On Riemann-Roch Graphs and Coverings over d-gonal Curves

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The purpose of this paper is to present a study of compact Riemann surfaces from two viewpoints. One is on the Weierstrass gap set of several points on a compact Riemann surface M and the other is on the d-gonality of M.

In Chapter I, we investigate the Weierstrass gap set $G(P_1, ..., P_n)$ of n distinct points $P_1, ..., P_n$ on M, and we introduce a certain type of graph for that purpose. In particular we obtain the lower bound on the cardinalities $\sharp G(P_1, ..., P_n)$ for an arbitrary n, and the upper bound on the cardinarities $\sharp G(P_1, P_2, P_3)$. We also show that these bounds are sharp.

In Chapter II and III, we focus on the d-gonality of a compact Riemann surface M. In Chapter II, we consider a covering map $\pi' : M' \to M$, where both M' and M are d-gonal curves. We will show that π' corresponds to a unique covering map $\pi : \mathbf{P}_1' \to \mathbf{P}_1$ with Riemann spheres $\mathbf{P}_1'$ and $\mathbf{P}_1$, and conversely every π' as above is obtained from a covering map $\pi : \mathbf{P}_1' \to \mathbf{P}_1$. As an application, we get the list of all ramification types of $\pi' : M' \to M$ when M is a cyclic d-gonal curve.

In Chapter III, we assume that M is a cyclic d-gonal curve, and give some remarks on M.

I On Riemann-Roch Graphs([7])

1

1 − 1 Let M be a compact Riemann surface of genus $g \geq 2$, and let $P_1, P_2, \ldots, P_n$ be distinct points on M. We define the Weierstrass gap set $G(P_1, P_2, .., P_n)$ by

$G(P_1, P_2, .., P_n) := \{(\gamma_1, \gamma_2, \ldots, \gamma_n) \in \mathbf{N}_0 \times \cdots \times \mathbf{N}_0 |\ \nexists$ meromorphic function f on M whose pole divisor $(f)_\infty$ is $\gamma_1 P_1 + \gamma_2 P_2 + \cdots + \gamma_n P_n\}$,

where $\mathbf{N}_0$ is the set of non-negative integers.

When $n = 1$, $G(P_1)$ is the set of Weierstrass gaps at P_1. One of the essential differences between the case $n = 1$ and the case $n \geq 2$ is that the cardinarity $\#G(P_1)$ is the constant g but $\#G(P_1, .., P_n)(n \geq 2)$ depends on the choice of M and the set of points $\{P_1, \ldots, P_n\}$ on M.

Kim has given formulas for $\#G(P_1, P_2)$ and shown the following inequalities

$$\frac{(g^2 + 3g)}{2} \leq \#G(P_1, P_2) \leq \frac{(3g^2 + g)}{2}.$$

Moreover he has proved that the upper bound $\frac{(3g^2+g)}{2}$ can be realized if and only if "M is hyperelliptic and $|2P_1| = |2P_2| = g_2^1$" ([13]). The lower bound $\frac{(g^2+3g)}{2}$ can be attained by taking general points P_1 and P_2 on arbitrary M. This is stated in [1] without proof, and has been proved by Homma([5]). He also has translated Kim's formulas into other practical ones, and added several interesting remarks in the case where M is a curve defined over a field of characteristic $p \geq 0$ ([5]). Through their works it seems to be helpful to use a certain type of graph $D^{(n)}$ defined as follows.

Definition 1 − 2(Riemann-Roch Graph) Fix positive integers g and n. Let $\mathbf{e}_i$ be the n-tuple $(0, .., 0, 1, 0, .., 0)$(i.e., the i-th component of $\mathbf{e}_i$ is 1) in $\mathbf{N}_0^n$. For an element $(\gamma_1, \ldots, \gamma_n) \in \mathbf{N}_0^n$, we also write $\sum_i \gamma_i \mathbf{e}_i$. Let $V^{(n)}$ denote the subset

$$\{\Gamma = (\gamma_1, \ldots, \gamma_n) | \gamma_i \in \mathbf{N}_0, 0 \leq \gamma_1 + \cdots + \gamma_n \leq 2g - 1\}$$

of $\mathbf{N}_0^n$.

For $\Gamma = \sum_i \gamma_i \mathbf{e}_i \in V^{(n)}$, define $deg\ \Gamma$ by

$$deg\ \Gamma := \sum_i \gamma_i .$$

Let $\Gamma = \sum_i \gamma_i \mathbf{e}_i$ and $\Gamma' = \sum_i \gamma'_i \mathbf{e}_i$ be in $V^{(n)}$. Then we write

$$\Gamma' \leq \Gamma \qquad \text{if} \quad \gamma'_i \leq \gamma_i \qquad \text{for } i = 1, 2, ..., n.$$

Let $E^{(n)}$ denote the subset

$$\{(\Gamma - \mathbf{e}_i)\Gamma | \Gamma \in V^{(n)} \text{ and } \Gamma - \mathbf{e}_i \in V^{(n)}\}$$

of $V^{(n)} \times V^{(n)}$, where $\Gamma - \mathbf{e}_i = (\gamma_1, ..., \gamma_i - 1, ..., \gamma_n)$ with $\Gamma = (\gamma_1, ..., \gamma_i, ..., \gamma_n)$.

Let $D^{(n)}$ denote the graph $\{V^{(n)}, E^{(n)}\}$ consisting of $V^{(n)}$ and $E^{(n)}$ as a set of vertices and a set of edges respectively. When $\Gamma' \leq \Gamma$, any chain of successive $(deg\ \Gamma - deg\ \Gamma')$ edges

$$\Gamma'\Gamma_1, \Gamma_1\Gamma_2, \Gamma_2\Gamma_3, ..., \Gamma_{deg\ \Gamma - deg\ \Gamma' - 1}\Gamma$$

is called a path from Γ' to Γ. Of course these paths are not unique even though Γ and Γ' are fixed, but we write $\Gamma'\Gamma$ for them abusively. Moreover, each edge is labeled "0" or "1" , which is called the weight of the edge, and the labeling has the following properties.

$*_n)$

$*_n - 1)$ Let $\Gamma = \sum_i \gamma_i \mathbf{e}_i$ and $\tilde{\Gamma} = \sum_i \tilde{\gamma}_i \mathbf{e}_i$ be in $V^{(n)}$. Assume $\tilde{\Gamma} \geq \Gamma$ and $\gamma_i = \tilde{\gamma}_i > 0$ with some i. If the edge $(\Gamma - \mathbf{e}_i)\Gamma$ is of weight 1, then so is the edge $(\tilde{\Gamma} - \mathbf{e}_i)\tilde{\Gamma}$.

$*_n - 2)$ Let $O = \sum_i 0\mathbf{e}_i$ and $\Gamma = \sum_i \gamma_i \mathbf{e}_i$ be in $V^{(n)}$ with $deg\ \Gamma = 2g - 1$. The number of edges of weight 1(resp.0) on any path $O\Gamma$ is $g - 1$(resp.g).

From now on, we will call the above type of graph $(D^{(n)}, *_n)$ a Riemann-Roch graph.

Definition 1 − 3 Define the gap set $G^{(n)}$ of $(D^{(n)}, *_n)$ by

$$G^{(n)} := \{\Gamma \in V^{(n)} | \exists i \text{ such that the edge } (\Gamma - \mathbf{e}_i)\Gamma \in E^{(n)} \text{ is of weight } 0\}.$$

$H^{(n)}$ denotes the compliment $V^{(n)} \setminus G^{(n)}$ of $G^{(n)}$ in $V^{(n)}$.

Remark $O = (0, ..., 0) \in H^{(n)}$.

1 − 4 Let M and $\{P_1, ..., P_n\}$ be as before . Then the following facts on an effective divisor $E = \gamma_1 P_1 + \gamma_2 P_2 + \cdots + \gamma_n P_n$ are known:

1) if $deg\ E = \gamma_1 + \cdots + \gamma_n = 2g - 1$, then $l(E) = h^0(\mathcal{O}(E)) = g$;
2) if P_i is not a base point of the linear system $|E|$, then P_i is not a base point of any linear system

$$|\tilde{\gamma}_1 P_1 + \tilde{\gamma}_2 P_2 + \cdots + \tilde{\gamma}_i P_i + \cdots + \tilde{\gamma}_n P_n|,$$

where $\tilde{\gamma}_k \geq \gamma_k (k = 1, ..., n)$ and $\tilde{\gamma}_i = \gamma_i$.

Identify each effective divisor $E = \sum_{i=1}^{n} \gamma_i P_i$ of degree $\leq 2g - 1$ with the vertex $\Gamma = \sum_{i=1}^{n} \gamma_i \mathbf{e}_i$, and give 1 to the edges $(\Gamma - \mathbf{e}_i)\Gamma$ if and only if P_i is not a base point of $|\sum_{i=1}^{n} \gamma_i P_i|$. Then we get a Riemann-Roch graph. $D_M(P_1, ..., P_n)$ denotes this graph. Then the gap set $G^{(n)}$ obtained from $D_M(P_1, ..., P_n)$ coincides with the Weierstrass gap set $G(P_1, ..., P_n)$ in 1-1.

1 − 5 In this paper, we start studying Riemann-Roch graphs $D^{(n)}$ and their gap sets $G^{(n)}$ in general(i.e., they are not necessarily obtained from M and $\{P_1, ..., P_n\}$).

In particular we will prove that

$$\#G^{(n)} \geq \binom{n+g}{g} - 1$$

and there is a unique graph $D^{(n)}$ satisfying $\#G^{(n)} = \binom{n+g}{g}$, where $\binom{a}{b} = a!/(a-b)!b!$ for integers $a \geq b \geq 0$(Theorem 2-3).

About upper bounds of $\#G^{(n)}$, we calculate in case $n = 3$, and show that

$$\#G^{(3)} \leq \frac{g(7g^2 + 6g + 5)}{6}$$

and there is a unique graph satisfying $\#G^{(3)} = \frac{g(7g^2+6g+5)}{6}$. Moreover this graph is exactly equal to $D_M(P_1, P_2, P_3)$, where M is hyperelliptic and P_1, P_2, P_3 are satisfying $|2P_1| = |2P_2| = |2P_3| = g_2^1$(Theorem 4-9).

Finally we try to replace $*_n)$ with another set of conditions in order to study a Riemann-Roch graph in detail(Appendix).

§2

Fix a Riemann-Roch graph $(D^{(n)}, *_n)$. Then we can easily have the following lemma.

Lemma 2 − 1 *The condition $* - 2)$ is equivalent to the following set*

$\{A), B), C)\}$ *of conditions.*

A) Let Γ *and* Γ' *be in* $V^{(n)}$ *with* $\Gamma \geq \Gamma'$. *Evry path from* Γ' *to* Γ *has the same number of edges of weight 1.*
We will write $[\Gamma'\Gamma]$ *for the number of edges of weight 1 on a path* $\Gamma'\Gamma$.

B) Let Γ, Γ' *and* Γ'' *be in* $V^{(n)}$ *with* $\Gamma' \leq \Gamma, \Gamma' \leq \Gamma''$, *and* $deg\ \Gamma = deg\ \Gamma'' = 2g-1$. *Then*

$$[\Gamma'\Gamma''] = [\Gamma'\Gamma].$$

C) Let $\Gamma = (2g-1)\mathbf{e}_1$ *and* $O = (0, ..., 0)$ *be in* $V^{(n)}$.
Then

$$[O\Gamma] = g-1.$$

Definition 2 − 2 For $\Gamma \in V^{(n)}$, define non-negative integers $l(\Gamma)$ and $i(\Gamma)$ by $l(\Gamma) := [O\Gamma]+1(\geq 1)$ and by $i(\Gamma) := l(\Gamma)-1+g-deg\,\Gamma(\geq 0)$ respectively.

Then we have:

Lemma 2 − 3 *If* Γ *and* Γ' *are in* $V^{(n)}$ *satisfying* $deg\ \Gamma = 2g-1$ *and* $\Gamma' \leq \Gamma$, *then* $i(\Gamma')$ *is equal to the number of edges of weight 0 on a path* $\Gamma'\Gamma$, *and this number does not depend on the choice of a path from* Γ' *to* Γ.

Let $(D^{(n-1)}, *_{n-1})$ be the subgraph of $(D^{(n)}, *_n)$ obtained by identifying $(\gamma_1, ..., \gamma_{n-1}) \in V^{(n-1)}$ with $(\gamma_1, ..., \gamma_{n-1}, 0) \in V^{(n)}$ and restricting $*_n$) to $V^{(n-1)}$. Then $G^{(n-1)}$ (resp. $H^{(n-1)}$) of this subgraph $(D^{(n-1)}, *_{n-1})$ is embedded in $G^{(n)}$ (resp. $H^{(n)}$) of $(D^{(n)}, *_n)$ by the same manner as above. We represent the element of $V^{(n-1)}$ by Γ_n (the index n of Γ_n suggests that Γ_n is obtained by omitting the n-th coordinate of some element Γ of $V^{(n)}$). For $\Gamma_n = (\gamma_1, ..., \gamma_{n-1}) \in V^{(n-1)}$ and $\gamma \in \mathbf{N}_0$, (Γ_n, γ) denotes $(\gamma_1, ..., \gamma_{n-1}, \gamma) \in \mathbf{N}_0^n$.

Definition 2 − 4 For $\Gamma_n = (\gamma_1, ..., \gamma_{n-1}) \in V^{(n-1)}$, define a subset Δ_{Γ_n} of $\mathbf{N}_0$ by

$$\Delta_{\Gamma_n} := \{\delta | \delta \in \mathbf{N}_0, (\Gamma_n, \delta) \in H^{(n)}\},$$

and define a non-negative integer δ^{Γ_n} by

$$\delta^{\Gamma_n} := \begin{cases} \min\{\delta | \delta \in \Delta_{\Gamma_n}\} & \text{if } \Delta_{\Gamma_n} \neq \emptyset \\ 2g - deg\ \Gamma_n (\geq 1) & \text{if } \Delta_{\Gamma_n} = \emptyset. \end{cases}$$

Lemma 2 − 5 *Let Δ_{Γ_n} and δ^{Γ_n} be as above. Then:*

i) *δ^{Γ_n} satisfies $0 \leq \delta^{\Gamma_n} \leq 2g - 1 - deg\ \Gamma_n (\leq 2g-1)$ if and only if $\Delta_{\Gamma_n} \neq \emptyset$;*

ii) *if $\Delta_{\Gamma_n} = \emptyset$, then $deg\ \Gamma_n > 0$ and $\delta^{\Gamma_n} = 2g - deg\ \Gamma_n \leq 2g - 1$;*

iii) *δ^{Γ_n} satisfies $\delta^{\Gamma_n} > 0$ if and only if $\Gamma_n \in G^{(n-1)}$.*

Moreover we have a surjective map

$$\{\Gamma_n | \Gamma_n \in G^{(n-1)}\} \to \{\gamma (> 0) | (O_n, \gamma) \in G^{(n)}\}$$

defined by $\Gamma_n \mapsto (O_n, \delta^{\Gamma_n})$, where $O_n = (0, .., 0) \in V^{(n-1)}$.

Proof) i) This follows from the fact that $\Delta_{\Gamma_n} \neq \emptyset$ is equivalent to $(\Gamma_n, \delta^{\Gamma_n}) \in V^{(n)}$.

ii) If $\Delta_{\Gamma_n} = \emptyset$, then $deg\ \Gamma_n \geq 1$. In fact, $deg\ \Gamma_n = 0$ means $\Gamma_n = O_n$. But O_n is in $H^{(n-1)}$ and $\delta^{O_n} = 0$. Therefore we get ii) by Definition 1-4.

iii) The first half of iii) follows from the fact that $\delta^{\Gamma_n} = 0$ is equivalent to $(\Gamma_n, 0) \in H^{(n)}$(i.e., $\Gamma_n \in H^{(n-1)}$).

We will prove that the map in iii) is well-defined, that is, $(O_n, \delta^{\Gamma_n}) \in G^{(n)}$ for $\Gamma_n \in G^{(n-1)}$.
Assume that there is a $\Gamma_n \in V^{(n-1)}$ satisfying

$$\delta^{\Gamma_n} > 0 \quad \text{and} \quad (O_n, \delta^{\Gamma_n}) \in H^{(n)}. \qquad \cdots\cdots 2-5-1)$$

Then $\qquad [(O_n, \delta^{\Gamma_n}) - \mathrm{e}_n, (O_n, \delta^{\Gamma_n})] = 1.$

Thus, by $*_n - 1)$, we have

$$[\{(\Gamma_n, \delta^{\Gamma_n}) - \mathrm{e}_i\} - \mathrm{e}_n, \{(\Gamma_n, \delta^{\Gamma_n}) - \mathrm{e}_i\}] = 1 \qquad \cdots\cdots 2-5-2)$$

for all i satisfying $\gamma_i > 0$ and $i \neq n$.

<u>case</u> $\Delta_{\Gamma_n} \neq \emptyset$

As $(\Gamma_n, \delta^{\Gamma_n}) \in H^{(n)}$, we have

$$[(\Gamma_n, \delta^{\Gamma_n}) - \mathrm{e}_i, (\Gamma_n, \delta^{\Gamma_n})] = 1 \qquad \cdots\cdots 2-5-3)$$

for all i satisfying $1 \le i \le n$ and $\gamma_i > 0$.
Define a subset Θ of $\mathbf{N}_0$ by

$$\Theta := \{\delta \in \mathbf{N}_0 | [(\Gamma_n, \delta) - \mathbf{e}_i, (\Gamma_n, \delta)] = 1 \text{ for all } i \text{ satisfying } \gamma_i > 0 \text{ and } i \ne n\}.$$

By 2-5-3, $\Theta \ni \delta^{\Gamma_n}$ and $\Theta \ne \emptyset$. Then we can define a non-negative integer $\tilde{\delta}$ by

$$\tilde{\delta} := \min\{\delta \in \mathbf{N}_0 | \delta \in \Theta\}.$$

On this $\tilde{\delta}$, we have

$$[(\Gamma_n, \tilde{\delta}) - \mathbf{e}_i, (\Gamma_n, \tilde{\delta})] = 1 \text{ for all } i \text{ satisfying } 1 \le i \le n \text{ and } \gamma_i > 0. \cdots\cdots 2-5-4)$$

(i.e., $\tilde{\delta} \in \Delta_{\Gamma_n}$.)

In fact, this is from the definition of Θ when $i = 1, ..., n-1$.
If $[(\Gamma_n, \tilde{\delta}) - \mathbf{e}_n, (\Gamma_n, \tilde{\delta})] = 0$, then $[\{(\Gamma_n, \tilde{\delta}) - \mathbf{e}_n\} - \mathbf{e}_i, \{(\Gamma_n, \tilde{\delta}) - \mathbf{e}_n\}] = 1$ for all i satisfying $i \ne n$ and $\gamma_i > 0$ by Lemma 2-1 A). Therefore $\tilde{\delta} - 1 \in \Theta$, and this contradicts to the definition of $\tilde{\delta}$. Hence 2-5-4) is correct when $i = n$. By 2-5-4) and the definiton of δ^{Γ_n}, we have $\tilde{\delta} \ge \delta^{\Gamma_n}$.
On the other hand, by Lemma 1-1 A), 1-5-2) and $(\Gamma_n, \delta^{\Gamma_n}) \in H^{(n)}$,

$$[\{(\Gamma_n, \delta^{\Gamma_n}) - \mathbf{e}_n\} - \mathbf{e}_i, \{(\Gamma_n, \delta^{\Gamma_n}) - \mathbf{e}_n\}] = 1$$

for all i satisfying $\gamma_i > 0$ and $i \ne n$.
Hence $\delta^{\Gamma_n} - 1 \in \Theta$ and $\tilde{\delta} \le \delta^{\Gamma_n} - 1$. This is a contradiction. Thus we get $(O_n, \delta^{\Gamma_n}) \in G^{(n)}$.

<u>case</u> $\Delta_{\Gamma_n} = \emptyset$

We have $\delta^{\Gamma_n} = 2g - deg\,\Gamma_n$ by Definition 1-4, and $(\Gamma_n, \delta^{\Gamma_n}) - \mathbf{e}_n \in V^{(n)}$.
Assume

$$[\{(\Gamma_n, \delta^{\Gamma_n}) - \mathbf{e}_n\} - \mathbf{e}_i, \{(\Gamma_n, \delta^{\Gamma_n}) - \mathbf{e}_n\}] = 1$$

for all i satisfying $\gamma_i > 0$ and $i \ne n$.

Then by the same way as in the case $\Delta_{\Gamma_n} \ne \emptyset$, we can find a positive integer $\tilde{\delta}$ satisfying $\tilde{\delta} \le 2g - 1 - deg\,\Gamma_n$ and $(\Gamma_n, \tilde{\delta}) \in H^{(n)}$. This contradicts to $\Delta_{\Gamma_n} = \emptyset$. So there is an i satisfying

$$[\{(\Gamma_n, \delta^{\Gamma_n}) - \mathbf{e}_n\} - \mathbf{e}_i, \{(\Gamma_n, \delta^{\Gamma_n}) - \mathbf{e}_n\}] = 0.$$

By Lemma 2-1 B),

$$[\{(\Gamma_n, \delta^{\Gamma_n}) - \mathbf{e}_i\} - \mathbf{e}_n, \{(\Gamma_n, \delta^{\Gamma_n}) - \mathbf{e}_i\}] = 0.$$

Then, by $*_n - 1)$,

$$[(O_n, \delta^{\Gamma_n}) - \mathbf{e}_n, (O_n, \delta^{\Gamma_n})] = 0 \text{ and } (O_n, \delta^{\Gamma_n}) \in G^{(n)}.$$

Thus our map is well-defined.

Next we will prove the surjectivity of our map.
Fix $(O_n, \gamma) \in G^{(n)}(\gamma > 0)$. Define a subset Δ of $\mathbf{N}_0$ and a positive integer $\tilde{\gamma}_1$ by

$$\Delta := \{\gamma_1 | (\gamma_1, 0, .., 0, \gamma) \in H^{(n)}\}$$

and by

$$\tilde{\gamma}_1 := \begin{cases} \min\{\gamma_1 | \gamma_1 \in \Delta\} & \text{if } \Delta \neq \emptyset \\ 2g - \gamma & \text{if } \Delta = \emptyset \end{cases}$$

respectively.
Let $\tilde{\Gamma}_n = (\tilde{\gamma}_1, 0, .., 0) \in V^{(n-1)}$. Let $\Delta_{\tilde{\Gamma}_n}$ and $\delta^{\tilde{\Gamma}_n}$ be as in Definition 1-4.

We will show $\delta^{\tilde{\Gamma}_n} = \gamma$.

<u>case</u> $\Delta \neq \emptyset$

Since $(\tilde{\Gamma}_n, \gamma)$ is in $H^{(n)}$, we have $\gamma \in \Delta_{\tilde{\Gamma}_n}$. Now assume that γ satisfies

$$\delta^{\tilde{\Gamma}_n} = \min\{\gamma' | \gamma' \in \Delta_{\tilde{\Gamma}_n}\} < \gamma.$$

Then, by $*_n - 1)$,

$$[\{(\tilde{\Gamma}_n, \gamma) - \mathbf{e}_n\} - \mathbf{e}_1, \{(\tilde{\Gamma}_n, \gamma) - \mathbf{e}_n\}] = 1. \qquad \cdots\cdots\cdots 2-5-5)$$

By 2-5-5), Lemma 1-1 A) and $(\tilde{\Gamma}_n, \gamma) \in H^{(n)}$, we have

$$[\{(\tilde{\Gamma}_n, \gamma) - \mathbf{e}_1\} - \mathbf{e}_n, \{(\tilde{\Gamma}_n, \gamma) - \mathbf{e}_1\}] = 1. \qquad \cdots\cdots\cdots 2-5-6)$$

Define

$$\Phi := \{\gamma_1 | [(\gamma_1, 0, .., 0, \gamma) - \mathbf{e}_n, (\gamma_1, 0, .., 0, \gamma)] = 1\}.$$

By 2-5-6), $\tilde{\gamma}_1 - 1 \in \Phi$, and we can define a positive integer $\tilde{\gamma}_1'$ by $\tilde{\gamma}_1'$ =min$\{\gamma_1 | \gamma_1 \in \Phi\}$. Then $\tilde{\gamma}_1' \leq \tilde{\gamma}_1 - 1$. But $(\tilde{\gamma}_1', 0, .., 0, \gamma) \in H^{(n)}$ by the minimality of $\tilde{\gamma}_1'$ and Lemma1-1 A). This is a contradiction. Thus we get $\delta^{\tilde{\Gamma}_n} = \gamma$.

<u>case</u> $\Delta = \emptyset$

If $\Delta_{\tilde{\Gamma}_n} = \emptyset$, then $\delta^{\tilde{\Gamma}_n} = 2g - deg\ \tilde{\Gamma}_n = 2g - \tilde{\gamma}_1 = \gamma$ by the definition of $\delta^{\tilde{\Gamma}_n}$ and $\tilde{\gamma}_1$. Then it is sufficient to show $\Delta_{\tilde{\Gamma}_n} = \emptyset$.
If $\Delta_{\tilde{\Gamma}_n} \neq \emptyset$, then there exists γ' such that $(\tilde{\Gamma}_n, \gamma') \in H^{(n)}$.
Because of $\gamma' < 2g - \tilde{\gamma}_1 = \gamma$ and $*_n - 1)$,

$$[\{(\tilde{\Gamma}_n, \gamma) - \mathbf{e}_n\} - \mathbf{e}_1, \{(\tilde{\Gamma}_n, \gamma) - \mathbf{e}_n\}] = 1.$$

By Lemma 2-1 B),

$$[\{(\tilde{\Gamma}_n,\gamma)-\mathbf{e}_1\}-\mathbf{e}_n,\{(\tilde{\Gamma}_n,\gamma)-\mathbf{e}_1\}]=1.$$

By the same argument in case $\Delta\neq\emptyset$, there exists an integer $\tilde{\gamma}_1'$ satisfying $\tilde{\gamma}_1'\leq\tilde{\gamma}_1-1$ and $(\tilde{\gamma}_1',0,..,0,\gamma)\in H^{(n)}$. This is a contradiction. Therefore we get $\Delta_{\tilde{\Gamma}_n}=\emptyset$. □

Definition 2 − 6 Let $\Gamma_n=(\gamma_1,...,\gamma_{n-1})\in V^{(n-1)}$. Assume $\Delta_{\Gamma_n}=\emptyset$. By the definition of δ^{Γ_n}, $deg\ \Gamma_n+\delta^{\Gamma_n}=2g$. Hence the n-tuple $(\Gamma_n,\delta^{\Gamma_n})$ is not in $V^{(n)}$. But we define $i(\Gamma_n,\delta^{\Gamma_n})$ and $l(\Gamma_n,\delta^{\Gamma_n})$

by $i(\Gamma_n,\delta^{\Gamma_n})=0$ and by $l(\Gamma_n,\delta^{\Gamma_n})=g+1$

respectively (See Definition 2-2).

Using the above notations we have the following equalities on $\#G^{(n)}$.

Theorem 2 − 7

(1)

$$\#G^{(n)}=\sum_{\Gamma_n\in H^{(n-1)}}i(\Gamma_n)+\sum_{\Gamma_n\in G^{(n-1)}}i(\Gamma_n,\delta^{\Gamma_n})+\sum_{\Gamma_n\in G^{(n-1)}}\delta^{\Gamma_n}.$$

(2)

$$\begin{aligned}\#G^{(n)}&=\sum_{\Gamma_n\in H^{(n-1)}}l(\Gamma_n)+\sum_{\Gamma_n\in G^{(n-1)}}l(\Gamma_n,\delta^{\Gamma_n})-\sum_{\Gamma_n\in V^{(n-1)}}deg\ \Gamma_n+(g-1)\times\#V^{(n-1)}\\&=\sum_{\Gamma_n\in H^{(n-1)}}l(\Gamma_n)+\sum_{\Gamma_n\in G^{(n-1)}}l(\Gamma_n,\delta^{\Gamma_n})-\sum_{k=0}^{2g-1}k\binom{n+k-2}{k}+(g-1)\binom{n+2g-2}{2g-1}.\end{aligned}$$

Proof) (1) Take $\Gamma_n=(\gamma_1,..,\gamma_{n-1})\in V^{(n-1)}$ and γ with $0\leq\gamma\leq 2g-1-deg\ \Gamma_n$.

Suppose $\Gamma_n\in H^{(n-1)}$ first. By $*_n-1)$, we can see that

$(\Gamma_n,\gamma)\in G^{(n)}$ if and only if "$\gamma>0$ and $[(\Gamma_n,\gamma)-\mathbf{e}_n,(\Gamma_n,\gamma)]=0$".

