Coherent Singular Complexes in Strong Shape Theory

by

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1. Introduction.

In [2] Borsuk introduced the concept of shape theory for compacta, and many authors investigated and extended the theory to more general spaces. Afterwards several authors introduced a stronger concept of the theory, which is called strong (or fine) shape theory. The origin may be found in Christie [7] or Quigley [31]. Various approaches were given by Edwards and Hastings [11], Bauer [1], Lisica [15], Kodama and Ono [13], Dydak and Segal [8], Calder and Hastings [4], Cathey and Segal [5], and Lisica [16]. In particular, [11], [1], [5] and [16] considered one for arbitrary spaces. Note that those approaches are equivalent for compacta.

Current interest stems from recent development in coherent prohomotopy theory, sometimes called Steenrod homotopy theory. The relation which strong shape theory bears to Steenrod homotopy theory is entirely analogous to that which shape theory bears to

Čech homotopy theory. Lisica and Mardešić [19] developed the coherent prohomotopy category CPHTOP, and described the strong shape category SSH of arbitrary spaces by using the category and ANR-resolutions defined by Mardešić [24] (see [18] for the summary).

In this paper we investigate the coherent prohomotopy category and construct the *coherent singular complex functor* S_c: CPHTOP $\longrightarrow$ KAN. For inverse systems $\underline{X}$ of spaces we define the canonical coherent maps $\tau_{\underline{X}}\colon |S_c(\underline{X})| \longrightarrow \underline{X}$ which have the property;

If $\underline{X}$ *is dominated by a CW-complex in* CPHTOP, *then* $\tau_{\underline{X}}$ *induces an isomorphism in* CPHTOP.

Hence we have the stability theorem in coherent prohomotopy theory. Corresponding a space X to the rudimentary system $\underline{X} = (X)$, the homotopy category can be considered as a full subcategory of CPHTOP. Then for each space X, we have the natural isomorphism $\phi_X\colon S_c(\underline{X}) \longrightarrow S(X)$ and the commutative triangle;

$$\begin{tikzcd}
{|S_c(\underline{X})|} \arrow[rr, "|\phi_X|"] \arrow[dr, "\tau_{\underline{X}}"'] & & {|S(X)|} \arrow[dl, "\omega_X"] \\
& \underline{X} = X &
\end{tikzcd},$$

where S is the usual singular complex functor and the canonical map $\omega_X\colon |S(X)| \longrightarrow X$ (see [27] and [28]).

The idea to consider $S_c(\underline{X})$ goes back to Bauer [1]. However, he used a less satisfactory coherent procategory.

Next, for inverse systems $(\underline{X},\underline{x})$ of pointed spaces we define the i-th <u>coherent</u> <u>homotopy</u> <u>groups</u> $\pi_i^c(\underline{X},\underline{x})$, which is an invariant in $CPHTOP_0$. Then the canonical coherent map $\tau_{(\underline{X},\underline{x})}$: $|S_c(\underline{X},\underline{x})| \longrightarrow (\underline{X},\underline{x})$ induces a weak equivalence. That is;

$$\tau_{(\underline{X},\underline{x})\#}: \pi_i(|S_c(\underline{X},\underline{x})|) \cong \pi_i^c(\underline{X},\underline{x}) \quad \text{for all } i \geq 0.$$

As another algebraic invariant in CPHTOP, we introduce the <u>coherent</u> <u>singular</u> <u>homology</u> <u>theory</u> H_*^c by using the functor S_c, and show that

$$\tau_{\underline{X}*}: H_i(|S_c(\underline{X})|:G) \cong H_i^c(\underline{X}:G) \quad \text{for all } i \geq 0,$$

where H_* is the usual singular homology theory. Hence we have the Hurewicz isomorphism theorem between coherent homotopy groups and coherent singular homology groups.

Moreover we show that coherent singular homology theory is different from Steenrod-Sitnikov's one even on inverse sequences of compact polyhedra. By the proof we can see that Bauer's assertion, [1], Theorem 7.7, is not valid, and that the S-C homology theory defined by Ono [30] is also different from the Steenrod-Sitnikov's one. The general description of Steenrod-Sitnikov homology theory was given by Lisica and Mardešić [17] (see [22] and [23] for more details).

Let $(\underline{X,x})$ be an inverse sequence of arcwise connected spaces. Modifying Edwards and Geoghegan's way [9], we introduce another construction of the pointed CW-complex $E(\underline{X,x})$ and the coherent map $\rho_{(\underline{X,x})}: E(\underline{X,x}) \longrightarrow (\underline{X,x})$ which also have the property;

$$\rho_{(\underline{X,x})\#}: \pi_i(E(\underline{X,x})) \cong \pi_i^c(\underline{X,x}) \quad \text{for all } i \geq 0.$$

Then we have the map $\psi_{(\underline{X,x})}: E(\underline{X,x}) \longrightarrow |S_c(\underline{X,x})|$ and the following triangle which is commutative up to coherent homotopy

$$\begin{tikzcd} E(\underline{X,x}) \arrow[rr, "\psi_{(\underline{X,x})}"] \arrow[dr, "\rho_{(\underline{X,x})}"'] & & {|S_c(\underline{X,x})|} \arrow[dl, "\tau_{(\underline{X,x})}"] \\ & (\underline{X,x}) & \end{tikzcd}$$

Hence $\psi_{(\underline{X,x})}$ is a weak equivalence.

Moreover our results in CPHTOP are summarized in strong shape theory.

In this paper we will assume that readers are familiar with shape theory and prohomotopy theory. [26] is a good reference for those theories. Throughout this paper spaces are topological spaces, and maps are continuous functions. ANR means an absolute neighborhood retract for metrizable spaces.

I wish to express my sincere appreciation and gratitude to Professor Y. Kodama.

Notations: For each $n \geq 0$, let Δ^n be the standard n-simplex, i.e.,

$$\Delta^n = \{(t_0,\ldots,t_n) \in R^{n+1} \mid t_i \geq 0 \text{ for every } i, \sum_{i=1}^{n} t_i = 1\}.$$

For each i, $0 \leq i \leq n$, let e_i be the i-th vertex of Δ^n.

If $n > 0$ and $0 \leq j \leq n$, the j-th face operator ∂_j^n: $\Delta^{n-1} \longrightarrow \Delta^n$ is defined by

$$\partial_j^n(t_0,\ldots,t_{n-1}) = (t_0,\ldots,t_{j-1},0,t_j,\ldots,t_{n-1}).$$

If $n \geq 0$ and $0 \leq j \leq n$, the j-th degeneracy operator σ_j^n: $\Delta^{n+1} \longrightarrow \Delta^n$ is defined by

$$\sigma_j^n(t_0,\ldots,t_{n+1}) = (t_0,\ldots,t_{j-1},t_j+t_{j+1},t_{j+2},\ldots,t_{n+1}).$$

I = the unit interval $[0,1]$.

2. Coherent prohomotopy.

Throughout this paper we consider only inverse systems of aces and maps $\underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ over directed cofinite sets.

In this section we shall introduce the coherent prohomotopy tegory defined by Lisica and Mardešić [19]. By a <u>system map</u> $\underline{X} \longrightarrow \underline{Y} = (Y_\mu, q_{\mu\mu'}, M)$ we mean an increasing function $M \longrightarrow \Lambda$ and a collection of maps $f_\mu : X_{\phi(\mu)} \longrightarrow Y_\mu$, $\in M$, satisfying

(1) $f_\mu p_{\phi(\mu)\phi(\mu')} = q_{\mu\mu'} f_{\mu'}$, for $\mu \leq \mu'$ in M.

A <u>coherent map</u> $f: \underline{X} \longrightarrow \underline{Y}$ is defined as follows; onsists of an increasing function $\phi: M \longrightarrow \Lambda$ and of maps $: \Delta^n \times X_{\phi(\mu_n)} \longrightarrow Y_{\mu_0}$, $\underline{\mu} = (\mu_0, \ldots, \mu_n) \in M^n$, $n \geq 0$, which tisfy

(2) $$f_{\underline{\mu}}(\partial_j^n(t), x) = \begin{cases} q_{\mu_0\mu_1} f_{\underline{\mu}_0}(t,x), & \text{if } j = 0, \\ f_{\underline{\mu}_j}(t,x), & \text{if } 0 < j < n, \\ f_{\underline{\mu}_n}(t, p_{\phi(\mu_{n-1})\phi(\mu_n)}(x)), & \text{if } j = n, \end{cases}$$

where $x \in X_{\phi(\mu_n)}$, $t \in \Delta^{n-1}$, $n > 0$,

(3) $f_{\underline{\mu}}(\sigma_j^n(t), x) = f_{\underline{\mu}^j}(t,x)$, for $0 \leq j \leq n$,

where $x \in X_{\phi(\mu_n)}$, $t \in \Delta^{n+1}$, $n \geq 0$, here M^n, $n \geq 0$ denotes the set of all increasing sequences $\underline{\mu} = (\mu_0, \ldots, \mu_n)$ in M, and $\underline{\mu}_j = (\mu_0, \ldots, \mu_{j-1}, \mu_{j+1}, \ldots, \mu_n)$ and $\underline{\mu}^j = (\mu_0, \ldots, \mu_j, \mu_j, \ldots, \mu_n)$ for $\underline{\mu} = (\mu_0, \ldots, \mu_n) \in M^n$ and $0 \leq j \leq n$. Every system map $\underline{f}$: $\underline{X} \longrightarrow \underline{Y}$ can be viewed as a coherent map from $\underline{X}$ to $\underline{Y}$ by putting $f_{\underline{\mu}}(t,x) = f_{\mu_0} p_{\phi(\mu_0)\phi(\mu_n)}(x)$ for $\underline{\mu} = (\mu_0, \ldots, \mu_n)$ and $(t,x) \in \Delta^n \times X_{\phi(\mu_n)}$.

A <u>coherent</u> <u>homotopy</u> <u>from</u> f <u>to</u> f' is a coherent map F: $\underline{X} \times I = (X_\lambda \times I, p_{\lambda\lambda'} \times 1, \Lambda) \longrightarrow \underline{Y}$, given by $\Phi \geq \phi, \phi'$, and $F_{\underline{\mu}}$ such that

(4) $F_{\underline{\mu}}(t,x,0) = f_{\underline{\mu}}(t, p_{\phi(\mu_n)\Phi(\mu_n)}(x))$,

$F_{\underline{\mu}}(t,x,1) = f'_{\underline{\mu}}(t, p_{\phi'(\mu_n)\Phi(\mu_n)}(x))$,

where $x \in X_{\Phi(\mu_n)}$, $t \in \Delta^n$, $n \geq 0$,

which is written by F: $f \simeq f'$.

Next we define the <u>composition</u> gf of f and g: $\underline{Y} \longrightarrow \underline{Z} = (Z_\nu, r_{\nu\nu'}, N)$. In the case f is a system map $\underline{f}$: $\underline{X} \longrightarrow \underline{Y}$,

(5) $(gf)_{\underline{\nu}}(t,x) = g_{\underline{\nu}}(t, f_{\psi(\nu_n)}(x))$, where

$\underline{\nu} = (\nu_0, \ldots, \nu_n) \in N^n$, $n \geq 0$, $x \in X_{\phi\psi(\nu_n)}$ and $t \in \Delta^n$.

Hence if $\underline{X}$ and $\underline{Y}$ are rudimentary systems (X) and (Y), respectively, and f is a map from X to Y, then $(gf)_{\underline{\nu}}(t,x) = g_{\underline{\nu}}(t, f(x))$, for $x \in X$, $t \in \Delta^n$, $\underline{\nu} \in N^n$, $n \geq 0$.

To define composition in the other case, one decomposes Δ^n into subpolyhedra

$P_i^n = \{(t_0,\ldots,t_n) \in \Delta^n \mid t_0+\ldots+t_{i-1} \leq \frac{1}{2} \leq t_0+\ldots+t_i\}$,

$0 \leq i \leq n$, and considers maps $\alpha_i^n\colon P_i^n \longrightarrow \Delta^{n-i}$,

$\beta_i^n\colon P_i^n \longrightarrow \Delta^i$, where $\alpha_i^n(t) = (\#, 2t_{i+1},\ldots,2t_n)$,

$\beta_i^n(t) = (2t_0,\ldots,2t_{i-1},\#)$, $\# = 1$ - sum of remaining terms.

Then

$$(6) \quad (gf)_{\underline{\nu}}(t,x) = g_{(\nu_0,\ldots,\nu_i)}(\beta_i^n(t), f_{(\psi(\nu_i),\ldots,\psi(\nu_n))}(\alpha_i^n(t),x))$$

where $\underline{\nu} = (\nu_0,\ldots,\nu_n) \in N^n$, $n \geq 0$, $x \in X_{\phi\psi(\nu_n)}$, $t \in P_i^n$, $0 \leq i \leq n$.

We define the <u>coherent identity map</u> $1_{\underline{X}}\colon \underline{X} \longrightarrow \underline{X}$ by putting for any $\underline{\lambda} = (\lambda_0,\ldots,\lambda_n) \in \Lambda^n$, $n \geq 0$,

$$(7) \quad \phi(\lambda_n) = \lambda_n ,$$

$$(8) \quad 1_{\underline{\lambda}}(t,x) = p_{\lambda_0\lambda_n}(x), \text{ where } x \in X_{\lambda_n} \text{ and } t \in \Delta^n.$$

In [19] Lisica and Mardešić showed that inverse systems of spaces and maps over directed cofinite sets and coherent homotopy classes of coherent maps construct a category. They call this category the <u>coherent prohomotopy category</u> and denote it by CPHTOP. We note that our definition of composition of coherent maps is slightly different from the original one in

[19], but by the proof of [19], Lemma I.9.7, the coherent homotopy class of our composition coincides with the one of the original composition. Hence we have the category CPHTOP.

Similarly, considering inverse systems of pointed spaces, pairs of spaces or pairs of pointed spaces and suitable maps, we have the suitable coherent prohomotopy categories. We denote those categories by $CPHTOP_0$, $CPHTOP_2$ and $CPHTOP_{2,0}$, respectively (c.f. [21]).

3. Coherent singular complexes.

Let $(\underline{X,X_0}) = ((X_\lambda, X_{0\lambda}), p_{\lambda\lambda'}, \Lambda)$ be an object of $CPHTOP_2$. Put $\underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ and $\underline{X}_0 = (X_{0\lambda}, p_{\lambda\lambda'} \mid X_{0\lambda'}, \Lambda)$. For each $i \geq 0$ let $S_i(\underline{X})$ be the set of all coherent maps from Δ^i to $\underline{X}$. For each $0 \leq k \leq i$, $i \geq 0$, we define the functions $d_k = d_k^i\colon S_i(\underline{X}) \longrightarrow S_{i-1}(\underline{X})$ and $s_k = s_k^i\colon S_i(\underline{X}) \longrightarrow S_{i+1}(\underline{X})$ by formulas;

(1) $d_k(h) = h\partial_k^i$, and $s_k(h) = h\sigma_k^i$ for $h \in S_i(\underline{X})$.

Then the triple $(S_i(\underline{X}), d_k, s_k)$ is a semi-simplicial complex, which is called the <u>coherent singular complex</u> of $\underline{X}$, and is denoted by $S_c(\underline{X})$. Similarly we have the coherent singular complex of $\underline{X}_0$. Then it is clear that $S_c(\underline{X}_0)$ is a subcomplex of $S_c(\underline{X})$. We denote the complex pair $(S_c(\underline{X}), S_c(\underline{X}_0))$ by $S_c(\underline{X,X_0})$. Then we have the following elementary facts.

3.1. Proposition. (1) <u>The complex pair</u> $S_c(\underline{X,X_0})$ <u>is a Kan complex</u>.

(2) <u>If</u> $(\underline{X,X_0})$ <u>is the rudimentary system</u> $((X,X_0))$, <u>then</u> $S_c(\underline{X,X_0})$ <u>is naturally isomorphic to the usual singular complex pair</u> $(S(X), S(X_0))$.

Proof. For convenience, we consider only the absolute case. (1) Let $f^0, f^1, \ldots, f^{j-1}, f^{j+1}, \ldots, f^{i+1}$ be i-simplexes of $S_c(\underline{X})$ such that $d_k f^\ell = d_{\ell-1} f^k$, $k < \ell$, $k \neq j$, $\ell \neq j$. Namely, by (1) and the definition of compositions, if $k < \ell$, $k \neq j$, and $\ell \neq j$,

$$f^{\ell}_{\underline{\lambda}}(1_{\Delta^n} \times \partial^i_k) = f^k_{\underline{\lambda}}(1_{\Delta^n} \times \partial^i_{\ell-1}) \quad \text{for every } \underline{\lambda} \in \Lambda^n.\ n \geq 0.$$

Hence, for each $\underline{\lambda} = (\lambda_0,\ldots,\lambda_n) \in \Lambda^n$, $n \geq 0$, we can define the map $\tilde{f}_{\underline{\lambda}}\colon \Delta^n \times \bigcup_{k\neq j} \partial^{i+1}_k(\Delta^i) \longrightarrow X_{\lambda_0}$ by

$$\tilde{f}_{\underline{\lambda}}(t, \partial^{i+1}_k(z)) = f^k_{\underline{\lambda}}(t,z) \quad \text{for } z \in \Delta^i \text{ and } t \in \Delta^n.$$

Then the collection of the maps $\tilde{f}_{\underline{\lambda}}$ induces a coherent map $\tilde{f}$: $\bigcup_{k\neq j} \partial^{i+1}_k(\Delta^i) \longrightarrow \underline{X}$. Therefore for a fixed retraction r: $\Delta^{i+1} \longrightarrow \bigcup_{k\neq j} \partial^{i+1}_k(\Delta^i)$, defining the coherent map $f\colon \Delta^{i+1} \longrightarrow \underline{X}$ by $f = \tilde{f}r$, we have an $(i+1)$-simplex f of $S_c(\underline{X})$ such that $d_k f = f_k$ for all $k \neq j$. That is, $S_c(\underline{X})$ is a Kan complex.

