathbf{R}^3)} + \|u\|^2_{\mathcal{H}} \quad \text{for } \forall u \in H^1(\mathbf{R}^3, \mathbf{C}^3).$$

This implies that the norm $\{u, u\}^{\frac{1}{2}}$ is equivalent to the norm $\|u\|_1$ defined by (1.2), and that $H^1(\mathbf{R}^3, \mathbf{C}^3)$ is also an Hilbert space (denoted by $\widetilde{H^1}(\mathbf{R}^3, \mathbf{C}^3)$) with the inner product $\{u, v\}$.

For any $f \in \mathcal{H}$, we consider the linear form on $\widetilde{H^1}(\mathbf{R}^3, \mathbf{C}^3)$:

$$\widetilde{H^1}(\mathbf{R}^3, \mathbf{C}^3) \ni v \mapsto (f, v) \in \mathbf{C}.$$

Since

$$|(f, v)| \leq \|f\|_{\mathcal{H}} \|v\|_{\mathcal{H}} \leq \|f\|_{\mathcal{H}} \{v, v\}^{\frac{1}{2}},$$

this linear form on $\widetilde{H^1}(\mathbf{R}^3, \mathbf{C}^3)$ is bounded. So by the Riesz representation theorem, there exists a $u \in \widetilde{H^1}(\mathbf{R}^3, \mathbf{C}^3)$ such that for all $v \in \widetilde{H^1}(\mathbf{R}^3, \mathbf{C}^3)$

$$(f, v) = \{u, v\}. \tag{1.9}$$

Next, we shall show $u \in D(A)$. By taking $v \in \mathcal{D}(\mathbf{R}^3, \mathbf{C}^3)$, the equality (1.9) can be written as follows:

$$< \rho(x_3)(f - u), v >= \left\langle -\sum_{i,j=1}^{3} M_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i} \right\rangle = \left\langle \sum_{i,j=1}^{3} M_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}, v \right\rangle,$$

where $< \cdot, \cdot >$ denotes the duality between $\mathcal{D}'$ and $\mathcal{D}$. This duality means

$$f - u = Mu = Au \in \mathcal{H}$$

in the distribution sense. Furthermore from (1.9)

$$(Mu, v) + \left(\frac{1}{\rho(x_3)} \sum_{i,j=1}^{3} M_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i} \right) = 0 \quad \text{for } \forall v \in H^1(\mathbf{R}^3, \mathbf{C}^3)$$

This means that u satisfies (1.3). Hence $u \in D(A)$. □

Lemma 1.6. *A function u belongs to $D(A)$ if and only if it belongs to the space $H^2(\mathbf{R}^3_-,\mathbf{C}^3)\oplus H^2(\mathbf{R}^3_+,\mathbf{C}^3)$ and satisfies the interface conditions (0.4) and (0.5) in the sense of trace on $x_3=0$.*

Proof. The implication ($\Leftarrow$) is trivial.

($\Rightarrow$) Since

$$H^1(\mathbf{R}^3,\mathbf{C}^3)\subset H^1(\mathbf{R}^3_-,\mathbf{C}^3)\oplus H^1(\mathbf{R}^3_+,\mathbf{C}^3),$$

every $u\in D(A)$ has a unique decomposition

$$u=u^I+u^{II},\quad u^I\in H^1(\mathbf{R}^3_-,\mathbf{C}^3),\ u^{II}\in H^1(\mathbf{R}^3_+,\mathbf{C}^3),$$

where u^I and u^{II} satisfy (1.3). We have the bilinear forms:

$$\left(\frac{1}{\rho_1}\sum_{i,j=1}^{3}M^I_{ij}\frac{\partial u^I}{\partial x_j},\frac{\partial v^I}{\partial x_i}\right)=-\left(M^Iu^I,v^I\right),$$
$$\left(\frac{1}{\rho_2}\sum_{i,j=1}^{3}M^{II}_{ij}\frac{\partial u^{II}}{\partial x_j},\frac{\partial v^{II}}{\partial x_i}\right)=-\left(M^{II}u^{II},v^{II}\right),$$

where $v^I\in H^1(\mathbf{R}^3_-,\mathbf{C}^3)$, $v^{II}\in H^1(\mathbf{R}^3_+,\mathbf{C}^3)$. Since we have by regularity theorem (see for example [1, Theorem 9.6]) ,if

$$-M^Iu^I\in L^2(\mathbf{R}^3_-,\mathbf{C}^3,\rho_1\,dx),$$
$$-M^{II}u^{II}\in L^2(\mathbf{R}^3_+,\mathbf{C}^3,\rho_2\,dx),$$

then it follows that

$$u^I\in H^2(\mathbf{R}^3_-,\mathbf{C}^3),\quad u^{II}\in H^2(\mathbf{R}^3_+,\mathbf{C}^3).$$

Note that for all $\omega\in C_0^\infty(\partial\mathbf{R}^3,\mathbf{C}^3)$, there exist $v\in C_0^\infty(\mathbf{R}^3,\mathbf{C}^3)$ such that $v|_{x_3=0}=\omega$. From (1.3) with this $v\in C_0^\infty(\mathbf{R}^3,\mathbf{C}^3)$ it follows that

$$\left\langle\sum_{i,j=1}^{3}M^I_{3j}\frac{\partial u^I}{\partial x_j}-\sum_{i,j=1}^{3}M^{II}_{3j}\frac{\partial u^{II}}{\partial x_j},\omega\right\rangle=0.$$

Since ω is arbitrary,

$$\sum_{i,j=1}^{3}M^I_{3j}\frac{\partial u^I}{\partial x_j}\bigg|_{x_3=0}=\sum_{i,j=1}^{3}M^{II}_{3j}\frac{\partial u^{II}}{\partial x_j}\bigg|_{x_3=0}.$$

(1.1) is equivalent to (0.5), so this means that u^I and u^{II} satisfy (0.5). □

Proof of Theorem 1.2. The fact that A is selfadjoint is a direct consequence of Lemmas 1.3 and 1.5 and standard results in the theory of linear operators in Hilbert space as follows:

$$A\subset A^*$$

$$\begin{array}{c} R(I+A)=\mathcal{H} \\ \Longrightarrow \\ A;\ \text{selfadjoint in } \mathcal{H} \end{array}$$

(see, for example, [13, Section 187, Theorem 2]). Lemma 1.4 shows that A is non-negative. The latter claim is a direct consequence of Lemma 1.6. □

As shown in Section 0, we transform A into a selfadjoint operator $\hat{A}(\eta')$ depending on a parameter η' and moreover we decompose $\hat{A}(\eta')$ as a direct sum of the simple selfadjoint operators $A_1(\eta')$ and $A_2(\eta')$ which is much easier to study (cf. [3] and [5]).

By a direct computation, we can easily prove the following proposition.

Proposition 1.7. *We have*

$$\hat{A}=\hat{A}(\eta')=\mathrm{UC}(A_1(\eta')\oplus A_2(\eta'))(\mathrm{UC})^{-1}\quad \text{for } \eta'\neq 0,$$

and

$$Au=F_{\eta'}^{-1}\mathrm{UC}(A_1(\eta')\oplus A_2(\eta'))(\mathrm{UC})^{-1}F_{x'}u\quad \text{for } u\in D(A), \tag{1.10}$$

where $A_1(\eta')$ and $A_2(\eta')$ are non-negative selfadjoint operators in $L^2(\mathbf{R},\mathbf{C}^2,\rho(x_3)\,dx_3)$ and $L^2(\mathbf{R},\mathbf{C},\rho(x_3)\,dx_3)$ defined respectively as follows:

$$D\left(A_1(\eta')\right)=\left\{\begin{pmatrix}u_1\\u_2\end{pmatrix}\in H^2(\mathbf{R}_-,\mathbf{C}^2)\oplus H^2(\mathbf{R}_+,\mathbf{C}^2);\right.$$
$$\left. u^I\big|_{x_3=0}=u^{II}\big|_{x_3=0},\ B_1^I(\eta')u^I\big|_{x_3=0}=B_1^{II}(\eta')u^{II}\big|_{x_3=0}\right\},$$

$$A_1\left(\eta',\frac{\mathrm{d}}{\mathrm{d}x_3}\right)\begin{pmatrix}u_1\\u_2\end{pmatrix}=\frac{1}{\rho}\begin{pmatrix}-\mu\frac{d^2}{dx_3^2}+(\lambda+2\mu)|\eta'|^2 & -i|\eta'|(\lambda+\mu)\frac{d}{dx_3}\\ -i|\eta'|(\lambda+\mu)\frac{d}{dx_3} & -(\lambda+2\mu)\frac{d^2}{dx_3^2}+\mu|\eta'|^2\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix},$$

$$B_1^I(\eta')\begin{pmatrix}u_1^I\\u_2^I\end{pmatrix}=\begin{pmatrix}\mu_1\frac{d}{dx_3} & i|\eta'|\mu_1\\ i|\eta'|\lambda_1 & (\lambda_1+2\mu_1)\frac{d}{dx_3}\end{pmatrix}\begin{pmatrix}u_1^I\\u_2^I\end{pmatrix},$$

$$B_1^{II}(\eta')\begin{pmatrix}u_1^{II}\\u_2^{II}\end{pmatrix}=\begin{pmatrix}\mu_2\frac{d}{dx_3} & i|\eta'|\mu_2\\ i|\eta'|\lambda_2 & (\lambda_2+2\mu_2)\frac{d}{dx_3}\end{pmatrix}\begin{pmatrix}u_1^{II}\\u_2^{II}\end{pmatrix},$$

$$D\left(A_2(\eta')\right)=\left\{u\in H^2(\mathbf{R}_-)\oplus H^2(\mathbf{R}_+);\right.$$
$$\left. u^I\big|_{x_3=0}=u^{II}\big|_{x_3=0},\ B_2^I(\eta')u^I\big|_{x_3=0}=B_2^{II}(\eta')u^{II}\big|_{x_3=0}\right\},$$

$$A_2\left(\eta',\frac{\mathrm{d}}{\mathrm{d}x_3}\right)u=-\frac{\mu(x_3)}{\rho(x_3)}\frac{d^2u}{dx_3^2}+\frac{\mu(x_3)}{\rho(x_3)}|\eta'|^2u,$$
$$B_2^I(\eta')u^I=\mu_1\frac{d}{dx_3}u^I,\quad B_2^{II}(\eta')u^{II}=\mu_2\frac{d}{dx_3}u^{II}.$$

Since $A_2(\eta')$ is an operator corresponding to the usual wave operator, from now on, we shall mainly treat the operator $A_1(\eta')$.

§ 2. The Green Function $G_1(x_3, y_3, \eta'; \zeta)$ of $A_1(\eta') - \zeta I$

In this section, we give an explicit representation of the Green function $G_1(x_3, y_3, \eta'; \zeta)$ for the operator $A_1(\eta') - \zeta I(\zeta \notin \mathbf{R})$ by using a standard technique (cf. [7], [9], [10], [16]) in order to define generalized eigenfunctions for the operator $A_1(\eta')$ in Section 4 below.

Denote by $R(\zeta; T)$ the resolvent $(T - \zeta)^{-1}$ of an operator T. The resolvent of the selfadjoint operator $A_1(\eta')$ has the kernel representation; that is, there exists the Green function $G_1(x_3, y_3, \eta'; \zeta)$ and for $f(\cdot, x_3) \in C_0^\infty(\mathbf{R} \setminus \{0\}, \mathbf{C}^2)$ we have

$$R(\zeta; A_1(\eta'))f(\eta', x_3) = \int_{\mathbf{R}} G_1(x_3, y_3, \eta'; \zeta) f(\eta', y_3)\, dy_3.$$

From the selfadjointness of $A_1(\eta')$, it follows that the resolvent kernel has the symmetry property

$$G_1(x_3, y_3, \eta'; \zeta)^* = G_1(y_3, x_3, \eta'; \bar{\zeta}).$$

In order to find the Green function, we consider the following problem:

$$\begin{aligned} &(A_1^I(\eta', D) - \zeta)v^I(\eta', x_3) = f(\eta', x_3), \quad x_3 < 0, \\ &(A_1^{II}(\eta', D) - \zeta)v^{II}(\eta', x_3) = f(\eta', x_3), \quad x_3 0, \end{aligned} \tag{2.1}$$

$$v^I(\eta', x_3)\big|_{x_3=0} = v^{II}(\eta', x_3)\big|_{x_3=0}, \tag{2.2}$$

$$B_1^I(\eta')v^I(\eta', x_3)\big|_{x_3=0} = B_1^{II}(\eta')v^{II}(\eta', x_3)\big|_{x_3=0}, \tag{2.3}$$

where $D = \frac{1}{i}\frac{d}{dx_3}$.

Let us seek the solutions $v^I(\eta', x_3)$ and $v^{II}(\eta', x_3)$ in the form

$$\begin{aligned} v^I(\eta', x_3) &= E^I(x_3, \eta'; \zeta) - K^I(x_3, \eta'; \zeta), \\ v^{II}(\eta', x_3) &= E^{II}(x_3, \eta'; \zeta) - K^{II}(x_3, \eta'; \zeta). \end{aligned}$$

Let

$$\begin{aligned} \mathcal{E}^I(x_3 - y_3, \eta', \zeta) &= \frac{1}{\sqrt{2\pi}} F_\xi^{-1}\left[(A^I(\eta', \xi) - \zeta)^{-1} e^{-iy_3\xi}\right], \\ \mathcal{E}^{II}(x_3 - y_3, \eta', \zeta) &= \frac{1}{\sqrt{2\pi}} F_\xi^{-1}\left[(A^{II}(\eta', \xi) - \zeta)^{-1} e^{-iy_3\xi}\right], \end{aligned} \tag{2.4}$$

where ξ is the dual variable of x_3 and F_ξ^{-1} denotes the inverse or conjugate Fourier transformation with respect to ξ :

$$(F_\xi^{-1} f)(\eta', x_3) = \underset{R\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{|\xi| \le R} e^{ix_3\xi} f(\xi)\, d\xi \quad \text{for } f \in \mathcal{H}.$$

$\mathcal{E}^I$ and $\mathcal{E}^{II}$ are fundamental solutions of $A_1^I(\eta') - \zeta$ and $A_1^{II}(\eta') - \zeta$, respectively, that is, $\mathcal{E}^I$ and $\mathcal{E}^{II}$ are distribution solutions of the equations

$$
\begin{aligned}
(A_1^I(\eta', D) - \zeta)\mathcal{E}^I(x_3 - y_3, \eta', \zeta) &= \delta(x_3 - y_3)I, \quad x_3 < 0, \\
(A_1^{II}(\eta', D) - \zeta)\mathcal{E}^{II}(x_3 - y_3, \eta', \zeta) &= \delta(x_3 - y_3)I, \quad x_3 > 0.
\end{aligned}
$$

Then we have in the sense of distributions

$$
\begin{aligned}
E^I(x_3, \eta'; \zeta) &= \int_{\mathbf{R}} \mathcal{E}^I(x_3 - y_3, \eta'; \zeta) f(\eta', y_3)\, dy_3, \\
E^{II}(x_3, \eta'; \zeta) &= \int_{\mathbf{R}} \mathcal{E}^{II}(x_3 - y_3, \eta'; \zeta) f(\eta', y_3)\, dy_3.
\end{aligned}
$$

$K^I(x_3, \eta'; \zeta) \in H^2(\mathbf{R}_-, \mathbf{C}^2)$ and $K^{II}(x_3, \eta'; \zeta) \in H^2(\mathbf{R}_+, \mathbf{C}^2)$ are solutions of the equations

$$
\begin{aligned}
(A_1^I(\eta', D) - \zeta)K^I(x_3, \eta', \zeta) &= 0, \quad x_3 < 0, \\
(A_1^{II}(\eta', D) - \zeta)K^{II}(x_3, \eta', \zeta) &= 0, \quad x_3 > 0,
\end{aligned}
\tag{2.5}
$$

and

$$
\begin{aligned}
E^I(x_3, \eta'; \zeta)\big|_{x_3=0} &- E^{II}(x_3, \eta'; \zeta)\big|_{x_3=0} \\
&= K^I(x_3, \eta'; \zeta)\big|_{x_3=0} - K^{II}(x_3, \eta'; \zeta)\big|_{x_3=0},
\end{aligned}
\tag{2.6}
$$

$$
\begin{aligned}
B_1^I(\eta')E^I(x_3, \eta'; \zeta)\big|_{x_3=0} &- B_1^{II}(\eta')\, E^{II}(x_3, \eta'; \zeta)\big|_{x_3=0} \\
&= B_1^I(\eta')K^I(x_3, \eta'; \zeta)\big|_{x_3=0} - B_1^{II}(\eta')\, K^{II}(x_3, \eta'; \zeta)\big|_{x_3=0}.
\end{aligned}
\tag{2.7}
$$

First, we find an explicit representation of the fundamental solution $\mathcal{E}^I$. The characteristic matrix $A_1^I(\eta', \xi)$ of $A_1^I(\eta', D)$ is a 2×2 Hermitian matrix with characteristic polynomial

$$
\det(A_1^I(\eta', \xi) - \zeta I) = \left(\zeta - \frac{\mu_1}{\rho_1}|\eta|^2\right)\left(\zeta - \frac{\lambda_1 + 2\mu_1}{\rho_1}|\eta|^2\right),
$$

where $\eta = (\eta', \xi) = (\eta_1, \eta_2, \xi)$ and $|\eta|^2 = |\eta'|^2 + \xi^2$. $(\mu_1/\rho_1)^{\frac{1}{2}}$ and $((\lambda_1 + 2\mu_1)/\rho_1)^{\frac{1}{2}}$ are the propagation speeds of shear and pressure waves, usually called S wave and P wave respectively by physicists and engineers. Thus, from now on, we use the following notation

$$
c_{s_1}^2 = \frac{\mu_1}{\rho_1}, \quad c_{p_1}^2 = \frac{\lambda_1 + 2\mu_1}{\rho_1}.
\tag{2.8}
$$

The distinct eigenvalues of $A_1^I(\eta', \xi)$ are

$$
\lambda_{s_1}(\eta) = c_{s_1}^2 |\eta|^2, \quad \lambda_{p_1}(\eta) = c_{p_1}^2 |\eta|^2.
\tag{2.9}
$$

Introducing the set of indices

$$M_1 = \{s_1, p_1\}, \tag{2.10}$$

the resolution of the identity for $A_1^I(\eta', \xi)$ is given by

$$I = \sum_{j \in M_1} P_j(\eta).$$

Here the $P_j(\eta) (j \in M_1)$ are mutually orthogonal projections defined by

$$P_j(\eta) = \frac{1}{2\pi i} \int_{|\lambda_j(\eta) - \zeta| = \delta} (\zeta I - A_1^I(\eta))^{-1} d\zeta, \quad j \in M_1,$$

where the integration goes over a small circle in the complex plane enclosing only the eigenvalue $\lambda_j(\eta)$ $(j \in M_1)$ in the positive direction.

Since $P_j(\eta)(j \in M_1)$ satisfy the following properties:

$$P_j^* = P_j, \quad \delta_{jk} P_j = P_j P_k,$$
$$A_1^I(\eta', \xi) P_j(\eta) = \lambda_j(\eta) P_j(\eta),$$

we have

$$(A_1^I(\eta', \xi) - \zeta I)^{-1} = \sum_{j \in M_1} \frac{1}{\lambda_j(\eta) - \zeta} P_j(\eta). \tag{2.11}$$

$P_{s_1}(\eta)$ and $P_{p_1}(\eta)$ have more explicit representations. In fact, the $\lambda_j(\eta)$ $(j \in M_1)$ are simple poles of $(\zeta I - A_1^I(\eta))^{-1}$

(2.12)

$$\begin{aligned} P_{s_1}(\eta) &= \lim_{\zeta \to \lambda_{s_1}(\eta)} (\zeta - \lambda_{s_1}(\eta))(\zeta I - A_1^I(\eta))^{-1} \\ &= \lim_{\zeta \to c_{s_1}^2 |\eta|^2} \frac{1}{\zeta - c_{p_1}^2 |\eta|^2} \begin{pmatrix} \zeta - (c_{p_1}^2 \xi^2 + c_{s_1}^2 |\eta'|^2) & |\eta'|(c_{p_1}^2 - c_{s_1}^2)\xi \\ |\eta'|(c_{p_1}^2 - c_{s_1}^2)\xi & \zeta - (c_{s_1}^2 \xi^2 + c_{p_1}^2 |\eta'|^2) \end{pmatrix} \\ &= \frac{1}{|\eta|^2} \begin{pmatrix} \xi^2 & -|\eta'|\xi \\ -|\eta'|\xi & |\eta'|^2 \end{pmatrix}, \end{aligned}$$

and similarly,

$$P_{p_1}(\eta) = \frac{1}{|\eta|^2} \begin{pmatrix} |\eta'|^2 & |\eta'|\xi \\ |\eta'|\xi & \xi^2 \end{pmatrix}. \tag{2.13}$$

From (2.4) and (2.11), we have

$$\begin{aligned} \mathcal{E}^I(x_3 - y_3, \eta'; \zeta) = \frac{1}{2\pi} \int_{\mathbf{R}} e^{i(x_3 - y_3)\xi} \frac{P_{s_1}(\eta', \xi)}{\lambda_{s_1}(\xi) - \zeta} d\xi \\ + \frac{1}{2\pi} \int_{\mathbf{R}} e^{i(x_3 - y_3)\xi} \frac{P_{p_1}(\eta', \xi)}{\lambda_{p_1}(\xi) - \zeta} d\xi, \quad x_3 > 0. \end{aligned} \tag{2.14}$$

Now, we calculate the two integrals on the right-hand side of (2.14). To do so, we change the real variable ξ to the complex variable $\tau = \xi + i\kappa$ and define

$$(2.15)\qquad \tau_{s_1} = \sqrt{\frac{\zeta}{c_{s_1}^2} - |\eta'|^2},\quad \operatorname{Im}\tau_{s_1} \geq 0,\quad \tau_{p_1} = \sqrt{\frac{\zeta}{c_{p_1}^2} - |\eta'|^2},\quad \operatorname{Im}\tau_{p_1} \geq 0.$$

Then the determinant $\det(A_1^I(\eta',\tau) - \zeta I)$ is equal to:

$$\det(A_1^I(\eta',\tau) - \zeta I) = c_{s_1}^2 c_{p_1}^2 (\tau + \tau_{s_1})(\tau - \tau_{s_1})(\tau + \tau_{p_1})(\tau - \tau_{p_1}).$$

Let us consider the first term in the right-hand side of (2.14). In the case where $x_3 - y_3 > 0$, we may deform the path of integration into the τ upper half plane as indicated in Figure 2.

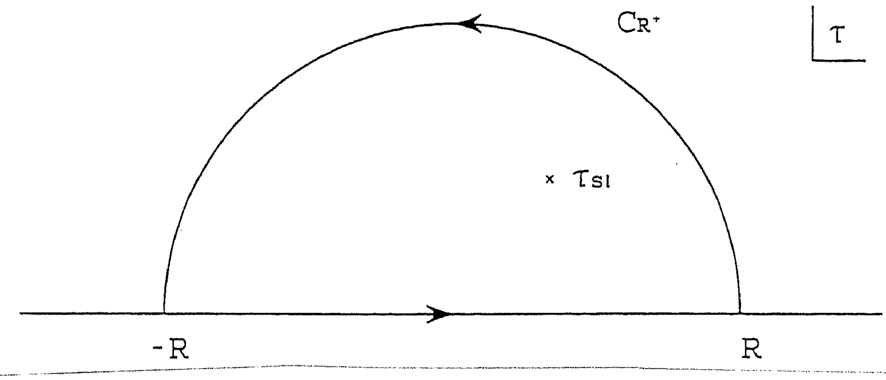

FIGURE 2 A PATH OF INTEGRATION

We obtain

$$\begin{aligned} 2\pi i \operatorname*{Res}_{\tau=\tau_{s_1}} &\left(e^{i(x_3-y_3)\tau} \frac{P_{s_1}(\eta',\tau)}{\lambda_{s_1}(\eta',\tau) - \zeta} \right) \\ &= \left(\int_{-R}^{R} + \int_{C_R^+} \right) e^{i(x_3-y_3)\tau} \frac{P_{s_1}(\eta',\tau)}{\lambda_{s_1}(\eta',\tau) - \zeta} \, d\tau. \end{aligned}$$

On the half-circle C_R^+, $\tau = Re^{i\theta} (0 \leq \theta \leq \pi)$, we have as $R \to \infty$

$$\left| \int_{C_R^+} e^{i(x_3-y_3)\tau} \frac{P_{s_1}(\eta',\tau)}{\lambda_{s_1}(\eta',\tau) - \zeta} \, d\tau \right| \leq \text{const.} \int_0^{\pi} \frac{e^{-(x_3-y_3)R\sin\theta}}{R^2} d\theta \to 0,$$

and so

$$\int_{-\infty}^{\infty} e^{i(x_3-y_3)\xi} \frac{P_{s_1}(\eta',\xi)}{\lambda_{s_1}(\eta',\xi) - \zeta} \, d\xi = 2\pi i \operatorname*{Res}_{\tau=\tau_{s_1}} \left(e^{i(x_3-y_3)\tau} \frac{P_{s_1}(\eta',\tau)}{\lambda_{s_1}(\eta',\tau) - \zeta} \right),$$

$$x_3 - y_3 > 0.$$

In the case where $x_3 - y_3 < 0$, we may deform the path of integration into the τ lower half plane, and so we have in a similar way

$$\int_{-\infty}^{\infty} e^{i(x_3-y_3)\xi} \frac{P_{s_1}(\eta',\xi)}{\lambda_{s_1}(\eta',\xi) - \zeta} \, d\xi = -2\pi i \operatorname*{Res}_{\tau=-\tau_{s_1}} \left(e^{i(x_3-y_3)\tau} \frac{P_{s_1}(\eta',\tau)}{\lambda_{s_1}(\eta',\tau) - \zeta} \right),$$

$x_3 - y_3 < 0.$

The second term in the right-hand side of (2.14) is also calculated similarly. Summing up, we have

$$\begin{aligned}
&\mathcal{E}^I(x_3 - y_3, \eta'; \zeta) \\
&= i \begin{cases} \operatorname{Res}_{\tau=\tau_{s_1}} \left(e^{i(x_3-y_3)\tau} \frac{P_{s_1}(\eta',\tau)}{\lambda_{s_1}(\eta',\tau)-\zeta} \right) + \operatorname{Res}_{\tau=\tau_{p_1}} \left(e^{i(x_3-y_3)\tau} \frac{P_{p_1}(\eta',\tau)}{\lambda_{p_1}(\eta',\tau)-\zeta} \right) \\ -\operatorname{Res}_{\tau=-\tau_{s_1}} \left(e^{i(x_3-y_3)\tau} \frac{P_{s_1}(\eta',\tau)}{\lambda_{s_1}(\eta',\tau)-\zeta} \right) - \operatorname{Res}_{\tau=-\tau_{p_1}} \left(e^{i(x_3-y_3)\tau} \frac{P_{p_1}(\eta',\tau)}{\lambda_{p_1}(\eta',\tau)-\zeta} \right) \end{cases} \\
&= \frac{i}{2} \begin{cases} \frac{e^{i(x_3-y_3)\tau_{s_1}}}{c_{s_1}^2 \tau_{s_1}} P_{s_1}(\eta', \tau_{s_1}) + \frac{e^{i(x_3-y_3)\tau_{p_1}}}{c_{p_1}^2 \tau_{p_1}} P_{p_1}(\eta', \tau_{p_1}), & x_3 - y_3 > 0, \\ \frac{e^{-i(x_3-y_3)\tau_{s_1}}}{c_{s_1}^2 \tau_{s_1}} P_{s_1}(\eta', -\tau_{s_1}) + \frac{e^{-i(x_3-y_3)\tau_{p_1}}}{c_{p_1}^2 \tau_{p_1}} P_{p_1}(\eta', -\tau_{p_1}), & x_3 - y_3 < 0. \end{cases}
\end{aligned}$$

So we have for $x_3 < 0$

(2.16)

$$\begin{aligned}
&E^I(x_3, \eta'; \zeta) \\
&= \frac{i}{2} \times \left(\int_{-\infty}^{x_3} \left(\frac{e^{i(x_3-y_3)\tau_{s_1}}}{c_{s_1}^2 \tau_{s_1}} P_{s_1}(\eta', \tau_{s_1}) + \frac{e^{i(x_3-y_3)\tau_{p_1}}}{c_{p_1}^2 \tau_{p_1}} P_{p_1}(\eta', \tau_{p_1}) \right) f(\eta', y_3)\, dy_3 \right. \\
&\left. + \int_{x_3}^{\infty} \left(\frac{e^{-i(x_3-y_3)\tau_{s_1}}}{c_{s_1}^2 \tau_{s_1}} P_{s_1}(\eta', -\tau_{s_1}) + \frac{e^{-i(x_3-y_3)\tau_{p_1}}}{c_{p_1}^2 \tau_{p_1}} P_{p_1}(\eta', -\tau_{p_1}) \right) f(\eta', y_3)\, dy_3 \right).
\end{aligned}$$

As to the $\mathcal{E}^{II}$, let

(2.17)

$$\begin{aligned}
&c_{s_2}^2 = \frac{\mu_2}{\rho_2}, \quad c_{p_2}^2 = \frac{\lambda_2 + 2\mu_2}{\rho_2}, \\
&\lambda_{s_2}(\eta) = c_{s_2}^2 |\eta|^2, \quad \lambda_{p_2}(\eta) = c_{p_2}^2 |\eta|^2, \\
&M_2 = \{s_2, p_2\}, \\
&\tau_{s_2} = \sqrt{\frac{\zeta}{c_{s_2}^2} - |\eta'|^2}, \quad \operatorname{Im} \tau_{s_2} \geq 0, \quad \tau_{p_2} = \sqrt{\frac{\zeta}{c_{p_2}^2} - |\eta'|^2}, \quad \operatorname{Im} \tau_{p_2} \geq 0. \\
&P_{s_2}(\eta) = \frac{1}{|\eta|^2} \begin{pmatrix} \xi^2 & -|\eta'|\xi \\ -|\eta'|\xi & |\eta'|^2 \end{pmatrix}, \quad P_{p_2}(\eta) = \frac{1}{|\eta|^2} \begin{pmatrix} |\eta'|^2 & |\eta'|\xi \\ |\eta'|\xi & \xi^2 \end{pmatrix}.
\end{aligned}$$

Using these notations, we have, by the same procedure as $\mathcal{E}^I$, for $x_3 > 0$

(2.18)

$$\begin{aligned}
&E^{II}(x_3, \eta'; \zeta) \\
&= \frac{i}{2} \times \left(\int_{-\infty}^{x_3} \left(\frac{e^{i(x_3-y_3)\tau_{s_2}}}{c_{s_2}^2 \tau_{s_2}} P_{s_2}(\eta', \tau_{s_2}) + \frac{e^{i(x_3-y_3)\tau_{p_2}}}{c_{p_2}^2 \tau_{p_2}} P_{p_2}(p, \tau_{p_2}) \right) f(\eta', y_3)\, dy_3 \right. \\
&\left. + \int_{x_3}^{\infty} \left(\frac{e^{-i(x_3-y_3)\tau_{s_2}}}{c_{s_2}^2 \tau_{s_2}} P_{s_2}(\eta', -\tau_{s_2}) + \frac{e^{-i(x_3-y_3)\tau_{p_2}}}{c_{p_2}^2 \tau_{p_2}} P_{p_2}(\eta', -\tau_{p_2}) \right) f(\eta', y_3)\, dy_3 \right).
\end{aligned}$$

From the equation (2.5), we may suppose that

$$K^I(x_3, \eta'; \zeta) = C_1 e^{-i\tau_{p_1} x_3} + C_2 e^{i\tau_{p_1} x_3} + C_3 e^{-i\tau_{s_1} x_3} + C_4 e^{i\tau_{s_1} x_3}, \quad x_3 < 0,$$

where $C_1, \ldots, C_4$ are (2×1) matrices. Since $x_3 < 0$ and (2.15), the hypothesis that $K^I(x_3, \eta'; \zeta) \in H^2(\mathbf{R}_-, \mathbf{C}^2)$ implies $C_2 = C_4 = 0$. Moreover, since K^I is a solution of (2.5), K^I can be represented as follows:

$$K^I(x_3, \eta'; \zeta) = \alpha_1 \begin{pmatrix} |\eta'| \\ -\tau_{p_1} \end{pmatrix} e^{-i\tau_{p_1} x_3} + \alpha_2 \begin{pmatrix} \tau_{s_1} \\ |\eta'| \end{pmatrix} e^{-i\tau_{s_1} x_3}, \quad x_3 < 0, \tag{2.19}$$

where $\alpha_1, \alpha_2 \in \mathbf{C}$. As for $K^{II}(x_3, \eta'; \zeta)$, we have the following representation:

$$K^{II}(x_3, \eta'; \zeta) = \beta_1 \begin{pmatrix} |\eta'| \\ \tau_{p_2} \end{pmatrix} e^{i\tau_{p_2} x_3} + \beta_2 \begin{pmatrix} -\tau_{s_2} \\ |\eta'| \end{pmatrix} e^{i\tau_{s_2} x_3}, \quad x_3 > 0, \tag{2.20}$$

where $\beta_1, \beta_2 \in \mathbf{C}$.

Let us determine the constants $\alpha_1, \alpha_2, \beta_1$ and β_2 so that K^I and K^{II} satisfy (2.6) and (2.7). Note that

$$P_{p_1}(\eta', \tau_{p_1}) = \frac{c_{p_1}^2}{\zeta} \begin{pmatrix} |\eta'|^2 & \tau_{p_1}|\eta'| \\ \tau_{p_1}|\eta'| & \tau_{p_1}^2 \end{pmatrix}, \quad P_{s_1}(\eta', \tau_{s_1}) = \frac{c_{s_1}^2}{\zeta} \begin{pmatrix} \tau_{s_1}^2 & -\tau_{s_1}|\eta'| \\ -\tau_{s_1}|\eta'| & |\eta'|^2 \end{pmatrix},$$
$$P_{p_2}(\eta', \tau_{p_2}) = \frac{c_{p_2}^2}{\zeta} \begin{pmatrix} |\eta'|^2 & \tau_{p_2}|\eta'| \\ \tau_{p_2}|\eta'| & \tau_{p_2}^2 \end{pmatrix}, \quad P_{s_2}(\eta', \tau_{s_2}) = \frac{c_{s_2}^2}{\zeta} \begin{pmatrix} \tau_{s_2}^2 & -\tau_{s_2}|\eta'| \\ -\tau_{s_2}|\eta'| & |\eta'|^2 \end{pmatrix}.$$

Then, if we multiply the both sides of (2.7) by $1/i$, then the equations on $\alpha_1, \alpha_2, \beta_1$ and β_2 can be written in the matrix form as follows:

$$\begin{pmatrix} |\eta'| & \tau_{s_1} & -|\eta'| & \tau_{s_2} \\ -\tau_{p_1} & |\eta'| & -\tau_{p_2} & -|\eta'| \\ -2\rho_1 c_{s_1}^2 \tau_{p_1}|\eta'| & -\rho_1 c_{s_1}^2(\tau_{s_1}^2 - |\eta'|^2) & -2\rho_2 c_{s_2}^2 \tau_{p_2}|\eta'| & \rho_2 c_{s_2}^2(\tau_{s_2}^2 - |\eta'|^2) \\ \rho_1 c_{s_1}^2(\tau_{s_1}^2 - |\eta'|^2) & -2\rho_1 c_{s_1}^2 \tau_{s_1}|\eta'| & -\rho_2 c_{s_2}^2(\tau_{s_2}^2 - |\eta'|^2) & -2\rho_2 c_{s_2}^2 \tau_{s_2}|\eta'| \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{pmatrix} \tag{2.21}$$

$$= \begin{pmatrix} |\eta'| \\ \tau_{p_1} \\ 2\rho_1 c_{s_1}^2 \tau_{p_1}|\eta'| \\ \rho_1 c_{s_1}^2(\tau_{s_1}^2 - |\eta'|^2) \end{pmatrix} g_1(\eta', \zeta) + \begin{pmatrix} -\tau_{s_1} \\ |\eta'| \\ -\rho_1 c_{s_1}^2(\tau_{s_1}^2 - |\eta'|^2) \\ 2\rho_1 c_{s_1}^2 \tau_{s_1}|\eta'| \end{pmatrix} g_2(\eta', \zeta)$$
$$+ \begin{pmatrix} -|\eta'| \\ -\tau_{p_2} \\ -2\rho_2 c_{s_2}^2 \tau_{p_2}|\eta'| \\ -\rho_2 c_{s_2}^2(\tau_{s_2}^2 - |\eta'|^2) \end{pmatrix} g_3(\eta', \zeta) + \begin{pmatrix} \tau_{s_2} \\ -|\eta'| \\ \rho_2 c_{s_2}^2(\tau_{s_2}^2 - |\eta'|^2) \\ -2\rho_2 c_{s_2}^2 \tau_{s_2}|\eta'| \end{pmatrix} g_4(\eta', \zeta)$$
$$+ \begin{pmatrix} |\eta'| \\ -\tau_{p_1} \\ -2\rho_1 c_{s_1}^2 \tau_{p_1}|\eta'| \\ \rho_1 c_{s_1}^2(\tau_{s_1}^2 - |\eta'|^2) \end{pmatrix} g_1'(\eta', \zeta) + \begin{pmatrix} \tau_{s_1} \\ |\eta'| \\ -\rho_1 c_{s_1}^2(\tau_{s_1}^2 - |\eta'|^2) \\ -2\rho_1 c_{s_1}^2 \tau_{s_1}|\eta'| \end{pmatrix} g_2'(\eta', \zeta)$$

$$+\begin{pmatrix} -|\eta'| \\ \tau_{p_2} \\ 2\rho_2 c_{s_2}^2 \tau_{p_2}|\eta'| \\ -\rho_2 c_{s_2}^2(\tau_{s_2}^2 - |\eta'|^2) \end{pmatrix} g_3'(\eta',\zeta) + \begin{pmatrix} -\tau_{s_2} \\ -|\eta'| \\ \rho_2 c_{s_2}^2(\tau_{s_2}^2 - |\eta'|^2) \\ 2\rho_2 c_{s_2}^2 \tau_{s_2}|\eta'| \end{pmatrix} g_4'(\eta',\zeta),$$

where

$$g_1(\eta',\zeta) = \frac{i}{2}\frac{1}{\tau_{p_1}\zeta}\int_{-\infty}^{0} e^{-i\tau_{p_1}y_3}(|\eta'|,\tau_{p_1})f(\eta',y_3)\,dy_3,$$
$$g_2(\eta',\zeta) = \frac{i}{2}\frac{1}{\tau_{s_1}\zeta}\int_{-\infty}^{0} e^{-i\tau_{s_1}y_3}(-\tau_{s_1},|\eta'|)f(\eta',y_3)\,dy_3,$$
$$g_3(\eta',\zeta) = \frac{i}{2}\frac{1}{\tau_{p_2}\zeta}\int_{-\infty}^{0} e^{-i\tau_{p_2}y_3}(|\eta'|,\tau_{p_2})f(\eta',y_3)\,dy_3,$$
$$g_4(\eta',\zeta) = \frac{i}{2}\frac{1}{\tau_{s_2}\zeta}\int_{-\infty}^{0} e^{-i\tau_{s_2}y_3}(-\tau_{s_2},|\eta'|)f(\eta',y_3)\,dy_3,$$
$$g_1'(\eta',\zeta) = \frac{i}{2}\frac{1}{\tau_{p_1}\zeta}\int_{0}^{\infty} e^{i\tau_{p_1}y_3}(|\eta'|,-\tau_{p_1})f(\eta',y_3)\,dy_3,$$
$$g_2'(\eta',\zeta) = \frac{i}{2}\frac{1}{\tau_{s_1}\zeta}\int_{0}^{\infty} e^{i\tau_{s_1}y_3}(\tau_{s_1},|\eta'|)f(\eta',y_3)\,dy_3,$$
$$g_3'(\eta',\zeta) = \frac{i}{2}\frac{1}{\tau_{p_2}\zeta}\int_{0}^{\infty} e^{i\tau_{p_2}y_3}(|\eta'|,-\tau_{p_2})f(\eta',y_3)\,dy_3,$$
$$g_4'(\eta',\zeta) = \frac{i}{2}\frac{1}{\tau_{s_2}\zeta}\int_{0}^{\infty} e^{i\tau_{s_2}y_3}(\tau_{s_2},|\eta'|)f(\eta',y_3)\,dy_3.$$

Put

(2.22) $\Delta(\eta',\zeta) =$

$$\begin{vmatrix} |\eta'| & \tau_{s_1} & -|\eta'| & \tau_{s_2} \\ -\tau_{p_1} & |\eta'| & -\tau_{p_2} & -|\eta'| \\ -2\rho_1 c_{s_1}^2\tau_{p_1}|\eta'| & -\rho_1 c_{s_1}^2(\tau_{s_1}^2 - |\eta'|^2) & -2\rho_2 c_{s_2}^2\tau_{p_2}|\eta'| & \rho_2 c_{s_2}^2(\tau_{s_2}^2 - |\eta'|^2) \\ \rho_1 c_{s_1}^2(\tau_{s_1}^2 - |\eta'|^2) & -2\rho_1 c_{s_1}^2\tau_{s_1}|\eta'| & -\rho_2 c_{s_2}^2(\tau_{s_2}^2 - |\eta'|^2) & -2\rho_2 c_{s_2}^2\tau_{s_2}|\eta'| \end{vmatrix}$$

$\Delta(\eta',\zeta)$ is called the Lopatinski determinant. $\Delta_{p_1}^{\ell}(\eta',\zeta)$, $\Delta_{s_1}^{\ell}(\eta',\zeta)$, $\Delta_{p_2}^{\ell}(\eta',\zeta)$, and $\Delta_{s_2}^{\ell}(\eta',\zeta)$ $(\ell = 1,2,3,4)$ denote the determinants respectively obtained from $\Delta(\eta',\zeta)$ by replacing the ith column by

$${}^t(|\eta'|,\quad \tau_{p_1},\quad 2\rho_1 c_{s_1}^2\tau_{p_1}|\eta'|,\quad \rho_1 c_{s_1}^2(\tau_{s_1}^2 - |\eta'|^2)),$$
$${}^t(-\tau_{s_1},\quad |\eta'|,\quad -\rho_1 c_{s_1}^2(\tau_{s_1}^2 - |\eta'|^2),\quad 2\rho_1 c_{s_1}^2\tau_{s_1}|\eta'|),$$
$${}^t(-|\eta'|,\quad \tau_{p_2},\quad 2\rho_2 c_{s_2}^2\tau_{p_2}|\eta'|,\quad -\rho_2 c_{s_2}^2(\tau_{s_2}^2 - |\eta'|^2)),$$

and by

$${}^t(-\tau_{s_2},\quad -|\eta'|,\quad \rho_2 c_{s_2}^2(\tau_{s_2}^2 - |\eta'|^2),\quad 2\rho_2 c_{s_2}^2\tau_{s_2}|\eta'|).$$

If $\Delta(\eta',\zeta)\neq 0$, then (2.21) has a unique solutions $(\alpha_1,\alpha_2,\beta_1,\beta_2)$ which are given in the following form:

$$\alpha_1=\frac{\Delta^1_{p_1}(\eta',\zeta)}{\Delta(\eta',\zeta)}g_1(\eta',\zeta)+\frac{\Delta^1_{s_1}(\eta',\zeta)}{\Delta(\eta',\zeta)}g_2(\eta',\zeta)+g_1'(\eta',\zeta) +\frac{\Delta^1_{p_2}(\eta',\zeta)}{\Delta(\eta',\zeta)}g_3'(\eta',\zeta)+\frac{\Delta^1_{s_2}(\eta',\zeta)}{\Delta(\eta',\zeta)}g_4'(\eta',\zeta),$$

$$\alpha_2=\frac{\Delta^2_{p_1}(\eta',\zeta)}{\Delta(\eta',\zeta)}g_1(\eta',\zeta)+\frac{\Delta^2_{s_1}(\eta',\zeta)}{\Delta(\eta',\zeta)}g_2(\eta',\zeta)+g_2'(\eta',\zeta) +\frac{\Delta^2_{p_2}(\eta',\zeta)}{\Delta(\eta',\zeta)}g_3'(\eta',\zeta)+\frac{\Delta^2_{s_2}(\eta',\zeta)}{\Delta(\eta',\zeta)}g_4'(\eta',\zeta),$$

$$\beta_1=\frac{\Delta^3_{p_1}(\eta',\zeta)}{\Delta(\eta',\zeta)}g_1(\eta',\zeta)+\frac{\Delta^3_{s_1}(\eta',\zeta)}{\Delta(\eta',\zeta)}g_2(\eta',\zeta)+g_3(\eta',\zeta) +\frac{\Delta^3_{p_2}(\eta',\zeta)}{\Delta(\eta',\zeta)}g_3'(\eta',\zeta)+\frac{\Delta^3_{s_2}(\eta',\zeta)}{\Delta(\eta',\zeta)}g_4'(\eta',\zeta),$$

$$\beta_2=\frac{\Delta^4_{p_1}(\eta',\zeta)}{\Delta(\eta',\zeta)}g_1(\eta',\zeta)+\frac{\Delta^4_{s_1}(\eta',\zeta)}{\Delta(\eta',\zeta)}g_2(\eta',\zeta)+g_4(\eta',\zeta) +\frac{\Delta^4_{p_2}(\eta',\zeta)}{\Delta(\eta',\zeta)}g_3'(\eta',\zeta)+\frac{\Delta^4_{s_2}(\eta',\zeta)}{\Delta(\eta',\zeta)}g_4'(\eta',\zeta).$$

Thus we have for $x_3<0$

$$\begin{aligned}
&v^I(\eta',x_3)\\
&=E^I(x_3,\eta';\zeta)-K^I(x_3,\eta';\zeta)\\
&=\frac{i}{2}\left(\int_{-\infty}^{x_3}\left(\frac{e^{i(x_3-y_3)\tau_{p_1}}}{c_{p_1}^2\tau_{p_1}}P_{p_1}(\eta',\tau_{p_1})+\frac{e^{i(x_3-y_3)\tau_{s_1}}}{c_{s_1}^2\tau_{s_1}}P_{s_1}(\eta',\tau_{s_1})\right)f(\eta',y_3)\,dy_3\right.\\
&\left.+\int_{x_3}^{\infty}\left(\frac{e^{-i(x_3-y_3)\tau_{p_1}}}{c_{p_1}^2\tau_{p_1}}P_{p_1}(\eta',-\tau_{p_1})+\frac{e^{-i(x_3-y_3)\tau_{s_1}}}{c_{s_1}^2\tau_{s_1}}P_{s_1}(\eta',-\tau_{s_1})\right)f(\eta',y_3)\,dy_3\right)\\
&-\left(\frac{\Delta^1_{p_1}(\eta',\zeta)}{\Delta(\eta',\zeta)}g_1(\eta',\zeta)+\frac{\Delta^1_{s_1}(\eta',\zeta)}{\Delta(\eta',\zeta)}g_2(\eta',\zeta)+g_1'(\eta',\zeta)\right.\\
&\qquad\left.+\frac{\Delta^1_{p_2}(\eta',\zeta)}{\Delta(\eta',\zeta)}g_3'(\eta',\zeta)+\frac{\Delta^1_{s_2}(\eta',\zeta)}{\Delta(\eta',\zeta)}g_4'(\eta',\zeta)\right)\begin{pmatrix}|\eta'|\\-\tau_{p_1}\end{pmatrix}e^{-i\tau_{p_1}x_3}\\
&-\left(\frac{\Delta^2_{p_1}(\eta',\zeta)}{\Delta(\eta',\zeta)}g_1(\eta',\zeta)+\frac{\Delta^2_{s_1}(\eta',\zeta)}{\Delta(\eta',\zeta)}g_2(\eta',\zeta)+g_2'(\eta',\zeta)\right.\\
&\qquad\left.+\frac{\Delta^2_{p_2}(\eta',\zeta)}{\Delta(\eta',\zeta)}g_3'(\eta',\zeta)+\frac{\Delta^2_{s_2}(\eta',\zeta)}{\Delta(\eta',\zeta)}g_4'(\eta',\zeta)\right)\begin{pmatrix}\tau_{s_1}\\|\eta'|\end{pmatrix}e^{-i\tau_{s_1}x_3},
\end{aligned}$$

and for $x_3 > 0$

$$
\begin{aligned}
&v^{II}(\eta', x_3) \\
&= E^{II}(x_3, \eta'; \zeta) - K^{II}(x_3, \eta'; \zeta) \\
&= \frac{i}{2}\left(\int_{-\infty}^{x_3}\left(\frac{e^{i(x_3-y_3)\tau_{p_2}}}{c_{p_2}^2\tau_{p_2}}P_{p_2}(\eta', \tau_{p_2}) + \frac{e^{i(x_3-y_3)\tau_{s_2}}}{c_{s_2}^2\tau_{s_2}}P_{s_2}(\eta', \tau_{s_2})\right) f(\eta', y_3)\, dy_3\right. \\
&\left. + \int_{x_3}^{\infty}\left(\frac{e^{-i(x_3-y_3)\tau_{p_2}}}{c_{p_2}^2\tau_{p_2}}P_{p_2}(\eta', -\tau_{p_2}) + \frac{e^{-i(x_3-y_3)\tau_{s_2}}}{c_{s_2}^2\tau_{s_2}}P_{s_2}(\eta', -\tau_{s_2})\right) f(\eta', y_3)\, dy_3\right) \\
&- \left(\frac{\Delta_{p_1}^3(\eta', \zeta)}{\Delta(\eta', \zeta)}g_1(\eta', \zeta) + \frac{\Delta_{s_1}^3(\eta', \zeta)}{\Delta(\eta', \zeta)}g_2(\eta', \zeta) + g_3(\eta', \zeta)\right. \\
&\qquad \left. + \frac{\Delta_{p_2}^3(\eta', \zeta)}{\Delta(\eta', \zeta)}g_3'(\eta', \zeta) + \frac{\Delta_{s_2}^3(\eta', \zeta)}{\Delta(\eta', \zeta)}g_4'(\eta', \zeta)\right)\begin{pmatrix}|\eta'| \\ \tau_{p_2}\end{pmatrix} e^{i\tau_{p_2}x_3} \\
&- \left(\frac{\Delta_{p_1}^4(\eta', \zeta)}{\Delta(\eta', \zeta)}g_1(\eta', \zeta) + \frac{\Delta_{s_1}^4(\eta', \zeta)}{\Delta(\eta', \zeta)}g_2(\eta', \zeta) + g_4(\eta', \zeta)\right. \\
&\qquad \left. + \frac{\Delta_{p_2}^4(\eta', \zeta)}{\Delta(\eta', \zeta)}g_3'(\eta', \zeta) + \frac{\Delta_{s_2}^4(\eta', \zeta)}{\Delta(\eta', \zeta)}g_4'(\eta', \zeta)\right)\begin{pmatrix}-\tau_{s_2} \\ |\eta'|\end{pmatrix} e^{i\tau_{s_2}x_3}.
\end{aligned}
$$