Then, by Lemma 2-3,

$\#\{\gamma|(\Gamma_n,\gamma)\in G^{(n)}\}=i(\Gamma_n)$ for $\Gamma_n\in H^{(n-1)}$. ⋯⋯2–7–1)

Next suppose $\Gamma_n\in G^{(n-1)}$.

If $\gamma\geq\delta^{\Gamma_n}$, then $[(\Gamma_n,\gamma)-\mathbf{e}_i,(\Gamma_n,\gamma)]=1$ for $i=1,...,n-1$. Thus we have

$$(\Gamma_n,\gamma)\in G^{(n)} \text{ if and only if } \begin{cases} \text{“}0\le\gamma<\delta^{\Gamma_n}\text{”} \\ \text{or} \\ \text{“}\gamma\ge\delta^{\Gamma_n} \text{ and } [(\Gamma_n,\gamma-1),(\Gamma_n,\gamma)]=0\text{”}. \end{cases}$$

Therefore, by Lemma 2-3,

$$\#\{\gamma|(\Gamma_n,\gamma)\in G^{(n)}\}=i(\Gamma_n,\delta^{\Gamma_n})+\delta^{\Gamma_n} \quad \text{for } \Gamma_n\in G^{(n-1)}. \qquad \cdots\cdots 2\text{–}7\text{–}2)$$

Thus we have the equation (1) by 2-7-1 and 2-7-2.

(2) This follows from $l(\Gamma)=i(\Gamma)+1+deg\ \Gamma-g$, $\#V^{(n-1)}=\binom{n+2g-2}{2g-1}$ and

$$\sum_{\Gamma_n\in V^{(n-1)}} deg\ \Gamma_n=\sum_{k=0}^{2g-1}k\binom{n+k-2}{k}. \qquad \square$$

§3 The lower bound of $\#\mathbf{G}^{(n)}$

In this section we will determine the lower bound of $\#G^{(n)}$, and show that there is a unique graph $(D^{(n)},*_n)$ which attains the lower bound of $\#G^{(n)}$.

Let the notation be as in §2. First we will prove the following lemma.

Lemma 3 − 1 *Let* $\Gamma=(\gamma_1,...,\gamma_n)\in V^{(n)}$. *Assume* $\gamma_i>0$ *and* $[\Gamma-\mathbf{e}_i,\Gamma]=1$ *for some* i. *Then there exists* $\Gamma'=(\gamma_1',\gamma_2',..,\gamma_n')\in H^{(n)}$ *that satisfies* $\Gamma'\le\Gamma$ *and* $\gamma_i'=\gamma_i$.

Proof) We may assume $i=1$. Define

$$\gamma_2':=\min\{\gamma|[(\gamma_1,\gamma,\gamma_3,..,\gamma_n)-\mathbf{e}_1,(\gamma_1,\gamma,\gamma_3,..,\gamma_n)]=1\}$$

for the above $\Gamma=(\gamma_1,\gamma_2,\gamma_3,..,\gamma_n)$.
Then $[(\gamma_1,\gamma_2',\gamma_3,..,\gamma_n)-\mathbf{e}_2,(\gamma_1,\gamma_2',\gamma_3,..,\gamma_n)]=1$.
In fact, if $[(\gamma_1,\gamma_2',\gamma_3,..,\gamma_n)-\mathbf{e}_2,(\gamma_1,\gamma_2',\gamma_3,..,\gamma_n)]=0$, then $[\{(\gamma_1,\gamma_2',\gamma_3,..,\gamma_n)-\mathbf{e}_2\}-\mathbf{e}_1,\{(\gamma_1,\gamma_2',\gamma_3,..,\gamma_n)-\mathbf{e}_2\}]=1$ by Lemma 2-1 A). This contradicts to the definition of γ_2'.
Next define

$$\gamma_3':=\min\Big\{\gamma\Big|[(\gamma_1,\gamma_2',\gamma,\gamma_4,..,\gamma_n)-\mathbf{e}_1,(\gamma_1,\gamma_2',\gamma,\gamma_4,..,\gamma_n)] = [(\gamma_1,\gamma_2',\gamma,\gamma_4,..,\gamma_n)-\mathbf{e}_2,(\gamma_1,\gamma_2',\gamma,\gamma_4,..,\gamma_n)]=1\Big\}.$$

Then

$$[(\gamma_1,\gamma_2',\gamma_3',\gamma_4,..,\gamma_n)-\mathbf{e}_3,(\gamma_1,\gamma_2',\gamma_3',\gamma_4,..,\gamma_n)]=1$$

by the same reason as above. After repeating these procedures, we get the Γ' that we want. □

Next we will define a filtration of $G^{(n)}$ by

$$G^{(n)}=A_0^{(n)}\supset A_1^{(n)}\supset A_2^{(n)}\supset\cdots\supset A_k^{(n)}\supset\cdots\supset A_{g-1}^{(n)}\supset A_g^{(n)}=\emptyset,$$

where

$$A_k^{(n)}:=\{\Gamma|i(\Gamma)\geq k,\Gamma\in G^{(n)}\}.$$

For each k, define subsets $B_k^{(n)}$ and $C_k^{(n)}$ of $A_k^{(n)}$ by

$$B_k^{(n)}=\{\ \Gamma\mid\Gamma=(\Gamma_n,\gamma)\in G^{(n)},\ \Gamma_n\in H^{(n-1)},\ i(\Gamma)\geq k\}$$

and by

$$C_k^{(n)}=\{\ \Gamma\mid\Gamma=(0_n,\gamma)\in G^{(n)},\ i(\Gamma)\geq k\}$$

respectively, where $0_n=(0,...,0)\in H^{(n-1)}$. Then we have

$$B_0^{(n)}\supset B_1^{(n)}\supset B_2^{(n)}\cdots\supset B_k^{(n)}\supset\cdots\supset B_{g-1}^{(n)}\supset B_g^{(n)},$$

$$C_0^{(n)}\supset C_1^{(n)}\supset C_2^{(n)}\cdots\supset C_k^{(n)}\supset\cdots\supset C_{g-1}^{(n)}\supset C_g^{(n)}$$

and

$$A_k^{(n)}\supset B_k^{(n)}\supset C_k^{(n)}(k=0,...,g).$$

$a_k^{(n)}$ and $b_k^{(n)}$ denote $\#A_k^{(n)}$ and $\#B_k^{(n)}$ respectively.

Then we have the following lemma.

Lemma 3 − 2 i) $b_k^{(n)}\geq g-k$ *for* $k=0,...,g$.
Moreover $b_k^{(n)}=g-k$ *if and only if* $B_k^{(n)}=C_k^{(n)}$.

ii) *The following conditions are equivalent:*

a) $b_0^{(n)}=g$;

b) $b_k^{(n)}=g-k$ *for* $k=0,1,...,g$;

c) $i(\Gamma_n)=0$ *for* $\Gamma_n\in H^{(n-1)}\backslash\{O_n\}$;

d) *take* $\tilde{\Gamma}_n\in V^{(n-1)}$ *with* $deg\ \tilde{\Gamma}_n=2g-1$. *Then the first* g *edges of any path from* O_n *to* $\tilde{\Gamma}_n$ *are of weight* 0;

e) $G^{(n-1)}=\{\Gamma_n\in V^{(n-1)}|0<deg\ \Gamma_n\leq g\}$.

Proof) i) By Lemma 2-3, we have $\#C_k^{(n)} = g - k (k = 0, ..., g)$. Then i) follows from $B_k\ (n) \supset C_k\ (n) (k = 1, ..., g)$.

ii) *a)* $\Longleftrightarrow$ *b)*

We can easily see that

$$\begin{aligned} b_0^{(n)} = g & \iff B_0^{(n)} = C_0^{(n)} \\ & \iff B_k^{(n)} = C_k^{(n)} (k = 0, .., g). \\ & \iff b_k^{(n)} = g - k. \end{aligned}$$

b) $\Longleftrightarrow$ *c)*

If $b_k^{(n)} > g - k$ for some k, then there exists $\Gamma = (\Gamma_n, \gamma) \in G^{(n)}$ with $\Gamma_n \in H^{(n-1)} \backslash \{0_n\}$ and $i(\Gamma) \geq k$. By Lemma 2-3, $i(\Gamma_n) \geq k + 1$. Thus we have *b)* $\Leftarrow$ *c)*, and vice versa.

c) $\Rightarrow$ *d)*

Suppose c) to be true. Fix a path $0_n \tilde{\Gamma}_n$ with $deg\ \tilde{\Gamma}_n = 2g - 1$. We denote this path by $\mathcal{P}$. Take a vertex $\Gamma_n = (\gamma_1, .., \gamma_i, .., \gamma_{n-1}) \neq 0_n$ on $\mathcal{P}$ that satisfies $\gamma_i > 0$ and $[\Gamma_n - e_i, \Gamma_n] = 1$ for some $1 \leq i \leq n - 1$. Then there exists $\Gamma'_n = (\gamma'_1, .., \gamma'_i, .., \gamma'_{n-1}) \in H^{(n-1)} \backslash \{0_n\}$ that satisfies $\Gamma'_n \leq \Gamma_n$ and $\gamma_i = \gamma'_i$ by Lemma 3-1.
Since $i(\Gamma'_n) = 0$ by c), there is no edge of weight 0 on any path $\Gamma'_n \tilde{\Gamma}_n$. So there is no edge of weight 0 between Γ_n and $\tilde{\Gamma}_n$ on $\mathcal{P}$. By $*_n - 2)$ we get d).

d) $\Rightarrow$ *e)*

By $*_n - 2)$, d) implies that $\Gamma_n \in G^{(n-1)}$ if and only if $deg\ \Gamma_n \leq g$.

e) $\Rightarrow$ *c)*

e) is equivalent to the fact that $\Gamma_n \in H^{(n-1)} \backslash 0_n$ if and only if $deg\ \Gamma_n > g$. This implies c). □

Now we will show the main theorem of this section.

Theorem 3 − 3 i) *For $n \geq 2$, the following conditions are equivalent:*

(1) $G^{(n)} = \{\Gamma | 0 < deg\ \Gamma \leq g\}$;

(2) $a_0^{(n)} = \#G^{(n)}$ *is minimal for all types of* $(D^{(n)}, *_n)$;

(3) *For each* $k (= 0, ..., g - 1)$,
$a_k^{(n)}$ *is minimal for all types of* $(D^{(n)}, *_n)$.

ii) *The lower bound of* $\#G^{(n)}$ *is*

$$\binom{n+g}{g} - 1,$$

which is only attainable by a unique graph defined by (1).

Proof) Let $(D^{(n)}, *_n)$ be an arbitrary Riemann-Roch graph, and let $(D^{(n-1)}, *_{n-1})$ be the subgraph of it as before. Since $i(\Gamma_n) = k$ for $\Gamma_n \in A_k^{(n-1)} \backslash A_{(k+1)}^{(n-1)}$, we have

$$\#\{\gamma > 0 | [(\Gamma_n, \gamma - 1), (\Gamma_n, \gamma)] = 0, deg\ \Gamma_n + \gamma \leq 2g - 1\} = k.$$

Of course $(\Gamma_n, \gamma) \in G^{(n)}$ if $[(\Gamma_n, \gamma - 1), (\Gamma_n, \gamma)] = 0$. Watching $(\Gamma_n, 0) \in G^{(n)}$ for $\Gamma_n \in G^{(n-1)}$, we have

$$I_k \quad \begin{aligned} &\#\{\gamma \geq 0 | i(\Gamma_n, \gamma) \geq 0, (\Gamma_n, \gamma) \in G^{(n)}\} = \#\{\gamma | (\Gamma_n, \gamma) \in G^{(n)}\} \geq k + 1 \\ &\#\{\gamma \geq 0 | i(\Gamma_n, \gamma) \geq 1, (\Gamma_n, \gamma) \in G^{(n)}\} \qquad \geq k \\ &\qquad \cdots\cdots \\ &\#\{\gamma \geq 0 | i(\Gamma_n, \gamma) \geq k, (\Gamma_n, \gamma) \in G^{(n)}\} \qquad \geq 1 \\ &\text{for } \Gamma_n \in A_k^{(n-1)} \backslash A_{(k+1)}^{(n-1)} \qquad (k = 0, 1, ..., g - 1). \end{aligned}$$

By using I_k for $k = 0, ..., g - 1$, we have

$$\begin{aligned} a_0^{(n)} &\geq (a_0^{(n-1)} - a_1^{(n-1)}) + 2(a_1^{(n-1)} - a_2^{(n-1)}) + \cdots + (g-1)(a_{g-2}^{(n-1)} - a_{g-1}^{(n-1)}) + g\, a_{g-1}^{(n-1)} + b_0^{(n)} \\ a_1^{(n)} &\geq (a_1^{(n-1)} - a_2^{(n-1)}) + \cdots + (g-2)(a_{g-2}^{(n-1)} - a_{g-1}^{(n-1)}) + (g-1) a_{g-1}^{(n-1)} + b_1^{(n)} \\ &\cdots\cdots\cdots \\ a_{g-1}^{(n)} &\geq a_{g-1}^{(n-1)} + b_{g-1}^{(n)}, \end{aligned}$$

and then

$$\text{II} \qquad a_k^{(n)} \geq a_k^{(n-1)} + \cdots\cdots + a_{g-1}^{(n-1)} + b_k^{(n)} \qquad (k = 0, 1, ..., g - 1).$$

Remark All the equalities of II) hold if and only if all the equalities of I_k hold for all $\Gamma_n \in G^{(n-1)}$.

To prove the theorem we use the follwing Lemma.

Lemma 3 − 4 (1) $b_0^{(n)}, ..., b_{g-1}^{(n)}$ *are minimal if and only if*

$$G^{(n-1)} = \{\Gamma_n | 0 < deg\ \Gamma_n \leq g\}.$$

(2) *Assume* $G^{(n-1)} = \{\Gamma_n | 0 < deg\ \Gamma_n \leq g\}$. *Then the following conditions are equivalent:*

a) *the first equality in each* $I_k (0 \leq k \leq g - 1)$ *holds;*

b) *all the equalities in each* $I_k (0 \le k \le g-1)$ *hold;*

c) $\delta^{\Gamma_n} = g+1-deg\ \Gamma_n$ *for* $\Gamma_n \in G^{(n-1)}$*;*

d) $G^{(n)} = \{\Gamma | 0 < deg\ \Gamma \le g\}$.

Proof) (1) This follows from Lemma 3-2.

(2) *b)* $\Rightarrow$ *c)*

Assume $\delta^{\Gamma_n} > g+1-deg\ \Gamma_n$ for some $\Gamma_n \in A_k^{(n-1)} \backslash A_{k+1}^{(n-1)}$. $i(\Gamma_n) = k \ge 0$. By Lemma 3-2 d), $i(\Gamma_n) = g - deg\ \Gamma_n$.
Hence there is $\tilde{\gamma}$ satisfying

$$[(\Gamma_n, \tilde{\gamma}-1), (\Gamma_n, \tilde{\gamma})] = 1 \quad \text{and} \quad 0 < \tilde{\gamma} \le g+1-deg\ \Gamma_n.$$

But $(\Gamma_n, \tilde{\gamma}) \in G^{(n)}$ because of $\delta^{\Gamma_n} > \tilde{\gamma}$. Then

$$\#\{\gamma | i(\Gamma_n, \gamma) \ge 0, (\Gamma_n, \gamma) \in G^{(n)}\} \ge k+2.$$

c) $\Rightarrow$ *d)*

Suppose c) to be true. By Lemma 2-5 iii) and $\{\delta^{\Gamma_n} | \Gamma_n \in G^{(n-1)}\} = \{1, .., g\}$, we have

$$(O_n, k) \in G^{(n)} \quad \text{if and only if} \quad 1 \le k \le g.$$

First we will show

$$[\Gamma - e_n, \Gamma] = 1$$

for $\Gamma = (\gamma_1, ..., \gamma_n) \in V^{(n)}$ with $deg\Gamma \ge g+1$ and $\gamma_n > 0$.
If $\gamma_n \ge g+1$, then $[\Gamma - e_n, \Gamma] = 1$ by $(O_n, \gamma_n) \in H^{(n)}$ and $*_n - 1)$.
When $\gamma_n \le g$, take $\Gamma' = (\gamma'_1, ..., \gamma'_{n-1}, \gamma_n) = (\Gamma'_n, \gamma_n)$ with $deg\ \Gamma' = g+1$ and $\Gamma' \le \Gamma$. Then $deg\ \Gamma'_n \le g$, $\Gamma'_n \in G^{(n-1)}$ and $\gamma_n = g+1-deg\ \Gamma'_n = \delta^{\Gamma'_n}$ by c). Also by $*_n - 1)$ and the definition of $\delta^{\Gamma'_n}$, we have $[\Gamma - e_n, \Gamma] = 1$.

Next we will show

$$[\Gamma - e_1, \Gamma] = 1$$

for $\Gamma = (\gamma_1, ..., \gamma_n) \in V^{(n)}$ with $deg\ \Gamma \ge g+1$ and $\gamma_1 > 0$.
When $\gamma_1 \ge g+1$, $[\Gamma - e_1, \Gamma] = 1$ as above. When $\gamma_1 \le g$, take $\Gamma' = (\gamma_1, \gamma'_2, ..., \gamma'_n)$ satisfying $\Gamma' \le \Gamma$ and $deg\ \Gamma' = g+1$. Put $\Gamma' = (\tilde{\Gamma}_n, \gamma'_n)$, then $\gamma'_n = \delta^{\tilde{\Gamma}_n}$ and $[\Gamma' - e_1, \Gamma'] = 1$. Thus we have $[\Gamma - e_1, \Gamma] = 1$ by $*_n - 1)$.
This argument is also effective when the index 1 is replaced with $i \ne 1$.
Thus if Γ satisfies $deg\ \Gamma \ge g+1$, then $[\Gamma - e_i, \Gamma] = 1 (0 \le i \le n)$.

The implications *d)* $\Rightarrow$ *a)* and *a)* $\Rightarrow$ *b)* are easy. □

Proof of Theorem 3-3 i)

We prove this theorem by induction on n.
Now we assume that

$a_k^{(n-1)}(k = 0, ..., g-1)$ are minimal if $G^{(n-1)} = \{\Gamma_n | 0 < deg\ \Gamma_n \leq g\} . \cdots \star_{n-1})$

By our assumption $\star_{n-1})$ and Lemma 3-4 (1), the right hand side of each inequality of II is minimal if and only if

$$G^{(n-1)} = \{\Gamma_n | 0 < deg\ \Gamma_n \leq g\}.$$

Moreover, when $G^{(n-1)} = \{\Gamma_n | 0 < deg\ \Gamma_n \leq g\}$, all the equalities of II hold if and only if

$$G^{(n)} = \{\Gamma | 0 < deg\ \Gamma \leq g\}$$

by Lemma 3-4 (2) and Remark before Lemma 3-4.
Thus $a_k^{(n)}(k = 0, ..., g-1)$ are minimal if and only if

$$G^{(n)} = \{\Gamma | 0 < deg\ \Gamma \leq g\}$$

under the assumption $\star_{n-1})$.

When $n = 2$, $\#G^{(1)} = g$ and $a_k^{(1)} = g - k(k = 0, .., g-1)$ for any type of $D^{(1)}$. Then the assumption $\star_1$ is satisfied, and we get Theorem 3-3. □

Example 3 – 5 Let M be a hyperelliptic curve and $P_1, P_2, ..., P_n$ be non-Wierestrass points satisfying $|P_i + P_j| \neq g_2^1 (1 \leq i, j \leq n)$. Then

$$G_M(P_1, ..., P_n) = \{\Gamma | 0 < deg\ \Gamma \leq g\}.$$

In fact this can be easily seen by the same calculation done by Kim([12]) in case $n = 2$.

§4 The upper bound of $\#G^{(3)}$

In this section we determine the upper bound of $\#G^{(3)}$.

Let $(D^{(n)}, *_n)$ be a Riemann-Roch graph and let $(D^{(n-1)}, *_{n-1})$ be its subgraph as in §1. The subsets of vertices

$$V^{(n)} \supset V^{(n-1)} \supset \cdots \supset V^{(1)},$$

$$G^{(n)} \supset G^{(n-1)} \supset \cdots \supset G^{(1)}$$

and

$$H^{(n)} \supset H^{(n-1)} \supset \cdots \supset H^{(1)}$$

are also as in §1.
Define

$$G_i := \{x | x e_i \in G^{(n)}\} \qquad \text{and} \qquad H_i := \{n | 0 \leq n \leq 2g-1\} \backslash G_i$$

respectively.

Remark H_1 and G_1 coincide with $H^{(1)}$ and $G^{(1)}$ respectively.

Lemma 4 − 1 *Fix a Riemann-Roch graph* $(D^{(2)}, *_2)$. *For* $\alpha \in V^{(1)}$, *let* $\beta(\alpha)$ *be the non-negative integer* δ^α *defined in 2-4*

$$\left(i.e., \beta(\alpha) = \delta^\alpha = \begin{cases} min\{\beta | (\alpha, \beta) \in H^{(2)}\} (\leq 2g-1-\alpha) & if \ \{\beta | (\alpha,\beta) \in H^{(2)}\} \neq \emptyset \\ 2g - \alpha & if \ \{\beta | (\alpha,\beta) \in H^{(2)}\} = \emptyset \end{cases} \right).$$

Then

i) *For* $\alpha \in G_1$, $\beta(\alpha)$ *is in* G_2. *Moreover the map* $\beta(*) : G_1 \to G_2$ *defined by* $\beta(\alpha)$ *is one to one.*

ii) *For* $\alpha \in G_1$, *we have*

$$\{\beta | \ [(\alpha-1, \beta), (\alpha, \beta)] = 1\} \neq \emptyset \quad \text{if and only if} \quad \{\beta | \ (\alpha, \beta) \in H^{(2)}\} \neq \emptyset$$

and

$$\beta(\alpha) = \begin{cases} \min\{\beta | \ [(\alpha-1,\beta),(\alpha,\beta)] = 1\} (\leq 2g-1-\alpha) & \text{if } \{\beta | \ [(\alpha-1,\beta),(\alpha,\beta)] = 1\} \neq \emptyset \\ 2g - \alpha & \text{if } \{\beta | \ [(\alpha-1,\beta),(\alpha,\beta)] = 1\} = \emptyset. \end{cases}$$

iii) *For* $\beta \in G_2$, *we have*

$$\{\alpha | \ [(\alpha, \beta-1), (\alpha, \beta)] = 1\} \neq \emptyset \quad \text{if and only if} \quad \{\alpha | \ (\alpha, \beta) \in H^{(2)}\} \neq \emptyset\}.$$

If $\alpha(*) : G_2 \to G_1$ *be the inverse map of* $\beta(*)$ *in* i), *then*

$$\alpha(\beta) =_{*)} \begin{cases} \min\{\alpha | (\alpha,\beta) \in H^{(2)}\} & \text{if} \quad \{\alpha | (\alpha,\beta) \in H^{(2)}\} \neq \emptyset \\ 2g - \beta & \text{if} \quad \{\alpha | (\alpha,\beta) \in H^{(2)}\} = \emptyset \end{cases}$$

$$=_{**)} \begin{cases} \min\{\alpha | \ [(\alpha,\beta-1),(\alpha,\beta)] = 1\} & \text{if} \quad \{\alpha | \ [(\alpha,\beta-1),(\alpha,\beta)] = 1\} \neq \emptyset \\ 2g - \beta & \text{if} \quad \{\alpha | [(\alpha,\beta-1),(\alpha,\beta)] = 1\} = \emptyset. \end{cases}$$

Proof) i) This follows from Lemma 2-5 iii) and $\#G_1 = \#G_2 = g$.

ii) Fix $\alpha \in G_1$.
Put

$$\beta' = \begin{cases} \min\{\beta| \, [(\alpha-1,\beta),(\alpha,\beta)] = 1\} (\leq 2g-1-\alpha) & \text{if } \{\beta| \, [(\alpha-1,\beta),(\alpha,\beta)] = 1\} \neq \emptyset \\ 2g-\alpha & \text{if } \{\beta| \, [(\alpha-1,\beta),(\alpha,\beta)] = 1\} = \emptyset. \end{cases}$$

Assume $\{\beta| \, [(\alpha-1,\beta),(\alpha,\beta)] = 1\} \neq \emptyset$.
Then we have

$$[(\alpha,\beta'-1),(\alpha,\beta')] = 1.$$

In fact, if $[(\alpha,\beta'-1),(\alpha,\beta')] = 0$, then

$$[(\alpha-1,\beta'-1),(\alpha,\beta'-1)] = 1$$

by 1-1 A). This contradicts to the definition of β'. Thus

$$\beta' \in \{\beta|(\alpha,\beta) \in H^{(2)}\}.$$

Consequently we have

$$\{\beta|(\alpha,\beta) \in H^{(2)}\} \neq \emptyset \quad \text{and} \quad \beta' \geq \beta(\alpha).$$

Conversely, if $\{\beta|(\alpha,\beta) \in H^{(2)}\} \neq \emptyset$, then obviously

$$\{\beta| \, [(\alpha-1,\beta),(\alpha,\beta)] = 1\} \neq \emptyset \quad \text{and} \quad \beta' \leq \beta(\alpha).$$

Thus we have

$$\{\beta| \, [(\alpha-1,\beta),(\alpha,\beta)] = 1\} \neq \emptyset \quad \text{if and only if} \quad \{\beta|(\alpha,\beta) \in H^{(2)}\} \neq \emptyset,$$

and

$$\beta(\alpha) = \beta'.$$

iii) Fix $\beta \in G_2$. By the same way as in ii), we have

$$\{\alpha|[(\alpha,\beta-1),(\alpha,\beta)] = 1\} \neq \emptyset \quad \text{if and only if} \quad \{\alpha|(\alpha,\beta) \in H^{(2)}\} \neq \emptyset,$$

and

$$\min\{\alpha|[(\alpha,\beta-1),(\alpha,\beta)] = 1\} = \min\{\alpha|(\alpha,\beta) \in H^{(2)}\}$$

if $\{\alpha|[(\alpha,\beta-1),(\alpha,\beta)] = 1\} \neq \emptyset$.
Thus we get the second equality **) .

Next we will show the first equality *).
Assume $\{\alpha| \, [(\alpha,\beta-1),(\alpha,\beta)] = 1\} \neq \emptyset$.
Put

$$\tilde{\alpha} = \min\{\alpha|[(\alpha,\beta-1),(\alpha,\beta)] = 1\} = \min\{\alpha|(\alpha,\beta) \in H^{(2)}\}.$$

Then $\tilde{\alpha} \leq 2g-1-\beta$ and $\beta(\tilde{\alpha}) \leq \beta$.
Now assume $\beta(\tilde{\alpha}) < \beta$. Then

$$[(\tilde{\alpha}-1,\beta-1),(\tilde{\alpha},\beta-1)] = 1$$

by $*_2-1)$, and

$$[(\tilde{\alpha}-1,\beta-1),(\tilde{\alpha}-1,\beta)] = 1$$

by Lemma 2-1 A) and $(\tilde{\alpha},\beta) \in H^{(2)}$.
This contradicts to the minimality of $\tilde{\alpha}$. Thus we have $\beta(\tilde{\alpha}) = \beta = \beta(\alpha(\beta))$. By i) of this lemma we get $\tilde{\alpha} = \alpha(\beta)$.

Next assume that $\{\alpha | [(\alpha,\beta-1),(\alpha,\beta)] = 1\} = \emptyset$.
If $2g-1-\alpha(\beta) \geq \beta = \beta(\alpha(\beta))$, then $(\alpha(\beta),\beta(\alpha(\beta))) \in H^{(2)}$. This contradicts to the above assumption. Since $\alpha(\beta)+\beta(\alpha(\beta)) \leq 2g$(Lemma 2-5), $\alpha(\beta) = 2g-\beta$.

Then we get the equality $*)$. □

Remark At first the map $\beta(*)$ was introduced by Kim in case $D^{(2)} = D_M(P,Q)$.

Formula (2) in Theorem 2-7 for $n=3$ and $n=2$ can be written as follows.

Lemma 4 − 2(Corollary of Theorem 2-7)
(1) Let $(\alpha,\beta) \in V^{(2)}$. We write $\delta^{\alpha\beta}$ for $(\alpha,\beta) \in V^{(2)}$. Then

$$\#G^{(3)} = \sum_{(\alpha,\beta)\in H^{(2)}} l(\alpha,\beta) + \sum_{(\alpha,\beta)\in G^{(2)}} l(\alpha,\beta,\delta^{\alpha\beta}) - \frac{g(2g+1)(g+1)}{3},$$

where $l(\alpha,\beta,\delta^{\alpha\beta}) = g+1$ if $\alpha+\beta+\delta^{\alpha\beta} = 2g$.
(2)

$$\#G^{(2)} = \frac{g(g-1)}{2} + \sum_{\alpha\in G_1} l(\alpha,\beta(\alpha)) \leq \frac{(3g^2+g)}{2},$$

where $l(\alpha,\beta(\alpha)) = g+1$ if $\alpha+\beta(\alpha) = 2g$.
Moreover $\#G^{(2)} = \frac{(3g^2+g)}{2}$ if and only if $\beta(\alpha) = 2g-\alpha$ for all $\alpha \in G_1$.