(2) Suppose that the rudimentary system (X) has the trivial index set $\Lambda_0 = \{\lambda_0\}$. Then for each $n \geq 1$, the set Λ^n_0 consists of only one degenerate element $(\lambda_0,\lambda_0,\ldots,\lambda_0)$. Hence for every i-simplex f of $S_c(\underline{X})$,

$$f_{\underline{\lambda}}(t,z) = f_{(\lambda_0)}(e_0,z) \quad \text{for all } (z,t) \in \Delta^i \times \Delta^n, \underline{\lambda} \in \Lambda^n_0,\ n \geq 0.$$

Hence if f corresponds to the map $\tilde{f}\colon \Delta^i \longrightarrow X$ given by $\tilde{f}(z) = f_{(\lambda_0)}(e_0,z)$, we have a natural isomorphism from $S_c(\underline{X})$ to $S(X)$.

In the latter part of this paper, if $(\underline{X,X_0})$ is the rudimentary system $((X,X_0))$, we frequently identify $S_c(\underline{X,X_0})$ with $(S(X),S(X_0))$ by the above isomorphism.

Let $f\colon (\underline{X,X_0}) \longrightarrow (\underline{Y,Y_0}) = ((Y_\mu,Y_{0\mu}),\ q_{\mu\mu'},M)$ be a coherent map. For each $i \geq 0$ we define the function $S_i(f)\colon S_i(\underline{X}) \longrightarrow S_i(\underline{Y})$ by

(2) $\quad S_i(f)(h) = fh \quad \text{for } h \in S_i(\underline{X}).$

Then $S_i(f)(S_i(\underline{X}_0)) \subset S_i(\underline{Y}_0)$ for every $i \geq 0$, and by the definition of composition the collection of $S_i(f)$, $i \geq 0$, is a semi-simplicial map from $S_c(\underline{X})$ to $S_c(\underline{Y})$. Hence we have the semi-simplicial map $S_c(f)\colon S_c(\underline{X,X_0}) \longrightarrow S_c(\underline{Y,Y_0})$. We call $S_c(f)$ the <u>semi-simplicial map induced by</u> f.

We have the following.

3.2. Theorem. <u>The corresponding</u> S_c <u>induces a functor from</u> $CPHTOP_2$ <u>to the category</u> KAN_2 <u>of Kan pairs and homotopy classes of semi-simplicial maps</u>.

The proof of Theorem 3.2 consists of the following three lemmas. The three lemmas are actually dependent on [19], §I. For convenience, in those lemmas we consider only inverse systems of absolute spaces. The proofs can be immediately applied to the relative case.

3.3. Lemma. Let $f, f'\colon \underline{X} \longrightarrow \underline{Y}$ be coherent maps. If $f \simeq f'$, then $S_c(f)$ is homotopic to $S_c(f')$.

Proof. Let $F\colon \underline{X} \times I \longrightarrow \underline{Y}$ be a coherent homotopy connecting f and f'. For each $h \in S_i(\underline{X})$, $i \geq 0$, we define the coherent map $R(h)\colon \Delta^i \times I \longrightarrow \underline{Y}$ by $R(h) = F(h \times 1)$. Then

(3) $\quad R(h)(\partial^i_k \times 1) = R(h\partial^i_k)$, and $R(h)(\sigma^i_k \times 1) = R(h\sigma^i_k)$.

For $s = 0, 1$, the map $\ell^i_s\colon \Delta^i \longrightarrow \Delta^i \times I$ is defined by $\ell^i_s(z) = (z,s)$ for $z \in \Delta^i$. Then by (3) the functions $g^i_s\colon S_i(\underline{X}) \longrightarrow S_i(\underline{Y})$ given by $g^i_s(h) = R(h)\ell^i_s$ induce the semi-simplicial map $g_s\colon S_c(\underline{X}) \longrightarrow S_c(\underline{Y})$.

For each k, $0 \leq k \leq i$, let $\theta_k\colon \Delta^{i+1} \longrightarrow \Delta^i \times I$ be the linear map given by

$$(4) \quad \theta_k(e_j) = \begin{cases} (e_j, 0) & \text{if } 0 \leq j \leq k, \\ (e_{j-1}, 1) & \text{if } k < j \leq i+1. \end{cases}$$

Defining functions $G^i_k\colon S_i(\underline{X}) \longrightarrow S_{i+1}(\underline{Y})$, $0 \leq k \leq i$, $i \geq 0$, as follows;

(5) $\quad G^i_k(h) = R(h)\theta_k \quad$ for $h \in S_i(\underline{X})$,

by (3), the collection $\{G^i_k\}$ gives the homotopy connecting g_0 and g_1 .

In the case h is a system map, $g_0(h) = fh$ and $g_1(h) = f'h$. That is, $\{G_k^i\}$ is the homotopy connecting $S_c(f)$ and $S_c(f')$.

Assume that h is not a system map,

(6) $g_0(h)_{\underline{\mu}}(t,z) = (F(h \times 1))_{\underline{\mu}}(t,z,0)$

$$= f_{(\mu_0,\ldots,\mu_j)}(\beta_j^n(t), p_{\phi(\mu_j)\phi(\mu_j)}h_{(\phi(\mu_j),\ldots,\phi(\mu_n))}(\alpha_j^n(t),z)).$$

(7) $(fh)_{\underline{\mu}}(t,z) = f_{(\mu_0,\ldots,\mu_j)}(\beta_j^n(t), h_{(\phi(\mu_j),\ldots,\phi(\mu_n))}(\alpha_j^n(t),z))$

where $\underline{\mu} = (\mu_0,\ldots,\mu_n) \in M_n$, $n \geq 0$, $z \in \Delta^i$, $t \in P_j^n$.

By the same way as [19], P.19 and P.20, we define a decomposition of $I \times \Delta^i$ into subpolyhedra T_k^i, $0 \leq k \leq i$, by putting $(s,t) \in T_k^i$ whenever

(8) $t_{k+1} + \ldots + t_i \leq s \leq t_k + \ldots + t_i$,

and define maps $\varepsilon_k^i \colon T_k^i \longrightarrow \Delta^{i+1}$ by

(9) $\varepsilon_k^i(s,t) = (t_0,\ldots,t_{k-1},(t_k + \ldots + t_n) - s,$

$$(t_0 + \ldots + t_k) - (1 - s),\ t_{k+1},\ldots,t_i).$$

Now we give the coherent map $P(h)\colon \Delta^i \times I \longrightarrow \underline{Y}$ by

(10) $P(h)_{\underline{\mu}}(t,z,s) = f_{(\mu_0,\ldots,\mu_j)}$

$$(\beta_j^n(t), h_{(\phi(\mu_j),\ldots,\phi(\mu_{j+k}),\phi(\mu_{j+k}),\ldots,\phi(\mu_n))}$$

$$(\varepsilon_k^{n-j}(s,\alpha_j^n(t)),z)),$$

where $\underline{\mu} = (\mu_0,\dots,\mu_n) \in M^n$, $n \geq 0$, $z \in \Delta^i$, $t \in P^n_j$, $(s,\alpha^n_j(t)) \in T^{n-j}_k$, $0 \leq k \leq n-j$, $0 \leq j \leq n$ (see [19], P.20). Then by the definition and [19], Lemma I.3.3,

(11) $P(h)(\partial^i_j \times 1) = P(h\partial^i_j)$ and $P(h)(\sigma^i_j \times 1) = P(h\sigma^i_j)$,

(12) $P(h)\ell_0 = fh = S_i(f)(h)$,

(13) $P(h)\ell_1 = g_0(h)$.

In the case h is a system map, $g_0(h) = fh$. Hence the coherent map $P(h)\colon \Delta^i \times I \longrightarrow \underline{Y}$ is defined by fh.

Then by the same way as the first part the correspondence P induces a homotopy connecting $S_c(f)$ and g_0.

Similarly we can find a homotopy connecting $S_c(f')$ and g_1. Therefore $S_c(f)$ is homotopic to $S_c(f')$.

3.4. Lemma. $S_c(1_{\underline{X}})$ _is homotopic to_ $1_{S_c(\underline{X})}$.

Proof. Consider the decomposition of $I \times \Delta^n$ defined in [19], P.28, which is formed by certain polyhedra $L^n_j \subset I \times \Delta^n$, $0 \leq j \leq n$:

$$L^n_j = \{(s,t) \in I \times \Delta^n \mid t_0 +\dots+ t_{j-1} \leq \frac{1-s}{2} \leq t_0 +\dots+ t_j\}.$$

We define maps $\gamma^n_j\colon L^n_j \longrightarrow \Delta^{n-j}$, $0 \leq j \leq n$, by putting

(14) $\gamma^n_j(s,t) = (\#, \frac{2}{1+s}t_{j+1},\dots,\frac{2}{1+s}t_n)$.

If $h \in S_i(\underline{X})$, $i \geq 0$, is not a system map, we define the coherent map $R(h)\colon \Delta^i \times I \longrightarrow \underline{X}$ by

(15) $R(h)_{\underline{\lambda}}(t,z,s) = p_{\lambda_0 \lambda_j} h_{(\lambda_j,\ldots,\lambda_n)}(\gamma_j^n(s,t),z))$,

where $\underline{\lambda} = (\lambda_0,\ldots,\lambda_n) \in \Lambda^n$, $z \in \Delta^i$, $(s,t) \in L_j^n$ (see [19], P.29).

Then by the definition and [19], §I.5,

(16) $R(h)(\partial_j^i \times 1) = R(h\partial_j^i)$, and $R(h)(\sigma_j^i \times 1) = R(h\sigma_j^i)$,

(17) $R(h)\ell_0 = 1_{\underline{X}}h = S_c(1_{\underline{X}})(h)$,

(18) $R(h)\ell_1 = h = 1_{S_c(\underline{X})}(h)$.

If h is a system map, $1_{\underline{X}}h = h$. Hence we define the coherent map $R(h)$ by h.

We define functions $G_k^i\colon S_i(\underline{X}) \longrightarrow S_{i+1}(\underline{X})$, $0 \leq k \leq i$, $i \geq 0$, by the formula;

(19) $G_k^i(h) = R(h)\theta_k$ for $h \in S_i(\underline{X})$.

Then the collection $\{G_k^i\}$ induces a homotopy connecting $S_c(1_{\underline{X}})$ and $1_{S_c(\underline{X})}$.

3.5. Lemma. Let $f\colon \underline{X} \longrightarrow \underline{Y}$ and $g\colon \underline{Y} \longrightarrow \underline{Z} = (Z_\nu, r_{\nu\nu'}, N)$ be coherent maps. Then $S_c(gf)$ is homotopic to $S_c(g)S_c(f)$.

Proof. By the same way as in [19], P.23, we define a decomposition of $I \times \Delta^n$ into subpolyhedra M^n_{jk}, $0 \leq j \leq k \leq n$, which consists of all points $(s,t) \in I \times \Delta^n$ satisfying

(20) $t_0 + \ldots + t_{j-1} \leq \frac{2-s}{4} \leq t_0 + \ldots + t_j$,

(21) $t_{k+1} + \ldots + t_n \leq \frac{1+s}{4} \leq t_k + \ldots + t_n$.

For each $0 \leq j \leq k \leq n$, define a map $\theta^n_{jk}\colon M^n_{jk} \longrightarrow \Delta^n$ by putting $\theta^n_{jk}(s,t) = t' = (t'_0, \ldots, t'_n)$, where

(22) $t'_0 = \frac{2}{2-s} t_0$, ..., $t'_{j-1} = \frac{2}{2-s} t_{j-1}$,

(23) $t'_j = t_j + \frac{s}{4} - \frac{s}{2-s}(t_0 + \ldots + t_{j-1})$,

(24) $t'_{j+1} = t_{j+1}$, ..., $t'_{k-1} = t_{k-1}$,

(25) $t'_{k+1} = \frac{1}{1+s} t_{k+1}$, ..., $t'_n = \frac{1}{1+s} t_n$,

(26) $t'_k = 1 - (t'_1 + \ldots + t'_{k-1}) - (t'_{k+1} + \ldots + t'_n)$.

If $h \in S_i(\underline{X})$, $i \geq 0$, is not a system map, we define the coherent map $R(h)\colon \Delta^i \times I \longrightarrow \underline{Z}$ as follows;

(27) $R(h)_{\underline{\nu}}(t,z,s) = (g(fh))_{\underline{\nu}}(\theta^n_{jk}(s,t),z)$,

where $\underline{\nu} = (\nu_0, \ldots, \nu_n) \in N^n$, $n \geq 0$, $z \in \Delta^i$, $(s,t) \in M^n_{jk}$ (see [19], P.27).

Then by the definition and [19], §I.4,

(28) $R(h)(\partial^i_j \times 1) = R(h\partial^i_j)$, and $R(h)(\sigma^i_j \times 1) = R(h\sigma^i_j)$,

(29) $R(h)\ell_0 = g(fh) = (S_c(g)S_c(f))(h)$,

(30) $R(h)\ell_1 = (gf)h = S_c(gf)(h)$.

If h is a system map, $g(fh) = (gf)h$. Hence we define the coherent map $R(h)$ by $g(fh)$.

Now we define functions $G_k^i\colon S_i(\underline{X}) \longrightarrow S_{i+1}(\underline{Z})$, $0 \leq k \leq i$, $i \geq 0$, by the formula;

(31) $G_k^i(h) = R(h)\theta_k$ for $h \in S_i(\underline{X})$.

Then by (28), (29) and (30) the collection $\{G_k^i\}$ induces a homotopy from $S_c(g)S_c(f)$ to $S_c(gf)$.

3.6. Remark. By the same way we have functors on CPHTOP and CPHTOP_0. We also denote those functors by S_c.

4. The canonical coherent map $\tau_{\underline{X}}\colon |S_c(\underline{X})| \longrightarrow \underline{X}$.

Let $|\cdot|\colon$ KAN $\longrightarrow$ CW be the geometric realization functor, where KAN is the category of Kan complexes and homotopy classes of semi-simplicial maps, and CW is the category of CW-complexes and homotopy classes of maps (see [27], Chapter III). Let $(\underline{X},\underline{X}_0) = ((X_\lambda, X_{0\lambda}), p_{\lambda\lambda'}, \Lambda)$ be an object of CPHTOP_2. Now the coherent map $\tau_{\underline{X}}\colon |S_c(\underline{X})| \longrightarrow \underline{X}$ is defined as follows:

For $\underline{\lambda} = (\lambda_0,\ldots,\lambda_n) \in \Lambda^n$, $n \geq 0$, the map $\tau_{\underline{\lambda}}\colon \Delta^n \times |S_c(\underline{X})| \longrightarrow X_{\lambda_0}$ is given by

(1) $\tau_{\underline{\lambda}}(t,|h,z|) = h_{\underline{\lambda}}(t,z)$, where $(h,z) \in S_i(\underline{X}) \times \Delta^i$, $i \geq 0$, $t \in \Delta^n$.

Indeed, for each $\underline{\lambda} \in \Lambda^n$, $n \geq 0$, $\tau_{\underline{\lambda}}$ is well-defined and continuous, and $\tau_{\underline{\lambda}}(|S_c(\underline{X}_0)|) \subset X_{0\lambda_0}$. Moreover,

$$
\begin{aligned}
(2)\quad \tau_{\underline{\lambda}}(\partial^n_j(t),|h,z|) &= h_{\underline{\lambda}}(\partial^n_j(t),z) \\
&= \begin{cases} p_{\lambda_0\lambda_1} h_{\underline{\lambda}_0}(t,z) \\ h_{\underline{\lambda}_j}(t,z) \end{cases} \\
&= \begin{cases} p_{\lambda_0\lambda_1} \tau_{\underline{\lambda}_0}(t,|h,z|) & \text{if } j = 0, \\ \tau_{\underline{\lambda}_j}(t,|h,z|) & \text{if } 0 < j \leq n, \end{cases}
\end{aligned}
$$

where $(h,z) \in S_i(\underline{X}) \times \Delta^i$, $i \geq 0$, $t \in \Delta^{n-1}$, $0 \leq j \leq n$,

(3) $\tau_{\underline{\lambda}}(\sigma^n_j(t),|h,z|) = h_{\underline{\lambda}}(\sigma^n_j(t),z) = h_{\underline{\lambda}^j}(t,z) = \tau_{\underline{\lambda}^j}(t,|h,z|)$,

where $(h,z) \in S_i(\underline{X}) \times \Delta^i$, $i \geq 0$, $t \in \Delta^{n+1}$, $0 \leq j \leq n$.