In summary, the Green functions $G_1^I(x_3, y_3, \eta'; \zeta)$ and $G_1^{II}(x_3, y_3, \eta'; \zeta)$ for $x_3 < 0$ and $x_3 > 0$ are given respectively in the following form:

(2.23)

$$
\begin{aligned}
&G_1^I(x_3, y_3, \eta'; \zeta) = \frac{i}{2}\times \\
&\Bigg[-H(-y_3)\left\{\frac{\Delta_{p_1}^1(\eta', \zeta)}{\Delta(\eta', \zeta)}e^{-i\tau_{p_1}x_3}e^{-i\tau_{p_1}y_3}\frac{1}{\tau_{p_1}\zeta}\begin{pmatrix}|\eta'| \\ -\tau_{p_1}\end{pmatrix}(|\eta'|\ \tau_{p_1})\right. \\
&\qquad \left. + \frac{\Delta_{s_1}^1(\eta', \zeta)}{\Delta(\eta', \zeta)}e^{-i\tau_{p_1}x_3}e^{-i\tau_{s_1}y_3}\frac{1}{\tau_{s_1}\zeta}\begin{pmatrix}|\eta'| \\ -\tau_{p_1}\end{pmatrix}(-\tau_{s_1}\ |\eta'|)\right\} \\
&-H(y_3)\left\{ e^{-i\tau_{p_1}x_3}e^{i\tau_{p_1}y_3}\frac{1}{\tau_{p_1}\zeta}\begin{pmatrix}|\eta'| \\ -\tau_{p_1}\end{pmatrix}(|\eta'|\ -\tau_{p_1})\right. \\
&\qquad + \frac{\Delta_{p_2}^1(\eta', \zeta)}{\Delta(\eta', \zeta)}e^{-i\tau_{p_1}x_3}e^{i\tau_{p_2}y_3}\frac{1}{\tau_{p_2}\zeta}\begin{pmatrix}|\eta'| \\ -\tau_{p_1}\end{pmatrix}(|\eta'|\ -\tau_{p_2}) \\
&\qquad \left. + \frac{\Delta_{s_2}^1(\eta', \zeta)}{\Delta(\eta', \zeta)}e^{-i\tau_{p_1}x_3}e^{i\tau_{s_2}y_3}\frac{1}{\tau_{s_2}\zeta}\begin{pmatrix}|\eta'| \\ -\tau_{p_1}\end{pmatrix}(\tau_{s_2}\ |\eta'|)\right\} \\
&-H(-y_3)\left\{\frac{\Delta_{p_1}^2(\eta', \zeta)}{\Delta(\eta', \zeta)}e^{-i\tau_{s_1}x_3}e^{-i\tau_{p_1}y_3}\frac{1}{\tau_{p_1}\zeta}\begin{pmatrix}\tau_{s_1} \\ |\eta'|\end{pmatrix}(|\eta'|\ \tau_{p_1})\right. \\
&\qquad \left. + \frac{\Delta_{s_1}^2(\eta', \zeta)}{\Delta(\eta', \zeta)}e^{-i\tau_{s_1}x_3}e^{-i\tau_{s_1}y_3}\frac{1}{\tau_{s_1}\zeta}\begin{pmatrix}\tau_{s_1} \\ |\eta'|\end{pmatrix}(-\tau_{s_1}\ |\eta'|)\right\} \\
&-H(y_3)\left\{ e^{-i\tau_{s_1}x_3}e^{i\tau_{s_1}y_3}\frac{1}{\tau_{s_1}\zeta}\begin{pmatrix}\tau_{s_1} \\ |\eta'|\end{pmatrix}(\tau_{s_1}\ |\eta'|)\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\Delta^2_{p_2}(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{-i\tau_{s_1}x_3}e^{i\tau_{p_2}y_3}\frac{1}{\tau_{p_2}\zeta}\begin{pmatrix}\tau_{s_1}\\ |\eta'|\end{pmatrix}\begin{pmatrix}|\eta'| & -\tau_{p_2}\end{pmatrix}\\
&+\frac{\Delta^2_{s_2}(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{-i\tau_{s_1}x_3}e^{i\tau_{s_2}y_3}\frac{1}{\tau_{s_2}\zeta}\begin{pmatrix}\tau_{s_1}\\ |\eta'|\end{pmatrix}\begin{pmatrix}\tau_{s_2} & |\eta'|\end{pmatrix}\Big\}\\
+H(x_3-y_3)\Big\{ &\qquad e^{i\tau_{p_1}(x_3-y_3)}\frac{1}{\tau_{p_1}\zeta}\begin{pmatrix}|\eta'|\\ \tau_{p_1}\end{pmatrix}\begin{pmatrix}|\eta'| & \tau_{p_1}\end{pmatrix}\\
+&\qquad e^{i\tau_{s_1}(x_3-y_3)}\frac{1}{\tau_{s_1}\zeta}\begin{pmatrix}-\tau_{s_1}\\ |\eta'|\end{pmatrix}\begin{pmatrix}-\tau_{s_1} & |\eta'|\end{pmatrix}\Big\}\\
+H(y_3-x_3)\Big\{ &\qquad e^{-i\tau_{p_1}(x_3-y_3)}\frac{1}{\tau_{p_1}\zeta}\begin{pmatrix}|\eta'|\\ -\tau_{p_1}\end{pmatrix}\begin{pmatrix}|\eta'| & -\tau_{p_1}\end{pmatrix}\\
+&\qquad e^{-i\tau_{s_1}(x_3-y_3)}\frac{1}{\tau_{s_1}\zeta}\begin{pmatrix}\tau_{s_1}\\ |\eta'|\end{pmatrix}\begin{pmatrix}\tau_{s_1} & |\eta'|\end{pmatrix}\Big\}\Big], \quad x_3<0,
\end{aligned}
$$

(2.24)

$$
G_1^{II}(x_3,y_3,\eta';\zeta)=\frac{i}{2}\times
$$

$$
\begin{aligned}
\Big[-H(-y_3)\Big\{ &\frac{\Delta^3_{p_1}(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{i\tau_{p_2}x_3}e^{-i\tau_{p_1}y_3}\frac{1}{\tau_{p_1}\zeta}\begin{pmatrix}|\eta'|\\ \tau_{p_2}\end{pmatrix}\begin{pmatrix}|\eta'| & \tau_{p_1}\end{pmatrix}\\
+&\frac{\Delta^3_{s_1}(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{i\tau_{p_2}x_3}e^{-i\tau_{s_1}y_3}\frac{1}{\tau_{s_1}\zeta}\begin{pmatrix}|\eta'|\\ \tau_{p_2}\end{pmatrix}\begin{pmatrix}-\tau_{s_1} & |\eta'|\end{pmatrix}\\
+&\qquad e^{i\tau_{p_2}x_3}e^{-i\tau_{p_2}y_3}\frac{1}{\tau_{p_2}\zeta}\begin{pmatrix}|\eta'|\\ \tau_{p_2}\end{pmatrix}\begin{pmatrix}|\eta'| & \tau_{p_2}\end{pmatrix}\Big\}\\
-H(y_3)\Big\{ &\frac{\Delta^3_{p_2}(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{i\tau_{p_2}x_3}e^{i\tau_{p_2}y_3}\frac{1}{\tau_{p_2}\zeta}\begin{pmatrix}|\eta'|\\ \tau_{p_2}\end{pmatrix}\begin{pmatrix}|\eta'| & -\tau_{p_2}\end{pmatrix}\\
+&\frac{\Delta^3_{s_2}(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{i\tau_{p_2}x_3}e^{i\tau_{s_2}y_3}\frac{1}{\tau_{s_2}\zeta}\begin{pmatrix}|\eta'|\\ \tau_{p_2}\end{pmatrix}\begin{pmatrix}\tau_{s_2} & |\eta'|\end{pmatrix}\Big\}\\
-H(-y_3)\Big\{ &\frac{\Delta^4_{p_1}(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{i\tau_{s_2}x_3}e^{-i\tau_{p_1}y_3}\frac{1}{\tau_{p_1}\zeta}\begin{pmatrix}-\tau_{s_1}\\ |\eta'|\end{pmatrix}\begin{pmatrix}|\eta'| & \tau_{p_1}\end{pmatrix}\\
+&\frac{\Delta^4_{s_1}(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{i\tau_{s_2}x_3}e^{-i\tau_{s_1}y_3}\frac{1}{\tau_{s_1}\zeta}\begin{pmatrix}-\tau_{s_2}\\ |\eta'|\end{pmatrix}\begin{pmatrix}-\tau_{s_1} & |\eta'|\end{pmatrix}\\
+&\qquad e^{i\tau_{s_2}x_3}e^{-i\tau_{s_2}y_3}\frac{1}{\tau_{s_2}\zeta}\begin{pmatrix}-\tau_{s_2}\\ |\eta'|\end{pmatrix}\begin{pmatrix}-\tau_{s_2} & |\eta'|\end{pmatrix}\Big\}\\
-H(y_3)\Big\{ &\frac{\Delta^4_{p_2}(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{i\tau_{s_2}x_3}e^{i\tau_{p_2}y_3}\frac{1}{\tau_{p_2}\zeta}\begin{pmatrix}-\tau_{s_2}\\ |\eta'|\end{pmatrix}\begin{pmatrix}|\eta'| & -\tau_{p_2}\end{pmatrix}\\
+&\frac{\Delta^4_{s_2}(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{i\tau_{s_2}x_3}e^{i\tau_{s_2}y_3}\frac{1}{\tau_{s_2}\zeta}\begin{pmatrix}-\tau_{s_2}\\ |\eta'|\end{pmatrix}\begin{pmatrix}\tau_{s_2} & |\eta'|\end{pmatrix}\Big\}\\
+H(x_3-y_3)\Big\{ &\qquad e^{i\tau_{p_2}(x_3-y_3)}\frac{1}{\tau_{p_2}\zeta}\begin{pmatrix}|\eta'|\\ \tau_{p_2}\end{pmatrix}\begin{pmatrix}|\eta'| & \tau_{p_2}\end{pmatrix}\\
+&\qquad e^{i\tau_{s_2}(x_3-y_3)}\frac{1}{\tau_{s_2}\zeta}\begin{pmatrix}-\tau_{s_2}\\ |\eta'|\end{pmatrix}\begin{pmatrix}-\tau_{s_2} & |\eta'|\end{pmatrix}\Big\}
\end{aligned}
$$

$$+H(y_3 - x_3)\Big\{ \quad e^{-i\tau_{p_2}(x_3-y_3)}\frac{1}{\tau_{p_2}\zeta}\begin{pmatrix} |\eta'| \\ -\tau_{p_2}\end{pmatrix}(|\eta'| \ -\tau_{p_2})$$
$$+ \quad e^{-i\tau_{s_2}(x_3-y_3)}\frac{1}{\tau_{s_2}\zeta}\begin{pmatrix} \tau_{s_2} \\ |\eta'|\end{pmatrix}(\tau_{s_2}\ |\eta'|)\Big\}\Big], \quad x_3 > 0.$$

Here $H(y_3)$ denotes the Heaviside function.

§ 3. Zeros of the Lopatinski Determinant of $A_1(\eta')$

In this section, we investigate the number and nature of the zeros of the Lopatinski determinant defined in Section 2 in order to obtain the speed of the Stoneley wave. Our Lopatinski determinant seems to be equivalent to Cagniard's one. But Cagniard expressed the solutions of the elastic equation in cylindrical coordinates by using the Bessel transformation, and investigated the existence of the Stoneley wave. So our parameters are different from Cagniard's one. For the sake of completeness, we present the proof by a method due to L. Cagniard [2, Section 4].

Put

$$(3.1)\qquad \begin{aligned} &z = \frac{\zeta}{|\eta'|^2}, \\ &a_1 = \sqrt{1-\frac{z}{c_{p_1}^2}}, \quad a_2 = \sqrt{1-\frac{z}{c_{p_2}^2}}, \quad b_1 = \sqrt{1-\frac{z}{c_{s_1}^2}}, \quad b_2 = \sqrt{1-\frac{z}{c_{s_2}^2}}, \end{aligned}$$

then

$$\tau_{p_1} = i|\eta'|a_1,\ \tau_{p_2} = i|\eta'|a_2,\ \tau_{s_1} = i|\eta'|b_1,\ \tau_{s_2} = i|\eta'|b_2.$$

With this notation, the Lopatinski determinant (2.22) is rewritten as follows:

$$\begin{aligned}\Delta(\eta',\zeta) &= |\eta'|^6 \begin{vmatrix} 1 & ib_1 & -1 & ib_2 \\ -ia_1 & 1 & -ia_2 & -1 \\ 2i\mu_1 a_1 & -\mu_1(b_1^2+1) & 2i\mu_2 a_2 & \mu_2(b_2^2+1) \\ \mu_1(b_1^2+1) & 2i\mu_1 b_1 & -\mu_2(b_2^2+1) & 2i\mu_2 b_2 \end{vmatrix} \\ &= |\eta'|^6 D(z),\end{aligned}$$

where

$$(3.2)\qquad \begin{aligned} D(z) = &\left(2(\mu_1-\mu_2) - \frac{\mu_1 z}{c_{s_1}^2} + \frac{\mu_2 z}{c_{s_2}^2}\right)^2 + 4(\mu_1-\mu_2)^2 a_1 a_2 b_1 b_2 \\ &- a_1 b_1\left(2(\mu_1-\mu_2) + \frac{\mu_2 z}{c_{s_2}^2}\right)^2 - a_2 b_2\left(2(\mu_1-\mu_2) - \frac{\mu_1 z}{c_{s_1}^2}\right)^2 \\ &- \frac{\mu_1\mu_2}{c_{s_1}^2 c_{s_2}^2}(a_1 b_2 + a_2 b_1) z^2. \end{aligned}$$

By (2.8) and (2.17), the propagation speeds of shear and pressure waves should satisfy

$$c_{s_1} < c_{p_1}, \quad c_{s_2} < c_{p_2},$$

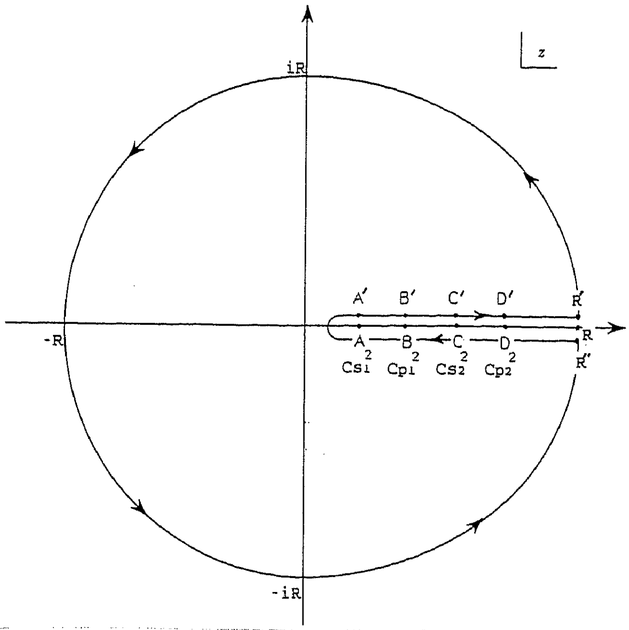

FIGURE 4 PATH γ IN THE z-PLANE

so there are six cases

$$\begin{aligned} &c_{s_1} < c_{p_1} \leqq c_{s_2} < c_{p_2}, \\ &c_{s_1} \leqq c_{s_2} \leqq c_{p_1} \leqq c_{p_2}, \\ &c_{s_1} \leqq c_{s_2} < c_{p_2} \leqq c_{p_1}, \\ &c_{s_2} < c_{p_2} \leqq c_{s_1} < c_{p_1}, \\ &c_{s_2} \leqq c_{s_1} \leqq c_{p_2} \leqq c_{p_1}, \\ &c_{s_2} \leqq c_{s_1} < c_{p_1} \leqq c_{p_2}, \end{aligned}$$

to consider. From now on, we have only to consider the standard case:

$$c_{s_1} < c_{p_1} < c_{s_2} < c_{p_2}, \tag{3.3}$$

since the other cases can be treated similarly.

Now, we examine the zeros of $D(z)$. For a_1 (resp. a_2, b_1, b_2), we make a branch cut on the real axis of the z-plane between the point $c_{p_1}^2$ and ∞ (resp. $c_{p_2}^2$ and ∞, $c_{s_1}^2$ and ∞, $c_{s_2}^2$ and ∞) as in Figure 3. To determine the number of roots of $D(z) = 0$, we make the path γ in the z-plane as in Figure 4. We let $A = c_{s_1}^2$, $B = c_{p_1}^2$, $C = c_{s_2}^2$, $D = c_{p_2}^2$, and take four points A', B', C', D' on the path γ to be near A, B, C, D, respectively. Moreover we take a real number R to be large enough, and a point R' on the path γ to be near R.

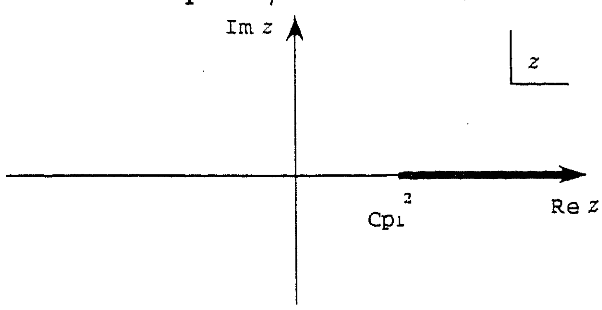

FIGURE 3 BRANCH CUT BETWEEN BRANCH POINTS $c_{p_1}^2$ AND ∞ IN THE z-PLANE

We discuss the image of $\frac{D(z)}{z^3}$ when z goes on the path above. When we change z to the complex conjugate of z, $D(z)$ changes to its complex conjugate; that is $D(\bar{z}) = \overline{D(z)}$. Therefore we can only consider the contour of the upper half plane. $D(A)$ denotes the image of A by $D(z)$.

1. If z is on A', we may consider that the sign of $D(A')$ is the same as that of $D(A)$, so $b_1 = 0$, and $a_1, a_2, b_2 < 0$, then we have

$$D(A) = \operatorname{Re} D(A) = \left(\mu_1 - 2\mu_2 + \frac{c_{s_1}^2}{c_{s_2}^2}\mu_2\right)^2 - a_2 b_2 (\mu_1 - 2\mu_2)^2 - a_1 b_2 \mu_1 \mu_2 \frac{c_{s_1}^2}{c_{s_2}^2} \gtreqless 0.$$

2. If z goes from A' to B', we may examine that z goes from A to B, so b_1 is pure imaginary such that $\operatorname{Im} b_1 > 0$, and $a_1, a_2, b_2 < 0$, then $\operatorname{Re} D(z)$ and $\operatorname{Im} D(z)$ may take positive and negative values or zero. In fact, we have

$$\operatorname{Re} D(z) = \left(2(\mu_1 - \mu_2) - \frac{\mu_1 z}{c_{s_1}^2} + \frac{\mu_2 z}{c_{s_2}^2}\right)^2 - a_2 b_2 \left(2(\mu_1 - \mu_2) - \frac{\mu_1 z}{c_{s_1}^2}\right)^2 - \frac{\mu_1 \mu_2}{c_{s_1}^2 c_{s_2}^2} a_1 b_2 z^2 \gtreqless 0,$$

$$\operatorname{Im} D(z) = \frac{1}{i}\left(4(\mu_1 - \mu_2)^2 a_1 a_2 b_1 b_2 - a_1 b_1 \left(2(\mu_1 - \mu_2) + \frac{\mu_2 z}{c_{s_2}^2}\right)^2 - \frac{\mu_1 \mu_2}{c_{s_1}^2 c_{s_2}^2} a_2 b_1 z^2\right) \gtreqless 0.$$

We shall show that there is no $z \in (c_{s_1}^2, c_{p_1}^2)$ such that

$$\operatorname{Re} D(z) = 0 \text{ and } \operatorname{Im} D(z) = 0,$$

more precisely, we shall show that

$$\operatorname{Re} D(z) = 0 \text{ and } \operatorname{Im} D(z) = 0 \implies z > c_{s_2}^2 + c_{p_2}^2.$$

For this purpose we consider $D(z) = D(a_1, a_2)$ as a function of a_1 and a_2 by regarding a_1 and a_2 as parameters. As in Figure 5, the equation $\operatorname{Im} D(a_1, a_2) = 0$ represents an equilateral hyperbola passing through the origin and with asymptotes parallel to the axes, intersecting in the third quadrant. And the equation $\operatorname{Re} D(a_1, a_2) = 0$ represents a tangent to one of the branches of the hyperbola at the point (a_1, a_2) where

$$(3.4)\qquad \begin{aligned} a_1 &= \frac{2(\mu_1 - \mu_2) - \frac{\mu_1 z}{c_{s_1}^2} + \frac{\mu_2 z}{c_{s_2}^2}}{2(\mu_1 - \mu_2) b_2}, \\ a_2 &= \frac{2(\mu_1 - \mu_2) - \frac{\mu_1 z}{c_{s_1}^2} + \frac{\mu_2 z}{c_{s_2}^2}}{2(\mu_1 - \mu_2) b_2} \cdot \frac{2(\mu_1 - \mu_2) + \frac{\mu_2 z}{c_{s_2}^2}}{2(\mu_1 - \mu_2) - \frac{\mu_1 z}{c_{s_1}^2}}. \end{aligned}$$

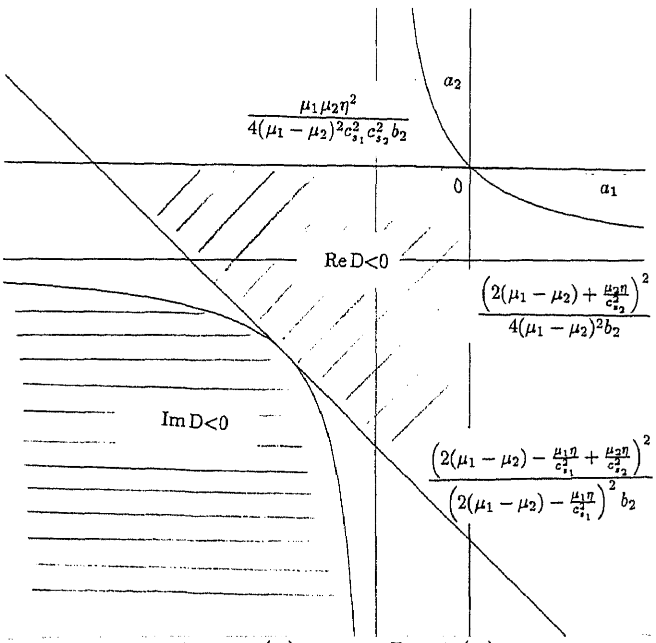

FIGURE 5 BEHAVIOR OF $\operatorname{Re} D(z)$ AND $\operatorname{Im} D(z)$ FOR VARIATIONS OF THE PARAMETERS a_1 AND a_2

In fact, we have in the second of (3.4)

$$a_2b_2 = \frac{2(\mu_1-\mu_2)-\frac{\mu_1 z}{c_{s_1}^2}+\frac{\mu_2 z}{c_{s_2}^2}}{2(\mu_1-\mu_2)-\frac{\mu_1 z}{c_{s_1}^2}} \cdot \frac{2(\mu_1-\mu_2)+\frac{\mu_2 z}{c_{s_2}^2}}{2(\mu_1-\mu_2)} > 1, \tag{3.5}$$

so

$$a_2b_2 = \sqrt{1-\frac{z}{c_{p_2}^2}}\sqrt{1-\frac{z}{c_{s_2}^2}} > 1,$$

from this inequality and $z>0$, we get

$$z > c_{s_2}^2 + c_{p_2}^2.$$

If we go back to the situation that a_1, a_2, b_1, and b_2 are functions of z, there is no solution z ($c_{s_1}^2 < z < c_{p_1}^2$) which satisfies (3.4). Since $\operatorname{Im} D(a_1,a_2)$ is negative in the lower region of the branch of the curve $\operatorname{Im} D(a_1,a_2)=0$ which does not pass the origin, and $\operatorname{Re} D(a_1,a_2)$ is negative in the upper region of the line $\operatorname{Re} D(a_1,a_2)=0$, $D(a_1(z),a_2(z))$ never goes into the third quadrant.

3. If z is on B', we may consider that the sign of $D(B')$ is the same as that of $D(B)$, so b_1 is pure imaginary such that $\operatorname{Im} b_1 > 0$, $a_1 = 0$, and $a_2, b_2 < 0$, then we have

$$\operatorname{Re} D(B) = \left(2(\mu_1-\mu_2)-\frac{\mu_1 c_{p_1}^2}{c_{s_1}^2}+\frac{\mu_2 c_{p_1}^2}{c_{s_2}^2}\right)^2 - a_2b_2\left(2(\mu_1-\mu_2)-\frac{\mu_1 c_{p_1}^2}{c_{s_1}^2}\right)^2 \gtreqless 0,$$

$$\operatorname{Im} D(B) = \frac{1}{i}\left(-\frac{\mu_1\mu_2}{c_{s_1}^2c_{s_2}^2}a_2b_1c_{p_1}^4\right) > 0.$$

4. If z goes from B' to C', we may examine that z goes from B to C, so a_1, b_1 are pure imaginary such that $\operatorname{Im} a_1, \operatorname{Im} b_1 > 0$ and $a_2, b_2 < 0$, then we have

$$\operatorname{Re} D(z) = \left(2(\mu_1-\mu_2)-\frac{\mu_1 z}{c_{s_1}^2}+\frac{\mu_2 z}{c_{s_2}^2}\right)^2 + 4(\mu_1-\mu_2)^2a_1a_2b_1b_2$$
$$- a_1b_1\left(2(\mu_1-\mu_2)+\frac{\mu_2 z}{c_{s_2}^2}\right)^2 - a_2b_2\left(2(\mu_1-\mu_2)-\frac{\mu_1 z}{c_{s_1}^2}\right)^2 \gtreqless 0,$$

$$\operatorname{Im} D(z) = \frac{1}{i}\left(-\frac{\mu_1\mu_2}{c_{s_1}^2 c_{s_2}^2}(a_1b_2 + a_2b_1)z^2\right) > 0.$$

5. If z is on C', we may consider that the sign of $D(C')$ is the same as that of $D(C)$, so a_1, b_1 are pure imaginary such that $\operatorname{Im} a_1, \operatorname{Im} b_1 > 0$, $b_2 = 0$ and $a_2 < 0$, then we have

$$\operatorname{Re} D(C) = \left(2\mu_1 - \frac{\mu_1 c_{s_2}^2}{c_{s_1}^2} - \mu_2\right)^2 - a_1b_1(2\mu_1 - \mu_2)^2 > 0,$$

$$\operatorname{Im} D(C) = \frac{1}{i}\left(-\frac{\mu_1\mu_2}{c_{s_1}^2} a_2 b_1 c_{s_2}^2\right) > 0.$$

6. If z goes from C' to D', we may examine that z goes from C to D, so a_1, b_1, b_2 are pure imaginary such that $\operatorname{Im} a_1, \operatorname{Im} b_1$, $\operatorname{Im} b_2 > 0$, and $a_2 < 0$, then we have

$$\operatorname{Re} D(z) = \left(2(\mu_1 - \mu_2) - \frac{\mu_1 z}{c_{s_1}^2} + \frac{\mu_2 z}{c_{s_2}^2}\right)^2 - a_1b_1\left(2(\mu_1 - \mu_2) + \frac{\mu_2 z}{c_{s_2}^2}\right)^2 > 0,$$

$$\operatorname{Im} D(z) = \frac{1}{i}\left(4(\mu_1 - \mu_2)^2 a_1a_2b_1b_2 - a_2b_2\left(2(\mu_1 - \mu_2) - \frac{\mu_1 z}{c_{s_1}^2}\right)^2 - \frac{\mu_1\mu_2}{c_{s_1}^2 c_{s_2}^2} a_2 b_1 z^2\right) > 0.$$

7. If z is on D', we may consider that the sign of $D(D')$ is the same as that of $D(D)$, so a_1, b_1, b_2 are pure imaginary such that $\operatorname{Im} a_1, \operatorname{Im} b_1, \operatorname{Im} b_2 > 0$, and $a_2 = 0$, then we have

$$D(D) = \operatorname{Re} D(D) = \left(2(\mu_1 - \mu_2) - \frac{\mu_1 c_{p_2}^2}{c_{s_1}^2} + \frac{\mu_2 c_{p_2}^2}{c_{s_2}^2}\right)^2 - a_1b_1\left(2(\mu_1 - \mu_2) + \frac{\mu_2 c_{p_2}^2}{c_{s_2}^2}\right)^2 - \frac{\mu_1\mu_2}{c_{s_1}^2 c_{s_2}^2} a_1 b_2 c_{p_2}^4 > 0.$$

8. If z goes from D' to R', we may examine that z goes from D to R, so a_1, a_2, b_1, and b_2 are all pure imaginary such that $\operatorname{Im} a_1, \operatorname{Im} a_2, \operatorname{Im} b_1, \operatorname{Im} b_2 > 0$, then we have

$$D(z) = \left(2(\mu_1 - \mu_2) - \frac{\mu_1 z}{c_{s_1}^2} + \frac{\mu_2 z}{c_{s_2}^2}\right)^2 + 4(\mu_1 - \mu_2)^2 a_1a_2b_1b_2 - a_1b_1\left(2(\mu_1 - \mu_2) + \frac{\mu_2 z}{c_{s_2}^2}\right)^2 - a_2b_2\left(2(\mu_1 - \mu_2) - \frac{\mu_1 z}{c_{s_1}^2}\right)^2 - \frac{\mu_1\mu_2}{c_{s_1}^2 c_{s_2}^2}(a_1b_2 + a_2b_1)z^2 > 0.$$

9. If z goes from R' to $-R'$, along the contour which the radius R is very large, then we have

$$D(z) = \left(\frac{\mu_2^2}{c_{p_1} c_{s_1} c_{s_2}^4} + \frac{\mu_1^2}{c_{p_2} c_{s_2} c_{s_1}^4} + \frac{\mu_1\mu_2}{c_{s_1}^2 c_{s_2}^3 c_{p_1}} + \frac{\mu_1\mu_2}{c_{s_1}^3 c_{s_2}^2 c_{p_2}}\right) z^3 + O(z^2),$$

and so

$$\frac{D(z)}{z^3} = \left(\frac{\mu_2^2}{c_{p_1}c_{s_1}c_{s_2}^4} + \frac{\mu_1^2}{c_{p_2}c_{s_2}c_{s_1}^4} + \frac{\mu_1\mu_2}{c_{s_1}^2 c_{s_2}^3 c_{p_1}} + \frac{\mu_1\mu_2}{c_{s_1}^3 c_{s_2}^2 c_{p_2}}\right) + O\left(\frac{1}{z}\right), \quad z \to \infty,$$

where

$$\frac{\mu_2^2}{c_{p_1}c_{s_1}c_{s_2}^4} + \frac{\mu_1^2}{c_{p_2}c_{s_2}c_{s_1}^4} + \frac{\mu_1\mu_2}{c_{s_1}^2 c_{s_2}^3 c_{p_1}} + \frac{\mu_1\mu_2}{c_{s_1}^3 c_{s_2}^2 c_{p_2}},$$

is real positive.

As mentioned above, there are two qualitatively different cases to be considered; in the first case $D(A)$ is negative, and in the second case $D(A)$ is positive.

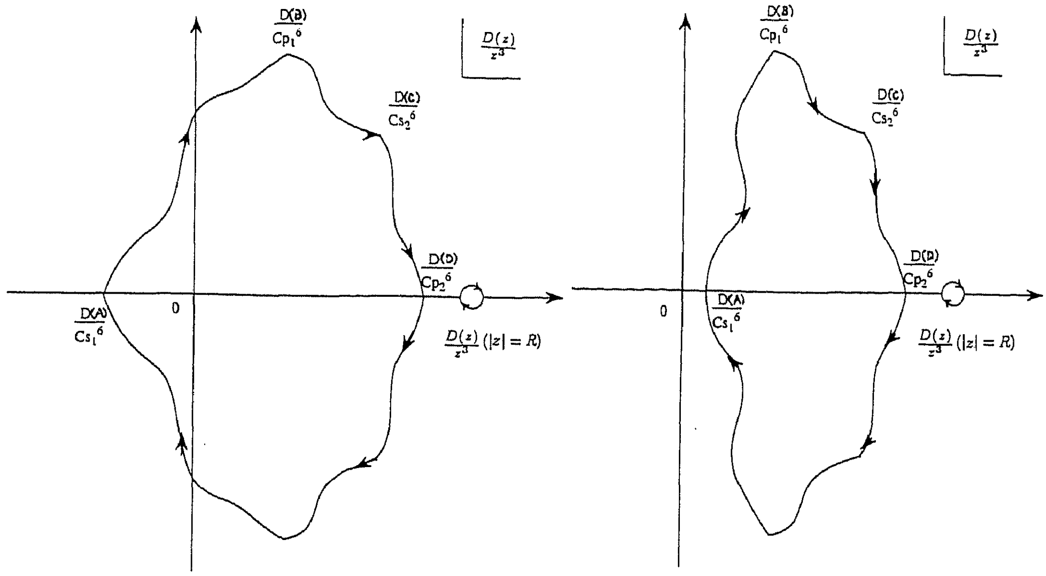

FIGURE 6 PATH OF THE IMAGE POINT IN THE $\frac{D(z)}{z^3}$-PLANE

Now, we have the following asymptotic expansions for sufficiently small z:

$$a_1 = \sqrt{1 - \frac{z}{c_{p_1}^2}} = 1 - \frac{z}{2c_{p_1}^2} - \frac{z^2}{8c_{p_1}^4} + O(z^3), \tag{3.6}$$

$$a_2 = \sqrt{1 - \frac{z}{c_{p_2}^2}} = 1 - \frac{z}{2c_{p_2}^2} - \frac{z^2}{8c_{p_2}^4} + O(z^3), \tag{3.7}$$

$$b_1 = \sqrt{1 - \frac{z}{c_{s_1}^2}} = 1 - \frac{z}{2c_{s_1}^2} - \frac{z^2}{8c_{s_1}^4} + O(z^3), \tag{3.8}$$

$$b_2 = \sqrt{1 - \frac{z}{c_{s_2}^2}} = 1 - \frac{z}{2c_{s_2}^2} - \frac{z^2}{8c_{s_2}^4} + O(z^3). \tag{3.9}$$

By the asymptotic expansions (3.6)-(3.9), it follows that

$$D(z) = z^2\left[\frac{(\mu_1-\mu_2)^2}{c_{p_1}^2 c_{p_2}^2} + \frac{\mu_1^2-\mu_2^2}{c_{p_1}^2 c_{s_2}^2} - \frac{\mu_1^2-\mu_2^2}{c_{p_2}^2 c_{s_1}^2} - \frac{(\mu_1+\mu_2)^2}{c_{s_1}^2 c_{s_2}^2} + O(z)\right], \tag{3.10}$$

$$z \to +0.$$

Since the quantity in brackets in (3.10) is equal to

$$\mu_1^2\left(\frac{1}{c_{p_2}^2}+\frac{1}{c_{s_2}^2}\right)\left(\frac{1}{c_{p_1}^2}-\frac{1}{c_{s_1}^2}\right)+\mu_2^2\left(\frac{1}{c_{p_1}^2}+\frac{1}{c_{s_1}^2}\right)\left(\frac{1}{c_{p_2}^2}-\frac{1}{c_{s_2}^2}\right) - 2\mu_1\mu_2\left(\frac{1}{c_{p_1}^2c_{p_2}^2}+\frac{1}{c_{s_1}^2c_{s_2}^2}\right),$$

it is always negative and never zero. So the order of pole of $\frac{D(z)}{z^3}$ at zero is one. Therefore, from principle of the argument:

$$N-P=C(0),$$

it follows that if $D(A)$ is negative, then the zeros of $D(z)$ do not exist, and if $D(A)$ is positive, there exists only one real zero with order 1 of $D(z)$ on $[0,c_{s_1}^2]$. Indeed, there is no $z\in\mathbf{C}$ such that

$$\operatorname{Re}D(z)=0,\quad \operatorname{Im}D(z)=0$$

outside the interval $[0,c_{s_1}^2]$. Here N, P denote, respectively, the number of zeros and the number of poles of $\frac{D(z)}{z^3}$ in the complex plane being countered with their proper multiplicities, and $C(0)$ the quotient by 2π of the variation of the argument of $\frac{D(z)}{z^3}$ when z described the closed path γ.

Hereafter we denote the real zero of $D(z)$ by c_{St}^2. Then the zero of $\Delta(\eta',\zeta)$ is $c_{St}^2|\eta'|^2$ and is the origin of the Stoneley wave propagating along the interface $x_3=0$ in the elastic space R^3, and c_{St} is its speed.

In conclusion, the conditions for the existence of zeros of the Lopatinski determinant $\Delta(\eta',\zeta)$ defined in (2.22) (the existence of the Stoneley waves) are given as follows: Let $D(z)$ be the polynomial of z defined by (3.2). Then $D(z)=\Delta(\eta',z|\eta'|^2)/|\eta'|^6$ is independent of $\eta'\neq 0$.

$$D(c_{s_1}^2)=\left(\mu_1-2\mu_2+\frac{c_{s_1}^2}{c_{s_2}^2}\mu_2\right)^2+\sqrt{\frac{c_{s_1}^2}{c_{p_2}^2}-1}\sqrt{\frac{c_{s_1}^2}{c_{s_2}^2}-1}(\mu_1-2\mu_2)^2 + \sqrt{\frac{c_{s_1}^2}{c_{p_1}^2}-1}\sqrt{\frac{c_{s_1}^2}{c_{s_2}^2}-1}\mu_1\mu_2\frac{c_{s_1}^2}{c_{s_2}^2}.$$

(i) $D(c_{s_1}^2)>0 \implies$ The zero $\zeta=c_{St}^2|\eta'|^2$ of $\Delta(\eta',\zeta)$ in ζ exists in $[0,\ c_{s_1}^2|\eta'|^2)$ with order 1. More precisely, we shall prove in the proof of Theorem 6.5 that $c_{St}\neq 0$.

(ii) $D(c_{s_1}^2)=0 \implies c_{St}=c_{s_1}$ and we shall consider this case under some restricted conditions (cf. Lemma 6.4).

(iii) $D(c_{s_1}^2)<0 \implies \Delta(\eta',\zeta)$ has no zero.

Remark. The minimum speed is either c_{s_1} or c_{s_2}, as is seen from the six cases mentioned above. If $c_{s_2} < c_{s_1}$, then we must replace $D(c_{s_1}^2)$ by $D(c_{s_2}^2)$.

§ 4. Generalized Eigenfunctions of $A_1(\eta')$

In this section, we give a family of generalized eigenfunctions for $A_1(\eta')$ by using the Green function $G_1(x_3, y_3, \eta'; \zeta)$ given in Section 2.

We define $\psi_{1j}(x_3, \eta; \zeta)$ $(j \in M = M_1 \cup M_2 = \{s_1, p_1, s_2, p_2\})$ as follows:

$$\psi_{1j}(x_3,\eta;\zeta) = \begin{cases} \psi_{1j}^{I}(x_3,\eta;\zeta), & x_3 < 0, \\ \psi_{1j}^{II}(x_3,\eta;\zeta), & x_3 > 0, \end{cases} \tag{4.1}$$

where

$$\begin{aligned} \psi_{1j}^{I}(x_3,\eta;\zeta) &= F_{y_3}^{-1}[G_1^{I}(x_3,y_3,\eta';\zeta)](\xi)(\lambda_j(\eta)-\zeta)P_j(\eta)\rho_1^{-1}, \quad x_3<0, \\ \psi_{1j}^{II}(x_3,\eta;\zeta) &= F_{y_3}^{-1}[G_1^{II}(x_3,y_3,\eta';\zeta)](\xi)(\lambda_j(\eta)-\zeta)P_j(\eta)\rho_2^{-1}, \quad x_3>0. \end{aligned} \tag{4.2}$$

and

$$F_{y_3}^{-1}[G_1^{m}(x_3,y_3,\eta';\zeta)] = \frac{1}{\sqrt{2\pi}}\int e^{iy_3\xi}G_1^{m}(x_3,y_3,\eta';\zeta)\,dy_3 \quad (m=I,II).$$

Here $\lambda_j(\eta)$ are the eigenvalues of $A_1(\eta')$ given by (2.9) and (2.17), and $P_j(\eta)$ are the mutually orthogonal projections of $A_1(\eta')$ given by (2.12), (2.13) and (2.17).

The motivation for these particular definitions (4.1) and (4.2) is shown in Section 6 (Lemma 6.1) below.

Lemma 4.1. *Let ζ be non-real. Then we have for $j \in M$*

$$(A_1^{I}(\eta')-\zeta I)\psi_{1j}^{I}(x_3,\eta;\zeta) = \frac{1}{\sqrt{2\pi}}e^{ix_3\xi}(\lambda_j(\eta)-\zeta)P_j(\eta), \tag{4.3}$$

$$(A_1^{II}(\eta')-\zeta I)\psi_{1j}^{II}(x_3,\eta;\zeta) = \frac{1}{\sqrt{2\pi}}e^{ix_3\xi}(\lambda_j(\eta)-\zeta)P_j(\eta), \tag{4.4}$$

$$\psi_{1j}^{I}(x_3,\eta;\zeta)\big|_{x_3=0} = \psi_{1j}^{II}(x_3,\eta;\zeta)\big|_{x_3=0}, \tag{4.5}$$

$$B_1^{I}(\eta')\psi_{1j}^{I}(x_3,\eta;\zeta)\big|_{x_3=0} = B_1^{II}(\eta')\psi_{1j}^{II}(x_3,\eta;\zeta)\big|_{x_3=0}. \tag{4.6}$$

Proof. Let $\phi \in C_0^\infty(\mathbf{R}_-, \mathbf{C}^2)$ and $\psi \in C_0^\infty(\mathbf{R}, \mathbf{C}^2)$. Then

$$\begin{aligned} &\langle (A_1^{I}(\eta')-\zeta I)F_{y_3}^{-1}[G_1^{I}(x_3,y_3,\eta';\zeta)](\xi), \phi(x_3)\psi(\xi)\rangle_{x_3,\xi} \\ &= \langle F_{y_3}^{-1}[G_1^{I}(x_3,y_3,\eta';\zeta)](\xi), (A_1^{I}(\eta')-\bar\zeta I)\phi(x_3)\psi(\xi)\rangle_{x_3,\xi} \\ &= \left\langle G_1^{I}(x_3,y_3,\eta';\zeta), (A_1^{I}(\eta')-\bar\zeta I)\phi(x_3)F_\xi^{-1}[\psi](y_3)\right\rangle_{x_3,y_3} \\ &= \left\langle \langle G_1^{I}(x_3,y_3,\eta';\zeta), (A_1^{I}(\eta')-\bar\zeta I)\phi(x_3)\rangle_{x_3}, F_\xi^{-1}[\psi](y_3)\right\rangle_{y_3} \\ &= \left\langle \langle (A_1^{I}(\eta')-\zeta I)G_1^{I}(x_3,y_3,\eta';\zeta), \phi(x_3)\rangle_{x_3}, F_\xi^{-1}[\psi](y_3)\right\rangle_{y_3} \end{aligned}$$

$$= \left\langle \langle \delta(x_3 - y_3)I, \phi(x_3)\rangle_{x_3}, F_\xi^{-1}[\psi](y_3) \right\rangle_{y_3}$$
$$= \int_{\mathbf{R}_-} \phi(x_3) F_\xi^{-1}[\psi](x_3)\rho_1^{-1}\, dx_3$$
$$= \int_{\mathbf{R}_-} \left(\frac{1}{\sqrt{2\pi}} \int e^{ix_3\xi}\phi(x_3)\psi(\xi)\, d\xi \right) dx_3$$
$$= \left\langle \frac{1}{\sqrt{2\pi}} e^{ix_3\xi} I, \phi(x_3)\psi(\xi) \right\rangle_{x_3,\xi},$$

where, for example,

$$\langle f, g\rangle_{x_3} = \int_{\mathbf{R}_-} f \cdot g\rho_1\, dx_3, \quad f \cdot g = \sum_{i=1}^{3} f_i \bar{g}_i.$$

In view of (4.2), the last equality implies formula (4.3).

Formula (4.4) can be proved in the same way. The interface conditions (4.5) and (4.6) are obvious because $G_1^I(x_3, y_3, \eta'; \zeta)$ and $G_1^{II}(x_3, y_3, \eta'; \zeta)$ satisfy the interface conditions. □

Now we shall introduce the generalized or improper eigenfunctions $\psi_{1j}(x_3, \eta)$ ($j \in M$) of $A_1(\eta')$, making use of the expressions (2.23) and (2.24) of the Green functions G_1^I and G_2^{II}.