Proof) (2) This follows from $\{l(\alpha) | \alpha \in H^{(1)} = H_1\} = \{1,2,...,g\}$. □

Definition 4 − 3 Let $(D^{(3)},*_3)$ be a Riemann-Roch graph. $(D^{(2)},*_2)$ is the subgraph of $(D^{(3)},*_3)$, and $(D^{(1)},*_1)$ is the subgraph of $(D^{(2)},*_2)$ as before. Define subsets S,T and R of $V^{(2)}$ as follows.

$S := \{(\alpha,\beta) \in G^{(2)} | (\alpha,\beta,\gamma) \in G^{(3)}$ for any $\gamma \leq 2g-1-\alpha-\beta\}$.
$T := \{(u,v) \in V^{(2)} | 0 \leq u+v \leq 2g-2, [(u,v),(u+1,v)] = [(u,v),(u,v+1)] = 0\}$.
$R := \{(a,b) \in V^{(2)} | 0 \leq a+b \leq 2g-2, [(a,b,2g-2-a-b),(a,b,2g-1-a-b)] = 0\}$.
(N.B., $(u+1,v) \in G_2$ and $(u,v+1) \in G_2$ for $(u,v) \in T$).

Lemma 4 − 4

(1)

$$\begin{aligned} R &= \{(a,b) \in V^{(2)} |\ [(a,b,2g-2-a-b),(a,b,2g-1-a-b)] = 0\} \\ &= \{(a,b) \in V^{(2)} |\ [(a,b,2g-2-a-b),(a+1,b,2g-2-a-b)] = 0\} \\ &= \{(a,b) \in V^{(2)} |\ [(a,b,2g-2-a-b),(a,b+1,2g-2-a-b)] = 0\}. \end{aligned}$$

(2)

$$S = \{(\alpha,\beta) \in G^{(2)} |\ l(\alpha,\beta,\delta^{\alpha\beta}) = g+1\} = \{(\alpha,\beta) \in G^{(2)} | \delta^{\alpha\beta} = 2g-\alpha-\beta\}.$$

Proof) (1) This follows from Lemma 2-1 B).
(2) This follows from the definition of S and Definition 2-6. □

By Lemma 4-4 (1), $[(a,b),(a,b+1)] = [(a,b),(a+1,b)] = 0$ for $(a,b) \in R$. Then there is a natural inclusion $\varphi : R \to T$(i.e., $(u,v) = \varphi(a,b) = (a,b)$) and $\#R \leq \#T$.

To estimate the cardinarities of S and T, we use the following number $r(\beta(*))$ defined by Homma.

Definition 4 − 5(Homma[5]) Let $G_1 = \{\alpha_1 < \alpha_2 < \cdots < \alpha_g\}$, and let $G_2 = \{\beta_1 < \beta_2 < \cdots < \beta_g\}$. Define a non-negative integer $r(\beta(*))$ by

$$r(\beta(*)) := \#\{(i,j) | \alpha_i < \alpha_j (\text{i.e.}, i < j) \text{ and } \beta(\alpha_i) > \beta(\alpha_j)\}.$$

Lemma 4 − 6 *Let $(D^{(3)}, *_3)$ be a Riemann-Roch graph, and let S and T be as above. Then*

(1)

$$T = \{(u,v) \in V^{(2)} | u+1 \in G_1, v+1 \in G_2, 0 \leq u+v \leq 2g-2, \beta(u+1) \geq v+1 \text{ and } \alpha(v+1) \geq u+1\}.$$

(2)

$$\#T = r(\beta(*)) + \#(G_1) = r(\beta(*)) + g \leq \frac{g(g+1)}{2}.$$

And the equality $\#T = \frac{g(g+1)}{2}$ *holds if and only if*

$$\beta(\alpha_i) = \beta_{g+1-i}, \quad 1 \le i \le g.$$

(3) $\#S \le g(g+1)$.
If the equality $\#S = g(g+1)$ *holds, then*

$$G_1 = G_2 = G_3 = \{1, 3, 5, ..., 2g-1\} \quad \textit{and} \quad \beta(\alpha) = 2g - \alpha.$$

In this case, $(D^{(2)}, *_2)$ *is defined by*

"$[(u-1, v), (u, v)] = 0$ *if and only if* u *is odd*"

and

"$[(u, v-1), (u, v)] = 0$ *if and only if* v *is odd.*"

Therefore we have $G^{(2)} = \{(u, v) \in V^{(2)} | u$ *or* v *is odd*$\}$ *and* $l(\alpha, \beta(\alpha)) = g + 1$ *for* $\alpha \in G_1$.

Proof) (1) By Lemma 4-1 ii),

"$[(u, v), (u+1, v)] = 0$ if and only if $v < \beta(u+1)$"

for $u + 1 \in G_1$, and by Lemma 3-1 iii),

"$[(u, v), (u, v+1)] = 0$ if and only if $u < \alpha(v+1)$"

for $v + 1 \in G_2$. Thus we get (1).
(2) For $(u, v) \in T$, put $x = u + 1$ and $y = v + 1$. Then $x \in G_1$, $y \in G_2$, $\beta(x) \ge y$ and $\alpha(y) \ge x$. Since $\alpha(*) = \beta^{-1}(*)$ on G_2, there exists a unique $x' \in G_1$ satisfying $\beta(x') = y$ and $\alpha(y) = x'$. Thus

$$\begin{aligned} \#T &= \#\{(x,y) | x \in G_1, y \in G_2, y < \beta(x) \text{ and } x < \alpha(y)\} + \#\{(x,y) | x \in G_1, \beta(x) = y\} \\ &= \#\{(x,x') | x \in G_1, x' \in G_1, x' > x, \beta(x') < \beta(x)\} + \#\{(x, \beta(x)) | x \in G_1\}, \end{aligned}$$

and we have $\#T = r(\beta(*)) + g$.
Homma([5]) has shown that

$$0 \le r(\beta(*)) \le \frac{g(g-1)}{2}$$

and

"$r(\beta(*)) = \dfrac{g(g-1)}{2}$ if and only if $\beta(\alpha_i) = \beta_{g+1-i} (1 \le i \le g)$".

Thus we get (2).

(3) Assume

$$\begin{aligned}&[(\alpha-1,\beta,2g-1-\alpha-\beta),(\alpha,\beta,2g-1-\alpha-\beta)]\\&=[(\alpha,\beta-1,2g-1-\alpha-\beta),(\alpha,\beta,2g-1-\alpha-\beta)]\\&=1.\end{aligned}$$

for $(\alpha,\beta)\in S$.
Let

$$\gamma_0 := \min\{\gamma | [(\alpha-1,\beta,\gamma),(\alpha,\beta,\gamma)] = [(\alpha,\beta-1,\gamma),(\alpha,\beta,\gamma)] = 1\}.$$

Then $\gamma_0 \le 2g-1$, and $[(\alpha,\beta,\gamma_0-1),(\alpha,\beta,\gamma_0)] = 1$ by Lemma 2-1 A) and the minimality of γ_0. This implies that (α,β,γ_0) is in $H^{(3)}$. This contradicts to $(\alpha,\beta)\in S$. Then for $(\alpha,\beta)\in S$, we have

$$[(\alpha-1,\beta,2g-1-\alpha-\beta),(\alpha,\beta,2g-1-\alpha-\beta)] = 0$$

$\mathfrak{b}$) or

$$[(\alpha,\beta-1,2g-1-\alpha-\beta),(\alpha,\beta,2g-1-\alpha-\beta)] = 0.$$

$\mathfrak{b}$) means that

$(\alpha-1,\beta)$ or $(\alpha,\beta-1)$ is in R for $(\alpha,\beta)\in S.\cdots\cdots 3-6-1)$

On the other hand, by Lemma 4-4 (1) and $*_3-1$),

$(a+1,b)$ and $(a,b+1)$ are in S for $(a,b)\in R.\cdots\cdots 4-6-2)$

Then we can consider the one-to-two correspndence $(a,b)\to\{(a+1,b),(a,b+1)\}$ from R to S by 4-6-2), and $\#S\le 2\times\#R$ by 4-6-1). Therefore, by (2) of this lemma, we have

$$\#S \le 2\times\#R \le 2\times\#T \le 2\times\frac{g(g+1)}{2} = g(g+1).$$

Thus we get the former half of (3).
Moreover we have

$$\#S = g(g+1) \text{ if and only if } \begin{cases} a)\#T = \#R = \frac{g(g+1)}{2} \\ b)\text{one and only one of } (\alpha-1,\beta) \text{ or } (\alpha,\beta-1) \\ \quad \text{is in } R \text{ for } (\alpha,\beta)\in S.\end{cases}$$

Now assume $\#S = g(g+1)$, and let $G_3 = \{\gamma_1 < \gamma_2, ..., < \gamma_g\}$. We will show that $\alpha_i + \beta(\alpha_i)(i = 1, ..., g)$ is constant.

Claim

$$\begin{aligned}\alpha_i + \beta(\alpha_i) &= \alpha(\beta_{g-i+1}) + \beta_{g-i+1} \\ &= 2g - \gamma_1 + 1 \quad \text{for all } i.\end{aligned}$$

Proof of Claim) By Lemma 4-1 ii) and $*_3 - 1)$, we have

$$[(\alpha_j - 1, \beta(\alpha_j) - 1), (\alpha_j, \beta(\alpha_j) - 1)] = [(\alpha_i - 1, \beta(\alpha_j) - 1), (\alpha_i, \beta(\alpha_j) - 1)] = 0. \cdots\cdots 4\text{–}6\text{–}3)$$

for $j \geq i$.
By (2) of this lemma, we have

$$\beta(\alpha_i) = \beta_{g+1-i} > \beta(\alpha_j) = \beta_{g+1-j} \quad \text{with} \quad j > i.$$

Since $[(\alpha_i - 1, \beta(\alpha_i) - 1), (\alpha_i - 1, \beta(\alpha_i))] = 0$,

$$[(\alpha_i - 1, \beta(\alpha_j) - 1), (\alpha_i - 1, \beta(\alpha_j)] = 0 \quad \text{for } j \geq i. \cdots\cdots 4 - 6 - 4)$$

By 4-6-3) and 4-6-4) $(\alpha_i - 1, \beta(\alpha_j) - 1) \in T = R$, and $(\alpha_i, \beta(\alpha_j) - 1) \in S$ for all $j \geq i$. Since $2g - \alpha - \beta = \delta^{\alpha\beta} \in G_3$ for $(\alpha, \beta) \in S$ by Lemma 4-4(2), we have

$$2g - \alpha_i - \beta(\alpha_j) + 1 \in G_3 \quad \text{with} \quad j \geq i.$$

As $\alpha_i < \alpha_j$ and $\beta(\alpha_i) > \beta(\alpha_j)$ $(j > i)$, we have

$$\gamma_k = 2g - \alpha_{g-i+1} - \beta(\alpha_{g-i+k}) + 1 \quad \text{with} \quad k = 1, .., i.$$

In particular

$$\gamma_1 = 2g - \alpha_{g-i+1} - \beta(\alpha_{g-i+1}) + 1.$$

Then Claim has been proved.

Assume $\alpha_{i+1} = \alpha_i + 1$, for some i. By Claim, $\beta(\alpha_i) = \beta(\alpha_{i+1}) + 1$. Then

$$(\alpha_i, \beta(\alpha_{i+1}) - 1) = (\alpha_{i+1} - 1, \beta(\alpha_{i+1}) - 1) \in T = R$$

and

$$(\alpha_i - 1, \beta(\alpha_{i+1})) = (\alpha_i - 1, \beta(\alpha_i) - 1) \in T = R.$$

But the condition $b)$ of $\#S = g(g+1)$ means that $(a+1, b-1)$ is not in R if (a, b) is in R. Then

$$\alpha_{i+1} \neq \alpha_i + 1 \quad \text{and} \quad \beta_{i+1} \neq \beta_i + 1 \text{ for all } i.$$

Since $\beta(\alpha_i) = \beta_{g-i+1}$, we also have

$$G_1 = \{\alpha_k = 2k - 1 | 1 \leq k \leq g - 1\}, \quad G_2 = \{\beta_k = 2k - 1 | 1 \leq k \leq g - 1\}$$

and $\beta(\alpha) = 2g - \alpha$ for $\alpha \in G_1$.
Using Lemma 4-1 ii),iii) and $*_3-1)$, we get the graph $(D^{(2)}, *_2)$ mentioned at the end of (3). □

Proposition 4 − 7 *Assume* $\#S = g(g+1)$.
Then $(D^{(3)}, *_3)$ *is defined by*

$$\alpha)\ \ [(\alpha-1,\beta,\gamma),(\alpha\,\beta,\gamma)] = 0 \textit{ if and only if } \begin{cases} \text{“}\alpha \textit{ is odd and } \alpha+\beta+\gamma \neq 2g-1\text{”} \\ \textit{or} \\ \text{“}\alpha+\beta+\gamma = 2g-1 \textit{ and } \beta,\gamma \textit{ are even”}, \end{cases}$$

$$\beta)\ \ [(\alpha,\beta-1,\gamma),(\alpha\,\beta,\gamma)] = 0 \textit{ if and only if } \begin{cases} \text{“}\beta \textit{ is odd and } \alpha+\beta+\gamma \neq 2g-1\text{”} \\ \textit{or} \\ \text{“}\alpha+\beta+\gamma = 2g-1 \textit{ and } \alpha,\gamma \textit{ are even”} \end{cases}$$

and

$$\gamma)\ \ [(\alpha,\beta,\gamma-1),(\alpha\,\beta,\gamma)] = 0 \textit{ if and only if } \begin{cases} \text{“}\gamma \textit{ is odd and } \alpha+\beta+\gamma \neq 2g-1\text{”} \\ \textit{or} \\ \text{“}\alpha+\beta+\gamma = 2g-1 \textit{ and } \alpha,\beta \textit{ are even”}. \end{cases}$$

In this case,

$$S = \{(\alpha,\beta) | 1 \leq \alpha+\beta \leq 2g-1 \textit{ and } \alpha+\beta \textit{ is odd }\}$$

and

$$G^{(2)}\backslash S = \{(\alpha,\beta) | 2 \leq \alpha+\beta \leq 2g-2, \alpha \textit{ and } \beta \textit{ are odd}\}.$$

Moreover, $\delta^{(\alpha\beta)} = 2g-1-\alpha-\beta$ *and* $l(\alpha,\beta,\delta^{(\alpha\beta)}) = g$ *for* $(\alpha,\beta) \in G^{(2)}\backslash S$.

Proof) By Lemma 4-6(3) and the proof of it , we can see that

$$R = T = \{(\alpha,\beta) \in V^{(2)} | \alpha \text{ and } \beta \text{ are even, } 0 \leq \alpha+\beta \leq 2g-2\},$$

$$S = \{(\alpha,\beta) | 1 \leq \alpha+\beta \leq 2g-1 \text{ and } \alpha+\beta \text{ odd}\}$$

and

$$G^{(2)}\backslash S = \{(\alpha,\beta) | 2 \leq \alpha+\beta \leq 2g-2, \alpha \text{ and } \beta \text{ are odd}\}.$$

Then, by Lemma 4-4(1),

$$(\alpha-1,\beta+1) \in R \quad \text{and} \quad [(\alpha-1,\beta+1,2g-2-\alpha-\beta),(\alpha,\beta+1,2g-2-\alpha-\beta)] = 0$$

for $(\alpha,\beta) \in G^{(2)}\backslash S$.
By $*_3 - 1)$,

$$[(\alpha-1,\beta,\gamma),(\alpha,\beta,\gamma)] = 0 \text{ (i.e., } (\alpha,\beta,\gamma) \in G^{(3)}) \qquad \cdots\cdots 4-7-1)$$

for every γ with $0 \le \gamma \le 2g-\alpha-\beta-2$ and $(\alpha,\beta) \in G^{(2)}\backslash S$.
Therefore we get $\delta^{\alpha\beta} \ge 2g-\alpha-\beta-1$. Since $(\alpha,\beta) \in G^{(2)}\backslash S$ and $\delta^{\alpha\beta} \le 2g-\alpha-\beta-1$, we have

$$\delta^{\alpha\beta} = 2g-\alpha-\beta-1 \text{ and } l(\alpha,\beta,\delta^{\alpha\beta}) = g.$$

Then we get the latter half of this lemma.

Let α and β be odd and even respectively. If $\tilde{\gamma} = 2g-1-\alpha-\beta \ge 0$, then $(\alpha,\beta) \in S$ and $(\alpha,\beta,\tilde{\gamma}) \in G^{(3)}$. But $[(\alpha,\beta-1,\tilde{\gamma}),(\alpha,\beta,\tilde{\gamma})] = [(\alpha,\beta,\tilde{\gamma}-1),(\alpha,\beta,\tilde{\gamma})] = 1$ because β and $\tilde{\gamma}$ are even. Then

$$[(\alpha-1,\beta,\gamma),(\alpha,\beta,\gamma)] = 0 \qquad \cdots\cdots 3-7-2)$$

for $0 \le \gamma \le 2g-1-\alpha-\beta$.
Let both α and β be odd. If $\tilde{\gamma} = 2g-1-\alpha-\beta \ge 0$, then $(\alpha,\beta) \in G^{(2)}\backslash S$ and $\delta^{\alpha\beta} = \tilde{\gamma}$. Hence $(\alpha,\beta,\tilde{\gamma}) \in H^{(3)}$ and

$$[(\alpha-1,\beta,\tilde{\gamma}),(\alpha,\beta,\tilde{\gamma})] = 1. \qquad \cdots\cdots 4-7-3)$$

By 4-7-1), 4-7-2), 4-7-3) and $*_3-1)$, we get the statement α). β) can be proved by the same way as in case α). The statement γ) follows from α), $*_3-2)$ and $*_3-1)$. □

Lemma 4 – 8 (1) *The first term* $\sum_{(\alpha\beta)\in H^{(2)}} l(\alpha,\beta)$ *of the equation of Lemma* 4-2(1) *satisfies*

$$\sum_{(\alpha\beta)\in H^{(2)}} l(\alpha,\beta) = \frac{g(g+1)(5g+1)}{6} + \frac{\sum_{\alpha\in G_1}\{-l(\alpha,\beta(\alpha))^2 + l(\alpha,\beta(\alpha))\}}{2}.$$

(2) *The second term* $\sum_{(\alpha\beta)\in G^{(2)}} l(\alpha,\beta,\delta^{\alpha\beta})$ *of* 4-2(1) *satisfies*

$$\sum_{(\alpha\beta)\in G^{(2)}} l(\alpha,\beta,\delta^{\alpha\beta}) \le g(g+1) + g \times \#G^{(2)},$$

and the equality holds if and only if $\#S = g(g+1)$.
(3)

$$\#G^{(3)} \le \frac{g(g+1)(g+5)}{6} + g \times \#G^{(2)} + \frac{\sum_{\alpha\in G_1}\{-l(\alpha,\beta(\alpha))^2 + l(\alpha,\beta(\alpha))\}}{2},$$

and the equality holds if and only if $\#S = g(g+1)$.

Proof) (1) Let

$$A = \sum_{\alpha\in H_1}\Big(\sum_{\beta \text{ s.t. } (\alpha\beta)\in H^{(2)}} l(\alpha,\beta)\Big) \text{ and } B = \sum_{\alpha\in G_1}\Big(\sum_{\beta \text{ s.t. } (\alpha\beta)\in H^{(2)}} l(\alpha,\beta)\Big).$$

Then

$$\sum_{(\alpha\beta)\in H^{(2)}} l(\alpha,\beta) = A + B.$$

We can calculate A and B as follows.

$$\begin{aligned}
A &= \sum_{\alpha\in H^{(1)}} \{l(\alpha,0) + (l(\alpha,0)+1) + \cdots + g\} \\
&= \sum_{\alpha\in H^{(1)}} \frac{(g - l(\alpha)+1)(g+l(\alpha))}{2} \\
&= \frac{\sum_{k=1}^{g}\{(g-k+1)(g+k)\}}{2} = \frac{g(g+1)(2g+1)}{6}.
\end{aligned}$$

$$\begin{aligned}
B &= \sum_{\alpha\in G^{(1)}} \Big(\sum_{\beta \text{ s.t. } (\alpha,\beta)\in H^{(2)}} l(\alpha,\beta) \Big) \\
&= \sum_{\alpha\in G^{(1)}} \{l(\alpha,\beta(\alpha)) + (l(\alpha,\beta(\alpha)+1) + \cdots + g\} \\
&= \frac{\sum_{\alpha\in G^{(1)}}\{-l(\alpha,\beta(\alpha))^2 + l(\alpha,\beta(\alpha))\}}{2} + \frac{g^2(g+1)}{2}.
\end{aligned}$$

Adding A and B, we get the equation in (1).

(2) Splitting $G^{(3)}$ into two subsets S and $G^{(3)}\backslash S$, we have

$$\begin{aligned}
\sum_{(\alpha\beta)\in G^{(2)}} l(\alpha,\beta,\delta^{\alpha\beta}) &= \sum_{(\alpha\beta)\in S} l(\alpha,\beta,\delta^{\alpha\beta}) + \sum_{(\alpha\beta)\in G^{(2)}\backslash S} l(\alpha,\beta,\delta^{\alpha\beta}) \\
&\le \#S \times (g+1) + (\#G^{(2)} - \#S) \times g \\
&\le g(g+1) + g \times \#G^{(2)} \qquad \text{(by Lemma 4-6 (3)).}
\end{aligned}$$

□

Theorem. 4 − 9 *Let $(D^{(3)}, *_3)$ be a Riemann-Roch graph, and let $G^{(3)}$ be its gap set.*
Then

$$\#G^{(3)} \le \frac{g(7g^2+6g+5)}{6},$$

and the equality holds if and only if $(D^{(3)}, *_3)$ is the graph defined as in Propodition 4-7.

Proof) Substituting (2) of Lemma 4-2 for $\#G^{(2)}$ in the inequality of lemma 4-8 (3), we have

$$\#G^{(3)} \le_{(1)} \frac{g(4g^2+3g+5)}{6} + \sum_{\alpha\in G_1}\{-l(\alpha,\beta(\alpha))^2 + (2g+1)l(\alpha,\beta(\alpha))\}$$

$$\leq_{(2)} \frac{g(7g^2+6g+5)}{6}.$$

As

$$-l(\alpha,\beta(\alpha))^2+(2g+1)l(\alpha,\beta(\alpha)) = -\{l(\alpha,\beta(\alpha))-(g+\frac{1}{2})\}^2+g^2+g+\frac{1}{4},$$

the second equality (2) holds if and only if $l(\alpha,\beta(\alpha))=g$ or $g+1$ for each $\alpha\in G_1$.

If the first equality (1) holds, then $\#S=g(g+1)$ and $(D^{(2)},*_2)$ is the graph defined in Lemma 4-6 (3). That is,

$$G_1=G_2=\{1,3,5,...,2g-1\},$$

$$G^{(2)}=\{(\alpha,\beta)|1\leq\alpha+\beta\leq 2g-1, \alpha \text{ or } \beta \text{ is odd }\},$$

$$\beta(\alpha)=2g-\alpha \quad\text{and}\quad l(\alpha,\beta(\alpha))=g+1 \quad\text{for}\quad \alpha\in G_1.$$

Thus the equality (1) implies the equality (2), and then $\#G^{(3)}=\frac{g(7g^2+6g+5)}{6}$ holds if and only if the equality (1) holds. So we have the graph discribed in Proposition 4-8. □

Example 4 − 10 The graph in Theorem 4-9 is exactly the graph $G_M(P_1,P_2,P_3)$ with hyperelliptic M and $|2P_1|=|2P_2|=|2P_3|=g^1_2$. This is also from the same calculation done by Kim in case $n=2$.

Remark 4 − 11 When $n=2$, the graph which attains the maximal value of $\#G^{(2)}$ is not unique. For example, if

$$G_1=\{\alpha_1,...,\alpha_g\}=\{1,2,3,..,g\},$$

$$G_2=\{\beta_1,...,\beta_g\}=\{g,g+1,...,2g-1\}\}$$

and $\beta(\alpha_i)=2g-\alpha_i$, then this graph attains the maximal value by Lemma 4-2, and this graph does not come from any Riemann surfaces.

§.Appendix.

Lemma 4-1 shows that a map $\beta(*):V_1\to V_2$ with some conditions completely determine a Riemann-Roch graph in case $n=2$. In this section we study the structure of $(D^{(n)},*_n)$ in detail when $n\geq 3$, and try to find some means, similar to $\beta(*)$, of construction of $(D^{(n)},*_n)$.

A − I

First we survey a given $(D^{(n)}, *_n)$.

Definition A − 1 Fix a Riemann-Roch graph $(D^{(n)}, *_n)$. Assume $n \geq 3$. Let i and $j (1 \leq i, j \leq n,\ i \neq j)$ be fixed. Take an (n-2)-tuple

$$\Gamma_{ij} = (\gamma_1, ..., \gamma_{i-1}, \gamma_{i+1}, ..., \gamma_{j-1}, \gamma_{j+1}, ..., \gamma_n) \in \mathbf{N}_0^{n-2},$$

and we identify Γ_{ij} with the n-tuple

$$\sum_{k \neq i,j} \gamma_k \mathbf{e}_k = (\gamma_1, ..., \gamma_{i-1}, 0, \gamma_{i+1}, ..., \gamma_{j-1}, 0, \gamma_{j+1}, ..., \gamma_n) \in \mathbf{N}_0^n.$$

We also write Γ_{ij} for this vertex.
For fixed Γ_{ij}, define a subset $G_i^{\Gamma_{ij}}$ of $\mathbf{N}_0$ by

$$G_i^{\Gamma_{ij}} := \{\gamma | \gamma > 0,\ \Gamma = \Gamma_{ij} + \gamma \mathbf{e}_i \in V^{(n)} \text{ and } [\Gamma - \mathbf{e}_i, \Gamma] = 0\}.$$

For $\gamma \in \mathbf{N}_0$ with $0 \leq \gamma \leq 2g - deg\ \Gamma_{ij} - 1$,
define a non-negative integer $\gamma_j^{\Gamma_{ij}}(\gamma)$ by:

i) for $\gamma \notin G_i^{\Gamma_{ij}}$,
$\gamma_j^{\Gamma_{ij}}(\gamma) := 0;$

\# ii) for $\gamma \in G_i^{\Gamma_{ij}}$,
a) $\gamma_j^{\Gamma_{ij}}(\gamma) := 2g - deg\ \Gamma_{ij} - \gamma (> 0)$ if $\Delta_j(\Gamma_{ij}, \gamma) = \emptyset$
b) $\gamma_j^{\Gamma_{ij}}(\gamma) := \min\{\alpha | \alpha \in \Delta_j(\Gamma_{ij}, \gamma)\} (> 0)$ if $\Delta_j(\Gamma_{ij}, \gamma) \neq \emptyset$,

where

$$\Delta_j(\Gamma_{ij}, \gamma) := \{\alpha | [(\Gamma - \mathbf{e}_i, \Gamma] = 1 \text{ with } \Gamma = \Gamma_{ij} + \gamma \mathbf{e}_i + \alpha \mathbf{e}_j \in V^{(n)} \text{ and } \gamma > 0\}.$$

Remark i) For $\gamma \in G_i^{\Gamma_{ij}}$, $1 \leq \gamma_j^{\Gamma_{ij}}(\gamma) \leq 2g - deg\ \Gamma_{ij} - 1$.
(see the proof of Lemma 4-1).
ii) If $\Gamma_{ij} = (0, ..., 0)$(write 0_{ij}), then $G_i^{0_{ij}} = \{\gamma | \gamma \mathbf{e}_i \in G^{(n)}\}$. We wrote G_i for $G_i^{0_{ij}}$ in §.4.