We note that maps $\tau_{\underline{\lambda}} \mid \Delta^n \times |S_c(\underline{X})|$ actually induce the coherent map $\tau_{\underline{X}_0}: |S_c(\underline{X}_0)| \longrightarrow \underline{X}_0$, and therefore $\tau_{\underline{X}}$ is the coherent map from $(|S_c(\underline{X})|,|S_c(\underline{X}_0)|)$ to $(\underline{X},\underline{X}_0)$ as pairs. We call $\tau_{\underline{X}}$ the <u>canonical coherent map</u> of $(\underline{X},\underline{X}_0)$.

For convenience, we denote the CW-pair $(|S_c(\underline{X})|,|S_c(\underline{X}_0)|)$ and the pointed CW-complex $(|S_c(\underline{X})|,|(\{c_n\})|)$ by $|S_c(\underline{X},\underline{X}_0)|$ and $|S_c(\underline{X},\underline{x})|$, respectively, where $c_n: \Delta^n \longrightarrow \underline{X}$, $n \geq 0$, is the constant coherent map.

4.1. Proposition. <u>Let</u> $f: (\underline{X},\underline{X}_0) \longrightarrow (\underline{Y},\underline{Y}_0)$ <u>be a coherent map</u>. <u>Then</u> $f\tau_{\underline{X}} = \tau_{\underline{Y}}|S_c(f)|$.

Proposition 4.1 is easily obtained by definitions. By Theorem 3.2 and Proposition 4.1 we have the following theorem, which is called the <u>stability theorem in coherent prohomotopy theory</u> (see [10] or [28]).

4.2. Theorem. <u>Let</u> $(\underline{X},\underline{X}_0)$ <u>be an object of</u> CPHTOP$_2$. <u>If</u> $(\underline{X},\underline{X}_0)$ <u>is dominated by a</u> CW-<u>pair</u> (P,P_0) <u>in</u> CPHTOP$_2$, <u>then the canonical coherent map</u> $\tau_{\underline{X}}: |S_c(\underline{X},\underline{X}_0)| \longrightarrow (\underline{X},\underline{X}_0)$ <u>induces an isomorphism in</u> CPHTOP$_2$.

Proof. Let $f\colon (\underline{X,X_0}) \longrightarrow (P,P_0)$ and $g\colon (P,P_0) \longrightarrow (\underline{X,X_0})$ be coherent maps such that $gf \simeq 1_{(\underline{X,X_0})}$. By Proposition 4.1 we consider the following diagram. We remark that if $\underline{Y}$ is the rudimentary system (Y), then $S_c(\underline{Y}) = S(Y)$ and $\tau_{\underline{Y}}$ is the canonical map $\omega_Y\colon |S(Y)| \longrightarrow Y$.

$$\begin{tikzcd}
{|S_c(\underline{X,X_0})|} \arrow[r, "|S_c(f)|"] \arrow[d, "\tau_{\underline{X}}"'] & {|S(P,P_0)|} \arrow[r, "|S_c(g)|"] \arrow[d, "\omega_P"'] & {|S_c(\underline{X,X_0})|} \arrow[d, "\tau_{\underline{X}}"'] \\
(\underline{X,X_0}) \arrow[r, "f"] & (P,P_0) \arrow[r, "g"] \arrow[u, dashed, bend right, "\rho"'] & (\underline{X,X_0})
\end{tikzcd}$$

Since (P,P_0) is a CW-pair, it is well-known that ω_P is a homotopy equivalence. Hence ω_P has a homotopy inverse ρ. Then we have

$$\begin{aligned}
(4)\quad (|S_c(g)|\rho f)\tau_{\underline{X}} &\simeq |S_c(g)|\rho\omega_P|S_c(f)| \\
&\simeq |S_c(g)||S_c(f)| \simeq 1_{|S_c(\underline{X,X_0})|},
\end{aligned}$$

$$(5)\quad \tau_{\underline{X}}(|S_c(g)|\rho f) \simeq g\omega_P\rho f \simeq gf \simeq 1_{(\underline{X,X_0})}.$$

Therefore $\tau_{\underline{X}}$ induces an isomorphism in $CPHTOP_2$.

4.3. Corollary. Let $(\underline{X,X_0})$ be an object of $CPHTOP_2$. Then the following are equivalent conditions;

(a) (X, X_0) *is dominated by a* CW-*pair in* CPHTOP_2,

(b) (X, X_0) *is equivalent to a* CW-*pair in* CPHTOP_2,

(c) (X, X_0) *is equivalent to a simplicial pair with weak topology in* CPHTOP_2,

(d) (X, X_0) *is equivalent to a simplicial pair with metric topology in* CPHTOP_2,

(e) (X, X_0) *is equivalent to an* ANR *pair in* CPHTOP_2.

5. Coherent prohomotopy groups $\pi_i^c(\underline{X,x})$.

Let $(\underline{X,x})$ be an object of $CPHTOP_0$. For each $i \geq 0$, we denote the set of all coherent homotopy classes of coherent maps from (S^i, s_0) to $(\underline{X,x})$ by $\pi_i^c(\underline{X,x})$. If $i \geq 1$, by using H-cogroup structure of S^i, $\pi_i^c(\underline{X,x})$ is a group. Indeed, if $n \geq 2$, $\pi_i^c(\underline{X,x})$ is an abelian group. We call $\pi_i^c(\underline{X,x})$ the *i-th coherent prohomotopy group* of $(\underline{X,x})$.

For a coherent map $f\colon (\underline{X,x}) \longrightarrow (\underline{Y,y})$ we define the function $f_\#\colon \pi_i^c(\underline{X,x}) \longrightarrow \pi_i^c(\underline{Y,y})$ by

(1) $\quad f_\#([\phi]) = [f\phi] \quad$ for each $[\phi] \in \pi_i^c(\underline{X,x})$.

Clearly $f_\#$ is a group-homomorphism for $i \geq 1$, and depends only on the coherent homotopy class of f. We call $f_\#$ the *homomorphism induced by* f.

Similarly, for an object $(\underline{X,A,x})$ of $CPHTOP_{2,0}$ and a coherent map $f\colon (\underline{X,A,x}) \longrightarrow (\underline{Y,B,y})$, we can define the i-th coherent prohomotopy group $\pi_i^c(\underline{X,A,x})$ of $(\underline{X,A,x})$ and the homomorphism $f_\#\colon \pi_i^c(\underline{X,A,x}) \longrightarrow \pi_i^c(\underline{Y,B,y})$ induced by f. Then we easily have the following.

5.1. Theorem. *The following statements hold;*

(a) *for* $i \geq 1$ *the correspondence* π_i^c *induces a functor from* $CPHTOP_0$ *to* GR, *and for* $i \geq 2$, *induces a functor from* $CPHTOP_{2.0}$ *to* GR, *where* GR *is the category of groups and homomorphisms*,

(b) <u>for an object</u> $(\underline{X},\underline{A},\underline{x})$ <u>of</u> $\mathrm{CPHTOP}_{2,0}$ <u>we have the following natural exact sequence</u>,

$$(2)\quad \cdots \longrightarrow \pi^c_{i+1}(\underline{X},\underline{A},\underline{x}) \xrightarrow{\partial^c} \pi^c_i(\underline{A},\underline{x}) \xrightarrow{\underline{i}_\#} \pi^c_i(\underline{X},\underline{x}) \xrightarrow{\underline{j}_\#} \pi^c_i(\underline{X},\underline{A},\underline{x}) \longrightarrow \cdots,$$

<u>where</u> $\underline{i}\colon (\underline{A},\underline{x}) \longrightarrow (\underline{X},\underline{x})$ <u>and</u> $\underline{j}\colon (\underline{X},\underline{x}) \longrightarrow (\underline{X},\underline{A},\underline{x})$ <u>are natural system maps induced by inclusions</u>, <u>and</u> ∂^c <u>is the boundary homomorphism given by the restriction</u>.

Next we will consider the relation between $\pi_i(|S_c(\underline{X},\underline{x})|)$ and $\pi^c_i(\underline{X},\underline{x})$. By Theorem 5.2, when we investigate coherent prohomotopy groups, we can widely use the usual homotopy theory. A direct application will appear in Theorem 6.2.

5.2. Theorem. <u>Let</u> $(\underline{X},\underline{x}) = ((X_\lambda, x_\lambda), p_{\lambda\lambda'}, \Lambda)$ <u>be an object of</u> CPHTOP_0. <u>Then the canonical coherent map</u> $\tau_{\underline{X}}\colon |S_c(\underline{X},\underline{x})| \longrightarrow (\underline{X},\underline{x})$ <u>induces isomorphisms</u>

$$(\tau_{\underline{X}})_\#\colon \pi_i(|S_c(\underline{X},\underline{x})|) \cong \pi^c_i(\underline{X},\underline{x}) \quad \underline{\text{for all}}\ i \geq 0.$$

The proof is given by a modification of [34], and is long but mechanical. Hence we show only the outline of the proof here.

<u>Outline of the proof</u>. Let (Z, z_0) be a compact polyhedron and let $T = (K,t)$ be its triangulation such that K is an ordered simplicial complex and z_0 is its vertex. For each

k-simplex $s = \langle v_{n(0)}, \ldots, v_{n(k)} \rangle$ of K, the linear homeomorphism $\rho_s\colon \Delta^k \longrightarrow |s|$ is defined by

(3) $\rho_s(e_j) = v_{n(j)}$ for each j, $0 \leq j \leq k$.

Let f: $(Z, z_0) \longrightarrow (\underline{X}, \underline{x})$ be a coherent map. The function $\Phi_f^T\colon (Z, z_0) \longrightarrow |S_c(\underline{X}, \underline{x})|$ is defined as follows;

For any point $z \in Z$ there is a simplex s of K such that $z \in |s|$, where $s = \langle v_{n(0)}, \ldots, v_{n(k)} \rangle$. Now we define

(4) $\Phi_f^T(z) = |f\rho_s, \rho_s^{-1}(z)|$.

Obviously Φ_f^T is well-defined and continuous, and $\Phi_f^T(z_0) = |(\{c_n\})|$. Moreover $\tau_{\underline{X}}\Phi_f^T = f$.

Indeed, for $\underline{\lambda} = (\lambda_0, \ldots, \lambda_n) \in \Lambda^n$, $n \geq 0$, $z \in |s| \subset Z$ and $t \in \Delta^n$,

$$(\tau_{\underline{X}}\Phi_f^T)_{\underline{\lambda}}(t,z) = \tau_{\underline{\lambda}}(t, \Phi_f^T(z)) = \tau_{\underline{\lambda}}(t, |f\rho_s, \rho_s^{-1}(z)|)$$

$$= (f\rho_s)_{\underline{\lambda}}(t, \rho_s^{-1}(z)) = f_{\underline{\lambda}}(t,z).$$

If $(Z, z_0) = (S^i, s_0)$, $i \geq 0$, by the above result, $(\tau_{\underline{X}})_\#\colon \pi_i(|S_c(\underline{X}, \underline{x})|) \longrightarrow \pi_i^c(\underline{X}, \underline{x})$ is surjective. The injectivity of $(\tau_{\underline{X}})_\#$ will be immediately obtained by the following two claims.

Claim 1. _Let_ $T = (K, t)$ _and_ $T' = (K', t')$ _be triangulations of_ (Z, z_0) _by ordered simplicial complexes such that_ K' _is a_ *subdivision of* K *and* z_0 *is a vertex of* K. *Let* $f, f': (Z, z_0) \longrightarrow (\underline{X}, x)$ *be coherent maps. If* $f \simeq f'$, *then* $\Phi_f^T \simeq \Phi_{f'}^{T'}$.

Claim 2. Let $T = (K, t)$ *be a triangulation of* (Z, z_0) *by an ordered simplicial complex* K *with its vertex* z_0. *Then the following holds;*

$$\Phi^T_{\tau_{\underline{X}} g} \simeq g \quad \textit{for every map } g: (Z, z_0) \longrightarrow |S_c(\underline{X}, x)|.$$

Proof of Claim 1. For each $s = 0, 1$, let $\ell_s: Z \longrightarrow Z \times I$ be the map defined by $\ell_s(z) = (z, s)$ for $z \in Z$. Then $T_0 = (K, \ell_0 t)$ and $T_1' = (K', \ell_1 t')$ are triangulations of $Z \times \{0\}$ and $Z \times \{1\}$, respectively Now we have the triangulation $T^* = (T^*, t^*)$ of $Z \times I$ which satisfies;

(i) every vertex of T^* is either one of T_0 or of T_1',

(ii) T^* contains both T_0 and T_1' as subcomplexes, and $\{z_0\} \times I$ as a 1-simplex.

Moreover T^* can be ordered such that;

(a) every vertex of T_0 is before any vertex of T_1',

(b) both $\ell_0 t$ and $\ell_1 t'$ are order-preserving.

Let $F: (Z, z_0) \times I \longrightarrow (\underline{X}, x)$ be a coherent homotopy connecting f and f'. Then we consider the map $\Phi_F^{T^*}: Z \times I \longrightarrow |S_c(\underline{X}, x)|$. By (ii) and the construction of $\Phi_F^{T^*}$,

$\Phi_F^{T*}(\{z_0\} \times I) = |(\{c_n\})|$. For any $z \in |s| = |v_{n(0)},\ldots,v_{n(k)}|$, by (i) and (ii), $(z,0) \in |s_0*| = |(v_{n(0)},0),\ldots,(v_{n(k)},0)|$. Moreover $\rho_{s_0*}\colon \Delta^k \longrightarrow |s_0*|$ is given by $\rho_{s_0*}(u) = (\rho_s(u),0)$ for $u \in \Delta^k$. Hence $\rho_s^{-1}(z) = \rho_{s_0*}^{-1}(z,0)$, and

$$(F\rho_{s_0*})_{\underline{\lambda}}(t,u) = F_{\underline{\lambda}}(t,\rho_s(u),0) = f_{\underline{\lambda}}(t,\rho_s(u)) = (f\rho_s)_{\underline{\lambda}}(t,u),$$

where $\underline{\lambda} = (\lambda_0,\ldots,\lambda_n) \in \Lambda^n$, $n \geq 0$, $u \in \Delta^k$, $t \in \Delta^n$. Therefore

$$\Phi_F^{T*}(z,0) = |F\rho_{s_0*}, \rho_{s_0*}^{-1}(z,0)| = |f\rho_s, \rho_s^{-1}(z)| = \Phi_f^T(z),$$

where $z \in |s| = |v_{n(0)},\ldots,v_{n(k)}| \subset Z$.

That is, $\Phi_F^{T*} | Z \times \{0\} = \Phi_f^T$.

Similarly we have that $\Phi_F^{T*} | Z \times \{1\} = \Phi_{f'}^{T'}$.

Therefore $\Phi_F^{T*}\colon \Phi_f^T \simeq \Phi_{f'}^{T'}$.

Before proving Claim 2, we introduce the concept of simpliciality of maps. Let $T = (K,t)$ be an ordered triangulation of a polyhedron Z and let $\phi\colon Z \longrightarrow |S_c(\underline{X})|$ be a map. If for each simplex s of K, there are $h_s \in S_c(\underline{X})$ and an order-preserving simplicial map $\alpha_s\colon |s| \longrightarrow \Delta^{q(s)}$, where $q(s) = \dim h_s$, such that

(5) $\phi(z) = |h_s, \alpha_s(z)|$ for each $z \in |s|$,

ϕ is said to be <u>simplicial with respect to</u> T. Then we have;

Claim 3. *If a map* $\phi\colon Z \longrightarrow |S_c(\underline{X})|$ *is simplicial with respect to* T, *then* $\Phi^T_{\tau_{\underline{X}}\phi} = \phi$.

Proof of Claim 3. Let s be any k-simplex of T. Then

$$(\tau_{\underline{X}}(\phi|\ |s|))_{\underline{\lambda}}(t,z) = \tau_{\underline{\lambda}}(t,\phi(z)) = \tau_{\underline{\lambda}}(t,|h_s,\alpha_s(z)|)$$

$$= (h_s)_{\underline{\lambda}}(t,\alpha_s(z)) = (h_s\alpha_s)_{\underline{\lambda}}(t,z),$$

where $\underline{\lambda} = (\lambda_0,\ldots,\lambda_n) \in \Lambda^n$, $n \geq 0$, $z \in |s|$, $t \in \Delta^n$. Moreover $\alpha_s\rho_s\colon \Delta^n \longrightarrow \Delta^{q(s)}$ is induced by a monotone function $\Delta[n] \longrightarrow \Delta[q(s)]$. Hence for every $z \in |s|$,

$$\Phi^T_{\tau_{\underline{X}}\phi}(z) = |(\tau_{\underline{X}}\phi)\rho_s,\rho_s^{-1}(z)| = |(h_s\alpha_s)\rho_s,\rho_s^{-1}(z)|$$

$$= |h_s(\alpha_s\rho_s),\rho_s^{-1}(z)| = |h_s,\alpha_s(z)| = \phi(z).$$

Therefore $\Phi^T_{\tau_{\underline{X}}\phi} = \phi$.

