SHRINKING PROPERTIES of OPEN COVERS

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THESIS

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CONTENTS

Conventions

All spaces are assumed to be Hausdorff topological spaces and all mappings are continuous.
An ordinal number is equal to the set of its predecessors and cardinal numbers are initial ordinals.
ω and ω_1 are used to denote the first infinite ordinal and the first uncountable ordinal respectively.
The letter I will always denote the closed unit interval [0, 1].
$|A|$ is the cardinality of a set A.
For a subset A of a space X, $cl_X A$ (or clA) denotes the closure of A in X.

Let $\mathcal{U}$ be a cover of X. A refinement $\mathcal{V}$ of $\mathcal{U}$ is a cover which refines $\mathcal{U}$.

Undefined notions and terminologies will follow R. Engelking [1989].

CHAPTER 0

INTRODUCTION

The notions of shrinkages of open covers have played an important role in the development of many areas of set-theoretic topology (R. Engelking [1989]).

An open cover $\mathfrak{U} = \{U_\alpha \mid \alpha \in A\}$ of a space X is said to be *shrinkable* if there exists an open cover $\{V_\alpha \mid \alpha \in A\}$ of $\mathfrak{U}$ such that $\mathrm{cl}V_\alpha \subset U_\alpha$ for each $\alpha \in A$.

The following theorem is the first one which shows the usefulness of shrinkages of some open covers in normal spaces and has an influence on related properties.

Recall that a space is countably paracompact if every countable open cover admits a locally finite open refinement.

0.1. **Theorem** (C. H. Dowker [1951])

The following conditions are equivalent for a normal space X:

(1) X is countably paracompact.

(2) $X \times I$ is normal.

(3) Every countable open cover of X is shrinkable.

If we do not assume the normality of X in the above theorem, the following one which contributed the development in covering properties is well-known and shows the values of shrinkages of increasing open covers.

A collection $\mathcal{A} = \{A_\lambda \mid \lambda \in \Lambda\}$ of subsets of a space X is *increasing* (resp. *decreasing*) if Λ is well-ordered and $A_\lambda \subset A_\mu$ (resp. $A_\lambda \supset A_\mu$) for any $\lambda, \mu \in A$ with $\lambda < \mu$.

0.2. **Theorem** (F. Ishikawa [1955] and J. Mack [1967])

The following conditions are equivalent for a space X:

(1) X is countably paracompact.

(2) Every countable increasing open cover of X is shrinkable.

(3) Every countable increasing open cover $\{U_n \mid n < \omega\}$ of X has an increasing open cover $\{V_n \mid n < \omega\}$ such that $\mathrm{cl}V_n \subset U_n$ for each $n < \omega$.

Under considerations of the above conditions (2) and (3) of Theorem 0.2, we have the following notions which have the rich developments in wide areas of topology:

0.3. **Definition**

Let X be a space and κ an infinite cardinal number.

Then X has *property* $\mathscr{D}(\kappa)$ (resp. *property* $\mathscr{B}(\kappa)$) if every increasing open cover $\{U_\alpha \mid \alpha < \kappa\}$ of X has an (resp. an increasing) open cover $\{V_\alpha \mid \alpha < \kappa\}$ such that $\mathrm{cl} V_\alpha \subset U_\alpha$ for each $\alpha < \kappa$.

If X has property $\mathscr{D}(\kappa)$ (resp. property $\mathscr{B}(\kappa)$) for any infinite cardinal number κ, X is said to have *property* $\mathscr{D}$ (resp. *property* $\mathscr{B}$).

Clearly property $\mathscr{B}(\kappa)$ implies property $\mathscr{D}(\kappa)$, and in case $\kappa = \omega$, properties $\mathscr{B}(\omega)$ and $\mathscr{D}(\omega)$ coincide with countable paracompactness.

Property $\mathscr{B}$ was introduced by P. Zenor [1970] and property $\mathscr{D}$ was defined in Yasui's paper [1972] under the name of *weak* $\mathscr{B}$*-property*.

It is difficult to check for a lot of spaces whether every (increasing) open cover is shrinkable or not, even if we know their normality. For instance the proof that every open cover of Σ-products of metric spaces, or compact or p-spaces with countable tightness is shrinkable is more involved and technical than that of normality (A. P. Kombarov [1978], K. Chiba [1982] and Y. Yajima [1984]).

Also we have another difficulty in finding spaces which are normal countably paracompact but not having property $\mathscr{D}(\kappa)$; indeed, the only such an example is obtained by M. E. Rudin

[1978].