From the elementary formulas:

$$F_{y_3}^{-1}\left[H(-y_3)e^{-i\tau y_3}\right](\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{i(\xi-\tau)y_3}\, dy_3 = \frac{1}{\sqrt{2\pi}} \frac{1}{i(\xi-\tau)},$$
$$F_{y_3}^{-1}\left[H(y_3)e^{i\tau y_3}\right](\xi) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{i(\xi+\tau)y_3}\, dy_3 = -\frac{1}{\sqrt{2\pi}} \frac{1}{i(\xi+\tau)}$$

for $\tau \in \mathbf{C}$, $\operatorname{Im}\tau \geq 0$, it follows that

$$(4.7) \quad F_{y_3}^{-1}\left[G_1^I(x_3, y_3, \eta'; \zeta)\right](\xi) = \frac{1}{2\sqrt{2\pi}} \times$$

$$\left[-\left\{ \frac{\Delta_{p_1}^1(\eta', \zeta)}{\Delta(\eta', \zeta)} e^{-i\tau_{p_1}x_3} \frac{1}{\xi - \tau_{p_1}} \frac{1}{\tau_{p_1}\zeta} \begin{pmatrix} |\eta'| \\ -\tau_{p_1} \end{pmatrix} (|\eta'| \ \tau_{p_1}) \right.\right.$$
$$\left. + \frac{\Delta_{s_1}^1(\eta', \zeta)}{\Delta(\eta', \zeta)} e^{-i\tau_{p_1}x_3} \frac{1}{\xi - \tau_{s_1}} \frac{1}{\tau_{s_1}\zeta} \begin{pmatrix} |\eta'| \\ -\tau_{p_1} \end{pmatrix} (-\tau_{s_1} \ |\eta'|) \right\}$$
$$+ \left\{ \qquad e^{-i\tau_{p_1}x_3} \frac{1}{\xi + \tau_{p_1}} \frac{1}{\tau_{p_1}\zeta} \begin{pmatrix} |\eta'| \\ -\tau_{p_1} \end{pmatrix} (|\eta'| \ -\tau_{p_1}) \right.$$
$$+ \frac{\Delta_{p_2}^1(\eta', \zeta)}{\Delta(\eta', \zeta)} e^{-i\tau_{p_1}x_3} \frac{1}{\xi + \tau_{p_2}} \frac{1}{\tau_{p_2}\zeta} \begin{pmatrix} |\eta'| \\ -\tau_{p_1} \end{pmatrix} (|\eta'| \ -\tau_{p_2})$$

$$+\frac{\Delta^1_{s_2}(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{-i\tau_{p_1}x_3}\frac{1}{\xi+\tau_{s_2}}\frac{1}{\tau_{s_2}\zeta}\begin{pmatrix}|\eta'|\\-\tau_{p_1}\end{pmatrix}(\tau_{s_2}\ |\eta'|)\Bigg\}$$

$$-\Bigg\{\frac{\Delta^2_{p_1}(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{-i\tau_{s_1}x_3}\frac{1}{\xi-\tau_{p_1}}\frac{1}{\tau_{p_1}\zeta}\begin{pmatrix}\tau_{s_1}\\|\eta'|\end{pmatrix}(|\eta'|\ \tau_{p_1})$$

$$+\frac{\Delta^2_{s_1}(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{-i\tau_{s_1}x_3}\frac{1}{\xi-\tau_{s_1}}\frac{1}{\tau_{s_1}\zeta}\begin{pmatrix}\tau_{s_1}\\|\eta'|\end{pmatrix}(-\tau_{s_1}\ |\eta'|)\Bigg\}$$

$$+\Bigg\{\qquad e^{-i\tau_{s_1}x_3}\frac{1}{\xi+\tau_{s_1}}\frac{1}{\tau_{s_1}\zeta}\begin{pmatrix}\tau_{s_1}\\|\eta'|\end{pmatrix}(\tau_{s_1}\ |\eta'|)$$

$$+\frac{\Delta^2_{p_2}(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{-i\tau_{s_1}x_3}\frac{1}{\xi+\tau_{p_2}}\frac{1}{\tau_{p_2}\zeta}\begin{pmatrix}\tau_{s_1}\\|\eta'|\end{pmatrix}(|\eta'|\ -\tau_{p_2})$$

$$+\frac{\Delta^2_{s_2}(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{-i\tau_{s_1}x_3}\frac{1}{\xi+\tau_{s_2}}\frac{1}{\tau_{s_2}\zeta}\begin{pmatrix}\tau_{s_1}\\|\eta'|\end{pmatrix}(\tau_{s_2}\ |\eta'|)\Bigg\}$$

$$+\Bigg\{\qquad e^{i\xi x_3}\frac{1}{\xi-\tau_{p_1}}\frac{1}{\tau_{p_1}\zeta}\begin{pmatrix}|\eta'|\\\tau_{p_1}\end{pmatrix}(|\eta'|\ \tau_{p_1})$$

$$+\qquad e^{i\xi x_3}\frac{1}{\xi-\tau_{s_1}}\frac{1}{\tau_{s_1}\zeta}\begin{pmatrix}-\tau_{s_1}\\|\eta'|\end{pmatrix}(-\tau_{s_1}\ |\eta'|)\Bigg\}$$

$$-\Bigg\{\qquad e^{i\xi x_3}\frac{1}{\xi+\tau_{p_1}}\frac{1}{\tau_{p_1}\zeta}\begin{pmatrix}|\eta'|\\-\tau_{p_1}\end{pmatrix}(|\eta'|\ -\tau_{p_1})$$

$$+\qquad e^{i\xi x_3}\frac{1}{\xi+\tau_{s_1}}\frac{1}{\tau_{s_1}\zeta}\begin{pmatrix}\tau_{s_1}\\|\eta'|\end{pmatrix}(\tau_{s_1}\ |\eta'|)\Bigg\}\Bigg],\quad x_3<0,$$

$$(4.8)\quad F^{-1}_{y_3}[G^{II}_1(x_3,y_3,\eta';\zeta)](\xi)=\frac{1}{2\sqrt{2\pi}}\times$$

$$\Bigg[-\Bigg\{\frac{\Delta^3_{p_1}(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{i\tau_{p_2}x_3}\frac{1}{\xi-\tau_{p_1}}\frac{1}{\tau_{p_1}\zeta}\begin{pmatrix}|\eta'|\\\tau_{p_2}\end{pmatrix}(|\eta'|\ \tau_{p_1})$$

$$+\frac{\Delta^3_{s_1}(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{i\tau_{p_2}x_3}\frac{1}{\xi-\tau_{s_1}}\frac{1}{\tau_{s_1}\zeta}\begin{pmatrix}|\eta'|\\\tau_{p_2}\end{pmatrix}(-\tau_{s_1}\ |\eta'|)$$

$$+\qquad e^{i\tau_{p_2}x_3}\frac{1}{\xi-\tau_{p_2}}\frac{1}{\tau_{p_2}\zeta}\begin{pmatrix}|\eta'|\\\tau_{p_2}\end{pmatrix}(|\eta'|\ \tau_{p_2})\Bigg\}$$

$$+\Bigg\{\frac{\Delta^3_{p_2}(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{i\tau_{p_2}x_3}\frac{1}{\xi+\tau_{p_2}}\frac{1}{\tau_{p_2}\zeta}\begin{pmatrix}|\eta'|\\\tau_{p_2}\end{pmatrix}(|\eta'|\ -\tau_{p_2})$$

$$+\frac{\Delta^3_{s_2}(\eta',\zeta)}{\Delta(\eta',\zeta)}e^{i\tau_{p_2}x_3}\frac{1}{\xi+\tau_{s_2}}\frac{1}{\tau_{s_2}\zeta}\begin{pmatrix}|\eta'|\\\tau_{p_2}\end{pmatrix}(\tau_{s_2}\ |\eta'|)\Bigg\}$$

$$
\begin{aligned}
&-\Bigg\{ \frac{\Delta^4_{p_1}(\eta',\zeta)}{\Delta(\eta',\zeta)} e^{i\tau_{s_2}x_3} \frac{1}{\xi-\tau_{p_1}} \frac{1}{\tau_{p_1}\zeta} \begin{pmatrix} -\tau_{s_2} \\ |\eta'| \end{pmatrix} (|\eta'| \ \tau_{p_1}) \\
&\quad + \frac{\Delta^4_{s_1}(\eta',\zeta)}{\Delta(\eta',\zeta)} e^{i\tau_{s_2}x_3} \frac{1}{\xi-\tau_{s_1}} \frac{1}{\tau_{s_1}\zeta} \begin{pmatrix} -\tau_{s_2} \\ |\eta'| \end{pmatrix} (-\tau_{s_1} \ |\eta'|) \\
&\quad + \qquad e^{i\tau_{s_2}x_3} \frac{1}{\xi-\tau_{s_2}} \frac{1}{\tau_{s_2}\zeta} \begin{pmatrix} -\tau_{s_2} \\ |\eta'| \end{pmatrix} (-\tau_{s_2} \ |\eta'|) \Bigg\} \\
&+\Bigg\{ \frac{\Delta^4_{p_2}(\eta',\zeta)}{\Delta(\eta',\zeta)} e^{i\tau_{s_2}x_3} \frac{1}{\xi+\tau_{p_2}} \frac{1}{\tau_{p_2}\zeta} \begin{pmatrix} -\tau_{s_2} \\ |\eta'| \end{pmatrix} (|\eta'| \ -\tau_{p_2}) \\
&\quad + \frac{\Delta^4_{s_2}(\eta',\zeta)}{\Delta(\eta',\zeta)} e^{i\tau_{s_2}x_3} \frac{1}{\xi+\tau_{s_2}} \frac{1}{\tau_{s_2}\zeta} \begin{pmatrix} -\tau_{s_2} \\ |\eta'| \end{pmatrix} (\tau_{s_2} \ |\eta'|) \Bigg\} \\
&+\Bigg\{ \qquad e^{i\xi x_3} \frac{1}{\xi-\tau_{p_2}} \frac{1}{\tau_{p_2}\zeta} \begin{pmatrix} |\eta'| \\ \tau_{p_2} \end{pmatrix} (|\eta'| \ \tau_{p_2}) \\
&\quad + \qquad e^{i\xi x_3} \frac{1}{\xi-\tau_{s_2}} \frac{1}{\tau_{s_2}\zeta} \begin{pmatrix} -\tau_{s_2} \\ |\eta'| \end{pmatrix} (-\tau_{s_2} \ |\eta'|) \Bigg\} \\
&-\Bigg\{ \qquad e^{i\xi x_3} \frac{1}{\xi+\tau_{p_2}} \frac{1}{\tau_{p_2}\zeta} \begin{pmatrix} |\eta'| \\ -\tau_{p_2} \end{pmatrix} (|\eta'| \ -\tau_{p_2}) \\
&\quad + \qquad e^{i\xi x_3} \frac{1}{\xi+\tau_{s_2}} \frac{1}{\tau_{s_2}\zeta} \begin{pmatrix} \tau_{s_2} \\ |\eta'| \end{pmatrix} (\tau_{s_2} \ |\eta'|) \Bigg\} \Bigg], \qquad x_3 > 0.
\end{aligned}
$$

Let us take the limits of the expressions (4.1) for $\zeta \to \lambda_j(\eta) \pm i0$. First, we note the following formulas as $\zeta \to \lambda_{p_1}(\eta) \pm i0$:

$$
\lim_{\zeta\to\lambda_{p_1}(\eta)\pm i0} \tau_{p_1} = \lim_{\zeta\to\lambda_{p_1}(\eta)\pm i0} \sqrt{\frac{\zeta}{c^2_{p_1}} - |\eta'|^2} = \pm|\xi|,
$$

$$
\lim_{\zeta\to\lambda_{p_1}(\eta)\pm i0} \tau_{s_1} = \xi_{s_1}(\eta',\lambda_{p_1}) = \pm\sqrt{\frac{c^2_{p_1}(|\eta'|^2+\xi^2)}{c^2_{s_1}} - |\eta'|^2},
$$

$$
\lim_{\zeta\to\lambda_{p_1}(\eta)\pm i0} \tau_{p_2} = \xi_{p_2}(\eta',\lambda_{p_1}) = \begin{cases} \pm\sqrt{\frac{c^2_{p_1}(|\eta'|^2+\xi^2)}{c^2_{p_2}} - |\eta'|^2} & (c^2_{p_1}|\eta|^2 > c^2_{p_2}|\eta'|^2) \\ i\sqrt{|\eta'|^2 - \frac{c^2_{p_1}(|\eta'|^2+\xi^2)}{c^2_{p_2}}} & (c^2_{p_1}|\eta|^2 < c^2_{p_2}|\eta'|^2), \end{cases}
$$

$$
\lim_{\zeta\to\lambda_{p_1}(\eta)\pm i0} \tau_{s_2} = \xi_{s_2}(\eta',\lambda_{p_1}) = \begin{cases} \pm\sqrt{\frac{c^2_{p_1}(|\eta'|^2+\xi^2)}{c^2_{s_2}} - |\eta'|^2} & (c^2_{p_1}|\eta|^2 > c^2_{s_2}|\eta'|^2) \\ i\sqrt{|\eta'|^2 - \frac{c^2_{p_1}(|\eta'|^2+\xi^2)}{c^2_{p_2}}} & (c^2_{p_1}|\eta|^2 < c^2_{s_2}|\eta'|^2). \end{cases}
$$

Moreover, we have for $\xi > 0$

$$
\lim_{\zeta\to\lambda_{p_1}(\eta)\pm i0} \frac{\lambda_{p_1}(\eta) - \zeta}{\xi \mp \tau_{p_1}(\eta',\zeta)} = \lim_{\zeta\to\lambda_{p_1}(\eta)\pm i0} \frac{c^2_{p_1}(|\eta'|^2+\xi^2) - \zeta}{\xi \mp \sqrt{\frac{\zeta}{c^2_{p_1}} - |\eta'|^2}}
$$

$$
\begin{aligned}
&= \lim_{\zeta\to\lambda_{p_1}(\eta)\pm i0} \frac{(c_{p_1}^2(|\eta'|^2+\xi^2)-\zeta)(\xi \pm \sqrt{\frac{\zeta}{c_{p_1}^2}-|\eta'|^2})}{\xi^2-(\frac{\zeta}{c_{p_1}^2}-|\eta'|^2)}\\
&= \lim_{\zeta\to\lambda_{p_1}(\eta)\pm i0} c_{p_1}^2(\xi \pm \sqrt{\frac{\zeta}{c_{p_1}^2}-|\eta'|^2})\\
&= 2c_{p_1}^2\xi,
\end{aligned}
$$

and also for $\xi < 0$

$$
\begin{aligned}
\lim_{\zeta\to\lambda_{p_1}(\eta)\pm i0} \frac{\lambda_{p_1}(\eta)-\zeta}{\xi \pm \tau_{p_1}(\eta',\zeta)} &= \lim_{\zeta\to\lambda_{p_1}(\eta)\pm i0} \frac{(c_{p_1}^2(|\eta'|^2+\xi^2)-\zeta)(\xi \mp \sqrt{\frac{\zeta}{c_{p_1}}-|\eta'|^2})}{\xi^2-(\frac{\zeta}{c_{p_1}^2}-|\eta'|^2)}\\
&= \lim_{\zeta\to\lambda_{p_1}(\eta)\pm i0} c_{p_1}^2(\xi \mp \sqrt{\frac{\zeta}{c_{p_1}^2}-|\eta'|^2})\\
&= 2c_{p_1}^2\xi.
\end{aligned}
$$

If $\xi > 0$, we have

$$
\begin{cases}
\psi_{1p_1}(x_3,\eta;\zeta) \to \psi_{1p_1}^{+}(x_3,\eta) & \text{as } \zeta \to \lambda_{p_1}(\eta)+i0,\\
\psi_{1p_1}(x_3,\eta;\zeta) \to 0 & \text{as } \zeta \to \lambda_{p_1}(\eta)-i0.
\end{cases}
$$

If $\xi < 0$, we have

$$
\begin{cases}
\psi_{1p_1}(x_3,\eta;\zeta) \to 0 & \text{as } \zeta \to \lambda_{p_1}(\eta)+i0,\\
\psi_{1p_1}(x_3,\eta;\zeta) \to \psi_{1p_1}^{-}(x_3,\eta) & \text{as } \zeta \to \lambda_{p_1}(\eta)-i0.
\end{cases}
$$

Here the limit functions $\psi_{1p_1}^{\pm}(x_3,\eta)$ are given respectively by the following:

$$
\psi_{1p_1}^{\pm}(x_3,\eta) = \begin{cases}
\psi_{1p_1}^{\pm I}(x_3,\eta), & x_3 < 0,\\
\psi_{1p_1}^{\pm II}(x_3,\eta), & x_3 > 0,
\end{cases}
$$

(4.9)

$$
\begin{aligned}
&\psi_{1p_1}^{\pm I}(x_3,\eta) \equiv \psi_{1p_1}^{I}(x_3,\eta;\lambda_{p_1}(\eta)\pm i0)\\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{|\eta|^2}\frac{1}{\rho_1}\Bigg[-\frac{\Delta_{p_1}^1(\eta',\lambda_{p_1})}{\Delta(\eta',\lambda_{p_1})}e^{-i\xi x_3}\begin{pmatrix} |\eta'|^2 & \xi|\eta'| \\ -\xi|\eta'| & -\xi^2 \end{pmatrix}\\
&\qquad -\frac{\Delta_{p_1}^2(\eta',\lambda_{p_1})}{\Delta(\eta',\lambda_{p_1})}e^{-i\xi_{s_1}(\eta',\lambda_{p_1})x_3}\begin{pmatrix} \xi_{s_1}(\eta',\lambda_{p_1})|\eta'| & \xi_{s_1}(\eta',\lambda_{p_1})\xi \\ |\eta'|^2 & \xi|\eta'| \end{pmatrix}\\
&\qquad + \qquad e^{i\xi x_3}\begin{pmatrix} |\eta'|^2 & \xi|\eta'| \\ \xi|\eta'| & \xi^2 \end{pmatrix}\Bigg],
\end{aligned}
$$

(4.10)

$$
\psi_{1p_1}^{\pm II}(x_3,\eta) \equiv \psi_{1p_1}^{II}(x_3,\eta;\lambda_{p_1}(\eta)\pm i0)
$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{|\eta|^2} \frac{1}{\rho_2} \Bigg[-\frac{\Delta^3_{p_1}(\eta', \lambda_{p_1})}{\Delta(\eta', \lambda_{p_1})} e^{i\xi_{p_2}(\eta',\lambda_{p_1})x_3} \begin{pmatrix} |\eta'|^2 & \xi|\eta'| \\ \xi_{p_2}(\eta', \lambda_{p_1})|\eta'| & \xi_{p_2}(\eta', \lambda_{p_1})\xi \end{pmatrix}$$
$$- \frac{\Delta^4_{p_1}(\eta', \lambda_{p_1})}{\Delta(\eta', \lambda_{p_1})} e^{i\xi_{s_2}(\eta',\lambda_{p_1})x_3} \begin{pmatrix} -\xi_{s_2}(\eta', \lambda_{p_1})|\eta'| & -\xi_{s_2}(\eta', \lambda_{p_1})\xi \\ |\eta'|^2 & \xi|\eta'| \end{pmatrix} \Bigg],$$

Next, we note the following formulas as $\zeta \to \lambda_j(\eta) \pm i0$ ($j \in \{s_1, p_2, s_2\}$).

$$\lim_{\zeta \to \lambda_{s_1}(\eta) \pm i0} \tau_{p_1} = \xi_{p_1}(\eta', \lambda_{s_1}) = \begin{cases} \pm\sqrt{\frac{c_{s_1}^2(|\eta'|^2+\xi^2)}{c_{p_1}^2} - |\eta'|^2} & (c_{s_1}^2|\eta|^2 > c_{p_1}^2|\eta'|^2) \\ i\sqrt{|\eta'|^2 - \frac{c_{s_1}^2(|\eta'|^2+\xi^2)}{c_{p_1}^2}} & (c_{s_1}^2|\eta|^2 < c_{p_1}^2|\eta'|^2), \end{cases}$$
$$\lim_{\zeta \to \lambda_{s_1}(\eta) \pm i0} \tau_{s_1} = \pm|\xi|,$$

$$\lim_{\zeta \to \lambda_{s_1}(\eta) \pm i0} \tau_{p_2} = \xi_{p_2}(\eta', \lambda_{s_1}) = \begin{cases} \pm\sqrt{\frac{c_{s_1}^2(|\eta'|^2+\xi^2)}{c_{p_2}^2} - |\eta'|^2} & (c_{s_1}^2|\eta|^2 > c_{p_2}^2|\eta'|^2) \\ i\sqrt{|\eta'|^2 - \frac{c_{s_1}^2(|\eta'|^2+\xi^2)}{c_{p_2}^2}} & (c_{s_1}^2|\eta|^2 < c_{p_2}^2|\eta'|^2), \end{cases}$$
$$\lim_{\zeta \to \lambda_{s_1}(\eta) \pm i0} \tau_{s_2} = \xi_{s_2}(\eta', \lambda_{s_1}) = \begin{cases} \pm\sqrt{\frac{c_{s_1}^2(|\eta'|^2+\xi^2)}{c_{s_2}^2} - |\eta'|^2} & (c_{s_1}^2|\eta|^2 > c_{s_2}^2|\eta'|^2) \\ i\sqrt{|\eta'|^2 - \frac{c_{s_1}^2(|\eta'|^2+\xi^2)}{c_{s_2}^2}} & (c_{s_1}^2|\eta|^2 < c_{s_2}^2|\eta'|^2), \end{cases}$$
$$\lim_{\zeta \to \lambda_{p_2}(\eta) \pm i0} \tau_{p_1} = \xi_{p_1}(\eta', \lambda_{p_2}) = \pm\sqrt{\frac{c_{p_2}^2(|\eta'|^2+\xi^2)}{c_{p_1}^2} - |\eta'|^2},$$
$$\lim_{\zeta \to \lambda_{p_2}(\eta) \pm i0} \tau_{s_1} = \xi_{s_1}(\eta', \lambda_{p_2}) = \pm\sqrt{\frac{c_{p_2}^2(|\eta'|^2+\xi^2)}{c_{s_1}^2} - |\eta'|^2},$$
$$\lim_{\zeta \to \lambda_{p_2}(\eta) \pm i0} \tau_{p_2} = \pm|\xi|,$$

$$\lim_{\zeta \to \lambda_{p_2}(\eta) \pm i0} \tau_{s_2} = \xi_{s_2}(\eta', \lambda_{p_2}) = \pm\sqrt{\frac{c_{p_2}^2(|\eta'|^2+\xi^2)}{c_{s_2}^2} - |\eta'|^2},$$
$$\lim_{\zeta \to \lambda_{s_2}(\eta) \pm i0} \tau_{p_1} = \xi_{p_1}(\eta', \lambda_{s_2}) = \pm\sqrt{\frac{c_{s_2}^2(|\eta'|^2+\xi^2)}{c_{p_1}^2} - |\eta'|^2},$$
$$\lim_{\zeta \to \lambda_{s_2}(\eta) \pm i0} \tau_{s_1} = \xi_{s_1}(\eta', \lambda_{s_2}) = \pm\sqrt{\frac{c_{s_2}^2(|\eta'|^2+\xi^2)}{c_{s_1}^2} - |\eta'|^2},$$
$$\lim_{\zeta \to \lambda_{s_2}(\eta) \pm i0} \tau_{p_2} = \xi_{p_2}(\eta', \lambda_{s_2}) = \begin{cases} \pm\sqrt{\frac{c_{s_2}^2(|\eta'|^2+\xi^2)}{c_{p_2}^2} - |\eta'|^2} & (c_{s_2}^2|\eta|^2 > c_{p_2}^2|\eta'|^2) \\ i\sqrt{|\eta'|^2 - \frac{c_{s_2}^2(|\eta'|^2+\xi^2)}{c_{p_2}^2}} & (c_{s_2}^2|\eta|^2 < c_{p_2}^2|\eta'|^2), \end{cases}$$
$$\lim_{\zeta \to \lambda_{s_2}(\eta) \pm i0} \tau_{s_2} = \pm|\xi|.$$

If $\xi > 0$, we have

$$\begin{cases} \psi_{1s_1}(x_3,\eta;\zeta) \to \psi_{1s_1}^{+}(x_3,\eta) & \text{as } \zeta \to \lambda_{s_1}(\eta)+i0, \\ \psi_{1s_1}(x_3,\eta;\zeta) \to 0 & \text{as } \zeta \to \lambda_{s_1}(\eta)-i0. \end{cases}$$
$$\begin{cases} \psi_{1p_2}(x_3,\eta;\zeta) \to 0 & \text{as } \zeta \to \lambda_{p_2}(\eta)+i0, \\ \psi_{1p_2}(x_3,\eta;\zeta) \to \psi_{1p_2}^{-}(x_3,\eta) & \text{as } \zeta \to \lambda_{p_2}(\eta)-i0. \end{cases}$$
$$\begin{cases} \psi_{1s_2}(x_3,\eta;\zeta) \to 0 & \text{as } \zeta \to \lambda_{s_2}(\eta)+i0, \\ \psi_{1s_2}(x_3,\eta;\zeta) \to \psi_{1s_2}^{-}(x_3,\eta) & \text{as } \zeta \to \lambda_{s_2}(\eta)-i0. \end{cases}$$

If $\xi < 0$, we have

$$\begin{cases} \psi_{1s_1}(x_3,\eta;\zeta) \to 0 & \text{as } \zeta \to \lambda_{s_1}(\eta)+i0, \\ \psi_{1s_1}(x_3,\eta;\zeta) \to \psi_{1s_1}^{-}(x_3,\eta) & \text{as } \zeta \to \lambda_{s_1}(\eta)-i0. \end{cases}$$
$$\begin{cases} \psi_{1p_2}(x_3,\eta;\zeta) \to \psi_{1p_2}^{+}(x_3,\eta) & \text{as } \zeta \to \lambda_{p_2}(\eta)+i0, \\ \psi_{1p_2}(x_3,\eta;\zeta) \to 0 & \text{as } \zeta \to \lambda_{p_2}(\eta)-i0. \end{cases}$$
$$\begin{cases} \psi_{1s_2}(x_3,\eta;\zeta) \to \psi_{1s_2}^{+}(x_3,\eta) & \text{as } \zeta \to \lambda_{s_2}(\eta)+i0, \\ \psi_{1s_2}(x_3,\eta;\zeta) \to 0 & \text{as } \zeta \to \lambda_{s_2}(\eta)-i0. \end{cases}$$

Here the limit functions $\psi_{1j}^{\pm}(x_3,\eta)$, $j \in \{s_1, p_2, s_2\}$, are given respectively by the following:

$$\psi_{1j}^{\pm}(x_3,\eta) = \begin{cases} \psi_{1j}^{\pm I}(x_3,\eta), & x_3 < 0, \\ \psi_{1j}^{\pm II}(x_3,\eta), & x_3 > 0, \end{cases} \qquad j \in \{s_1, p_2, s_2\},$$

(4.11)

$$\begin{aligned}
\psi_{1s_1}^{\pm I}(x_3,\eta) &\equiv \psi_{1s_1}^{I}(x_3,\eta;\lambda_{s_1}(\eta)\pm i0) \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{|\eta|^2}\frac{1}{\rho_1}\Bigg[-\frac{\Delta_{s_1}^1(\eta',\lambda_{s_1})}{\Delta(\eta',\lambda_{s_1})}e^{-i\xi_{p_1}(\eta',\lambda_{s_1})x_3}\begin{pmatrix} -\xi|\eta'| & |\eta'|^2 \\ \xi_{p_1}(\eta',\lambda_{s_1})\xi & -\xi_{p_1}(\eta',\lambda_{s_1})|\eta'| \end{pmatrix} \\
&\qquad - \frac{\Delta_{s_1}^2(\eta',\lambda_{s_1})}{\Delta(\eta',\lambda_{s_1})}e^{-i\xi x_3}\begin{pmatrix} -\xi^2 & \xi|\eta'| \\ -\xi|\eta'| & |\eta'|^2 \end{pmatrix} \\
&\qquad + e^{i\xi x_3}\begin{pmatrix} \xi^2 & -\xi|\eta'| \\ -\xi|\eta'| & |\eta'|^2 \end{pmatrix}\Bigg],
\end{aligned}$$

(4.12)

$$\begin{aligned}
\psi_{1s_1}^{\pm II}(x_3,\eta) &\equiv \psi_{1s_1}^{II}(x_3,\eta;\lambda_{s_1}(\eta)\pm i0) \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{|\eta|^2}\frac{1}{\rho_2}\Bigg[-\frac{\Delta_{s_1}^3(\eta',\lambda_{s_1})}{\Delta(\eta',\lambda_{s_1})}e^{i\xi_{p_2}(\eta',\lambda_{s_1})x_3}\begin{pmatrix} -\xi|\eta'| & |\eta'|^2 \\ -\xi_{p_2}(\eta',\lambda_{s_1})\xi & \xi_{p_2}(\eta',\lambda_{s_1})|\eta'| \end{pmatrix} \\
&\qquad - \frac{\Delta_{s_1}^4(\eta',\lambda_{s_1})}{\Delta(\eta',\lambda_{s_1})}e^{i\xi_{s_2}(\eta',\lambda_{s_1})x_3}\begin{pmatrix} \xi_{s_2}(\eta',\lambda_{s_1})\xi & -\xi_{s_2}(\eta',\lambda_{s_1})|\eta'| \\ -\xi|\eta'| & |\eta'|^2 \end{pmatrix}\Bigg],
\end{aligned}$$

(4.13)

$$\psi_{1p_2}^{\mp I}(x_3,\eta) \equiv \psi_{1p_2}^{I}(x_3,\eta;\lambda_{p_2}(\eta)\mp i0)$$

$$= \frac{1}{\sqrt{2\pi}}\frac{1}{|\eta|^2}\frac{1}{\rho_1}\left[\frac{\Delta^1_{p_2}(\eta',\lambda_{p_2})}{\Delta(\eta',\lambda_{p_2})}e^{-i\xi_{p_1}(\eta',\lambda_{p_2})x_3}\begin{pmatrix}|\eta'|^2 & \xi|\eta'| \\ -\xi_{p_1}(\eta',\lambda_{p_2})|\eta'| & -\xi_{p_1}(\eta',\lambda_{p_2})\xi\end{pmatrix}\right.$$
$$\left. + \frac{\Delta^2_{p_2}(\eta',\lambda_{p_2})}{\Delta(\eta',\lambda_{p_2})}e^{-i\xi_{s_1}(\eta',\lambda_{p_2})x_3}\begin{pmatrix}\xi_{s_1}(\eta',\lambda_{p_2})|\eta'| & \xi_{s_1}(\eta',\lambda_{p_2})\xi \\ |\eta'|^2 & \xi|\eta'|\end{pmatrix}\right],$$

(4.14)

$$\psi^{\mp II}_{1p_2}(x_3,\eta) \equiv \psi^{II}_{1p_2}(x_3,\eta;\lambda_{p_2}(\eta)\mp i0)$$
$$= \frac{1}{\sqrt{2\pi}}\frac{1}{|\eta|^2}\frac{1}{\rho_2}\left[\frac{\Delta^3_{p_2}(\eta',\lambda_{p_2})}{\Delta(\eta',\lambda_{p_2})}e^{-i\xi x_3}\begin{pmatrix}|\eta'|^2 & \xi|\eta'| \\ -\xi|\eta'| & -\xi^2\end{pmatrix}\right.$$
$$+ \frac{\Delta^4_{p_2}(\eta',\lambda_{p_2})}{\Delta(\eta',\lambda_{p_2})}e^{i\xi_{s_2}(\eta',\lambda_{p_2})x_3}\begin{pmatrix}-\xi_{s_2}(\eta',\lambda_{p_2})|\eta'| & -\xi_{s_2}(\eta',\lambda_{p_2})\xi \\ |\eta'|^2 & \xi|\eta'|\end{pmatrix}$$
$$\left. - e^{i\xi x_3}\begin{pmatrix}|\eta'|^2 & \xi|\eta'| \\ \xi|\eta'| & \xi^2\end{pmatrix}\right],$$

(4.15)

$$\psi^{\mp I}_{1s_2}(x_3,\eta) \equiv \psi^{I}_{1s_2}(x_3,\eta;\lambda_{s_2}(\eta)\mp i0)$$
$$= \frac{1}{\sqrt{2\pi}}\frac{1}{|\eta|^2}\frac{1}{\rho_1}\left[\frac{\Delta^1_{s_2}(\eta',\lambda_{s_2})}{\Delta(\eta',\lambda_{s_2})}e^{-i\xi_{p_1}(\eta',\lambda_{s_2})x_3}\begin{pmatrix}-\xi|\eta'| & |\eta'|^2 \\ \xi_{p_1}(\eta',\lambda_{s_1})\xi & -\xi_{p_1}(\eta',\lambda_{s_2})|\eta'|\end{pmatrix}\right.$$
$$\left. + \frac{\Delta^2_{s_2}(\eta',\lambda_{s_2})}{\Delta(\eta',\lambda_{s_2})}e^{-i\xi_{s_1}(\eta',\lambda_{s_2})x_3}\begin{pmatrix}-\xi_{s_1}(\eta',\lambda_{s_2})\xi & \xi_{s_1}(\eta',\lambda_{s_2})|\eta'| \\ -\xi|\eta'| & |\eta'|^2\end{pmatrix}\right],$$

(4.16)

$$\psi^{\mp II}_{1s_2}(x_3,\eta) \equiv \psi^{II}_{1s_2}(x_3,\eta;\lambda_{s_2}(\eta)\mp i0)$$
$$= \frac{1}{\sqrt{2\pi}}\frac{1}{|\eta|^2}\frac{1}{\rho_2}\left[\frac{\Delta^3_{s_2}(\eta',\lambda_{s_2})}{\Delta(\eta',\lambda_{s_2})}e^{i\xi_{p_2}(\eta',\lambda_{s_2})x_3}\begin{pmatrix}-\xi|\eta'| & |\eta'|^2 \\ -\xi_{p_2}(\eta',\lambda_{s_2})\xi & \xi_{p_2}(\eta',\lambda_{s_2})|\eta'|\end{pmatrix}\right.$$
$$+ \frac{\Delta^4_{s_2}(\eta',\lambda_{s_2})}{\Delta(\eta',\lambda_{s_2})}e^{-i\xi x_3}\begin{pmatrix}-\xi^2 & \xi|\eta'| \\ -\xi|\eta'| & |\eta'|^2\end{pmatrix}$$
$$\left. - e^{i\xi x_3}\begin{pmatrix}\xi^2 & -\xi|\eta'| \\ -\xi|\eta'| & |\eta'|^2\end{pmatrix}\right].$$

From Lemma 4.1, we get for $j \in M$,

$$A^I_1(\eta')\psi^{\pm I}_{1j}(x_3,\eta) = \lambda_j(\eta)\psi^{\pm I}_{1j}(x_3,\eta), \quad x_3 < 0,$$
$$A^{II}_1(\eta')\psi^{\pm II}_{1j}(x_3,\eta) = \lambda_j(\eta)\psi^{\pm II}_{1j}(x_3,\eta), \quad x_3 > 0,$$
$$\psi^{\pm I}_{1j}(x_3,\eta)\big|_{x_3=0} = \psi^{\pm II}_{1j}(x_3,\eta)\big|_{x_3=0},$$
$$B^I_1(\eta')\psi^{\pm I}_{1j}(x_3,\eta)\big|_{x_3=0} = B^{II}_1(\eta')\psi^{\pm II}_{1j}(x_3,\eta)\big|_{x_3=0}.$$

This shows that $\psi^{\pm}_{1j}(x_3,\eta)(j \in M)$ are generalized eigenfunctions for $A_1(\eta')$.

Next we define $\psi^{St}_{1j}(x_3,\eta;\zeta)(j \in M)$ as follows:

$$\psi^{St}_{1j}(x_3,\eta;\zeta) = \begin{cases}\psi^{StI}_{1j}(x_3,\eta;\zeta), & x_3 < 0,\\ \psi^{StII}_{1j}(x_3,\eta;\zeta), & x_3 > 0,\end{cases}$$

$$\psi_{1j}^{StI}(x_3,\eta;\zeta) = \frac{\zeta - c_{St}^2|\eta'|^2}{\zeta - \lambda_j(\eta)}\psi_{1j}^I(x_3,\eta;\zeta), \quad x_3 < 0,$$
$$\psi_{1j}^{StII}(x_3,\eta;\zeta) = \frac{\zeta - c_{St}^2|\eta'|^2}{\zeta - \lambda_j(\eta)}\psi_{1j}^{II}(x_3,\eta;\zeta), \quad x_3 > 0.$$

From the expression of $\psi_{1j}(x_3,\eta;\zeta)(j \in M)$, we see that the limits

$$\psi_{1j}^{StI}(x_3,\eta) = \lim_{\zeta \to c_{St}^2|\eta'|^2} \psi_{1j}^{StI}(x_3,\eta;\zeta)$$
$$\psi_{1j}^{StII}(x_3,\eta) = \lim_{\zeta \to c_{St}^2|\eta'|^2} \psi_{1j}^{StII}(x_3,\eta;\zeta)$$

exist. Moreover, by Lemma 4.1, $\psi_{1j}^{StI}(x_3,\eta)$ and $\psi_{1j}^{StII}(x_3,\eta)$ satisfy for $j \in M$

$$A_1^I(\eta')\psi_{1j}^{StI}(x_3,\eta) = c_{St}^2|\eta'|^2\psi_{1j}^{StI}(x_3,\eta), \quad x_3 < 0,$$
$$A_1^{II}(\eta')\psi_{1j}^{StII}(x_3,\eta) = c_{St}^2|\eta'|^2\psi_{1j}^{StII}(x_3,\eta), \quad x_3 > 0,$$
$$\psi_{1j}^{StI}(x_3,\eta)\big|_{x_3=0} = \psi_{1j}^{StII}(x_3,\eta)\big|_{x_3=0},$$
$$B_1^I(\eta')\psi_{1j}^{StI}(x_3,\eta)\big|_{x_3=0} = B_1^{II}(\eta')\psi_{1j}^{StII}(x_3,\eta)\big|_{x_3=0}.$$

This shows that $\psi_{1j}^{St}(x_3,\eta)$ $(j \in M)$ are generalized eigenfunctions for $A_1(\eta')$ corresponding to the Stoneley wave. Let us give an expression of $\psi_{1j}^{St}(x_3,\eta)$ for each $j \in M$. It suffices to consider the case where the Lopatinski determinant $\Delta(\eta',\zeta)$ has real zero. If the Lopatinski determinant has no zero, we consider $\psi_{1j}^{St}(x_3,\eta) \equiv 0$ $(j \in M)$. Put

$$\Delta(\eta',\zeta) = (\zeta - c_{St}^2|\eta'|^2)\Delta_0(\eta',\zeta),$$

and noting that $c_{St} \le c_{s_1} < c_{p_1} < c_{s_2} < c_{p_2}$, we have

$$\lim_{\zeta \to c_{St}^2|\eta'|^2} \tau_{p_1} = \lim_{\zeta \to c_{St}^2|\eta'|^2} \sqrt{\frac{\zeta}{c_{p_1}^2} - |\eta'|^2} = i\sqrt{|\eta'|^2 - \frac{c_{St}^2}{c_{p_1}^2}|\eta'|^2} = i\xi_{p_1}^{St},$$

$$\lim_{\zeta \to c_{St}^2|\eta'|^2} \tau_{s_1} = \lim_{\zeta \to c_{St}^2|\eta'|^2} \sqrt{\frac{\zeta}{c_{s_1}^2} - |\eta'|^2} = i\sqrt{|\eta'|^2 - \frac{c_{St}^2}{c_{s_1}^2}|\eta'|^2} = i\xi_{s_1}^{St},$$

$$\lim_{\zeta \to c_{St}^2|\eta'|^2} \tau_{p_2} = \lim_{\zeta \to c_{St}^2|\eta'|^2} \sqrt{\frac{\zeta}{c_{p_2}^2} - |\eta'|^2} = i\sqrt{|\eta'|^2 - \frac{c_{St}^2}{c_{p_2}^2}|\eta'|^2} = i\xi_{p_2}^{St},$$

$$\lim_{\zeta \to c_{St}^2|\eta'|^2} \tau_{s_2} = \lim_{\zeta \to c_{St}^2|\eta'|^2} \sqrt{\frac{\zeta}{c_{s_2}^2} - |\eta'|^2} = i\sqrt{|\eta'|^2 - \frac{c_{St}^2}{c_{s_2}^2}|\eta'|^2} = i\xi_{s_2}^{St},$$

where $\xi_{p_1}^{St}$, $\xi_{s_1}^{St}$, $\xi_{p_2}^{St}$, and $\xi_{s_2}^{St}$ are all real. So we have for $x_3 < 0$

(4.17)

$$\psi_{1j}^{StI}(x_3,\eta) = \frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{1}{c_{St}^2|\eta'|^2}\frac{1}{\rho_1}\frac{1}{i}$$

$$
\begin{aligned}
&\Bigg[\frac{\Delta^1_{p_1}(\eta',c^2_{St}|\eta'|^2)}{\Delta_0(\eta',c^2_{St}|\eta'|^2)}e^{\xi^{St}_{p_1}x_3}\frac{1}{\xi^{St}_{p_1}}\frac{1}{\xi-i\xi^{St}_{p_1}}\begin{pmatrix}|\eta'|^2 & i\xi^{St}_{p_1}|\eta'|\\ -i\xi^{St}_{p_1}|\eta'| & \xi^{St2}_{p_1}\end{pmatrix}\\
&+\frac{\Delta^1_{s_1}(\eta',c^2_{St}|\eta'|^2)}{\Delta_0(\eta',c^2_{St}|\eta'|^2)}e^{\xi^{St}_{p_1}x_3}\frac{1}{\xi^{St}_{s_1}}\frac{1}{\xi-i\xi^{St}_{s_1}}\begin{pmatrix}-i\xi^{St}_{s_1}|\eta'| & |\eta'|^2\\ -\xi^{St}_{p_1}\xi^{St}_{s_1} & -i\xi^{St}_{p_1}|\eta'|\end{pmatrix}\\
&-\frac{\Delta^1_{p_2}(\eta',c^2_{St}|\eta'|^2)}{\Delta_0(\eta',c^2_{St}|\eta'|^2)}e^{\xi^{St}_{p_1}x_3}\frac{1}{\xi^{St}_{p_2}}\frac{1}{\xi-i\xi^{St}_{p_2}}\begin{pmatrix}|\eta'|^2 & -i\xi^{St}_{p_2}|\eta'|\\ -i\xi^{St}_{p_1}|\eta'| & -\xi^{St}_{p_1}\xi^{St}_{p_2}\end{pmatrix}\\
&-\frac{\Delta^1_{s_2}(\eta',c^2_{St}|\eta'|^2)}{\Delta_0(\eta',c^2_{St}|\eta'|^2)}e^{\xi^{St}_{p_1}x_3}\frac{1}{\xi^{St}_{s_2}}\frac{1}{\xi-i\xi^{St}_{s_2}}\begin{pmatrix}i\xi^{St}_{s_2}|\eta'| & |\eta'|^2\\ \xi^{St}_{p_1}\xi^{St}_{s_2} & -i\xi^{St}_{p_1}|\eta'|\end{pmatrix}\\
&+\frac{\Delta^2_{p_1}(\eta',c^2_{St}|\eta'|^2)}{\Delta_0(\eta',c^2_{St}|\eta'|^2)}e^{\xi^{St}_{s_1}x_3}\frac{1}{\xi^{St}_{p_1}}\frac{1}{\xi-i\xi^{St}_{p_1}}\begin{pmatrix}i\xi^{St}_{s_1}|\eta'| & -\xi^{St}_{s_1}\xi^{St}_{p_1}\\ |\eta'|^2 & i\xi^{St}_{p_1}|\eta'|\end{pmatrix}\\
&+\frac{\Delta^2_{s_1}(\eta',c^2_{St}|\eta'|^2)}{\Delta_0(\eta',c^2_{St}|\eta'|^2)}e^{\xi^{St}_{s_1}x_3}\frac{1}{\xi^{St}_{s_1}}\frac{1}{\xi-i\xi^{St}_{s_1}}\begin{pmatrix}\xi^{St2}_{s_1} & i\xi^{St}_{s_1}|\eta'|\\ -i\xi^{St}_{s_1}|\eta'| & |\eta'|^2\end{pmatrix}\\
&-\frac{\Delta^2_{p_2}(\eta',c^2_{St}|\eta'|^2)}{\Delta_0(\eta',c^2_{St}|\eta'|^2)}e^{\xi^{St}_{s_1}x_3}\frac{1}{\xi^{St}_{p_2}}\frac{1}{\xi-i\xi^{St}_{p_2}}\begin{pmatrix}i\xi^{St}_{s_1}|\eta'| & \xi^{St}_{s_1}\xi^{St}_{p_2}\\ |\eta'|^2 & -i\xi^{St}_{p_1}|\eta'|\end{pmatrix}\\
&-\frac{\Delta^2_{s_2}(\eta',c^2_{St}|\eta'|^2)}{\Delta_0(\eta',c^2_{St}|\eta'|^2)}e^{\xi^{St}_{s_1}x_3}\frac{1}{\xi^{St}_{s_2}}\frac{1}{\xi-i\xi^{St}_{s_2}}\begin{pmatrix}-\xi^{St}_{s_1}\xi^{St}_{s_2} & i\xi^{St}_{s_1}|\eta'|\\ i\xi^{St}_{s_2}|\eta'| & |\eta'|^2\end{pmatrix}\Bigg]P_j(\eta),
\end{aligned}
$$

and for $x_3 > 0$

(4.18)

$$
\begin{aligned}
\psi^{StII}_{1j}(x_3,\eta)&=\frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{1}{c^2_{St}|\eta'|^2}\frac{1}{\rho_2}\frac{1}{i}\\
&\Bigg[\frac{\Delta^3_{p_1}(\eta',c^2_{St}|\eta'|^2)}{\Delta_0(\eta',c^2_{St}|\eta'|^2)}e^{-\xi^{St}_{p_2}x_3}\frac{1}{\xi^{St}_{p_1}}\frac{1}{\xi-i\xi^{St}_{p_1}}\begin{pmatrix}|\eta'|^2 & i\xi^{St}_{p_1}|\eta'|\\ i\xi^{St}_{p_2}|\eta'| & -\xi^{St}_{p_1}\xi^{St}_{p_2}\end{pmatrix}\\
&+\frac{\Delta^3_{s_1}(\eta',c^2_{St}|\eta'|^2)}{\Delta_0(\eta',c^2_{St}|\eta'|^2)}e^{-\xi^{St}_{p_2}x_3}\frac{1}{\xi^{St}_{s_1}}\frac{1}{\xi-i\xi^{St}_{s_1}}\begin{pmatrix}-i\xi^{St}_{s_1}|\eta'| & |\eta'|^2\\ \xi^{St}_{s_1}\xi^{St}_{p_2} & i\xi^{St}_{p_2}|\eta'|\end{pmatrix}\\
&-\frac{\Delta^3_{p_2}(\eta',c^2_{St}|\eta'|^2)}{\Delta_0(\eta',c^2_{St}|\eta'|^2)}e^{-\xi^{St}_{p_2}x_3}\frac{1}{\xi^{St}_{p_2}}\frac{1}{\xi-i\xi^{St}_{p_2}}\begin{pmatrix}|\eta'|^2 & i\xi^{St}_{p_2}|\eta'|\\ i\xi^{St}_{p_2}|\eta'| & -\xi^{St2}_{p_2}\end{pmatrix}\\
&-\frac{\Delta^3_{s_2}(\eta',c^2_{St}|\eta'|^2)}{\Delta_0(\eta',c^2_{St}|\eta'|^2)}e^{-\xi^{St}_{p_2}x_3}\frac{1}{\xi^{St}_{s_2}}\frac{1}{\xi-i\xi^{St}_{s_2}}\begin{pmatrix}i\xi^{St}_{s_2}|\eta'| & |\eta'|^2\\ -\xi^{St}_{p_2}\xi^{St}_{s_2} & i\xi^{St}_{p_2}|\eta'|\end{pmatrix}\\
&+\frac{\Delta^4_{p_1}(\eta',c^2_{St}|\eta'|^2)}{\Delta_0(\eta',c^2_{St}|\eta'|^2)}e^{-\xi^{St}_{s_2}x_3}\frac{1}{\xi^{St}_{p_1}}\frac{1}{\xi-i\xi^{St}_{p_1}}\begin{pmatrix}-i\xi^{St}_{s_2}|\eta'| & \xi^{St}_{s_2}\xi^{St}_{p_1}\\ |\eta'|^2 & i\xi^{St}_{p_1}|\eta'|\end{pmatrix}\\
&+\frac{\Delta^4_{s_1}(\eta',c^2_{St}|\eta'|^2)}{\Delta_0(\eta',c^2_{St}|\eta'|^2)}e^{-\xi^{St}_{s_2}x_3}\frac{1}{\xi^{St}_{s_1}}\frac{1}{\xi-i\xi^{St}_{s_1}}\begin{pmatrix}-\xi^{St}_{s_1}\xi^{St}_{s_2} & -i\xi^{St}_{s_2}|\eta'|\\ -i\xi^{St}_{s_1}|\eta'| & |\eta'|^2\end{pmatrix}\\
&-\frac{\Delta^4_{p_2}(\eta',c^2_{St}|\eta'|^2)}{\Delta_0(\eta',c^2_{St}|\eta'|^2)}e^{-\xi^{St}_{s_2}x_3}\frac{1}{\xi^{St}_{p_2}}\frac{1}{\xi-i\xi^{St}_{p_2}}\begin{pmatrix}-i\xi^{St}_{s_2}|\eta'| & -\xi^{St}_{s_2}\xi^{St}_{p_2}\\ |\eta'|^2 & -i\xi^{St}_{p_2}|\eta'|\end{pmatrix}\\
&-\frac{\Delta^4_{s_2}(\eta',c^2_{St}|\eta'|^2)}{\Delta_0(\eta',c^2_{St}|\eta'|^2)}e^{-\xi^{St}_{s_2}x_3}\frac{1}{\xi^{St}_{s_2}}\frac{1}{\xi-i\xi^{St}_{s_2}}\begin{pmatrix}\xi^{St2}_{s_2} & -i\xi^{St}_{s_2}|\eta'|\\ i\xi^{St}_{s_2}|\eta'| & |\eta'|^2\end{pmatrix}\Bigg]P_j(\eta),
\end{aligned}
$$

where $j \in M = \{p_1, s_1, p_2, s_2\}$.

In conclusion, $\{\psi_{1j}^{\pm}(x_3, \eta)\}_{j\in M}$ are generalized eigenfunctions corresponding to the roots of the characteristic equation of $A_1(\eta')$. $\{\psi_{1j}^{St}(x_3, \eta)\}_{j\in M}$ are generalized eigenfunctions corresponding to the zero of the Lopatinski determinant of $A_1(\eta')$. (4.9)-(4.16) and (4.17)-(4.18) are explicit formulas of these generalized eigenfunctions for $A_1(\eta')$.