By a slight modification of [34], P.103, we have the following.

Fact. *Let* $T = (K,t)$ *be an ordered triangulation of a compact polyhedron* (Z,z_0) *such that* z_0 *is a vertex of* K. *Then for any map* $g\colon (Z,z_0) \longrightarrow |S_c(\underline{X,x})|$ *there are a subdivision* T' *of* T *and a map* $\phi\colon (Z,z_0) \longrightarrow |S_c(\underline{X,x})|$ *such that*

(i) ϕ *is simplicial with respect to* T,

(ii) $g \simeq \phi$ *as maps* $(Z,z_0) \longrightarrow |S_c(\underline{X,x})|$.

Proof of Claim 2. For any map $g\colon (Z,z_0) \longrightarrow |S_c(\underline{X,x})|$, by Fact, there are a subdivision T' of T and a map $\phi\colon (Z,z_0) \longrightarrow |S_c(\underline{X,x})|$ such that

(6) ϕ is simplicial with respect to T',

(7) $g \simeq \phi$ rel. z_0.

By (7) and Claim 1,

(8) $\Phi^{T}_{\tau_{\underline{X}}} g \simeq \Phi^{T'}_{\tau_{\underline{X}}} \phi.$

By (6) and Claim 3,

(9) $\Phi^{T'}_{\tau_{\underline{X}}} \phi = \phi.$

Hence by (8), (9) and (7), we have

(10) $\Phi^{T}_{\tau_{\underline{X}}} g \simeq g.$

Therefore the proof of Claim 2 is completed. That is, we complete the proof of Theorem 5.2.

By the same way as the proof of Theorem 5.2 we can show the next result, which is the relative version of Theorem 5.2.

5.3. Corollary. *For an object $(\underline{X,A,x})$ of $\mathrm{CPHTOP}_{2,0}$, the canonical coherent map $\tau_{\underline{X}}\colon |S_c(\underline{X,A,x})| \longrightarrow (\underline{X,A,x})$ induces isomorphisms*

$$(\tau_{\underline{X}})_{\#}\colon \pi_i(|S_c(\underline{X,A,x})|) \cong \pi_i^c(\underline{X,A,x}) \quad \textit{for all } i \geq 1.$$

6. Coherent singular homology groups $H_*^c(\underline{X}:G)$.

Let $(\underline{X,A}) = ((X_\lambda, A_\lambda), p_{\lambda\lambda'}, \Lambda)$ be an object of $CPHTOP_2$ and let G be an abelian group. For each $i \geq 0$, we define

(1) $H_i^c(\underline{X,A}:G) = H_i(S_c(\underline{X,A}):G)$,

which is called the i-th <u>coherent singular homology group of</u> $(\underline{X,A})$ <u>with the coefficient group</u> G. If G is the additive group of all integers Z, then we denote $H_*^c(\underline{X,A}:Z)$ by $H_*^c(\underline{X,A})$.

Let $f: (\underline{X,A}) \longrightarrow (\underline{Y,B}) = ((Y_\mu, B_\mu), q_{\mu\mu'}, M)$ be a coherent map. Then we have the homomorphism $f_*: H_*^c(\underline{X,A}:G) \longrightarrow H_*^c(\underline{Y,B}:G)$, defined by

(2) $f_* = S_c(f)_*$.

We call f_* the <u>homomorphism induced</u> by f.

Considering CPHTOP as a full subcategory of $CPHTOP_2$, we define $H_*^c(\underline{X}:G)$ and $f_*: H_*^c(\underline{X}:G) \longrightarrow H_*^c(\underline{Y}:G)$ for a coherent map $f: \underline{X} \longrightarrow \underline{Y}$. We note that if $(\underline{X,A})$ is a rudimentary system $((X,A))$, then $H_*^c(\underline{X,A}:G)$ is the usual singular homology group $H_*(X,A:G)$.

6.1. Theorem. (a) <u>The correspondence</u> H_*^c <u>induces a functor from</u> $CPHTOP_2$ <u>to</u> GR.

(b) <u>For an object</u> $(\underline{X,A})$ <u>of</u> $CPHTOP_2$, <u>the following is a natural exact sequence</u>;

$$(3) \quad \cdots \longrightarrow H^c_{i+1}(\underline{X,A}:G) \xrightarrow{\partial^c} H^c_i(\underline{A}:G) \xrightarrow{\underline{i}_*} H^c_i(\underline{X}:G)$$

$$\xrightarrow{\underline{j}_*} H^c_i(\underline{X,A}:G) \longrightarrow \cdots .$$

(c) <u>For an object</u> $(\underline{X,A})$ <u>of</u> CPHTOP_2, <u>the canonical coherent map</u> $\tau_{\underline{X}}: |S_c(\underline{X,A})| \longrightarrow (\underline{X,A})$ <u>induces isomorphisms</u>;

$$(\tau_{\underline{X}})_*: H_i(|S_c(\underline{X,A})|:G) \cong H^c_i(\underline{X,A}:G) \quad \text{for every } i \geq 0$$

<u>and every abelian group</u> G.

Proof. Both (a) and (b) are immediate consequences of Theorem 3.2 and well-known results. We will show only (c). A semi-simplicial map $\eta_{(\underline{X,A})}: S_c(\underline{X,A}) \longrightarrow S(|S_c(\underline{X,A})|)$ is given by

$$(4) \quad \eta_i(h)(u) = |h,u| \quad \text{for } h \in S_i(\underline{X}) \text{ and } u \in \Delta^i.$$

Then by [27], Proposition 16.2,

$$(5) \quad (\eta_{(\underline{X,A})})_*: H^c_i(\underline{X,A}:G) \cong H_i(|S_c(\underline{X,A})|:G).$$

On the other hand, for any $h \in S_i(\underline{X})$, $i \geq 0$, $\underline{\lambda} \in \Lambda^n$, $n \geq 0$,

$$\begin{aligned}
(S_c(\tau_{\underline{X}})\eta_{(\underline{X,A})})(h)_{\underline{\lambda}}(t,u) &= (S_c(\tau_{\underline{X}})\eta_{(\underline{X,A})}(h))_{\underline{\lambda}}(t,u)\\
&= (\tau_{\underline{X}}\eta_{(\underline{X,A})}(h))_{\underline{\lambda}}(t,u)\\
&= \tau_{\underline{\lambda}}(t,\eta_{(\underline{X,A})}(h)(u))\\
&= \tau_{\underline{\lambda}}(t,|h,u|)\\
&= h_{\underline{\lambda}}(t,u),
\end{aligned}$$

where $(t,u) \in \Delta^n \times \Delta^i$.

Hence $s_c(\tau_{\underline{X}})\eta_{(\underline{X},\underline{A})} = 1_{S_c(\underline{X},\underline{A})}$. Therefore by (5) we have

$$(\tau_{\underline{X}})_*\colon H_i(|S_c(\underline{X},\underline{A})|:G) \cong \check{H}_i^c(\underline{X},\underline{A}:G) \quad \text{for every } i \geq 0.$$

For an object $(\underline{X},\underline{x})$ of CPHTOP_0 we define the function $\Phi^i_{(\underline{X},\underline{x})}\colon \pi_i^c(\underline{X},\underline{x}) \longrightarrow H_i^c(\underline{X})$ by the formula;

(6) $\Phi^i_{(\underline{X},\underline{x})}([h]) = h_*(1)$ for every $[h] \in \pi_i^c(\underline{X},\underline{x})$.

Then the following square is commutative;

$$\begin{tikzcd}
\pi_i(|S_c(\underline{X},\underline{x})|) \arrow[r, "(\tau_{\underline{X}})_\#"] \arrow[d, "\Phi^i_{|S_c(\underline{X},\underline{x})|}"'] & \pi_i^c(\underline{X},\underline{x}) \arrow[d, "\Phi^i_{(\underline{X},\underline{x})}"] \\
H_i(|S_c(\underline{X})|) \arrow[r, "(\tau_{\underline{X}})_*"'] & H_i^c(\underline{X})
\end{tikzcd}$$

where $\Phi^i_{|S_c(\underline{X},\underline{x})|}$ is the usual Hurewicz homomorphism of $|S_c(\underline{X},\underline{x})|$.

We will can $\Phi^i_{(\underline{X},\underline{x})}$ the i-*th* *coherent* *Hurewicz* *homomorphism* of $(\underline{X},\underline{x})$. By Theorem 5.2 and Theorem 6.1(c) we have

6.2. Theorem (Hurewicz isomorphism theorem in coherent prohomotopy). *Let* $(\underline{X},\underline{x})$ *be an object of* CPHTOP_0. *Then*

(a) if $\pi_k^c(\underline{X,x}) = 0$ <u>for</u> <u>every</u> k, $0 \leq k \leq i-1$, <u>where</u> $i \geq 2$,

$\Phi^i_{(\underline{X,x})}\colon \pi_i^c(\underline{X,x}) \cong H_i^c(\underline{X})$,

<u>and</u> $\Phi^{i+1}_{(\underline{X,x})}$ <u>is</u> <u>an</u> <u>epimorphism</u>,

(b) <u>if</u> $\pi_0^c(\underline{X,x}) = 0$, <u>then</u> $\Phi^1_{(\underline{X,x})}$ <u>is</u> <u>an</u> <u>epimorphism</u> <u>and</u> <u>its</u> <u>kernel</u> <u>is</u> <u>the</u> <u>commutator</u> <u>subgroup</u> <u>of</u> $\pi_1^c(\underline{X,x})$.

Similarly, for an object $(\underline{X,A,x})$ of $CPHTOP_{2,0}$, we can define the i-th coherent Hurewicz homomorphism $\Phi^i_{(\underline{X,A,x})}$: $\pi_i^c(\underline{X,A,x}) \longrightarrow H_i^c(\underline{X,A}:G)$. Then we have the following, which is the relative version of Theorem 6.2.

6.3. Theorem (relative Hurewicz isomorphism theorem in coherent prohomotopy). <u>Let</u> $(\underline{X,A,x})$ <u>be</u> <u>an</u> <u>object</u> <u>of</u> $CPHTOP_{2,0}$. <u>Then</u> <u>if</u> $\pi_k^c(\underline{X,A,x}) = 0$ <u>for</u> <u>every</u> k, $0 \leq k \leq i-1$, <u>where</u> $i \geq 2$, <u>and</u> $\pi_1^c(\underline{A,x}) = 0$, <u>then</u>

$$\Phi^i_{(\underline{X,A,x})}\colon \pi_i^c(\underline{X,A,x}) \cong H_i^c(\underline{X,A}).$$

In [17], Lisica and Mardešić defined a <u>strong</u> <u>homology</u> of inverse systems, which is an invariant of coherent prohomotopy: For an abelian group G they associate with $\underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ a chain complex $C_\#(\underline{X}:G)$, defined as follows. A <u>strong</u> p-<u>chain</u> of $\underline{X}$, $p \geq 0$, is a function x, which assigns to every $\underline{\lambda} \in \Lambda^n$ a singular (p+n)-chain $x_{\underline{\lambda}} \in C_{p+n}(X_{\lambda_0}:G)$. The boundary operator $d\colon C_{p+1}(\underline{X}:G) \longrightarrow C_p(\underline{X}:G)$ is defined by the formula

$$(7) \quad (-1)^n(dx)_{\underline{\lambda}} = \partial(x_{\underline{\lambda}}) - p_{\lambda_0\lambda_1\#}(x_{\underline{\lambda}_0}) - \sum_{j=0}^{n}(-1)^j x_{\underline{\lambda}_j};$$

here ∂ denotes boundary of singular chains. By definition,

$$(8) \quad H_p^s(\underline{X}:G) = H_p(C_\#(\underline{X}:G)),$$

which is called the p-<u>th strong homology group of X with the coefficient group</u> G.

With a coherent map $f: \underline{X} \longrightarrow \underline{Y} = (Y_\mu, q_{\mu\mu'}, M)$ they associate a chain map $f_\#: C_\#(\underline{X}:G) \longrightarrow C_\#(\underline{Y}:G)$, given by

$$(9) \quad (f_\#(x))_{\underline{\mu}} = \sum_{i=0}^{n} f_{(\mu_0,\ldots,\mu_i)\#}(x_{(\phi(\mu_i),\ldots,\phi(\mu_n))} \times \Delta^i)$$

where $\underline{\mu} \in M^n$, $n \geq 0$, $x \in C_p(\underline{X}:G)$.
Then $f_\#$ induces a homomorphism $f_*: H_p^s(\underline{X}:G) \longrightarrow H_p^s(\underline{Y}:G)$ for each $p \geq 0$. f_* is called the <u>homomorphism induced by</u> f. For more details, see [22] and [23].

For every coherent singular i-simplex $h \in S_i(\underline{X})$, $i \geq 0$, we define a strong i-chain $\xi(h)$ of $\underline{X}$ by the formula

$$\xi(h)_{\underline{\lambda}} = h_{\underline{\lambda}\#}(\Delta^n \times \Delta^i) \quad \text{for } \underline{\lambda} \in \Lambda^n,\ n \geq 0,$$

here $\Delta^n \times \Delta^i$ is the singular (i+n)-chain of $\Delta^n \times \Delta^i$ described in [23], §3 (c.f. [32], §5.3). Then the correspondence ξ induce a homomorphism from $C_i(S_c(\underline{X}))$ to $C_i(\underline{X})$, which is also denoted by ξ. The homomorphism ξ have the following property.

6.4. Proposition. $d^s\xi(h) = \xi(d^c h)$ *for* $h \in S_i(\underline{X})$, $i \geq 1$, *where* d^s *and* d^c *are boundary operators of strong chain complex and coherent singular complex, respectively.*

Proof. For every $\underline{\lambda} = (\lambda_0, \ldots, \lambda_n) \in \Lambda^n$, $n \geq 0$,

$$\begin{aligned}
\partial(\xi(h)_{\underline{\lambda}}) &= \partial(h_{\underline{\lambda}\#}(\Delta^n \times \Delta^i)) = h_{\underline{\lambda}\#}(\partial(\Delta^n \times \Delta^i)) \\
&= h_{\underline{\lambda}\#}(\partial\Delta^n \times \Delta^i + (-1)^n\Delta^n \times \partial\Delta^i) \\
&= h_{\underline{\lambda}\#}\Big(\sum_{j=0}^{n}(-1)^j\partial_j^n(\Delta^{n-1}) \times \Delta^i + (-1)^n\sum_{k=0}^{i}(-1)^k\Delta^n \\
&\qquad\qquad \times \partial_k^i(\Delta^{i-1})\Big) \\
&= \sum_{j=0}^{n}(-1)^j h_{\underline{\lambda}\#}(\partial_j^n(\Delta^{n-1}) \times \Delta^i) \\
&\quad + (-1)^n\sum_{k=0}^{i}(-1)^k h_{\underline{\lambda}}(\Delta^n \times \partial_k^i(\Delta^{i-1}))) \\
&= p_{\lambda_0\lambda_1\#}h_{\underline{\lambda}_0\#}(\Delta^{n-1} \times \Delta^i) + \sum_{j=1}^{n}(-1)^j h_{\underline{\lambda}_j\#}(\Delta^{n-1} \times \Delta^i) \\
&\quad + (-1)^n\sum_{k=0}^{i}(-1)^k(h\partial_k^i)_{\underline{\lambda}}(\Delta^n \times \Delta^{i-1}) \\
&= p_{\lambda_0\lambda_1\#}\xi(h)_{\underline{\lambda}_0} + \sum_{j=1}^{n}(-1)^j\xi(h)_{\underline{\lambda}_j} \\
&\quad + (-1)^n\sum_{k=0}^{i}(-1)^k\xi(h\partial_k^i)_{\underline{\lambda}}.
\end{aligned}$$

Hence by (7) and the definitions,

$$(-1)^n d^s(\xi(h))_{\underline{\lambda}} = (-1)^n \sum_{k=0}^{i} (-1)^k \xi(h\partial_k^i)_{\underline{\lambda}}$$

$$= (-1)^n \xi(\sum_{k=0}^{i} (-1)^k h\partial_k^i)_{\underline{\lambda}} = (-1)^n \xi(d^c(h))_{\underline{\lambda}}.$$

By Proposition 6.4, we have the natural homomorphism

$\xi_{\underline{X}}^i \colon H_i^c(\underline{X}:G) \longrightarrow H_i^s(\underline{X}:G)$ for each $i \geq 0$.

Then the following natural problem is posed;

Problem 1. <u>Under what conditions of</u> X <u>and</u> G <u>is the homomorphism</u> $\xi_{\underline{X}}^i$ <u>isomorphism</u>?

We note that there is an inverse sequence $\underline{X}$ of compact polyhedra such that $\xi_{\underline{X}}^1 \colon H_1^c(\underline{X}) \longrightarrow H_1^s(\underline{X})$ is not surjective. The details will be discussed in the next section.