Lemma A − 2 *Fix* Γ_{ij}. *For* γ *with* $0 \leq \gamma \leq 2g - deg\Gamma_{ij} - 1$, *put* $\tilde{\gamma} = \gamma_j^{\Gamma_{ij}}(\gamma)$ *and* $\Gamma = \Gamma_{ij} + \gamma \mathbf{e}_i + \tilde{\gamma} \mathbf{e}_j$.
If $0 < \tilde{\gamma} < 2g - deg\Gamma_{ij} - \gamma$, *then*

$$\gamma > 0\ , \qquad [\Gamma - \mathbf{e}_j, \Gamma] = 1 \qquad \textit{and} \qquad [\{\Gamma - \mathbf{e}_i\} - \mathbf{e}_j, \{\Gamma - \mathbf{e}_i\}] = 0.$$

Proof) As $\tilde{\gamma} > 0$, γ must be positive. By the definition of $\tilde{\gamma} = \gamma_j^{\Gamma_{ij}}(\gamma)$,

$$[\Gamma - \mathbf{e}_i, \Gamma] = 1 \quad \text{and} \quad [\{\Gamma - \mathbf{e}_j\} - \mathbf{e}_i, \{\Gamma - \mathbf{e}_j\}] = 0.$$

By Lemma 2-1 A), we get this lemma. □

The system of maps

$$\left\{\tilde{\gamma}_j^{\Gamma_{ij}} : \{\gamma | 0 \le \gamma \le 2g-1-deg\,\Gamma_{ij}\} \to \{\gamma | 0 \le \gamma \le 2g-1-deg\,\Gamma_{ij}\} \middle| \Gamma_{ij} \in V^{(n)}, 1 \le i,j \le n\right\}$$

have the following properties.

Lemma A − 3 *Fix a Riemann-Roch graph* $(D^{(n)}, *_n)$. *Let* Γ_{ij} *be as in Definition A-1. Then*

i) $$\#G_i^{\Gamma_{ij}} = \#G_j^{\Gamma_{ij}} = i(\Gamma_{ij}).$$

ii) $\gamma_j^{\Gamma_{ij}}$ *induces a bijection from* $G_i^{\Gamma_{ij}}$ *to* $G_j^{\Gamma_{ij}}$, *and its inverse map is*

$$(\gamma_j^{\Gamma_{ij}})^{-1} = \gamma_i^{\Gamma_{ij}}.$$

iii) *Let* $\Gamma'_{ij} = \sum_{k \ne i,j} \gamma'_k e_k$ *be another* $(n-2)-$*tuple with* $\Gamma_{ij} \le \Gamma'_{ij}$, *then*

$$G_i^{\Gamma_{ij}} \supset G_i^{\Gamma'_{ij}}$$

and

$$\gamma_j^{\Gamma_{ij}}(\gamma) \ge \gamma_j^{\Gamma'_{ij}}(\gamma)$$

for γ *with* $0 \le \gamma \le 2g - 1 - deg\ \Gamma'_{ij}$.
Moreover if $G_i^{\Gamma_{ij}} = G_i^{\Gamma'_{ij}}$, *then*

$$\gamma_j^{\Gamma_{ij}} = \gamma_j^{\Gamma'_{ij}}.$$

Proof) i) This can be easily proved by Lemma 2-3.

ii) Put $\tilde{\gamma} = \gamma_j^{\Gamma_{ij}}(\gamma)$ and $\Gamma = \Gamma_{ij} + \gamma e_i + \tilde{\gamma} e_j$ for $\gamma \in G_i^{\Gamma_{ij}}$. Then $\gamma > 0$ and $\tilde{\gamma} > 0$.

First we will show $\tilde{\gamma} \in G_j^{\Gamma_{ij}}$.

Assume $\tilde{\gamma} \notin G_j^{\Gamma_{ij}}$.

<u>case</u> $\Delta_j(\Gamma_{ij}, \gamma) \ne \emptyset (i.e., \Gamma \in V^{(n)})$
By $\tilde{\gamma} \notin G_j^{\Gamma_{ij}}$, we have $[\{\Gamma_{ij} + \tilde{\gamma} e_j\} - e_j, \{\Gamma_{ij} + \tilde{\gamma} e_j\}] = 1$.
Then, by $*_n - 1)$,

$$[\{\Gamma - e_i\} - e_j, \{\Gamma - e_i\}] = [\Gamma - e_j, \Gamma] = 1.$$

On the other hand $[\Gamma - e_i, \Gamma] = 1$ by # ii-b).
Thus, by Lemma 2-1 A), we have

$$[\{\Gamma - e_j\} - e_i, \{\Gamma - e_j\}] = 1.$$

But this contradicts to the definition # ii-b).

<u>case</u> $\Delta_j(\Gamma_{ij}, \gamma) = \emptyset (i.e., \Gamma \notin V^{(n)})$
deg $(\Gamma - e_j) = 2g - 1$ and then $\Gamma - e_j \in V^{(n)}$. We have

$$[\{\Gamma - e_j\} - e_i, \{\Gamma - e_j\}] = 0.$$

On the other hand, by $\tilde{\gamma} \notin G_j^{\Gamma_{ij}}$ and $*_n - 1)$,

$$[\{\Gamma_{ij} + \tilde{\gamma}e_j\} - e_j, \{\Gamma_{ij} + \tilde{\gamma}e_j\}] = [\{\Gamma - e_i\} - e_j, \{\Gamma - e_i\}] = 1.$$

Then, by Lemma 2-1 B),

$$[\{\Gamma - e_j\} - e_i, \{\Gamma - e_j\}] = 1.$$

This is also a contradiction. Thus $\tilde{\gamma} \in G_j^{\Gamma_{ij}}$ in any case.

Next we will show $(\gamma_i^{\Gamma_{ij}})^{-1} = \gamma_j^{\Gamma_{ij}}$.

<u>case</u> $\Delta_j(\Gamma_{ij}, \gamma) \neq \emptyset$
By Lemma A-2 and $*_n - 1)$, we have

$$[\{\Gamma_{ij} + \delta e_i + \tilde{\gamma}e_j\} - e_j, \{\Gamma_{ij} + \delta e_i + \tilde{\gamma}e_j\}] = 0$$

for any δ with $0 \leq \delta \leq \gamma - 1$, and $\Delta_i(\Gamma_{ij}, \tilde{\gamma}) \ni \gamma$. Thus we have

$$\gamma_i^{\Gamma_{ij}}(\tilde{\gamma}) = \gamma = (\gamma_j^{\Gamma_{ij}})^{-1}(\tilde{\gamma})$$

by the definiton of $\gamma_i^{\Gamma_{ij}}(\tilde{\gamma})$.

<u>case</u> $\Delta_j(\Gamma_{ij}, \gamma) = \emptyset$
Using Lemma 2-1 B) and # ii-a), we also have $\Delta_i(\Gamma_{ij}, \tilde{\gamma}) = \emptyset$ and

$$\gamma_i^{\Gamma_{ij}}(\tilde{\gamma}) = 2g - deg\ \Gamma_{ij} - \tilde{\gamma} = \gamma = (\gamma_j^{\Gamma_{ij}})^{-1}(\tilde{\gamma}).$$

iii) $G_i^{\Gamma_{ij}} \supset G_i^{\Gamma'_{ij}}$ and $\gamma_j^{\Gamma_{ij}}(\gamma) \geq \gamma_j^{\Gamma'_{ij}}(\gamma)$ follow from $*_n - 1)$.

Next assume $G_i^{\Gamma_{ij}} = G_i^{\Gamma'_{ij}}$. Then $i(\Gamma_{ij}) = i(\Gamma'_{ij})$. By Lemma 2-3 and $*_n - 1)$, we have

$$[\Gamma_{ij} + \alpha e_i + \beta e_j, \Gamma'_{ij} + \alpha e_i + \beta e_j] = deg\ \Gamma'_{ij} - deg\ \Gamma_{ij} \cdots\cdots ♮$$

for $\alpha \geq 0$ and $\beta \geq 0$.
Fix γ with $1 \leq \gamma \leq 2g - 1 - deg\ \Gamma'_{ij}$.
Put $\tilde{\gamma}' = \gamma_j^{\Gamma'_{ij}}(\gamma)$, $\tilde{\Gamma} = \Gamma_{ij} + \gamma e_i + \tilde{\gamma}' e_j$ and $\tilde{\Gamma}' = \Gamma'_{ij} + \gamma e_i + \tilde{\gamma}' e_j$.
Then $\tilde{\Gamma} \leq \tilde{\Gamma}'$ and $[\tilde{\Gamma}' - e_i, \tilde{\Gamma}'] = 1$.
Therefore, by ♮,

$$[\tilde{\Gamma} - e_i, \tilde{\Gamma}'] = [\tilde{\Gamma} - e_i, \tilde{\Gamma}' - e_i] + [\tilde{\Gamma}' - e_i, \tilde{\Gamma}'] = (deg\ \Gamma'_{ij} - deg\ \Gamma_{ij}) + 1 = [\tilde{\Gamma}, \tilde{\Gamma}'] + 1.$$

On the other hand, since

$$[\tilde{\Gamma}-\mathbf{e}_i,\tilde{\Gamma}'] = [\tilde{\Gamma}-\mathbf{e}_i,\tilde{\Gamma}] + [\tilde{\Gamma},\tilde{\Gamma}'],$$

we have $\quad [\tilde{\Gamma}-\mathbf{e}_i,\tilde{\Gamma}] = 1 \qquad$ and $\qquad \gamma_j^{\Gamma_{ij}}(\gamma) \leq \tilde{\gamma}' = \gamma_j^{\Gamma'_{ij}}(\gamma). \qquad \square$

Also we can have the following proposition from $*_n - 1)$.

Proposition A − 4 *Let* $\Gamma = \sum_{i=1}^{n} \gamma_i \mathbf{e}_i$ *be in* $V^{(n)}$. *Let* $\Gamma_{kn} (k \neq n)$ *be the* $(n-2)$*-tuple that satisfies* $\Gamma = \Gamma_{kn} + \gamma_k \mathbf{e}_k + \gamma_n \mathbf{e}_n$.

i) *Assume* $\gamma_i > 0$ *for some* $i (\neq n)$. *Then*

$$[\Gamma - \mathbf{e}_i, \Gamma] = 1 \qquad \textit{if and only if} \qquad \gamma_n^{\Gamma_{in}}(\gamma_i) \leq \gamma_n.$$

ii) *Assume* $\gamma_n > 0$. *Then, for any* $k (\neq n)$,

$$[\Gamma - \mathbf{e}_n, \Gamma] = 1 \qquad \textit{if and only if} \qquad \gamma_k^{\Gamma_{kn}}(\gamma_n) \leq \gamma_k.$$

This proposition and Proposition A-3 ii) imply that $D^{(n)}$ with $*_n$ is exactly decided by the system

$$\left\{ \{\gamma_n^{\Gamma_{in}} | \{\gamma | 0 \leq \gamma \leq 2g-1-deg\ \Gamma_{in}\} \rightarrow \{\gamma_n | 0 \leq \gamma \leq 2g-1-deg\ \Gamma_{in}\}\} \middle| \ \Gamma_{in} \in V^{(n)}, 1 \leq i \leq n-1 \right\}.$$

A − II

Let $D^{(n)}$ be as before, but we do not assume the condition $*_n$ on it. Regarding $D^{(n-1)}$ as the subgraph of $D^{(n)}$ by the natural way (i.e., $(\gamma_1, .., \gamma_{n-1}) \longleftrightarrow (\gamma_1, .., \gamma_{n-1}, 0)$), and assume that $D^{(n-1)}$ is equipped with the condition $*_{n-1}$. We will investigate how we can build up $*_n$, which induces the given $*_{n-1}$.

Definition A − 5 i) Let Γ_{in} and Γ'_{in} be as in Definition A-1. We define a subset $\tilde{G}_i^{\Gamma_{in}}$ of $\{\gamma | 1 \leq \gamma \leq 2g-1-deg\ \Gamma_{in}\}$ by

$$\tilde{G}_i^{\Gamma_{in}} := \{\gamma |\ [\{\Gamma_{in} + \gamma \mathbf{e}_i\} - \mathbf{e}_i, \{\Gamma_{in} + \gamma \mathbf{e}_i\}] = 0 \text{ by } *_{n-1}\}.$$

If $\Gamma_{in} \leq \Gamma'_{in}$, then we can see from $*_{n-1} - 1)$ that

C-0) $$\tilde{G}_i^{\Gamma_{in}} \supseteq \tilde{G}_i^{\Gamma'_{in}}.$$

ii) Assume that there is a system of maps

$$\left\{ \tilde{\gamma}_n^{\Gamma_{in}} : \{\gamma | 0 \le \gamma \le 2g-1-deg\,\Gamma_{in}\} \to \{\gamma | 0 \le \gamma \le 2g-1-deg\,\Gamma_{in}\} \middle| \Gamma_{in} \in V^{(n)}, 1 \le i \le n-1 \right\}$$

satisfying

$\alpha)$ $\quad \tilde{\gamma}_n^{\Gamma_{in}}(\gamma) = 0 \qquad$ if $\quad \gamma \notin \tilde{G}_i^{\Gamma_{in}}$.

$\beta)$ $\quad \tilde{\gamma}_n^{\Gamma_{in}}$ is an injective map from $\tilde{G}_i^{\Gamma_{in}}$ into $\{\gamma | 1 \le \gamma \le 2g-1-deg\,\Gamma_{in}\}$.

Define a map $\tilde{\gamma}_i^{\Gamma_{in}}$ on $\{\gamma | 0 \le \gamma \le 2g-1-deg\,\Gamma_{in}\}$ by

$$\gamma) \quad \begin{aligned} &\tilde{\gamma}_i^{\Gamma_{in}}(\gamma) = (\tilde{\gamma}_n^{\Gamma_{in}})^{-1}(\gamma) \qquad &&\text{for} \quad \gamma \in \tilde{\gamma}_n^{\Gamma_{in}}(\tilde{G}_i^{\Gamma_{in}}) \\ &\tilde{\gamma}_i^{\Gamma_{in}}(\gamma) = 0 &&\text{for} \quad \gamma \notin \tilde{\gamma}_n^{\Gamma_{in}}(\tilde{G}_i^{\Gamma_{in}}). \end{aligned}$$

Moreover they assume to be satisfied the following conditions (C-1), C-2), C-3)).

C-1) If $\Gamma_{in} \le \Gamma'_{in}$, then
$\tilde{\gamma}_n^{\Gamma_{in}}(\gamma) \ge \tilde{\gamma}_n^{\Gamma'_{in}}(\gamma) \quad$ on $\{\gamma | 0 \le \gamma \le 2g-1-deg\,\Gamma'_{in}\}$.
(N.B. C-1) is equivalent to $\quad \tilde{\gamma}_n^{\Gamma_{in}}(\gamma) \ge \tilde{\gamma}_n^{\Gamma'_{in}}(\gamma)$ on $\tilde{G}_i^{\Gamma'_{in}}$
by C-0), α) and β).)

C-2) $\tilde{\gamma}_i^{\Gamma_{in}}(\gamma) \ge \tilde{\gamma}_i^{\Gamma'_{in}}(\gamma) \qquad$ on $\quad \{\gamma | 0 \le \gamma \le 2g-1-deg\,\Gamma'_{in}\}$.
(N.B. C-2) is equivalent to

$$\begin{cases} \tilde{\gamma}_n^{\Gamma_{in}}(\tilde{G}_i^{\Gamma_{in}}) \supset \tilde{\gamma}_n^{\Gamma'_{in}}(\tilde{G}_i^{\Gamma'_{in}}) \\ \text{and} \\ \tilde{\gamma}_i^{\Gamma_{in}}(\gamma) \ge \tilde{\gamma}_i^{\Gamma'_{in}}(\gamma) \text{ on } \tilde{\gamma}_n^{\Gamma'_{in}}(\tilde{G}_i^{\Gamma'_{in}}). \end{cases}$$

In fact, if C-2) holds and there exists $\gamma \in \tilde{G}_i^{\Gamma'_{in}}$ satisfying $\tilde{\gamma}_n^{\Gamma'_{in}}(\gamma) \notin \tilde{\gamma}_n^{\Gamma_{in}}(\tilde{G}_i^{\Gamma_{in}})$, then $\tilde{\gamma}_i^{\Gamma'_{in}}\tilde{\gamma}_n^{\Gamma'_{in}}(\gamma) \le \tilde{\gamma}_i^{\Gamma_{in}}\tilde{\gamma}_n^{\Gamma'_{in}}(\gamma) = 0$ by γ). Hence $\gamma \le 0$. This is a contradiction.)

C-3) For $\Gamma = \sum_{i=1}^{n} \gamma_i \mathbf{e}_i \in V^{(n)}$ and $1 \le k,\ l \le n-1$, Γ_{kn} and Γ_{ln} are as in Proposition A-4. Then

$$\gamma_k < \tilde{\gamma}_k^{\Gamma_{kn}}(\gamma_n) \quad \text{if and only if} \quad \gamma_l < \tilde{\gamma}_l^{\Gamma_{ln}}(\gamma_n).$$

Now we put the weight 0 or 1 on each edge in $E^{(n)}$ according to the following set R of rules $R1),..,Rn)$.

$R)$

$R-i)(i=1,...,n-1)$ Let $\Gamma=(\gamma_1,..,\gamma_n)\in V^{(n)}$ with $\gamma_i>0$.

$$[\Gamma-\mathbf{e}_i,\ \Gamma]=1 \iff \tilde{\gamma}_n^{\Gamma_{in}}(\gamma_i)\le\gamma_n.$$

$R-n)$ Let $\Gamma=(\gamma_1,..,\gamma_n)\in V^{(n)}$ with $\gamma_n>0$.

$$[\Gamma-\mathbf{e}_n,\ \Gamma]=1 \iff \tilde{\gamma}_k^{\Gamma_{kn}}(\gamma_n)\le\gamma_k \text{ for some } k\ne n.$$

$$(\qquad \iff \tilde{\gamma}_k^{\Gamma_{kn}}(\gamma_n)\le\gamma_k \text{ for all } k\ne n \quad \text{by C-3))}.$$

Because of C-3), the weight of each edge is well defined by $R-i)$.

Definition A − 6 $(D^{(n)},R)$ denotes the graph such that each edge has weight 0 or 1 according to $R)$, and define $G_i^{\Gamma_{ik}}$ and $\gamma_k^{\Gamma_{ik}}(*)$ by the same way as in Definition A-1.

(i.e., Let $\Gamma=(\gamma_1,...,\gamma_n)\in V^{(n)}$ and put $\Gamma=\Gamma_{ij}+\gamma_i\mathbf{e}_i+\gamma_j\mathbf{e}_j$ for fixed i and $j(1\le i,j\le n, i\ne j)$, then

$$G_i^{\Gamma_{ij}}:=\{\gamma|0<\gamma\le 2g-deg\,\Gamma_{ij}-1 \text{ and } [\{\Gamma_{ij}+\gamma\mathbf{e}_i\}-\mathbf{e}_i,\{\Gamma_{ij}+\gamma\mathbf{e}_i\}]=0 \text{ by } R\}.$$

For $0\le\ \gamma\le\ 2g-deg\ \Gamma_{ij}-1$, we define a non-negative integer $\gamma_j^{\Gamma_{ij}}(\gamma)$ by

i) For $\gamma\notin G_i^{\Gamma_{ij}}$, $\gamma_j^{\Gamma_{ij}}(\gamma)=0$.

#′ ii) For $\gamma\in G_i^{\Gamma_{ij}}$,

a) $\gamma_j^{\Gamma_{ij}}(\gamma):=2g-deg\ \Gamma_{ij}-\gamma(\ge 1)$ if $\Delta_j(\Gamma_{ij},\gamma)=\emptyset$

b) $\gamma_j^{\Gamma_{ij}}(\gamma):=\min\{\alpha|\alpha\in\Delta_j(\Gamma_{ij},\gamma)\}$ if $\Delta_j(\Gamma_{ij},\gamma)\ne\emptyset$,

where

$$\Delta_j(\Gamma_{ij},\gamma)=\{\alpha|[\{\Gamma_{ij}+\gamma\mathbf{e}_i+\alpha\mathbf{e}_j\}-\mathbf{e}_i,\{\Gamma_{ij}+\gamma\mathbf{e}_i+\alpha\mathbf{e}_j\}]=1 \text{ by } R\}.)$$

Lemma A − 7 (1) *For* $1\le i\le n-1$, *we have*

$$\tilde{G}_i^{\Gamma_{in}}=G_i^{\Gamma_{in}},\quad \tilde{\gamma}_n^{\Gamma_{in}}(\tilde{G}_i^{\Gamma_{in}})=G_n^{\Gamma_{in}},\quad \tilde{\gamma}_n^{\Gamma_{in}}(*)=\gamma_n^{\Gamma_{in}}(*)\quad \textit{and}\quad \tilde{\gamma}_i^{\Gamma_{in}}(*)=\gamma_i^{\Gamma_{in}}(*).$$

(2) *Let* $1\le i,\ k\le n-1$ *and* $i\ne k$.
For $\Gamma=\Gamma_{ik}+\gamma_i\mathbf{e}_i+\gamma_k\mathbf{e}_k\in V^{(n)}$ *with* $\gamma_i>0$,

$$\gamma_k^{\Gamma_{ik}}(\gamma_i)\le\gamma_k \quad \textit{if and only if} \quad [\Gamma-\mathbf{e}_i,\Gamma]=1.$$

Proof) (1) By α) and β) in Definition A-5, $\gamma \in \tilde{G}_i^{\Gamma_{in}}$ is equivalent to $\tilde{\gamma}_n^{\Gamma_{in}}(\gamma) > 0$. And, by $R-i)(i \neq n)$, $\tilde{\gamma}_n^{\Gamma_{in}}(\gamma) > 0$ is equivalent to $\gamma \in G_i^{\Gamma_{in}}$. Thus $\tilde{G}_i^{\Gamma_{in}} = G_i^{\Gamma_{in}}(i \neq n)$. By $R-i)(i \neq n)$, we also have $\Delta_n(\Gamma_{in}, \gamma) = \{\alpha | \tilde{\gamma}_n^{\Gamma_{in}}(\gamma) \leq \alpha\}$.
Then $\quad \tilde{\gamma}_n^{\Gamma_{in}}(*) = \gamma_n^{\Gamma_{in}}(*) \quad$ and $\quad \tilde{\gamma}_i^{\Gamma_{in}}(*) = \gamma_i^{\Gamma_{in}}(*)$.

Next we will prove $\tilde{\gamma}_n(\tilde{G}_i^{\Gamma_{in}}) = G_n^{\Gamma_{in}}$.
Take $\gamma \in \tilde{\gamma}_n^{\Gamma_{in}}(\tilde{G}_i^{\Gamma_{in}}) = \gamma_n^{\Gamma_{in}}(G_i^{\Gamma_{in}})$. Then

$$\tilde{G}_i^{\Gamma_{in}} \ni (\tilde{\gamma}_n^{\Gamma_{in}})^{-1}(\gamma) = \tilde{\gamma}_i^{\Gamma_{in}}(\gamma) > 0.$$

Thus, by $R-n$),

$$[\{\Gamma_{in} + \gamma \mathbf{e}_n\} - \mathbf{e}_n, \{\Gamma_{in} + \gamma \mathbf{e}_n\}] = 0$$

and $\quad \gamma \in G_n^{\Gamma_{in}}$.

Conversely, if $\gamma \in G_n^{\Gamma_{in}}$, then $\tilde{\gamma}_i^{\Gamma_{in}}(\gamma) > 0$ by $R-n$). And we have $\gamma \in \tilde{\gamma}_n^{\Gamma_{in}}(\tilde{G}_i^{\Gamma_{in}})$ by γ) in Definition A-5.

(2) Assume $\gamma_k^{\Gamma_{ik}}(\gamma_i)) \leq \gamma_k$, and put $\Gamma' = \Gamma_{ik} + \gamma_i \mathbf{e}_i + \gamma_k^{\Gamma_{ik}}(\gamma_i)\mathbf{e}_k$. Then $[\Gamma' - \mathbf{e}_i, \Gamma'] = 1$. Let Γ'_{in} be the $(n-2)$-tuple satisfying $\Gamma' = \Gamma'_{in} + \gamma_i \mathbf{e}_i + \gamma_n \mathbf{e}_n$. Then, by $R-i$),

$$\gamma_n^{\Gamma'_{in}}(\gamma_i) \leq \gamma_n.$$

Since $\Gamma'_{in} \leq \Gamma_{in}$,

$$\gamma_n^{\Gamma_{in}}(\gamma_i) \leq \gamma_n^{\Gamma'_{in}}(\gamma_i) \qquad \text{by C-2).}$$

Hence $\gamma_n^{\Gamma_{in}}(\gamma_i) \leq \gamma_n$. We proved that $[\Gamma - \mathbf{e}_i, \Gamma] = 1$ if $\gamma_k^{\Gamma_{ik}}(\gamma_i)) \leq \gamma_k$.
Conversely if $[\Gamma - \mathbf{e}_i, \Gamma] = 1$, then $\gamma_k \in \Delta_k(\Gamma_{ik}, \gamma_i) \neq \emptyset$ and $\gamma_k^{\Gamma_{ik}}(\gamma_i)$ is equal to $\min\{\alpha | \alpha \in \Delta_k(\Gamma_{ik}, \gamma_i)\}$. Thus $\gamma_k^{\Gamma_{ik}}(\gamma_i)) \leq \gamma_k$. □

By Lemma A-7(2) and R), we can see easily that the graph $(D^{(n)}, R)$ satisfies the condition $*_n - 1$).

Now we add the following assumption so that the graph $(D^{(n)}, R)$ satisfies $*_n - 2$).

C-4) Let $1 \leq i, k \leq n-1$, $\gamma_k^{\Gamma_{ik}}$ is a bijection from $G_i^{\Gamma_{ik}}$ to $G_k^{\Gamma_{ik}}$ so that

$$(\gamma_k^{\Gamma_{ik}})^{-1}(*) = \gamma_i^{\Gamma_{ik}}(*) \qquad \text{on} \qquad G_k^{\Gamma_{ik}}.$$

Theorem A − 8 *Assume that*

$$(V^{(n-1)}, *_{n-1}) \quad \textit{and} \quad \{\tilde{\gamma}_n^{\Gamma_{in}} | 1 \le i \le n-1, \Gamma_{in} \in V^{(n-2)}\}$$

satisfy the conditions $C-1) \sim C-4)$. *Then the graph* $(D^{(n)}, R)$ *is equipped with* $*_n$ *which induces the given* $*_{n-1}$.

Proof) We only have to show that $*_n - 2)$ is satisfied.
Let $\Gamma = \sum_{k=1}^n \gamma_k \mathbf{e}_k \in V^{(n)}$ satisfying $\gamma_i > 0$ and $\gamma_j > 0$ for some i and $j(1 \le i, j \le n,\ i \ne j)$. Let Γ_{ij} be the $(n-2)$-tuple satisfying $\Gamma = \Gamma_{ij} + \gamma_i \mathbf{e}_i + \gamma_j \mathbf{e}_j$.
By $*_n - 1)$ and $(\gamma_j^{\Gamma_{ij}})^{-1}(*) = \gamma_i^{\Gamma_{ij}}(*)$($C-4)$ and γ)), the following two cases can be happened.

$$\dagger) \begin{cases} i) \ [\{\Gamma - \mathbf{e}_j\} - \mathbf{e}_i, \{\Gamma - \mathbf{e}_j\}] = [\Gamma - \mathbf{e}_i, \Gamma] \text{ and } [\{\Gamma - \mathbf{e}_i\} - \mathbf{e}_j, \{\Gamma - \mathbf{e}_i\}] = [\Gamma - \mathbf{e}_j, \Gamma]. \\ ii) \ [\{\Gamma - \mathbf{e}_j\} - \mathbf{e}_i, \{\Gamma - \mathbf{e}_j\}] = [\{\Gamma - \mathbf{e}_i\} - \mathbf{e}_j, \{\Gamma - \mathbf{e}_i\}] = 0 \text{ and } [\Gamma - \mathbf{e}_i, \Gamma] = [\Gamma - \mathbf{e}_j, \Gamma] = 1. \end{cases}$$

(In fact, for example, if $[\Gamma - \mathbf{e}_i, \Gamma] = 1$ and $[\{\Gamma - \mathbf{e}_j\} - \mathbf{e}_i, \{\Gamma - \mathbf{e}_j\}] = 0$, then $\gamma_j^{\Gamma_{ij}}(\gamma_i) = \gamma_j$. As $\gamma_i = (\gamma_j^{\Gamma_{ij}})^{-1}(\gamma_j) = \gamma_i^{\Gamma_{ij}}(\gamma_j)$, $[\Gamma - \mathbf{e}_j, \Gamma] = 1$ and $[\{\Gamma - \mathbf{e}_i\} - \mathbf{e}_j, \{\Gamma - \mathbf{e}_i\}] = 0$. This is the case †) *ii*).)
†) implies the condition A) of Lemma 2-1.