§ 5. Generalized Eigenfunctions of $A_2(\eta')$

In this section, we give an explicit representation of the Green function $G_2(x_3, y_3, \eta'; \zeta)$ for the operator $A_2(\eta') - \zeta I (\zeta \notin \mathbf{R})$ by the same method as getting $G_1(x_3, y_3, \eta'; \zeta)$, in order to define generalized eigenfunctions for the operator $A_2(\eta')$.

Consider the following interface problem:

$$(5.1)\qquad \begin{aligned} &(A_2^I(\eta', D) - \zeta)u^I(\eta', x_3) = f(\eta', x_3), \qquad x_3 < 0,\\ &(A_2^{II}(\eta', D) - \zeta)u^{II}(\eta', x_3) = f(\eta', x_3), \qquad x_3 > 0, \end{aligned}$$

$$(5.2)\qquad u^I(\eta', x_3)\big|_{x_3=0} = u^{II}(\eta', x_3)\big|_{x_3=0},$$

$$(5.3)\qquad B_2^I(\eta')u^I(\eta', x_3)\big|_{x_3=0} = B_2^{II}(\eta')u^{II}(\eta', x_3)\big|_{x_3=0},$$

where $f(\cdot, x_3) \in C_0^\infty(\mathbf{R} \setminus \{0\})$. Let us seek the solutions $u^I(\eta', x_3)$ and $u^{II}(\eta', x_3)$ in the form

$$\begin{aligned} u^I(\eta', x_3) &= E_2^I(x_3, \eta'; \zeta) - K_2^I(x_3, \eta'; \zeta),\\ u^{II}(\eta', x_3) &= E_2^{II}(x_3, \eta'; \zeta) - K_2^{II}(x_3, \eta'; \zeta). \end{aligned}$$

The expressions of E_2^I and E_2^{II} corresponding to (2.16) and (2.18) in Section 2 are given in the following form:

$$\begin{aligned} E_2^I(x_3, \eta'; \zeta) &= \frac{i}{2}\left(\int_{-\infty}^{x_3} \frac{e^{i(x_3-y_3)\tau_{s_1}}}{c_{s_1}^2\tau_{s_1}} f(\eta', y_3)\,dy_3 \right.\\ &\qquad \left. + \int_{x_3}^{\infty} \frac{e^{-i(x_3-y_3)\tau_{s_1}}}{c_{s_1}^2\tau_{s_1}} f(\eta', y_3)\,dy_3\right), \quad x_3 < 0,\\ E_2^{II}(x_3, \eta'; \zeta) &= \frac{i}{2}\left(\int_{-\infty}^{x_3} \frac{e^{i(x_3-y_3)\tau_{s_2}}}{c_{s_2}^2\tau_{s_2}} f(\eta', y_3)\,dy_3 \right.\\ &\qquad \left. + \int_{x_3}^{\infty} \frac{e^{-i(x_3-y_3)\tau_{s_2}}}{c_{s_2}^2\tau_{s_2}} f(\eta', y_3)\,dy_3\right), \quad x_3 > 0, \end{aligned}$$

where

$$\tau_{s_1} = \sqrt{\frac{\zeta}{c_{s_1}^2} - |\eta'|^2}, \quad \operatorname{Im}\tau_{s_1} \ge 0, \qquad \tau_{s_2} = \sqrt{\frac{\zeta}{c_{s_2}^2} - |\eta'|^2}, \quad \operatorname{Im}\tau_{s_2} \ge 0.$$

On the other hand, put

$$K_2^I(x_3,\eta';\zeta) = \alpha e^{-i\tau_{s_1}x_3}, \qquad x_3 < 0,$$
$$K_2^{II}(x_3,\eta';\zeta) = \beta e^{i\tau_{s_2}x_3}, \qquad x_3 > 0,$$

where α and β are determined so that u^I and u^{II} satisfy the interface conditions (5.2) and (5.3). Then the equations on α and β can be written in the matrix form as follows:

$$\begin{pmatrix} 1 & -1 \\ \rho_1 c_{s_1}^2 \tau_{s_1} & \rho_2 c_{s_2}^2 \tau_{s_2} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{i}{2} \begin{pmatrix} h_1(\eta',\zeta) \\ h_2(\eta',\zeta) \end{pmatrix},$$

where

$$\begin{aligned} h_1(\eta',\zeta) = & \int_{-\infty}^0 \frac{e^{-i\tau_{s_1}y_3}}{c_{s_1}^2\tau_{s_1}} f(\eta',y_3)\,dy_3 + \int_0^\infty \frac{e^{i\tau_{s_1}y_3}}{c_{s_1}^2\tau_{s_1}} f(\eta',y_3)\,dy_3 \\ & - \int_{-\infty}^0 \frac{e^{-i\tau_{s_2}y_3}}{c_{s_2}^2\tau_{s_2}} f(\eta',y_3)\,dy_3 - \int_0^\infty \frac{e^{i\tau_{s_2}y_3}}{c_{s_2}^2\tau_{s_2}} f(\eta',y_3)\,dy_3, \\ h_2(\eta',\zeta) = & \rho_1 \left(-\int_{-\infty}^0 e^{-i\tau_{s_1}y_3} f(\eta',y_3)\,dy_3 + \int_0^\infty e^{i\tau_{s_1}y_3} f(\eta',y_3)\,dy_3 \right) \\ & + \rho_2 \left(\int_{-\infty}^0 e^{-i\tau_{s_2}y_3} f(\eta',y_3)\,dy_3 - \int_0^\infty e^{i\tau_{s_2}y_3} f(\eta',y_3)\,dy_3 \right). \end{aligned}$$

The Lopatinski determinant for the problem (5.1),(5.2), and (5.3)

$$\Delta'(\eta',\zeta) = \rho_1 c_{s_1}^2 \tau_{s_1} + \rho_2 c_{s_2}^2 \tau_{s_2}$$

has no zero with respect to ζ for $|\eta'| \neq 0$. Therefore

$$\begin{aligned} u^I(x_3,\eta';\zeta) = & E_2^I(x_3,\eta';\zeta) - K_2^I(x_3,\eta';\zeta) \\ = & \frac{i}{2}\left(\int_{-\infty}^{x_3} \frac{e^{i(x_3-y_3)\tau_{s_1}}}{c_{s_1}^2\tau_{s_1}} f(\eta',y_3)\,dy_3 + \int_{x_3}^\infty \frac{e^{-i(x_3-y_3)\tau_{s_1}}}{c_{s_1}^2\tau_{s_1}} f(\eta',y_3)\,dy_3 \right) \\ & - \frac{1}{\Delta'(\eta',\zeta)} \frac{i}{2} e^{-i\tau_{s_1}x_3} \times \\ & \left[\rho_2 c_{s_2}^2 \tau_{s_2} \left(\int_{-\infty}^0 \frac{e^{-i\tau_{s_1}y_3}}{c_{s_1}^2\tau_{s_1}} f(\eta',y_3)\,dy_3 + \int_0^\infty \frac{e^{i\tau_{s_1}y_3}}{c_{s_1}^2\tau_{s_1}} f(\eta',y_3)\,dy_3 \right) \right. \\ & + \rho_1 \left(-\int_{-\infty}^0 e^{-i\tau_{s_1}y_3} f(\eta',y_3)\,dy_3 + \int_0^\infty e^{i\tau_{s_1}y_3} f(\eta',y_3)\,dy_3 \right) \\ & \left. -2\rho_2 \int_0^\infty e^{i\tau_{s_2}y_3} f(\eta',y_3)\,dy_3 \right], \qquad x_3 < 0, \end{aligned}$$

$$\begin{aligned} u^{II}(x_3,\eta';\zeta) = & E_2^{II}(x_3,\eta';\zeta) - K_2^{II}(x_3,\eta';\zeta) \\ = & \frac{i}{2}\left(\int_{-\infty}^{x_3} \frac{e^{i(x_3-y_3)\tau_{s_2}}}{c_{s_2}^2\tau_{s_2}} f(\eta',y_3)\,dy_3 + \int_{x_3}^\infty \frac{e^{-i(x_3-y_3)\tau_{s_2}}}{c_{s_2}^2\tau_{s_2}} f(\eta',y_3)\,dy_3 \right) \\ & - \frac{1}{\Delta'(\eta',\zeta)} \frac{i}{2} e^{i\tau_{s_2}x_3} \times \end{aligned}$$

$$\Big[\rho_1 c_{s_1}^2 \tau_{s_1} \Big(\int_{-\infty}^{0} \frac{e^{-i\tau_{s_2} y_3}}{c_{s_2}^2 \tau_{s_2}} f(\eta', y_3)\, dy_3 + \int_0^{\infty} \frac{e^{i\tau_{s_2} y_3}}{c_{s_2}^2 \tau_{s_2}} f(\eta', y_3)\, dy_3 \Big)$$
$$+ \rho_2 \Big(\int_{-\infty}^{0} e^{-i\tau_{s_2} y_3} f(\eta', y_3)\, dy_3 - \int_0^{\infty} e^{i\tau_{s_2} y_3} f(\eta', y_3)\, dy_3 \Big)$$
$$-2\rho_1 \int_{-\infty}^{0} e^{-i\tau_{s_1} y_3} f(\eta', y_3)\, dy_3 \Big], \quad x_3 > 0.$$

So, the Green function of $A_2(\eta') - \zeta$ is given by the following form:

$$G_2(x_3, y_3, \eta'; \zeta) = \begin{cases} G_2^I(x_3, y_3, \eta'; \zeta), & x_3 < 0, \\ G_2^{II}(x_3, y_3, \eta'; \zeta), & x_3 > 0, \end{cases}$$

where

$$G_2^I(x_3, y_3, \eta'; \zeta) = \frac{i}{2} \times$$
$$\Big[H(x_3 - y_3) \frac{e^{i(x_3 - y_3)\tau_{s_1}}}{c_{s_1}^2 \tau_{s_1}} + H(y_3 - x_3) \frac{e^{-i(x_3 - y_3)\tau_{s_1}}}{c_{s_1}^2 \tau_{s_1}}$$
$$- \frac{1}{\Delta'(\eta', \zeta)} e^{-i\tau_{s_1} x_3} \times$$
$$\Big\{ \rho_2 c_{s_2}^2 \tau_{s_2} \Big(H(-y_3) \frac{e^{-i\tau_{s_1} y_3}}{c_{s_1}^2 \tau_{s_1}} + H(y_3) \frac{e^{i\tau_{s_1} y_3}}{c_{s_1}^2 \tau_{s_1}} \Big)$$
$$+ \rho_1 \left(-H(-y_3) e^{-i\tau_{s_1} y_3} + H(y_3) e^{i\tau_{s_1} y_3} \right)$$
$$- 2\rho_2 H(y_3) e^{i\tau_{s_2} y_3} \Big\} \Big], \quad x_3 < 0,$$

$$G_2^{II}(x_3, y_3, \eta'; \zeta) = \frac{i}{2} \times$$
$$\Big[H(x_3 - y_3) \frac{e^{i(x_3 - y_3)\tau_{s_2}}}{c_{s_2}^2 \tau_{s_2}} + H(y_3 - x_3) \frac{e^{-i(x_3 - y_3)\tau_{s_2}}}{c_{s_2}^2 \tau_{s_2}}$$
$$- \frac{1}{\Delta'(\eta', \zeta)} e^{i\tau_{s_2} x_3} \times$$
$$\Big\{ \rho_1 c_{s_1}^2 \tau_{s_1} \Big(H(-y_3) \frac{e^{-i\tau_{s_2} y_3}}{c_{s_2}^2 \tau_{s_2}} - H(y_3) \frac{e^{i\tau_{s_2} y_3}}{c_{s_2}^2 \tau_{s_2}} \Big)$$
$$+ \rho_2 \left(H(-y_3) e^{-i\tau_{s_2} y_3} - H(y_3) e^{i\tau_{s_2} y_3} \right)$$
$$- 2\rho_1 H(-y_3) e^{-i\tau_{s_1} y_3} \Big\} \Big], \quad x_3 > 0.$$

Now we define $\psi_{2k}(x_3, \eta; \zeta) (k \in N = \{s_1, s_2\})$ by

(5.4) $$\psi_{2k}(x_3, \eta; \zeta) = \begin{cases} \psi_{2k}^I(x_3, \eta; \zeta), & x_3 < 0, \\ \psi_{2k}^{II}(x_3, \eta; \zeta), & x_3 > 0. \end{cases}$$

and

(5.5) $$\begin{aligned} \psi_{2k}^I(x_3, \eta; \zeta) &= F_{y_3}^{-1}[G_2^I(x_3, y_3, \eta'; \zeta)](\xi)(\lambda_k(\eta) - \zeta)\rho_1^{-1}, \quad x_3 < 0, \\ \psi_{2k}^{II}(x_3, \eta; \zeta) &= F_{y_3}^{-1}[G_2^{II}(x_3, y_3, \eta'; \zeta)](\xi)(\lambda_k(\eta) - \zeta)\rho_2^{-1}, \quad x_3 > 0, \end{aligned}$$

where $\lambda_k(\eta)(k \in N = \{s_1, s_2\})$ are the eigenvalues of $A_2(\eta')$ which have concrete expressions $\lambda_{s_1}(\eta)(= c_{s_1}^2|\eta|^2)$, $\lambda_{s_2}(\eta)(= c_{s_2}^2|\eta|^2)$.

The motivation for these particular definitions (5.4) and (5.5) is shown in Section 6 (Lemma 6.1) below. Then we can see that the limit

$$\psi_{2k}(x_3, \eta) = \lim_{\zeta \to \lambda_k(\eta) \pm i0} \psi_{2k}(x_3, \eta; \zeta)$$

exist. Moreover applying Lemma 4.1 to $A_2(\eta')$, $\psi_{2k}^I(x_3, \eta)$ and $\psi_{2k}^{II}(x_3, \eta)$ satisfy the equations

$$\begin{aligned}
&A_2^I(\eta')\psi_{2k}^{\pm I}(x_3, \eta) = \lambda_k(\eta)\psi_{2k}^{\pm I}(x_3, \eta), \qquad x_3 < 0,\\
&A_2^{II}(\eta')\psi_{2k}^{\pm II}(x_3, \eta) = \lambda_k(\eta)\psi_{2k}^{\pm II}(x_3, \eta), \qquad x_3 > 0,\\
&\psi_{2k}^{\pm I}(x_3, \eta)\big|_{x_3=0} = \psi_{2k}^{\pm II}(x_3, \eta)\big|_{x_3=0},\\
&B_2^I(\eta')\psi_{2k}^{\pm I}(x_3, \eta)\big|_{x_3=0} = B_2^{II}(\eta')\psi_{2k}^{\pm II}(x_3, \eta)\big|_{x_3=0},
\end{aligned}$$

where $k \in N = \{s_1, s_2\}$.

We note the following relations.

$$\lim_{\zeta \to \lambda_{s_1}(\eta) \pm i0} \tau_{s_1} = \pm|\xi|,$$

$$\lim_{\zeta \to \lambda_{s_1}(\eta) \pm i0} \tau_{s_2} = \xi_{s_2}(\eta', \lambda_{s_1}) = \begin{cases} \pm\sqrt{\frac{c_{s_1}^2(|\eta'|^2+\xi^2)}{c_{s_2}^2} - |\eta'|^2} & (c_{s_1}^2|\eta|^2 > c_{s_2}^2|\eta'|^2) \\ i\sqrt{|\eta'|^2 - \frac{c_{s_1}^2(|\eta'|^2+\xi^2)}{c_{s_2}^2}} & (c_{s_1}^2|\eta|^2 < c_{s_2}^2|\eta'|^2), \end{cases}$$

$$\lim_{\zeta \to \lambda_{s_1}(\eta) \pm i0} \tau_{s_2} = \xi_{s_1}(\eta', \lambda_{s_2}) = \pm\sqrt{\frac{c_{s_2}^2(|\eta'|^2+\xi^2)}{c_{s_1}^2} - |\eta'|^2},$$

$$\lim_{\zeta \to \lambda_{s_2}(\eta) \pm i0} \tau_{s_2} = \pm|\xi|.$$

We have for $\xi > 0$

$$\lim_{\zeta \to \lambda_k(\eta) \pm i0} \frac{\lambda_k(\eta) - \zeta}{\xi \mp \tau_k(\eta', \zeta)} = 2c_k^2\xi,$$

and also for $\xi < 0$,

$$\lim_{\zeta \to \lambda_k(\eta) \pm i0} \frac{\lambda_k(\eta) - \zeta}{\xi \pm \tau_k(\eta', \zeta)} = 2c_k^2\xi,$$

where $c_k (k \in N)$ are defined by

$$c_{s_1}^2 = \frac{\mu_1}{\rho_1}, \quad c_{s_2}^2 = \frac{\mu_2}{\rho_2},$$

and $\tau_k(\eta', \zeta)$ are defined by

$$\tau_{s_1} = \sqrt{\frac{\zeta}{c_{s_1}^2} - |\eta'|^2}, \quad \operatorname{Im}\tau_{s_1} \geq 0, \quad \tau_{s_2} = \sqrt{\frac{\zeta}{c_{s_2}^2} - |\eta'|^2}, \quad \operatorname{Im}\tau_{s_2} \geq 0.$$

If $\xi > 0$, we have the following expressions:

(5.6)
$$\psi^{\pm I}_{2s_1}(x_3,\eta) \equiv \psi^{I}_{2s_1}(x_3,\eta;\lambda_{s_1}(\eta) \pm i0)$$
$$= \frac{1}{\sqrt{2\pi}}\frac{1}{\rho_1}\left[e^{i\xi x_3} + \frac{1}{\Delta'(\eta',\lambda_{s_1})}e^{-i\xi x_3}(\rho_1 c^2_{s_1}\xi - \rho_2 c^2_{s_2}\xi_{s_2}(\eta',\lambda_{s_1})\right],$$

(5.7)
$$\psi^{\pm II}_{2s_1}(x_3,\eta) \equiv \psi^{II}_{2s_1}(x_3,\eta;\lambda_{s_1}(\eta) \pm i0)$$
$$= \frac{1}{\sqrt{2\pi}}\frac{\rho_1}{\rho_2}\frac{1}{\Delta'(\eta',\lambda_{s_1})}2e^{i\xi_{s_2}(\eta',\lambda_{s_1})x_3},$$

(5.8)
$$\psi^{\mp I}_{2s_1}(x_3,\eta) \equiv \psi^{I}_{2s_1}(x_3,\eta;\lambda_{s_1}(\eta) \mp i0)$$
$$= \frac{1}{\sqrt{2\pi}}\frac{1}{\rho_1}\left[e^{i\xi x_3} + \frac{1}{\Delta'(\eta',\lambda_{s_1})}e^{-i\xi x_3}(\rho_1 c^2_{s_1}\xi + \rho_2 c^2_{s_2}\xi_{s_2}(\eta',\lambda_{s_1})\right],$$

(5.9)
$$\psi^{\mp I}_{2s_1}(x_3,\eta) \equiv \psi^{I}_{2s_1}(x_3,\eta;\lambda_{s_1}(\eta) \mp i0)$$
$$= 0,$$

(5.10)
$$\psi^{\pm I}_{2s_2}(x_3,\eta) \equiv \psi^{I}_{2s_2}(x_3,\eta;\lambda_{s_2}(\eta) \pm i0)$$
$$= 0,$$

(5.11)
$$\psi^{\pm II}_{2s_2}(x_3,\eta) \equiv \psi^{II}_{2s_2}(x_3,\eta;\lambda_{s_2}(\eta) \pm 0)$$
$$= \frac{1}{\sqrt{2\pi}}\frac{1}{\rho_2}\left[e^{i\xi x_3} - \frac{1}{\Delta'(\eta',\lambda_{s_2})}e^{i\xi x_3}(\rho_1 c^2_{s_1}\xi_{s_1}(\eta',\lambda_{s_2}) + \rho_2 c^2_{s_2}\xi)\right],$$

(5.12)
$$\psi^{\mp I}_{2s_2}(x_3,\eta) \equiv \psi^{I}_{2s_2}(x_3,\eta;\lambda_{s_2}(\eta) \mp i0)$$
$$= -\frac{1}{\sqrt{2\pi}}\frac{\rho_2}{\rho_1}\frac{1}{\Delta'(\eta',\lambda_{s_2})}2e^{-i\xi_{s_1}(\eta',\lambda_{s_2})x_3},$$

(5.13)
$$\psi^{\mp II}_{2s_2}(x_3,\eta) \equiv \psi^{II}_{2s_2}(x_3,\eta;\lambda_{s_2}(\eta) \mp i0)$$
$$= \frac{1}{\sqrt{2\pi}}\frac{1}{\rho_2}\left[e^{i\xi x_3} - \frac{1}{\Delta'(\eta',\lambda_{s_2})}e^{i\xi x_3}(-\rho_1 c^2_{s_1}\xi_{s_1}(\eta',\lambda_{s_2}) + \rho_2 c^2_{s_2}\xi)\right].$$

If $\xi < 0$, the eigenfunctions $\psi_{2k}(x_3,\eta,\lambda_k(\eta) \pm i0)$ coincide with the eigenfunctions $\psi_{2k}(x_3,\eta,\lambda_k(\eta) \mp i0)$ in the formula above.

In conclusion, $\{\psi^{\pm}_{2k}(x_3,\eta)\}_{k\in N}$ are generalized eigenfunctions corresponding to the roots of the characteristic equation of $A_2(\eta')$. (5.6)-(5.13) are the explicit formulas of these generalized eigenfunctions for $A_2(\eta')$. Since the Lopatinski determinant of $A_2(\eta')$ has no zero, we need not consider other eigenfunctions.

§ 6. Construction of the Spectral Family of A

In this section, we construct the spectral family of A by means of the generalized eigenfunctions of $A_1(\eta')$ and $A_2(\eta')$ defined in Section 4 and Section 5, respectively. Then we define the Fourier transforms of $f \in \mathcal{H}$ with respect to these generalized eigenfunctions of A and we prove the corresponding Parseval formula (Theorem 6.5). The key lemma is Lemma 6.4 below, which justifies to pass to the limit under the integral sign over $\mathbf{R}^3$.

Using $\psi_{1j}(x_3,\eta;\zeta)$ $(j \in M)$ defined in Section 4, we define $\psi_{1j}(x,\eta;\zeta)$ $(j \in M)$ by

$$\psi_{1j}(x,\eta;\zeta) = \begin{cases} \psi_{1j}^I(x,\eta;\zeta), & x \in \mathbf{R}^3_-, \\ \psi_{1j}^{II}(x,\eta;\zeta), & x \in \mathbf{R}^3_+, \end{cases} \tag{6.1}$$

where

$$\psi_{1j}^I(x,\eta;\zeta) = \frac{1}{2\pi} e^{i(x_1\eta_1+x_2\eta_2)}\mathrm{UC}(\psi_{1j}^I(x_3,\eta;\zeta) \oplus O_{1\times 1}),$$
$$\psi_{1j}^{II}(x,\eta;\zeta) = \frac{1}{2\pi} e^{i(x_1\eta_1+x_2\eta_2)}\mathrm{UC}(\psi_{1j}^{II}(x_3,\eta;\zeta) \oplus O_{1\times 1}).$$

$O_{n\times n}$ denotes the $n \times n$ zero matrix. Using $\psi_{2k}(x_3,\eta;\zeta)$ $(k \in N)$ defined in Section 5, we define $\psi_{2k}(x,\eta;\zeta)$ $(k \in N)$ by

$$\psi_{2k}(x,\eta;\zeta) = \begin{cases} \psi_{2k}^I(x,\eta;\zeta), & x \in \mathbf{R}^3_-, \\ \psi_{2k}^{II}(x,\eta;\zeta), & x \in \mathbf{R}^3_+, \end{cases} \tag{6.2}$$

where

$$\psi_{2k}^I(x,\eta;\zeta) = \frac{1}{2\pi} e^{i(x_1\eta_1+x_2\eta_2)}\mathrm{UC}(O_{2\times 2} \oplus \psi_{2k}^I(x_3,\eta;\zeta)),$$
$$\psi_{2k}^{II}(x,\eta;\zeta) = \frac{1}{2\pi} e^{i(x_1\eta_1+x_2\eta_2)}\mathrm{UC}(O_{2\times 2} \oplus \psi_{2k}^{II}(x_3,\eta;\zeta)),$$

and $\eta = (\eta',\xi) = (\eta_1,\eta_2,\xi)$. Further we define for $f \in C_0^\infty(\mathbf{R}^3,\mathbf{C}^3)$

$$\hat{f}_{1j}(\eta;\zeta) = \int_{\mathbf{R}^3} \psi_{1j}(x,\eta;\zeta)^* f(x)\rho(x_3)\,dx, \quad j \in M, \tag{6.3}$$

$$\hat{f}_{2k}(\eta;\zeta) = \int_{\mathbf{R}^3} \psi_{2k}(x,\eta;\zeta)^* f(x)\rho(x_3)\,dx, \quad k \in N. \tag{6.4}$$

Next lemma shows the motivation of definitions (4.1), (4.2), (5.4) and (5.5).

Lemma 6.1. *Let $f \in C_0^\infty(\mathbf{R}^3,\mathbf{C}^3)$. Then we have*

$$(\mathrm{UC})^{-1}F_x[R(\bar{\zeta})f] = \sum_{j\in M} \frac{\hat{f}_{1j}(\eta;\zeta)}{\lambda_j(\eta)-\bar{\zeta}} + \sum_{k\in N} \frac{\hat{f}_{2k}(\eta;\zeta)}{\lambda_k(\eta)-\bar{\zeta}}, \tag{6.5}$$

in the distribution sense.

Proof. Let $\psi \in C_0^\infty(\mathbf{R}^3,\mathbf{C}^3)$. We denote by $< \ , \ >_{\eta_1,\eta_2,\xi}$ the duality between $\mathcal{S}$ and $\mathcal{S}'$ where $\mathcal{S}$ is the space of rapidly decreasing C^∞ functions and $\mathcal{S}'$ the space of temperate distributions. F_x denotes the Fourier transformation with respect to $x=(x_1,x_2,x_3)$. Then from the Parseval equality

$$\begin{aligned}
&\langle (\mathrm{UC})^{-1}(F_{x\to(\eta',\xi)}\psi)(\eta), (\mathrm{UC})^{-1}(F_{x\to(\eta',\xi)}[R(\bar\zeta)f])(\eta)\rangle_{\eta_1,\eta_2,\xi}\\
&=\langle \psi(x), R(\bar\zeta)f(x)\rangle_{x_1,x_2,x_3}\\
&=\langle R(\zeta)\psi(x), f(x)\rangle_{x_1,x_2,x_3}\\
&=\langle (A-\zeta)^{-1}\psi(x), f(x)\rangle_{x_1,x_2,x_3}.(*)
\end{aligned}$$

From (1.10), it follows that

(6.6)

$$\begin{aligned}
&(A-\zeta)^{-1}\psi(x)\\
&\quad=F^{-1}_{\eta'\to x'}[\mathrm{UC}((A_1(\eta')-\zeta)^{-1}\oplus(A_2(\eta')-\zeta)^{-1})(\mathrm{UC})^{-1}(F_{x'\to\eta'}\psi)(\eta',x_3)](x)\\
&\quad=F^{-1}_{\eta'\to x'}[\mathrm{UC}((A_1(\eta')-\zeta)^{-1}\oplus O_{1\times1})(\mathrm{UC})^{-1}(F_{x'\to\eta'}\psi)(\eta',x_3)](x)\\
&\qquad+F^{-1}_{\eta'\to x'}[\mathrm{UC}(O_{2\times2}\oplus(A_2(\eta')-\zeta)^{-1})(\mathrm{UC})^{-1}(F_{x'\to\eta'}\psi)(\eta',x_3)](x),
\end{aligned}$$

so that from (4.1), (4.2), (5.4), (5.5), (6.1)-(6.4) and the Parseval equality

$$\begin{aligned}
(*)&=\Big\langle F^{-1}_{\eta'}[\mathrm{UC}((A_1(\eta')-\zeta)^{-1}\oplus(A_2(\eta')-\zeta)^{-1})(\mathrm{UC})^{-1}(F_{x'}\psi)](x), f(x)\Big\rangle_{x_1,x_2,x_3}\\
&=\langle \mathrm{UC}((A_1(\eta')-\zeta)^{-1}\oplus O_{1\times1})(\mathrm{UC})^{-1}(F_{x'}\psi)(\eta',x_3),(F_{x'}f)(\eta',x_3)\rangle_{\eta_1,\eta_2,x_3}\\
&\quad+\langle \mathrm{UC}(O_{2\times2}\oplus(A_2(\eta')-\zeta)^{-1})(\mathrm{UC})^{-1}(F_{x'}\psi)(\eta',x_3),(F_{x'}f)(\eta',x_3)\rangle_{\eta_1,\eta_2,x_3}\\
&=\langle \mathrm{UC}(G_1(x_3,y_3,\eta';\zeta)\oplus O_{1\times1})(\mathrm{UC})^{-1}(F_{x'}\psi)(\eta',y_3),\\
&\qquad\qquad (F_{x'}f)(\eta',x_3)\rangle_{\eta_1,\eta_2,x_3,y_3}\\
&\quad+\langle \mathrm{UC}(O_{2\times2}\oplus G_2(x_3,y_3,\eta';\zeta))(\mathrm{UC})^{-1}(F_{x'}\psi)(\eta',y_3),\\
&\qquad\qquad (F_{x'}f)(\eta',x_3)\rangle_{\eta_1,\eta_2,x_3,y_3}\\
&=\langle\langle \mathrm{UC}(G_1(x_3,y_3,\eta';\zeta)\oplus O_{1\times1})(\mathrm{UC})^{-1},\\
&\qquad\qquad (F_{x'}\psi)(\eta',y_3)\rangle_{y_3},(F_{x'}f)(\eta',x_3)\rangle_{\eta_1,\eta_2,x_3}\\
&\quad+\langle\langle \mathrm{UC}(O_{2\times2}\oplus G_2(x_3,y_3,\eta';\zeta))(\mathrm{UC})^{-1},\\
&\qquad\qquad (F_{x'}\psi)(\eta',y_3)\rangle_{y_3},(F_{x'}f)(\eta',x_3)\rangle_{\eta_1,\eta_2,x_3}\\
&=\langle\langle \mathrm{UC}(F^{-1}_{y_3}[G_1(x_3,y_3,\eta';\zeta)\oplus O_{1\times1}])(\xi)(\mathrm{UC})^{-1},\\
&\qquad\qquad (F_{y_3}F_{x'}\psi)(\eta)\rangle_{\xi},(F_{x'}f)(\eta',x_3)\rangle_{\eta_1,\eta_2,x_3}\\
&\quad+\langle\langle \mathrm{UC}(F^{-1}_{y_3}[O_{2\times2}\oplus G_2(x_3,y_3,\eta';\zeta)])(\xi)(\mathrm{UC})^{-1},\\
&\qquad\qquad (F_{y_3}F_{x'}\psi)(\eta)\rangle_{\xi},(F_{x'}f)(\eta',x_3)\rangle_{\eta_1,\eta_2,x_3}\\
&=\langle \mathrm{UC}(F^{-1}_{y_3}[G_1(x_3,y_3,\eta';\zeta)\oplus O_{1\times1}])(\xi)(\mathrm{UC})^{-1}(F_{y_3}F_{x'}\psi)(\eta),\\
&\qquad\qquad (F_{x'}f)(\eta',x_3)\rangle_{\eta_1,\eta_2,\xi,x_3}\\
&\quad+\langle \mathrm{UC}(F^{-1}_{y_3}[O_{2\times2}\oplus G_2(x_3,y_3,\eta';\zeta)])(\xi)(\mathrm{UC})^{-1}(F_{y_3}F_{x'}\psi)(\eta),\\
&\qquad\qquad (F_{x'}f)(\eta',x_3)\rangle_{\eta_1,\eta_2,\xi,x_3}
\end{aligned}$$

$$= \sum_{j\in M} \langle \mathrm{UC}(F_{y_3}^{-1}[G_1(x_3,y_3,\eta';\zeta)\oplus O_{1\times 1}])(\xi)(\lambda_j(\eta)-\zeta)P_j(\eta)\rho(x_3)^{-1}\rho(x_3)$$

$$(\mathrm{UC})^{-1}(F_{y_3}F_{x'}\psi)(\eta), (\lambda_j(\eta)-\bar\zeta)^{-1}(F_{x'}f)(\eta',x_3)\rangle_{\eta_1,\eta_2,\xi,x_3}$$

$$+ \sum_{k\in N} \langle \mathrm{UC}(F_{y_3}^{-1}[O_{2\times 2}\oplus G_2(x_3,y_3,\eta';\zeta)])(\xi)(\lambda_k(\eta)-\zeta)\rho(x_3)^{-1}\rho(x_3)$$

$$(\mathrm{UC})^{-1}(F_{y_3}F_{x'}\psi)(\eta), (\lambda_k(\eta)-\bar\zeta)^{-1}(F_{x'}f)(\eta',x_3)\rangle_{\eta_1,\eta_2,\xi,x_3}$$

$$= \sum_{j\in M} \langle \mathrm{UC}(\psi_{1j}(x_3,\eta;\zeta)\oplus O_{1\times 1})\rho(x_3)$$

$$(\mathrm{UC})^{-1}(F_{y_3}F_{x'}\psi)(\eta), (\lambda_j(\eta)-\bar\zeta)^{-1}(F_{x'}f)(\eta',x_3)\rangle_{\eta_1,\eta_2,\xi,x_3}$$

$$+ \sum_{k\in N} \langle \mathrm{UC}(O_{2\times 2}\oplus \psi_{2k}(x_3,\eta;\zeta))\rho(x_3)$$

$$(\mathrm{UC})^{-1}(F_{y_3}F_{x'}\psi)(\eta), (\lambda_k(\eta)-\bar\zeta)^{-1}(F_{x'}f)(\eta',x_3)\rangle_{\eta_1,\eta_2,\xi,x_3}$$

$$= \sum_{j\in M} \Big\langle (\mathrm{UC})^{-1}(F_{y_3}F_{x'}\psi)(\eta), (\psi_{1j}(x_3,\eta;\zeta)^*\oplus O_{1\times 1})\rho(x_3)(\mathrm{UC})^{-1}$$

$$\times (\lambda_j(\eta)-\bar\zeta)^{-1}\frac{1}{2\pi}\int e^{-i(x_1\eta_1+x_2\eta_2)}f(x)\,dx'\Big\rangle_{\eta_1,\eta_2,\xi,x_3}$$

$$+ \sum_{k\in N} \Big\langle (\mathrm{UC})^{-1}(F_{y_3}F_{x'}\psi)(\eta), (O_{1\times 1}\oplus \psi_{2k}(x_3,\eta;\zeta)^*)\rho(x_3)(\mathrm{UC})^{-1}$$

$$\times (\lambda_k(\eta)-\bar\zeta)^{-1}\frac{1}{2\pi}\int e^{-i(x_1\eta_1+x_2\eta_2)}f(x)\,dx'\Big\rangle_{\eta_1,\eta_2,\xi,x_3}$$

$$= \sum_{j\in M} \Big\langle (\mathrm{UC})^{-1}(F_{y_3}F_{x'}\psi)(\eta), (\lambda_j(\eta)-\bar\zeta)^{-1}\int \psi_{1j}(x,\eta;\zeta)^* f(x)\rho(x_3)\,dx\Big\rangle_{\eta_1,\eta_2,\xi}$$

$$+ \sum_{k\in N} \Big\langle (\mathrm{UC})^{-1}(F_{y_3}F_{x'}\psi)(\eta), (\lambda_k(\eta)-\bar\zeta)^{-1}\int \psi_{2k}(x,\eta;\zeta)^* f(x)\rho(x_3)\,dx\Big\rangle_{\eta_1,\eta_2,\xi}$$

$$= \Big\langle (\mathrm{UC})^{-1}(F_{y_3\to\xi}F_{x'\to\eta'}\psi)(\eta), \sum_{j\in M}\frac{\hat f_{1j}(\eta;\zeta)}{(\lambda_j(\eta)-\bar\zeta)} + \sum_{k\in N}\frac{\hat f_{2k}(\eta;\zeta)}{(\lambda_k(\eta)-\bar\zeta)}\Big\rangle_{\eta_1,\eta_2,\xi}$$

$$= \Big\langle (\mathrm{UC})^{-1}(F_{x_3\to\xi}F_{x'\to\eta'}\psi)(\eta), \sum_{j\in M}\frac{\hat f_{1j}(\eta;\zeta)}{(\lambda_j(\eta)-\bar\zeta)} + \sum_{k\in N}\frac{\hat f_{2k}(\eta;\zeta)}{(\lambda_k(\eta)-\bar\zeta)}\Big\rangle_{\eta_1,\eta_2,\xi}.$$

This completes the proof of (6.5). □

The selfadjoint operator A admits a uniquely determined spectral resolution:

$$A = \int_{-\infty}^{\infty} \lambda\, d\pi(\lambda)$$

where $\{\pi(\lambda)\}_{-\infty<\lambda<\infty}$ denotes the right-continuous spectral family of A. The representation of $\pi(\lambda)$ is based on the well-known theorem of Stone (see, e. g., [14]):

$$\begin{aligned}(6.7)\qquad &\frac{\pi(b)+\pi(b-)}{2}-\frac{\pi(a)+\pi(a-)}{2}\\ &= s-\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_a^b [R(\lambda+i\varepsilon)-R(\lambda-i\varepsilon)]\,d\lambda, \qquad a<b.\end{aligned}$$

From (6.5) and (6.7), we obtain the following.

Lemma 6.2. *Let $f\in C_0^\infty(\mathbf{R}^3,\mathbf{C}^3)$ and $0<a<b<\infty$. Then we have*

$$\begin{aligned}(6.8)\qquad &\left(\left(\frac{\pi(b)+\pi(b-)}{2}-\frac{\pi(a)+\pi(a-)}{2}\right)f,f\right)\\ &=\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\left(\sum_{j\in M}\int_a^b d\lambda\int_{\mathbf{R}^3}\frac{\varepsilon}{(\lambda_j(\eta)-\lambda)^2+\varepsilon^2}|\hat f_{1j}(\eta;\lambda\pm i\varepsilon)|^2\,d\eta\right.\\ &\qquad\left.+\sum_{k\in N}\int_a^b d\lambda\int_{\mathbf{R}^3}\frac{\varepsilon}{(\lambda_k(\eta)-\lambda)^2+\varepsilon^2}|\hat f_{2k}(\eta;\lambda\pm i\varepsilon)|^2\,d\eta\right)\\ &=\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\left(\sum_{j\in M}\int_{\mathbf{R}^3} d\eta\int_a^b\frac{\varepsilon}{(\lambda_j(\eta)-\lambda)^2+\varepsilon^2}|\hat f_{1j}(\eta;\lambda\pm i\varepsilon)|^2\,d\lambda\right.\\ &\qquad\left.+\sum_{k\in N}\int_{\mathbf{R}^3} d\eta\int_a^b\frac{\varepsilon}{(\lambda_k(\eta)-\lambda)^2+\varepsilon^2}|\hat f_{2k}(\eta;\lambda\pm i\varepsilon)|^2\,d\lambda\right).\end{aligned}$$

Proof. From (6.5) and the resolvent identity

$$R(\zeta)-R(\zeta')=(\zeta-\zeta')R(\zeta)R(\zeta')=(\zeta-\zeta')R(\zeta')R(\zeta),$$

we have

$$\begin{aligned}&([R(\lambda+i\varepsilon)-R(\lambda-i\varepsilon)]f,f)\\ &=(2i\varepsilon R(\lambda\pm i\varepsilon)R(\lambda\mp i\varepsilon)f,f)\\ &=2i\varepsilon(R(\lambda\mp i\varepsilon)f,R(\lambda\mp i\varepsilon)f)\\ &=2i\varepsilon((\mathrm{UC})^{-1}F[R(\lambda\mp i\varepsilon)f],(\mathrm{UC})^{-1}F[R(\lambda\mp i\varepsilon)f])\\ &=2i\varepsilon\left(\sum_{j\in M}\left(\frac{\hat f_{1j}(\cdot;\lambda\pm i\varepsilon)}{\lambda_j(\eta)-(\lambda\mp i\varepsilon)},\frac{\hat f_{1j}(\cdot;\lambda\pm i\varepsilon)}{\lambda_j(\eta)-(\lambda\mp i\varepsilon)}\right)\right.\\ &\qquad\left.+\sum_{k\in N}\left(\frac{\hat f_{2k}(\cdot;\lambda\pm i\varepsilon)}{\lambda_k(\eta)-(\lambda\mp i\varepsilon)},\frac{\hat f_{2k}(\cdot;\lambda\pm i\varepsilon)}{\lambda_k(\eta)-(\lambda\mp i\varepsilon)}\right)\right)\\ &=2i\left(\sum_{j\in M}\int_{\mathbf{R}^3}\frac{\varepsilon}{(\lambda_j(\eta)-\lambda)^2+\varepsilon^2}|\hat f_{1j}(\eta;\lambda\pm i\varepsilon)|^2\,d\eta\right.\end{aligned}$$

$$+\sum_{k\in N}\int_{\mathbf{R}^3}\frac{\varepsilon}{(\lambda_k(\eta)-\lambda)^2+\varepsilon^2}|\hat{f}_{2k}(\eta;\lambda\pm i\varepsilon)|^2\,d\eta\Bigg),$$

and hence by (6.7) for any interval $(a,b)\subset\mathbf{R}_+$

$$\left(\left(\frac{\pi(b)+\pi(b-)}{2}-\frac{\pi(a)+\pi(a-)}{2}\right)f,f\right)$$
$$=\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_a^b([R(\lambda+i\varepsilon)-R(\lambda-i\varepsilon)]\,f,f)\,d\lambda$$
$$=\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_a^b d\lambda\Bigg(\sum_{j\in M}\int_{\mathbf{R}^3}\frac{\varepsilon}{(\lambda_j(\eta)-\lambda)^2+\varepsilon^2}|\hat{f}_{1j}(\eta;\lambda\pm i\varepsilon)|^2\,d\eta$$
$$+\sum_{k\in N}\int_a^b d\lambda\int_{\mathbf{R}^3}\frac{\varepsilon}{(\lambda_k(\eta)-\lambda)^2+\varepsilon^2}|\hat{f}_{2k}(\eta;\lambda\pm i\varepsilon)|^2\,d\eta\Bigg)$$
$$=\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\Bigg(\sum_{j\in M}\int_{\mathbf{R}^3}d\eta\int_a^b\frac{\varepsilon}{(\lambda_j(\eta)-\lambda)^2+\varepsilon^2}|\hat{f}_{1j}(\eta;\lambda\pm i\varepsilon)|^2\,d\lambda$$
$$+\sum_{j\in M}\int_{\mathbf{R}^3}d\eta\int_a^b\frac{\varepsilon}{(\lambda_k(\eta)-\lambda)^2+\varepsilon^2}|\hat{f}_{2k}(\eta;\lambda\pm i\varepsilon)|^2\,d\lambda\Bigg),$$

since from (6.5) $\sum_{j\in M}\frac{\hat{f}_{1j}(\eta;\zeta)}{\lambda_j(\eta)-\zeta}+\sum_{k\in N}\frac{\hat{f}_{2k}(\eta;\zeta)}{\lambda_k(\eta)-\zeta}$ is a continuous L^2-valued function of ζ for $\mathrm{Im}\zeta\neq 0$. □

Using the generalized eigenfunctions $\psi_{1j}^{\pm}(x_3,\eta)$ (given by (4.9)-(4.16)) and $\psi_{1j}^{St}(x_3,\eta)$ (given by (4.17)-(4.18)) ($j\in M$) for $A_1(\eta')$ and $\psi_{2k}^{\pm}(x_3,\eta)$ (given by (5.6) -(5.13)) ($k\in N$) for $A_2(\eta')$, we define the generalized eigenfunctions $\psi_{1j}^{\pm}(x,\eta)$, $\psi_{1j}^{St}(x,\eta)$ ($j\in M$) and $\psi_{2k}^{\pm}(x,\eta)$ ($k\in N$) for A by

$$\psi_{1j}^{\pm}(x,\eta)=\begin{cases}\psi_{1j}^{\pm I}(x,\eta), & x\in\mathbf{R}^3_-,\\ \psi_{1j}^{\pm II}(x,\eta), & x\in\mathbf{R}^3_+,\end{cases}\tag{6.9}$$

where

$$\psi_{1j}^{\pm I}(x,\eta)=\frac{1}{2\pi}e^{i(x_1\eta_1+x_2\eta_2)}\mathrm{UC}(\psi_{1j}^{\pm I}(x_3,\eta)\oplus O_{1\times 1}),\quad x\in\mathbf{R}^3_-,$$
$$\psi_{1j}^{\pm II}(x,\eta)=\frac{1}{2\pi}e^{i(x_1\eta_1+x_2\eta_2)}\mathrm{UC}(\psi_{1j}^{\pm II}(x_3,\eta)\oplus O_{1\times 1}),\quad x\in\mathbf{R}^3_+,$$

$$\psi_{1j}^{St}(x,\eta)=\begin{cases}\psi_{1j}^{StI}(x,\eta), & x\in\mathbf{R}^3_-,\\ \psi_{1j}^{tII}(x,\eta), & x\in\mathbf{R}^3_+,\end{cases}\tag{6.10}$$

where

$$\psi_{1j}^{StI}(x,\eta)=\frac{1}{2\pi}e^{i(x_1\eta_1+x_2\eta_2)}\mathrm{UC}(\psi_{1j}^{StI}(x_3,\eta)\oplus O_{1\times 1}),\quad x\in\mathbf{R}^3_-,$$

$$\psi_{1j}^{StII}(x,\eta) = \frac{1}{2\pi} e^{i(x_1\eta_1+x_2\eta_2)} \mathrm{UC}(\psi_{1j}^{StII}(x_3,\eta) \oplus O_{1\times 1}), \quad x \in \mathbf{R}^3_+,$$

and

$$\psi_{2k}^{\pm}(x,\eta) = \begin{cases} \psi_{2k}^{\pm I}(x,\eta), & x \in \mathbf{R}^3_-, \\ \psi_{2k}^{\pm II}(x,\eta), & x \in \mathbf{R}^3_+, \end{cases} \tag{6.11}$$

where

$$\psi_{2k}^{\pm I}(x,\eta) = \frac{1}{2\pi} e^{i(x_1\eta_1+x_2\eta_2)} \mathrm{UC}(O_{2\times 2} \oplus \psi_{2k}^{\pm I}(x_3,\eta)),\ x \in \mathbf{R}^3_-,$$
$$\psi_{2k}^{\pm II}(x,\eta) = \frac{1}{2\pi} e^{i(x_1\eta_1+x_2\eta_2)} \mathrm{UC}(O_{2\times 2} \oplus \psi_{2k}^{\pm II}(x_3,\eta)),\ x \in \mathbf{R}^3_+.$$

Here we consider the case where the Stoneley wave exists, i.e., $D(c_{s_1}^2) > 0$ if $c_{s_1} < c_{s_2}$ and that $D(c_{s_2}^2) > 0$ if $c_{s_2} < c_{s_1}$ as shown in Section 3. Note that there is no term $\psi_{1j}^{St}(x,\eta)$ $(j \in M)$, if Stoneley wave does not exist.

Then we easily have the following proposition.

Proposition 6.3. *Let $x \in \mathbf{R}^3$ and $\eta \in \mathbf{R}^3$ $(\eta \neq 0)$. Then we have:*
(1) $\psi_{1j}^{\pm}(x,\eta)(j \in M)$ belong to $\mathcal{H}_{loc} = L^2_{loc}(\mathbf{R}^3, \mathbf{C}^3, \rho(x_3)\,dx)$ and

$$\begin{aligned} &A\psi_{1j}^{\pm}(x,\eta) = \lambda_j(\eta)\psi_{1j}^{\pm}(x,\eta), \\ &\psi_{1j}^{\pm I}(x,\eta)\big|_{x_3=0} = \psi_{1j}^{\pm II}(x,\eta)\big|_{x_3=0}, \\ &\sigma_{i3}(\psi_{1j}^{\pm I}(x,\eta))\big|_{x_3=0} = \sigma_{i3}(\psi_{1j}^{\pm II}(x,\eta))\big|_{x_3=0}. \end{aligned}$$

(2) $\psi_{1j}^{St}(x,\eta)(j \in M)$ belong to $\mathcal{H}_{loc}$ and

$$\begin{aligned} &A\psi_{1j}^{St}(x,\eta) = c_{St}^2|\eta'|^2\psi_{1j}^{St}(x,\eta), \\ &\psi_{1j}^{StI}(x,\eta)\big|_{x_3=0} = \psi_{1j}^{StII}(x,\eta)\big|_{x_3=0}, \\ &\sigma_{i3}(\psi_{1j}^{StI}(x,\eta))\big|_{x_3=0} = \sigma_{i3}(\psi_{1j}^{StII}(x,\eta))\big|_{x_3=0}. \end{aligned}$$

(3) $\psi_{2k}^{\pm}(x,\eta)(k \in N)$ belong to $\mathcal{H}_{loc}$ and

$$\begin{aligned} &A\psi_{2k}^{\pm}(x,\eta) = \lambda_k(\eta)\psi_{2k}^{\pm}(x,\eta), \\ &\psi_{2k}^{\pm I}(x,\eta)\big|_{x_3=0} = \psi_{2k}^{\pm II}(x,\eta)\big|_{x_3=0}, \\ &\sigma_{i3}(\psi_{2k}^{\pm I}(x,\eta))\big|_{x_3=0} = \sigma_{i3}(\psi_{2k}^{\pm II}(x,\eta))\big|_{x_3=0}. \end{aligned}$$

Proposition 6.3 means that $\{\psi_{1j}^{+}(x,\eta),\ \psi_{1j}^{St}(x,\eta)(j \in M),\ \psi_{2k}^{+}(x,\eta)(k \in N)\}$ and $\{\psi_{1j}^{-}(x,\eta),\ \psi_{1j}^{St}(x,\eta)(j \in M),\ \psi_{2k}^{-}(x,\eta)(k \in N)\}$ are two families of generalized eigenfunctions for the operator A. One is a family of outgoing eigenfunctions, and the other is a family of incoming eigenfunctions. We shall show later the completeness of each family.