7. Fundamental singular complexes.

Let $\underline{X} = (X_n, p_{nn+1})$ be an inverse sequence of spaces and maps. For each $i = 1,2,\ldots$, let $K_i(\underline{X})$ be the set of all strong fundamental sequences from Δ^i to $\underline{X}$ in the sense of Lisica [15]. That is, every $\underline{h} \in K_i(\underline{X})$ consists of maps h_m: $\Delta^i \longrightarrow Y_m$ and of maps h_{mm+1}: $I \times \Delta^i \longrightarrow Y_m$ such that

(1) $h_{mm+1}(0,u) = h_m(u)$,

(2) $h_{mm+1}(1,u) = p_{mm+1}h_{m+1}(u)$.

For each $i \geq 0$ and each k, $0 \leq k \leq i$, the k-th degeneracy operator $d_k = d_k^i$: $K_i(\underline{X}) \longrightarrow K_{i-1}(\underline{X})$ and the k-th face operator $s_k = s_k^i$: $K_i(\underline{X}) \longrightarrow K_{i+1}(\underline{X})$ are defined as follows;

(3) $d_k(\underline{h})_m = h_m\partial_k^i$ and $d_k(\underline{h})_{mm+1} = h_{mm+1}(1 \times \partial_k^i)$,

(4) $s_k(\underline{h})_m = h_m\sigma_k^i$ and $s_k(\underline{h})_{mm+1} = h_{mm+1}(1 \times \sigma_k^i)$.

Then we have a semi-simplicial complex $K(\underline{X}) = (K_i(\underline{X}), d_k, s_k)$. The proof of Proposition 3.1 (1) essentially shows that $K(\underline{X})$ is also a Kan complex. We call $K(\underline{X})$ the <u>fundamental singular complex of $\underline{X}$</u>. Moreover by the same way as §4, a <u>canonical strong fundamental sequence</u> $\underline{\nu}_X$: $|K(\underline{X})| \longrightarrow \underline{X}$ can be defined by

(5) $\nu_m(|\underline{h},u|) = h_m(u)$,

(6) $\nu_{mm+1}(t,(\underline{h},u|) = h_{mm+1}(t,u)$,

where $(\underline{h},u) \in K_i(\underline{X}) \times \Delta^i$, $i \geq 0$, $t \in I$.

With every $h \in S_i(\underline{X})$, $i \geq 0$, we associate a strong fundamental sequence $\underline{h} \in K_i(\underline{X})$ by considering maps h_m: $\Delta^i \longrightarrow X_m$ and h_{mm+1}: $I \times \Delta^i \longrightarrow X_m$, where we have identified I with Δ^1 by identifying $t \in I$ with $(1-t,t) \in \Delta^1$. Then by (3.1), (3) and (4) the above correspondence induces a semi-simplicial map f: $S_c(\underline{X}) \longrightarrow K(\underline{X})$. Using the method of [20], we can show the following.

7.1. Theorem. The semi-simplicial map f induces an isomorphism in KAN.

In order to prove Theorem 7.1 we rewrite from [20], the proof of Lemma 1.1, the concept of the standard extension h' of a strong fundamental sequence $\underline{h}$: $\Delta^i \longrightarrow \underline{X}$. Let $L_i \subset \Delta^n$ denote the 1-simplex connecting e_{i-1} to e_i and let

$$L^n = L_1 \cup L_2 \cup \dots \cup L_n \subset \Delta^n.$$

Then there are a retraction r^n: $\Delta^n \longrightarrow L^n$ and a homotopy D^n: $I \times \Delta^n \longrightarrow \Delta^n$ such that

(7) $D^n(0,t) = t$,

(8) $D^n(1,t) = r^n(t)$,

(9) $D^n(1_I \times \partial^n_j) = \partial^n_j D^{n-1}$, $j = 0,n$.

By induction on $n \geq 0$ we will define maps $h'_{\underline{m}}: \Delta^n \times \Delta^i \longrightarrow X_{m_0}$ for $\underline{m} = (m_0,\ldots,m_n) \in N^n$.

Assume that $\underline{m}$ is non-degenerate. For each j, $0 \leq j \leq k$, let $w^k_j: I \longrightarrow L^k \subset \Delta^k$ be the linear map which takes 0 to e_{j-1} and 1 to e_j. Put $\ell(\underline{m}) = m_n - m_0$. We define a map $h_{\underline{m}}: L^{\ell(\underline{m})} \times \Delta^i \longrightarrow X_{m_0}$ by

$$(10)\quad h_{\underline{m}}(w_j^{\ell(\underline{m})}(t),u) = p_{m_0,m_0+j-1}h_{m_0+j-1,m_0+j}(t,u),$$
$$t \in I,\ 1 \leq j \leq \ell(\underline{m}).$$

We consider the linear map $v_{\underline{m}}: \Delta^n \longrightarrow \Delta^{\ell(\underline{m})}$ which takes the vertex e_i of Δ^n to the vertex $e_{m_i-m_0}$ of $\Delta^{\ell(\underline{m})}$. Now the map $h'_m: \Delta^n \times \Delta^i \longrightarrow X_{m_0}$ is defined by

$$(11)\quad h'_{\underline{m}}(t,u) = h_{\underline{m}}(r^{\ell(\underline{m})}v_{\underline{m}}(t),u),\ u \in \Delta^i,\ t \in \Delta^n.$$

If $\underline{m}$ is degenerate, $\underline{m} = \underline{k}^j$ for some $\underline{k} \in N^{n-1}$ and some j, $0 \leq j \leq n-1$. Then we define

$$(12)\quad h'_{\underline{m}}(t,u) = h'_{\underline{k}}(\sigma_j^{n-1}(t),u),\ u \in \Delta^i,\ t \in \Delta^n.$$

Proof of Theorem 7.1. By definitions, the standard extension of strong fundamental sequences induces the semi-simplicial map $g: K(\underline{X}) \longrightarrow S_c(\underline{X})$. By the definition,

$$(13)\quad fg = 1_{K(\underline{X})}.$$

For any $h \in S_i(\underline{X})$, $i \geq 0$, $g_i f_i(h)$ is the standard extension of the strong fundamental sequence $f_i(h)$ associated with h. Now we define the coherent map $R(h)\colon \Delta^i \times I \longrightarrow \underline{X}$ as follows (see [20], Lemma 1.2);

(i) if $\underline{m} = (m_0,\ldots,m_n)$ is non-degenerate, we define $R(h)_{\underline{m}}$ by

(14) $R(h)_{\underline{m}}(t,u,s) = h_{\underline{m}*}(D^{\ell(\underline{m})}(d,v_{\underline{m}}(t)),u)$,

where $\underline{m}* = (m_0,m_0+1,m_0+2,\ldots,m_1,m_1+1,\ldots,m_n)$.

(ii) if $\underline{m}$ is degenerate, $\underline{m} = \underline{k}^j$ for some $\underline{k} \in N^{n-1}$ and some j, $0 \leq j \leq n-1$. Then we put

(15) $R(h)_{\underline{m}}(t,u,s) = R(h)_{\underline{k}}(\sigma_j^{n-1}(t),u,s)$, $t \in \Delta^n$.

Then by the definition of composition of coherent maps

$$R(h)(\partial_j^n \times 1) = R(h\partial_j^n) \text{ and } R(h)(\sigma_j^n \times 1) = R(h\sigma_j^n).$$

Moreover,

$$R(h)\ell_0 = h = 1_{S_i(\underline{X})}(h) \text{ and } R(h)\ell_1 = f_i(h)' = g_i f_i(h).$$

Hence by the same way as §3, the correspondence $R\colon S_n(\underline{X}) \longrightarrow S_{n+1}(\underline{X})$ induces a homotopy connecting $1_{S_c(\underline{X})}$ and gf.

Similarly we have the relative and the pointed versions of Theorem 7.1. We denote the Kan pairs of an inverse sequence $(\underline{X,x})$ of pointed spaces and an inverse sequence $(\underline{X,X_0})$ of pairs by $K(\underline{X,x})$ and $K(\underline{X,X_0})$, respectively.

7.2. Corollary. $f_*: H^c_*(\underline{X}:G) \longrightarrow H_*(K(\underline{X}):G)$ *for every abelian group* G.

7.3. Corollary. *The map* $|f|: |S_c(\underline{X})| \longrightarrow |K(\underline{X})|$ *is the homotopy equivalence with* $|g|$ *as its homotopy inverse.*

7.4. Corollary. *The canonical fundamental sequence* $\underline{\nu}_{(\underline{X},x)}: |K(\underline{X},x)| \longrightarrow (\underline{X},x)$ *induces isomorphisms*

$$(\underline{\nu}_{(\underline{X},x)})_\#: \pi_i(|K(\underline{X},x)|) \cong \overline{\pi}_i(\underline{X},x) \text{ for every } i \geq 0,$$

where $\overline{\pi}_i(\underline{X},x)$ *is the i-th strong homotopy group of* $(\underline{X},x)$ *defined by* Lisica [15].

Proof. Let $\underline{h}: (S^i,s_0) \longrightarrow (\underline{X},x)$ be a strong fundamental sequence and $h': (S^i,s_0) \longrightarrow (\underline{X},x)$ be its standard extension. Then by Theorem 5.2 there is a map $\Phi_{h'}: (S^i,s_0) \longrightarrow |S_c(\underline{X},x)|$ such that

$$(16) \quad \tau_{\underline{X}}\Phi_{h'} = h'.$$

Since $\underline{\nu}_{(\underline{X},x)}|f|$ is the strong fundamental sequence associated with $\tau_{(\underline{X},x)}$, the strong fundamental sequence $(\underline{\nu}_{(\underline{X},x)}|f|)\Phi_{h'} = \underline{\nu}_{(\underline{X},x)}(|f|\Phi_{h'})$ is associated with $\tau_{\underline{X}}\Phi_{h'}$. Hence by (16)

$$\underline{\nu}_{\underline{X}}(|f|\Phi_{h'}) = h.$$

Let $\alpha,\beta: (S^i,s_0) \longrightarrow |K(\underline{X},x)|$ be maps such that

(17) $\underline{\nu}_{(\underline{X},x)}\alpha \simeq \underline{\nu}_{(\underline{X},x)}\beta$.

We note that $\tau_{\underline{X}}|g|$ is the standard extension of $\underline{\nu}_{(\underline{X},x)}$. Hence $\tau_{\underline{X}}|g|\alpha$ and $\tau_{\underline{X}}|g|\beta$ are standard extensions of $\underline{\nu}_{(\underline{X},x)}\alpha$ and $\underline{\nu}_{(\underline{X},x)}\beta$, respectively. By (17) and Theorem 5.2,

(18) $|g|\alpha \simeq |g|\beta$.

Hence by Corollary 7.3 we have that

$$\alpha \simeq |f||g|\alpha \simeq |f||g|\beta \simeq \beta.$$

Therefore we have Corollary 7.4.

Next we introduce the Steenrod-Sitnikov homology of an inverse sequence $\underline{X} = (X_m, p_{mm+1})$ (c.f. [16] and [22]). For each $i \geq 0$, a s-s i-chain is a function x which assigns to every m a singular i-chain $x(m) \in C_i(X_m:G)$ and to every $(m,m+1)$ a singular $(i+1)$-chain $x(m,m+1) \in C_{i+1}(X_m:G)$. The set of those s-s i-chains is denoted by $C_i^{s-s}(\underline{X}:G)$. The boundary operator $d: C_{i+1}^{s-s}(\underline{X}:G) \longrightarrow C_i^{s-s}(\underline{X}:G)$ is defined by the formula;

(19) $(dx)(m) = \partial(x(m))$, and

(20) $(dx)(m,m+1) = p_{mm+1\#}(x(m+1)) - x(m) - \partial(x(m,m+1))$,

where ∂ denotes the boundary of singular chains. Then we define

(21) $H_i^{s-s}(\underline{X}:G) = H_i(C_{\#}^{s-s}(\underline{X}:G))$ for each $i \geq 0$.

For each $m \geq 1$, let α_m: $C_i^{s-s}(\underline{X}:G) \longrightarrow C_i(X_m:G)$ be the chain map given by $\alpha_m(x) = x(m)$. Then the family $\{\alpha_m\}$ induces a homomorphism α: $H_i^{s-s}(\underline{X}:G) \longrightarrow \varprojlim(H_i(X_m), p_{mm+1*})$. Concerning α we have the following.

7.5. Proposition. <u>α is an epimorphism and there is an isomorphism</u> β: $\varprojlim^1(H_{i+1}(X_m), p_{mm+1*}) \longrightarrow \mathrm{Ker}\,\alpha$. <u>That is, the following sequence is exact;</u>

$$0 \longrightarrow \varprojlim{}^1(H_{i+1}(X_m), p_{mm+1*}) \xrightarrow{\beta} H_i^{s-s}(\underline{X}:G)$$
$$\xrightarrow{\alpha} \varprojlim(H_i(X_m), p_{mm+1*}) \longrightarrow 0.$$

Each strong fundamental sequence $\underline{h}$: $\Delta^i \longrightarrow \underline{X}$ can be identified with the s-s i-chain of $\underline{X}$ by the formula;

$\underline{h}(m) = h_m$, and $\underline{h}(m,m+1) = h_{mm+1}$.

Then we have a natural homomorphism $\zeta_{\underline{X}}^i$: $H_i(K(\underline{X})) \longrightarrow H_i^{s-s}(\underline{X}:G)$.

On the other hand, each strong i-chain of $\underline{X}$ can be considered as a s-s i-chain of $\underline{X}$. Hence there is the natural homomorphism $\theta_{\underline{X}}^i$: $H_i^s(\underline{X}:G) \longrightarrow H_i^{s-s}(\underline{X}:G)$. By the definitions

$\theta_{\underline{X}}^i \xi_{\underline{X}}^i = \zeta_{\underline{X}}^i f_*$.

$$\begin{tikzcd}
H_i(S_c(\underline{X}):G) \arrow[r, "f_*"] \arrow[d, "\xi^i_{\underline{X}}"'] & H_i(K(\underline{X}):G) \arrow[d, "\zeta^i_{\underline{X}}"] \\
H^s_i(\underline{X}:G) \arrow[r, "\theta^i_{\underline{X}}"'] & H^{s-s}_i(\underline{X}:G)
\end{tikzcd}$$

By [22], §8, $\theta^i_{\underline{X}}$ is an isomorphism. Hence by Corollary 7.2, if $\zeta^i_{\underline{X}}$ is not surjective, the $\xi^i_{\underline{X}}$ is also not surjective. Next we will show that there is an inverse system $\underline{X}$ of 1-dimensional compact polyhedra such that $\zeta^1_{\underline{X}}$ is not surjective.

7.6. Example. Let $\{S_i\}$ be a collection of pairwise disjoint copies of the 1-sphere S^1. Let x_0 be a point which does not belong to $\bigcup_{i \geq 1} S_i$. For each $m \geq 1$, put

(22) $X_m = \{x_0\} \cup S_1 \cup \dots \cup S_m$,

and define the map $p_{mm+1}: X_{m+1} \longrightarrow X_m$ by

(23) $p_{mm+1} \mid X_m = 1_{X_m}$ and $p_{mm+1}(S_{m+1}) = \{x_0\}$.

We will show that the inverse system $\underline{X} = (X_m, p_{mm+1})$ has the required property. We note that $\underline{X}$ is movable, and

(24) $\varprojlim(H_1(X_m), p_{mm+1*}) = \prod_{m \geq 1} H_1(S_m)$,

for $H_1(X_m) = H_1(S_1) \times \dots \times H_1(S_m)$ for each $m \geq 1$.

Assume that $\zeta^1_{\underline{X}}$ is surjective. Let $z = (z_m) \in \prod_{m \geq 1} H_1(S_m)$ be an element such that $z_m \neq 0$ for all $m \geq 1$. Then by Proposition 7.5 there is an element $x \in H_1(K(\underline{X}))$ such that

(25) $\alpha\zeta^1_{\underline{X}}(x) = z.$

That is,

(26) $\alpha_m\zeta^1_{\underline{X}}(x) = (z_1,\ldots,z_m)$ for every $m \geq 1$.

Take integers a_i, $1 \leq i \leq n$, and $\underline{h}^i \in K_1(\underline{X})$, $1 \leq i \leq n$, such that

(27) $a_1\underline{h}^1+\ldots+a_n\underline{h}^n \in Z_1(C_\#(K(\underline{X})))$ represents x in $H_1(K(\underline{X}))$.

Then

(28) $\alpha_{n+1}\zeta^1_{\underline{X}}(x) = [a_1h^1_{n+1}+\ldots+a_nh^n_{n+1}]$,

where $[h]$ is the homology class of $h \in Z_1(X_{n+1})$.
In fact, since Δ^1 is connected, for each i, $1 \leq i \leq n$, there is $k(i) \in \{1,2,\ldots,n+1\}$ such that

(29) $h^i_{n+1} \in Z_1(S_{k(i)})$.

By (28) and (29), we have that

(30) $\alpha_{n+1}\zeta^1_{\underline{X}}(x) \in \prod_{i=1}^{n} H_1(S_{k(i)}) \subsetneq \prod_{j=1}^{n+1} H_1(S_j) = H_1(X_{n+1})$.