Let $\Gamma = \sum_{k=1}^n \gamma_k \mathbf{e}_k \in V^{(n)}$ with $deg\ \Gamma = 2g-2$. Then we have

$$\dagger\dagger) \qquad [\Gamma, \Gamma + \mathbf{e}_i] = [\Gamma, \Gamma + \mathbf{e}_n] \quad \text{for } 1 \le i \le n-1.$$

(In fact, $[\Gamma, \Gamma + \mathbf{e}_i] = 0$ is equivalent to $\gamma_n^{\Gamma_{in}}(\gamma_i + 1) = \gamma_n + 1$ by $R - i$).
As $(\gamma_n^{\Gamma_{in}})^{-1} = \gamma_i^{\Gamma_{in}}$, we have $\gamma_i^{\Gamma_{in}}(\gamma_n + 1) = \gamma_i + 1$. This is equivalent to $[\Gamma, \Gamma + \mathbf{e}_n] = 0$ by $R - n$).)
†) and ††) imply The condition B) in Lemma 2-1.

When $\Gamma_{kn} = (0, ..., 0)$(write O_{kn}) for $k \ne n$, the subset $\tilde{\gamma}_n(G_k^{0_{kn}}) = G_n^{O_{kn}}$(Lemma A-7 (1)) of $\{\gamma | 1 \le \gamma \le 2g-1\}$ is uniquely determined whichever k we may take(by C-3 and $R - n$)). We denote this set by $\tilde{G}$. Then $\tilde{G} = \{\gamma | \gamma \mathbf{e}_n \in V^{(n)}, [(\gamma - 1)\mathbf{e}_n, \gamma \mathbf{e}_n] = 0\}$ and $\#\tilde{G} = \#(G_k^{O_{kn}}) = g$. This means that the condition C) in Lemma 2-1 is satisfied. □

Remark A − 9 The non-negative integer δ^{Γ_n} defined in §.2 can be re-defined by

$$\delta^{\Gamma_n} := max\{\gamma_n^{\Gamma_{in}}(\gamma_i) | 1 \le i \le n-1\},$$

where $\Gamma = (\gamma_1, ..., \gamma_n) = (\Gamma_n, \gamma_n) = \Gamma_{in} + \gamma_i \mathbf{e}_i + \gamma_n \mathbf{e}_n$ for $1 \le i \le n-1$.

Example A − 10 Let $(V^{(3)}, *_3)$ be the graph in Theorem 4-3. Let $\Gamma =$

$(\gamma_1, \gamma_2, \gamma_3) \in V^{(3)}$. Then $\Gamma_{23} = \gamma_1$ and $\Gamma_3 = (\gamma_1, \gamma_2)$. If $\gamma_1 = 2k-1$ or $2k$ with $1 \leq k \leq g-1$, then

$$G_2^{\Gamma_{23}} = G_3^{\Gamma_{23}} = \{1, 3, 5, ..., 2(g-k)-1\}$$

and

$$\gamma_3^{\Gamma_{23}}(\gamma_2) = \begin{cases} 0 & \text{if } \gamma_2 \text{ is even} \\ 2(g-k)-\gamma_2 & \text{if } \gamma_2 \text{ is odd.} \end{cases}$$

Then, for $(\gamma_1, \gamma_2) \in V^{(2)}$,

$$\delta^{\Gamma_3} = \begin{cases} 2g-1-\gamma_1-\gamma_2 & \text{if } \gamma_1 \text{ and } \gamma_2 \text{ are odd} \\ 2g-\gamma_1-\gamma_2 & \text{if } \gamma_1 \text{ is odd(resp.even) and } \gamma_2 \text{ is even(resp.odd)} \\ 0 & \text{if } \gamma_1 \text{ and } \gamma_2 \text{ are even.} \end{cases}$$

(In fact, if $\gamma_1 = 2k-1$ and $\gamma_2 = 2l-1$ ($0 \leq k, l \leq g-1$, $k+l \leq g$) then

$$\begin{aligned} \delta^{\Gamma_3} &= max\{\gamma_3^{\Gamma_{13}}(\gamma_1) = 2(g-l)-\gamma_1,\ \gamma_3^{\Gamma_{23}}(\gamma_2) = 2(g-k)-\gamma_2\} \\ &= 2g-1-\gamma_1-\gamma_2. \end{aligned}$$

If $\gamma_1 = 2k-1$ and $\gamma_2 = 2l$($0 \leq k, l \leq g-1$, $k+l \leq g$), then

$$\begin{aligned} \delta^{\Gamma_3} &= max\{\gamma_3^{\Gamma_{13}}(\gamma_1) = 2(g-l)-\gamma_1,\ \gamma_3^{\Gamma_{23}}(\gamma_2) = 0\} \\ &= 2g-\gamma_1-\gamma_2. \end{aligned}$$

(See Proposition 4-7).)

II Covering over d-gonal Curves([8])

1

Let M be a compact Riemann surface and f be a meromorphic function on M. Let (f) be the principal divisor associated to f and $(f)_\infty$ be the polar divisor of f. We call f a meromorphic function of degree d if $d =$ degree $(f)_\infty$. If d is the minimal integer in which a meromorphic function of degree d exists on M, then we call M a d-gonal curve.

Now we assume that M is d-gonal, and consider a covering map $\pi' : M' \to M$ that M' still remains d-gonal. The purpose of this chapter is to show how such π' can be characterized.

The case that π' is a normal covering and $d = 2$ (i.e., M is hyperelliptic) has been already studied ([3],[6],[11] and [15]). In this case the existence of the hyperellptic involution v' on M' plays an important role. More precisely, as v' commutes with each element of the Galois group $G = Gal(M'/M)$, v' induces the hyperelliptic involution v on M and we can reduce π' to a normal covering $\pi : \mathbf{P}'_1 \to \mathbf{P}_1$ with Galois group G, where $\mathbf{P}'_1$ and $\mathbf{P}_1$ are Riemann spheres isomorphic to quotient Riemann surfaces $M'/ < v' >$ and $M/ < v >$ respectively. On the other hand it is known that finite subgroups of the linear transformation group are cyclic, dihedral, tetrahedral, octahedral and icosahedral. Horiuchi [6] decided all the different normal coverings $\pi' : M' \to M$ over a hyperelliptic curve M that M' still remains a hyperelliptic curve by investigating each of above five types.

Let M be a d-gonal curve. In this chapter we will show at first that a covering map $\pi' : M' \to M$(not necessarily normal) with d-gonal M' canonically induces some covering map $\pi : \mathbf{P}'_1 \to \mathbf{P}_1$ (Theorem 2.1 §2). Moreover if both M and M' have a unique linear system g^1_d and π' is normal, then we can see that π is also normal(Cor.2.3).

In §3, §4 and §5 we assume that M is a cyclic p-gonal curve for a prime number p. We will determine all ramification types of normal coverings $\pi' : M' \to M$ with p-gonal M' by the same way as Horiuchi did in case $p = 2$(§4), and we give some results about unramified coverings $\pi' : M' \to M$, where π' is not necessarily normal (§5).

2

Let $\pi' : M' \to M$ be a covering over an arbitrary compact Riemann surface M. Let $\mathbf{C}(M)$ and $\mathbf{C}(M')$ be the function fields of M and M' respectively and $Nm_{\pi'} = Nm : \mathbf{C}(M') \to \mathbf{C}(M)$ be the norm map. For a divisor $D = \sum_{i=1} n_i Q_i (n_i \in \mathbf{Z})$ on M', we define a divisor $Nm_{\pi'}\ D = Nm\ D$ on M by

$$Nm_{\pi'}\ D = \sum n_i \pi'(Q_i).$$

Then the following equation of principal divisors holds([2] Appendix B):

$$Nm_{\pi'}((f)) = (Nm\ f).$$

If two divisors D' and E' are linearly equivalent, write $D' \sim E'$, the above equation means that $Nm\ D' \sim NmE'$.

Let $\pi'^* P$ denote a divisor on M' obtained by the inverse image of a point $P \in M$ with ramification points counted according to multiplicity. For a divisor $D = \sum n_i P_i$, $\pi'^* D := \sum n_i \pi'^* P_i$. $|D|$ is the complete linear system of D and $\mathcal{L}(D)$ is the $\mathbf{C}$-vector space consisting of 0 and meromorphic functions f satisfying $(f) + D > 0$. $l(D)$ is the dimension of $\mathcal{L}(D)$ over $\mathbf{C}$.

From now on we assume that M is d-gonal. Then there exists a positive divisor D of degree d on M satisfying $l(D) \geq 2$, and $l(E) = 1$ for any positive divisor E of degree less than d. Actually on this D we can easily see that $l(D) = 2$, and then the linear system $|D|$ defines a covering map of degree d;

$$\psi_{|D|} = \psi : M \to \mathbf{P}_1$$

, where $\mathbf{P}_1$ is a Riemann sphere. Explicitely $\psi(P)$ is defined by $\psi(P) = h(P) \in \mathbf{C} \cup \{\infty\}$ for $P \in M$, where h is a non-trivial meromorphic function in $\mathcal{L}(D)$. ψ is defined uniquely up to linear transformations of $\mathbf{P}_1$. By the minimality of d, a divisor $\psi^*\psi(P)$ is uniquely determined not corresponding to the choice of h. For distinct points P and P' on M, $\psi^*\psi(P)$ and $\psi^*\psi(P')$ are linearly equivalent and having no common point in their supports.

Let $\pi' : M' \to M$ be a covering of degree n over M that M' still remains d-gonal. Let D' be a positive divisor on M' of degree d satisfying $l(D') = 2$. then we have;

Theorem 2.1 *Put* $D = Nm_{\pi'} D'$. *Then*

i) There exists a covering map $\pi : \mathbf{P}_1' \to \mathbf{P}_1$ *satisfying the following diagram;*

$$\begin{array}{ccc} & \psi_{|D'|} = \psi' & \\ M' & \rightarrow & \mathbf{P}_1' \\ \pi' \downarrow & & \downarrow \pi \\ M & \rightarrow & \mathbf{P}_1 \\ & \psi_{|D|} = \psi & \end{array}$$

with $deg\pi' = deg\pi = n$ *and deg* $\psi' = deg\psi = d$.

ii) Let $\mathbf{C}(M')$, $\mathbf{C}(M)$, $\mathbf{C}(\mathbf{P}_1')$ *and* $\mathbf{C}(\mathbf{P}_1)$ *be the function fields. Then* $\mathbf{C}(M) \cap \mathbf{C}(\mathbf{P}_1') = \mathbf{C}(\mathbf{P}_1)$ *in* $\mathbf{C}(M')$ *and* $\mathbf{C}(M') = \mathbf{C}(M) \bigotimes_{\mathbf{C}(\mathbf{P}_1)} \mathbf{C}(\mathbf{P}_1')$.

To prove this Theorem we prepare some lemmas. Put $D' = \sum_{i=1}^{d} P_i (P_i$ are not necessarily distinct), and $\pi'^* \pi' P_i = \sum_{k=1}^{n} P_i^{(k)}$

Lemma 2.1.1. *For each* i,

$$Nm_{\pi'} \psi'^* \psi'(P_i^{(k)}) = Nm_{\pi'} \psi'^* \psi'(P_i) = Nm_{\pi'} D', \quad k = 1, 2, ..., n.$$

Proof. $Nm_{\pi'} \psi'^* \psi'(P_i^{(k)})$ and $Nm_{\pi'} \psi'^* \psi'(P_i)$ are divisors of degree d on M, and they have a common point $\pi'(P_i^{(k)}) = \pi'(P_i)$. But they are linearly equivalent as $\psi'^* \psi'(P_i^{(k)}) \sim \psi'^* \psi'(P_i)$. Then we have $Nm_{\pi'} \psi'^* \psi'(P_i^{(k)}) = Nm_{\pi'} \psi'^* \psi'(P_i)$ by the minimality of d. □

As $l(D') > 1$, we may assume that $D' = \sum_{i=1}^{d} P_i (= \psi'^* \psi'(P_1))$ satisfies the following conditions $*$);

$$*) \qquad \begin{array}{c} P_i \text{ are distinct, } \pi' \text{ is unramified over } \pi'(P_i), 1 \leq i \leq d, \\ \text{and } \psi' \text{ is unramified over } \psi'(P_1^{(k)}), 1 \leq k \leq n. \end{array}$$

Let $Nm_{\pi'} D' = d_1 R_1 + d_2 R_2 + \cdots d_t R_t, d_1 + d_2 + \cdots d_t = d$, where R_i are distinct points in M and $\pi'(P_1) = R_1$. Changing the indeces of P_i, we may assume that

$$\begin{array}{rcl} \pi'(P_1) & = & \cdots = \pi'(P_{d_1}) = R_1 \\ \pi'(P_{d_1+1}) & = & \cdots = \pi'(P_{d_1+d_2}) = R_2, \\ \cdots\cdots & & \\ \pi'(P_{d_1+d_2+\cdots+d_{t-1}+1}) & = & \cdots = \pi'(P_{d_1+\cdots+d_t}) = R_t. \end{array}$$

Lemma 2.1.2 $d_1|n$, $d_1|d$ *and* $d_1 = d_2 = \cdots = d_t$.

Proof. Put $\pi'^* R_i = \pi'^* \pi'(P_{d_1+\cdots d_{i-1}+s_i}) = A_i^{(1)} + \cdots + A_i^{(n)}$, $s_i = 1, \cdots, d_i, i =$

$1, \cdots, t$. Then $A_i^{(k)} (k = 1, \cdots, n)$ are distinct by $*$). By Lemma 2.1.1 $Nm_{\pi'}\psi'^*\psi'(A_i^{(k)}) = d_1 R_1 + \cdots + d_t R_t$. For ψ' is unramified over $\psi'(A_1^{(k)})$, $\psi'^*\psi'(A_1^{(1)})$ also consists of distinct d points. Changing the induces k of $A_j^{(k)}$ for each j, we may write;

$$\psi'^*\psi'(A_1^{(1)}) = (A_1^{(1)} + \cdots A_1^{(d_1)}) + (A_2^{(1)} + \cdots A_2^{(d_2)}) + \cdots + (A_t^{(1)} + \cdots A_t^{(d_t)}).$$

Especially $d_1 \leq n$. By the minimality of d, $\psi'^*\psi'(A_1^{(1)}) = \ldots = \psi'^*\psi'(A_1^{(d_1)})$. If $d_1 < n$, then take a point over R_1, namely $A_1^{(d_1+1)}$, not equal to $A_1^{(k)}, 1 \leq k \leq d_1$. Then we may write;

$$\psi'^*\psi'(A_1^{(d_1+1)}) = (A_1^{(d_1+1)} + \cdots + A_1^{(2d_1)}) + \cdots + (A_t^{(d_t+1)} + \cdots A_t^{(2d_t)})$$

and $\psi'^*\psi'(A_1^{(d_1+1)}) = \ldots = \psi'^*\psi'(A_1^{(2d_1)})$. If still $2d_1 < n$, then repeat the same manner as above and finally we have the following $sd_1 + 1$ equations of divisors;

$$\begin{cases} \psi'^*\psi'(A_1^{(1)}) = (A_1^{(1)} + \cdots + A_1^{(d_1)}) + \cdots + (A_t^{(1)} + \cdots + A_t^{(d_t)}) & (1.1) \\ \cdots\cdots & \\ \psi'^*\psi'(A_1^{(d_1)}) = (A_1^{(1)} + \cdots + A_1^{(d_1)}) + \cdots + (A_t^{(1)} + \cdots + A_t^{(d_t)}) & (1.d_1) \end{cases}$$

$$\begin{cases} \psi'^*\psi'(A_1^{(d_1+1)}) = (A_1^{(d_1+1)} + \cdots + A_1^{(2d_1)}) + \cdots + (A_t^{(d_t+1)} + \cdots + A_t^{(2d_t)}) & (2.1) \\ \cdots\cdots & \\ \psi'^*\psi'(A_1^{(2d_1)}) = (A_1^{(d_1+1)} + \cdots + A_1^{(2d_1)}) + \cdots + (A_t^{(d_t+1)} + \cdots + A_t^{(2d_t)}) & (2.d_1) \end{cases}$$

$$\cdots\cdots$$

$$\begin{cases} \psi'^*\psi'(A_1^{((s_1-1)d_1+1)}) = (A_1^{((s-1)d_1+1)} + \cdots + A_1^{(sd_1)}) + \cdots + (A_t^{((s-1)d_t+1)} + \cdots + A_t^{(sd_t)}) & (s.1) \\ \cdots\cdots & \\ \psi'^*\psi'(A_1^{(sd_1)}) = (A_1^{((s-1)d_1+1)} + \cdots + A_1^{(sd_1)}) + \cdots + (A_t^{((s-1)d_t+1)} + \cdots + A_t^{(sd_t)}) & (s.d_1) \end{cases}$$

and

$$\pi'^* R_1 = (A_1^{(1)} + \cdots + A_1^{(d_1)}) + \cdots + (A_1^{((s-1)d_1+1)} + \cdots + A_1^{(sd_1)}). \qquad (**)$$

Then $n = d_1 \cdot s$. If $d_1 > d_t$, then $n = d_1 \cdot s > d_t \cdot s$. There exists a point over R_t, namely $A_t^{(n)}$, never appears in the right hand sides of the above equations $(1.1) \sim s.d_1)$. On the other hand $\psi'^*\psi'(A_t^n)$ has $A_1^{(k)}$ for some k in its support by Lemma 2.1.1. For the minimality of d, $\psi'^*\psi'(A_t^{(n)}) = \psi'^*\psi'(A_t^{(k)})$. This is a contradiction. If $d_1 < d_t$, then $n = d_1 \cdot s < d_t \cdot s$. This also can not be happened. □

By Lemma 2.1.2, and the above equations $(1.1) \sim (s.d_1)$, $*, *)$, we have;

Lemma 2.1.3.

$$\sum_{k=1}^{n} \psi'^*\psi'(P_1^{(k)}) = \sum_{i=1}^{d} \pi'^*\pi'(P_i) = \pi'^* Nm(D').$$

Proof of Theorem 2.1.

Let $E' = \sum Q_i$ and $E'' = \sum S_i$ be in $|D'|$ satisfying the conditions $*$). Let h' be a non-constant function in $\mathcal{L}(D')$ and $h = Nm\ h'$.

$$\begin{aligned}
div(h \circ \pi') &= \pi'^* Nm\ E' - \pi'^* Nm\ E'' \\
&= \sum_{k=1}^{n} \psi'^* \psi'(Q_1^{(k)}) - \sum_{k=1}^{n} \psi'^* \psi'(S_1^{(k)}) \quad \text{by Lemma 2.1.3} \\
&= \sum_{k=1}^{n} [\{\psi'^* \psi'(Q_1^{(k)}) - \psi'^* \psi'(P_1)\} - \{\psi'^* \psi'(S_1^{(k)}) - \psi'^* \psi'(P_1)\}] \\
&= \sum_{k=1}^{n} [\{\psi'^* \psi'(Q_1^{(k)}) - D'\} - \{\psi'^* \psi'(S_1^{(k)}) - D'\}] \\
&= \sum_{k=1}^{n} \{(a_k h' + b_k) - (c_k h' + d_k)\} \\
&= (\prod_{k=1}^{n} \frac{a_k h' + b_k}{c_k h' + d_k}).
\end{aligned}$$

Then $h \circ \pi'$ is in $\mathbf{C}(h') = \mathbf{C}(\mathbf{P}'_1)$ and we have

$$\begin{array}{ccc}
\mathbf{C}(M') & \supset & \mathbf{C}(\mathbf{P}'_1) \\
\cup & & \cup \\
\mathbf{C}(M) & \supset & \mathbf{C}(\mathbf{P}_1)
\end{array}$$

with

$[\mathbf{C}(M') : \mathbf{C}(M)] = [\mathbf{C}(\mathbf{P}'_1) : \mathbf{C}(\mathbf{P}_1)] = n$

and $[\mathbf{C}(M) : \mathbf{C}(\mathbf{P}_1)] = [\mathbf{C}(M') : \mathbf{C}(\mathbf{P}'_1)] = d$.

As

$$[\mathbf{C}(M) \bigotimes_{\mathbf{C}(\mathbf{P}_1)} \mathbf{C}(\mathbf{P}'_1) : \mathbf{C}(\mathbf{P}'_1)] = [\mathbf{C}(M') : \mathbf{C}(\mathbf{P}'_1)]$$

, we have ii). □

Conversely we have;

Remark 2.2. Let $\psi : M \to \mathbf{P}_1$ be a d-gonal curve with a d-th covering ψ over a Riemann sphere $\mathbf{P}_1$. Let $\pi' : \mathbf{P}'_1 \to \mathbf{P}_1$ be an arbitrary covering. Then function fields $\mathbf{C}(M)$ and $\mathbf{C}(\mathbf{P}'_1)$ are linearly disjoint over $\mathbf{C}(\mathbf{P}_1)$, and the Riemann surface M' obtained from the function field $\mathbf{C}(M) \otimes_{\mathbf{C}(\mathbf{P}_1)} \mathbf{C}(\mathbf{P}'_1) = \mathbf{C}(M) \cdot \mathbf{C}(\mathbf{P}'_1)$ is d-gonal.

Proof. Consider the canonical surjective map $\mathbf{C}(M) \otimes_{\mathbf{C}(\mathbf{P}_1)} \mathbf{C}(\mathbf{P}'_1) \to \mathbf{C}(M) \cdot \mathbf{C}(\mathbf{P}'_1)$. Put $d' = [\mathbf{C}(M) \cdot \mathbf{C}(\mathbf{P}'_1) : \mathbf{C}(\mathbf{P}'_1)]$. If $d' \leq d$, then M should be d''-gonal for some $d'' \leq d'$. This is a contradiction. □

Concerning about the diagram in Theorem 2.1, π is not necessarily normal even if π' is normal. But we have;

Corollary 2.3. *If M' has a unique linear system g_d^1 and π' is normal, then π is normal and $Gal(M'/M) \simeq Gal(\mathbf{P}_1'/\mathbf{P}_1)$.*

Proof. Let σ be an automorphism on M'. For the uniqueness of g_d^1 there is an automorphism $\tilde{\sigma}$ on $\mathbf{P}_1'$ satisfying the following diagram;

$$\begin{array}{ccc} M' & \rightarrow & \mathbf{P}_1' \\ \pi \downarrow & & \downarrow \tilde{\pi} \\ M' & \rightarrow & \mathbf{P}_1 \end{array}$$

As $\mathbf{C}(M) \cap \mathbf{C}(\mathbf{P}_1') = \mathbf{C}(\mathbf{P}_1)$, $Gal(M'/M) \simeq Gal(\mathbf{P}_1'/\mathbf{P}_1)$. □

Remark 2.4. Under the two assumptions of Corollary 2.3, (i.e., π' is normal and the uniqess of g_d^1, we can prove Theorem 2.1. i) easier. In fact $Gal(M'/M)$ acts on $\mathbf{P}_1'$ as the proof of Corollary 2.3, and the fixed subfield of $\mathbf{C}(\mathbf{P}_1')$ by the action of $Gal(M'/M)$ is $\mathbf{C}(M) \cap \mathbf{C}(\mathbf{P}_1')$. This field is a function field of genus 0, and $[\mathbf{C}(M) : \mathbf{C}(M) \cap \mathbf{C}(\mathbf{P}_1')] = d$ for the minimality of d.

Remark 2.5. The condition that M' has a unique g_d^1 is satisfied in the following case:

> M' is p-gonal of genus $\geq (p-1)^2 + 1$ for a prime number p([17]),Cor.2.4.5), especially M' is defined by the equation $D(u, v) = 0$(§3(1)) with $m \geq 2p + 1$([17],[16],[12]).

Remark 2.6. Let p be a prime number. We assume that M has a p-th covering over $\mathbf{P}_1$. Then the condition that M is p-gonal is satisfied when genus of $M > (p-1)(p-2)$ ([17], Cor.2.4.5).

3

Let p be a prime number . We assume that M is a Riemann surface defined by the equation

$$D(\underline{y}, v) := y^p - (u - \alpha_1)^{k_1} \cdots (u - \alpha_m)^{k_m} = 0 \qquad (1)$$

,where α_i $(1 \leq i \leq m)$ are distinct and k_i are integers satisfying $1 \leq k_i \leq p-1$ and $\sum k_i \equiv 0 \bmod p$. Let $\psi : M \rightarrow \mathbf{P}_1$ be the cyclic normal covering of degree p over $\mathbf{P}_1$ defined by $(u, y) \mapsto u$. The branch points of ψ are $\alpha_i \in \mathbf{P}_1$, and ψ is completely ramified over α_i. Put $S = \{\alpha_i | 1 \leq i \leq m\}$. The genus of M is $\frac{(p-1)(m-2)}{2}$. Sometimes we use another equation $D'(u, y)$ for M;

$$D'(y, v) := y^p - (u - \beta_1)^{k_1} \cdots (u - \beta_{m-1})^{k_{m-1}} = 0 \qquad (2)$$

with $1 \leq k_i \leq p-1$ and $\sum k_i \not\equiv 0 \bmod p$. ψ is defined as above and the set S of the branch points of $\psi = \{\beta_1, ..., \beta_{m-1}, \infty\}$. In this case let $k_m > 0$ denote a minimal integer satisfying $k_m \equiv -\sum_{i=1}^{m-1} k_i \bmod p$, then we can get an equation of type (1) birationally equinalent to (2).

We call M a cyclic p-gonal if M is p-gonal and defined by (1) or (2). Hereafter we assume that M is cyclic p-gonal and having a unique g_d^1. If $m \geq 2p-1$, then M is p-gonal by Remark 2.6. If $m \geq 2p+1$, then M has a unique g_d^1 by Remark 2.5. $\pi' : M' \to M$ always means a covering map with p-gonal M'. Then a covering map $\pi' : M' \to M$ corresponds to a covering map $\pi' : \mathbf{P}'_1 \to \mathbf{P}_1$ by Theorem 2.1.

In this section we show the method how to get the equation of M' and π' explicitely from the equation of M and π. Put $\mathbf{P}_1 = Proj\ \mathbf{C}[z_0, z_1]$, $\mathbf{P}'_1 = Proj\ \mathbf{C}[u_0, u_1]$, $z = z_1/z_0$ and $u = u_1/u_0$. Assume that π' is defined by $(z_0, z_1) \mapsto (F_0(z_0, z_1) : F_1(z_0, z_1))$, where $F_i (i = 1, 2)$ are relatively prime homogeneous polynomials of same degree n. $V = Spec\ \mathbf{C}[z]_{F_0(1:z)}$ and $U = Spec\ \mathbf{C}[u]$ are affine open subsets of $\mathbf{P}'_1$ and $\mathbf{P}_1$ respectively. Then $\pi' : V \to U$ is represented by $z \to u = \frac{F_1(1:z)}{F_0(1:z)}$. Put $f = \frac{F_1(1:z)}{F_0(1:z)}$. Assume M is defined by the equation $D(u, v) = 0$ with $\alpha_i \in U = \mathbf{C}$ for all i. Put $A = \mathbf{C}[u, y]/(D(u, y))$. By Theorem 2.1,

$$\begin{aligned}
\mathbf{C}(M') &= \mathbf{C}(M) \otimes_{\mathbf{C}(\mathbf{P}_1)} \mathbf{C}(\mathbf{P}'_1) \\
&= A \otimes_{\mathbf{C}[u]} \mathbf{C}(\mathbf{P}'_1) \\
&\supset A \otimes_{\mathbf{C}[u]} \mathbf{C}[z]_{(F_0(1:z))} \\
&= \mathbf{C}[z]_{(F_0(1:z))}[y]/(D(\frac{F_1(1:z)}{F_0(1:z)}, y)).
\end{aligned}$$

Put $B = \mathbf{C}[z]_{(F_0(1:z))}[y]/(D(\frac{F_1(1:z)}{F_0(1:z)}, y))$. Then $Spec\ B = V \times_U A$. If we have factorizations;

$$F_1(1:z) - F_0(1:z)\alpha_i = c_i \prod_{t=1}^{l^{(i)}} (z - a_t^{(i)})^{e_t^{(i)}}$$

with some constants c_i, $a_t^{(i)} \in \mathbf{C}$ and $e_t^{(i)} \in \mathbf{N}$ satisfying $\sum_{t=1}^{l^{(i)}} e_t^{(i)} \leq n$, then $spec\ A \times_{\mathbf{P}_1} Spec\ \mathbf{C}[z]$ is defined by the equation

$$F_0(1:z)^{\sum_{i=1}^m k_i} y^p - \prod_{i=1}^m (c_i \prod_{t=1}^{l^{(i)}} (z - a_t^{(i)})^{e_t^{(i)}})^{k_i} = 0.$$

Put

$$G(z) = F_0(1:z)^{(\sum_{i=1}^m k_i)/p} \cdot \prod_{i=1}^m \cdot \prod_{t=1}^{l^{(i)}} (z - a_t^{(i)})^{-[e_t^{(i)} k_i/p]} (\prod_{i=1}^m c_i)^{(-\sum_{i=1}^m k_i)/p}$$

, where $[a/b]$ is Gauss symbol. Changing $G(z)y$ by y we have an equation of type (1) for M'

$$y^p - \prod_{i=1}^m \cdot \prod_{t=1}^{l^{(i)}} (z - a_t^{(i)})^{f_t^{(i)}} = 0 \qquad (3)$$

, where $f_t^{(i)}$ are positive integers satisfying $0 \le f_t^{(i)} < p$ and $f_t^{(i)} \equiv e_t^{(i)} k_i \mod p$. π' is defined by

$$(y, z) \mapsto (G(z))^{-1} y, F_1(1,z)/F_0(1,z)).$$

Let f_∞ be the integer satisfying $\sum_{i,t} f_t^{(i)} + f_\infty \equiv 0 \mod p$ and $0 \le f_\infty < p$. The set S' of branch points of ψ' consists of $a_t^{(i)}$ with $f_t^{(i)} \ne 0$ and ∞ if $f_\infty^{(i)} \ne 0$.