In order to obtain the desired representation of the spectral family, it remains to pass to the limit under the integral sign over $\mathbf{R}^3$ in (6.8) and evaluate the limit of the integral over (a,b). For $f \in \mathcal{H}$, we define the Fourier components with respect to the generalized eigenfunctions $\psi_{1j}^{\pm}(x,\eta)$, $\psi_{1j}^{St}(x,\eta)$ $(j \in M)$ and $\psi_{2k}^{\pm}(x,\eta)(k \in N)$ for A by

$$\hat{f}_{1j}^{\pm}(\eta) = \underset{R\to\infty}{\mathrm{l.i.m.}} \int_{|x|\le R} \psi_{1j}^{\pm}(x,\eta)^* f(x)\rho(x_3)\,dx, \quad j \in M, \tag{6.12}$$

$$\hat{f}_{1j}^{St}(\eta) = \underset{R\to\infty}{\mathrm{l.i.m.}} \int_{|x|\le R} \psi_{1j}^{St}(x,\eta)^* f(x)\rho(x_3)\,dx, \quad j \in M, \tag{6.13}$$

$$\hat{f}_{2k}^{\pm}(\eta) = \underset{R\to\infty}{\mathrm{l.i.m.}} \int_{|x|\le R} \psi_{2k}^{\pm}(x,\eta)^* f(x)\rho(x_3)\,dx, \quad k \in N. \tag{6.14}$$

Then the mapping $f \mapsto (\hat{f}_{1j}^{\pm},\ \hat{f}_{1j}^{St},\ \hat{f}_{2k}^{\pm})$ may be considered as the generalized Fourier transform of f.

The following lemma gives the representation of the spectral family of A by means of the generalized eigenfunctions of A.

Lemma 6.4. *We assume that $D(c_{s_1}^2) > 0$ if $c_{s_1} < c_{s_2}$ and that $D(c_{s_2}^2) > 0$ if $c_{s_2} < c_{s_1}$. Let $f \in C_0^\infty(\mathbf{R}^3, \mathbf{C}^3)$ and $0 < a < b < \infty$. Then we have*

$$\begin{aligned}
&\left(\left(\frac{\pi(b)+\pi(b-)}{2} - \frac{\pi(a)+\pi(a-)}{2}\right) f, f\right) \\
&= \sum_{j\in M}\left(\int_{a\le\lambda_j(\eta)\le b} |\hat{f}_{1j}^{\pm}(\eta)|^2\,d\eta + \int_{a\le c_{St}^2|\eta'|^2\le b} |\hat{f}_{1j}^{St}(\eta)|^2\,d\eta\right) \\
&+ \sum_{k\in N}\int_{a\le\lambda_k(\eta)\le b} |\hat{f}_{2k}^{\pm}(\eta)|^2\,d\eta.
\end{aligned} \tag{6.15}$$

Remark. Under the following two conditions (i. e., Stoneley waves does not exist), the formula (6.15) holds without the second terms on the right hand side:

(i) If $c_{s_1} < c_{s_2}$, then either $D(c_{s_1}^2) < 0$ or $D(c_{s_1}^2) = 0$ and the expression (6.23) below does not vanish.

(ii) If $c_{s_2} < c_{s_1}$, then either $D(c_{s_2}^2) < 0$ or $D(c_{s_2}^2) = 0$ and the expression (6.23), exchanged with c_{s_1} and c_{s_2}, does not vanish.

Proof. The essential part of the proof of this lemma is to justify the passage to the limit under the integral sign over $\mathbf{R}^3$ in (6.8).

First of all, from the definition of $\psi_{1j}(x,\eta;\zeta)(j \in M)$

(6.16)

$$\begin{aligned}
\hat{f}_{1j}(\eta;\zeta) &= \int_{\mathbf{R}^3} \psi_{1j}(x,\eta;\zeta)^* f(x)\rho(x_3)\,dx \\
&= \frac{1}{2\pi}\int_{\mathbf{R}^3_-} e^{-i(x_1\eta_1+x_2\eta_2)}(\psi_{1j}^I(x_3,\eta;\zeta)^* \oplus O_{1\times1})(\mathrm{UC})^{-1} f(x)\rho_1\,dx
\end{aligned}$$

$$+\frac{1}{2\pi}\int_{\mathbf{R}^3_+} e^{-i(x_1\eta_1+x_2\eta_2)}(\psi^{II}_{1j}(x_3,\eta;\zeta)^*\oplus O_{1\times 1})(\mathrm{UC})^{-1}f(x)\rho_2\,dx$$
$$=\int_{\mathbf{R}_-}(\psi^{I}_{1j}(x_3,\eta;\zeta)^*\oplus O_{1\times 1})(\mathrm{UC})^{-1}(F_{x'}f)(\eta',x_3)\rho_1\,dx_3$$
$$+\int_{\mathbf{R}_+}(\psi^{II}_{1j}(x_3,\eta;\zeta)^*\oplus O_{1\times 1})(\mathrm{UC})^{-1}(F_{x'}f)(\eta',x_3)\rho_2\,dx_3.$$

By the expressions (4.2),(4.7), and (4.8), the integrands in the last two integrals of the right-hand side of (6.16) are summable with respect to x_3. Furthermore, using the inequality $|\alpha||\beta|\le(|\alpha|^2+|\beta|^2)/2$, we have

(6.17)
$$|\hat{f}_{1j}(\eta;\zeta)|^2\le C\sum_{\ell=1}^{4}\sum_{m\in M}\left(\left|\frac{1}{\tau_m(\eta)\zeta}\frac{\lambda_m(\eta)-\zeta}{\xi-\tau_m(\eta)}\right|^2\right.$$
$$\left.+\left|\frac{\Delta^{\ell}_m(\eta',\zeta)}{\Delta(\eta',\zeta)}\frac{1}{\tau_m(\eta)\zeta}\frac{\lambda_m(\eta)-\zeta}{\xi-\tau_m(\eta)}\right|^2\right)|g(\eta')|^2,$$

where C is a positive constant and $g(\eta')$ is a rapidly decreasing function with respect to η'.

Next, we consider justifying the passage to the limit under the integral sign over $\mathbf{R}^3$. We separate the integral as follows:

$$\int_{\mathbf{R}^3}\left(\int_a^b\frac{\varepsilon}{(\lambda_j(\eta)-\lambda)^2+\varepsilon^2}|\hat{f}_{1j}(\eta;\lambda\pm i\varepsilon)|^2\,d\lambda\right)d\eta$$
$$\int_{\{|\eta|<3R\}\cap\{|\xi|>\delta\}}\int_a^b d\lambda\,d\eta+\int_{\{|\eta|<3R\}\cap\{|\xi|<\delta\}}\int_a^b d\lambda\,d\eta+\int_{\{|\eta|>3R\}}\int_a^b d\lambda\,d\eta$$
$$=I^1_{R_j}(\varepsilon)+I^2_{R_j}(\varepsilon)+I^3_{R_j}(\varepsilon).$$

We divide the proof into three steps. Without loss of generality, we can assume that $c_{s_1}<c_{s_2}$, replacing c_{s_1} by c_{s_2} when $c_{s_2}<c_{s_1}$.
Step 1. First we consider $I^1_{R_j}(\varepsilon)$. Since

$$c^2_{St}|\eta'|^2<c^2_{s_1}(|\eta'|^2+\xi^2)=\lambda_{s_1}(\eta)$$

for $|\xi|>\delta$, the limits $\hat{f}_{1j}(\eta;\lambda\pm i0)$ exist when $\lambda=\lambda_j(\eta)$ and are continuous in η. So, for any ε such that $0<\varepsilon\le\varepsilon_\circ$

$$\int_{\lambda_j(\eta)-\delta_1}^{\lambda_j(\eta)+\delta_1}\frac{\varepsilon}{(\lambda_j(\eta)-\lambda)^2+\varepsilon^2}|\hat{f}_{1j}(\eta;\lambda\pm i\varepsilon)|^2\,d\lambda$$
$$\le c_1\int_{\lambda_j(\eta)-\delta_1}^{\lambda_j(\eta)+\delta_1}\frac{\varepsilon}{(\lambda_j(\eta)-\lambda)^2+\varepsilon^2}\,d\lambda$$
$$\le c_1\int_{-\frac{\delta_1}{\varepsilon}}^{\frac{\delta_1}{\varepsilon}}\frac{1}{x^2+1}\,dx$$

$$\leq c_1 \pi,$$

where c_1 is a positive constant and independent of ε and η. Since

$$\begin{aligned} \hat{f}^{St}_{1j}(\eta;\zeta) &= \int_{\mathbf{R}^3} \psi^{St}_{1j}(x,\eta;\zeta)^* f(x)\rho(x_3)\,dx \\ &= \frac{\bar{\zeta} - c^2_{St}|\eta'|^2}{\bar{\zeta} - \lambda_j(\eta)} \hat{f}_{1j}(\eta;\zeta), \end{aligned}$$

the limits $\hat{f}^{St}_{1j}(\eta;\lambda \pm i0)$ exists when $\lambda = c^2_{St}|\eta'|^2$ and are continuous in η. So

$$\begin{aligned} &\int_{c^2_{St}|\eta'|^2-\delta_2}^{c^2_{St}|\eta'|^2+\delta_2} \frac{\varepsilon}{(\lambda_j(\eta)-\lambda)^2+\varepsilon^2} |\hat{f}_{1j}(\eta;\lambda \pm i\varepsilon)|^2\,d\lambda \\ &\quad= \int_{c^2_{St}|\eta'|^2-\delta_2}^{c^2_{St}|\eta'|^2+\delta_2} \frac{\varepsilon}{(c^2_{St}|\eta'|^2-\lambda)^2+\varepsilon^2} |\hat{f}^{St}_{1j}(\eta;\lambda \pm i\varepsilon)|^2\,d\lambda \\ &\quad\leq c_2 \int_{c^2_{St}|\eta'|^2-\delta_2}^{c^2_{St}|\eta'|^2+\delta_2} \frac{\varepsilon}{(c^2_{St}|\eta'|^2-\lambda)^2+\varepsilon^2}\,d\lambda \\ &\quad\leq c_2\pi, \end{aligned}$$

where c_2 is a positive constant and independent of ε and η. It follows from the well-known formula of Cauchy (see, e. g., [15]) that

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\pi} \int_a^b \frac{\varepsilon}{(\lambda-\mu)^2+\varepsilon^2} f(\lambda)\,d\lambda = \chi_{(a,b)}(\mu) f(\mu),$$

that

$$\begin{aligned} &\lim_{\varepsilon\downarrow 0} \frac{1}{\pi} \int_a^b \frac{\varepsilon}{(\lambda_j(\eta)-\lambda)^2+\varepsilon^2} |\hat{f}_{1j}(\eta;\lambda \pm i\varepsilon)|^2\,d\lambda \\ &= \chi_{(a,b)}(\lambda_j(\eta)) \lim_{\varepsilon\downarrow 0} \frac{1}{\pi} \int_{\lambda_j(\eta)-\delta_1}^{\lambda_j(\eta)+\delta_1} \frac{\varepsilon}{(\lambda_j(\eta)-\lambda)^2+\varepsilon^2} |\hat{f}_{1j}(\eta;\lambda \pm i\varepsilon)|^2\,d\lambda \\ &+ \chi_{(a,b)}(c^2_{St}|\eta'|^2) \lim_{\varepsilon\downarrow 0} \frac{1}{\pi} \int_{c^2_{St}|\eta'|^2-\delta_2}^{c^2_{St}|\eta'|^2+\delta_2} \frac{\varepsilon}{(c^2_{St}|\eta'|^2-\lambda)^2+\varepsilon^2} |\hat{f}^{St}_{1j}(\eta;\lambda \pm i\varepsilon)|^2\,d\lambda \\ &= \chi_{(a,b)}(\lambda_j(\eta)) |\hat{f}_{1j}(\eta;\lambda_j(\eta) \pm i0)|^2 \\ &\quad + \chi_{(a,b)}(\lambda_j(\eta)) \lim_{\varepsilon\downarrow 0} \max_{\lambda\in(\lambda_j(\eta)-\delta_1,\lambda_j(\eta)+\delta_1)} |\hat{f}_{1j}(\eta;\lambda \pm i\varepsilon) - \hat{f}_{1j}(\eta;\lambda \pm i0)|^2 \\ &+ \chi_{(a,b)}(c^2_{St}|\eta'|^2) |\hat{f}^{St}_{1j}(\eta;c^2_{St}|\eta'|^2 \pm i0)|^2 \\ &\quad + \chi_{(a,b)}(c^2_{St}|\eta'|^2) \lim_{\varepsilon\downarrow 0} \max_{\lambda\in(c^2_{St}|\eta'|^2-\delta_2,c^2_{St}|\eta'|^2+\delta_2)} |\hat{f}^{St}_{1j}(\eta;\lambda \pm i\varepsilon) - \hat{f}^{St}_{1j}(\eta;\lambda \pm i0)|^2 \\ &= \chi_{(a,b)}(\lambda_j(\eta)) |\hat{f}^{\pm}_{1j}(\eta)|^2 + \chi_{(a,b)}(c^2_{St}|\eta'|^2) |\hat{f}^{St}_{1j}(\eta)|^2. \end{aligned}$$

By the Lebesgue bounded convergence theorem, we have

$$\lim_{\varepsilon\downarrow 0} I^1_{R_j}(\varepsilon)$$

$$= \int_{\{|\eta|<3R\}\cap\{|\xi|>\delta\}} \left(\lim_{\varepsilon\downarrow 0} \frac{1}{\pi} \int_a^b \frac{\varepsilon}{(\lambda_j(\eta)-\lambda)^2+\varepsilon^2} |\hat{f}_{1j}(\eta;\lambda\pm i\varepsilon)|^2 \, d\lambda \right) d\eta$$

$$= \int_{\{|\eta|<3R\}\cap\{|\xi|>\delta\}\cap\{a\le\lambda_j(\eta)\le b\}} |\hat{f}_{1j}^{\pm}(\eta)|^2 \, d\eta$$

$$+ \int_{\{|\eta|<3R\}\cap\{|\xi|>\delta\}\cap\{a\le c_{St}^2|\eta'|^2\le b\}} |\hat{f}_{1j}^{St}(\eta)|^2 \, d\eta.$$

Step 2. Next we consider the $I_{Rj}^2(\varepsilon)$. The principal difficulty in interchanging limit and integration occurs when the zeros of the denominators of $\frac{\varepsilon}{|\zeta-\lambda_j(\eta)|^2}$ and the zero of the Lopatinski determinant $\Delta(\eta',\zeta)$ nearly coincide, that is, when $c_{St}=c_{s_1}$. This is the case where $D(c_{s_1}^2)=0$.

1) Consider the first term of the right-hand side of (6.17). If sign ξ is the same as sign τ_{s_1}, then

$$\frac{1}{\tau_{s_1}(\xi-\tau_{s_1})} = \frac{2}{(\xi-\tau_{s_1})(\xi+\tau_{s_1})} + \frac{1}{\tau_{s_1}(\xi+\tau_{s_1})},$$

we have

$$\frac{1}{|\tau_{s_1}||\xi-\tau_{s_1}|} \le \frac{2}{|\lambda_{s_1}(\eta)-\zeta|} + \frac{1}{|\tau_{s_1}|^2},$$

hence

$$\lim_{\varepsilon\downarrow 0} \int_a^b \frac{\varepsilon}{|\zeta-\lambda_{s_1}(\eta)|^2} \left| \frac{1}{\tau_{s_1}\zeta} \frac{\lambda_{s_1}(\eta)-\zeta}{\xi-\tau_{s_1}} \right|^2 |g(\eta')|^2 \, d\lambda$$

$$\le c' \lim_{\varepsilon\downarrow 0} \left(\int_a^b \frac{1}{\lambda^2} \frac{\varepsilon}{(\lambda_{s_1}(\eta)-\lambda)^2+\varepsilon^2} |g(\eta')|^2 \, d\lambda \right.$$

$$\left. + \int_a^b \frac{1}{\lambda^2} \frac{\varepsilon}{(c_{s_1}^2|\eta'|^2-\lambda)^2+\varepsilon^2} |g(\eta')|^2 \, d\lambda \right)$$

$$\le c_3\pi.$$

If sign ξ is different from sign τ_{s_1}, then

$$\frac{1}{|\tau_{s_1}||\xi-\tau_{s_1}|} \le \frac{1}{|\tau_{s_1}|^2},$$

hence

$$\lim_{\varepsilon\downarrow 0} \int_a^b \frac{\varepsilon}{|\zeta-\lambda_{s_1}(\eta)|^2} \left| \frac{1}{\tau_{s_1}\zeta} \frac{\lambda_{s_1}(\eta)-\zeta}{\xi-\tau_{s_1}} \right|^2 |g(\eta')|^2 \, d\lambda \le c_4\pi.$$

Here c_3, c_4 are positive constants and independent of ε and η.

2) Consider the second term of the right-hand side of (6.17). By the change of variable $z=\frac{\zeta}{|\eta'|^2}$, we have

$$\frac{\Delta_{s_1}^{\ell}(\eta',\zeta)}{\Delta(\eta',\zeta)} = \frac{D_{s_1}^{\ell}(z)}{D(z)}, \quad \ell=1,2,3,4,$$

since $\Delta(\eta',\zeta) = |\eta'|^6 D(z)$ and $\Delta^{\ell}_{s_1}(\eta',\zeta) = |\eta'|^6 D^{\ell}_{s_1}(z)$, where

$$(6.18)\quad D^1_{s_1}(z) = 2ib_1\left[2(\mu_1-\mu_2)a_2b_2\left(2(\mu_1-\mu_2)-\frac{\mu_1 z}{c^2_{s_1}}\right)\right. \\ \left. -\left(2(\mu_1-\mu_2)-\frac{\mu_1 z}{c^2_{s_1}}+\frac{\mu_2 z}{c^2_{s_2}}\right)\left(2(\mu_1-\mu_2)+\frac{\mu_2 z}{c^2_{s_2}}\right)\right],$$

$$(6.19)\quad D^2_{s_1}(z) = \left(2(\mu_1-\mu_2)-\frac{\mu_1 z}{c^2_{s_1}}+\frac{\mu_2 z}{c^2_{s_2}}\right)^2 - 4(\mu_1-\mu_2)^2 a_1a_2b_1b_2 \\ + a_1b_1\left(2(\mu_1-\mu_2)+\frac{\mu_2 z}{c^2_{s_2}}\right)^2 - a_2b_2\left(2(\mu_1-\mu_2)-\frac{\mu_1 z}{c^2_{s_1}}\right)^2 \\ - \frac{\mu_1\mu_2}{c^2_{s_1}c^2_{s_2}}(a_1b_2-a_2b_1)z^2,$$

$$(6.20)\quad D^3_{s_1}(z) = 2ib_1\left[2a_1b_2\mu_1(\mu_2-\mu_1)\left(4-\frac{z}{c^2_{s_1}}\right)\right. \\ \left. -\frac{\mu_1 z}{c^2_{s_1}}\left(2(\mu_1-\mu_2)-\frac{\mu_1 z}{c^2_{s_1}}+\frac{\mu_2 z}{c^2_{s_2}}\right)\right],$$

$$(6.21)\quad D^4_{s_1}(z) = -2b_1\frac{\mu_1 z}{c^2_{s_1}}(a_1+a_2)\left(2(\mu_1-\mu_2)-\frac{\mu_1 z}{c^2_{s_1}}\right).$$

We have the following asymptotic formulas for a_1, b_1, a_2, and b_2 as functions of $t = \sqrt{z-c^2_{s_1}}$, respectively,

$$a_1 = \frac{1}{c_{p_1}}\sqrt{-t^2+c^2_{p_1}-c^2_{s_1}} = \frac{\sqrt{c^2_{p_1}-c^2_{s_1}}}{c^2_{p_1}}\left(1-\frac{t^2}{2(c^2_{p_1}-c^2_{s_1})}+O(t^4)\right)\ (t\to 0),$$
$$b_1 = \frac{1}{c_{s_1}}it \qquad (t\to 0),$$
$$a_2 = \frac{1}{c_{p_2}}\sqrt{-t^2+c^2_{p_2}-c^2_{s_1}} = \frac{\sqrt{c^2_{p_2}-c^2_{s_1}}}{c^2_{p_2}}\left(1-\frac{t^2}{2(c^2_{p_2}-c^2_{s_1})}+O(t^4)\right)\ (t\to 0),$$
$$b_2 = \frac{1}{c_{s_2}}\sqrt{-t^2+c^2_{s_2}-c^2_{s_1}} = \frac{\sqrt{c^2_{s_2}-c^2_{s_1}}}{c^2_{s_2}}\left(1-\frac{t^2}{2(c^2_{s_2}-c^2_{s_1})}+O(t^4)\right)\ (t\to 0).$$

If we put $\tilde{D}(t) = D(z(t))$, $\tilde{D}(0) = 0$ by assumption, that is,

$$(6.22)\quad (\mu_1-\mu_2)^2 - \frac{\sqrt{c^2_{p_2}-c^2_{s_1}}}{c_{p_2}}\frac{\sqrt{c^2_{s_2}-c^2_{s_1}}}{c_{s_2}}(\mu_1-2\mu_2)^2 \\ - \mu_1\mu_2\frac{\sqrt{c^2_{p_1}-c^2_{s_1}}}{c_{p_1}}\frac{\sqrt{c^2_{s_2}-c^2_{s_1}}}{c_{s_2}}\frac{c^2_{s_1}}{c^2_{s_2}} = 0,$$

and the coefficient of t in $\tilde{D}(t)$ is given in the form

$$\text{(6.23)}\quad \left.\frac{\tilde{D}(t)}{t}\right|_{t=0} = \tilde{D}_1(t) = 4(\mu_1-\mu_2)^2\frac{\sqrt{c_{p_1}^2-c_{s_1}^2}}{c_{p_1}}\frac{\sqrt{c_{p_2}^2-c_{s_1}^2}}{c_{p_2}}\frac{\sqrt{c_{s_2}^2-c_{s_1}^2}}{c_{s_2}}\frac{1}{c_{s_1}} - \frac{\sqrt{c_{p_1}^2-c_{s_1}^2}}{c_{p_1}}\frac{1}{c_{s_1}}\left(2(\mu_1-\mu_2)+\frac{c_{s_1}^2}{c_{s_2}^2}\mu_2\right)^2 - \mu_1\mu_2\frac{\sqrt{c_{p_1}^2-c_{s_1}^2}}{c_{p_1}}\frac{1}{c_{s_1}}\frac{c_{s_1}^2}{c_{s_2}^2}.$$

From now on, we shall consider only the case where $\tilde{D}_1(0)$ is not 0. On the other hand, we can see that

$$\text{(6.24)}\quad \tilde{D}_{s_1}^{\ell}(t) = D_{s_1}^{\ell}(z(t)) = \text{const.}\times t + O(t^2) \quad \text{as } t\to 0 \quad (\ell = 1,2,3,4),$$

hence the functions $D_{s_1}^{\ell}(z)/D(z)$ are bounded when z varies near $c_{s_1}^2$. Thus

$$\lim_{\varepsilon\downarrow 0}\int_a^b \frac{\varepsilon}{|\zeta-\lambda_{s_1}(\eta)|^2}\left|\frac{\Delta_{s_1}^{\ell}(\eta';\zeta)}{\Delta(\eta';\zeta)}\frac{1}{\tau_{s_1}\zeta}\frac{\lambda_{s_1}(\eta)-\zeta}{\xi-\tau_{s_1}}\right|^2 |g(\eta')|^2\,d\lambda \le c_5\pi,$$

where c_5 is a positive constant and independent of ε and η. This means that in the case where $D(c_{s_1}^2) = 0$, the Lopatinski determinant has no zeros under the condition that the expression (6.23) does not equal 0. Therefore, by the Lebesgue bounded convergence theorem, we have

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}I_{R_j}^2(\varepsilon) &= \int_{\{|\eta|<3R\}\cap\{|\xi|<\delta\}} c|g(\eta')|^2\,d\eta \\ &\le \int_{|\eta'|<3R} c\delta|g(\eta')|^2\,d\eta' \\ &\to 0 \qquad\qquad \text{as}\quad \delta\to 0.\end{aligned}$$

Step 3. Finally we consider $I_{R_j}^3(\varepsilon)$. Divide the domain of integration as follows:

$$\text{(6.25)}\quad I_{R_j}^3(\varepsilon) = \int_{\{|\eta|>3R\}\cap\{|\eta'|>2R\}}\int_a^b d\lambda\,d\eta + \int_{\{|\eta|>3R\}\cap\{|\eta'|<2R\}}\int_a^b d\lambda\,d\eta.$$

1) Let us consider the first term of the right-hand side of (6.25). Take R such that $R > \max\{\frac{1}{3c_{1_j}^2}, \frac{1}{4c_{S_t}^2}\}$. Since $R > \frac{1}{3c_{1_j}^2}$, we have

$$|\xi\pm\tau_j| \ge |\xi\pm iR| = \sqrt{\xi^2+R^2},$$

and

$$\left|\frac{1}{\tau_j\zeta}\frac{1}{\xi-\tau_j}\right|^2 \le \left|\frac{1}{Ra}\frac{1}{\sqrt{\xi^2+R^2}}\right|^2 \le \frac{1}{R}\frac{k}{\xi^2+R^2} \quad (\frac{1}{Ra}\le k).$$

Since $g(\eta')$ is a rapidly decreasing function of η', it follows that

$$\begin{aligned}
&\int_{\{|\eta|>3R\}\cap\{|\eta'|>2R\}} \int_a^b \frac{\varepsilon}{|\zeta-\lambda_j(\eta)|^2} \left|\frac{1}{\tau_j\zeta}\frac{\lambda_j(\eta)-\zeta}{\xi-\tau_j}\right|^2 |g(\eta')|^2\,d\lambda\,d\eta \\
&\leq \int_{\{|\eta|>3R\}\cap\{|\eta'|>2R\}} \int_a^b \frac{k}{R(\xi^2+R^2)} |g(\eta')|^2\,d\lambda\,d\eta \\
&\leq \varepsilon \int_{|\eta'|>2R} \left(\int_R \frac{k}{(\xi^2+R^2)}\,d\xi\right) |g(\eta')|^2\,d\eta' \\
&\leq \varepsilon \frac{c_6}{R}\pi,
\end{aligned}$$

where c_6 is a positive constant and independent of ε and η. Since $R > \frac{1}{4c_{St}^2}$, the zero of $\Delta(\eta', \zeta)$ does not exist in $0 < \lambda < R$ ($\zeta = \lambda + i\varepsilon$).
2) As to the second term of the right-hand side of (6.25). Since $|\xi| > R$ and we have, taking $R > \frac{1}{3c_j^2}$,

$$|\xi-\tau_j|^2 \geq \left|\xi - \sqrt{\frac{\lambda}{c_j^2} - |\eta'|^2}\right|^2 \geq \left|\frac{\xi^2 - \frac{\xi^2}{2}}{\xi + \sqrt{\frac{\lambda}{c_j^2} - |\eta'|^2}}\right|^2 \geq \left|\frac{\frac{\xi^2}{2}}{\xi + \sqrt{\frac{R^2}{2}}}\right|^2 \geq c\xi^2,$$

it follows that

$$\begin{aligned}
&\int_{\{|\eta|>3R\}\cap\{|\eta'|<2R\}} \int_a^b \frac{\varepsilon}{|\zeta-\lambda_j(\eta)|^2} \left|\frac{1}{\tau_j\zeta}\frac{\lambda_j(\eta)-\zeta}{\xi-\tau_j}\right|^2 |g(\eta')|^2\,d\lambda\,d\eta \\
&\leq \int_{\{|\eta|>3R\}\cap\{|\eta'|<2R\}} \frac{c_j^2}{c\xi^2}\frac{1}{a^2} \left(\int_a^b \frac{\varepsilon}{\lambda - c_j^2|\eta'|^2}\,d\lambda\right) |g(\eta')|^2\,d\eta \\
&\leq \int_{|\eta'|<2R} c'\varepsilon\frac{1}{R^2}\log(b-a)|g(\eta')|^2\,d\eta' \\
&\leq \varepsilon\frac{c_7}{R^2},
\end{aligned}$$

where c_7 is a positive constant and independent of ε and η. The terms having $\Delta(\eta', \zeta)$ as the denominator may be estimated in the same way as in $I^2_{R_j}(\varepsilon)$.

Thus we have

$$\lim I^3_{R_j}(\varepsilon) \to 0 \qquad \text{as } R \to \infty.$$

The second term in the right hand side of (6.15) can be handled in the same way. The proof of Lemma 6.4 is now complete. □

We can easily extend the equation (6.15) for all $f \in \mathcal{H}$ and obtain $\pi(a) = \pi(a-)$, $a \neq 0$, making $b \downarrow a$ in (6.15).

Theorem 6.5. *We assume that $D(c_{s_1}^2) > 0$ if $c_{s_1} < c_{s_2}$ and that $D(c_{s_2}^2) > 0$ if*

$c_{s_2} < c_{s_1}$. *Let* $f,\ g \in \mathcal{H}$ *and* $0 < a < b < \infty$. *Then we have*

$$(6.26)\quad \begin{aligned}&((\pi(b)-\pi(a))f,g)\\ &= \sum_{j\in M}\left(\int_{a\le\lambda_j(\eta)\le b} \hat f^{\pm}_{1j}(\eta)\cdot \hat g^{\pm}_{1j}(\eta)\,d\eta + \int_{a\le c^2_{St}|\eta'|^2\le b} \hat f^{St}_{1j}(\eta)\cdot \hat g^{St}_{1j}(\eta)\,d\eta\right)\\ &\quad+\sum_{k\in N}\int_{a\le\lambda_k(\eta)\le b}\hat f^{\pm}_{2k}(\eta)\cdot\hat g^{\pm}_{2k}(\eta)\,d\eta,\end{aligned}$$

and the Parseval formula

$$(6.27)\quad \begin{aligned}(f,g) &= \sum_{j\in M}\left(\int_{\mathbf{R}^3}\hat f^{\pm}_{1j}(\eta)\cdot\hat g^{\pm}_{1j}(\eta)\,d\eta + \int_{\mathbf{R}^3}\hat f^{St}_{1j}(\eta)\cdot\hat g^{St}_{1j}(\eta)\,d\eta\right)\\ &\quad+\sum_{k\in N}\int_{\mathbf{R}^3}\hat f^{\pm}_{2k}(\eta)\cdot\hat g^{\pm}_{2k}(\eta)\,d\eta.\end{aligned}$$

Proof. It suffices to prove that $0 \ne \sigma_p(A)$, where $\sigma_p(A)$ denotes the point spectrum of A. The characteristic polynomials associated with $A^I_1(\eta', D)$ and $A^{II}_1(\eta', D)$ are $c^2_{p_1}c^2_{s_1}(\xi-|\eta'|)^2(\xi+|\eta'|)^2$ and $c^2_{p_2}c^2_{s_2}(\xi-|\eta'|)^2(\xi+|\eta'|)^2$, respectively, so

$$u^I = \begin{pmatrix}\alpha_1\\ -i\alpha_1\end{pmatrix} x_3 e^{|\eta'|x_3} + \begin{pmatrix}\alpha_2\\ i\frac{c^2_{p_1}+c^2_{s_1}}{(c^2_{p_1}-c^2_{s_1})|\eta'|}\alpha_1 - i\alpha_2\end{pmatrix} e^{|\eta'|x_3},\quad x_3<0,$$

$$u^{II} = \begin{pmatrix}\alpha_3\\ i\alpha_3\end{pmatrix} x_3 e^{-|\eta'|x_3} + \begin{pmatrix}\alpha_4\\ i\frac{c^2_{p_2}+c^2_{s_2}}{(c^2_{p_2}-c^2_{s_2})|\eta'|}\alpha_3 + i\alpha_4\end{pmatrix} e^{-|\eta'|x_3},\quad x_3>0.$$

Since u^I and u^{II} should satisfy the interface conditions, we have

$$\begin{pmatrix} 0 & 1 & 0 & -1\\ \frac{c^2_{p_1}+c^2_{s_1}}{(c^2_{p_1}-c^2_{s_1})|\eta'|} & -1 & -\frac{c^2_{p_2}+c^2_{s_2}}{(c^2_{p_2}-c^2_{s_2})|\eta'|} & -1\\ \frac{\rho_1 c^4_{s_1}}{c^2_{p_1}-c^2_{s_1}} & -\rho_1 c^2_{s_1}|\eta'| & -\frac{\rho_2 c^4_{s_2}}{c^2_{p_2}-c^2_{s_2}} & -\rho_2 c^2_{s_2}|\eta'|\\ \frac{\rho_1 c^2_{s_1}c^2_{p_1}}{c^2_{p_1}-c^2_{s_1}} & \rho_1 c^2_{s_1}|\eta'| & -\frac{\rho_2 c^2_{s_2}c^2_{p_2}}{c^2_{p_2}-c^2_{s_2}} & -\rho_2 c^2_{s_2}|\eta'| \end{pmatrix}\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{pmatrix} = 0.$$

The determinant of the 4×4 matrix above

$$\begin{aligned}\Delta = (\rho_1 c^2_{s_1})^2(c^2_{p_2}+c^2_{s_2}) + (\rho_2 c^2_{s_2})^2(c^2_{p_1}+c^2_{s_1})(c^2_{s_2}-c^2_{p_2})\\ - 2\rho_1 c^2_{s_1}\rho_2 c^2_{s_2}(c^2_{p_1}c^2_{p_2}+c^2_{s_1}c^2_{s_2})\end{aligned}$$

is negative, so $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ have only the trivial solution $(0,0,0,0)$. As to $A_2(\eta')$, $0\ne\sigma_p(A_2(\eta'))$, because the Lopatinski determinant for the problem (5.1), (5.2) and (5.3) has no zero with respect to ζ for $|\eta'|\ne 0$ as showed in Section 5.

§ 7. Eigenfunction Expansions for A

In this section, we prove the eigenfunction expansion theorem for A. To this end, we use the representation of the spectral family of A developed in Section 6. Throughout this section, we assume that $D(c_{s_1}^2) > 0$ if $c_{s_1} < c_{s_2}$ and that $D(c_{s_2}^2) > 0$ if $c_{s_2} < c_{s_1}$. Note that, under the following two conditions (i. e., Stoneley waves does not exist), the theorems in this section hold without the terms corresponding to the Stoneley waves:

(i) If $c_{s_1} < c_{s_2}$, then either $D(c_{s_1}^2) < 0$ or $D(c_{s_1}^2) = 0$ and the expression (6.23) does not vanish.

(ii) If $c_{s_2} < c_{s_1}$, then either $D(c_{s_2}^2) < 0$ or $D(c_{s_2}^2) = 0$ and the expression (6.23), exchanged with c_{s_1} and c_{s_2}, does not vanish.

Let us begin with definition of mappings needed to formulate and prove the expansion theorem.

Lemma 7.1. *We define the mappings by*

$$\Phi_{1j}^{\pm} : \mathcal{H} \ni f \to \hat{f}_{1j}^{\pm}(\eta) \in L^2(\mathbf{R}_{\pm}^3, \mathbf{C}^3),\ j \in M$$
$$\Phi_{1j}^{St} : \mathcal{H} \ni f \to \hat{f}_{1j}^{St}(\eta) \in L^2(\mathbf{R}^3, \mathbf{C}^3),\ j \in M$$
$$\Phi_{2k}^{\pm} : \mathcal{H} \ni f \to \hat{f}_{2k}^{\pm}(\eta) \in L^2(\mathbf{R}_{\pm}^3, \mathbf{C}^3),\ k \in N,$$

and put

$$\Phi^{\pm} = \sum_{j\in M} \Phi_{1j}^{\pm} \oplus \sum_{j\in M} \Phi_{1j}^{St} \oplus \sum_{k\in N} \Phi_{2k}^{\pm}.$$

Then we have

$$(P_j \oplus O_{1\times 1})\Phi_{1j}^{\pm} = \Phi_{1j}^{\pm}, \quad j \in M, \tag{7.1}$$
$$\Phi_{1j}^{\pm *}\Phi_{1l}^{\pm} = 0, \quad if \quad j \neq l, \tag{7.2}$$
$$(P_j \oplus O_{1\times 1})\Phi_{1j}^{St} = \Phi_{1j}^{St}, \quad j \in M, \tag{7.3}$$
$$\Phi_{1j}^{St *}\Phi_{1l}^{St} = 0, \quad if \quad j \neq l, \tag{7.4}$$
$$(O_{2\times 2} \oplus 1)\Phi_{2k}^{\pm} = \Phi_{2k}^{\pm}, \quad k \in N, \tag{7.5}$$
$$\Phi_{2k}^{\pm *}\Phi_{2i}^{\pm} = 0, \quad if \quad k \neq i, \tag{7.6}$$

Moreover $\Phi^{\pm}$ is an isometry, that is,

$$\Phi^{\pm *}\Phi^{\pm} = I_{\mathcal{H}}, \quad \Phi^{\pm}\Phi^{\pm *} = P^{\pm}, \tag{7.7}$$

where $P^{\pm}$ is the orthogonal projection in $L^2(\mathbf{R}_{\pm}^3, \mathbf{C}^3) \times L^2(\mathbf{R}^3, \mathbf{C}^3) \times L^2(\mathbf{R}_{\pm}^3, \mathbf{C}^3)$ whose range is equal to $R(\Phi^{\pm})$.

Proof. The formulas (7.1)-(7.6) follow immediately from the definition of $\Phi_{1j}^{\pm}$, Φ_{1j}^{St} and $\Phi_{2k}^{\pm}$, while the formula (7.7) follows from Theorem 6.5. □

The first half of next theorem expresses the Fourier inversion formula with respect to generalized eigenfunctions. The latter half gives the canonical form for A.

Theorem 7.2. *Let $f \in \mathcal{H}$.*

(1) The following expansion formula holds:

$$(7.8)\qquad f(x) = \sum_{j\in M} \operatorname*{l.i.m.}_{R\to\infty} \int_{|\eta|\le R} \left(\psi_{1j}^{\pm}(x,\eta)\hat{f}_{1j}^{\pm}(\eta) + \psi_{1j}^{St}(x,\eta)\hat{f}_{1j}^{St}(\eta)\right) d\eta$$
$$+ \sum_{k\in N} \operatorname*{l.i.m.}_{R\to\infty} \int_{|\eta|\le R} \psi_{2k}^{\pm}(x,\eta)\hat{f}_{2k}^{\pm}(\eta)\, d\eta.$$

(2) $f \in D(A)$ if and only if we have

$$\lambda_j(\eta)\hat{f}_{1j}^{\pm}(\eta), \in (P_j(\eta)\oplus O_{1\times1})L^2(\mathbf{R}^3_{\pm},\mathbf{C}^3),$$
$$c_{St}^2|\eta'|^2\hat{f}_{1j}^{St}(\eta) \in (P_j(\eta)\oplus O_{1\times1})L^2(\mathbf{R}^3,\mathbf{C}^3),$$
$$\lambda_k(\eta)\hat{f}_{2k}^{\pm}(\eta) \in (O_{2\times2}\oplus 1)L^2(\mathbf{R}^3_{\pm},\mathbf{C}^3).$$

Moreover, in this case, we have the following formulas:

(7.9)
$$Af(x) = \sum_{j\in M} \operatorname*{l.i.m.}_{R\to\infty} \int_{|\eta|\le R} \left(\lambda_j(\eta)\psi_{1j}^{\pm}(x,\eta)\hat{f}_{1j}^{\pm}(\eta) + c_{St}^2|\eta'|^2\psi_{1j}^{St}(x,\eta)\hat{f}_{1j}^{St}(\eta)\right) d\eta$$
$$+ \sum_{k\in N} \operatorname*{l.i.m.}_{R\to\infty} \int_{|\eta|\le R} \lambda_k(\eta)\psi_{2k}^{\pm}(x,\eta)\hat{f}_{2k}^{\pm}(\eta)\, d\eta,$$

and

$$(7.10)\qquad (\widehat{Af})_{1j}^{\pm}(\eta) = \lambda_j(\eta)\hat{f}_{1j}^{\pm}(\eta), \quad j\in M,$$
$$(7.11)\qquad (\widehat{Af})_{1j}^{St}(\eta) = c_{St}^2|\eta'|^2\hat{f}_{1j}^{St}(\eta), \quad j\in M,$$
$$(7.12)\qquad (\widehat{Af})_{2k}^{\pm}(\eta) = \lambda_k(\eta)\hat{f}_{2k}^{\pm}(\eta), \quad k\in N.$$

Proof. Let $f \in \mathcal{H}$ have a compact support, and $g \in C_0^\infty(\mathbf{R}^3_{\pm},\mathbf{C}^3)$. We have

$$\begin{aligned}
&\left(f, \Phi_{1j}^{\pm *} g\right)_{\mathcal{H}} \\
&= \left(\Phi_{1j}^{\pm} f, g\right)_{L^2(\mathbf{R}^3_{\pm},\mathbf{C}^3)} \\
&= \left(\psi_{1j}^{\pm *}(x,\eta)f(x), g(\eta)\right)_{\mathcal{H}\times L^2(\mathbf{R}^3_{\pm},\mathbf{C}^3)} \\
&= \left(f(x), \psi_{1j}^{\pm}(x,\eta)g(\eta)\rho(x_3)\right)_{\mathcal{H}\times L^2(\mathbf{R}^3_{\pm},\mathbf{C}^3)} \\
&= \int_{\mathbf{R}^3} f(x)\cdot\left(\int_{\mathbf{R}^3_{\pm}} \psi_{1j}^{\pm}(x,\eta)g(\eta)\,d\eta\right)\rho(x_3)\,dx,
\end{aligned}$$

and so

$$(7.13)\qquad \Phi_{1j}^{\pm *} g(x) = \int_{\mathbf{R}^3_{\pm}} \psi_{1j}^{\pm}(x,\eta)g(\eta)\,d\eta.$$

By virtue of the boundedness of $\Phi_{1j}^{\pm}$ and $\Phi_{1j}^{\pm *}$, we find that (7.13) holds for all $g \in L^2(\mathbf{R}_{\pm}^3, \mathbf{C}^3)$, where the the integrals are taken in the sense of the limit in the mean.

Similarly, we can verify that

$$\Phi_{1j}^{St*}g(x) = \operatorname*{l.i.m.}_{R\to\infty} \int_{|\eta|\le R} \psi_{1j}^{St}(x,\eta)g(\eta)\,d\eta, \quad \text{for} \quad g \in L^2(\mathbf{R}^3, \mathbf{C}^3),$$
$$\Phi_{2k}^{\pm *}g(x) = \operatorname*{l.i.m.}_{R\to\infty} \int_{|\eta|\le R} \psi_{2k}^{\pm}(x,\eta)g(\eta)\,d\eta, \quad \text{for} \quad g \in L^2(\mathbf{R}_{\pm}^3, \mathbf{C}^3).$$

Thus (7.8) follows from (7.2), (7.4), (7.6), and (7.7).

Next we prove the diagonal representation of A. From Theorem 6.5, we have

$$(\pi(\lambda)f,g) = \sum_{j\in M}\left(\int_{\lambda_j(\eta)\le\lambda} \hat{f}_{1j}^{\pm}(\eta)\cdot\hat{g}_{1j}^{\pm}(\eta)\,d\eta + \int_{c_{St}^2|\eta'|^2\le\lambda} \hat{f}_{1j}^{St}(\eta)\cdot\hat{g}_{1j}^{St}(\eta)\,d\eta\right)$$
$$+\sum_{k\in N}\int_{\lambda_k(\eta)\le\lambda} \hat{f}_{2k}^{\pm}(\eta)\cdot\hat{g}_{2k}^{\pm}(\eta)\,d\eta$$

for $f,\, g \in \mathcal{H}$. It is well known that $f \in D(A)$ if and only if

$$\int_{-\infty}^{\infty} \lambda^2\, d(\pi(\lambda)f,f) < \infty,$$

(e.g., [4]). Thus it is easy to see that $f \in D(A)$ if and only if

$$\hat{f}_{1j}^{\pm}(\eta),\ \lambda_j(\eta)\hat{f}_{1j}^{\pm}(\eta) \in (P_j(\eta)\oplus O_{1\times1})L^2(\mathbf{R}_{\pm}^3,\mathbf{C}^3), \quad j\in M,$$
$$\hat{f}_{1j}^{St}(\eta),\ c_{St}^2|\eta'|^2\hat{f}_{1j}^{St}(\eta) \in (P_j(\eta)\oplus O_{1\times1})L^2(\mathbf{R}^3,\mathbf{C}^3), \quad j\in M,$$
$$\hat{f}_{2k}^{\pm}(\eta),\ \lambda_k(\eta),\hat{f}_{2k}^{\pm}(\eta) \in (O_{2\times2}\oplus 1)L^2(\mathbf{R}_{\pm}^3,\mathbf{C}^3), \quad k\in N.$$

Let $\alpha_r(x)$ be a C^∞ real valued function such that $\alpha_r(x) = 1$ for $|x| < r$, $= 0$ for $|x| > r+1$. Let $f \in D(A)$,

$$\begin{aligned}(\widehat{Af})_{1j}^{\pm}(\eta) &= \operatorname*{l.i.m.}_{r\to\infty}\int_{\mathbf{R}^3} \psi_{1j}^{\pm}(x,\eta)^*\alpha_r(x)(Af)(x)\rho(x_3)\,dx \\
&= \operatorname*{l.i.m.}_{r\to\infty}\int_{\mathbf{R}_-^3} [A^I(\alpha_r(x)\psi_{1j}^{\pm I}(x,\eta))]^* f(x)\rho_1\,dx \\
&\qquad + \operatorname*{l.i.m.}_{r\to\infty}\int_{\mathbf{R}_+^3} [A^{II}(\alpha_r(x)\psi_{1j}^{\pm II}(x,\eta))]^* f(x)\rho_2\,dx \\
&= \operatorname*{l.i.m.}_{r\to\infty}\int_{\mathbf{R}_-^3} A^I\psi_{1j}^{\pm I}(x,\eta)^*\alpha_r(x)f(x)\rho_1\,dx \\
&\qquad + \operatorname*{l.i.m.}_{r\to\infty}\int_{\{x\in\mathbf{R}_-^3,\, r\le|x|\le r+1\}} [(A^I\alpha_r(x))\psi_{1j}^{\pm I}(x,\eta)]^* f(x)\rho_1\,dx \\
&\quad + \operatorname*{l.i.m.}_{r\to\infty}\int_{\mathbf{R}_+^3} A^{II}\psi_{1j}^{\pm II}(x,\eta)^*\alpha_r(x)f(x)\rho_2\,dx\end{aligned}$$

$$
\begin{aligned}
&+ \operatorname*{l.i.m.}_{r\to\infty} \int_{\{x\in \mathbf{R}^3_+, r\le |x|\le r+1\}} [(A^{II}\alpha_r(x))\psi_{1j}^{\pm II}(x,\eta)]^* f(x)\rho_2\,dx \\
&= \operatorname*{l.i.m.}_{r\to\infty} \int_{\mathbf{R}^3} A\psi_{1j}^{\pm}(x,\eta)^*\alpha_r(x) f(x)\rho(x_3)\,dx \\
&= \operatorname*{l.i.m.}_{r\to\infty} \int_{\mathbf{R}^3} \lambda_j \psi_{1j}^{\pm}(x,\eta)^*\alpha_r(x) f(x)\rho(x_3)\,dx \\
&= \lambda_j(\eta)\hat{f}_{1j}^{\pm}(\eta),
\end{aligned}
$$

where

$$
Au(x) = \begin{cases} A^I u(x) = M^I u(x), & x_3 < 0, \\ A^{II} u(x) = M^{II} u(x), & x_3 > 0, \end{cases}
$$

for $u \in D(A)$. This proves (7.10).