But it contradicts the assumption that $z_m \neq 0$ for all $m \geq 1$.

It follows that $\zeta^1_{\underline{X}}$ is not surjective. Note that $H_1(K(\underline{X})) = \oplus Z$ and $H_1^{s-s}(\underline{X}) = \Pi Z$.

Finally we will consider a condition under which $\zeta^i_{\underline{X}}$ is an isomorphism.

7.7. Theorem. <u>Let</u> $(\underline{X,x})$ <u>be an inverse sequence of pointed compact polyhedra</u>. <u>If</u> $\bar{\pi}_k(\underline{X,x}) = 0$ <u>for every</u> k, $0 \leq k \leq i-1$, <u>where</u> $i \geq 2$, <u>then</u>

$$\zeta^i_{\underline{X}}: H_i(K(\underline{X})) \cong H_i^{s-s}(\underline{X}).$$

Proof. We may assume that X_1 is a singleton. The following sequare is commutative.

$$\begin{array}{ccc}
\pi_i(|K(\underline{X,x})|) & \xrightarrow{\phi^i_{|K(\underline{X,x})|}} & H_i(K(\underline{X})) \\
\Big\downarrow{\scriptstyle (\underline{\nu}_{(\underline{X,x})})_\#} & & \Big\downarrow{\scriptstyle \zeta^i_{\underline{X}}} \\
\bar{\pi}_i(\underline{X,x}) & \xrightarrow{\psi^i_{(\underline{X,x})}} & H_i^{s-s}(\underline{X})
\end{array},$$

where $\phi^i_{|K(\underline{X,x})|}$ is the Hurewicz homomorphism of $|K(\underline{X,x})|$ and $\psi^i_{(\underline{X,x})}$ is the homomorphism defined in [12]. Since $\bar{\pi}_k(\underline{X,x}) = 0$ for every k, $0 \leq k \leq i-1$, by Corollary 7.4, the usual Hurewicz isomorphism theorem and [12], Corollary 3, both $\phi^i_{|K(\underline{X,x})|}$ and $\psi^i_{(\underline{X,x})}$ are isomorphisms. Therefore $\zeta^i_{\underline{X}}$ is an isomorphism.

7.8. Remark. By [33] and [14], the condition $\bar{\pi}_k(\underline{X,x}) = 0$ for every k, $0 \leq k \leq i-1$, is equivalent to the condition that $(\underline{X,x})$ is approximatively (i-1)-connected and pointed S^i-movable. Hence we may call [12], Corollary 3 the <u>Hurewicz isomorphism theorem in strong shape theory</u>.

7.9. Remark. Our definitions of $K(\underline{X})$ and H_*^{s-s} are slight generalizations of [1] and [30]. But our method may be more useful in order to generalize the construction to more general spaces and investigate its algebraic properties.

Problem 1'. <u>What condition of $\underline{X}$ implies that $\zeta^i_{\underline{X}}$ is an isomorphism?</u>

8. The CW-complex $E(\underline{X,x})$.

In this section we assume that $(\underline{X,x}) = ((X_m, x_m), p_{mm+1})$ is an inverse sequence of pointed arcwise connected spaces. Then by the way of Edwards and Geoghegan [9], we construct a pointed CW-complex $E(\underline{X,x})$ and a strong fundamental sequence $\rho_{(\underline{X,x})}$: $E(\underline{X,x}) \longrightarrow (\underline{X,x})$ as follows;

By [9], Lemma 2.2, we have the following diagram;

$$\begin{tikzcd}
(X_1,x_1) \arrow[d, equal, "t_1"] & (X_2,x_2) \arrow[l, "p_{1,2}"'] \arrow[d, "t_2"] & (X_3,x_3) \arrow[l, "p_{2,3}"'] \arrow[d, "t_3"] & \cdots \arrow[l, "p_{3,4}"'] \\
(Y_1,y_1)=(X_1,x_1) & (Y_2,y_2) \arrow[l, "q_{1,2}"'] & (Y_3,y_3) \arrow[l, "q_{2,3}"'] & \cdots \arrow[l, "q_{3,4}"']
\end{tikzcd}$$

such that for each $m=1,2,\ldots,$

(1) $t_m p_{mm+1} = q_{mm+1} t_{m+1}$,

(2) t_m is a homotopy equivalence, and

(3) q_{mm+1} is a fibration (see [32], Theorem 2.8.9).

For each $m \geq 1$, let u_m: $(Y_m, y_m) \longrightarrow (X_m, x_m)$ be a homotopy inverse of t_m. Then by (1), there is a homotopy u_{mm+1}: $I \times (Y_{m+1}, y_{m+1}) \longrightarrow (X_m, x_m)$ such that

(4) u_{mm+1}: $u_m q_{mm+1} \simeq p_{mm+1} u_{m+1}$.

The collections $\{u_m\}$ and $\{u_{mm+1}\}$ induce the strong fundamental sequence $\underline{u}$: $(\underline{Y,y}) = ((Y_m,y_m),q_{mm+1}) \longrightarrow (\underline{X,x})$.

Let S: TOP $\longrightarrow$ KAN and $|\cdot|$: KAN $\longrightarrow$ CW be the usual singular-complex and geometric realization functors (see [27]). Then we have the inverse sequence $S(\underline{Y,y}) = (S(Y_m,y_m),S(q_{mm+1}))$ and $|S(\underline{Y,y})| = (|S(Y_m,y_m)|,|S(q_{mm+1})|)$, and the strong fundamental sequence $\underline{w}$: $|S(\underline{Y,y})| \longrightarrow (\underline{Y,y})$, which is induced by canonical maps w_m: $|S(Y_m,y_m)| \longrightarrow (Y_m,y_m)$. Let $\underline{\tilde{q}} = \{\underline{\tilde{q}}_m\}$: $\varprojlim S(\underline{Y,y}) \longrightarrow S(\underline{Y,y})$ be the projection.

Now we define

(5) $E(\underline{X,x}) = |\varprojlim S(\underline{Y,y})|$, and

(6) $\rho(\underline{X,x}) = \underline{uw}|\underline{\tilde{q}}|$: $E(\underline{X,x}) \longrightarrow (\underline{X,x})$, where $|\underline{\tilde{q}}| = \{|\underline{\tilde{q}}_m|\}$.

In [9], Edwards and Geoghegan proved that, if $(\underline{X,x})$ is dominated by a pointed CW-complex and each (X_m,x_m) has a homotopy type of a pointed CW-complex, then $\rho_{(\underline{X,x})}$ induces an isomorphism in pro-HTOP_0. In this section, without an additional assumption of $(\underline{X,x})$, we shall show the following property of $\rho_{(\underline{X,x})}$.

8.1. Theorem. $(\rho_{(\underline{X,x})})_{\#}$: $\pi_i(E(\underline{X,x})) \cong \bar{\pi}_i(\underline{X,x})$ for all $i \geq 0$.

The other pointed CW-complex and strong fundamental sequence having the same property were obtained in Corollary 7.4

by a quite different way. But $\rho_{(\underline{X},\underline{x})}$ is more constructive than $\underline{\nu}_{(\underline{X},\underline{x})}$, and may be effective for calculating $\bar{\pi}_i(\underline{X},\underline{x})$. A comparison of the two constructions will be discussed in the next section. The key tools of the proof of Theorem 8.1 are the following two lemmas.

8.2. Lemma ([6]). Let $(\underline{Z},\underline{z}) = ((Z_m,z_m),r_{mm+1})$ be an inverse sequence such that every r_{mm+1} is a Serre fibration. Then there is the following short exact sequence;

$$* \longrightarrow \varprojlim{}^1(\pi_{n+1}(Z_m,z_m)r_{mm+1\#}) \xrightarrow{\beta} \pi_n(\varprojlim(\underline{Z},\underline{z}))$$

$$\xrightarrow{\alpha} \varprojlim(\pi_n(Z_m,z_m),r_{mm+1\#}) \longrightarrow * .$$

In particular, in the case $n = 0$, $\beta : \varprojlim{}^1(\pi_1(Z_m,z_m),r_{mm+1\#}) \cong \operatorname{Ker}\theta$.

8.3. Lemma ([33]). There is the following short exact sequence;

$$* \longrightarrow \varprojlim{}^1(\pi_{n+1}(X_m,x_m),p_{mm+1\#}) \xrightarrow{\Xi} \bar{\pi}_n(\underline{X},\underline{x})$$

$$\xrightarrow{\theta} \varprojlim(\pi_n(X_m,x_m),p_{mm+1\#}) \longrightarrow * .$$

In particular, in the case $n = 0$, $\Xi : \varprojlim{}^1(\pi_1(X_m,x_m),p_{mm+1\#}) \cong \operatorname{Ker}\theta$.

Concerning the relation between exact sequences of Lemma 8.2 and Lemma 8.3 we have the next result.

8.4. Lemma. *Let* $(\underline{Z},\underline{z}) = ((Z_m, z_m) r_{mm+1})$ *be an inverse sequence such that every* r_{mm+1} *is a Serre fibration. Then the following diagram is commutative;*

$$\begin{tikzcd}
* \arrow[r] & \varprojlim^1(\pi_{n+1}(Z_m, z_m)) \arrow[r, "\beta"] \arrow[d, equal] & \pi_n(\varprojlim(\underline{Z},\underline{z})) \arrow[r, "\alpha"] \arrow[d, "\underline{r}_{\#}"] & \varprojlim(\pi_n(Z_m, z_m)) \arrow[r] \arrow[d, equal] & * \\
* \arrow[r] & \varprojlim^1(\pi_{n+1}(Z_m, z_m)) \arrow[r, "\Xi"] & \bar{\pi}_n(\underline{Z},\underline{z}) \arrow[r, "\theta"] & \varprojlim(\pi_n(Z_m, z_m)) \arrow[r] & *
\end{tikzcd},$$

where $\underline{r}$ *is the strong fundamental sequence induced by the projections* $r_m\colon \varprojlim(\underline{Z},\underline{z}) \longrightarrow (Z_m, z_m)$, $m \geq 1$.

Proof. Since $\theta\,\underline{r}_{\#} = \alpha$ clearly holds, we will show only the equation $\underline{r}_{\#}\beta = \Xi$. Identify S^{n+1} with $S^n \times I/S^n \times \{0,1\} \cup \{s_0\} \times I$, where s_0 is the base point of S^n. Let $([f_m])$ be a given element of $\prod_{m=1}^{\infty} \pi_{n+1}(Z_m, z_m)$. For each $m \geq 1$, put

(7) $g_m = f_m \mid S^n \times \{1/2\}\colon (S^n, s_0) \longrightarrow (Z_m, z_m)$

(8) $G_m = (f_m \mid S^n \times [1/2, 1]) * (r_{mm+1} f_{m+1} \mid S^n \times [0, 1/2])\colon$
$(S^n, s_0) \times I \longrightarrow (Z_m, z_m)$.

Then G_m: $g_m \simeq r_{mm+1} g_{m+1}$ rel. s_0 for every $m \geq 1$. Now put

(9) $\hat{g}_1 = g_1$, and

(10) $\hat{G}_1$: $(S^n, s_0) \times I \longrightarrow (Z_1, z_1)$ by $\hat{G}_1(x,t) = g_1(X)$.

Assume that we have already defined maps $\hat{g}_k$: $(S^n, s_0) \longrightarrow (Z_k, z_k)$, $\hat{G}_k$: $(S^n, s_0) \times I \longrightarrow (Z_k, z_k)$ and $\hat{G}_{k-1,k}$: $(S^n, s_0) \times I \times I \longrightarrow (Z_{k-1}, z_{k-1})$ for all $k \leq i$, which satisfy the followings;

$(i)_k$ $\hat{G}_k$: $\hat{g}_k \simeq g_k$ rel. s_0, and

$(ii)_{k-1}$ $\hat{G}_{k-1,k}$: $\hat{G}_{k-1} \simeq r_{k,k-1}\hat{G}_{k-1}$,

$\hat{G}_{k-1,k}(x,0,t) = \hat{g}_{k-1}(t)$ and $\hat{G}_{k-1,k}(x,1,t) = \hat{G}_{k-1}(x,t)$.

Note that $(i)_1$ holds. Since $\hat{G}_i * G_i$: $\hat{g}_i \simeq r_{ii+1} g_{i+1}$ rel. s_0 and r_{ii+1} is a Serre fibration, there is a homotopy $\hat{G}_{i+1}$: $(S^n, s_0) \times I \longrightarrow (Z_{i+1}, z_{i+1})$ such that

(11) $r_{ii+1}\hat{G}_{i+1} = \hat{G}_i * G_i$,

(12) $\hat{G}_{i+1} \mid S^n \times \{1\} = g_{i+1}$.

Now we define the map $\hat{g}_{i+1}$: $(S^n, s_0) \longrightarrow (Z_{i+1}, z_{i+1})$ by

(13) $\hat{g}_{i+1} = \hat{G}_{i+1} \mid S^n \times \{0\}$.

Moreover it is easily see that there is a homotopy $\hat{G}_{i,i+1}$: $(S^n, s_0) \times I \times I \longrightarrow (Z_i, z_i)$ satisfying the condition $(ii)_i$.

Hence we have maps $\hat{g}_i$, $\hat{G}_i$ and $\hat{G}_{i,i+1}$ for all $i \geq 1$, satisfying conditions $(i)_i$ and $(ii)_i$.

Thus we have strong fundamental sequences $\underline{g} = (g_m, G_m)$ and $\hat{\underline{g}} = (\hat{g}_m, \hat{G}_{m,m+1} \mid S^n \times \{0\} \times I)\colon (S^n, s_0) \longrightarrow (\underline{Z,z})$ such that $\underline{g} \simeq \hat{\underline{g}}$ and $r_{ii+1}\hat{g}_{i+1} = \hat{g}_i$ for every $i \geq 1$.
Then by definitions

$$(14)\quad \beta(\{([f_m])\}) = [\varprojlim\{\hat{g}_m\}],$$

$$(15)\quad \Xi(\{([f_m])\}) = [\underline{g}] = [\hat{\underline{g}}], \text{ where } \{([f_m])\}$$

is the equivalence class of $([f_m])$ in $\varprojlim^1(\pi_{n+1}(z_m, z_m), p_{mm+1\#})$.
Hence

$$\underline{r}_{\#}\beta(\{([f_m])\}) = \underline{r}_{\#}([\varprojlim\{\hat{g}_m\}]) = [(\hat{g}_m, G'_m)] = [\hat{\underline{g}}] = \Xi(\{([f_m])\}),$$

where $G'_m\colon (S^n, s_0) \times I \longrightarrow (Z_m, z_m)$ is the homotopy given by $G'_m(X,t) = \hat{g}_m(x)$. Therefore $\underline{r}_{\#}\beta = \Xi$.

Proof of Theorem 8.1. By Lemma 8.3 and the five Lemma, we have

$$(16)\quad \underline{u}_{\#}\colon \bar{\pi}_n(\underline{Y,y}) \cong \bar{\pi}_n(\underline{X,x}) \quad \text{for every } n \geq 0, \text{ and}$$

$$(17)\quad \underline{w}_{\#}\colon \bar{\pi}_n(|S(\underline{Y,y})|) \cong \bar{\pi}_n(\underline{Y,y}) \quad \text{for every } n \geq 0.$$

Let $\underline{r} = \{r_m\}\colon \varprojlim |S(\underline{Y,y})| \longrightarrow |S(\underline{Y,y})|$ be the projection. Since $|S(q_{mm+1})||\tilde{q}_{m+1}| = |\tilde{q}_m|$ for every $m \geq 1$, there is the map $s\colon |\varprojlim S(\underline{Y,y})| \longrightarrow \varprojlim |S(\underline{Y,y})|$ such that

(18) $r_m s = |\tilde{q}_m|$ for every $m \geq 1$.

For each $n \geq 0$, we consider the following diagram, where the third row is the Cohen's exact sequence in KAN_2 (see [3], Theorem IX.3.1), and Ψ and Ψ_m are natural isomorphisms defined by [27], Lemma 16.3.

See the diagram in the next page. Then by Lemma 8.4, the upper squares of the diagram are commutative. Hence

(19) $\underline{r}_{\#}: \pi_n(\varprojlim |S(\underline{Y,y})|)) \cong \overline{\pi}_n(\underline{Y,y})$ for every $n \geq 0$.

Since $\underline{r}_{\#}\beta(\varprojlim{}^1\Psi_m) = |\tilde{\underline{q}}|_{\#}\Psi\tilde{\beta} = \underline{r}_{\#}s_{\#}\Psi\tilde{\beta}$, by (19), $\beta(\varprojlim{}^1\Psi_m) = s_{\#}\Psi\tilde{\beta}$. By (18), $\alpha s_{\#}\Psi = (\varprojlim \Psi_m)\tilde{\alpha}$. Hence we have that

(20) $s_{\#}\Psi: \pi_n(\varprojlim S(\underline{Y,y})) \cong \pi_n(\varprojlim |S(\underline{Y,y})|)$ for every $n \geq 0$,

Since $\Psi: \pi_n(\varprojlim S(\underline{Y,y})) \cong \pi_n(|\varprojlim S(\underline{Y,y})|)$ for every $n \geq 0$, by (18), (19) and (20),

(21) $|\tilde{\underline{q}}|_{\#}: \pi_n(|\varprojlim S(\underline{Y,y})| \cong \overline{\pi}_n(|S(\underline{Y,y})|)$ for every $n \geq 0$.