Next assume that M is defined by the equation $D'(u, y) = 0$ in (2) and we have factorizations;

$$F_1(1:z) - \beta_i F_0(1:z) = c_i \prod_{t=1}^{l^{(i)}} (z - b_t^{(i)})^{e_t^{(i)}} \qquad (1 \le i \le m-1)$$

and

$$F_0(1:z) = c_m (z - \gamma_1)^{r_1} \cdots (z - \gamma_s)^{r_s}, \qquad r_1 + \cdots + r_s \le n.$$

Let $f_t^{(i)}$ $(1 \le i \le m-1)$ be numbers satisfying $e_t^{(i)} \cdot k_i \equiv f_i^{(i)} \mod p$ and $0 \le f_t^{(i)} < p$. Let g_j $(1 \le j \le s)$ be numbers satisfying $r_j \cdot k_m \equiv g_j \mod p$ and $0 \le g_j < p$, where k_m is defined as before. By the same way as above we have an equation of M';

$$y^p - (\prod_{i=1}^{m-1} \prod_{t=1}^{l^{(i)}} (z - b_t^{(i)})^{f_t^{(i)}})(z - \gamma_1)^{g_1} \cdots (z - \gamma_s)^{g_s} = 0. \qquad (4)$$

π is defined by

$$(y, z) \mapsto (G'(z)^{-1} y, F_1(1:z)/F_0(1:z))$$

,where

$$G'(z) = F_0(1:z)^{(\sum_{i=1}^{m} k_i)/p}(\prod_{i}^{m} c_i)^{(-\sum_{i=1}^{m} k_i)/p} \cdot \prod_{i=1}^{m-1} \cdot \prod_{t=1}^{l^{(i)}} (z - b_t^{(i)})^{-[e_i^{(i)} \cdot k_i/p]} \prod_{j=1}^{s} (z - \gamma_j)^{-[r_j \cdot k_m/p]}.$$

Let f_∞ be the integer satisfying $\sum_{i,t} f_t^{(i)} + \sum_j g_j + f_\infty \equiv 0 \pmod p$ and $0 \le f_\infty < p$. The set S' of branch points of ψ' consists of $b_t^{(i)}$ ($f_t^{(i)} \neq 0$), γ_j ($g_j \neq 0$) and ∞ if $f_\infty \neq 0$.

Lemma 3.1. *For a point $P \in M'$, put $\psi'(P) = a$ and $\pi \circ \psi'(P) = \alpha$. e (resp.e') denotes the ramification index of π (resp. π') at a(resp. P).*

(1) *Assume α is a branch point of ψ. Then $e' = e/p$ if $p|e$, and $e = e'$ if $p \not| e'$.*

(2) *Assume α is not a branch point of ψ. Then $e' = e$.*

Proof. we may assume that M is defined by the equation (1). If α is a branch point of ψ , then $\alpha = \alpha_i$, $a = a_t^{(i)}$ and $e = e_t^{(i)}$ for some i and t. If $p|e$, then $f_t^{(i)} = 0$ and a is not a branch point of ψ' by (3). On the other hand the ramification index of ψ over α_i is p. As $\psi \circ \pi' = \pi \circ \psi'$, $p \cdot e' = e$. If $p \not| e$, then $f_t^{(i)} \neq 0$ and $a = a_t^{(i)}$ is a branch point of ψ. Then $e = e'$.

4

Let $\pi' : M' \to M$ be as in §3. Moreover we assume that π' is normal with Galois group G. By Corollary 2.3 in §2, π induced by π' is also normal with Galois group G. Then we use the following lemma to determine π';

Lemma 4.1. ([14],[6]) *By choosing suitable coordinates z and u for $\mathbf{P}_1'$ and $\mathbf{P}_1$ respectively, any normal coverings $\pi' : \mathbf{P}_1' \to \mathbf{P}_1$ ($z \mapsto u = f(z)$) are one of the following five types;*

group	$\sharp G$	$u = f(x)$	$\left\{\begin{matrix} \text{ramification indeces} \\ \text{branch points} \end{matrix}\right\}$.
I *cyclic* C_n	n	$u = z^n$	$\left\{\begin{matrix} n & n \\ 0 & \infty \end{matrix}\right\}$.
II *dihedral* D_ν	2ν	$u = \frac{(z^\nu+1)^2}{4z^\nu}$	$\left\{\begin{matrix} 2 & 2 & \nu \\ 0 & 1 & \infty \end{matrix}\right\}$.
III *tetrahedral* A_4	12	$u = \frac{(z^4-2\sqrt{3}iz^2+1)^3}{-12\sqrt{3}iz^2(z^4-1)^2}$	$\left\{\begin{matrix} 3 & 3 & 2 \\ 0 & 1 & \infty \end{matrix}\right\}$.
IV *octahedral* S_4	24	$u = \frac{(z^8+14z^4+1)^3}{108z^4(z^4-1)^4}$	$\left\{\begin{matrix} 3 & 2 & 4 \\ 0 & 1 & \infty \end{matrix}\right\}$.

V *isosahedral* A_5 60 $u = \frac{\{-(z^{20}+1)+228(z^{15}-z^5)-494z^{10}\}^3}{1728z^5(z^{10}+11z^5-1)^5}$ $\left\{ \begin{matrix} 3 & 2 & 5 \\ 0 & 1 & \infty \end{matrix} \right\}$.

, *where the symbol* $\left\{ \begin{matrix} n_1 & n_2 \cdots \\ \alpha_1 & \alpha_2 \cdots \end{matrix} \right\}$ *means that* π' *is ramified over* α_i *with a ramification index* n_i.

Now we determine all tamification types of normal coverings $\pi' : M' \to M$ for an arbitrary prime number p as Horiuchi did in case $p = 2$.

As notations we use $P, P', P'', \cdots$ for ramification points of ψ, and $Q_1, Q_2, \cdots, Q_P$ $(Q'_1, \cdots Q'_p; Q''_1, \cdots, Q''_p; \cdots)$ mean p distinct points with $\psi(Q_1) = \cdots \psi(Q_p)$ $(\psi(Q'_1) = \cdots \psi(Q'_p); \psi(Q''_1) = \cdots \psi(Q''_p); \cdots)$. The symbol $\left\{ \begin{matrix} m & \cdots \\ R & \cdots \end{matrix} \right\}$ means that π' is ramified over R with ramification index m.

Proposition 4.2. *All the ramification types of normal coverings* π' *with Galois group* $G \simeq C_n$ *are as follows;*

i) *If* $p \not| n$, *then*

$$\text{a)} \left\{ \begin{matrix} n & n \\ P & P' \end{matrix} \right\} \quad \text{b)} \left\{ \begin{matrix} n & n & \cdots & n \\ P & Q_1 & \cdots & Q_p \end{matrix} \right\} \quad \text{c)} \left\{ \begin{matrix} n & \cdots & n & n & \cdots & n \\ Q_1 & \cdots & Q_p & Q'_1 & \cdots & Q'_p \end{matrix} \right\}$$

ii) *If* $p|n$ *and* $p \neq n$, *then*

$$\text{a)} \left\{ \begin{matrix} n/p & n/p \\ P & P' \end{matrix} \right\} \quad \text{b)} \left\{ \begin{matrix} n/p & n & \cdots & n \\ P & Q_1 & \cdots & Q_p \end{matrix} \right\} \quad \text{c)} \left\{ \begin{matrix} n & \cdots & n & n & \cdots & n \\ Q_1 & \cdots & Q_p & Q'_1 & \cdots & Q'_p \end{matrix} \right\}$$

iii) *If* $p = n$, *then*

a) *unramified* $\text{b)} \left\{ \begin{matrix} n & \cdots & n \\ Q_1 & \cdots & Q_p \end{matrix} \right\}$ $\text{c)} \left\{ \begin{matrix} n & \cdots & n & n & \cdots & n \\ Q_1 & \cdots & Q_p & Q'_1 & \cdots & Q'_p \end{matrix} \right\}$.

Proof. We may assume that the ramification type of π is $\left\{ \begin{matrix} n & n \\ 0 & \infty \end{matrix} \right\}$. Let S be the set of branch points of $\psi : M \to \mathbf{P}_1$. When $S \cap \{0, \infty\} = \{0, \infty\}$, we have i, ii , iii-a) by Lemma 3.1. When $S \cap \{0, \infty\} = \{0\}$ or $\{\infty\}$, we have i, ii, iii-b). When $S \cap \{0, \infty\} = \emptyset$, we have i, ii, iii-c). □

Proposition 4.3. *All the ramification types of normal coverings* π' *with Galois group* $G \simeq D_\nu$ *are as follows;*

i) *If* $p \not| 2\nu$, *then*

a) $\left\{\begin{matrix} 2 & 2 & \nu \\ P & P' & P'' \end{matrix}\right\}$. b) $\left\{\begin{matrix} 2 & 2 & \nu & \cdots & \nu \\ P & P' & Q_1 & \cdots & Q_p \end{matrix}\right\}$. c) $\left\{\begin{matrix} 2 & \cdots & 2 & 2 & \nu \\ Q_1 & \cdots & Q_p & P & P' \end{matrix}\right\}$.

d) $\left\{\begin{matrix} 2 & 2 & \cdots & 2 & \nu & \cdots & \nu \\ P & Q_1 & \cdots & Q_p & Q_1' & \cdots & Q_p' \end{matrix}\right\}$. e) $\left\{\begin{matrix} 2 & \cdots & 2 & 2 & \cdots & 2 & \nu \\ Q_1 & \cdots & Q_p & Q_1' & \cdots & Q_p' & P \end{matrix}\right\}$.

f) $\left\{\begin{matrix} 2 & \cdots & 2 & 2 & \cdots & 2 & \nu & \cdots & \nu \\ Q_1 & \cdots & Q_p & Q_1' & \cdots & Q_p' & Q_1'' & \cdots & Q_p'' \end{matrix}\right\}$.

ii) *If $p|\nu$, $p \neq \nu$ and ν odd, then*

a) $\left\{\begin{matrix} 2 & 2 & \nu/p \\ P & P' & P'' \end{matrix}\right\}$. b) $\left\{\begin{matrix} 2 & 2 & \nu & \cdots & \nu \\ P & P' & Q_1 & \cdots & Q_p \end{matrix}\right\}$. c) $\left\{\begin{matrix} 2 & \cdots & 2 & 2 & \nu/p \\ Q_1 & \cdots & Q_p & P & P' \end{matrix}\right\}$.

d) $\left\{\begin{matrix} 2 & 2 & \cdots & 2 & \nu & \cdots & \nu \\ P & Q_1 & \cdots & Q_p & Q_1' & \cdots & Q_p' \end{matrix}\right\}$. e) $\left\{\begin{matrix} 2 & \cdots & 2 & 2 & \cdots & 2 & \nu/p \\ Q_1 & \cdots & Q_p & Q_1' & \cdots & Q_p' & P \end{matrix}\right\}$.

f) $\left\{\begin{matrix} 2 & \cdots & 2 & 2 & \cdots & 2 & \nu & \cdots & \nu \\ Q_1 & \cdots & Q_p & Q_1' & \cdots & Q_p' & Q_1'' & \cdots & Q_p'' \end{matrix}\right\}$.

iii) *If $p = \nu$ and ν is odd, then*

a) $\left\{\begin{matrix} 2 & 2 \\ P & P' \end{matrix}\right\}$. b) $\left\{\begin{matrix} 2 & 2 & \nu & \cdots & \nu \\ P & P' & Q_1 & \cdots & Q_p \end{matrix}\right\}$. c) $\left\{\begin{matrix} 2 & \cdots & 2 & 2 \\ Q_1 & \cdots & Q_p & P \end{matrix}\right\}$.

d) $\left\{\begin{matrix} 2 & 2 & \cdots & 2 & \nu & \cdots & \nu \\ P & Q_1 & \cdots & Q_p & Q_1' & \cdots & Q_p' \end{matrix}\right\}$. e) $\left\{\begin{matrix} 2 & \cdots & 2 & 2 & \cdots & 2 \\ Q_1 & \cdots & Q_p & Q_1' & \cdots & Q_p' \end{matrix}\right\}$.

f) $\left\{\begin{matrix} 2 & \cdots & 2 & 2 & \cdots & 2 & \nu & \cdots & \nu \\ Q_1 & \cdots & Q_p & Q_1' & \cdots & Q_p' & Q_1'' & \cdots & Q_p'' \end{matrix}\right\}$.

iv) *If $p = 2$ and ν is odd, then*

a) $\left\{\begin{matrix} \nu \\ P \end{matrix}\right\}$. b) $\left\{\begin{matrix} \nu & \nu \\ Q_1 & Q_2 \end{matrix}\right\}$. c) $\left\{\begin{matrix} 2 & 2 & \nu \\ Q_1 & Q_2 & P \end{matrix}\right\}$.

d) $\left\{\begin{matrix} 2 & 2 & \nu & \nu \\ Q_1 & Q_2 & Q_1' & Q_2 \end{matrix}\right\}$. e) $\left\{\begin{matrix} 2 & 2 & 2 & 2 & \nu \\ Q_1 & Q_2 & Q_1' & Q_2' & P \end{matrix}\right\}$.

f) $\left\{\begin{matrix} 2 & 2 & 2 & 2 & \nu & \nu \\ Q_1 & Q_2 & Q'_1 & Q'_2 & Q''_1 & Q''_p \end{matrix}\right\}$.

v) *If $p = 2$ and ν is even ≥ 4, then*

a) $\left\{\begin{matrix} \nu/2 \\ P \end{matrix}\right\}$. b) $\left\{\begin{matrix} \nu & \nu \\ Q_1 & Q_2 \end{matrix}\right\}$. c) $\left\{\begin{matrix} 2 & 2 & \nu/2 \\ Q_1 & Q_2 & P \end{matrix}\right\}$.

d) $\left\{\begin{matrix} 2 & 2 & \nu & \nu \\ Q_1 & Q_2 & Q'_1 & Q'_2 \end{matrix}\right\}$. e) $\left\{\begin{matrix} 2 & 2 & 2 & 2 & \nu/2 \\ Q_1 & Q_2 & Q'_1 & Q'_2 & P \end{matrix}\right\}$.

f) $\left\{\begin{matrix} 2 & 2 & 2 & 2 & \nu & \nu \\ Q_1 & Q_2 & Q'_1 & Q'_2 & Q''_1 & Q''_p \end{matrix}\right\}$.

vi) *If $p = \nu = 2$ (Theorem 2'[6],[11]), then*

a) *unramified* b) $\left\{\begin{matrix} 2 & 2 \\ Q_1 & Q_1 \end{matrix}\right\}$. c) $\left\{\begin{matrix} 2 & 2 & 2 & 2 \\ Q_1 & Q_2 & Q'_1 & Q'_2 \end{matrix}\right\}$.

d) $\left\{\begin{matrix} 2 & 2 & 2 & 2 & 2 & 2 \\ Q_1 & Q_2 & Q'_1 & Q'_2 & Q''_1 & Q''_2 \end{matrix}\right\}$.

Proof. The ramification type of π is $\left\{\begin{matrix} 2 & 2 & \nu \\ 0 & 1 & \infty \end{matrix}\right\}$. The cases i~v *a*), i~v *b*), i~v *c*), i~v *d*), i~v *e*) and i~v *f*) are coresponding to $S \cap \{0, 1, \infty\} =^{a)}$ $\{0, 1, \infty\}$, $^{b}\{0, 1\}$, $^{c)}\{0, \infty\}$ or $\{1, \infty\}$, $^{d)}\{0\}$ or $\{1\}$, $^{e)}\{\infty\}$ and $^{f)}\emptyset$ respectively. In case vi), a), b), c) and d) are corresponding to $S \cap \{0, 1, \infty\} =^{a)}$ $\{0, 1, \infty\}$, $^{b}\{0, 1\}$ or $\{0, \infty\}$ or $\{1, \infty\}$, $^{c)}\{0\}$ or $\{1\}$ or $\{\infty\}$ and $^{d)}\emptyset$ respectively. □

Proposition 4.4. *All the ramification types of normal coverings π' with Galois group $G \simeq A_4$ are as follows;*

i) *If $p \geq 5$, then*

a) $\left\{\begin{matrix} 3 & 3 & 2 \\ P & P' & P'' \end{matrix}\right\}$. b) $\left\{\begin{matrix} 3 & 3 & 2 & \cdots & 2 \\ P & P' & Q_1 & \cdots & Q_p \end{matrix}\right\}$. c) $\left\{\begin{matrix} 3 & \cdots & 3 & 3 & 2 \\ Q_1 & \cdots & Q_p & P & P' \end{matrix}\right\}$.

d) $\left\{\begin{matrix} 3 & 3 & \cdots & 3 & 2 & \cdots & 2 \\ P & Q_1 & \cdots & Q_p & Q'_1 & \cdots & Q'_p \end{matrix}\right\}$. e) $\left\{\begin{matrix} 3 & \cdots & 3 & 3 & \cdots & 3 & 2 \\ Q_1 & \cdots & Q_p & Q'_1 & \cdots & Q'_p & P \end{matrix}\right\}$.

f) $\left\{\begin{matrix} 3 & \cdots & 3 & 3 & \cdots & 3 & 2 & \cdots & 2 \\ Q_1 & \cdots & Q_p & Q'_1 & \cdots & Q'_p & Q''_1 & \cdots & Q''_p \end{matrix}\right\}$.

ii) *If $p = 3$, then*

a) $\begin{Bmatrix} 2 \\ P \end{Bmatrix}$. b) $\begin{Bmatrix} 2 & 2 & 2 \\ Q_1 & Q_2 & Q_3 \end{Bmatrix}$. c) $\begin{Bmatrix} 3 & 3 & 3 & 2 \\ Q_1 & Q_2 & Q_3 & P \end{Bmatrix}$.

d) $\begin{Bmatrix} 3 & 3 & 3 & 2 & 2 & 2 \\ Q_1 & Q_2 & Q_3 & Q'_1 & Q'_2 & Q'_3 \end{Bmatrix}$. e) $\begin{Bmatrix} 3 & 3 & 3 & 3 & 3 & 3 & 2 \\ Q_1 & Q_2 & Q_3 & Q'_1 & Q'_2 & Q'_3 & P \end{Bmatrix}$.

f) $\begin{Bmatrix} 3 & 3 & 3 & 3 & 3 & 3 & 2 & 2 & 2 \\ Q_1 & Q_2 & Q_3 & Q'_1 & Q'_2 & Q'_3 & Q''_1 & Q''_2 & Q''_3 \end{Bmatrix}$.

iii) *If $p = 2$, then*

a) $\begin{Bmatrix} 3 & 3 \\ P & P' \end{Bmatrix}$. b) $\begin{Bmatrix} 3 & 3 & 2 & 2 \\ P & P' & Q_1 & Q_2 \end{Bmatrix}$. c) $\begin{Bmatrix} 3 & 3 & 3 \\ P & Q_1 & Q_2 \end{Bmatrix}$.

d) $\begin{Bmatrix} 3 & 3 & 3 & 2 & 2 \\ P & Q_1 & Q_2 & Q'_1 & Q'_2 \end{Bmatrix}$.

e) $\begin{Bmatrix} 3 & 3 & 3 & 3 \\ Q_1 & Q_2 & Q'_1 & Q'_2 \end{Bmatrix}$.

f) $\begin{Bmatrix} 3 & 3 & 3 & 3 & 2 & 2 \\ Q_1 & Q_2 & Q'_1 & Q'_2 & Q''_1 & Q''_2 \end{Bmatrix}$.

Proof. The ramification type of π is $\begin{Bmatrix} 3 & 3 & 2 \\ 0 & 1 & \infty \end{Bmatrix}$. The cases i~iii *a*), i~iii *b*), i~iii *c*), i~iii *d*), i~iii *e*) and i~iii *f*) are coresponding to $S \cap \{0,1,\infty\} =^{a)} \{0,1,\infty\}$, $^{b}\{0,1\}$, $^{c)}\{0,\infty\}$ or $\{1,\infty\}$, $^{d}\{0\}$ or $\{1\}$, $^{e)}\{\infty\}$ and $^{f)}\emptyset$ respectively. □

Proposition 4.5. *All the ramification types of normal coverings π' with Galois group $G \simeq S_4$ are as follows;*

i) *If $p \geq 5$, then*

a) $\begin{Bmatrix} 3 & 2 & 4 \\ P & P' & P'' \end{Bmatrix}$. b) $\begin{Bmatrix} 3 & 2 & 4 & \cdots & 4 \\ P & P' & Q_1 & \cdots & Q_p \end{Bmatrix}$. c) $\begin{Bmatrix} 3 & 2 & \cdots & 2 & 4 \\ P & Q_1 & \cdots & Q_p & P' \end{Bmatrix}$.

d) $\begin{Bmatrix} 3 & \cdots & 3 & 2 & 4 \\ Q_1 & \cdots & Q_p & P & P' \end{Bmatrix}$. e) $\begin{Bmatrix} 3 & 2\cdots & 2 & 4 & \cdots & 4 \\ P & Q_1\cdots & Q_p & Q'_1 & \cdots & Q'_p \end{Bmatrix}$.

f) $\begin{Bmatrix} 3 & \cdots & 3 & 2 & 4 & \cdots & 4 \\ Q_1 & \cdots & Q_p & P & Q'_1 & \cdots & Q'_p \end{Bmatrix}$.

g) $\left\{\begin{array}{cccccccc} 3 & \cdots & 3 & 2 & \cdots & 2 & 4 \\ Q_1 & \cdots & Q_p & Q'_1 & \cdots & Q'_p & P \end{array}\right\}.$

h) $\left\{\begin{array}{ccccccccc} 3 & \cdots & 3 & 2 & \cdots & 2 & 4 & \cdots & 4 \\ Q_1 & \cdots & Q_p & Q'_1 & \cdots & Q'_p & Q''_1 & \cdots & Q''_p \end{array}\right\}.$

ii) *If $p = 3$, then*

a) $\left\{\begin{array}{cc} 2 & 4 \\ P & P' \end{array}\right\}.$ b) $\left\{\begin{array}{cccc} 2 & 4 & 4 & 4 \\ P & Q_1 & Q_2 & Q_3 \end{array}\right\}.$ c) $\left\{\begin{array}{cccc} 2 & 2 & 2 & 4 \\ Q_1 & Q_2 & Q_3 & P \end{array}\right\}.$

d) $\left\{\begin{array}{ccccc} 3 & 3 & 3 & 2 & 4 \\ Q_1 & Q_2 & Q_3 & P & P' \end{array}\right\}.$ e) $\left\{\begin{array}{cccccc} 2 & 2 & 2 & 4 & 4 & 4 \\ Q_1 & Q_2 & Q_3 & Q'_1 & Q'_2 & Q'_3 \end{array}\right\}.$

f) $\left\{\begin{array}{ccccccc} 3 & 3 & 3 & 2 & 4 & 4 & 4 \\ Q_1 & Q_2 & Q_3 & P & Q'_1 & Q'_2 & Q'_3 \end{array}\right\}.$

g) $\left\{\begin{array}{ccccccc} 3 & 3 & 3 & 2 & 2 & 2 & 4 \\ Q_1 & Q_2 & Q_3 & Q'_1 & Q'_2 & Q'_3 & P \end{array}\right\}.$

h) $\left\{\begin{array}{ccccccccc} 3 & 3 & 3 & 2 & 2 & 2 & 4 & 4 & 4 \\ Q_1 & Q_2 & Q_3 & Q'_1 & Q'_2 & Q'_3 & Q''_1 & Q''_2 & Q''_3 \end{array}\right\}.$

iii) *If $p = 2$, then*

a) $\left\{\begin{array}{cc} 3 & 2 \\ P & P' \end{array}\right\}.$ b) $\left\{\begin{array}{ccc} 3 & 4 & 4 \\ P & Q_1 & Q_2 \end{array}\right\}.$ c) $\left\{\begin{array}{cccc} 3 & 2 & 2 & 2 \\ P & Q_1 & Q_2 & P' \end{array}\right\}.$

d) $\left\{\begin{array}{ccc} 3 & 3 & 2 \\ Q_1 & Q_2 & P \end{array}\right\}.$

e) $\left\{\begin{array}{ccccc} 3 & 2 & 2 & 4 & 4 \\ P & Q_1 & Q_2 & Q'_1 & Q'_2 \end{array}\right\}.$

f) $\left\{\begin{array}{cccc} 3 & 3 & 4 & 4 \\ Q_1 & Q_2 & Q'_1 & Q'_2 \end{array}\right\}.$

g) $\left\{\begin{array}{ccccc} 3 & 3 & 2 & 2 & 2 \\ Q_1 & Q_2 & Q'_1 & Q'_2 & P \end{array}\right\}.$

h) $\left\{\begin{array}{cccccc} 3 & 3 & 2 & 2 & 4 & 4 \\ Q_1 & Q_2 & Q'_1 & Q'_2 & Q''_1 & Q''_2 \end{array}\right\}.$

Proof. The ramification type of π is $\left\{\begin{matrix} 3 & 2 & 4 \\ 0 & 1 & \infty \end{matrix}\right\}$. The cases i~iii $a)$, i~iii $b)$, i~iii $c)$, i~iii $d)$, i~iii $e)$, i~iii $f)$ and i~iii $g)$ and i~iii $h)$ are coresponding to $S \cap \{0,1,\infty\} =^{a)} \{0,1,\infty\}$, $^{b)}\{0,1\}$, $^{c)}\{0,\infty\}$, $^{d)}\{1,\infty\}$, $^{e)}\{0\}$, $^{f)}\{1\}$, $^{g)}\{\infty\}$ and $^{h)}\emptyset$ respectively. □

Proposition 4.6. *All the ramification types of normal coverings π' with Galois group $G \simeq A_5$ are as follows;*

i) *If $p \geq 7$, then*

a) $\left\{\begin{matrix} 3 & 2 & 5 \\ P & P' & P'' \end{matrix}\right\}$. b) $\left\{\begin{matrix} 3 & 2 & 5 & \cdots & 5 \\ P & P' & Q_1 & \cdots & Q_p \end{matrix}\right\}$. c) $\left\{\begin{matrix} 3 & 2 & \cdots & 2 & 5 \\ P & Q_1 & \cdots & Q_p & P' \end{matrix}\right\}$.

d) $\left\{\begin{matrix} 3 & \cdots & 3 & 2 & 5 \\ Q_1 & \cdots & Q_p & P & P' \end{matrix}\right\}$. e) $\left\{\begin{matrix} 3 & 2 & \cdots & 2 & 5 & \cdots & 5 \\ P & Q_1 & \cdots & Q_p & Q'_1 & \cdots & Q'_p \end{matrix}\right\}$.

f) $\left\{\begin{matrix} 3 & \cdots & 3 & 2 & 5 & \cdots & 5 \\ Q_1 & \cdots & Q_p & P & Q'_1 & \cdots & Q'_p \end{matrix}\right\}$.

g) $\left\{\begin{matrix} 3 & \cdots & 3 & 2 & \cdots & 2 & 5 \\ Q_1 & \cdots & Q_p & Q'_1 & \cdots & Q'_p & P \end{matrix}\right\}$.

h) $\left\{\begin{matrix} 3 & \cdots & 3 & 2 & \cdots & 2 & 5 & \cdots & 5 \\ Q_1 & \cdots & Q_p & Q'_1 & \cdots & Q'_p & Q''_1 & \cdots & Q''_p \end{matrix}\right\}$.

ii) *If $p = 5$, then*

a) $\left\{\begin{matrix} 3 & 2 \\ P & P' \end{matrix}\right\}$. b) $\left\{\begin{matrix} 3 & 2 & 5 & \cdots & 5 \\ P & P' & Q_1 & \cdots & Q_p \end{matrix}\right\}$. c) $\left\{\begin{matrix} 3 & 2 & \cdots & 2 \\ P & Q_1 & \cdots & Q_p \end{matrix}\right\}$.

d) $\left\{\begin{matrix} 3 & \cdots & 3 & 2 \\ Q_1 & \cdots & Q_p & P \end{matrix}\right\}$. e) $\left\{\begin{matrix} 3 & 2 & \cdots & 2 & 5 & \cdots & 5 \\ P & Q_1 & \cdots & Q_p & Q'_1 & \cdots & Q'_p \end{matrix}\right\}$.

f) $\left\{\begin{matrix} 3 & \cdots & 3 & 2 & 5 & \cdots & 5 \\ Q_1 & \cdots & Q_p & P & Q'_1 & \cdots & Q'_p \end{matrix}\right\}$.

g) $\left\{\begin{matrix} 3 & \cdots & 3 & 2 & \cdots & 2 \\ Q_1 & \cdots & Q_p & Q'_1 & \cdots & Q'_p \end{matrix}\right\}$.

h) $\left\{\begin{matrix} 3 & \cdots & 3 & 2 & \cdots & 2 & 5 & \cdots & 5 \\ Q_1 & \cdots & Q_p & Q_1' & \cdots & Q_p' & Q_1'' & \cdots & Q_p'' \end{matrix}\right\}$.

iii) *If* $p = 3$, *then*

a) $\left\{\begin{matrix} 2 & 5 \\ P & P' \end{matrix}\right\}$. b) $\left\{\begin{matrix} 2 & 5 & 5 & 5 \\ P & Q_1 & Q_2 & Q_3 \end{matrix}\right\}$. c) $\left\{\begin{matrix} 2 & 2 & 2 & 5 \\ Q_1 & Q_2 & Q_3 P & \end{matrix}\right\}$.

d) $\left\{\begin{matrix} 3 & 3 & 3 & 2 & 5 \\ Q_1 & Q_2 & Q_3 & P & P' \end{matrix}\right\}$.

e) $\left\{\begin{matrix} 2 & 2 & 2 & 5 & 5 & 5 \\ Q_1 & Q_2 & Q_3 & Q_1' & Q_2' & Q_3' \end{matrix}\right\}$.

f) $\left\{\begin{matrix} 3 & 3 & 3 & 2 & 5 & 5 & 5 \\ Q_1 & Q_2 & Q_3 & P & Q_1' & Q_2' & Q_3' \end{matrix}\right\}$.

g) $\left\{\begin{matrix} 3 & 3 & 3 & 2 & 2 & 2 & 5 \\ Q_1 & Q_2 & Q_3 & Q_1' & Q_2' & Q_3' & P \end{matrix}\right\}$.

h) $\left\{\begin{matrix} 3 & 3 & 3 & 2 & 2 & 2 & 5 & 5 & 5 \\ Q_1 & Q_2 & Q_3 & Q_1' & Q_2' & Q_3' & Q_1'' & Q_2'' & Q_3'' \end{matrix}\right\}$.

iv) *If* $p = 2$, *then*

a) $\left\{\begin{matrix} 3 & 5 \\ P & P' \end{matrix}\right\}$. b) $\left\{\begin{matrix} 3 & 5 & 5 \\ P & Q_1 & Q_2 \end{matrix}\right\}$. c) $\left\{\begin{matrix} 3 & 2 & 2 & 5 \\ P & Q_1 & Q_2 & P' \end{matrix}\right\}$.

d) $\left\{\begin{matrix} 3 & 3 & 5 \\ Q_1 & Q_2 & P \end{matrix}\right\}$.

e) $\left\{\begin{matrix} 2 & 3 & 3 & 5 & 5 \\ P & Q_1 & Q_2 & Q_1' & Q_2' \end{matrix}\right\}$.

f) $\left\{\begin{matrix} 3 & 3 & 5 & 5 \\ Q_1 & Q_2 & Q_1' & Q_2' \end{matrix}\right\}$.

g) $\left\{\begin{matrix} 3 & 3 & 2 & 2 & 5 \\ Q_1 & Q_2 & Q_1' & Q_2' & P \end{matrix}\right\}$.

h) $\left\{\begin{matrix} 3 & 3 & 2 & 2 & 5 & 5 \\ Q_1 & Q_2 & Q_1' & Q_2' & Q_1'' & Q_2'' \end{matrix}\right\}$.