Similarly we can show (7.11) and (7.12), and thereby (7.9) follows. The proof of Theorem 7.2 is now complete. □

The following theorem gives an explicit expression of the ranges $R(\Phi^{\pm})$, $R(\Phi_{1j}^{\pm})$, $R(\Phi_{1j}^{St})$ and $R(\Phi_{2k}^{\pm})$.

Theorem 7.3. *(1) For $R(\Phi^{\pm})$, we have*

(7.14)

$$
\begin{aligned}
R(\Phi^{\pm}) = &\sum_{j\in M} \oplus (P_j(\eta) \oplus O_{1\times 1}) L^2(\mathbf{R}^3_{\pm}, \mathbf{C}^3) \oplus \sum_{j\in M} \oplus (P_j(\eta) \oplus O_{1\times 1}) L^2(\mathbf{R}^3, \mathbf{C}^3) \\
&\oplus \sum_{k\in N} \oplus (O_{2\times 2} \oplus 1) L^2(\mathbf{R}^3_{\pm}, \mathbf{C}^3).
\end{aligned}
$$

(2) For $R(\Phi_{1j}^{\pm})$, $R(\Phi_{1j}^{St})$ and $R(\Phi_{2k}^{\pm})$, we have

(7.15)

$$
R(\Phi_{1j}^{\pm}) = (P_j(\eta) \oplus O_{1\times 1}) L^2(\mathbf{R}^3_{\pm}, \mathbf{C}^3), \quad \Phi_{1j}^{\pm}\Phi_{1l}^{\pm *} = 0,\ j \ne l,
$$

(7.16)

$$
R(\Phi_{1j}^{St}) = (P_j(\eta) \oplus O_{1\times 1}) L^2(\mathbf{R}^3, \mathbf{C}^3), \quad \Phi_{1j}^{St}\Phi_{1l}^{St *} = 0,\ j \ne l,
$$

(7.17)

$$
R(\Phi_{2k}^{\pm}) = (O_{2\times 2} \oplus 1) L^2(\mathbf{R}^3_{\pm}, \mathbf{C}^3), \quad \Phi_{2k}^{\pm}\Phi_{2i}^{\pm *} = 0,\ k \ne i,
$$

The mappings $\Phi_{1j}^{\pm}$, Φ_{1j}^{St} and $\Phi_{2k}^{\pm}$ are partial isometries.

This implies that $\Phi^{\pm}$ are unitary operators in $\mathcal{H}$, and that the system of generalized eigenfunctions $\{\psi_{1j}^{\pm}, \psi_{1j}^{St}, \psi_{2k}^{\pm}\}$ is complete.

Proof. It suffices to prove that:

$$
\begin{aligned}
&g \in N(\Phi^{\pm *}) \cap \Big(\sum_{j\in M} \oplus (P_j(\eta) \oplus O_{1\times 1}) L^2(\mathbf{R}^3_{\pm}, \mathbf{C}^3) \\
&\qquad \oplus \sum_{j\in M} \oplus (P_j(\eta) \oplus O_{1\times 1}) L^2(\mathbf{R}^3, \mathbf{C}^3) \oplus \sum_{k\in N} \oplus (O_{2\times 2} \oplus 1) L^2(\mathbf{R}^3_{\pm}, \mathbf{C}^3) \Big) \\
&\Longrightarrow\ g \equiv 0.
\end{aligned}
$$

Let

$$g(\eta) \equiv g^{\pm}_{1s_1}(\eta) \oplus \cdots \oplus g^{\pm}_{1p_2}(\eta) \oplus g^{St}_{1s_1}(\eta) \oplus \cdots \oplus g^{St}_{1p_2}(\eta) \oplus g^{\pm}_{2s_1}(\eta) \oplus g^{\pm}_{2s_2}(\eta)$$
$$\in N(\Phi^{\pm *}) \cap \left(\sum_{j\in M} \oplus (P_j(\eta) \oplus O_{1\times 1}) L^2(\mathbf{R}^3_{\pm}, \mathbf{C}^3) \right.$$
$$\left. \oplus \sum_{j\in M} \oplus (P_j(\eta) \oplus O_{1\times 1}) L^2(\mathbf{R}^3, \mathbf{C}^3) \oplus \sum_{k\in N} \oplus (O_{2\times 2} \oplus 1) L^2(\mathbf{R}^3_{\pm}, \mathbf{C}^3) \right).$$

Then it follows that

$$0 = \Phi^{\pm *} g = \operatorname*{l.i.m.}_{N\to\infty} \sum_{j\in M} \int_{\mathbf{R}^3_{\pm}} \psi^{\pm}_{1j}(x,\eta) g_N(\eta)\, d\eta$$
$$+ \operatorname*{l.i.m.}_{N\to\infty} \sum_{j\in M} \int_{\mathbf{R}^3} \psi^{St}_{1j}(x,\eta) g_N(\eta)\, d\eta$$
$$+ \operatorname*{l.i.m.}_{N\to\infty} \sum_{k\in N} \int_{\mathbf{R}^3_{\pm}} \psi^{\pm}_{2k}(x,\eta) g_N(\eta)\, d\eta$$

where $g_N(\eta) = g(\eta)$ for $|\eta| < N$, $= 0$ for $|\eta| > N$. Hence, for non-real ζ, we have

(7.18) $$(\mathrm{UC})^{-1} F_{x'} (A - \zeta)^{-1} \Phi^{\pm *} g_N \to 0 \quad \text{in } L^2(\mathbf{R}^3, \mathbf{C}^3) \text{ as } N \to \infty.$$

Let $f \in L^2(\mathbf{R}^3, \mathbf{C}^3)$ such that $F^{-1}_{\eta'} f \in C_0^\infty(\mathbf{R}^3, \mathbf{C}^3)$. By (7.18) and (6.6), we have

$$\begin{aligned}
&\left(f, (\mathrm{UC})^{-1} F_{x'} (A - \zeta)^{-1} \Phi^{\pm *} g_N\right) \\
&= \left(f, ((A_1(\eta') - \zeta)^{-1} \oplus O_{1\times 1})(\mathrm{UC})^{-1} F_{x'} \Phi^{\pm *}_{1j} g_N\right) \\
&\quad + \left(f, ((A_1(\eta') - \zeta)^{-1} \oplus O_{1\times 1})(\mathrm{UC})^{-1} F_{x'} \Phi^{St *}_{1j} g_N\right) \\
&\quad + \left(f, (O_{2\times 2} \oplus (A_2(\eta') - \zeta)^{-1})(\mathrm{UC})^{-1} F_{x'} \Phi^{\pm *}_{2k} g_N\right) \\
&= \left(\Phi^{\pm}_{1j} F^{-1}_{\eta'} (\mathrm{UC})((A_1(\eta') - \bar\zeta)^{-1} \oplus O_{1\times 1}) f, g_N\right) \\
&\quad + \left(\Phi^{St}_{1j} F^{-1}_{\eta'} (\mathrm{UC})((A_1(\eta') - \bar\zeta)^{-1} \oplus O_{1\times 1}) f, g_N\right) \\
&\quad + \left(\Phi^{\pm}_{2k} F^{-1}_{\eta'} (\mathrm{UC})(O_{2\times 2} \oplus (A_2(\eta') - \bar\zeta)^{-1}) f, g_N\right) \\
&= \left(\sum_{j\in M} (\lambda_j(\eta) - \bar\zeta)^{-1} \Phi^{\pm}_{1j} F^{-1}_{\eta'} (\mathrm{UC}) f, g_N\right) \\
&\quad + \left(\sum_{j\in M} (c^2_{St} |\eta'|^2 - \bar\zeta)^{-1} \Phi^{St}_{1j} F^{-1}_{\eta'} (\mathrm{UC}) f, g_N\right) \\
&\quad + \left(\sum_{k\in N} (\lambda_k(\eta) - \bar\zeta)^{-1} \Phi^{\pm}_{2k} F^{-1}_{\eta'} (\mathrm{UC}) f, g_N\right) \\
&\to 0 \qquad\qquad \text{as } N \to \infty.
\end{aligned}$$

Thus

$$\begin{aligned}0 = \sum_{j\in M}\int_{\mathbf{R}^3_\pm} \frac{1}{\lambda_j(\eta)-\zeta}\,\Phi^{\pm}_{1j}F^{-1}_{\eta'}(\mathrm{UC})f\cdot g\,d\eta \\ + \sum_{j\in M}\int_{\mathbf{R}^3} \frac{1}{c^2_{St}|\eta'|^2-\zeta}\,\Phi^{St}_{1j}F^{-1}_{\eta'}(\mathrm{UC})f\cdot g\,d\eta \\ + \sum_{k\in N}\int_{\mathbf{R}^3_\pm} \frac{1}{\lambda_k(\eta)-\zeta}\,\Phi^{\pm}_{2k}F^{-1}_{\eta'}(\mathrm{UC})f\cdot g\,d\eta,\end{aligned}$$

and hence we obtain that

$$\begin{aligned}0 &= \sum_{j\in M}\int_{\mathbf{R}^3_\pm}\frac{\varepsilon}{\pi}\left(\int_a^b \frac{1}{\lambda_j(\eta)-\lambda}\,\Phi^{\pm}_{1j}F^{-1}_{\eta'}(\mathrm{UC})f\cdot g\,d\lambda\right)d\eta \\ &\quad + \sum_{j\in M}\int_{\mathbf{R}^3}\frac{\varepsilon}{\pi}\left(\int_a^b \frac{1}{c^2_{St}|\eta'|^2-\lambda}\,\Phi^{St}_{1j}F^{-1}_{\eta'}(\mathrm{UC})f\cdot g\,d\lambda\right)d\eta \\ &\quad + \sum_{k\in N}\int_{\mathbf{R}^3_\pm}\frac{\varepsilon}{\pi}\left(\int_a^b \frac{1}{\lambda_k(\eta)-\lambda}\,\Phi^{\pm}_{2k}F^{-1}_{\eta'}(\mathrm{UC})f\cdot g\,d\lambda\right)d\eta \\ &= \sum_{j\in M}\left(\int_{a\le\lambda_j(\eta)\le b}\Phi^{\pm}_{1j}F^{-1}_{\eta'}(\mathrm{UC})f\cdot g\,d\eta + \int_{a\le c^2_{St}|\eta'|^2\le b}\Phi^{St}_{1j}F^{-1}_{\eta'}(\mathrm{UC})f\cdot g\,d\eta\right) \\ &\quad + \sum_{k\in N}\int_{a\le\lambda_k(\eta)\le b}\Phi^{\pm}_{2k}F^{-1}_{\eta'}(\mathrm{UC})f\cdot g\,d\eta \\ &= \sum_{j\in M}\left(\left(\Phi^{\pm}_{1j}F^{-1}_{\eta'}(\mathrm{UC})f,\ g^{\pm}_{1j}(\Delta)\right) + \left(\Phi^{St}_{1j}F^{-1}_{\eta'}(\mathrm{UC})f,\ g^{St}_{1j}(\Delta)\right)\right) \\ &\quad + \sum_{k\in N}\left(\Phi^{\pm}_{2k}F^{-1}_{\eta'}(\mathrm{UC})f,\ g^{\pm}_{2k}(\Delta)\right) \\ &= \sum_{j\in M}\left((f,(\mathrm{UC})^{-1}F_{x'}\Phi^{\pm*}_{1j}g^{\pm}_{1j}(\Delta)) + (f,(\mathrm{UC})^{-1}F_{x'}\Phi^{St*}_{1j}g^{St}_{1j}(\Delta))\right) \\ &\quad + \sum_{k\in N}\left(f,(\mathrm{UC})^{-1}F_{x'}\Phi^{\pm*}_{2k}g^{\pm}_{2k}(\Delta)\right),\end{aligned}$$

where

$$\begin{aligned}&g^{\pm}_{1j}(\Delta) = g(\eta) \quad \text{for} \quad a\le\lambda_j(\eta)\le b,\ g^{\pm}_{1j}(\Delta)=0 \text{ otherwise.}\\ &g^{St}_{1j}(\Delta) = g(\eta) \quad \text{for} \quad a\le c^2_{St}|\eta'|^2\le b,\ g^{\pm}_{1j}(\Delta)=0 \text{ otherwise.}\\ &g^{\pm}_{2k}(\Delta) = g(\eta) \quad \text{for} \quad a\le\lambda_k(\eta)\le b,\ g^{\pm}_{2k}(\Delta)=0 \text{ otherwise.}\end{aligned}$$

So we have

$$0 = \sum_{j\in M}\left((\mathrm{UC})^{-1}F_{x'}\Phi^{\pm*}_{1j}g^{\pm}_{1j}(\Delta) + (\mathrm{UC})^{-1}F_{x'}\Phi^{St*}_{1j}g^{St}_{1j}(\Delta)\right)$$

$$+\sum_{k\in N}(\mathrm{UC})^{-1}F_{x'}\Phi_{2k}^{\pm *}g_{2k}^{\pm}(\Delta)$$

$$=\sum_{j\in M}(\mathrm{UC})^{-1}F_{x'}\left[\int_{\mathbf{R}^3_\pm}\frac{1}{2\pi}e^{i(x_1\eta_1+x_2\eta_2)}\mathrm{UC}(\psi_{1j}^{\pm}(x_3,\eta)\oplus O_{1\times 1})g_{1j}^{\pm}(\Delta)\,d\eta\right]$$

$$+\sum_{j\in M}(\mathrm{UC})^{-1}F_{x'}\left[\int_{\mathbf{R}^3}\frac{1}{2\pi}e^{i(x_1\eta_1+x_2\eta_2)}\mathrm{UC}(\psi_{1j}^{St}(x_3,\eta)\oplus O_{1\times 1})g_{1j}^{St}(\Delta)\,d\eta\right]$$

$$+\sum_{k\in N}(\mathrm{UC})^{-1}F_{x'}\left[\int_{\mathbf{R}^3_\pm}\frac{1}{2\pi}e^{i(x_1\eta_1+x_2\eta_2)}\mathrm{UC}(O_{2\times 2}\oplus\psi_{2k}^{\pm}(x_3,\eta))g_{2k}^{\pm}(\Delta)\,d\eta\right]$$

$$=\sum_{j\in M}\left(\int_{\mathbf{R}_\pm}(\psi_{1j}^{\pm}(x_3,\eta)\oplus O_{1\times 1})g_{1j}^{\pm}(\Delta)\,d\xi+\int_{\mathbf{R}}(\psi_{1j}^{St}(x_3,\eta)\oplus O_{1\times 1})g_{1j}^{St}(\Delta)\,d\xi\right)$$

$$+\sum_{k\in N}\int_{\mathbf{R}_\pm}(O_{2\times 2}\oplus\psi_{2k}^{\pm}(x_3,\eta))g_{2k}^{\pm}(\Delta)\,d\xi$$

$$=\sum_{j\in M}\left(\int_{a\le\lambda_j(\eta)\le b}(\psi_{1j}^{\pm}(x_3,\eta)\oplus O_{1\times 1})g_{1j}^{\pm}(\eta)\,d\xi\right.$$

$$\left.+\int_{a\le c_{St}^2|\eta'|^2\le b}(\psi_{1j}^{St}(x_3,\eta)\oplus O_{1\times 1})g_{1j}^{St}(\eta)\,d\xi\right)$$

$$+\sum_{k\in N}\int_{a\le\lambda_k(\eta)\le b}(O_{2\times 2}\oplus\psi_{2k}^{\pm}(x_3,\eta))g_{2k}^{\pm}(\eta)\,d\xi.$$

By the linear independence of $\psi_{1j}^{\pm}(x_3,\eta)$, $\psi_{1j}^{St}(x_3,\eta)$ $(j\in M)$ and $\psi_{2k}^{\pm}(x_3,\eta)$ $(k\in N)$, it follows that

$$g_{1j}^{\pm}(\eta)=0\ \text{ in } (P_j(\eta)\oplus O_{1\times 1})L^2(\mathbf{R}^3_\pm,\mathbf{C}^3),$$
$$g_{1j}^{St}(\eta)=0\ \text{ in } (P_j(\eta)\oplus O_{1\times 1})L^2(\mathbf{R}^3,\mathbf{C}^3),$$
$$g_{2k}^{\pm}(\eta)=0\ \text{ in } (O_{2\times 2}\oplus 1)L^2(\mathbf{R}^3_\pm,\mathbf{C}^3).$$

Thus

$$g(\eta)=0\ \text{ in }\sum_{j\in M}\oplus(P_j(\eta)\oplus O_{1\times 1})L^2(\mathbf{R}^3_\pm,\mathbf{C}^3)\oplus\sum_{j\in M}\oplus(P_j(\eta)\oplus O_{1\times 1})L^2(\mathbf{R}^3,\mathbf{C}^3)$$
$$\oplus\sum_{k\in N}\oplus(O_{2\times 2}\oplus 1)L^2(\mathbf{R}^3_\pm,\mathbf{C}^3).$$

This completes the proof of Theorem 7.3. □

REFERENCES

1. Agmon, S., *Lectures on elliptic boundary value problems*, Van Nostrand, Math. Studies, Vol. 2, 1965.
2. Cagniard, L., *Reflection and Refraction of Progressive Seismic Waves*, McGraw-Hill Book Company, New York, 1962.
3. Dermenjian, Y. and Guillot, J. C., *Scattering of elastic waves in a perturbed isotropic half space with a free boundary. The limiting absorption principle*, Math. Meth. in the Appl. Sci. **10** (1988), 87–124.
4. Fujita, H., Kuroda, S. T. and Ito, S., *Functional analysis*, Iwanami-Shuppan, Tokyo, 1991. (Japanese)
5. Guillot, J. C., *Existence and uniqueness of a Rayleigh surface wave propagation along the free boundary of a transversely isotropic elastic half space*, Math. Meth. in the Appl. Sci. **8** (1986), 289–310.
6. Ito, H., *Extended Korn's inequalities and the associated best possible constants*, J.of Elasticity **24** (1990), 43–78.
7. Matsumura, M., *Comportement asymptotique de solutions de certains problèms mixtes pour des systèmes hyperboliques symétriques à coefficients constants*, Publ. RIMS, Kyoto Univ. **5** (1970), 301–360.
8. Mochizuki, K., *Spectral and scattering theory for symmetric hyperbolic systems in an exterior domain*, Publ. RIMS, Kyoto Univ. **5** (1969), 219–258.
9. Schulenberger, J. R., *Elastic waves in the half space* R^2_+, J.Diff.Eq. **29** (1978), 405–438.
10. ______, *Boundary waves on perfect conductors*, J.Math.Anal.Appl. **66** (1978), 514–549.
11. Schulenberger, J. R. and Wilcox, C. H., *Eigenfunction expansions and scattering theory for wave propagation problems of classical physics*, Arch. Rational Mech. Anal. **46** (1972), 280–320.
12. Shibata, Y. and Soga, H., *Scattering theory for the elastic wave equation*, Publ. RIMS, Kyoto Univ. **25** (1989), 861–887.
13. Smirnov, V. I., *A course of higher mathematics*, ; English translation: Pergamon Press, Oxford (1964); Japanese translation: Kyoritsu-Shuppan, Tokyo, (1962), vol. 5, Moscow, 1957 (Russian).
14. Stone, M. H., *Linear transformations in Hilbert space and their applications to analysis*, Amer. Math. Soc. Colloquium Publ., vol. XV, 1932.
15. Titchmarsh, E. C., *Introduction to the theory of Fourier integrals*, University Press, Oxford, 1937.
16. Wakabayashi, S., *Eigenfunction expansions for symmetric systems of first order in the half-space* R^n_+, Publ. RIMS, Kyoto Univ. **11** (1975), 67–147.
17. Wilcox, C. H., *Spectral Analysis of the Pekeris Operator in the Theory of Acoustic Wave Propagation in Shallow Water*, Arch. Rational Mech. Anal. **60** (1976), 259–300.
18. ______, *Scattering Theory for the d'Alembert Equation in Exterior Domains. Springer Lecture Notes in Mathematics*, vol. 442, Springer-Verlag, 1975.

CHAPTER II

ENERGY DISTRIBUTION OF THE SOLUTIONS OF ELASTIC WAVE PROPAGATION PROBLEMS IN STRATIFIED MEDIA $\mathbf{R}^3$

Abstract

This paper deals with the asymptotic energy distributions for large times of the solutions of elastic wave propagation problems in stratified media $\mathbf{R}^3$. We construct asymptotic wave functions which approximate the solutions for large times and calculate the asymptotic energy of the solutions using these asymptotic wave functions. In particular, it is shown that the energy of Stoneley wave is asymptotically concentrated along the interface.

1991 Mathematical Subject Classification: Primary 35P20; Secondary 73C35.

Key words and phrases: Elastic wave propagation, Stoneley wave, asymptotic energy distribution

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

§ 1. Introduction

Energy distribution of the solutions of various wave propagation problems has been studied by C. H. Wilcox ([10], [11], [12], [13]). He constructed asymptotic wave functions which approximate the solutions in the sense of L^2 for large times and calculated asymptotic energy distributions of the solutions in several domain by making use of these asymptotic wave functions.

The construction of asymptotic wave functions is based on an eigenfunction expansion theorem which is proved by the same author and on the method of stationary phase. J.C.Guillot [3] studied a Rayleich surface propagating along the free boundary of a transversely isotropic elastic half space and showed that the energy of the Rayleich component of energy solution with finite energy is asymptotically concentrated along the boundary.

In this paper we shall derive energy distribution of the solutions of elastic wave propagation problems in plane-stratified media $\mathbf{R}^3$ using methods due to Wilcox. We construct asymptotic wave functions by using spectral integral representations of the solutions and the method of stationary phase. The integral representations are based on an eigenfunction expansion theory which was proved by the author [8] using methods due to S. Wakabayashi [9]. We calculate asymptotic energy of the solutions for large times of the interface problems for elastic waves and show that the energy of the Stoneley components of the solutions with finite energy is asymptotically concentrated along the interface.

We start with the mathematical formulation of the elastic wave propagation problem.

Consider the plane stratified medium $\mathbf{R}^3 = \{x = (x_1, x_2, x_3); x_i \in \mathbf{R}\}$ with the planar interface $x_3 = 0$, which is defined by

$$(\lambda(x_3), \mu(x_3), \rho(x_3)) = \begin{cases} (\lambda_1, \mu_1, \rho_1), & x_3 < 0, \\ (\lambda_2, \mu_2, \rho_2), & x_3 > 0. \end{cases}$$

Here λ_1, λ_2, μ_1, μ_2 are certain quantities called the Lamé constants and ρ_1, $\rho_2 > 0$ are the densities.

We shall denote the lower halfspace $\mathbf{R}^3_- = \{x \in \mathbf{R}^3; x_3 < 0\}$ by *medium I* and the upper halfspace $\mathbf{R}^3_+ = \{x \in \mathbf{R}^3; x_3 > 0\}$ by *medium II*, respectively, as in Figure 1.

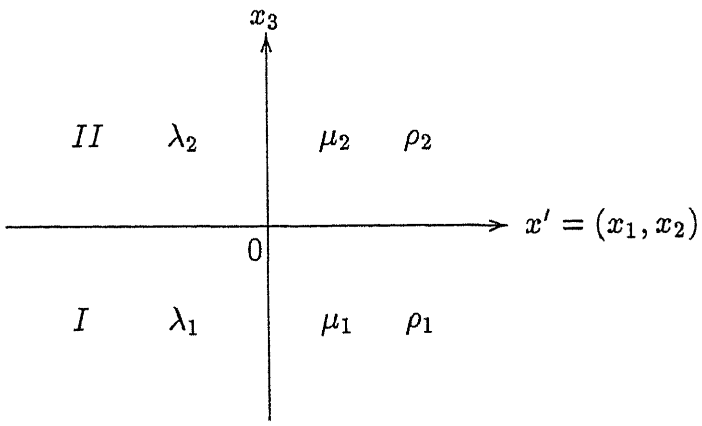

Figure 1 Stratified media I and II

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX

The propagation problem of elastic waves in the stratified medium is formulated as the following mixed initial and interface value problem:

$$\frac{\partial^2 u}{\partial t^2}(t,x) + Mu(t,x) = 0, \tag{1.1}$$

$$u(t,x)|_{x_3=-0} = u(t,x)|_{x_3=+0}, \tag{1.2}$$

$$\sigma_{k3}u(t,x)|_{x_3=-0} = \sigma_{k3}u(t,x)|_{x_3=+0}, \tag{1.3}$$

$$u(0,x) = f(x), \quad \frac{\partial u}{\partial t}(0,x) = g(x), \tag{1.4}$$

where

(1.5)

$$Mu = -\frac{\lambda(x_3)+\mu(x_3)}{\rho(x_3)}\nabla(\nabla\cdot u) - \frac{\mu(x_3)}{\rho(x_3)}\Delta u = \frac{1}{\rho(x_3)}\sum_{k,j=1}^{3} M_{kj}\frac{\partial^2 u}{\partial x_k \partial x_j},$$

(1.6)

$$\sigma_{kj}u = \lambda(x_3)(\nabla\cdot u)\delta_{kj} + 2\mu(x_3)\varepsilon_{kj}u,$$

(1.7)

$$\varepsilon_{kj}u = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k}\right).$$

(1.2) and (1.3) are called interface conditions, and (1.4) is called an initial condition.

The $c^I_{ki\ell j}, c^{II}_{ki\ell j}(i,j,k,\ell = 1,2,3)$ are the stress-strain tensors given by

$$\begin{aligned} c^I_{ki\ell j} &= \lambda_1\delta_{ki}\delta_{\ell j} + \mu_1(\delta_{k\ell}\delta_{ij} + \delta_{kj}\delta_{i\ell}), \\ c^{II}_{ki\ell j} &= \lambda_2\delta_{ki}\delta_{\ell j} + \mu_2(\delta_{k\ell}\delta_{ij} + \delta_{kj}\delta_{i\ell}) \end{aligned} \tag{1.8}$$

with the properties

$$\begin{aligned} c^I_{ki\ell j} &= c^I_{ik\ell j} = c^I_{kij\ell} = c^I_{\ell jki}, \\ c^{II}_{ki\ell j} &= c^{II}_{ik\ell j} = c^{II}_{kij\ell} = c^{II}_{\ell jki}, \end{aligned}$$

and δ_{ki} is the Kronecker delta. We assume that the constants $c^I_{ki\ell j}$, $c^{II}_{ki\ell j}$ satisfy the following stability conditions

$$\lambda_i + \mu_i > 0, \quad \mu_i > 0, \quad (i = 1,2), \tag{1.9}$$

which are equivalent to the conditions

$$\begin{aligned} \sum_{k,i,\ell,j=1}^{3} c^I_{ki\ell j} s_{\ell j}\overline{s_{ki}} &\geq \exists\delta_1 \sum_{k,i=1}^{3} |s_{ki}|^2, \qquad \delta_1 > 0, \\ \sum_{k,i,\ell,j=1}^{3} c^{II}_{ki\ell j} s_{\ell j}\overline{s_{ki}} &\geq \exists\delta_2 \sum_{k,i=1}^{3} |s_{ki}|^2, \qquad \delta_2 > 0, \end{aligned} \tag{1.9'}$$

for all complex symmetric 3×3 matrices (s_{ki}), $s_{ki} = s_{ik} \in \mathbf{C}$ (cf. [4]).

We introduce the Hilbert space

$$\mathcal{H} = L^2(\mathbf{R}^3, \mathbf{C}^3, \rho(x_3)\,dx) \tag{1.10}$$

with inner product

$$(u, v) = \int_{\mathbf{R}^3} u \cdot v \rho(x_3)\,dx,$$

where $u \cdot v$ denotes the usual scalar product in $\mathbf{C}^3$: $u \cdot v = \sum_{i=1}^3 u_i \bar{v}_i$. It was shown in [8 Theorem 1.2] that the operator A on $\mathcal{H}$ with domain

$$\begin{aligned} D(A) = \{u \in H^2(\mathbf{R}^3_-, \mathbf{C}^3) \oplus H^2(\mathbf{R}^3_+, \mathbf{C}^3);\ & \\ & u \text{ satisfies the interface conditions (1.2) and (1.3)} \\ & \text{in the sense of trace on } x_3 = 0\} \end{aligned}$$

and action defined by

$$Au = Mu, \quad u \in D(A) \tag{1.11}$$

is a selfadjoint operator on $\mathcal{H}$. Here

$$H^2(\mathbf{R}^3_\pm, \mathbf{C}^3) = \{u(x);\ D_x^\alpha u \in L^2(\mathbf{R}^3_\pm) \quad \text{for} \quad 0 \le \alpha \le 2\}$$

is a Hilbert space with inner product

$$(u, v)_2 = \int_{\mathbf{R}^3_\pm} \sum_{|\alpha| \le m} D^\alpha u(x) \cdot D^\alpha v(x)\,dx.$$

Every $u \in D(A)$ satisfies the interface conditions (1.2) and (1.3), so the mixed problem (1.1)-(1.4) may be reformulated as the problem of finding a function u : $\mathbf{R} \to \mathcal{H}$ such that

$$\frac{\mathrm{d}^2 u}{\mathrm{d}t^2} + Au = 0 \quad \text{for } \forall t \in \mathbf{R}, \tag{1.12}$$

$$u(0) = f, \quad \frac{\mathrm{d}u}{\mathrm{d}t}(0) = g. \tag{1.13}$$

The operator A is non-negative [8, Lemma 1.4] and the spectral theorem for selfadjoint operators (cf. [2]) implies that (1.12) and (1.13) has a (generalized) solution given by

$$u(t) = \left(\cos tA^{\frac{1}{2}}\right) f + \left(A^{-\frac{1}{2}} \sin tA^{\frac{1}{2}}\right) g, \quad t \in \mathbf{R} \tag{1.14}$$

for every pair f, $g \in \mathcal{H}$. u has derivatives $\frac{\mathrm{d}u}{\mathrm{d}t}$ and $\frac{\mathrm{d}^2 u}{\mathrm{d}t^2}$ and is a strict solution of (1.12) if and only if $f \in D(A)$, $g \in D(A^{\frac{1}{2}})$.

Next we define the energy of solution u on a set $K \subset \mathbf{R}^3$ at time t for the elastic wave propagation problem by

$$(1.15) \qquad E(u, K, t) = \int_K \left(\left| \frac{\partial u}{\partial t} \right|^2 \rho(x_3) - \sum_{k,j=1}^{3} M_{kj} \frac{\partial u}{\partial x_j} \cdot \frac{\partial u}{\partial x_k} \right) dx.$$

If u is a solution of (1.1)-(1.4), u satisfies the conservation laws of energy:

$$E(u, \mathbf{R}^3, t) = E(u, \mathbf{R}^3, 0) = \text{const.} \quad \text{for } \forall t \in \mathbf{R},$$

where the constant may be finite or infinite. If one defines a sesquilinear form B in $\mathcal{H}$ by

$$D(B) = H^1(\mathbf{R}^3, \mathbf{C}^3) \subset \mathcal{H}$$

and

$$B(u, v) = -\sum_{k,j=1}^{3} \int_{\mathbf{R}^3} M_{kj} \frac{\partial u}{\partial x_j} \cdot \frac{\partial u}{\partial x_k} dx,$$

then it is easy to verify that B is closed and non-negative, and that A is the unique selfadjoint non-negative operator in $\mathcal{H}$ associated with B (cf. [5]). Then $D(A^{\frac{1}{2}}) = H^1(\mathbf{R}^3, \mathbf{C}^3)$ and for all $u \in D(A^{\frac{1}{2}})$ one has

$$\left\| A^{\frac{1}{2}} u \right\|^2 = B(u, u) = -\sum_{k,j=1}^{3} \int_{\mathbf{R}}^{3} M_{kj} \frac{\partial u}{\partial x_j} \cdot \frac{\partial u}{\partial x_k} dx,$$

where $\|\cdot\|$ is the norm in $\mathcal{H}$. It follows that

$$(1.16) \qquad E(u, \mathbf{R}^3, t) = \left\| \frac{\mathrm{d}u}{\mathrm{d}t} \right\|^2 + \left\| A^{\frac{1}{2}} u \right\|^2 = \|u\|_{\mathcal{E}}^2 .$$

Here the norm $\|u\|_{\mathcal{E}}$ is called the energy norm. If $f \in D(A^{\frac{1}{2}})$, $g \in \mathcal{H}$, then $u(t) \in D(A^{\frac{1}{2}})$, $\frac{\mathrm{d}u}{\mathrm{d}t} \in \mathcal{H}$ for all $t \in \mathbf{R}$ and $u(t)$ satisfies

$$(1.17) \qquad \|u(t)\|_{\mathcal{E}}^2 = \|u(0)\|_{\mathcal{E}}^2 < \infty \quad \text{for } \forall\, t \in \mathbf{R}.$$

Therefore a necessary and sufficient condition for u to have this property is that the initial state f, g has finite energy:

$$(1.18) \qquad f \in D(A^{\frac{1}{2}}), \quad g \in \mathcal{H}.$$

Hereafter we consider only solutions with finite total energy.

When

$$f \in \mathcal{H}, \quad g \in D(A^{-\frac{1}{2}}),$$

the solution u of the elastic wave propagation problem in $\mathcal{H}$, defined by (1.12) and (1.13), satisfies

$$u(t, x) = Re\{v(t, x)\},$$

where

$$v(t,\cdot) = e^{-itA^{\frac{1}{2}}}h, \quad h = f + iA^{-\frac{1}{2}}g,$$

then $v(t,x)$ has the following representation (see Section 2):

$$v(t,x) = \sum_{j\in M} v_{1j}^{\pm}(t,x) + \sum_{j\in M} v_{1j}^{St}(t,x) + \sum_{k\in N} v_{2k}^{\pm}(t,x) \in \mathcal{H}.$$

$v_{1j}^{\pm}(t,x)(j \in \{p_1,p_2\})$ are called Pressure (P) components, $v_{1j}^{\pm}(t,x)(j \in \{s_1,s_2\})$ are called Shear Vertical (SV) components, $v_{1j}^{St}(t,x)(j \in M = \{s_1,p_1,s_2,p_2\})$ are called Stoneley components and $v_{2k}^{\pm}(t,x)(k \in N = \{s_1,s_2\})$ are called Shear Horizontal (SH) components. We remark that if

$$Dis(c_{s_\iota}^2) > 0, \tag{1.19}$$

then the Stoneley components exist. Here $c_{s_\iota} = \min\{c_{s_1}, c_{s_2}\}$ and $Dis(z)$ is defined by (2.6) below (cf. Section 2, [8, Section 3]). This condition is determined by Lamé constants λ_i, μ_i and densities ρ_i $(i = 1,2)$.

Our main results are the following theorems. Theorem 1.1 shows that the energy of the Stoneley components $v_{1j}^{St}(t,x)(j \in M)$ of v is asymptotically concentrated along the interface $x_3 = 0$.

Theorem 1.1. *We assume that*

$$f \in D(A^{\frac{1}{2}}) \cap \mathcal{H}, \quad g \in \mathcal{H} \cap D(A^{-\frac{1}{2}}), \quad Dis(c_{s_\iota}^2) > 0,$$

then

$$\lim_{t\to\infty} E(v_{1j}^{St}, (C^-(\theta) \cup C^+(\theta)) \cap B(t,\vartheta(t)), t) = E(v_{1j}^{St}, \mathbf{R}^3, 0), \quad j \in M,$$

where

$$\begin{aligned}
&C^-(\theta) = \{x \in \mathbf{R}_-^3;\ -\theta(|x'|) < x_3 < 0\},\\
&C^+(\theta) = \{x \in \mathbf{R}_-^3;\ 0 < x_3 < \theta(|x'|)\},\\
&B(t,\vartheta(t)) = \{x \in \mathbf{R}^3;\ c_{St}t - \vartheta(t) \le |x'| \le c_{St}t + \vartheta(t),\ x_3 \in \mathbf{R}\},\\
&\vartheta(t) : \lim_{t\to\infty} \vartheta(t) = \infty,\ |\vartheta(t)| < 2c_{St}t,\\
&\theta(|x'|) : \lim_{|x'|\to\infty} \theta(|x'|) = \infty,\ \text{monotone increasing function},\\
&c_{St} : \text{propagation speed of Stoneley wave.}
\end{aligned}$$

The next theorem shows that the P, SV, SH components $v_{1j}^{\pm}(t,x)(j \in M)$, $v_{2k}^{\pm}(t,x)(k \in N)$ behave like free waves.

Theorem 1.2. *We assume that*

$$f \in D(A^{\frac{1}{2}}) \cap \mathcal{H}, \quad g \in \mathcal{H} \cap D(A^{-\frac{1}{2}}),$$

then

$$\lim_{t\to\infty} E(v_{1j}^{\pm},\ S_{s_1}(t,\vartheta)\cup S_{p_1}(t,\vartheta)\cup S_{s_2}(t,\vartheta)\cup S_{p_2}(t,\vartheta),t) = E(v_{1j}^{\pm},\mathbf{R}^3,0), \quad j\in M,$$
$$\lim_{t\to\infty} E(v_{2k}^{\pm},\ S_{s_1}(t,\vartheta)\cup S_{s_2}(t,\vartheta),t) = E(v_{2k}^{\pm},\mathbf{R}^3,0), \qquad k\in N,$$

where

$$\begin{aligned}
&S_{s_1}(t,\vartheta(t)) = \{x\in\mathbf{R}^3_-;\ c_{s_1}t-\vartheta(t)\le|x|\le c_{s_1}t+\vartheta(t)\}\\
&S_{p_1}(t,\vartheta(t)) = \{x\in\mathbf{R}^3_-;\ c_{p_1}t-\vartheta(t)\le|x|\le c_{p_1}t+\vartheta(t)\}\\
&S_{s_2}(t,\vartheta(t)) = \{x\in\mathbf{R}^3_+;\ c_{s_2}t-\vartheta(t)\le|x|\le c_{s_2}t+\vartheta(t)\}\\
&S_{p_2}(t,\vartheta(t)) = \{x\in\mathbf{R}^3_+;\ c_{p_2}t-\vartheta(t)\le|x|\le c_{p_2}t+\vartheta(t)\}\\
&\vartheta(t) : \lim_{t\to\infty}\vartheta(t)=\infty,\\
&c_{p_1},c_{p_2} : \text{propagation speeds of P waves},\\
&c_{s_1},c_{s_2} : \text{propagation speeds of SV and SH waves}.
\end{aligned}$$

These theorems are obtained calculating the energy of the asymptotic wave functions $v_{1j}^{St\infty}(t,x)$, $v_{1j}^{\pm\infty}(t,x)$ $(j\in M)$, $v_{2k}^{\pm\infty}(t,x)$ $(k\in N)$ which defined by means of the stationary phase method.

The remainder of this paper is organized as follows. In Section 2, we give spectral integral representations of the solutions of the propagation problem by using the eigenfunction expansion theorem for A developed in [8]. In Section 3, we construct asymptotic wave functions of the Stoneley components by means of the method of stationary phase. We construct asymptotic wave functions of the P, SV, SH components in Section 4. In Section 5, we calculate the asymptotic energy distributions of the solutions for large times.

§ 2. Eigenfunction Expansions for A

The eigenfunction expansion theorem for A was developed in [8]. In this section it is applied to give spectral integral representations of the solutions of the elastic propagation problem. This section begins with a brief review of the structure and properties of the eigenfunctions and the expansion theorem.

Let $\eta'=(\eta_1,\eta_2)\in\mathbf{R}^2$ be the dual variables of $x'=(x_1,x_2)$ and let $F_{x'}$ denote the partial Fourier transformation with respect to x';

$$\hat u(\eta',x_3)=(F_{x'}u)(\eta',x_3)=\operatorname*{l.i.m.}_{R\to\infty}\frac{1}{2\pi}\int_{|x'|\le R}e^{-i(x_1\eta_1+x_2\eta_2)}u(x)\,dx'$$

for u in $\mathcal{H}$. Let

$$\begin{aligned}
&D(\hat A)=F_{x'}D(A)=\{\hat u; u\in D(A)\},\\
&\hat A\hat u = F_{x'}AF_{\eta'}^{-1}\hat u, \qquad \hat u\in D(\hat A).
\end{aligned}$$

For every $\eta'\neq 0$, let

$$\text{(2.1)}\qquad \mathrm{U}=\frac{1}{|\eta'|}\begin{pmatrix}\eta_1 & -\eta_2 & 0\\ \eta_2 & \eta_1 & 0\\ 0 & 0 & |\eta'|\end{pmatrix},\quad \mathrm{C}=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix},$$

where U and C are unitary matrices and $|\eta'| = (\eta_1^2 + \eta_2^2)^{\frac{1}{2}}$. Then we have

$$(2.2) \qquad Au = F_{\eta'}^{-1}\mathrm{UC}(A_1(\eta') \oplus A_2(\eta'))(\mathrm{UC})^{-1}F_{x'}u \quad \text{for } u \in D(A),$$

where $A_1(\eta')$ and $A_2(\eta')$ are non-negative selfadjoint operators (see [8, Proposition 1.7], [1], [3]).

We can get an explicit representation of the Green function $G_1(x_3, y_3, \eta'; \zeta)$ for the operator $A_1(\eta') - \zeta I$ ($\zeta \notin \mathbf{R}$) from the expression of the solution for the following problem:

$$(2.3) \qquad (A_1(\eta', D) - \zeta)v(\eta', x_3) = f(\eta', x_3),$$

$$(2.4) \qquad v(\eta', x_3)|_{x_3=-0} = v(\eta', x_3)|_{x_3=+0},$$

$$(2.5) \qquad B_1(\eta')v(\eta', x_3)|_{x_3=-0} = B_1(\eta')v(\eta', x_3)|_{x_3=+0}.$$

Here (2.4) and (2.5) are the interface conditions for $A_1(\eta', D)$ corresponding to (1.2) and (1.3). $A_1(\eta', D)$ ($D = \frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}x_3}$) is the differential operators corresponding to the selfadjoint operator $A_1(\eta')$. Since the solution v of (2.3) should satisfy the interface conditions (2.4) and (2.5), the denominator of v has the Lopatinski determinant $\Delta(\eta', \zeta)$ as follows:

$$\Delta(\eta', \zeta) = |\eta'|^6 Dis(z),$$

(2.6)

$$\begin{aligned} Dis(z) = &\left(2(\mu_1 - \mu_2) - \frac{\mu_1 z}{c_{s_1}^2} + \frac{\mu_2 z}{c_{s_2}^2}\right)^2 + 4(\mu_1 - \mu_2)^2 a_1 a_2 b_1 b_2 \\ &- a_1 b_1 \left(2(\mu_1 - \mu_2) + \frac{\mu_2 z}{c_{s_2}^2}\right)^2 - a_2 b_2 \left(2(\mu_1 - \mu_2) - \frac{\mu_1 z}{c_{s_1}^2}\right)^2 \\ &- \frac{\mu_1 \mu_2}{c_{s_1}^2 c_{s_2}^2}(a_1 b_2 + a_2 b_1) z^2, \end{aligned}$$

where

$$z = \frac{\zeta}{|\eta'|^2},$$

$$a_1 = \sqrt{1 - \frac{z}{c_{p_1}^2}}, \quad a_2 = \sqrt{1 - \frac{z}{c_{p_2}^2}}, \quad b_1 = \sqrt{1 - \frac{z}{c_{s_1}^2}}, \quad b_2 = \sqrt{1 - \frac{z}{c_{s_2}^2}}.$$

The squares of propagation speeds of shear(SV, SH) and pressure(P) waves are given by

$$(2.7) \qquad c_{s_i}^2 = \frac{\mu_i}{\rho_i}, \quad c_{p_i}^2 = \frac{\lambda_i + 2\mu_i}{\rho_i}, \quad (i = 1, 2),$$

respectively. From the conditions (1.9), the minimum speed of $\{c_{s_1}, c_{p_1}, c_{s_2}, c_{p_2}\}$ is either c_{s_1} or c_{s_2}.

We can see that $Dis(z)$ has the only one real zero when $Dis(z)$ has zeros. Denote by c_{St}^2 its real zero. Then the zero of $\Delta(\eta', \zeta)$ is $c_{St}^2|\eta\prime|^2$ and is the origin of the

Stoneley wave propagating along the interface $x_3 = 0$ in the elastic space R^3, and c_{St} is its speed.

By virtue of principle of the argument, the conditions for the existence of zeros of the Lopatinski determinant $\Delta(\eta', \zeta) = |\eta'|^6 Dis(z)$ (the existence of the Stoneley waves) are given as follows:
If $c_{s_1} < c_{s_2}$, then

(i) $Dis(c_{s_1}^2) > 0 \implies$ The zero $\zeta = c_{St}^2|\eta'|^2$ of $\Delta(\eta', \zeta)$ in ζ exists in $[0,\ c_{s_1}^2|\eta'|^2)$ with order 1. More precisely, we shall prove in the proof of [8, Theorem 6.5] that $c_{St} \neq 0$.

(ii) $Dis(c_{s_1}^2) = 0 \implies c_{St} = c_{s_1}$ and we shall consider this case under some restricted conditions (cf. [8, Lemma 6.4]).

(iii) $Dis(c_{s_1}^2) < 0 \implies \Delta(\eta', \zeta)$ has no zero.

If $c_{s_2} < c_{s_1}$, then we must replace $Dis(c_{s_1}^2)$ by $Dis(c_{s_2}^2)$.

We also obtain an explicit representation of the Green function $G_2(x_3, y_3, \eta'; \zeta)$ for the operator $A_2(\eta') - \zeta I$ ($\zeta \notin \mathbf{R}$) by the same method as $G_1(x_3, y_3, \eta'; \zeta)$. The Lopatinski determinant corresponding to the operator $A_2(\eta') - \zeta I$ ($\zeta \notin \mathbf{R}$) has no zero. By using the Green functions $G_1(x_3, y_3, \eta'; \zeta)$ and $G_2(x_3, y_3, \eta'; \zeta)$, we define

$$\psi_{1j}(x_3, \eta; \zeta) = F_{y_3}^{-1}[G_1(x_3, y_3, \eta'; \zeta)](\xi)(\lambda_j(\eta) - \zeta)P_j(\eta)\rho(x_3)^{-1}, \quad j \in M,$$
$$\psi_{1j}^{St}(x_3, \eta; \zeta) = \frac{\zeta - c_{St}^2|\eta'|^2}{\zeta - \lambda_j(\eta)}\psi_{1j}(x_3, \eta; \zeta), \quad j \in M,$$
$$\psi_{2k}(x_3, \eta; \zeta) = F_{y_3}^{-1}[G_2(x_3, y_3, \eta'; \zeta)](\xi)(\lambda_j(\eta) - \zeta)\rho(x_3)^{-1}, \quad k \in N.$$

Here $\eta = (\eta_1, \eta_2, \xi) = (\eta', \xi)$, $\lambda_j(\eta) = c_j^2|\eta|^2$ are the eigenvalues of $A_1(\eta')$, $P_j(\eta)$ are mutually orthogonal projections for $A_1(\eta')$, $\lambda_k(\eta) = c_k^2|\eta|^2$ are the eigenvalues of $A_2(\eta')$, $M = \{s_1, p_1, s_2, p_2\}$ and $N = \{s_1, s_2\}$. When $\zeta \to \lambda_j(\eta) \pm i0$, $\zeta \to c_{St}^2|\eta|^2$, and $\zeta \to \lambda_k(\eta) \pm i0$, the limits $\psi_{1j}^{\pm}(x_3, \eta)$, $\psi_{1j}^{St}(x_3, \eta)$, and $\psi_{2k}^{\pm}(x_3, \eta)$ exist and these limit functions are generalized eigenfunctions for $A_1(\eta')$, $A_2(\eta')$, respectively.

Using these generalized eigenfunctions for $A_1(\eta')$, $A_2(\eta')$, we define generalized eigenfunctions for A as follows:

$$\psi_{1j}^{\pm}(x, \eta) = \frac{1}{2\pi}e^{i(x_1\eta_1 + x_2\eta_2)}\mathrm{UC}(\psi_{1j}^{\pm}(x_3, \eta) \oplus O_{1\times 1}), \quad j \in M, \tag{2.8}$$

$$\psi_{1j}^{St}(x, \eta) = \frac{1}{2\pi}e^{i(x_1\eta_1 + x_2\eta_2)}\mathrm{UC}(\psi_{1j}^{St}(x_3, \eta) \oplus O_{1\times 1}), \quad j \in M, \tag{2.9}$$

$$\psi_{2k}^{\pm}(x, \eta) = \frac{1}{2\pi}e^{i(x_1\eta_1 + x_2\eta_2)}\mathrm{UC}(O_{2\times 2} \oplus \psi_{2k}^{\pm}(x_3, \eta)), \quad k \in N. \tag{2.10}$$

where $O_{n\times n}$ denotes the $n \times n$ zero matrix.