Therefore by (16), (17), (21) and (6) we have shown that $\rho_{(\underline{X,x})}$ satisfies the desired property.

$$
\begin{tikzcd}
* \arrow[r] & \varprojlim^1(\pi_{n+1}(|S(Y_m,y_m)|)) \arrow[r, "\Xi"] \arrow[d, equal] & \bar{\pi}_n(|S(\underline{Y},\underline{y})|) \arrow[r, "\theta"] & \varprojlim(\pi_n(|S(Y_m,y_m)|)) \arrow[r] \arrow[d, equal] & * \\
* \arrow[r] & \varprojlim^1(\pi_{n+1}(|S(Y_m,y_m)|)) \arrow[r, "\beta"] & \pi_n(\varprojlim|S(\underline{Y},\underline{y})|) \arrow[r, "\alpha"] \arrow[u, "\underline{r}_\#"'] & \varprojlim(\pi_n(|S(Y_m,y_m)|)) \arrow[r] & * \\
 & & \pi_n(|\varprojlim S(\underline{Y},\underline{y})|) \arrow[u, "s_\#"'] \arrow[uu, bend left=60, "|\underline{\tilde{q}}|_\#"] & & \\
* \arrow[r] & \varprojlim^1(\pi_{n+1}(S(Y_m,y_m))) \arrow[r, "\tilde{\beta}"] \arrow[uu, "\varprojlim^1\Psi_m"] & \pi_n(\varprojlim S(\underline{Y},\underline{y})) \arrow[r, "\tilde{\alpha}"] \arrow[u, "\Psi"'] & \varprojlim(\pi_n(S(Y_m,y_m))) \arrow[r] \arrow[uu, "\varprojlim \Psi_m"'] & *
\end{tikzcd}
$$

9. The comparison of $E(\underline{X,x})$ with $|K(\underline{X,x})|$.

In this section we will consider only an inverse sequence $(\underline{X,x}) = ((X_m, x_m), p_{mm+1})$ of pointed arcwise connected spaces. The purpose of this section is to define a weak homotopy equivalence $f: E(\underline{X,x}) \longrightarrow |K(\underline{X,x})|$.

For each $i \geq 0$, every element $x \in (\underleftarrow{\lim}\, S(\underline{Y,y}))_i$ is a collection of maps $x_m: \Delta^i \longrightarrow Y_m$ such that $q_{mm+1}x_{m+1} = x_m$ for every $m \geq 1$. Hence we may consider x as a strong fundamental sequence $\underline{x}: \Delta^i \longrightarrow \underline{Y}$. That is, the correspondence induces a function $G_i: (\underleftarrow{\lim}\, S(\underline{Y,y}))_i \longrightarrow K(\underline{Y})_i$. Then it is clear that

$$G_i d_k^{i+1} = d_k^i G_{i+1} \quad \text{and} \quad G_{i+1} s_k^i = s_k^i G_i \quad \text{for every } k,\ 0 \leq k \leq i.$$

Hence we have a semi-simplicial map

$$G = \{G_i\}: \underleftarrow{\lim}\, S(\underline{Y,y}) \longrightarrow K(\underline{Y}).$$

Let c^i be an element of $(\underleftarrow{\lim}\, S(\underline{Y,y}))_i$ such that $c_m^i(\Delta^i) = \{y_m\}$ for every $m \geq 1$. Then by definition, $G_i(c^i) = \underline{c}_i \in K(\underline{Y})_i$ for every $i \geq 0$. Hence G is a semi-simplicial map from $\underleftarrow{\lim}\, S(\underline{Y,y})$ to $K(\underline{Y,y})$. Therefore G induces a map

$$(1) \quad g = |G|: E(\underline{X,x}) = |\underleftarrow{\lim}\, S(\underline{Y,y})| \longrightarrow |K(\underline{Y,y})|.$$

9.1. Lemma. $\underline{\nu}_{(\underline{Y,y})}g = \underline{w}|\underline{\tilde{q}}|$.

$$\begin{tikzcd}[column sep=huge] E(\underline{X},\underline{x}) = |\varprojlim S(\underline{Y},\underline{y})| \arrow[r, "g"] \arrow[d, "|\underline{\tilde{q}}|"'] & |K(\underline{Y},\underline{y})| \arrow[d, "\underline{\nu}(\underline{Y},\underline{y})"] \\ |S(\underline{Y},\underline{y})| \arrow[r, "\underline{w}"'] & (\underline{Y},\underline{y}) \end{tikzcd}$$

Proof. For any $x \in (\varprojlim S(Y,y))_i$, $i \geq 0$ and any $t \in \Delta^i$, $n \geq 0$,

$(\underline{\nu}_{(\underline{Y},\underline{y})}g)_n(|x,t|) = \nu_n(|G_i(x),t|) = (G_i(X))_n(t) = x_n(t)$, and

$(\underline{w}|\underline{\tilde{q}}|)_n(|x,t|) = w_n(|\tilde{q}_n(x),t|) = w_n(|x_n,t|) = x_n(t)$.

Hence $(\underline{\nu}_{(\underline{Y},\underline{y})}g)_n = (\underline{w}|\underline{\tilde{q}}|)_n$ for every $n \geq 1$.

Similarly we can see that $(\underline{\nu}_{(\underline{Y},\underline{y})}g)_{nn+1} = (\underline{w}|\underline{\tilde{q}}|)_{nn+1}$ for every $n \geq 1$. Therefore $\underline{\nu}_{(\underline{Y},\underline{y})}g = \underline{w}|\underline{\tilde{q}}|$.

We define a function $K(\underline{u})\colon K(\underline{Y},\underline{y}) \longrightarrow K(\underline{X},\underline{x})$ by

(2) $K(\underline{u})_i(\underline{h}) = \underline{u}\underline{h}$ for every $\underline{h} \in K(\underline{Y})_i$ and every $i \geq 0$.

Then it is easily seen that $K(\underline{u})$ is a semi-simplicial map and $\underline{\nu}_{(\underline{X},\underline{x})}|K(\underline{u})| = \underline{u}\underline{\nu}_{(\underline{Y},\underline{y})}$. Therefore, defining the map

(3) $f = |K(\underline{u})|g\colon E(\underline{X},\underline{x}) \longrightarrow |K(\underline{X},\underline{x})|$,

by Corollary 7.4, Lemma 9.1 and the proof of Theorem 8.1 we have the following.

9.2. Theorem. *The map* $f\colon E(\underline{X},\underline{x}) \longrightarrow |K(\underline{X},\underline{x})|$ *is a weak homotopy equivalence.*

$$\begin{array}{ccccc}
E(\underline{X},\underline{x}) & \xrightarrow{\ g\ } & |K(\underline{Y},\underline{y})| & \xrightarrow{\ |K(\underline{u})|\ } & |K(\underline{X},\underline{x})| \\
\downarrow{\scriptstyle |\tilde{\underline{q}}|} & & \downarrow{\scriptstyle \underline{\nu}(\underline{Y},\underline{y})} & & \downarrow{\scriptstyle \underline{\nu}(\underline{X},\underline{x})} \\
|S(\underline{Y},\underline{y})| & \xrightarrow[\ \underline{w}\ ]{} & (\underline{Y},\underline{y}) & \xrightarrow[\ \underline{u}\ ]{} & (\underline{X},\underline{x})
\end{array}$$

9.3. Corollary. *If* $\bar{\pi}_0(\underline{X},\underline{x}) = 0$, *then the map* $f\colon E(\underline{X},\underline{x}) \longrightarrow |K(\underline{X},\underline{x})|$ *is a homotopy equivalence.*

By Corollary 9.3 and [14], Corollary, the following is obtained.

9.4. Corollary. *Let* $(\underline{X},\underline{x}) = ((X_m,x_m),p_{mm+1})$ *be an inverse sequence of pointed compact connected polyhedra. If* $(\underline{X},\underline{x})$ *is pointed 1-movable, then the map* $f\colon E(\underline{X},\underline{x}) \longrightarrow |K(\underline{X},\underline{x})|$ *is a homotopy equivalence.*

Problem 2. *For every inverse sequence* $(\underline{X},\underline{x}) = ((X_m,x_m), p_{mm+1})$, *is the map* $f\colon E(\underline{X},\underline{x}) \longrightarrow |K(\underline{X},\underline{x})|$ *a homotopy equivalence?*

10. Summary in strong shape theory.

In [25], Mardešić defined resolutions of pairs of spaces. A system map $\underline{p} = \{p_\lambda\}: (X,A) \longrightarrow (\underline{X,A}) = ((X_\lambda, A_\lambda), p_{\lambda\lambda'}, \Lambda)$ is a <u>resolution of the pair</u> (X,A) provided that the following conditions are satisfied for any ANR-pair (P,Q), that is, a pair of ANR's such that Q is a closed subset of P, and for any open covering V of P;

(R1) for every map $f: (X,A) \longrightarrow (P,Q)$, there are $\lambda \in \Lambda$ and a map $g: (X_\lambda, A_\lambda) \longrightarrow (P,Q)$ such that gp_λ and f are V-near maps,

(R2) there exists an open covering V' of P such that whenever $\lambda \in \Lambda$ and $g, g': (X_\lambda, A_\lambda) \longrightarrow (P,Q)$ are maps such that the maps gp_λ and $g'p_\lambda$ are V'-near, then there exists $\lambda' \geq \lambda$ in Λ such that $gp_{\lambda\lambda'}$ and $g'p_{\lambda\lambda'}$ are V-near maps.

If all (X_λ, A_λ) are ANR-pairs, $\underline{p}$ is called an ANR-<u>resolution</u> of (X,A).

If A, A_λ and Q are all empty sets or singletons, from the above definition, we have the definitions of (ANR-) resolutions $\underline{p}: X \longrightarrow \underline{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ of a single space X or $\underline{p}: (X,a) \longrightarrow ((X_\lambda, a_\lambda), p_{\lambda\lambda'}, \Lambda)$ of a pointed space (X,a), respectively (c.f. [24]).

In [19], Lisica and Mardešić defined a strong shape category SSH whose objects are all spaces. Morphisms $F: X \longrightarrow Y$ are given by triples $(\underline{p}, \underline{q}, [f])$, where $\underline{p}$ and $\underline{q}$ are

ANR-resolutions of X and Y, respectively, $[f]$ is a morphism in CPHTOP. Two triples $(\underline{p},\underline{q},[f])$ and $(\underline{p}',\underline{q}',[f'])$ are <u>equivalent</u> if

(1) $[f][i] = [j][f']$,

where $[i]: \underline{X} \longrightarrow \underline{X}'$ and $[j]: \underline{Y} \longrightarrow \underline{Y}'$ are unique morphisms in CPHTOP such that $[i][\underline{p}] = [\underline{p}']$ and $[j][\underline{q}] = [\underline{q}']$. We define F the equivalence class of $(\underline{p},\underline{q},[f])$.

Let F and G be morphisms in SSH given by triples $(\underline{p},\underline{q},[f])$ and $(\underline{q}',\underline{r},[g])$, respectively. Then the composition GF is given by the triple $(\underline{p},\underline{r},[gjf])$, where $[j]$ is the unique morphism in CPHTOP such that $[j][\underline{q}] = [\underline{q}']$. Note that we may assume that $\underline{q} = \underline{q}'$.

The identity morphism on X is defined by $(\underline{p},\underline{p},[1_{\underline{X}}])$.

Lisica and Mardešić [21] investigated $CPHTOP_2$, and defined a strong shape category of pairs by using ANR-resolution of pairs. In this paper, although we use their results, we leave the details.

In this section we will summarize our results in strong shape theory. First, by §3 and §4 we have the followings.

10.1. Theorem. <u>If a space</u> X <u>is dominated by a</u> CW-<u>complex in</u> SSH, <u>then</u> X <u>is equivalent to the</u> CW-<u>complex</u> $|S_c(\underline{X})|$ <u>in</u> SSH, <u>where</u> $\underline{p}$: $X \longrightarrow \underline{X}$ <u>is an</u> ANR-<u>resolution of</u> X.

10.2. Corollary. The following are equivalent conditions;

(a) a space X is dominated by a CW-complex in SSH,

(b) X is equivalent to a CW-complex in SSH,

(c) X is equivalent to a simplicial complex in SSH,

(d) X is equivalent to an ANR in SSH.

For a pointed space (X,x) we define the strong shape group $\pi_i^s(X,x)$, $i \geq 0$, by

(2) $\pi_i^s(X,x) = \pi_i^c(\underline{X,x})$,

where $\underline{p}$: $(X,x) \longrightarrow (\underline{X,x})$ is an ANR-resolution of (X,x). The morphism F: $(X,x) \longrightarrow (Y,y)$ given by a triple $(\underline{p},\underline{q},[\underline{f}])$ defines the homomorphism $F_\#$: $\pi_i^s(X,x) \longrightarrow \pi_i^s(Y,y)$ by

(3) $F_\# = \underline{f}_\#$: $\pi_i^c(\underline{X,x}) \longrightarrow \pi_i^c(\underline{Y,y})$.

$F_\#$ is called the homomorphism induced by F.

Then by §5, π_i^s is a functor from SSH_0 to GR. Similarly, by using ANR-resolutions of pairs, we can define the relative strong shape group $\pi_i^s(X,A,x)$. If A is P-embedded in X, by [25], Theorem 3, there exists an ANR-resolution $\underline{p}$: $(X,A,x) \longrightarrow (\underline{X,A,x})$ such that $\underline{p} \mid (A,x)$: $(A,x) \longrightarrow (\underline{A,x})$ is an ANR-resolution of (A,x). Hence if A is P-embedded in X, the following sequence is exact;

$$\cdots \longrightarrow \pi^s_{i+1}(X,A,x) \xrightarrow{\partial} \pi^s_i(A,x) \xrightarrow{i_\#} \pi^s_i(X,x)$$

$$\xrightarrow{j_\#} \pi^s_i(X,A,x) \longrightarrow \cdots ,$$

where $i\colon (A,x) \longrightarrow (X,x)$ and $j\colon (X,x) \longrightarrow (X,A,x)$ are inclusion maps.

Let $\underline{p}\colon (X,x) \longrightarrow (\underline{X,x})$ be an ANR-resolution of (X,x). Then we call the strong shape morphism given by the triple $(1,\underline{p},[\tau_{(\underline{X,x})}])$) the <u>canonical strong shape morphism</u>, and denote by $\tau_{(X,x)}\colon |S_c(X,x)| \longrightarrow (X,x)$. Similarly, we can define the canonical strong shape morphisms of an absolute space or a pair of spaces. By Theorem 5.2 the next theorem is obtained.

10.3. Theorem. <u>The canonical strong shape morphism of a pointed space</u> (X,x) <u>induces isomorphisms</u>;

$$\tau_{(X,x)\#}\colon \pi_i(|S_c(X,x)|) \cong \pi^s_i(X,x) \quad \text{for all } i \geq 0.$$

We note that, if (X,x) is a pointed compactum, the strong shape group $\pi^s_i(X,x)$ is naturally isomorphic to the approching group $\underline{\underline{\pi}}_i(X,x)$ defined by Quigley [31].

For each space X we define the <u>coherent singular homology group of</u> X by

(4) $H^c_i(X,G) = H^c_i(\underline{X}:G)$ for an abelian group G,

where $\underline{p}$: $X \longrightarrow \underline{X}$ is an ANR-resolution of X.

The morphism $F: X \longrightarrow Y$ given by a triple $(\underline{p},\underline{q},[\underline{f}])$ admits the homomorphism $F_*: H_i^c(X:G) \longrightarrow H_i^c(Y:G)$ defined by

(5) $F_* = f_*: H_i^c(\underline{X}:G) \longrightarrow H_i^c(\underline{Y}:G)$.

We call F_* the <u>homomorphism</u> <u>induced</u> by F. Then by §6, H_i^c is a functor from SSH to GR.

Similarly we can define the relative coherent singular homology group $H_*^c(X,A:G)$ of a pair (X,A) of spaces. Then if A is P-embedded in X, the following sequence is clearly exact;

$$\cdots \longrightarrow H_{i+1}^c(X,A:G) \xrightarrow{\partial} H_i^c(A:G) \xrightarrow{i_*} H_i^c(X:G)$$

$$\xrightarrow{j_*} H_i^c(X,A:G) \longrightarrow \cdots .$$

Moreover by Theorem 6.1 we have the following.

10.4. Theorem. <u>The canonical strong shape morphism of a pair</u> (X,A) <u>induces isomorphisms</u>;

$$\tau_{(X,A)*}: H_i(|S_c(X,A)|:G) \cong H_i^c(X,A:G) \quad \text{for all } i \geq 0.$$

Let $\underline{p}$: $(X,x) \longrightarrow (\underline{X},\underline{x})$ be an ANR-resolution of a pointed space (X,x). The <u>Hurewicz homomorphism</u> $\Phi_i: \pi_i^s(X,x) \longrightarrow H_i^c(X)$ is defined by

(6) $\Phi_i = \Phi^i_{(\underline{X},\underline{x})}: \pi_i^c(\underline{X},\underline{x}) \longrightarrow H_i^c(\underline{X})$.