Proof. The ramification type of π is $\left\{\begin{matrix} 3 & 2 & 5 \\ 0 & 1 & \infty \end{matrix}\right\}$. The cases i~iv

a), i∼iv b), i∼iv c), i∼iv d), i∼iv e), i∼iv f), i∼iv g) and i∼iv h) are coresponding to $S\cap\{0,1,\infty\} =^{a)} \{0,1,\infty\}$, $^{b)}\{0,1\}$, $^{c)}\{0,\infty\}$, $^{d)}\{1,\infty\}$, $^{e)}\{0\}$, $^{f)}\{1\}$, $^{g)}\{\infty\}$ and $^{h)}\emptyset$ respectively. □

Remark 4.7. There exists a unique covering π' that attains each type in proposition 4.2 ∼ 4.6, if we appoint branch points $P, P', \cdots; Q_1, Q_2, \cdots; Q_1', Q_2', \cdots$. By Lemma 4.1, §3.(4) and Proposition 4.2∼ 4.6 we have;

Theorem 4.8. *Let M be a cyclic p-gonal curve. All the unramified normal coverings $\pi' : M' \to M$ with a p-gonal curve M' are obtained by the following manners;*

i) *Let p be an arbitrary prime number. Take two ramification points P, P' of $\psi : M \to \mathbf{P}_1$. Let $\pi : \mathbf{P}_1' \to \mathbf{P}_1$ be a normal covering with Galois grooup C_p ramified over $\psi(P)$ and $\psi(P')$. Then π' as in Theorem 2.1 is unramified. Moreover if M and π are defined by*

$$y^p - u^{m_1}(u-a_2)^{m_2}\cdots(u-a_{r-1})^{m_{r-1}} = 0 \ (a_i \in \mathbf{C}-\{0\}, \sum_i \not\equiv 0 \ mod \ p)$$

and $\pi : z \mapsto z^p$, *then M' and π' are defined by*

$$y^p - (z^p - a_2)^{m_2}\cdots(z^p - a_{r-1})^{m_{r-1}} = 0 \quad and \quad \pi' : (z,y) \mapsto (z^p, z^{-m_1}y).$$

ii) $p = 2$([6],[11]). *Take three ramification points P, P', P'' of ψ and a normal covering π of degree 4 with Galois group D_2 ramified over $\psi(P), \psi(P'), \psi(P'')$. Then π' is unramified. Moreover if M and ψ are defined by*

$$y^2 - u(u-1)(u-a_3)\cdots(u-a_{r-1}) = 0, \ r-1 \not\equiv \ 0 \ mod \ 2, \ a_i \in \mathbf{C}-\{0\}$$

and $\pi : z \mapsto u = (z^2+1)^2/4z^2$, then M' and ψ' are defined by

$$y^2 - \{(z^2+1)^2 - 4a_3z^2\}\cdots\{(z^2+1)^2 - 4a_{r-1}z^2\} = 0$$

and

$$\pi' : (z,y) \mapsto (\frac{(z^2+1)^2}{4z^2}, \frac{(z^2+1)(z^2-1)}{(2z)^{r-1}}y).$$

5

Let M be a cyclic p-gonal curve with $m \geq 2p+1$ and $\pi' : M' \to M$ be as before, but we do not assume that π' is normal. We consider the condition that π' is unramified (if π' is normal, all unramified π' are obtained by Theorem 4.8). By Lemma 3.1 we have;

Lemma 5.1. *Let $\pi : \mathbf{P}_1' \to \mathbf{P}_1$ and $\psi : M \to \mathbf{P}_1$ be as in Theorem 2.1. Then the following are equivalent;*

i) *π' is unramified.*

ii) *Any branch points of π are also branch points of ψ and any ramification indeces of π are equal to p.*

Finally we give an example of an unramified covering π' that is not normal.

Example 5.2. Let $\pi : \mathbf{P}_1' \to \mathbf{P}_1$ be defined by

$$z \to \frac{(z-1)^2(z-k)^2}{z^2}, \quad \text{where } k \neq 0, \pm 1.$$

Then the ramification points ($\in \mathbf{P}_1'$) of π are $1, k, 0, \infty$ and $\pm\sqrt{k}$ with ramification index p. $\pi(1) = \pi(k) = 0, \pi(0) = \pi(\infty) = \infty, \pi(\sqrt{k}) = (1-\sqrt{k})^4$ and $\pi(-\sqrt{k}) = (1+\sqrt{k})^4$. Thus π is not normal. Let M be a hyperelliptic curve defined by

$$y^2 - u\{u-(1-\sqrt{k})^4\}\{u-(1+\sqrt{k})^4\}(u-a_5)\cdots(u-a_{2g+2}) = 0.$$

Then $\pi' : M' \to M$ as in Theorem 2.1 is unramified. Explicitely M' and π' are represented by

$$\begin{aligned} &y^2 - \{z^2-(2-2\sqrt{k}+2k)z+k\}\cdot\{z^2-(2+2\sqrt{k}+2k)z+k\} \\ &\cdot\{(z-1)^2(z-k)^2 - a_5z^2\}\cdots\{(z-1)^2(z-k)^2 - a_{2g+2}z^2\} = 0 \end{aligned}$$

and

$$\pi' : (z,y) \mapsto \left(\frac{(z-1)^2(z-k)^2}{z^2}, z^{-(2g+2)}(z+\sqrt{k})(z-\sqrt{k})y\right).$$

III Remarks on d-gonal Curves([9])

Let M be a compact Riemann surface and f be a meromorphic function on M. We denote the principal divisor associated to f by (f) and the polar divisor of f by $(f)_\infty$. If d is the degree of divisor $(f)_\infty$, then we call f a meromorphic function of degree d. If d is the minimal integer in which a non-trivial meromorphic function f of degree d exists on M, then we call M a d-gonal curve. In this case the complete linear system $|(f)_\infty|$ has projective dimension one. Moreover if f defines a cyclic covering $M \to \mathbf{P}_1$ over a Riemann sphere $\mathbf{P}_1$, then we call M a cyclic d-gonal curve.

Now we assume that M is a p-gonal curve of genus g with a prime number p. Then Namba has shown that M has a unique linear system g_p^1 of projective dimension one and degree p provided $g > (p-1)^2$ ([17]). For example if M is defined by an equation $y^p - (x-a_1)^{r_1} \cdots (x-a_s)^{r_s} = 0$ with $(p, r_i) = 1$, $\sum r_i \equiv 0$ $(mod\ p)$ and $s \geq 2p+1$, then M is p-gonal and having a unique g_p^1 ([16]).

In this chapter, we treat a compact Riemann surface M defined by an equation;

$$y^d - (x-a_1)^{r_1} \cdots (x-a_s)^{r_s} = 0 \qquad *)$$

with $\sum r_i \equiv 0 \ \ mod\, d \ \ and \ \ 1 \leq \ r_i < d,$
where d is not necessarily a prime number.

In § 2, we will show that M is d-gonal with the function x of degree d if there are enough r_i's relatively prime to p for each prime number p dividing d. In this case we call M a cyclic d-gonal curve. We will also show that M has a unique g_d^1 if there are more sufficient such r_i's are as above.

In § 3, let M be a cyclic d-gonal curve defined by $*)$ having unique g_d^1, and let M' be a compact Riemann surface defined by

$$y^d - (x-b_1)^{t_1} \dots (x-b_s)^{t_s} = 0.$$

We will study the relations among a_i, b_i, r_i and t_i $(1 \leq i \leq s)$ in the case M and M' are conformaly equivlent. Namba[16] and Kato[12] have already studied this problem in the case d is a prime number. We will give similar results for an arbitrary d.

In § 4, we cosider a covering map $\pi' : M' \to M$, where M is a cyclic d-gonal curve with a unique g_d^1 and M' is a d'-gonal curve. In the case $d = d'$, we can apply the same methods in Chapter II, and we will see that M' is also cyclic d-gonal. Moreover if π' is normal and $d = d'$, then the covering groop of π' is isomorphic to cyclic, dehederal, tetrahedral, octahedral or icosahedral. For a general case $d \leq d'$, we will show some relations between d and d'.

In § 5, we will give some remarks about coverings $M \to N$ with a clyclic d-gonal curve M having a unique g^1_d.

Finally we determine the equation $*$), which defines the curve M (with a unique g^1_d) having an automorphism $V(\notin < T >)$ of order N, where T is the automorphism defined by $T^*x = x$ and $T^*y = e^{2\pi i/d}y$ (§6).

1

At first we give several results on the existence of meromorphic functions on a compact Riemann surface M of genus g following Accola and Namba.

Lemma 1.1 *(Accola [1]) Let M be a compact Riemann surface of genus g. Let f_1 and f_2 be two meromorphic functions on M of degree n_1 and n_2 respectively. If f_1 and f_2 generate the full field $\mathbf{C}(M)$ of meromorphic functions on M, then $g \leq (n_1 - 1)(n_2 - 1)$.*

The following lemma by Namba is easily obtained from Lemma 1.1.

Lemma 1.2 *(Namba [17]) Let M be a compact Riemann surface of genus g and f be a meromorphic function of degree p on M with a prime number p.*

(1) If h is a meromorphic function of degree n on M satisfying

$$(p-1)((n-1) \leq g-1$$

, then p divides n and $h = r(f)$, where $r(x)$ is a rational function of degree n/p.

(2) If $(p-1)^2 \leq g-1$, then M is p-gonal and having a unique linear system g^1_p of degree p and dimension 1.

Proof. (1) By Lemma 1.1, the subfield $\mathbf{C}(f,h)$ of $\mathbf{C}(M)$ generated by f and h is not equal to $\mathbf{C}(M)$. As $p = [\mathbf{C}(M) : \mathbf{C}(f)]$ is a prime number, $\mathbf{C}(f) = \mathbf{C}(f,h)$.

(2) If h is any meromorphic function of degree p, then $\mathbf{C}(h) = \mathbf{C}(f)$ by (1). □.

Next we give some results concerning covering maps. Let $\pi : M' \to M$ be an arbitrary covering with compact Riemann surfaces M and M'. For a divisor $D = \sum n_i Q_i$ $(n_i \in \mathbf{Z}, Q_i \in M')$ we define a divisor $Nm_\pi D = Nm\ D$ by

$$\sum n_i \pi(Q_i).$$

On the other hand , for a meromorphic function f on M' we denote by $Nm[f]$ the meromorphic function on M obtained by the norm map $Nm : \mathbf{C}(M') \to \mathbf{C}(M)$. It is well known that the equation of principal divisors $Nm_\pi(f) = (Nm[f])$ holds([2]). When the divisor $Nm(f)$ is trivial, we can choose a consant c such that the divisor $Nm(f+c)$ is non trivial. This means that $d' \geq d$ if M' and M are d'-gonal and d-gonal respectively.

We already studied the case $d = d'$ in Chapter II.

2

Let M be a compact Riemann surface of genus g that has two meromorphic functions h and h' of degree d and d' respectively. Let $\mathbf{C}(h,h')$ be a subfield of $\mathbf{C}(M)$ generated by h and h', and $\tilde{M}$ be the compact Riemann surface of genus $\tilde{g}$ whose function field is isomorphic to $\mathbf{C}(h,h')$. Put $[\mathbf{C}(M), \mathbf{C}(h,h')] = t$. Then $\tilde{M}$ has meromorphic functions of degree d/t and d'/t induced by h and h' respectively. By Lemma 1.1 we have;

Lemma 2.1

$$\tilde{g} \leq (d/t - 1)(d'/t - 1)$$

From now on we assume;

M is defined by the equation *), T is the automorphism of M defined by

$$(x,y) \mapsto (x, \zeta_d y), \text{ where } \zeta_d = exp(2\pi i/d)$$

, and h is the canonical map $M \to M/<T>= \mathbf{P}_1$.

We denote by g_k the genus of the quotient compact Riemann surface $M/<T^k>$ for a positive integer k dividing d and $k \neq d$. Moreover if $k = q$ is a prime number, we denote by s_q the number of branch points of the canonical map $M/<T^q>\to M/<T>\simeq \mathbf{P}_1$. s_q is equal to the number of r_i's prime to q and we have $g_q = (q-1)(s_q-2)/2$ (for $\sum r_i \equiv 0 \bmod d$).

Lemma 2.2 *Assume that M has a meromorphic function h' of degree d'. Let q_0 be the smallest prime number dividing G.C.D.$(d,d') = (d,d')$. If d'*

satisfies the inequalities

$$g_q > (d/q_0 - 1)(d'/q_0 - 1) \ldots\ldots **)$$

for any prime q dividing G.C.D.(d, d'),
then $t = d$ or 1. Especially when $(r_i, d) = 1$ for all $1 \leq i \leq s, t = d$ or 1 provided $g_{q_0} > (d/q_0 - 1)(d'/q_0 - 1)$.

Proof. Assume $t \neq d, 1$. As $< T^{d/t} >$ is a unique subgroup of order t in $< T >$, $\tilde{M}$ should be isomorhic to $M/ < T^{d/t} >$ and $\tilde{g} = g_{d/t}$. For any prime number q dividing $d/t (\neq 1)$, we have $< T^q > \supset < T^{d/t} >$ and $\tilde{g} - 1 \geq g_q - 1 \geq (d/q_0 - 1)(d'/q_0 - 1) \geq (d/t - 1)(d'/t - 1)$. This contradicts to Lemma 2.1. If $(r_i, d) = 1$ for all $i = 1, ..., s$, then $s = s_q = s_{q_0}$ and $g_q \geq g_{q_0}$ for any prime number q dividing (d, d'). Thus the latter part of this lemma is reduced to the first part. □

Proposition 2.3 *Assume M is a compact Riemann surface of genus g defined by the equation $*$). Let d' be a positive integer satisfying the inequalities $**$) in Lemma 2.2 and$(d-1)(d'-1) \leq g - 1$. Then ;*

(1) If d does not divide d', then there is no meromorphic function of degree d'.

(2) If d divides d', then every meromorphic function h' of degree d' is obtained by $r(h)$, where r is some rational function of degree d'/d and h is the canonical map $M \to M/ < T >$.

Proof. Let h' be a meromorphic function of degree d'. $(d-1)(d'-1) \leq g-1$ means $t \neq 1$ by Lemma 1.1. Thus $\mathbf{C}(h, h') = \mathbf{C}(h)$ by Lemma 2.2 and $h' = r(h)$ for some rational function r □.

Remark If $d = p$ is a prime number, then this proposition is exactly same as Lemma 1.2(1).

Theorem 2.4 *Let M be a compact Riemann surface of genus g defined by $*$) and q_0 be the smallest prime number dividing d.*

(1) Assume $(d-1)(d-2) \leq g - 1$ and $(d/q_0 - 1)(d/q_0 - 2) \leq g_q - 1$ for any prime q dividing d. Then M is d-gonal.

(2) Assume $(d-1)^2 \leq g - 1$ and $(d/q_0 - 1)^2 \leq g_q - 1$ for any prime q dividing d. Then M is d-gonal and having a unique g^1_d.

Proof. (1) Assume that there is a meromorphic function h' of degree d' with $d' \leq d-1$. By $(d-1)(d-2) \leq g-1$ and Lemma 1.1, $t = [\mathbf{C}(M) : \mathbf{C}(h, h')] \neq 1$. As $t | (d, d')$ and $d' < d$, we have $d' \leq d - t$. Thus $d'/q_0 \leq d/q_0 - 1$ and

$(d/q_0-1)(d'/q_0-1) \leq (d/q_0-1)(d/q_0-2) \leq g_q - 1$ for any prime number q dividing d. Hence the assumptions in Proposition 2.3 are satisfied. This is a contradiction.

(2) Let h' be a meromorphic function of degree d. By the same way as in (1) and Proposition 2.3(2), we have $\mathbf{C}(h,h') = \mathbf{C}(h)$. Thus M has a unique g_d^1.

When $(r_i, d) = 1$ for all $i = 1, ..., s$, we can restate Theorem 2.4 as follows;

Theorem2.4′. (1) *If* $(d-1)(d-2) \leq g-1$ *and* $(d/q_0-1)(d/q_0-2) \leq g_{q_0}-1$, *then* M *is d-gonal.*

(2) *If* $(d-1)^2 \leq g-1$ *and* $(d/q_0-1)^2 \leq g_{q_0}-1$, *then* M *is d-gonal and having a unique* g_d^1.

Proof. Use the latter part of Lemma 2.2. □

Example 2.5. Let M be a compact Riemann surface defined by

$$y^4 - x(x-a_1)(x-a_2)(x-a_3)\{(x-a_4)(x-a_5)(x-a_6)(x-a_7)\}^2 = 0$$

,where a_i $(1 \leq i \leq 7)$ are distinct non-zero numbers, then $g = 7$. Put $N = M/ < T^2 >$. N is defined by

$$y^2 - x(x-a_1)(x-a_2)(x-a_3) = 0, \text{ i.e., } g_2 = 1$$

M satisfies the conditions of Theorem 2.4(1), and then M is 4-gonal. On the other hand M has infinitely many g_4^1. In fact if g_2^1 and $g_2^{1\prime}$ are two distinct linear systems on N, then $\pi^* g_2^1$ and $\pi^* g_2^{1\prime}$ are distinct linear systems of degree 4 and dimension 1 on M, where $\pi : M \to N$ is canonical map. Thus M has infinitely many g_4^1.

Example 2.6. For prime numbers p and q with $p \geq q$, let M be defined by

$$y^{pq} - (x-a_1)^{r_1}(x-a_2)^{r_2}\dots(x-a_s)^{r_s} = 0$$

with $\sum r_i \equiv 0 \mod pq$ and $(r_i, pq) = 1 (1 \leq i \leq s)$.
If s satisfies $s \geq 2pq-1$ and $(p-1)(p-2) < (q-1)(s-2)/2$, then M is pq-gonal. If s satisfies $s \geq 2pq+1$ and $(p-1)^2 < (q-1)(s-2)/2$, then M is pq-gonal and having a unique g_{pq}^1.

Proof. These results are easily from $g = (pq-1)(s-1)/2$, $g_p = (p-1)(s-2)/2$, $g_q = (q-1)(s-1)/2$ and Theorem 2.4′. □

Example 2.7. Let M be defined by

$$y^4 - x^2(x-a_1)(x-a_2)(x-a_3) = 0$$

,where a_1, a_2 and a_3 are distinct non-zero numbers. The covering map $x : M \to \mathbf{P_1}$ is completely ramified at A_1, A_2, A_3 and Q with $x(A_i) = a_i (i = 1, 2, 3)$ and $x(Q) = \infty$ respectively. Also x is ramified at two points P_1 and P_2 with ramification index 2 and $x(P_1) = x(P_2) = 0$. Thus $g = 4 (< (4-1)(4-2))$ and $g_2 = 1$. Then this M does not satisfy the conditions in Theorem 2.4(1). In fact M is trigonal with a principal divisor $(x/y) = P_1 + P_2 + Q - A_1 - A_2 - A_3$, and not a hyperelliptic curve by Lemma 1.2(1).

Remark M in Example 2.7 does not satisfy the condition of Lemma 1.2(2) for $p = 3$. But M has a unique g_3^1, because M has canonical divisor $(dx/y) = 2A_1 + 2A_2 + 2A_3$ and by [4](III.8.7).

3

In the following sections we give some applications of our results in § 2. At first we will prove the following Theorem, Which has been obtained by Namba [16] and improved by Kato [12] in the case $d = p$ a prime number.

Theorem 3.1 *Let M and M' be defined by the following equations;*

$$y^d - (x - a_1)^{r_1} \dots (x - a_s)^{r_s} = 0 \quad \dots\dots\dots i)$$

and

$$\tilde{y}^d - (\tilde{x} - b_1)^{t_1} \dots (\tilde{x} - b_s)^{t_s} = 0 \quad \dots\dots\dots ii)$$

respectively, where $1 \le r_i \le d-1$, $1 \le t_i \le d-1$ and $\sum r_i \equiv \sum t_i \equiv 0 \bmod d$. Assume M satisfies the conditions in Theorem 2.4(2), and M and M' are birationally equivalent. Then, by changing the indeces suitably, we have;

(1) there exists $A \in Aut(\mathbf{P}_1)$ satisfying $b_i = Aa_i (1 \le i \le s)$, and

$$\sharp) \begin{cases} ord_p\ t_i = ord_p r_i & if \quad ord_p\ r_i < ord_p d, \quad or \\ ord_p\ t_i \ge ord_d & if \quad ord_p\ r_i \ge ord_p d \quad (i \le i \le s) \end{cases}$$

for each prime number p dividing d.

(2) if $(r_1, d) = 1$, then $r_1/t_1 \in (\mathbf{Z}/d\mathbf{Z})^\times$ and $(r_1/t_1)t_i \equiv r_i \bmod d (1 \le i \le s)$.

(3) if d is square free, then $r_1 t_i \equiv t_1 r_i \bmod d (2 \le i \le s)$.

Proof. (1) The proof owes to the uniqueness of g_d^1(Theorem 2.4(2)), and goes almost same way as in the proof of Theorem 1.1 in [16]. Let $\varphi : M \to M'$ be the birational map. As M has a unique g_d^1, there exists $A \in Aut\mathbf{P}_1$

satisfying a commutative diagram;

$$\begin{array}{ccc} & \varphi & \\ M & \rightarrow & M' \\ x\downarrow & & \downarrow\tilde{x} \\ \mathbf{P}_1 & \rightarrow & \mathbf{P}_1. \\ & A & \end{array}$$

Thus we may assume $Aa_i = b_i$ for $i = 1, ..., s$. Let M'' be a curve defined by

$$z^d - (u - A^{-1}b_1)^{t_1} \ldots (u - A^{-1}b_s)^{t_s} = 0$$

,and $\psi_A = \psi$ be a birational map from M' to M'' defined by

$$(\tilde{x}, \tilde{y}) \mapsto (u, z) = (A^{-1}\tilde{x}, c\tilde{y}/(\tilde{x} - \gamma)^{k'})$$

,where c is a suitable constant, $\gamma = A(\infty)$ and $k' = (\sum t_\nu)/d$ ([17]). Put $w = z \cdot \psi \cdot \varphi$, which is a meromorphic function on M. Then M is also defined by

$$w^d - (x - a_1)^{t_1} \cdots (x - a_s)^{t_s} = 0 \cdots\cdots \text{i}').$$

As both i) and i') define the ramification type of the same cyclic covering $x : M \to \mathbf{P}_1$, we can see ♮) by considering a covering map $M/ < T^{p^{ord_p d}} > \to \mathbf{P}_1$ induced by x.

(2), (3) Put $v = w^{r_1}/y^{t_1}$, then we have;

$$v^d - (x - a_2)^{r_1t_2 - r_2t_1} \ldots (x - a_s)^{r_1t_s - r_st_1} = 0 \ldots\ldots iii).$$

Put $[\mathbf{C}(M) : \mathbf{C}(x, v)] = t$. As $\mathbf{C}(M) \supset \mathbf{C}(x, v) \supset \mathbf{C}(x)$ are cyclic extensions, $v^{d/t}$ is in $\mathbf{C}(x)$ and $r_1t_i - t_1r_i \equiv 0 \ mod \ t$ $(2 \le i \le s)$ by iii). Moreover we can see that s numbers $(r_1t_i - t_1r_i)/t$ $(2 \le i \le s)$ and d/t have no common divisor and $G.C.D.(r_1, t_1, d) = (r_1, t_1, d)$ divides t. On the other hand $\mathbf{C}(x, v)$ is the function field of the curve $M/ < T^{d/t} >$. Assume $d \ne t$, and take prime number q dividing d/t. Then the curve $M/ < T^q >$ is defined by the following two equations simultaneously;

$$y^q - (x - a_1)^{r_1} \ldots (x - a_s)^{r_s} = 0 \qquad \ldots\ldots\ldots \qquad A)$$

and

$$v^q - (x - a_2)^{(r_1t_2 - r_2t_1)/t} \ldots (x - a_s)^{(r_1t_s - r_st_1)/t} = 0 \quad \ldots \quad B).$$

Now we will show $r_1 \not\equiv 0 \ mod \ q$. In fact this is obvious when $(r_1, d) = 1$.

Next we consider the case d is square free. From ♮) we have $(r_1, t_1, d) = (r_1, d)$. As d is square free and $(r_1, t_1, d)|t$, $(d/t, r_1, d) = (d/t, r_1) = 1$ and

$(r_1, q) = 1$. Thus a_1 is a branch point of the covering $x : M/ < T^q > \rightarrow \mathbf{P}_1$ by A). But this contradicts to B). So we have $t = d$ and

$$r_1 t_i - t_1 r_i \equiv 0 \ \ mod\ d\ (2 \le i \le s).$$

When $(r_1, d) = 1$, then $(t_1, d) = 1$ by $\sharp$, and we get (2). □

Remark Conversely if there exists $A \in Aut(\mathbf{P}_1)$ as in (1) and we have $(r_1/t_1)t_i \equiv r_i \ \ mod\ d(2 \le i \le s)$, then M and M' are birationally equivalent([17]).