Now we define the Fourier transform of $f \in \mathcal{H}$ with respect to these generalized eigenfunctions: $f \mapsto (\hat{f}^{\pm}_{1j},\ \hat{f}^{St}_{1j},\ \hat{f}^{\pm}_{2k})$,

$$\hat{f}^{\pm}_{1j}(\eta) = \operatorname*{l.i.m.}_{R\to\infty} \int_{|x|\le \mathbf{R}} \psi^{\pm}_{1j}(x,\eta)^* f(x)\rho(x_3)\,dx, \quad j \in M, \tag{2.11}$$

$$\hat{f}^{St}_{1j}(\eta) = \operatorname*{l.i.m.}_{R\to\infty} \int_{|x|\le \mathbf{R}} \psi^{St}_{1j}(x,\eta)^* f(x)\rho(x_3)\,dx, \quad j \in M, \tag{2.12}$$

$$\hat{f}^{\pm}_{2k}(\eta) = \operatorname*{l.i.m.}_{R\to\infty} \int_{|x|\le \mathbf{R}} \psi^{\pm}_{2k}(x,\eta)^* f(x)\rho(x_3)\,dx, \quad k \in N. \tag{2.13}$$

Theorem 2.1 corresponds to the Parseval and Plancherel formulas.

Theorem 2.1. *We assume that $Dis(c^2_{s_i}) > 0$. Let $f,\ g \in \mathcal{H}$ and $0 < a < b < \infty$. Then we have*

$$(f,g) = \sum_{j\in M}\left(\int_{\mathbf{R}^3} \hat{f}^{\pm}_{1j}(\eta)\cdot \hat{g}^{\pm}_{1j}(\eta)\,d\eta + \int_{\mathbf{R}^3} \hat{f}^{St}_{1j}(\eta)\cdot \hat{g}^{St}_{1j}(\eta)\,d\eta\right) + \sum_{k\in N}\int_{\mathbf{R}^3} \hat{f}^{\pm}_{2k}(\eta)\cdot \hat{g}^{\pm}_{2k}(\eta)\,d\eta.$$

The first half of Theorem 2.2 expresses the Fourier inversion formula with respect to generalized eigenfunctions. The latter half gives the canonical form for A.

Theorem 2.2. *We assume the same assumption as Theorem 2.1.*

(1) For $f \in \mathcal{H}$,

$$f(x) = \sum_{j\in M} \operatorname*{l.i.m.}_{R\to\infty} \int_{|\eta|\le R} \left(\psi^{\pm}_{1j}(x,\eta)\hat{f}^{\pm}_{1j}(\eta) + \psi^{St}_{1j}(x,\eta)\hat{f}^{St}_{1j}(\eta)\right) d\eta + \sum_{k\in N} \operatorname*{l.i.m.}_{R\to\infty} \int_{|\eta|\le R} \psi^{\pm}_{2k}(x,\eta)\hat{f}^{\pm}_{2k}(\eta)\,d\eta.$$

(2) For $f \in D(A)$,

$$Af(x) = \sum_{j\in M} \operatorname*{l.i.m.}_{R\to\infty} \int_{|\eta|\le R} \left(\lambda_j(\eta)\psi^{\pm}_{1j}(x,\eta)\hat{f}^{\pm}_{1j}(\eta) + c^2_{St}|\eta'|^2\psi^{St}_{1j}(x,\eta)\hat{f}^{St}_{1j}(\eta)\right) d\eta + \sum_{k\in N} \operatorname*{l.i.m.}_{R\to\infty} \int_{|\eta|\le R} \lambda_k(\eta)\psi^{\pm}_{2k}(x,\eta)\hat{f}^{\pm}_{2k}(\eta)\,d\eta,$$

and

$$(\widehat{Af})^{\pm}_{1j}(\eta) = \lambda_j(\eta)\hat{f}^{\pm}_{1j}(\eta), \quad j \in M,$$
$$(\widehat{Af})^{St}_{1j}(\eta) = c^2_{St}|\eta'|^2\hat{f}^{St}_{1j}(\eta), \quad j \in M,$$
$$(\widehat{Af})^{\pm}_{2k}(\eta) = \lambda_k(\eta)\hat{f}^{\pm}_{2k}(\eta), \quad k \in N.$$

Theorem 2.3 gives an explicit expression of the ranges $R(\Phi^{\pm})$.

Theorem 2.3. *Assume the same assumption as Theorem 2.1. We define the mappings by*

$$\Phi^{\pm}_{1j} : \mathcal{H} \ni f \to \hat{f}^{\pm}_{1j}(\eta) \in L^2(\mathbf{R}^3_{\pm}, \mathbf{C}^3), \; j \in M$$
$$\Phi^{St}_{1j} : \mathcal{H} \ni f \to \hat{f}^{St}_{1j}(\eta) \in L^2(\mathbf{R}^3, \mathbf{C}^3), \; j \in M$$
$$\Phi^{\pm}_{2k} : \mathcal{H} \ni f \to \hat{f}^{\pm}_{2k}(\eta) \in L^2(\mathbf{R}^3_{\pm}, \mathbf{C}^3), \; k \in N,$$

and put

$$\Phi^{\pm} = \sum_{j\in M} \Phi^{\pm}_{1j} \oplus \sum_{j \in M} \Phi^{St}_{1j} \oplus \sum_{k\in N} \Phi^{\pm}_{2k}.$$

Then we have

$$R(\Phi^{\pm}) = \sum_{j\in M} \oplus (P_j(\eta) \oplus O_{1\times 1}) L^2(\mathbf{R}^3_{\pm}, \mathbf{C}^3) \oplus \sum_{j\in M} \oplus (P_j(\eta) \oplus O_{1\times 1}) L^2(\mathbf{R}^3, \mathbf{C}^3)$$
$$\oplus \sum_{k\in N} \oplus (O_{2\times 2} \oplus 1) L^2(\mathbf{R}^3_{\pm}, \mathbf{C}^3).$$

This implies that $\Phi^{\pm}$ are unitary operators in $\mathcal{H}$, and that the systems of generalized eigenfunctions $\{\psi^+_{1j}, \psi^{St}_{1j}, \psi^+_{2k}\}$ and $\{\psi^-_{1j}, \psi^{St}_{1j}, \psi^-_{2k}\}$ are complete, respectively.

The next theorem shows the utility of the eigenfunction expansion theorem for the operator A.

Theorem 2.4. *Let $\Psi(\lambda)$ be a complex-valued bounded Lebesgue measurable function on $\sigma(A) = \{\lambda : \lambda \geq 0\}$ and let $\Psi(A)$ be the corresponding operator defined by means of the spectral theorem.*

Then we have

$$(\widehat{\Psi(A)f})^{\pm}_{1j}(\eta) = \Psi(c_j^2|\eta|^2)\hat{f}^{\pm}_{1j}(\eta) \in (P_j(\eta) \oplus 0_{1\times 1}) L^2(\mathbf{R}^3_{\pm}, \mathbf{C}^3), \quad j \in M,$$
$$(\widehat{\Psi(A)f})^{St}_{1j}(\eta) = \Psi(c_{St}^2|\eta'|^2)\hat{f}^{St}_{1j}(\eta) \in (P_j(\eta) \oplus 0_{1\times 1}) L^2(\mathbf{R}^3, \mathbf{C}^3), \quad j \in M,$$
$$(\widehat{\Psi(A)f})^{\pm}_{2k}(\eta) = \Psi(c_k^2|\eta|^2)\hat{f}^{\pm}_{2k}(\eta) \in (0_{2\times 2} \oplus 1) L^2(\mathbf{R}^3_{\pm}, \mathbf{C}^3), \quad k \in N.$$

It will be convenient to rewrite the solution (1.12)-(1.13) in the following form.

Theorem 2.5. *Let f and g be real-valued functions such that*

$$f \in \mathcal{H}, \quad g \in D(A^{-\frac{1}{2}}), \tag{2.14}$$

and define

$$h = f + iA^{-\frac{1}{2}} g \in \mathcal{H}. \tag{2.15}$$

Then the solution in $\mathcal{H}$ defined by (1.14) satisfies

$$u(t,x) = Re\{v(t,x)\}, \tag{2.16}$$

where $v(t,x)$ is the complex-valued solution in $\mathcal{H}$ defined by

$$v(t,\cdot) = e^{-itA^{\frac{1}{2}}} h. \tag{2.17}$$

The proof of Theorem 2.5 is due to Wilcox [10, Theorem 2.3]. This theorem implies that the solution $u(t,x)$ of (1.12) and (1.13) is determined by $v(t,x)$.

Combining Theorem 2.4 and Theorem 2.5, we have the following:

Theorem 2.6. *We assume that*

$$f \in \mathcal{H}, \quad g \in D(A^{-\frac{1}{2}}), \quad Dis(c_{s_t}^2) > 0.$$

Then the solution of the elastic wave propagation problem, defined by (1.12) and (1.13) has the representation

$$v(t,x) = \sum_{j\in M} v_{1j}^{\pm}(t,x) + \sum_{j\in M} v_{1j}^{St}(t,x) + \sum_{k\in N} v_{2k}^{\pm}(t,x) \in \mathcal{H}, \tag{2.18}$$

where

$$v_{1j}^{\pm}(t,x) = \underset{R\to\infty}{\text{l.i.m.}} \int_{|\eta|\le R} e^{-itc_j|\eta|}\psi_{1j}^{\pm}(x,\eta)\hat{h}_{1j}^{\pm}(\eta)\,d\eta,\ j\in M, \tag{2.19}$$

$$v_{1j}^{St}(t,x) = \underset{R\to\infty}{\text{l.i.m.}} \int_{|\eta|\le R} e^{-itc_{St}|\eta'|}\psi_{1j}^{St}(x,\eta)\hat{h}_{1j}^{St}(\eta)\,d\eta,\ j\in M, \tag{2.20}$$

$$v_{2k}^{\pm}(t,x) = \underset{R\to\infty}{\text{l.i.m.}} \int_{|\eta|\le R} e^{-itc_k|\eta|}\psi_{2k}^{\pm}(x,\eta)\hat{h}_{2k}^{\pm}(\eta)\,d\eta,\ k\in N, \tag{2.21}$$

and

$$\hat{h}_{1j}^{\pm}(\eta) = \hat{f}_{1j}^{\pm}(\eta) + i\frac{1}{c_j|\eta|}\hat{g}_{1j}^{\pm}(\eta) \in (P_j(\eta)\oplus 0_{1\times 1})L^2(\mathbf{R}_{\pm}^3,\mathbf{C}^3), \tag{2.22}$$

$$\hat{h}_{1j}^{St}(\eta) = \hat{f}_{1j}^{St}(\eta) + i\frac{1}{c_{St}|\eta'|}\hat{g}_{1j}^{St}(\eta) \in (P_j(\eta)\oplus 0_{1\times 1})L^2(\mathbf{R}^3,\mathbf{C}^3), \tag{2.23}$$

$$\hat{h}_{2k}^{\pm}(\eta) = \hat{f}_{2k}^{\pm}(\eta) + i\frac{1}{c_k|\eta|}\hat{g}_{2k}^{\pm}(\eta) \in (0_{2\times 2}\oplus 1)L^2(\mathbf{R}_{\pm}^3,\mathbf{C}^3). \tag{2.24}$$

§ 3. Transient Guided (Stoneley) Waves

This section deals with the Stoneley components $v_{1j}^{St}(t,x)$ $(j\in M)$ defined by (2.20) and (2.23). It is shown in Section 5 below that these components are transient, in the sense that the energy in any bounded region tends to zero for large t, and are guided, in the sense that the energy concentrate near the boundary $x_3 = 0$. The proofs are based on asymptotic approximations for $v_{1j}^{St}(t,x)$ $(j\in M)$ for large t which are derived in this section.

In this section it is assumed that the initial data $f(x)$ and $g(x)$ are real-valued functions and $f\in\mathcal{H}$, $g\in D(A^{-\frac{1}{2}})$, and that the condition $Dis(c_{s_t}^2) > 0$ (i.e. existence of the Stoneley wave) is satisfied.

Substituting (2.9) in (2.20), we can represent the Stoneley components $v_{1j}^{St}(t,x)$ $(j \in M)$ in the form
(3.1)
$$v_{1j}^{St}(t,x) = \operatorname*{l.i.m.}_{R\to\infty} \left(\frac{1}{2\pi}\right) \int_{|\eta|\le R} e^{i(x'\cdot\eta' - tc_{St}|\eta'|)} U(\eta') C(\psi_{1j}^{St}(x_3,\eta) \oplus 0_{1\times 1}) \hat{h}_{1j}^{St}(\eta)\, d\eta,$$

where

$$\mathrm{U} = \frac{1}{|\eta'|}\begin{pmatrix} \eta_1 & -\eta_2 & 0 \\ \eta_2 & \eta_1 & 0 \\ 0 & 0 & |\eta'| \end{pmatrix}, \quad \mathrm{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \eta = (\eta',\xi) = (\eta_1,\eta_2,\xi),$$

and $\psi_{1j}^{St}(x_3,\eta)$ is a generalized eigenfunction for the operator $A_1(\eta')$ (given by [8, (4.17) and (4.18)]).

The function $\psi_{1j}^{St}(x_3,\eta)$ and $\hat{h}_{1j}^{St}(\eta)$ $(j \in M)$ can be written in the form.

$$\psi_{1j}^{St}(x_3,\eta) = \frac{|\eta'|}{\xi - ic_{0j}|\eta'|} e^{-c_{0j}|\eta'||x_3|} \phi_{1j}^{St}(\eta') P_j(\eta), \tag{3.2}$$

$$\hat{h}_{1j}^{St}(\eta) = \frac{\sqrt{|\eta'|}}{\xi + ic_{0j}|\eta'|} k_{1j}^{St}(\eta'), \tag{3.3}$$

where

$$c_{0j} = \sqrt{1 - \frac{c_{St}^2}{c_j^2}} \quad (0 < c_{0j} < 1).$$

Here $\phi_{1j}^{St}(\eta')$ is a bounded continuous function (see [8, (4.17) and (4.18)]) and $k_{1j}^{St}(\eta') \in L^2(\mathbf{R}^2, \mathbf{C}^3)$ because

$$\begin{aligned} \|\hat{h}_{1j}^{St}(\eta)\|_{L^2(\mathbf{R}^3)}^2 &= \int_{\mathbf{R}^2}\int_{\mathbf{R}} \left| \frac{\sqrt{|\eta'|}}{\xi + ic_{0j}|\eta'|} k_{1j}^{St}(\eta') \right|^2 d\xi d\eta' \\ &= \int_{\mathbf{R}^2} |k_{1j}^{St}(\eta')|^2 \left(\int_{-\infty}^{\infty} \frac{|\eta'|}{\xi^2 + c_{0j}^2|\eta'|} d\xi \right) d\eta' \\ &= \frac{\pi}{c_{0j}} \|k_{1j}^{St}(\eta')\|_{L^2(\mathbf{R}^2)}^2. \end{aligned} \tag{3.4}$$

Then the integral in (3.1) is rewritten

$$\begin{aligned} v_{1j}^{St}(t,x) = \operatorname*{l.i.m.}_{R\to\infty} \left(\frac{1}{2\pi}\right) \int_{|\eta'|\le R} & e^{i(x'\cdot\eta' - tc_{St}|\eta'|) - c_{0j}|\eta'||x_3|} \\ & \times U(\eta') C(\phi_{1j}^{St}(\eta') \oplus 0_{1\times 1}) Q(\eta') \sqrt{|\eta'|} k_{1j}^{St}(\eta')\, d\eta', \ j \in M, \end{aligned} \tag{3.5}$$

where

$$Q(\eta') = \int_{\xi^2 \le R^2 - |\eta'|^2} \frac{|\eta'|}{\xi^2 + c_{0j}^2|\eta'|^2} (P_j(\eta',\xi) \oplus 0_{1\times 1})\, d\xi \tag{3.6}$$

$$P_{s_1}(\eta) = P_{s_2}(\eta) = \frac{1}{|\eta|^2}\begin{pmatrix} \xi^2 & -|\eta'|\xi \\ -|\eta'|\xi & |\eta'|^2 \end{pmatrix}, \tag{3.7}$$
$$P_{p_1}(\eta) = P_{p_2}(\eta) = \frac{1}{|\eta|^2}\begin{pmatrix} |\eta'|^2 & |\eta'|\xi \\ |\eta'|\xi & \xi^2 \end{pmatrix}.$$

We note that $U(\eta')C(\phi_{1j}^{St}(\eta') \oplus O_{1\times 1})Q(\eta')$ is a bounded continuous function of η' in $\mathbf{R}^2$, because

$$|Q(\eta')| \le \int_{\xi^2 \le R^2 - |\eta'|^2} \frac{|\eta'|}{\xi^2 + c_{0j}^2|\eta'|^2}\, d\xi \le \frac{2}{c_{0j}} \int_0^\infty \frac{1}{\theta^2+1}\, d\theta = \frac{\pi}{c_{0j}}.$$

Now we consider the following integral
(3.8)
$$w(t,x) = \frac{1}{2\pi}\int_{\mathbf{R}^2} e^{i(x'\cdot\eta' - tc_1|\eta'|) - c_2|\eta'||x_3|}\sqrt{|\eta'|}\phi(\eta')\, d\eta', \quad \phi \in \mathcal{D}(\mathbf{R}^2) = \mathcal{D}(\mathbf{R}^2, \mathbf{C})$$

where c_1 and c_2 are positive constants and $\mathcal{D}(\mathbf{R}^2)$ denotes the usual Schwartz space.
Introducing plane polar coordinates (ν, ω) for η', we find

$$\begin{aligned} w(t,x) &= \frac{1}{2\pi}\int_0^\infty \int_{S^1} e^{i\nu(x'\cdot\omega - c_1 t) - \nu c_2|x_3|}\nu^{\frac{3}{2}}\phi(\nu\omega)\, d\omega d\nu \\ &= \frac{1}{2\pi}\int_0^\infty e^{-i\nu c_1 t - \nu c_2|x_3|}\nu^{\frac{3}{2}} J(x',\nu)\, d\nu, \end{aligned} \tag{3.9}$$

where

$$J(x',\nu) = \int_{S^1} e^{i\nu x'\cdot\omega}\phi(\nu\omega)\, d\omega. \tag{3.10}$$

In order to find the asymptotic behavior of $w(t,x)$ for $t \to \infty$, we calculate the asymptotic behavior of $J(x',\nu)$ for $|x'| \to \infty$ making use of the method of stationary phase.

The following theorem is a version of the method of stationary phase and give the asymptotic formula at infinity of the Fourier transform of a measure (with smooth density) concentrated on the hypersurface S^{n-1} (see [6, Section 5], [7, Section 4] for general C^∞ hypersurface S).

Theorem 3.1. *Let S be the unit sphere of $\mathbf{R}^n$ $(n \ge 2)$, μ be a C^∞ function defined on S. Then we have the following asymptotic formula:*

(3.11)
$$\begin{aligned} I(x) &= \int_S e^{i<x,s>}\mu(s)\, dS \\ &= \mu(\theta)\left(\frac{2\pi}{|x|}\right)^{\frac{n-1}{2}} e^{i(|x| - \frac{\pi}{4}(n-1))} + \mu(-\theta)\left(\frac{2\pi}{|x|}\right)^{\frac{n-1}{2}} e^{-i(|x|-\frac{\pi}{4}(n-1))} + q(x) \end{aligned}$$

as $|x| \to \infty$ along the ray $x = |x|\theta$, where $q(x)$ satisfies

$$\left(\frac{\partial}{\partial x}\right)^\nu q(x) = O\left(|x|^{-\frac{n+1}{2}}\right) \quad \text{as} \quad |x| \to \infty \tag{3.12}$$

for each multi-index ν.

Applying this theorem to (3.10) we find

$$J(x',\nu) = \left(\frac{2\pi}{i\nu r}\right)^{\frac{1}{2}} e^{i\nu r}\phi(\nu\theta) + \left(\frac{2\pi}{-i\nu r}\right)^{\frac{1}{2}} e^{-i\nu r}\phi(-\nu\theta) + q_0(x',\nu) \tag{3.13}$$

where

$$x' = r\theta, \quad r = |x'| \geq 0, \quad \theta \in S^1, \tag{3.14}$$

and we get

$$|q_0(x',\nu)| \leq M_0|\nu x'|^{-\frac{3}{2}} \quad \text{for} \quad |x'| \geq 1 \tag{3.15}$$

Here $M_0 = M_0(\phi)$ is a positive constant which is independent of x', $\theta \in S^1$ and $\nu > 0$. In (3.13) the square roots are defined by the convention that if $z = \pm i|z|$ then $z^{\frac{1}{2}} = e^{\pm i\frac{\pi}{4}}|z|^{\frac{1}{2}}$ with $|z|^{\frac{1}{2}} \geq 0$.

Substituting (3.13) in (3.9), we obtain

$$\begin{aligned} w(t,x) = {} & (2\pi i r)^{-\frac{1}{2}} \int_0^\infty e^{i\nu(r-c_1t)-\nu c_2|x_3|}\nu\phi(\nu\theta)\,d\nu \\ & + (-2\pi i r)^{-\frac{1}{2}} \int_0^\infty e^{-i\nu(r-c_1t)-\nu c_2|x_3|}\nu\phi(-\nu\theta)\,d\nu \\ & + q_1(t,x) \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} q_1(t,x) &= (2\pi)^{-1} \int_0^\infty e^{-i\nu c_1 t-\nu c_2|x_3|}\nu^{\frac{3}{2}} q_0(x',\nu)\,d\nu, \\ |q_1(t,x)| &\leq (2\pi)^{-1} M_0 |x'|^{-\frac{3}{2}} \int_0^\infty e^{-\nu c_2|x_3|}\,d\nu \\ &= (2\pi)^{-1} M_0 c_2^{-1} |x'|^{-\frac{3}{2}} |x_3|^{-1} \end{aligned} \tag{3.17}$$

From (3.16), it follows that $q_1(t,x)$ is a continuous function of $x = (x', x_3)$. Therefore we have

$$|q_1(t,x)| \leq M(1+|x'|^{\frac{3}{2}})^{-1}(1+|x_3|)^{-1} \quad \text{for} \quad x = (x',x_3) \in \mathbf{R}^3 \tag{3.18}$$

where $M = \max((2\pi)^{-1}M_0c_2^{-1},\ \max_{t,|x|\leq 1}|q_1(t,x)|)$ is independent of t.

Let us define the functions $G_\phi^\pm(\tau,\theta,x_3)$ by

$$G_\phi^\pm(\tau,\theta,x_3) = (\pm 2\pi i)^{-\frac{1}{2}} \int_0^\infty e^{\pm i\nu\tau-\nu c_2|x_3|}\nu\phi(\pm\nu\theta)\,d\nu, \qquad \tau, x_3 \in \mathbf{R}, \theta \in S^1. \tag{3.19}$$

Then we have

$$w(t,x) = \frac{G_\phi^+(r-c_1t,\theta,x_3)}{\sqrt{r}} + \frac{G_\phi^-(r+c_1t,\theta,x_3)}{\sqrt{r}} + q_1(t,x) \tag{3.20}$$

$$x' = r\theta, \quad r = |x'| \geq 0, \quad \theta \in S^1, \quad x_3 \in \mathbf{R}.$$

We prepare the following four lemmas.

Lemma 3.2. *For every $\phi(\eta') \in L^2(\mathbf{R}^2)$, $G_\phi^+(\tau, \theta, x_3)$ can be define and we have*

$$\|G_\phi^+(\tau,\theta,x_3)\|_{L^2(\mathbf{R}\times S\times \mathbf{R})} = \frac{1}{\sqrt{c_2}}\|\phi(\eta')\|_{L^2(\mathbf{R}^2)}, \tag{3.21}$$

Proof. By Fubini's theorem, we have

$$\begin{aligned}
&\|G_\phi^+(\tau,\theta,x_3)\|^2_{L^2(\mathbf{R}\times S\times \mathbf{R})}\\
&= \int_{\mathbf{R}}\int_{S^1}\int_{\mathbf{R}} |G_\phi^+(\tau,\theta,x_3)|^2 d\tau d\theta dx_3 = \int_{\mathbf{R}}\int_{S^1}\left(\int_{\mathbf{R}} |G_\phi^+(\tau,\theta,x_3)|^2 d\tau\right) d\theta dx_3\\
&= \int_{\mathbf{R}}\int_{S^1}\left(\int_{\mathbf{R}} |F_{\tau\to\nu}[G_\phi^+(\tau,\theta,x_3)|^2 d\nu\right) d\theta dx_3 \quad \text{(by Parseval's formula)}\\
&= \int_{\mathbf{R}}\int_{S^1}\left(\int_{-\infty}^{\infty} e^{-2\nu c_2|x_3|}|\nu\phi(\nu\theta)|^2 d\nu\right) d\theta dx_3\\
&= \int_{S^1}\int_{\mathbf{R}}\left(2\int_0^{\infty} e^{-2\nu c_2 x_3}\nu\, dx_3\right)|\phi(\nu\theta)|^2\nu\, d\nu d\theta\\
&= \frac{1}{c_2}\int_{S^1}\int_{\mathbf{R}} |\phi(\nu\theta)|^2\nu\, d\nu d\theta\\
&= \frac{1}{c_2}\|\phi(\eta')\|^2_{L^2(\mathbf{R}^2)}, \quad \eta' = \nu\theta. \quad \square
\end{aligned}$$

Lemma 3.3. *For every $\phi \in L^2(\mathbf{R}^2)$, we define $w_\phi^\infty(t,x)$ by*

$$w_\phi^\infty(t,x) = \frac{G_\phi^+(|x'| - c_1 t, \frac{x'}{|x'|}, x_3)}{\sqrt{|x'|}}. \tag{3.22}$$

Then we have

$$\begin{aligned}
\|w_\phi^\infty(t,\cdot)\|^2_{L^2(\mathbf{R}^3)} &\le \|G_\phi^+(\tau,\theta,x_3)\|^2_{L^2(\mathbf{R}\times S^1\times \mathbf{R})}\\
&= \frac{1}{c_2}\|\phi(\eta')\|^2_{L^2(\mathbf{R}^2)}.
\end{aligned} \tag{3.23}$$

Proof.

$$\begin{aligned}
\|w_\phi^\infty(t,\cdot)\|^2_{L^2(\mathbf{R}^3)} &= \int_{\mathbf{R}}\int_{S^1}\int_0^{\infty} |G_\phi^+(r - c_1 t,\theta,x_3)|^2 dr d\theta dx_3\\
&= \int_{\mathbf{R}}\int_{S^1}\int_{-c_1 t}^{\infty} |G_\phi^+(\tau,\theta,x_3)|^2 d\tau d\theta dx_3\\
&\le \|G_\phi^+(\tau,\theta,x_3)\|^2_{L^2(\mathbf{R}\times S^1\times \mathbf{R})}. \quad \square
\end{aligned}$$

Lemma 3.4. *The function $w(t,x)$ defined by (3.8) for $\phi \in \mathcal{D}(\mathbf{R}^2)$ can also be defined for any $\phi \in L^2(\mathbf{R}^2)$ and we have*

$$\|w(t,\cdot)\|_{L^2(\mathbf{R}^3)} = \frac{1}{\sqrt{c_2}}\|\phi(\eta')\|_{L^2(\mathbf{R}^2)}. \tag{3.24}$$

Proof. In fact,

$$\begin{aligned}
\|w(t,\cdot)\|^2_{L^2(\mathbf{R}^3)} &= \int_{\mathbf{R}}\left(\int_{\mathbf{R}^2}|w(t,x)|^2dx'\right)dx_3\\
&= \int_{\mathbf{R}}\left(\int_{\mathbf{R}^2}|e^{-itc_1|\eta'|-c_2|\eta'||x_3|}\sqrt{|\eta'|}\phi(\eta')|^2d\eta'\right)dx_3\\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(Parseval's formula)}\\
&= \int_{\mathbf{R}^2}\left(2\int_0^\infty|\eta'|e^{-2c_2|\eta'|x_3}dx_3\right)|\phi(\eta')|^2\,d\eta'\\
&= \frac{1}{c_2}\|\phi(\eta')\|^2_{L^2(\mathbf{R}^2)}. \quad \square
\end{aligned}$$

Lemma 3.5. *(See Wilcox* [10, Lemma 2.7]*) Assume that $u(t,x)$ has the properties*

$$u(t,\cdot)\in L^2(\mathbf{R}^n) \quad \text{for every } t>t_0, \tag{3.25}$$

$$\lim_{t\to\infty}\|u(t,\cdot)\|_{L^2(K)} = 0 \text{ for every compact } K\subset\mathbf{R}^n, \tag{3.26}$$

$$|u(t,x)|\le g(x)\in L^2(\mathbf{R}^n). \tag{3.27}$$

where t_0 is a constant. Then

$$\lim_{t\to\infty}\|u(t,\cdot)\|_{L^2(\mathbf{R}^n)} = 0. \tag{3.28}$$

Theorem 3.6. *Let $w_\phi = w$ and w_ϕ^∞ be the functions defined by (3.8) and (3.22) for $\phi\in L^2(\mathbf{R}^2)$, respectively. Then*

$$\lim_{t\to\infty}\|w_\phi(t,\cdot)-w_\phi^\infty(t,\cdot)\|_{L^2(\mathbf{R}^3)} = 0. \tag{3.29}$$

Proof. First we consider the case where $\phi\in\mathcal{D}(\mathbf{R}^2)$. Putting

$$u(t,x) = w_\phi(t,x) - w_\phi^\infty(t,x), \tag{3.30}$$

we verify that (3.25)-(3.27) hold for $u(t,x)$. From (3.30), Lemma 3.3 and Lemma 3.4, $u(t,\cdot)\in L^2(\mathbf{R}^3)$ for every $t\in\mathbf{R}$. Next consider

$$w_\phi(t,x) = \frac{1}{2\pi}\int_{S^1}\int_0^\infty e^{-ic_1\nu t}\psi(x',\nu,\omega)d\nu d\omega,$$

where

$$\psi(x,\nu,\omega) = e^{i\nu x'\cdot\omega - \nu c_2|x_3|}\nu^{\frac{3}{2}}\phi(\nu\omega).$$

Noting that ψ is a C^1 function of ν in $[0,\infty)$ for fixed (x',ω), we perform an integration by parts with respect to ν. Then we get the estimate

$$\sup_{x\in K}|w_\phi(t,x)| \le \frac{M_K}{|t|},$$

where M_K is a positive constant which depends on K and ϕ but does not depend on t. As for $w_\phi^\infty(t,x)$, we have for any $d>0$

$$\begin{aligned}\|w_\phi^\infty(t,\cdot)\|^2_{L^2(B_d)} &\le \int_{-d}^{d}\int_{S^1}\int_0^d |G_\phi^+(r-c_1t,\theta,x_3)|^2 dr d\theta dx_3\\ &= \int_{-d}^{d}\int_{S^1}\int_{-\infty}^{\infty}\chi_{[-c_1t,d-c_1t]}(s)|G_\phi^+(s,\theta,x_3)|^2 ds d\theta dx_3,\end{aligned}$$

where $B_d=\{x;|x|\le d\}$ and $\chi_{[a,b]}$ denotes the characteristic function of the interval $[a,b]$. The last integral tends to zero when $t\to\infty$ by Lebesgue's dominated convergence theorem. Thus

$$\lim_{t\to\infty}\|u(t,\cdot)\|_{L^2(B_d)} = 0 \quad \text{as} \quad t\to\infty.$$

From (3.20), (3.22) and (3.30), it follows that

$$u(t,x) = \frac{G_\phi^-(|x'|+c_1t,\frac{x'}{|x'|},x_3)}{\sqrt{|x'|}} + q_1(t,x), \quad x'=r\theta. \tag{3.31}$$

An integration by parts in (3.19) gives

$$G_\phi^-(\tau,\theta,x_3) = \frac{1}{\sqrt{-2\pi i}}\frac{1}{-(i\tau+c_2|x_3|)}\int_0^\infty e^{-i\nu\tau-\nu c_2|x_3|}\frac{\partial}{\partial\nu}(\nu\phi(-\nu\theta))d\nu.$$

From this we deduce the estimate

$$\left|\frac{G_\phi^-(|x'|+c_1t,\frac{x'}{|x'|},x_3)}{\sqrt{|x'|}}\right| \le g(x)\in L^2(\mathbf{R}^3) \quad \text{for} \quad |t|\ge 1, \tag{3.32}$$

where

$$g(x) = \begin{cases} \frac{M}{(c_1+|x'|+c_2|x_3|)\sqrt{|x'|}} & \text{for} \quad |x'|\ge 1\\ \frac{M}{(c_1+c_2|x_3|)\sqrt{|x'|}} & \text{for} \quad |x'|\le 1,\end{cases} \tag{3.33}$$

and M is a suitable constant. From (3.18), (3.31) and (3.32) with (3.33), we see that (3.27) holds for this $u(t,x)$.

Now we show (3.29) for general $\phi \in L^2(\mathbf{R}^2)$. For arbitrary $\varepsilon > 0$, there exists $\phi_0 \in \mathcal{D}(\mathbf{R}^2)$ such that $\|\phi-\phi_0\|_{L^2(\mathbf{R}^2)} < \varepsilon$, because $\mathcal{D}(\mathbf{R}^2)$ is dense in $L^2(\mathbf{R}^2)$. Then

$$\begin{aligned}
&\|w_\phi(t,\cdot) - w_\phi^\infty(t,\cdot)\|_{L^2(\mathbf{R}^3)} \\
&\le \|w_\phi(t,\cdot) - w_{\phi_0}(t,\cdot)\|_{L^2(\mathbf{R}^3)} + \|w_\phi^\infty(t,\cdot) - w_{\phi_0}^\infty(t,\cdot)\|_{L^2(\mathbf{R}^3)} \\
&\qquad + \|w_{\phi_0}(t,\cdot) - w_{\phi_0}^\infty(t,\cdot)\|_{L^2(\mathbf{R}^3)} \\
&= \frac{2}{\sqrt{c_2}}\|\phi - \phi_0\|_{L^2(\mathbf{R}^2)} + \|w_{\phi_0}(t,\cdot) - w_{\phi_0}^\infty(t,\cdot)\|_{L^2(\mathbf{R}^3)} \\
&\le \frac{2\varepsilon}{\sqrt{c_2}} + \|w_{\phi_0}(t,\cdot) - w_{\phi_0}^\infty(t,\cdot)\|_{L^2(\mathbf{R}^3)}.
\end{aligned}$$

Since $\phi_0 \in \mathcal{D}(\mathbf{R}^2)$, there exists $t_0 > 0$ such that for any $t \ge t_0$

$$\|w_{\phi_0}(t,\cdot) - w_{\phi_0}^\infty(t,\cdot)\|_{L^2(\mathbf{R}^3)} < \varepsilon.$$

Thus (3.29) holds for any $\phi \in L^2(\mathbf{R}^2)$. This completes the proof of Theorem 3.6. □

In order to state our main theorem in this section, we recall some relations. When $f \in \mathcal{H}$ and $g \in D(A^{-\frac{1}{2}})$, $h = f + iA^{-\frac{1}{2}}g \in \mathcal{H}$. Let $\hat{f}_{1j}^{St}$ and $\hat{g}_{1j}^{St}$ be the Fourier transforms of f and g with respect to the generalized eigenfunction ψ_{1j}^{St} of A, respectively. Then

$$\hat{h}_{1j}^{St}(\eta) = \hat{f}_{1j}^{St}(\eta) + i\frac{1}{c_{St}|\eta'|}\hat{g}_{1j}^{St}(\eta) \in L^2(\mathbf{R}^3, \mathbf{C}^3)$$

and the Stoneley components $v_{1j}^{St}(t,\cdot) \in L^2(\mathbf{R}^3, \mathbf{C}^3)$ $(j \in M)$ of the solution $v(t,x)$ of the elastic wave propagation problem defined by (1.12) and (1.13) can be represented in the form (3.5):

$$\begin{aligned}
v_{1j}^{St}(t,x) = \operatorname*{l.i.m.}_{R\to\infty} \left(\frac{1}{2\pi}\right) \int_{|\eta'|\le R} & e^{i(x'\cdot\eta' - tc_{St}|\eta'|) - c_{0j}|\eta'||x_3|} \\
& \times U(\eta')C(\phi_{1j}^{St}(\eta') \oplus 0_{1\times 1})Q(\eta')\sqrt{|\eta'|}k_{1j}^{St}(\eta')\,d\eta',\ j \in M,
\end{aligned}$$

where $\phi_{1j}^{St}(\eta')$ and $k_{1j}^{St}(\eta')$ be the functions defined by (3.2) and (3.3), respectively, i.e.,

$$\begin{aligned}
\psi_{1j}^{St}(x_3,\eta) &= \frac{|\eta'|}{\xi - ic_{0j}|\eta'|}e^{-c_{0j}|\eta'||x_3|}\phi_{1j}^{St}(\eta')P_j(\eta), \\
\hat{h}_{1j}^{St}(x_3,\eta) &= \frac{\sqrt{|\eta'|}}{\xi + ic_{0j}|\eta'|}k_{1j}^{St}(\eta').
\end{aligned}$$

By (3.4),

$$k_{1j}^{St}(\eta') \in L^2(\mathbf{R}^2, \mathbf{C}^3),$$

and

$$U(\eta')C(\phi_{1j}^{St}(\eta') \oplus 0_{1\times 1})Q(\eta')k_{1j}^{St}(\eta') \in L^2(\mathbf{R}^2, \mathbf{C}^3).$$

Taking as ϕ in Theorem 3.6 each component of the matrix function

$$U(\eta')C(\phi_{1j}^{St}(\eta') \oplus O_{1\times 1})Q(\eta')\hat{h}_{1j}^{St}(\eta'),$$

then we obtain the following main theorem in this section.

Theorem 3.7. *We assume that*

$$f \in D(A^{\frac{1}{2}}) \cap \mathcal{H}, \quad g \in \mathcal{H} \cap D(A^{-\frac{1}{2}}), \quad Dis(c_{s_1}^2) > 0.$$

Let $v_{1j}^{St\infty}(t,x)$ $(j \in M)$ *be the functions defined by*

$$v_{1j}^{St\infty}(t,x) = \frac{G_{St}(r - c_{St}t, \theta, x_3)}{\sqrt{r}} \quad x' = r\theta, \ r = |x'| \geq 0, \ \theta \in S^1, \tag{3.34}$$

where

$$G_{St}(\tau, \theta, x_3) = \underset{R\to\infty}{\mathrm{l.i.m.}} \frac{1}{\sqrt{2\pi i}} \int_0^R e^{i\nu\tau - c_{0j}\nu|x_3|} \times U(\nu\theta) C(\phi_{1j}^{St}(\nu\theta) \oplus O_{1\times 1}) Q(\nu\theta) \sqrt{\nu} \hat{k}_{1j}^{St}(\nu\theta) \frac{1}{\rho(x_3)}\, d\nu. \tag{3.35}$$

Then we have

$$\lim_{t\to\infty} \|v_{1j}^{St}(t,\cdot) - v_{1j}^{St\infty}(t,\cdot)\|_{\mathcal{H}} = 0. \tag{3.36}$$

$v_{1j}^{St\infty}(t,x) \in \mathcal{H}$ *is called asymptotic wave function for Stoneley component* $v_{1j}^{St}(t,x)$ *of the solution* $v(t,x)$.

§ 4. Transient Free (P, SV, SH) Waves

This section deals with the P, SV components $v_{1j}^{\pm}(t,x)$ $(j \in M)$ and the SH components $v_{2k}^{\pm}(t,x)$ $(k \in N)$ defined by (2.19) and (2.22), (2.21) and (2.24), respectively. It is shown in Section 5 below that each such components are transient and free in the sense that they behave like a diverging cylindrical wave when $t \to \infty$. The proofs of these statements are based on asymptotic approximations of $v_{1j}^{\pm}(t,x)$ $(j \in M)$ and $v_{2k}^{\pm}(t,x)$ $(k \in N)$ for large t which are derived in this section.

In this section it is assumed as in Section 3 that the initial data $f(x)$ and $g(x)$ are real-valued functions. We study mainly the asymptotic behavior for large times of the component $v_{1p_1}^{+}(t,x)$, because the other components $v_{1p_1}^{-}(t,x)$, $v_{1j}^{\pm}(t,x)$ $(j \in \{s_1, p_2, s_2\})$ and $v_{2k}^{\pm}(t,x)$ $(k \in N = \{s_1, s_2\})$ can be handled in a quite similar way.

If $f \in \mathcal{H}$, $g \in D(A^{-\frac{1}{2}})$, the component $v_{1p_1}^{+}(t,x)$ has the following spectral integral representation

(4.1)

$$v_{1p_1}^{+}(t,x) = \underset{R\to\infty}{\mathrm{l.i.m.}} \left(\frac{1}{2\pi}\right) \int_{|\eta|\leq R} e^{i(x'\cdot\eta' - tc_{p_1}|\eta|)} U(\eta') C(\psi_{1p_1}^{+}(x_3,\eta) \oplus 0_{1\times 1}) \hat{h}_{1p_1}^{+}(\eta)\, d\eta.$$

Here U(η'), C are the matrices defined by (2.1), $\hat{h}_{1p_1}^{+}(\eta)$ is defined by (2.22), and $\psi_{1p_1}^{+}(x_3,\eta)$ is a generalized eigenfunction of $A_1(\eta')$.

We now recall that $\psi_{1p_1}^{+}(x_3,\eta)$ has the following form.

$$\psi_{1p_1}^{+}(x_3,\eta) = \begin{cases} \psi_{1p_1}^{I+}(x_3,\eta), & x_3 < 0, \\ \psi_{1p_1}^{II+}(x_3,\eta), & x_3 > 0, \end{cases} \tag{4.2}$$

$$(4.3)\quad \psi_{1p_1}^{I+}(x_3,\eta)=\begin{cases} e^{i\xi x_3}\alpha_1(\eta)+e^{-i\xi x_3}\alpha_2(\eta)+e^{-i\xi_{s_1}(\eta,\lambda_{p_1})x_3}\alpha_3(\eta), & \xi>0,\\ 0, & \xi<0,\end{cases}$$

$$(4.4)\quad \psi_{1p_1}^{II+}(x_3,\eta)=\begin{cases} e^{i\xi_{p_2}(\eta,\lambda_{p_1})x_3}\alpha_4(\eta)+e^{i\xi_{s_2}(\eta,\lambda_{p_1})x_3}\alpha_5(\eta), & \xi>0,\\ 0, & \xi<0.\end{cases}$$

Here $\alpha_i(\eta)(i=1,\cdots,5)$ are bounded continuous 2×2 matrix functions of $\eta=(\eta',\xi)=(\eta_1,\eta_2,\xi)$ and

$$(4.5)\quad \xi_{s_1}(\eta,\lambda_{p_1})=\begin{cases} \pm\sqrt{\frac{c_{p_1}^2}{c_{s_1}^2}|\eta|^2-|\eta'|^2}, & c_{p_1}|\eta|>c_{s_1}|\eta'|,\\ i\sqrt{|\eta'|^2-\frac{c_{p_1}^2}{c_{s_1}^2}|\eta|^2}, & c_{p_1}|\eta|<c_{s_1}|\eta'|,\end{cases}$$

(cf. [8, (4.9), (4.10)]).

Then $v_{1p_1}^+(t,x)$ has for $x_3<0$ the form

$$(4.6)\quad v_{1p_1}^+(t,x)=\operatorname*{l.i.m.}_{R\to\infty}\left(\frac{1}{2\pi}\right)\int_{|\eta|\le R,\ \xi>0} e^{i(x'\cdot\eta'-tc_{p_1}|\eta|)} \times U(\eta')C(\psi_{1p_1}^{I+}(x_3,\eta)\oplus 0_{1\times1})\hat{h}_{1p_1}^+(\eta)\,d\eta$$

and the decomposition

(4.7)

$$\begin{aligned}
&=\operatorname*{l.i.m.}_{R\to\infty}\left(\frac{1}{2\pi}\right)\int_{|\eta|\le R,\ \xi>0} e^{i(x'\cdot\eta'+\xi x_3-tc_{p_1}|\eta|)}U(\eta')C(\alpha_1(\eta)\oplus 0_{1\times1})\hat{h}_{1p_1}^+(\eta)\,d\eta\\
&+\operatorname*{l.i.m.}_{R\to\infty}\left(\frac{1}{2\pi}\right)\int_{|\eta|\le R,\ \xi>0} e^{i(x'\cdot\eta'-\xi x_3-tc_{p_1}|\eta|)}U(\eta')C(\alpha_2(\eta)\oplus 0_{1\times1})\hat{h}_{1p_1}^+(\eta)\,d\eta\\
&+\operatorname*{l.i.m.}_{R\to\infty}\left(\frac{1}{2\pi}\right)\int_{|\eta|\le R,\ \xi>0} e^{i(x'\cdot\eta'-\xi_{s_1}(\eta,\lambda_{p_1})x_3-tc_{p_1}|\eta|)}\\
&\qquad\qquad \times U(\eta')C(\alpha_3(\eta)\oplus 0_{1\times1})\hat{h}_{1p_1}^+(\eta)\,d\eta\\
&=V_1(t,x)+V_2(t,x)+V_3(t,x),\qquad \text{if}\quad x_3<0.
\end{aligned}$$

Since we can decompose $v_{1p_1}^+(t,x)$ $(x_3>0)$ into a sum of integral expression of type $V_3(t,x)$ using (4.4), we consider $v_{1p_1}^+(t,x)$ only in $\mathbf{R}_-^3=\{x=(x',x_3),\ x'\in\mathbf{R}^2, x_3<0\}$.