Then we have the following Hurewicz isomorphism theorem between strong shape groups and coherent singular homology groups.

10.5. Theorem. (a) *If* $\pi_k^S(X,x) = 0$ *for all* $0 \leq k \leq i-1$, *where* $i \geq 2$, *then* $\Phi_i\colon \pi_i^S(X,x) \cong H_i^c(X)$, *and* Φ_{i+1} *is an epimorphism*.

(b) *If* $\pi_0^S(X,x) = 0$, *then* $\Phi_1\colon \pi_1^S(X,x) \longrightarrow H_1^c(X)$ *is surjective and its kernel is the commutator subgroup of* $\pi_1^S(X,x)$.

Let X be a compactum and let $\underline{X} = (X_m, p_{mm+1})$ be an inverse sequence of compact polyhedra whose limit is X. Then the collection $\underline{p} = \{p_m\}$ of projections is clearly an ANR-resolution of X. Hence by Theorem 7.1 and Corollary 7.2

10.6. Theorem. *There is a natural isomorphism from* $H_*^c(X:G)$ *to* $H_*(K(\underline{X}):G)$ *for every abelian group* G.

Therefore we identify $H_*^c(X:G)$ with $H_*(K(\underline{X}):G)$. Moreover, it is known that $H_*^{S-S}(\underline{X}:G)$ is the Steenrod-Sitnikov homology group $H_{*+1}^{S-S}(X:G)$. It follows that the natural homomorphism $\zeta_X^i\colon H_i^c(X:G) \longrightarrow H_{i+1}^{S-S}(X:G)$ is given by

(7) $\zeta_X^i = \zeta_{\underline{X}}^i$.

Then by Example 7.6 ζ_X^i is not even an epimorphism, in general.

More exactly, using Example 7.6, we will show that H_*^c is different from the Steenrod-Sitnikov homology theory.

10.7. Exapmle. For each n=1,2,..., define

$S_n = \{(x,y) \in R^2 \mid (x-\frac{1}{n})^2 + y^2 = \{\frac{1}{2n(n+1)}\}^2\}$, and

$X_n = \{(0,0)\} \cup S_n$.

Then we have the planar 1-dimensional compactum

$$X = \bigcup_{n \geq 1} X_n.$$

Moreover $\varprojlim_n \operatorname{diam}(X_n) = 0$. Hence if H_*^c is the Steenrod-Sitnikov homology theory, by [29], it must be that the homomorphism $w\colon H_*^c(x) \longrightarrow \prod_{n=1}^{\infty} H_*^c(X_n) = \prod_{n=1}^{\infty} H_*(X_n)$ given by

(8) $w(a) = (r_{1*}(a), r_{2*}(a), \ldots)$ for each $a \in H_*^c(X)$,

where $r_n\colon X \longrightarrow X_n$ is the retraction such that $r_n(\bigcup_{j \neq n} S_j) = \{(0,0)\}$, is an isomorphism.

On the other hand, if * = 1, the homomorphism w is equal to the homomorphism $\alpha\zeta_{\underline{X}}^1$ defined in Example 7.6. Hence w is not an epimorphism. That is, H_*^c is not the Steenrod-Sitnikov homology theory.

10.8. Remark. In [30], Ono defined the S-C homology theory on the class of compacta. By Example 10.7, we easily see that the S-C homology theory is different from the Steenrod-Sitnikov homology theory. Similarly, Example 10.7 shows that Bauer's result, [1], Theorem 7.7 is not valid.

Concerning the natural homomorphism ζ_X^i we have the next result by Theorem 7.7.

10.9. Theorem. *If* $\pi_k^S(X,x) = 0$ *for all* k, $0 \leq k \leq i-1$, *where* $i \geq 2$, *then* $\zeta_X^i\colon H_i^C(X) \cong H_{i+1}^{S-S}(X)$.

Problem 1". *Under what condition of* X *does it hold that* ζ_X^i *is an isomorphism?*

Finally we consider a pointed contunuum (X,x). Let $(\underline{X,x}) = ((X_n,x_n),p_{nn+1})$ be an inverse sequence of pointed compact connected polyhedra, whose limit is (X,x). Properties of $E(\underline{X,x})$ are summarized as follows;

10.10. Theorem. (a) *There is a strong shape morphism* $\rho_{(\underline{X,x})}\colon E(\underline{X,x}) \longrightarrow (X,x)$ *such that*

$$\rho_{(X,x)\#}\colon \pi_i(E(\underline{X,x})) \cong \pi_i^S(X,x) \quad \text{for all } i \geq 0.$$

(b) *There is a weak homotopy equivalence* $F\colon E(\underline{X},\underline{x}) \longrightarrow |S_c(X,x)|$.

In particular, if (X,x) *is pointed* 1-*movable*, F *is a homotopy equivalence*.

Related to Theorem 10.10 (b), we pose the following problem.

Problem 2'. *Is the map* $F\colon E(\underline{X},\underline{x}) \longrightarrow |S_c(X,x)|$ *a homotopy equivalence*?

11. Problems in the coherent singular homology theory.

Steenrod-Sitnikov and Čech homology groups of a k-dimensional compactum vanish in dimensions greater than k. In this section, for each integer $k \geq 2$, we will construct a k-dimensional movable continuum $X(k)$ such that $H^c_{2k-1}(X(k):Q) \neq 0$. Hence we can also see that the coherent singular homology theory is different from Steenrod-Sitnikov and Čech ones (c.f. Example 10.7). First, we will show the next lemma.

11.1. Lemma. Let $(\underline{X},x)$ be an object of $CPHTOP_0$ such that $\pi^c_0(\underline{X},x) = \pi^c_1(\underline{X},x) = 0$. Then for $\alpha \in \pi^c_i(\underline{X},x)$, where $i \geq 2$, $\Phi^i_{(\underline{X},x)}(\alpha) = 0$ if and only if there exist a pointed finite polyhedron (K,k) of dim $K < i$ and a coherent map $f: (K,k) \longrightarrow (\underline{X},x)$ such that $\alpha \in f_\#(\pi_i((K,k)))$.

Proof. By Theorem 5.2, there exists a map $h: (S^i,s_0) \longrightarrow |S_c(\underline{X},x)|$ such that $\alpha = (\tau_X)_\#([h])$. Then

$$(\tau_{\underline{X}})_*\Phi^i_{|S_c(\underline{X},x)|}([h]) = \Phi^i_{(\underline{X},x)}(\tau_{\underline{X}})_\#([h]) = \Phi^i_{(\underline{X},x)}(\alpha) = 0.$$

Hence by Theorem 6.1(c), $\Phi^i_{|S_c(\underline{X},x)|}([h]) = 0$. Therefore there exists a map $g: (S^i,s_0) \longrightarrow |S_c(X,x)|$ such that

$$g \simeq h \text{ rel. } s_0, \text{ and } g(S^i) \subset |S_c(\underline{X})|^{(i-1)}.$$

Take a finite subcomplex $K \subset |S_c(\underline{X})|^{(i-1)}$ including $g(S^i)$.

Then the pair $(K,|(\{c_n\})|)$ and the coherent map $\tau_{\underline{X}} \mid K$: $(K,|(\{c_n\})|) \longrightarrow (\underline{X,x})$ satisfy the desired condition.

Let $(A_m,*)$ and $(B_m,*)$, $m=1,2,3,\ldots$, be simply connected compact ANRs with the base point satisfying the followings;

(i) if $m \neq m'$, then $A_m \cap A_{m'} = \{*\} = B_m \cap B_{m'}$, and

(ii) $(\bigcup_{m \geq 1} A_m) \cap (\bigcup_{m \geq 1} B_m) = \{*\}$.

For each $m \geq 1$, define the simply connected comapct ANR

$$(X_m,*) = ((A_1,*) \vee (B_1,*)) \vee \ldots \vee ((A_m,*) \vee (B_m,*)),$$

and the map p_{mm+1}: $(X_{m+1},*) \longrightarrow (X_m,*)$ given by

$$p_{mm+1}(x) = x \text{ for } x \in X_m, \text{ and } p_{mm+1}(A_{m+1} \vee B_{m+1}) = \{*\}.$$

Thus, we have a movable inverse sequence $(\underline{X,*}) = ((X_m,*),p_{mm+1})$ of simply connected compact ANRs. Then by Lemma 8.3, θ: $\bar{\pi}_n(\underline{X,*}) \cong \underleftarrow{\lim}(\pi_n(X_m,*),p_{mm+1\#})$ for every $n \geq 0$.

Let k, $\ell > 1$ be fixed integers and let $i = k + \ell - 1$. We will use the following notation: for $\alpha \in \pi_k(Y,y)$ and $\beta \in \pi_\ell(Y,y)$, $[\alpha,\beta] \in \pi_i(Y,y)$ is the <u>Whitehead product</u> of α and β. For every $m \geq 1$, let $\alpha_m \in \pi_k(A_m,*)$, $\beta_m \in \pi_\ell(B_m,*)$ and $\gamma_m = [\alpha_1,\beta_1] + \ldots + [\alpha_m,\beta_m] \in \pi_i(X_m,*)$. Then $(\gamma_m) \in \underleftarrow{\lim}(\pi_i(X_m,*),p_{mm+1\#})$ and there is the unique element $\gamma \in \bar{\pi}_i(\underline{X,x})$ such that $\theta(\gamma) = (\gamma_m)$. By using Lemma 11.1 and the analogous way of Barratt and Milnor (Proc. Amer. Math. Soc. 13 (1962), 293-297), Theorem 2, we have the following;

11.2. Theorem. $\Phi_Q(\gamma) \neq 0$ if $\phi_Q(\alpha_m) \neq 0$ and $\phi_Q(\beta_m) \neq 0$ for infinitely many $m \geq 1$, where Φ_Q and ϕ_Q are compositions

$$\bar{\pi}_i(\underline{X},*) \xrightarrow{\Phi(\underline{X},\underline{x})} H_i^c(\underline{X}:Z) \longrightarrow H_i^c(\underline{X}:Q), \text{ and}$$

$$\pi_j(Y,y) \xrightarrow{\phi(Y,y)} H_i(Y:Z) \longrightarrow H_i(Y:Q)$$

of suitable Hurewicz homomorphisms and coefficient homomorphisms induced by the inclusion $Z \longrightarrow Q$, respectively.

We note that in the proof of Theorem 11.2, we need the notion of the coherent singular cohomology groups $H_c^*(\underline{X}:G)$ and the functional cup-product of coherent maps.

11.3. Example. Let $k > 1$ be a fixed integer. For each $m \geq 1$, let

$$X(k,m) = \{(x_1,\ldots,x_{k+1}) \in R^{k+1} \mid (x_1-\tfrac{1}{n})^2 + x_2{}^2 + \ldots + x_{k+1}{}^2 = \frac{1}{n^2}\}.$$

Defining

$$X(k) = \bigcup_{m \geq 1} X(k,m) \quad \text{and} \quad x_k = (0,\ldots,0) \in X(k),$$

we have the k-dimensional pointed movable continuum $(X(k),x_k)$

in R^{k-1}. Then by Theorem 11.2, we can easily see that $H^c_{2k-1}(X(k):Q) \neq 0$.

Example 11.3 also shows that the coherent singular homology theory H^c_* does not satisfy the wedge axiom. Concerning axioms of homology theories, the following problems are posed.

Problem 4. _Does the theory_ H^c_* _satisfy the excision axiom?_

In particular, does a relative homeomorphism between compacta induce an isomorphism of H^c_* _for each dimension?_

Problem 5. _Let_ X_1, X_2 _be closed subsets of a compactum_ X _with_ $X = X_1 \cup X_2$, _and let_ $X_0 = X_1 \cap X_2$. _Then is the following_ Mayer-Vietoris _sequence exact?_

$$\cdots \longrightarrow H^c_i(X_0:G) \xrightarrow{(k^1{}_*,k^2{}_*)} H^c_i(X_1:G) \oplus H^c_i(X_2:G)$$

$$\xrightarrow{\ell^1{}_*-\ell^2{}_*} H^c_i(X:G) \xrightarrow{\partial} H^c_{i-1}(X_0:G) \longrightarrow \cdots,$$

where $k^i: X_0 \longrightarrow X_i$ _and_ $\ell^i: X_i \longrightarrow X$ $(i=1,2)$ _are suitable inclusion maps, and_ ∂ _is the boundary homomorphism of the pair_ (X,X_0).

References

[1] Bauer, F., A shape theory with singular homology, Pacific J. Math. 64(1976), 25-65.

[2] Borsuk, K., Concerning homotopy properties of compacta, Fund. Math. 62(1968), 223-254.

[3] Bousfield, A. and Kan, D. M., Homotopy limits, completions and localizations, Lecture Notes in Math. 304, Springer-Verlag, Berlin, 1972.

[4] Calder A. and Hastings, H. M., Realizing strong shape equivalences, J. Pure and Appl. Alg. 20(1981), 129-156.

[5] Cathey, F. and Segal, J., Strong shape theory and resolutions, Top. and its Appl. 15(1983), 119-130.

[6] Cohen, J. M., Homotopy groups of inverse limits, Proc. Adv. Study Inst. Aarhus, Alg. Top. 1(1970), 29-43.

[7] Christie, D., Net homotopy for compacta, Trans. Amer. Math. Soc. 56(1944), 275-308.

[8] Dydak, J. and Segal, J., Strong shape theory, Dissertationes Math. 192(1981).

[9] Edwards, D. A. and Geoghegan, R., Shape of complexes, ends of manifolds, homotopy limits and the Wall obstruction, Ann. of Math. 101(1975), 521-535.

[10] Edwards, D. A. and Geoghegan, R., Stability theorems in shape and prohomotopy, Trans. Amer. Math. Soc. 222(1976), 389-403.

[11] ______________ and Hastings, H. M., Čech and Steenrod homotopy theories with applications to geometric topology, Lecture Notes in Math. 542, Springer-Verlag, Berlin, 1976.

[12] Kodama, Y. and Koyama, A., Hurewicz isomorphism theorem for Steenrod homology, Proc. Amer. Math. Soc. 74(1979), 363-367.

[13] __________ and Ono, J., On fine shape theory, Fund. Math. 105(1979), 29-39.

[14] Koyama, A., Ono, J. and Tsuda, K., An algebraic characterization of pointed S^n-movability, Bull. Acad. Pol. 25(1977), 1249-1252.

[15] Lisica, Ju. T., On exactness of the spectral homotopy group sequence in shape theory, Dokl. Acad. Nauk. SSSR. 236(1977), 23-26.

[16] ______________, Strong shape theory and the Steenrod-Sitnikov homology (Russian), Sibirsk. Mat. Z. 24(1983), 81-99.

[17] ______________ and Mardešić, S., Steenrod-Sitnikov homology for arbitary spaces, Bull. Amer. Math. Soc. 9(1983), 207-210.

[18] Lisica, Ju. T. and Mardešić, S., Coherent prohomotopy and a strong shape category of topological spaces, International Topology Conference, Leningrad, 1982, Lecture Notes in Math. Springer-Verlarg, Berlin, 1984 (to appear).

[19] ______________________, Coherent prohomotopy and strong shape, to appear in Glasnik Mat.

[20] ______________________, Coherent prohomotopy and strong shape of metric compacta, to appear in Glasnik Mat.

[21] ______________________, Coherent prohomotopy and strong shape for pairs, to appear in Glasnik Mat.

[22] ______________________, Strong homology of inverse systems of spaces I, to appear in Top. and its Appl.

[23] ______________________, Strong homology of inverse systems of spaces II, to appear in Top. and its Appl.

[24] Mardešić, S., Approximative polyhedra, resolutions of maps and shape fibrations, Fund. Math. 114(1981), 53-78.

[25] ________, On resolutions for pairs of spaces, Tsukuba J. Math. 8(1984), 81-94.

[26] Mardešić, S. and Segal, J., Shape theory, North-Holland Publ. Co. Amsterdam, 1982.

[27] May, J. P., Simplicial objects in algebraic topology, Van Nostrand Math. Study 11, 1967.

[28] Milnor, J., On spaces having the homotopy type of a CW-complex, Trans. Amer. Math. Soc. 90(1959), 272-280.

[29] __________, On the Steenrod homology theory, Mimeographed Notes, Berkeley, 1960.

[30] Ono, J., A homology group construced by Čech representatives of simplexes, Reports of Liberal Arts, Sizuoka University, (Science), 18(1982), 7-11.

[31] Quigley, J. B., An exact sequence from the n-th to (n-1)-st fundamental groups, Fund. Math. 77(1973), 195-210.

[32] Spanier, E. H., Algebraic Topology, McGraw-Hill, New York, 1966.

[33] Watanabe, T., On a problem of Y. Kodama, Bull. Acad. Pol. 25(1977), 981-985.

[34] Whitehead, J. H. C., A certain exact sequence, Ann. of Math. 52(1950), 51-110.