4

Next we consider a covering map $\pi' : M' \rightarrow M$ with a cyclic d-gonal curve M defined by $*$) of genus g and a d'-gonal curve M' of genus g'.

Theorem 4.1 *Assume $d = d'$. Then;*

(1) M' is also a cyclic d-gonal curve.

(2) If M satisfies the conditions of Theorem 2.4(2) and π' is normal, then the Galois group of π' is cyclic, dehedral, tetrahedral ,octahedral or isosahedral.

Proof. (1) Easily from II Theorem 2.1.

(2) Let T (resp. T') be the automorphism of order d on M (resp. M') as in § 2. By the commutative diagram in II Theorem 2.1 and the uniqueness of g^1_d on M, we may assume that T' induces T. For each prime number q dividing d, we have a commutative diagram;

$$\begin{array}{ccc} M' & \rightarrow & M'/ < T'^q > \\ \downarrow & & \downarrow \\ M' & \rightarrow & M'/ < T^q >. \end{array}$$

Let g'_q be genus of $M'/ < T'^q >$. As $g \le g'$ and $g_q \le g'_q$, M' is also satisfying the conditions in Theorem 2.4(2). Then M' has a unique g^1_d. By II Corollary 2.3, we have our results. □

Theorem 4.2 *Assume $d \le d'$. If d and d' satisfy the conditions of Proposition 2.3. on M, then d divides d'.*

Proof. Let D' be a positive divisor of degree d' on M' such that $|D'|$ has projective dimension 1. Assume $Nm_\pi D'$ has some common point with $Nm_\pi E$ for $E \in |D'|$. Then each $E \in |D'|$ has some common point with $\pi^* NmD'$. On the other hand if E and E' in $|D'|$ have common points, then $E = E'$ by the minimality of d'. Hence $|D'|$ should be a finite set. This is a contradiction. Thus there is a meromorphic function h of degree d' on M' and $Nm[h]$ is also of degree d' on M . By Proposition 2.3 we have $d|d'$. □

Corollary 4.3 *Let $\pi' : M' \to M$ be an unramified covering of degree q with a cyclic p-gonal curve M of genus g, where p and q are distinct prime numbers.*
Assume $g > p^2q - 2p + 1$. Then;

(a) M' is a pq-gonal curve with a unique g^1_{pq}.

(b) Let $\psi : M' \to \mathbf{P}'_1$ be the covering map defined by g^1_{pq} in (a), then;

(b-i) ψ is not cyclic(i.e., M' is not a cyclic pq-gonal curve).

(b-ii) if $p \not| q-1$, then ψ is not normal.

Proof. (a) Let $h : M \to \mathbf{P}_1$ be the covering map of degree p, then $h \circ \pi'$ is a meromorphic function of degree pq on M'. For $g > p^2q - 2p + 1 > (pq-1)(p-1)$, M' is pm-gonal$(1 \leq m \leq q-1)$ or pq-gonal by Theorem 4.2(see the remark of Proposition 2.3). Now we assume that M' is pq-gonal. Let ψ be a meromorphic function of degree pq on M'. Put $K = \mathbf{C}(\psi, h \circ \pi')$ and $[\mathbf{C}(M') : K] = t$. As the genus g' of M' is $q(g-1)+1$, we have $g' > (pq-1)^2$ and $t \neq 1$. Consider the following diagram;

$$\begin{array}{ccc} \mathbf{C}(M') \supset K & \supset & \mathbf{C}(\psi) \\ \cup \quad\quad \cup & & \\ \mathbf{C}(M) \supset \mathbf{C}(h \circ \pi'). & & \end{array}$$

If $t = q$, then $[K : \mathbf{C}(h \circ \pi')] = p$ and genus of $K = g$(for π' is unramified and $(p,q) = 1$). For $g > (p-1)^2, K = \mathbf{C}(h \circ \pi')$. This is a contradiction. If $t = p$, then $K \supset \mathbf{C}(h \circ \pi')$ is an unramified extension. As $\mathbf{C}(h \circ \pi')$ is of genus 0, this is a contradiction. Thus we have $t = pq$ and M' has a unique g^1_{pq}. If M' is pm-gonal $(1 \leq m \leq q-1)$ and ψ is a meromorphic function of degree pm on M', then $[\mathbf{C}(M') : \mathbf{C}(\psi, h \circ \pi')] = p$ by $(p,q) = 1$ and $g' > (pm-1)(pq-1)$. This is a contradiction.

(b-i) We may assume $h \circ \pi' = \psi$ by (a). If ψ is cyclic, then there exists an automorphism T' on M' of order p, and we have a commutative diagram;

$$\begin{array}{ccc} M' & \to & M'/<T'> \\ \pi' \downarrow & & \downarrow \pi \\ M & \to & M/<T> = \mathbf{P}_1 \\ & h & \end{array}$$

,where π' is unramified.
For $(p,q) = 1$, π is unramified. This is a contradiction. □

(b-ii) Assume ψ is normal with Galois group G. If $p < q$ and $p \not| q-1$, then it is well known that G is cyclic. But this can not happened by (a). If

$p > q$, then G has a unique normal subgroup $< T' >$ of index q generated by T'. Thus we have a same commutative diagram as in the proof of (b-i). This is also a contradiction. □

5

We consider a covering $\pi' : M \to N$, where M is cyclic d-gonal and N is e-gonal. Put $deg\ \pi' = n$ and $d' = ne$.

Theorem 5.1 *Assume d and d' satisfy the conditions of Proposition 2.3. Then e divides d. Moreover if $u : M \to M/ < T^e >$ is the canonical map, then there exists a covering map $v : M/ < T^e > \to N$ satisfying $\pi' = v \circ u$. Especially when $d = d' = ne$, N is isomorphic to $M/ < T^e >$.*

Proof. Let $\psi_N \to \tilde{\mathbf{P}}_1$ be the covering over Riemann sphere $\tilde{\mathbf{P}}_1$ of degree e. Then $\psi_N \circ \pi'$ is a meromorphic function on M of degree $d' = ne$. By Proposition 2.3, d divides $ne = d'$, and we have a commutative diagram;

$$\begin{array}{ccc} & h & \\ M & \to & \mathbf{P}_1 = M/ < T > \\ \pi' \downarrow & & \downarrow \tilde{\pi} \\ N & \to & \tilde{\mathbf{P}}_1 \\ & \psi_N & \end{array}$$

with a rational function $\tilde{\pi}$ of degree d'/d and the canonical map h. The function field $\mathbf{C}(N)$ and $\mathbf{C}(\mathbf{P}_1)$ are linearly independent over $\mathbf{C}(\tilde{\mathbf{P}}_1)$ for the minimality of e. Then there exists a e-gonal curve $\tilde{M}$ with a function field $\mathbf{C}(\tilde{M})$ isomorphic to $\mathbf{C}(\mathbf{P}_1) \bigotimes_{\mathbf{C}(\tilde{\mathbf{P}}_1)} \mathbf{C}(N)$. By the universal property of $\mathbf{C}(\tilde{M})$, we have the following commutative diagram;

$$\begin{array}{cccc} M & & h & \\ & \tilde{M} & \to & \mathbf{P}_1 = M/ < T > \\ & & \tilde{\psi} & \\ \pi' & \downarrow & & \downarrow \tilde{\pi} \\ & N & \to & \tilde{\mathbf{P}}_1 \\ & & \psi_N & \end{array}$$

,where $deg\tilde{\psi} = e$ and $deg\tilde{\pi} = ne/d$. We can see that e divides d. As h is a cyclic extension, $\tilde{M} \simeq M/ < T^e >$. □

Example 5.2. Let M be the cyclic pq-gonal curve defined in Example 2.6 with $p \geq q$, $s \geq 2pq + 1$ and $(p-1)^2 < (q-1)(s-2)/2$. Then any covering

$\pi : M \to N$ of degree p (resp.q) with a q (resp. p)-gonal curve N is birational to the cyclic q (resp.p)-gonal curve defined by $y^q - (x-a_1)^{r_1}\cdots(x-a_s)^{r_s} = 0$(resp. $y^p - (x-a_1)^{r_1}\cdots(x-a_s)^{r_s} = 0$).

6

Let M be a cyclic d-gonal curve with a unique g_d^1 defined by

$$y^d-(x-a_1)^{r_1}\cdots(x-a_s)^{r_s} = 0,\ \sum r_i \equiv 0\ mod\, d, (r_i, d) = 1(i = 1, ..., s)\ldots *)$$

, here we can take ∞ as on of a_i's.

Let T be the automorphism of order d as in § 2, and $\psi : M \to M/ < T >$ be the canonical map. We will determine the equation $*$), which defines M having an automorphism $V(\notin < T >)$ of order N.

For the uniqueness of g_d^1 we have $V < T > V^{-1} =< T >$ and V induces an automorphism $\tilde{V}$ on $M/ < T >= \mathbf{P}_1(x)$. Let $\mathbf{C}(x)$ and $\mathbf{C}(u)$ be the function fields of $M/ < T >$ and $M/ < T, V >$ respectively. Then π' : $M/ < T >\to M/ < T, V >$ is a cyclic covering of order N' (N' divides N) and we may assume $\pi'^* u = x^{N'}$.

Before considering generally, we study the following two cases;

Case 1) $< T > \cap < V >=< T >$, Case 2) $< T > \cap < V >= \{1\}$.

Case 1) $< T > \cap < V >=< T >$

We can see that $d|N$ and $N' = N/d$. We may assume $V^{N/d} = T$ and $\tilde{V}^* x = \zeta' x$ with a primitive N'-th root ζ' of 1. We denote the set {fixed point of $\tilde{V}$} by $F(\tilde{V})$. Then $\sharp F(\tilde{V}) = 2$.

Case 1-a) $\sharp\{F(\tilde{V}) \cap \{a_1, ..., a_s\}\} = 2$

We may assume that two elements of the above set are $a_{s-1} = 0$ and $a_s = \infty$. As $\tilde{V}$ acts on $\{a_1, ..., a_{s-2}\}$ faithfully, M can be defined by;

$$A) \qquad y^d = x\{\prod_{t=1}^{k}\prod_{j=1}^{N/d}(x - \zeta'^{j-1}c_t)^{m_{N/d\cdot(t-1)+j}}\}$$

$$1 + \sum_{t=1}^{k}\sum_{j=1}^{N/d} m_{N/d\cdot(t-1)+j} \not\equiv 0\ \ mod\ d$$

,where $(m_{N/d\cdot(t-1)+j}, d) = 1$ $(1 \le j \le N/d, 1 \le t \le k)$, and c_t $(\neq 0)$ are distinct complex numbers satisfying

$$\{\zeta'^{j-1}c_t | 1 \le j \le N/d\} \cap \{\zeta'^{j-1}c_s | 1 \le j \le N/d\} = \emptyset \ \text{ for }\ t \neq s.$$

By acting V^* on both sides of A), we have;

$$B) \quad (V^* y)^d = \zeta'^M \{\prod_{t=1}^{k}\prod_{j=1}^{N/d}(x - \zeta'^{j-2} c_t)^{m_{N/d\cdot(t-1)+j}}\} x$$

, where $M = 1 + \sum_{t=1}^{k}\sum_{j=1}^{N/d} m_{N/d\cdot(t-1)+j}$.
By the proof of Theorem 3.1 and comparing A) and B), there exists a positive integer v $(1 \le v < d,\ (v,d) = 1)$ satisfying

$$v \cdot m_{N/d\cdot(t-1)+j} \equiv m_{N/d\cdot(t-1)+j+1} \ mod\ d, (1 \le j \le N/d - 1)$$

,and

$$v \cdot m_{N/d\cdot t} \equiv m_{N/d\cdot(t-1)+1} \ mod\ d.$$

But in this case, $v \cdot 1 \equiv 1 \ \ mod\ d$. Thus we have $v = 1$ and $m_{N/d\cdot(t-1)+1} = \cdots = m_{N/d\cdot t}(t = 1, \cdots, k)$. Put $r_t = m_{N/d\cdot t}$. The equation A) is;

$$I) \qquad y^d = x\{\prod_{t=1}^{k}\prod_{j=1}^{N/d}(x - \zeta'^{j-1} c_t)^{r_t}\} = x \cdot \prod_{t=1}^{r_t}(x^{N/d} - b_t)^{r_t}.$$

As $V^* y^d = \zeta' y^d$ and V is of order N, we have $V^* y = \eta y$, where η satisfies $\eta^d = \zeta'$ and $\eta^{N'}$ is a primitive $N/N'(= d)$-th root of 1.

Proposition 6.1 a) *Case 1-a) happens if and only if M is defined by I) with $d|N$, $(r_t, d) = 1(t = 1, ..., k)$ and $N/d \cdot \sum_{t=1}^{k} r_t + 1 \not\equiv 0 \ \ mod\ d$. V is defined by*

$$V^* x = \zeta' x \quad and \quad V^* y = \eta y, \cdots\cdots\cdots 1)$$

, where ζ' is a primitive N'-th root of 1, η satisfies $\eta^d = \zeta'$ and $\eta^{N'}$ is a primitive d-th root of 1 (for example, $\eta = e^{2\pi i/N}$ and $\zeta' = e^{2\pi i/N'}$ satisfy these conditions).

Case 1-b) $\quad \sharp\{F(\tilde{V}) \cap \{a_1, ..., a_s\}\} = 1$

We may assume that $F(\tilde{V}) \cap \{a_1, ..., a_s\} = \{a_s\}$. There exists a point $P \in M$ such that $\psi(P) \notin \{a_1, ..., a_s\}$ and $V(P) \in < T > P = < V^{N/d} > P$. Then $V^d(P) = P$. If $(d, N/d) = r \neq 1$, then $T^{d/r} P = V^{N/d\cdot d/r} P = P$. This contradicts to $\psi(P) \notin \{a_1, ..., a_s\}$. Thus $(d, N/d) = 1$ and $< V^d > \cap < V^{N/d} > = \{1\}$. We have

$$\mathbf{C}(M) = \mathbf{C}(M/ < V^{N/d} >) \bigotimes_{\mathbf{C}(M/<V>)} \mathbf{C}(M/ < V^d >).$$

Assume $\psi(P) = \infty$, $a_s = 0$ and $\pi^* u = x^{N/d}$.
As $M/ < V^d > \to M/ < V > = \mathbf{P}_1(u)$ is cyclic of degree d, $\mathbf{C}(M/ < V^d >)$

is defined by $y^d = u\prod_{t=1}^{k}(u-b_t)^{n_t}$, with $(n_t, d) = 1$ $(t = 1,..,e)$ and $1+n_1+\cdots+n_k \not\equiv 0 \mod d$. Then M is defined by

$$y^d = x^{N/d}\cdot(x^{N/d}-b_1)^{n_1}\cdots(x^{N/d}-b_k)^{n_k}.$$

For $(d, N/d) = 1$, M can be defined by the following equation;

$$II) \qquad y^d = x(x^{N/d}-b_1)^{r_1}\cdots(x^{N/d}-b_k)^{r_k}$$

with $1+\sum r_t \not\equiv 0 \mod d$.
After all, we have;

Proposition 6.1 b) *Case 1-b) happens if and only if* $(N/d, d) = 1$ *and* M *is defined by II) with* $(r_t, d) = 1$ *and* $1+\sum_{t=1}^{e} r_t \not\equiv 0 \mod d$.
V is defined by;

$$V^*x = \zeta' x \quad \text{and} \quad V^*y = \eta y \cdots\cdots\cdots 2)$$

, *where ζ' is primitive N'-th root of 1, η satisfies $\eta^d = \zeta'$ and $\eta^{N'}$ is a primitive d-th root of 1.*

Case 1-c) $\sharp\{F(\tilde{V})\cap\{a_1, ..., a_s\}\} = \emptyset$
By the same way as in Case 1-b), we have;

Proposition 6.1 c) *Case 1-c) happened if and only if* $(N/d, d) = 1$ *and* M *is defined by;*

$$III) \qquad y^d = (x^{N/d}-b_1)^{r_1}\cdots(x^{N/d}-b_k)^{r_k}$$

with $(r_t, d) = 1$ *and* $\sum_{t=1}^{k} r_t \equiv 0 \mod d$.
V is defined by;

$$V^*x = \zeta' x \quad \textit{and} \quad V^*y = \zeta'' y \cdots\cdots\cdots 3)$$

, *where ζ' (resp.ζ'') is a primitive N' (resp. d)-th root of 1.*

Case 2) $<T>\cap<V>=\{1\}$
The automorphism $\tilde{V}$ on $M/<T>$ induced by V is of order N, and we may assume that $\tilde{V}^*x = \zeta x$ with a primitive N-th root *zeta* of 1.

Case 2-a) $\#\{\{a_1, ..., a_s\}\cap F(\tilde{V})\} = 2$
and
Case 2-b) $\#\{\{a_1, ..., a_s\}\cap F(\tilde{V})\} = 1$

By the same way as in Case 1-a), M can be defined by

$$IV) \qquad y^d = x\prod_{t=1}^{k}(x^N-b_t)^{r_t}, \quad \textit{with} \quad (r_i, N) = 1.$$

In Case 2-a)(resp.2-b), $N\sum_{t=1}^{k} r_t + 1 \not\equiv$(resp.$\equiv 0$) *mod d*. As V satisfies $V^* y^d = \zeta \cdot y^d$ and V is of order N, V is defined by;

$$V^* x = \zeta x \qquad and \qquad V^* y = \xi \cdot y \cdots\cdots\cdots 4)$$

, where ξ is a N-th root of 1 satisfying $\xi^d = \zeta$. Then $(N,d) = 1$ and ξ is also a primitive N-th root of 1. After all we have;

Proposition 6.2. *Case 2-a) (resp. 2-b)), happens if and only if $(N,d) = 1$ and M is birational to the curve defined by IV) with $(r_t, N) = 1$ and $N\sum_{t=1}^{k} r_t + 1 \not\equiv 0$(resp.$\equiv 0$) mod d.*
V is defined by 4) with a a primitive N-th root ξ of 1 and $\zeta = \xi^d$.

Case 2-c) $\#\{\{a_1, ..., a_s\} \cap F(\tilde{V})\} = \emptyset$
By the same way as in Case 1-a), M is birational to the curve defined by

$$y^d = \{\prod_{t=1}^{k}\prod_{j=1}^{N}(x - \zeta^{j-1}b_t)^{m_{N(t-1)+j}}\}$$

with $\sum_{t=1}^{k}\sum_{j=1}^{N} m_{N(t-1)+j} \equiv 0 \bmod d$ and $(m_{N(t-1)+j}, d) = 1$ $(j = 1, ..., N)$.
Moreover there exists a positive integer v $(1 \leq v \leq d-1, (v,d) = 1)$ satisfying $v\, m_{N(t-1)+j} \equiv m_{N(t-1)+j+1} \bmod d$ $(1 \leq j \leq N-1)$, and $v\, m_{Nt} \equiv m_{N(t-1)+1} \bmod d$. We see $v^N \equiv 1 \bmod d$. Thus M is defined by

$$V) \qquad y^d = \prod_{t=1}^{k}\prod_{j=1}^{N}(x - \zeta^{j-1}b_t)^{n_t v^{j-1}}$$

with positive integers n_t satisfying $\sum_{t=1}^{k}\sum_{j=1}^{N} n_t v^{j-1} \equiv 0 \bmod d$ and $(n_t, d) = 1 (t = 1, ..., k)$.
Put $R = \sum n_t$ and $S = \sum v^{j-1}$. Then $RS \equiv 0 \bmod d$. By acting V^* on the both sides of V again, we have

$$\begin{aligned}
(V^* y)^d &= \prod_{t=1}^{k}\prod_{j=1}^{N}(\zeta x - \zeta^{j-1}b_t)^{n_t v^{j-1}} \\
&= \zeta^{RS}\prod_{t=1}^{k}\prod_{j=1}^{N}(x - \zeta^{j-2}b_t)^{n_t v^{j-1}} \\
&= \begin{cases} \zeta^{RS} y^{vd} / \prod_{t=1}^{k}(x - \zeta^{N-1}b_t)^{n_t(v^N-1)},\ \zeta^{NR} \neq 1 & (\text{if } RS \not\equiv 0 \bmod N) \\ or & \\ y^{vd} / \prod_{t=1}^{k}(x - \zeta^{N-1}b_t)^{n_t(v^N-1)} & (\text{if } RS \equiv 0 \bmod N) \end{cases}
\end{aligned}$$

Then we have;

$$V^*y = \begin{cases} \eta\zeta^{RS/d}y^v/\prod_{t=1}^k(x-\zeta^{N-1}b_t)^{n_t(v^N-1)/d} & (\text{if } RS \not\equiv\ 0 \bmod N)\cdots V-i) \\ or & \\ \eta y^v/\prod_{t=1}^k(x-\zeta^{N-1}b_t)^{n_t(v^N-1)/d} & (\text{if } RS \equiv\ 0 \bmod N)\cdots V-ii) \end{cases}$$

, where η is some d-th root (not necessarily primitive) of 1.

Assume $RS \not\equiv 0 \mod N$. Using V-i) respectively, we have;

$$\begin{aligned} & V^{*N}y \\ = \ & \eta^S\zeta^{(RS/d)S}y^{v^N}/[\{\prod_{l=0}^{N-1}\prod_{t=1}^{k}(\zeta^l x-\zeta^{N-1}b_t)^{n_t}\}^{v^{N-1-l}}]^{(v^N-1)/d} \\ = \ & \eta^S\zeta^{(RS/d)S}y^{v^N}/\zeta^{R(v^{N-2}+2v^{N-3}+\cdots+(N-1)v^0)}[\{\prod_{l=0}^{N-1}\prod_{t=1}^{k}(x-\zeta^{N-l-1}b_t)^{n_t}\}^{v^{N-1-l}}]^{(v^N-1)/d} \\ = \ & \eta^S\zeta^{(RS/d)S-R(S^2-NS)/d}y^{v^N}/(y^d)^{(v^N-1)/d} \\ = \ & \eta^S\zeta^{RNS/d}y \\ = \ & \eta^S y \quad (\text{for } RS \equiv 0 \mod d). \end{aligned}$$

For $V^{*N}y = y$, $\eta^S = 1$ should be held.

When $RS \equiv 0 \mod N$, by the same way as above, we have;

$$\begin{aligned} V^{*N}y &= \eta^S\zeta^{-R(S^2-NS)/d}y^{v^N}/(y^d)^{(v^N-1)/d} \\ &= \eta^S\zeta^{-RS^2/d}y. \end{aligned}$$

Thus η should satisfy $\eta^S = \zeta^{RS^2/d}$.

Proposition 6.3. *Case 2-c) happens if and only if M is birationl to the curve defined by V) with $v^N \equiv 1 \bmod d$ and $RS \equiv 0 \bmod d$.*
If $RS \not\equiv 0$(resp.$RS \equiv 0$) mod N, V is defined by $V^*x = \zeta x$ and V-i) (resp. V-ii) with d-th root η of 1 satisfying $\eta^S = 1$ ($\eta^S = \zeta^{RS^2/d}$), here η is not necessarily primitive (for example, $\eta = 1$ (resp. $\eta^{RS/d}$) satisfies $\eta^S = 1$(resp. $\eta^S = \zeta^{RS^2/d}$).

General case $< T > \cap < V' > = < V^{N'} > = < T^{d'} >$.

We can obtain the equations of M and V as follows. We may assume that $N'|N$ and $d'|d$, then $d/d' = N/N'$.

The case $d' = 1$ is exactly same as the case 1)(Propositions 6-a)~ c)).

When $d' > 1$, put $M' = M/ < T > \cap < V >$. Then M' is d'-gonal with a unique $g^1_{d'}$ having an automorphism $V'(= V \mod < V^{d'} >)$ of order d'.

We can apply Proposition 6.2 or 6.3, and M' is defined by an equation of type IV) or V).

For example, assume M' is defined by;

$$y'^{d'} = \prod_{t=1}^{k'}\prod_{j=1}^{N'}(x - \zeta'^{j-1}b'_t)^{n'_t v'^{j-1}} \qquad (cf.V)$$

with $(n'_t, d') = (v', d') = 1 (1 \le t \le k')$, $1 \le v' \le d' - 1$, and $R'S' \equiv 0 \mod d'$, where $R' = \sum_{t=1}^{k'} n'_t$, $S' = \sum_{j=1}^{N'} v'^{j-1}$ and a primitive N'-th root ζ' of 1.

Moreover assume $R'S' \not\equiv 0 \mod N'$. Then V' is defined by;

$$\begin{cases} V'^{*}x = \zeta' x \\ V'^{*}y' = \eta' \zeta'^{R'S'/d'} y'^{v'} / \prod_{t=1}^{k'}(x - \zeta'^{N'-1}b'_t)^{n'_t(v'^{N'}-1)/d'} \end{cases} \qquad (cf.V-i)$$

with d'-th root η' (not necessarily primitive) of 1 satisfying $\eta'^{S'} = 1$. Put $y' = y^{d/d'}$, we can have the equation of M;

$$VI) \qquad y^d = \prod_{t=1}^{k'}\prod_{j=1}^{N'}(x - \zeta'^{j-1}b'_t)^{n'_t v'^{j-1}}$$

As M is defined by $*$, we have $R'S' \equiv 0 \mod d$, $(n'_t, d) = (v', d) = 1$ $(t = 1, \cdots, k')$ and $v'^N \equiv 1 \mod d$. Thus V on M is defined by;

$$\begin{cases} V^{*}x = \zeta' x \\ V^{*}y = \eta \zeta'^{R'S'/d} y^{v'} / \prod_{t=1}^{k'}(x - \zeta'^{N'-1}b'_t)^{n'_t(v'^{N'}-1)/d} \end{cases}$$

, where η satisfies $\eta^{d/d'} = \eta'$. We can see $V^{*N'}y = \eta^{S'}y$. As V is of order N, $\eta'^{S'}$ should be a primitive $N/N'(= d/d')$ root of 1. When $(S', d/d') = 1$, $\eta' = 1$ and $\eta = exp(2\pi i d/d')$ satisfies these conditions,

Considering the other cases, we finally have;

Theorem 6.4. *Let M be a cyclic d-gonal curve with a unique g_d^1 defined by $*$) with an autimorphism $V(\notin < T >)$ of order N. Then M and V are determined as the following types;*

I) *Let $d'(> 1)$ and $N'(> 1)$ be two integers satisfying $d'|d$, $N'|N$ and $d/d' = N/N' \neq 1$.*

I-i) *M is a curve defined by the equation*

$$VI) \qquad y^d = \prod_{t=1}^{k'}\prod_{j=1}^{N'}(x - \zeta'^{j-1}b_t)^{n'_t v'^{j-1}}$$

with $1 \le v' \le d' - 1, (n'_t, d) = (v', d) = 1 (t = 1, \cdots, k')$, and $R'S' \equiv 0 \mod d$.

If $S'R' \not\equiv 0$ mod N', then V is defined by

$$\begin{cases} V^*x = \zeta' x \\ V^*y = \eta\zeta'^{R'S'/d} y^{v'} / \prod_{t=1}^{k'} (x - \zeta'^{N'-1} b'_t)^{n'_t(v'^{N'}-1)/d} \end{cases}$$

, where η is a d-th root(not necessarily primitive) of 1 such that $\eta^{S'}$ is a primitive d/d'-th root of 1(for example, when $(S', d/d') = 1$, $e^{2\pi i d'/d}$ can be taken as η).

If $S'R' \equiv 0 \mathrm{mod} N'$, V is defined by

$$\begin{cases} V^*x = \zeta' x \\ V^*y = \eta y^{v'} / \prod_{t=1}^{k'} (x - \zeta'^{N'-1} b'_t)^{n'_t(v'^{N'}-1)/d} \end{cases}$$

,where η is a d-th root (not necessarily primitive) of 1 such that $\eta\zeta'^{-R'S'^2/d}$ is a primitive d/d'-root of 1(for example, when $(S', d/d') = 1$, we can take $\zeta'^{R'S'/d}\zeta_{d/d'}$ as η, where $\zeta_{d/d'}$ is a primitive d/d'-th root of 1) (cf. Proposition 6.3).

I-ii) *If $(d', N') = 1$, then we have an additional type;*

$$y^d = x \prod_{t=1}^{k} (x^{N'} - b_t)^{r_t}$$

with $(r_t, N) = 1$. In this case V is defined by;

$$V^*y = \xi y \qquad \text{and} \qquad V^*x = \xi^d x$$

, where ξ is a primitive N-th root of 1(cf. Proposition 6.2).

II) *In case of $d|N$, in addition to 1), we have other types of M and V as follows;*

II-i) *M and V in Proposition 6.1. a).*

II-ii) *In addition to II-i), M and V in Proposition 6.1 b) and 6.1 c) provided $(d, N/d) = 1$.*

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