First we consider $V_1(t,x)$. Let $Y(\xi)$ be the Heaviside function of ξ (i.e. $Y(\xi)=1$ for $\xi>0$ and $=0$ for $\xi<0$) and put

$$(4.8)\qquad \Phi(\eta)=Y(\xi)U(\eta')C(\alpha_1(\eta)\oplus 0_{1\times1})\hat{h}_{1p_1}^+(\eta).$$

Then $\Phi\in L^2(\mathbf{R}_+^3,\mathbf{C}^3)$. As in Section 3, we can extend the result obtained for $\Phi\in\mathcal{D}(\mathbf{R}_+^3,\mathbf{C}^3)$ to the result for $\Phi\in L^2(\mathbf{R}_+^3,\mathbf{C}^3)$ by using the fact that $\mathcal{D}(\mathbf{R}_+^3,\mathbf{C}^3)$ is dense in $L^2(\mathbf{R}_+^3,\mathbf{C}^3)$. Therefore it suffices to consider the integral

(4.9)

$$W_1(t,x)=\frac{1}{2\pi}\int_{\mathbf{R}^3} e^{i(x'\cdot\eta'+x_3\xi-tc_1|\eta|)}\phi(\eta)\,d\eta,\ \phi\in\mathcal{D}(\mathbf{R}_+^3)=\mathcal{D}(\mathbf{R}_+^3,\mathbf{C})\subset\mathcal{D}(\mathbf{R}^3,\mathbf{C}).$$

Then

$$\frac{\partial}{\partial \xi}(x' \cdot \eta' + x_3\xi - tc_1\sqrt{|\eta'|^2+\xi^2}) = x_3 - tc_1\frac{\xi}{\sqrt{|\eta'|^2+\xi^2}} < 0$$
$$\text{if} \quad x_3 < 0,\ \xi > 0 \quad \text{and} \quad t > 0.$$

This means that the phase function has no stationary point on supp$\phi(\eta)$ and therefore we can see integrating by parts with respect to ξ that $W_1(t,x)$ tends to zero when $t \to \infty$ for fixed x and uniformly an each compact set $K \subset \mathbf{R}^3_-$. In order to find the asymptotic behavior of $W_1(t,x)$ as $|x| \to \infty$, we introduce spherical coordinates

$$\eta = \nu\omega, \quad \nu = |\eta| \geq 0, \quad \omega \in S^2.$$

We find

$$W_1(t,x) = \frac{1}{2\pi}\int_0^\infty \nu^2 e^{-ic_1t\nu} J(x,\nu)\,d\nu, \tag{4.10}$$

where

$$J(x,\nu) = \int_{S^2} e^{i\nu x\cdot\omega}\phi(\nu\omega)\,d\omega. \tag{4.11}$$

By Theorem 3.1 we have

$$J(x,\nu) = \left(\frac{2\pi}{i\nu r}\right)e^{i\nu r}\phi(\nu\theta) + \left(\frac{2\pi}{-i\nu r}\right)e^{-i\nu r}\phi(-\nu\theta) + q(x,\nu), \tag{4.12}$$

where

$$x = r\theta, \quad r = |x| \geq 0, \quad \theta \in S^2,$$

and

$$|q(x,\nu)| \leq \frac{M_0}{|\nu r|^2} \quad \text{for} \quad |\nu x| > 0. \tag{4.13}$$

Note that if $x \in \mathbf{R}^3_-$, $\phi(\nu\theta) = 0$ because $\theta_3 < 0$ and supp$\phi \subset \mathbf{R}^3_+$. Now we define following Wilcox's procedure [10]

$$G_1^\pm(\tau,\theta) = \int_0^\infty e^{\pm i\nu\tau}(\pm i\nu)\phi(\pm\nu\theta)\,d\nu, \tag{4.14}$$

then we have

$$W_1(t,x) = \frac{G_1^-(r+c_1t,\theta)}{r} + q_1(t,x), \tag{4.15}$$

where

$$q_1(t,x) = \frac{1}{2\pi}\int_0^\infty \nu^2 e^{-ic_1t\nu} q(x,\nu)\,d\nu. \tag{4.16}$$

From (4.13), we get the estimate

$$|q_1(t,x)| \le \frac{M_1}{|x|^2}, \quad \text{for} \quad |x| \ge 1. \tag{4.17}$$

By the same argument as in Section 3 which is due to Wilcox, one shows that

$$\lim_{t\to\infty} \|W_1(t,\cdot)\|_{L^2(\mathbf{R}^3_-)} = 0. \tag{4.18}$$

As for $V_2(t,x)$, it suffices to consider $W_1(t,x',-x_3)$ for $x_3 < 0$. In fact,

$$\begin{aligned} W_2(t,x) &= \frac{1}{2\pi}\int_{\mathbf{R}^3_+} e^{i(x'\cdot\eta'-x_3\xi-tc_1|\eta|)}\phi(\eta)\,d\eta, \quad \phi\in\mathcal{D}(\mathbf{R}^3_+) \\ &= \frac{1}{2\pi}\int_{\mathbf{R}^3} e^{i(x'\cdot\eta'+x_3\xi-tc_1|\eta|)}\phi(\eta',-\xi)\,d\eta'd\xi \\ &= W_1(t,x',-x_3) \end{aligned} \tag{4.19}$$

Note that if $x\in\mathbf{R}^3_-$ i.e. $x = r\theta$, $\theta_3 < 0$, then $\phi(-\nu\theta',-(-\nu\theta_3)) = 0$. Hence we find

$$W_2(t,x) = \frac{G_2^+(r-c_1t,\theta)}{r} + q_1(t,x',-x_3), \tag{4.20}$$

where

$$G_2^+(\tau,\theta) = \int_0^\infty e^{i\nu\tau}(-i\nu)\phi(\nu\theta',-\nu\theta_3)\,d\nu. \tag{4.21}$$

In this case, we can also show that

$$\lim_{t\to\infty} \|W_2(t,\cdot) - W_2^\infty(t,\cdot)\|_{L^2(\mathbf{R}^3_-)} = 0, \tag{4.22}$$

where

$$W_2^\infty(t,x) = \frac{G_2^+(r-c_1t,\theta)}{r}, \quad x = r\theta. \tag{4.23}$$

Next we consider the following integral

$$W_3(t,x) = \frac{1}{2\pi}\int_{\mathbf{R}^3_+} e^{i(x'\cdot\eta'-x_3\zeta(\eta)-tc_1|\eta|)}\phi(\eta)\,d\eta, \quad \phi\in\mathcal{D}(\mathbf{R}^3_+), \tag{4.24}$$

where

$$\zeta(\eta) = \begin{cases} \pm\sqrt{c_2^2|\eta|^2-|\eta'|^2}, & c_2|\eta| > |\eta'|, \\ i\sqrt{|\eta'|^2-c_2^2|\eta|^2}, & c_2|\eta| < |\eta'|. \end{cases} \tag{4.25}$$

We take a C^∞ partition of unity $\{\chi_1,\chi_2,\chi_3\}$ in a neighborhood of suppϕ ($\bar{\Omega}\subset\mathbf{R}^3_+$) such that

$$0 \le \chi_j(\eta) \le 1 \quad (j=1,2,3), \quad \chi_j\in C_0^\infty(\Omega_j), \tag{4.26}$$

$$\chi_1(\eta)+\chi_2(\eta)+\chi_3(\eta)=1 \quad \text{on} \quad \Omega,$$
$$\Omega_1=\left\{\eta;|\eta|^2>\frac{|\eta'|^2}{(c_2^2-\varepsilon)}\right\},$$
$$\Omega_2=\left\{\eta;\frac{|\eta'|^2}{(c_2^2+2\varepsilon)}<|\eta|^2<\frac{|\eta'|^2}{(c_2^2-2\varepsilon)}\right\},$$
$$\Omega_3=\left\{\eta;|\eta|^2<\frac{|\eta'|^2}{(c_2^2+\varepsilon)}\right\},$$

where ε is a sufficiently small positive constant. Using this partition of unity, we decompose $W_3(t,x)$ as follows:

$$(4.27) \qquad \begin{aligned} W_3(t,x)&=\sum_{j=1}^{3}\frac{1}{2\pi}\int_{\Omega_j} e^{i(x'\cdot\eta'-x_3\zeta(\eta)-tc_1|\eta|)}\chi_j(\eta)\phi(\eta)\,d\eta \\ &=\sum_{j=1}^{3}W_{3j}(t,x), \quad \text{respectively.}\end{aligned}$$

First consider

$$(4.28) \qquad W_{31}^{\pm}(t,x)=\frac{1}{2\pi}\int_{\Omega_1} e^{i(x'\cdot\eta'\pm x_3\sqrt{c_2^2|\eta|-|\eta'|}-tc_1|\eta|)}\chi_1(\eta)\phi(\eta)\,d\eta.$$

Making the change of variables $(\eta',\xi)\to(\eta',\lambda)$, $\lambda=\sqrt{c_2^2|\eta|^2-|\eta'|^2}$, we get

$$(4.29) \quad \begin{aligned} W_{31}^{\pm}(t,x)=\frac{1}{2\pi}\int_{\{(\eta',\lambda);\eta'\in\mathbf{R}^2,\lambda>0\}} & e^{i(x'\cdot\eta'\pm x_3\lambda-t\frac{c_1}{c_2}\sqrt{|\eta'|^2+\lambda^2})} \\ & \times J(\eta',\lambda)\chi_1(\eta',\xi(\eta',\lambda))\phi(\eta',\xi(\eta',\lambda))\,d\eta'd\lambda,\end{aligned}$$

where

$$(4.30) \qquad J(\eta',\lambda)=\frac{\partial(\eta_1,\eta_2,\xi)}{\partial(\eta_1,\eta_2,\lambda)}=\frac{\lambda}{c_2^2\xi}, \quad \xi=\xi(\eta',\lambda)=\frac{1}{c_2}\sqrt{\lambda^2-(c_2^2-1)|\eta'|^2}.$$

This transformation is non-singular on a suitable neighborhood of $\text{supp}\chi_1\phi$. Noting that

$$J(\eta',\lambda)\chi_1(\eta',\xi(\eta',\lambda))\phi(\eta',\xi(\eta',\lambda))\in\mathcal{D}(\{(\eta',\lambda);\eta'\in\mathbf{R}^2,\lambda>0\}),$$

we see that (4.29) is an integral of the same type of (4.9) and (4.19). Therefore we can show that

$$(4.31) \qquad \lim_{t\to\infty}\|W_{31}^{+}(t,\cdot)\|_{L^2(\mathbf{R}^3_-)}=0$$

and

$$(4.32) \qquad \lim_{t\to\infty}\|W_{31}^{-}(t,\cdot)-W_{31}^{-\infty}(t,\cdot)\|_{L^2(\mathbf{R}^3_-)}=0.$$

Here

$$(4.33) \qquad W_{31}^{-\infty}(t,x) = \frac{G_{31}^{+}(r - \frac{c_1}{c_2}t, \theta)}{r}, \quad x = r\theta,$$

and

$$(4.34) \quad G_{31}^{+}(\tau,\theta) = \int_0^\infty e^{i\nu\tau}(-i\nu)J(\nu\theta', -\nu\theta_3) \times \chi_1(\nu\theta', \xi(\nu\theta', -\nu\theta_3))\phi(\nu\theta', \xi(\nu\theta', -\nu\theta_3))\, d\nu.$$

Next consider

$$(4.35) \qquad W_{33}(t,x) = \frac{1}{2\pi}\int_{\mathbf{R}_+^3} e^{i(x'\cdot\eta' - tc_1|\eta|) + x_3\sqrt{|\eta'|^2 - c_2^2|\eta|^2}}\chi_3(\eta)\phi(\eta)\, d\eta.$$

Making use of spherical coordinates in (t,x)-space:

$$(4.36) \qquad t = r\theta_0, \quad x_j = r\theta_j \ (j = 1,2,3) \quad r = \sqrt{t^2 + |x|^2} \geq 0, \quad \theta \in S^3.$$

We write the integral in (4.35) as follows:

$$(4.37) \qquad W_{33}(t,x) = \frac{1}{2\pi}\int_{\mathbf{R}_+^3} e^{ir(\theta'\cdot\eta' - c_1\theta_0|\eta|) + r\theta_3\sqrt{|\eta'|^2 - c_2^2|\eta|^2}}\chi_3(\eta)\phi(\eta)\, d\eta.$$

Then

$$(4.38) \qquad p(\theta,\eta) = \theta'\cdot\eta' - c_1\theta_0|\eta| - i\theta_3\sqrt{|\eta'|^2 - c_2^2|\eta|^2}, \quad \eta = (\eta', \xi),$$

is a complex phase function such that $\mathrm{Im}p(\theta,\eta) > 0$ on $\mathrm{supp}\chi_3(\eta)\phi(\eta)$ when $\theta_3 < 0$. Since

$$\frac{\partial p}{\partial \eta_k} = \theta_k - c_1\theta_0\frac{\eta_k}{|\eta|} - i\theta_3\frac{(1 - c_2^2)\eta_k}{\sqrt{|\eta'|^2 - c_2^2|\eta|^2}}, \quad k = 1,2,$$
$$\frac{\partial p}{\partial \xi} = -c_1\theta_0\frac{\xi}{|\eta|} + i\theta_3\frac{c_2^2\xi}{\sqrt{|\eta'|^2 - c_2^2|\eta|^2}}.$$

We find that

$$(4.39) \quad \sum_{k=1}^2\left|\frac{\partial p}{\partial \eta_k}\right|^2 + \left|\frac{\partial p}{\partial \xi}\right|^2 = \sum_{k=1}^2\left(\theta_k - c_1\theta_0\frac{\eta_k}{|\eta|}\right)^2 + c_1^2\theta_0^2\frac{\xi^2}{|\eta|^2} + \theta_3^2\frac{(1 - c_2^2)^2|\eta'|^2 + c_2^4\xi^2}{(1 - c_2^2)|\eta'|^2 - c_2^2\xi^2} \geq \exists\delta > 0$$

on a suitable neighborhood of $\mathrm{supp}\chi_3\phi$. Then for the operator L

$$(4.40) \qquad L = \left(i\sum_{k=1}^2\left|\frac{\partial p}{\partial \eta_k}\right|^2 + \left|\frac{\partial p}{\partial \xi}\right|^2\right)^{-1}\left(\sum_{k=1}^m \frac{\partial \bar{p}}{\partial \eta_k}\frac{\partial}{\partial \eta_k} + \frac{\partial \bar{p}}{\partial \xi}\frac{\partial}{\partial \xi}\right),$$

we have

$$e^{irp(\theta,\eta)} = \frac{1}{r}L[e^{irp(\theta,\eta)}], \tag{4.41}$$

where $\bar{p}$ denotes the complex conjugate of p. Using repeatedly this relation, we find

$$\begin{aligned}W_{33}(t,x) &= \frac{1}{2\pi r}\int_{\mathbf{R}^3_+} L[e^{irp(\theta,\eta)}]\chi_3(\eta)\phi(\eta)\,d\eta \\ &= \frac{1}{2\pi r}\int_{\mathbf{R}^3_+} e^{irp(\theta,\eta)\,t}L(\chi_3(\eta)\phi(\eta))\,d\eta \\ &= \dots \\ &= \frac{1}{2\pi r^l}\int_{\mathbf{R}^3_+} e^{irp(\theta,\eta)}({}^tL)^l(\chi_3(\eta)\phi(\eta))\,d\eta.\end{aligned}$$

Here ${}^tL$ denotes the transpose operator of L. From this expression, we get the estimate

$$|W_{33}(t,x)| \le \frac{M_{p,\phi,l}}{(t^2+|x|^2)^{\frac{l}{2}}}, \tag{4.42}$$

where $M_{p,\phi,l}$ is a positive constant. Taking $l > [\frac{n}{2}]+1$, we deduce from the estimate (4.42) that

$$\lim_{t\to\infty} \|W_{33}(t,\cdot)\|_{L^2(\mathbf{R}^3_-)} = 0. \tag{4.43}$$

Now consider $W_{32}(t,x)$. From (2.19), we see that the linear operator

$$L^2(\mathbf{R}^3_+,\mathbf{C}^3) \ni Y(\xi)\hat{h}^+_{1p_1}(\eta) \mapsto v^+_{1p_1}(t,\cdot) \in L^2(\mathbf{R}^3_-,\mathbf{C}^3)$$

is continuous uniformly in $t \in \mathbf{R}$. From (4.9) and (4.19), we have

$$\begin{aligned}\|W_1(t,\cdot)\|_{L^2(\mathbf{R}^3_-)} &= \sqrt{2\pi}\|e^{-itc_1|\eta|}\phi(\eta)\|_{L^2(\mathbf{R}^3_+)}, \\ \|W_2(t,\cdot)\|_{L^2(\mathbf{R}^3_-)} &= \sqrt{2\pi}\|e^{-itc_1|\eta|}\phi(\eta)\|_{L^2(\mathbf{R}^3_+)}.\end{aligned}$$

From these relations, it follows that the linear operators

$$\begin{aligned}L^2(\mathbf{R}^3_+,\mathbf{C}^3) \ni \hat{h}^+_{1p_1}(\eta) &\mapsto V_1(t,\cdot) \in L^2(\mathbf{R}^3_-,\mathbf{C}^3), \\ L^2(\mathbf{R}^3_+,\mathbf{C}^3) \ni \hat{h}^+_{1p_1}(\eta) &\mapsto V_2(t,\cdot) \in L^2(\mathbf{R}^3_-,\mathbf{C}^3),\end{aligned}$$

are continuous uniformly in $t \in \mathbf{R}$. Hence the linear operator

$$L^2(\mathbf{R}^3_+,\mathbf{C}^3) \ni \hat{h}^+_{1p_1}(\eta) \mapsto V_3(t,\cdot) \in L^2(\mathbf{R}^3_-,\mathbf{C}^3)$$

is also continuous uniformly in $t \in \mathbf{R}$ and we have

$$V_3(t,x) = W_{32}(t,x) \quad \text{for} \quad \phi(\eta) = \chi_3(\eta)U(\eta')C(\alpha_3(\eta)\oplus O_{1\times 1})\hat{h}^+_{1p_1}(\eta).$$

Thus for arbitrary $\delta > 0$, there exists a $R > 0$ for which

$$\|W_{32}(t,x)\|_{L^2(\mathbf{R}^3_-\cap\{x;|x|\geq R\},\mathbf{C}^3)} < \delta \quad \text{for } \forall t \in \mathbf{R}.$$

Taking ε small enough, we have from (4.26) and (4.27)

$$\|W_{32}(t,\cdot)\|_{L^2(\mathbf{R}^3_-\cap\{x;|x|\leq R\},\mathbf{C}^3)} < \delta.$$

Note that

$$\Phi(\eta) = Y(\xi)U(\eta')C(\alpha_i(\eta)\oplus 0_{1\times 1})\hat{h}^+_{1p_1}(\eta) \in \mathcal{D}(\{\eta \in \mathbf{R}^3_+;\eta' \neq 0\},\mathbf{C}^3)$$

when

$$\hat{h}^+_{1p_1}(\eta) \in \mathcal{D}(\{\eta \in \mathbf{R}^3_+;\eta' \neq 0\},\mathbf{C}^3),$$

and define $v^{+\infty}_{1p_1}(t,x)$ by

$$v^{+\infty}_{1p_1}(t,x) = \frac{G^+_2(r-c_{p_1}t,\theta)}{r} + \frac{G^+_{31}(r-c_{s_1}t,\theta)}{r}, \quad x \in \mathbf{R}^3_-,$$

where G^+_2 and G^+_{31} are the functions defined by (4.21) and (4.34) for $\phi(\eta) = \Phi(\eta) \in \mathcal{D}(\{\eta \in \mathbf{R}^3_+;\eta' \neq 0\},\mathbf{C}^3)$. Then we conclude that

$$\lim_{t\to\infty} \|v^+_{1p_1}(t,\cdot) - v^{+\infty}_{1p_1}(t,\cdot)\|_{L^2(\mathbf{R}^3_-,\mathbf{C}^3)} = 0 \tag{4.44}$$

when $\hat{h}^+_{1p_1}(\eta) \in \mathcal{D}(\{\eta \in \mathbf{R}^3_+;\eta' \neq 0\},\mathbf{C}^3)$.

For general $\hat{h}^+_{1p_1}(\eta) \in L^2(\mathbf{R}^3_+,\mathbf{C}^3)$, we can show that (4.44) also holds. In fact, from the continuity of linear operators

$$L^2(\mathbf{R}^3_+,\mathbf{C}^3) \ni \hat{h}^+_{1p_1}(\eta) \to v^+_{1p_1}(t,\cdot) \in L^2(\mathbf{R}^3_-,\mathbf{C}^3),$$
$$L^2(\mathbf{R}^3_+,\mathbf{C}^3) \ni \Phi(\eta) \to v^{+\infty}_{1p_1}(t,\cdot) \in L^2(\mathbf{R}^3_-,\mathbf{C}^3),$$
$$L^2(\mathbf{R}^3_+,\mathbf{C}^3) \ni \hat{h}^+_{1p_1}(\eta) \to \Phi(\eta) = Y(\xi)U(\eta')C(\alpha_i(\eta)\oplus 0_{1\times 1})\hat{h}^+_{1p_1}(\eta) \in L^2(\mathbf{R}^3_-,\mathbf{C}^3),$$

and from the fact that $\mathcal{D}(\{\eta \in \mathbf{R}^3_+;\eta' \neq 0\},\mathbf{C}^3)$ is dense in $L^2(\mathbf{R}^3_+,\mathbf{C}^3)$ by the same argument in Section 3.

Therefore, the principal result of this section states as follows:

Theorem 4.1. *We assume that*

$$f \in D(A^{\frac{1}{2}}) \cap \mathcal{H}, \quad g \in \mathcal{H} \cap D(A^{-\frac{1}{2}}).$$

Let $v^{+\infty}_{1j}(t,x)$ $(j \in M)$ be the functions defined by

$$v^{+\infty}_{1j}(t,x) = \begin{cases} \frac{G^+_{s_1}(r-c_{s_1}t,\theta)}{r} + \frac{G^+_{p_1}(r-c_{p_1}t,\theta)}{r}, & x_3 < 0, \\ \frac{G^+_{s_2}(r-c_{s_2}t,\theta)}{r} + \frac{G^+_{p_2}(r-c_{p_2}t,\theta)}{r}, & x_3 > 0, \end{cases} \tag{4.45}$$

for $t \in \mathbf{R}$, $x = r\theta$, $r = |x| \geq 0$, $\theta \in S^2$, *where if* $l = j$, *then*

$$(4.46)\quad G_l^+(\tau,\theta) = \underset{R\to\infty}{\mathrm{l.i.m.}} \int_0^R e^{i\nu\tau}(-i\nu)Y(-\nu\theta_3)U(\nu\theta')C \times (\alpha(\nu\theta',-\nu\theta_3) \oplus 0_{1\times 1})\hat{h}_{1j}^+(\nu\theta',-\nu\theta_3)\frac{1}{\rho(x_3)}\,d\nu,$$

and if $l \neq j$, *then*

$$(4.47)\quad G_l^+(\tau,\theta) = \underset{R\to\infty}{\mathrm{l.i.m.}} \int_0^R e^{i\nu\tau}(-i\nu)J(\nu\theta',-\nu\theta_3)\chi_1(\nu\theta',\xi(\nu\theta',-\nu\theta_3)) \times Y(\xi(\nu\theta',-\nu\theta_3))U(\nu\theta')C(\alpha(\nu\theta',\xi(\nu\theta',-\nu\theta_3)) \oplus 0_{1\times 1}) \times \hat{h}_{1j}^+(\nu\theta',\xi(\nu\theta',-\nu\theta_3))\frac{1}{\rho(x_3)}\,d\nu$$

for $\eta = \nu\omega$, $\nu \geq 0$, $\omega \in S^2$. *Here* α*'s are bounded continuous* 2×2 *matrix functions,*

$$(4.48)\qquad J(\eta',\lambda) = \frac{c_l\lambda}{c_j\sqrt{\lambda^2 - (\frac{c_j^2}{c_l^2} - 1)|\eta'|^2}},$$

and χ_1 *satisfies*

$$(4.49)\qquad 0 \leq \chi_1(\eta) \leq 1, \quad \chi_1 \in C_0^\infty(\Omega_1), \quad \Omega_1 = \left\{\eta; |\eta|^2 > \frac{|\eta'|^2}{(\frac{c_j^2}{c_l^2} - \varepsilon)}\right\}.$$

Then we have

$$(4.50)\qquad \lim_{t\to\infty} \|v_{1j}^+(t,\cdot) - v_{1j}^{+\infty}(t,\cdot)\|_{\mathcal{H}} = 0.$$

$v_{1j}^{+\infty}(t,x) \in \mathcal{H}$ *are called asymptotic wave functions for P, SV components* $v_{1j}^+(t,x)$ *of the solution* $v(t,x)$.

Moreover let $v_{2k}^{+\infty}(t,x)$ $(k \in N)$ *be the functions defined by*

$$(4.51)\qquad v_{2k}^{+\infty}(t,x) = \begin{cases} \frac{G_{s_1}^+(r-c_{s_1}t,\theta)}{r} & x_3 < 0, \\ \frac{G_{s_2}^+(r-c_{s_2}t,\theta)}{r} & x_3 > 0, \end{cases}$$

for $t \in \mathbf{R}$, $x = r\theta$, $r = |x| \geq 0$, $\theta \in S^2$, *where if* $l = k$, *then*

$$(4.52)\quad G_l^+(\tau,\theta) = \underset{R\to\infty}{\mathrm{l.i.m.}} \int_0^R e^{i\nu\tau}(-i\nu)Y(-\nu\theta_3)U(\nu\theta')C \times (0_{2\times 2} \oplus \beta(\nu\theta',-\nu\theta_3))\hat{h}_{2k}^+(\nu\theta',-\nu\theta_3)\frac{1}{\rho(x_3)}\,d\nu,$$

and if $l \neq k$, then

$$(4.53)\quad G_l^{+}(\tau,\theta) = \underset{R\to\infty}{\mathrm{l.i.m.}} \int_0^R e^{i\nu\tau}(-i\nu)J(\nu\theta', -\nu\theta_3)\chi_1(\nu\theta', \xi(\nu\theta', -\nu\theta_3)) \times Y(\xi(\nu\theta', -\nu\theta_3))U(\nu\theta')C(0_{2\times 2}\oplus \beta(\nu\theta', \xi(\nu\theta', -\nu\theta_3))) \times \hat{h}_{2k}^{+}(\nu\theta', \xi(\nu\theta', -\nu\theta_3))\frac{1}{\rho(x_3)}\,d\nu$$

for $\eta = \nu\omega$, $\nu \geq 0$, $\omega \in S^2$. Here β's are bounded continuous functions, and J and χ_1 are defined by (4.48) and (4.49), respectively. Then we have

$$(4.54)\qquad \lim_{t\to\infty} \|v_{2k}^{+}(t,\cdot) - v_{2k}^{+\infty}(t,\cdot)\|_{\mathcal{H}} = 0.$$

$v_{2k}^{+\infty}(t,x) \in \mathcal{H}$ are called asymptotic wave functions for SH component $v_{2k}^{+}(t,x)$ of the solution $v(t,x)$.

Remark. As to $v_{1j}^{-\infty}(t,x)$ $(j \in M)$ and $v_{2k}^{-\infty}(t,x)$ $(k \in N)$, we obtain similar asymptotic wave functions by obvious modification.

Proof of Theorem 4.1 is the same as the proof of Theorem 3.7.

§ 5. The Asymptotic Energy Distributions for Large Times

In this section we calculate asymptotic energy distributions of the solutions of the elastic propagation problem when $t \to \infty$, by using the asymptotic wave functions $v_{1j}^{St\infty}(t,x)$, $v_{1j}^{\pm\infty}(t,x)$ $(j \in M)$, $v_{2k}^{\pm\infty}(t,x)$ $(k \in N)$ which constructed in Section 3 and 4.

In this section, as in Section 3 and 4, it is assumed that $f(x)$ and $g(x)$ are real-valued functions.

Theorem 5.1. *Suppose that the solution $u(t)$ of (1.12) and (1.13) defined by (1.14) has the property*

$$\lim_{t\to\infty} \|u(t)\|_{\mathcal{H}} = 0,$$

for any initial data $f \in D(A^{\frac{1}{2}}) \cap \mathcal{H}$, $g \in \mathcal{H} \cap D(A^{-\frac{1}{2}})$. Then, for the solution $u(t)$ of (1.12) and (1.13) with initial data

$$(5.1)\qquad f \in D(A^{\frac{1}{2}}) \cap \mathcal{H}, \quad g \in \mathcal{H} \cap D(A^{-\frac{1}{2}}),$$

we have

$$(5.2)\qquad \lim_{t\to\infty} E(u, \mathbf{R}^3, t) = \lim_{t\to\infty} \|u(t)\|_E = 0.$$

Proof. From the condition (5.1) and (2.8),

$$A^{\frac{1}{2}}u(t) = A^{\frac{1}{2}}e^{-itA^{\frac{1}{2}}}(f + iA^{-\frac{1}{2}}g) = e^{-itA^{\frac{1}{2}}}(A^{\frac{1}{2}}f + ig) \in \mathcal{H},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = -iA^{\frac{1}{2}}e^{-itA^{\frac{1}{2}}}(f + iA^{-\frac{1}{2}}g) = -ie^{-itA^{\frac{1}{2}}}(A^{\frac{1}{2}}f + ig) \in \mathcal{H}.$$

Thus $\frac{\mathrm{d}}{\mathrm{d}t}u(t)$ is the solution of (1.12) for $f' = A^{\frac{1}{2}}f \in \mathcal{H}$ and $g' = A^{\frac{1}{2}}g \in D(A^{-\frac{1}{2}})$. Then by assumption

$$\lim_{t\to\infty} \|A^{\frac{1}{2}}u(t)\|_{\mathcal{H}} = 0 \quad \text{and} \quad \lim_{t\to\infty} \left\|\frac{\mathrm{d}}{\mathrm{d}t}u(t)\right\|_{\mathcal{H}} = 0.$$

Hence

$$\lim_{t\to\infty} E(u, \mathbf{R}^3, t) = \lim_{t\to\infty} \left(\left\|\frac{\mathrm{d}}{\mathrm{d}t}u(t)\right\|_{\mathcal{H}}^2 + \|A^{\frac{1}{2}}u(t)\|_{\mathcal{H}}^2 \right) = 0. \quad \square$$

Let $f \in D(A^{\frac{1}{2}}) \cap \mathcal{H}$, $g \in \mathcal{H} \cap D(A^{-\frac{1}{2}})$,$Dis(c_{s_t}^2) > 0$, and $v(t,x)$ be the corresponding solution of (1.12) and (1.13). We define the asymptotic wave functions $v_{1jl}^{St\infty}(t,x)$ $(l = 0,1,2,3)$ by

$$v_{1jl}^{St\infty}(t,x) = \frac{G_{St}^l(x' - c_{St}t, \frac{x'}{|x'|}, x_3)}{|x'|^{\frac{1}{2}}}, \quad x' \neq 0, \tag{5.3}$$

$$G_{St}^0(\tau,\theta,x_3) = \operatorname*{l.i.m.}_{R\to\infty} \frac{1}{\sqrt{2\pi i}} \int_0^R e^{i\nu\tau - c_{0j}\nu|x_3|}(-ic_{St}\nu) \times U(\nu\theta)C(\phi_{1j}^{St}(\nu\theta) \oplus O_{1\times1})Q(\nu\theta)\sqrt{\nu}\hat{h}_{1j}^{St}(\nu\theta)\frac{1}{\rho(x_3)}\,d\nu, \tag{5.4}$$

$$G_{St}^l(\tau,\theta,x_3) = \operatorname*{l.i.m.}_{R\to\infty} \frac{1}{\sqrt{2\pi i}} \int_0^R e^{i\nu\tau - c_{0j}\nu|x_3|}(i\nu\theta_l) \times U(\nu\theta)C(\phi_{1j}^{St}(\nu\theta) \oplus O_{1\times1})Q(\nu\theta)\sqrt{\nu}\hat{h}_{1j}^{St}(\nu\theta)\frac{1}{\rho(x_3)}\,d\nu, \quad (l = 1,2) \tag{5.5}$$

$$G_{St}^3(\tau,\theta,x_3) = \operatorname*{l.i.m.}_{R\to\infty} \frac{1}{\sqrt{2\pi i}} \int_0^R e^{i\nu\tau - c_{0j}\nu|x_3|}(\sigma c_{0j}\nu) \times U(\nu\theta)C(\phi_{1j}^{St}(\nu\theta) \oplus O_{1\times1})Q(\nu\theta)\sqrt{\nu}\hat{h}_{1j}^{St}(\nu\theta)\frac{1}{\rho(x_3)}\,d\nu, \tag{5.6}$$

where σ is 1 or -1 according as $x_3 < 0$ or $x_3 > 0$. Then we have

Theorem 5.2. *Assume that*

$$f \in D(A^{\frac{1}{2}}) \cap \mathcal{H}, \quad g \in \mathcal{H} \cap D(A^{-\frac{1}{2}}), \quad Dis(c_{s_t}^2) > 0.$$

Then

$$\lim_{t\to\infty} \left\| \frac{\partial}{\partial t} v_{1j}^{St}(t,\cdot) - v_{1j0}^{St\infty}(t,\cdot) \right\|_{\mathcal{H}} = 0, \quad j \in M, \tag{5.7}$$

$$\lim_{t\to\infty} \left\| \frac{\partial}{\partial x_k} v_{1j}^{St}(t,\cdot) - v_{1jl}^{St\infty}(t,\cdot) \right\|_{\mathcal{H}} = 0, \; l = 1,2,3, \; j \in M. \tag{5.8}$$

The proof of Theorem 5.2 is the same as that of Theorem 3.2 except for obvious modifications.

The calculation of asymptotic energy distributions are based on the next lemma.

Lemma 5.3. *Assume that*

$$f \in D(A^{\frac{1}{2}}) \cap \mathcal{H}, \quad g \in \mathcal{H} \cap D(A^{-\frac{1}{2}}), \quad Dis(c_{s_t}^2) > 0.$$

Let

$$B(t, \vartheta(t)) = \{x \in \mathbf{R}^3;\ c_{St}t - \vartheta(t) \le |x'| \le c_{St}t + \vartheta(t),\ x_3 \in \mathbf{R}\}, \tag{5.9}$$

where $\vartheta(t)$ is any functions of $t \in \mathbf{R}$ which satisfy

$$0 \le \vartheta(t) \le \infty, \quad \text{for } \forall t \in \mathbf{R}. \tag{5.10}$$

Then we have

(5.11)

$$\begin{aligned} E(v_{1j}^{St\infty}, &B(t,\vartheta(t)), t) \\ &= \int_{\mathbf{R}} \int_{-\vartheta(t)}^{\vartheta(t)} \|G_{St}^0(r,\cdot,x_3)\|_{L^2(S^1)}^2 dr \rho(x_3)\, dx_3 \\ &- \int_{\mathbf{R}} \int_{-\vartheta(t)}^{\vartheta(t)} \int_{S^1} \sum_{k,l=1}^{3} M_{kl} G_{St}^k(r,\theta,x_3) \cdot G_{St}^l(r,\theta,x_3)\, d\theta dr dx_3. \end{aligned}$$

Proof. From the definition of the energy (1.15)

$$E(v_{1j}^{St\infty}, B, t) = \int_B \left(|v_{1j0}^{St\infty}|\rho(x_3) - \sum_{k,l=1}^{3} M_{kl} v_{1jk}^{St\infty} \cdot v_{1jl}^{St\infty} \right) dx. \tag{5.12}$$

By the change of variable $r - c_{St} = r'$, the first term of the right-hand side of (5.12) is

$$\begin{aligned} \|v_{1j0}^{St\infty}&(t,\cdot)\|_{\mathcal{H}(B(t,\vartheta(t))}^2 \\ &= \int_{-\infty}^{\infty} \int_{c_{St}t-\vartheta(t)}^{c_{St}t+\vartheta(t)} \int_{S^1} |G_{St}^0(r - c_{St}t, \theta, x_3)|^2\, d\theta dr \rho(x_3) dx_3 \\ &= \int_{-\infty}^{\infty} \int_{-\vartheta(t)}^{\vartheta(t)} \int_{S^1} |G_{St}^0(r,\theta,x_3)|^2\, d\theta dr \rho(x_3) dx_3. \end{aligned}$$

By introducing the spherical coordinates $x' = r\theta$, $r = |x'| \ge 0$, $\theta \in S^1$, and by the change of variable $r - c_{St}t = r'$, the second term of the right-hand side of (5.12) is

$$\begin{aligned} &- \sum_{k,l=1}^{3} \int_{B(t,\vartheta(t))} M_{kl} v_{1jk}^{St\infty}(t,x) \cdot v_{1jl}^{St\infty}(t,x)\, dx \\ &= - \sum_{k,l=1}^{3} \int_{c_{St}t-\infty}^{\infty} \int_{c_{St}t-\vartheta(t)}^{\vartheta(t)} \int_{S^1} M_{kl} G_{St}^k(r - c_{St}t, \theta, x_3) \\ &\qquad\qquad\qquad\qquad \cdot G_{St}^l(r - c_{St}t, \theta, x_3) d\theta dr dx_3 \\ &= - \sum_{k,l=1}^{3} \int_{-\infty}^{\infty} \int_{-\vartheta(t)}^{\vartheta(t)} \int_{S^1} M_{kl} G_{St}^k(r,\theta,x_3) \cdot G_{St}^l(r,\theta,x_3) d\theta dr dx_3. \end{aligned}$$

Thus we have (5.11). □

The following theorem shows that asymptotic energy distributions concern the asymptotic concentration of energy in expanding spherical region $B(t, \vartheta(t))$.

Theorem 5.4. *Assume that*

$$f \in D(A^{\frac{1}{2}}) \cap \mathcal{H}, \quad g \in \mathcal{H} \cap D(A^{-\frac{1}{2}}), \quad Dis(c_{s_i}^2) > 0.$$

Let $\vartheta(t)$ satisfy (5.10) and also

$$\lim_{t\to\infty} \vartheta(t) = \infty. \tag{5.13}$$

Then

$$\lim_{t\to\infty} E(v_{1j}^{St}, B(t,\vartheta(t)), t) = E(v_{1j}^{St}, \mathbf{R}^3, 0), \tag{5.14}$$

and hence

$$\lim_{t\to\infty} E(v_{1j}^{St}, \mathbf{R}^3 \setminus B(t,\vartheta(t)), t) = 0. \tag{5.15}$$

Proof. From the triangle inequality

$$\left| E(v_{1j}^{St}, B, t)^{\frac{1}{2}} - E(v_{1j}^{St\infty}, B, t)^{\frac{1}{2}} \right| \le E(v_{1j}^{St} - v_{1j}^{St\infty}, B, t)^{\frac{1}{2}}.$$

Theorem 5.1 implies

$$\lim_{t\to\infty} E(v_{1j}^{St} - v_{1j}^{St\infty}, B, t)^{\frac{1}{2}} \le \lim_{t\to\infty} \|v_{1j}^{St} - v_{1j}^{St\infty}\|_E = 0.$$

Lemma 5.3 implies

$$\begin{aligned}
\lim_{t\to\infty} E(v_{1j}^{St\infty}, B, t)^{\frac{1}{2}} &= \int_{\mathbf{R}} \int_{-\infty}^{\infty} \|G_{St}^0(r,\cdot,x_3)\|_{L^2(S^1)}^2 dr\rho(x_3)\, dx_3 \\
&\quad - \int_{\mathbf{R}} \int_{-\infty}^{\infty} \int_{S^1} \sum_{k,l=1}^{3} M_{kl} G_{St}^k(r,\theta,x_3) \cdot G_{St}^l(r,\theta,x_3)\, d\theta dr dx_3 \\
&= E(v_{1j}^{St\infty}, \mathbf{R}^3, 0).
\end{aligned}$$

This gives (5.14) and (5.15). □

The next corollary shows the transiency of the Stoneley components $v_{1j}^{St}(t,x)$ ($j \in M$) in the sense that the energy in any bounded region tends to 0 for $t \to \infty$.

Corollary 5.5. *Assume that*

$$f \in D(A^{\frac{1}{2}}) \cap \mathcal{H}, \quad g \in \mathcal{H} \cap D(A^{-\frac{1}{2}}), \quad Dis(c_{s_i}^2) > 0.$$

Let $K \subset \mathbf{R}^3$ be any bounded set. Then we have

$$\lim_{t\to\infty} E(v_{1j}^{St}, K, t) = 0. \tag{5.16}$$

Proof. By the boundedness of $K \subset \mathbf{R}^3$, there exists $r > 0$ such that

$$K \subset \Omega_r = \{x \in \mathbf{R}^3, |x| \leq r\}.$$

In theorem 5.4, if we take

$$-\vartheta(t) = r - c_{St}t \geq -c_{St}t,$$

then

$$K \subset \Omega_r \subset \mathbf{R}^3 \setminus B(t, \vartheta(t)) \quad \text{for } \forall t \in \mathbf{R}.$$

Hence

$$0 \leq E(v_{1j}^{St}, K, t) \leq E(v_{1j}^{St}, \mathbf{R}^3 \setminus B(t, \vartheta(t)), t),$$

so (5.16) follows from Theorem 5.4. □

The main result of this paper is the following theorem. This theorem shows that the energy of the Stoneley components $v_{1j}^{St}(t,x)(j \in M)$ of v is asymptotically concentrated along the interface $x_3 = 0$.

Theorem 5.6. *Assume that*

$$f \in D(A^{\frac{1}{2}}) \cap \mathcal{H}, \quad g \in \mathcal{H} \cap D(A^{-\frac{1}{2}}), \quad Dis(c_{s_t}^2) > 0.$$

Then we have

$$\lim_{t\to\infty} E(v_{1j}^{St}, (C^-(\theta) \cup C^+(\theta)) \cap B(t, \vartheta(t)), t) = E(v_{1j}^{St}, \mathbf{R}^3, 0), \quad j \in M, \tag{5.17}$$

where

$$C^-(\theta) = \{x \in \mathbf{R}_-^3;\ -\theta(|x'|) < x_3 < 0\}, \tag{5.18}$$

$$C^+(\theta) = \{x \in \mathbf{R}_-^3;\ 0 < x_3 < \theta(|x'|)\}, \tag{5.19}$$

$$B(t, \vartheta(t)) = \{x \in \mathbf{R}^3;\ c_{St}t - \vartheta(t) \leq |x'| \leq c_{St}t + \vartheta(t),\ x_3 \in \mathbf{R}\}, \tag{5.20}$$

$$\vartheta(t) : \lim_{t\to\infty} \vartheta(t) = \infty,\ |\vartheta(t)| < 2c_{St}t, \tag{5.21}$$

$$\theta(|x'|) : \lim_{|x'|\to\infty} \theta(|x'|) = \infty,\ \text{monotone increasing function.} \tag{5.22}$$

Proof. It suffices to show that

$$\lim_{t\to\infty} E\left(v_{1j}^{St\infty}, \mathbf{R}^3 \setminus \left(((C^-(\theta) \cup C^+(\theta)) \cap B(t, \vartheta(t))\right), t\right) = 0. \tag{5.23}$$

Because the triangle inequality and Theorem 5.1 imply

$$\lim_{t\to\infty} \left| E(v_{1j}^{St}, K, t)^{\frac{1}{2}} - E(v_{1j}^{St\infty}, K, t)^{\frac{1}{2}} \right| = 0$$

for any $K \subset \mathbf{R}^3$. Note that

(5.24)
$$\begin{aligned}
&\mathbf{R}^3 \setminus (((C^-(\theta) \cup C^+(\theta)) \cap B(t, \vartheta(t)), t) \\
&= \{(\{x \in \mathbf{R}^3_-, x_3 \le -\theta(|x'|)\} \cup \{x \in \mathbf{R}^3_+, \theta(|x'|) \le x_3\}) \cap B(t, \vartheta(t))\} \\
&\cup \{|x'| \le c_{St}t - \vartheta(t), c_{St}t + \vartheta(t) \le |x'|, \ x_3 \in \mathbf{R}\} \\
&= G_1 \cup G_2,
\end{aligned}$$

and

(5.25)
$$E(v_{1j}^{St\infty}, G_i, t) = \int_{G_i} \left(|v_{1j0}^{St\infty}| \rho(x_3) - \sum_{k,l=1}^{3} M_{kl} v_{1jk}^{St\infty} \cdot v_{1jl}^{St\infty} \right) dx, \quad i = 1, 2.$$

We consider the first term of the right-hand side of (5.25). By the change of variable $r' = r - c_{St}t$,

$$\begin{aligned}
&\|v_{1j0}^{St\infty}(t, \cdot)\|^2_{\mathcal{H}(G_1)} \\
&= \int_{c_{St}t - \vartheta(t)}^{c_{St}t + \vartheta(t)} \int_{S^1} \left(\int_{\theta(r)}^{\infty} + \int_{-\infty}^{-\theta(r)} \right) |G^0_{St}(r - c_{St}t, \theta, x_3)|^2 \rho(x_3) dx_3 d\theta dr \\
&= \int_{-\vartheta(t)}^{+\vartheta(t)} \int_{S^1} \left(\int_{\theta(r + c_{St}t)}^{\infty} + \int_{-\infty}^{-\theta(r + c_{St}t)} \right) |G^0_{St}(r, \theta, x_3)|^2 \rho(x_3) dx_3 d\theta dr.
\end{aligned}$$

The conditions (5.21) and (5.22) implies

$$\lim_{t \to \infty} \theta(r + c_{St}t) \ge \lim_{t \to \infty} \theta(-\vartheta(t) + 2_{St}t) = \infty.$$

Hence

$$\lim_{t \to \infty} \|v_{1j0}^{St\infty}(t, \cdot)\|^2_{\mathcal{H}(G_1)} = 0.$$

By the change of variable $r' = r - c_{St}t$,

$$\begin{aligned}
&\|v_{1j0}^{St\infty}(t, \cdot)\|^2_{\mathcal{H}(G_2)} \\
&= \int_{\mathbf{R}} \left(\int_{c_{St}t + \theta(r)}^{\infty} + \int_{-\infty}^{c_{St}t - \theta(r)} \right) \int_{S^1} |G^0_{St}(r - c_{St}t, \theta, x_3)|^2 d\theta dr \rho(x_3) dx_3 \\
&= \int_{\mathbf{R}} \left(\int_{-\theta(r)}^{\infty} + \int_{-\infty}^{-\theta(r)} \right) \int_{S^1} |G^0_{St}(r, \theta, x_3)|^2 d\theta dr \rho(x_3) dx_3.
\end{aligned}$$

From the condition (5.21), we have

$$\lim_{t \to \infty} \|v_{1j0}^{St\infty}(t, \cdot)\|^2_{\mathcal{H}(G_2)} = 0.$$

The second term of the right-hand side of (5.25) can be treated similarly. This completes the proof of Theorem 5.6. □

Finally, we consider the P, SV, SH components $v_{1j}^{\pm}(t, x)(j \in M)$, $v_{2k}^{\pm}(t, x)(k \in N)$. The Next theorem shows that the P, SV, SH components behave like free waves.

Theorem 5.7. *Assume that*

$$f \in D(A^{\frac{1}{2}}) \cap \mathcal{H}, \quad g \in \mathcal{H} \cap D(A^{-\frac{1}{2}}).$$

Then we have

(5.26)
$$\lim_{t\to\infty} E(v_{1j}^{\pm},\ S_{s_1}(t,\vartheta) \cup S_{p_1}(t,\vartheta) \cup S_{s_2}(t,\vartheta) \cup S_{p_2}(t,\vartheta), t) = E(v_{1j}^{\pm}, \mathbf{R}^3, 0), \quad j \in M,$$
(5.27)
$$\lim_{t\to\infty} E(v_{2k}^{\pm},\ S_{s_1}(t,\vartheta) \cup S_{s_2}(t,\vartheta), t) = E(v_{2k}^{\pm}, \mathbf{R}^3, 0), \qquad k \in N,$$

where

$$S_{s_1}(t,\vartheta(t)) = \{x \in \mathbf{R}_-^3;\ c_{s_1}t - \vartheta(t) \le |x| \le c_{s_1}t + \vartheta(t)\}, \tag{5.28}$$
$$S_{p_1}(t,\vartheta(t)) = \{x \in \mathbf{R}_-^3;\ c_{p_1}t - \vartheta(t) \le |x| \le c_{p_1}t + \vartheta(t)\}, \tag{5.29}$$
$$S_{s_2}(t,\vartheta(t)) = \{x \in \mathbf{R}_+^3;\ c_{s_2}t - \vartheta(t) \le |x| \le c_{s_2}t + \vartheta(t)\}, \tag{5.30}$$
$$S_{p_2}(t,\vartheta(t)) = \{x \in \mathbf{R}_+^3;\ c_{p_2}t - \vartheta(t) \le |x| \le c_{p_2}t + \vartheta(t)\}, \tag{5.31}$$
$$\vartheta(t) : \lim_{t\to\infty} \vartheta(t) = \infty. \tag{5.32}$$

The proof of this theorem is obtained by using Theorem 4.2 and modified Theorem 5.4.

The next corollary shows the transiency of the P, SV components $v_{1j}^{\pm}(t,x)$ ($j \in M$) and the SH components $v_{2k}^{\pm}(t,x)$ ($k \in N$) in the sense that the energy in any bounded region tends to 0 for $t \to \infty$.

Corollary 5.8. *Assume that*

$$f \in D(A^{\frac{1}{2}}) \cap \mathcal{H}, \quad g \in \mathcal{H} \cap D(A^{-\frac{1}{2}}).$$

Let $K \subset \mathbf{R}^3$ be any bounded set. Then we have

$$\lim_{t\to\infty} E(v_{1j}^{\pm}, K, t) = 0, \tag{5.33}$$
$$\lim_{t\to\infty} E(v_{2k}^{\pm}, K, t) = 0. \tag{5.34}$$

References

1. Dermenjian, Y. and Guillot, J. C., *Scattering of elastic waves in a perturbed isotropic half space with a free boundary. The limiting absorption principle*, Math. Meth. in the Appl. Sci. **10** (1988), 87–124.
2. Fujita, H., Kuroda, S. T. and Ito, S., *Functional analysis*, Iwanami-Shuppan, Tokyo, 1991. (Japanese)
3. Guillot, J. C., *Existence and uniqueness of a Rayleigh surface wave propagation along the free boundary of a transversely isotropic elastic half space*, Math. Meth. in the Appl. Sci. **8** (1986), 289–310.
4. Ito, H., *Extended Korn's inequalities and the associated best possible constants*, J.of Elasticity **24** (1990), 43–78.

5. Kato, T., *Perturbation for linear operators*, Springer, 1966.
6. Matsumura, M., *Asymptotic behavior at infinity for Green's functions of first order systems with characteristics of nonuniform multiplicity*, Publ. RIMS, Kyoto Univ. **12** (1976), 317–377.
7. Matsumura, M., *On properties of fundamental solutions for stationary and non stationary wave propagation problems*, functional equation **24**. (Japanese)
8. Shimizu, S., *Eigenfunction expansions for elastic wave propagation problems in stratified media R^3*, Tsukuba J. Math. (to appear).
9. Wakabayashi, S., *Eigenfunction expansions for symmetric systems of first order in the half-space R^n_+*, Publ. RIMS, Kyoto Univ. **11** (1975), 67–147.
10. Wilcox, C. H., *Scattering Theory for the d'Alembert Equation in Exterior Domains. Springer Lecture Notes in Mathematics*, vol. 442, Springer-Verlag, 1975.
11. Wilcox, C. H., *Transient electromagnetic wave propagation in a dielectric waveguide*, Istituto Nazionale di Alta Mathematica, Symposia Mathematica **XVIII** (1976), 239–277.
12. Wilcox, C. H., *Sound propagation in stratified fluids*, Applied Mathematical Sciences, vol. 50, Springer-Verlag New York Berlin Heidelberg Tokyo, 1984.
13. Wilcox, C. H., *Spectral and asymptotic analysis of acoustic wave propagation*, Reidei, 1977.
14. Wilcox, C. H., *Spectral Analysis of the Pekeris Operator in the Theory of Acoustic Wave Propagation in Shallow Water*, Arch. Rational Mech. Anal. **60** (1976), 259–300.

Acknowledgement

The author would like to take this opportunity to thank Professor Mutsuhide Matsumura for his generous support, invaluable advice and enlightening guidance throughout this research. The author also would like to thank Professor Seiichiro Wakabayashi for his invaluable suggestions.