Therefore to study the shrinkability of (increasing) open covers is one of the subjects which we cannot fail to notice.

Most covering properties are defined by refining arbitrary open cover with an appropriate open cover such as compactness, paracompactness, Lindelöfness, subparacompactness, submetacompactness (see Definitions 4.2 and 4.3).

One of our purposes is to characterize property $\mathscr{B}$ along this line without using increasing open covers. Indeed we shall prove that a space X has a property $\mathscr{B}$ if and only if every infinite open cover $\mathfrak{U}$ of X has an open refinement $\mathscr{V}$ with the following property: Each $x \in X$ has a nbd O such that the cardinality of $\{V \in \mathscr{V} \mid O \cap V \neq \phi\}$ is less than $|\mathfrak{U}|$ (Theorem 3.3).

This characterization is useful when we apply property $\mathscr{B}$ and study the relations among covering properties (Corollaries 3.4 and 3.5).

To study the normality of product spaces is one of basic and difficult problems in general topology.

Next we will characterize property $\mathscr{B}(\kappa)$ in terms of normality of product spaces. As is known, for a normal space X the product $X \times I^{\kappa}$ is normal if and only if X is κ-paracompact (K. Morita [1961]).

Since every normal κ-paracompact space has property $\mathscr{B}(\kappa)$

(see Theorem 1.1), it seems to be desirable to find a specific space Y with the property that for a normal space X, X×Y is normal for this space Y if and only if X has property $\mathscr{B}(\kappa)$.

In Chapter 3 we will construct such a space which is denoted by I_κ. That is, I_κ is a test space for property $\mathscr{B}(\kappa)$.

Several compact spaces that are test spaces for κ-paracompactness or κ-collectionwise normality have been considered (see C. H. Dowker [1951], k. Morita [1961] and O. T. Alas [1971]). We note that I_κ is not compact and besides these compact spaces and our I_κ other specific spaces that can be test spaces for other covering properties are not yet obtained.

As to shrinkability of (not necessarily increasing) open covers, there has not been any other equivalent condition.

In 1984, K. Chiba obtained a sufficient one that if a space is either normal subparacompact or perfectly normal, then every open cover is shrinkable. Our Theorem 4.7 will extend this result to normal submetacompact spaces. Notice that a class of submetacompact spaces includes both subparacompact spaces and perfectly normal spaces, and that a wider class of spaces than that of our case is not known for which the same result is true.

Another purpose is to study shrinkability of a certain kind of open cover. In [1983], Gruenhage and Michael showed that every cover of a regular space by open subsets with Lindelöf boundaries is shrinkable. In their paper they posed the following problem: "Is every cover of a regular space by open subsets with metrizable closures shrinkable?"

We shall show the following theorem on shrinkability (Theorem 4.8): "Every cover of a space by open subsets with perfectly normal closures is shrinkable." This theorem contains a positive answer to the problem above.

Finally we shall discuss on the product spaces of countably many spaces having property $\mathscr{B}$.

CHAPTER 1

RELATIONS AMONG VARIOUS COVERING PROPERTIES

In this chapter we see how our properties $\mathcal{B}$ and $\mathcal{D}$ relate to other covering ones.

Basically the following implications hold:

1.1. Theorem

Paracompactness $\Longrightarrow$ *property* $\mathcal{B}$ $\Longrightarrow$ *property* $\mathcal{D}$

$\Longrightarrow$ *countable paracompactness.*

Main purpose of this chapter is to see that all the reverse implications of Theorem 1.1 do not hold.

Before we list the spaces in illustration of their gaps, we observe a proposition for later use; the proof is easy and omitted.

1.2. Proposition

Let X be a space and κ an infinite cardinal number. Then X has property $\mathcal{D}(\kappa)$ (resp. property $\mathcal{B}(\kappa)$) if and only if, for any decreasing collection $\{F_\alpha \mid \alpha < \kappa\}$ of closed subsets of X

with $\bigcap_{\alpha<\kappa} F_\alpha = \phi$, *there is a (resp. a decreasing) collection* $\{G_\alpha \mid \alpha < \kappa\}$ *of open subsets of X such that*

(1) $F_\alpha \subset G_\alpha$ *for each* $\alpha < \kappa$

and

(2) $\bigcap_{\alpha<\kappa} \mathrm{cl} G_\alpha = \phi$.

As the first example, we shall show the following:

1.3. **Example** (Y. Yasui [1972])

Let ω_1 *be the first uncountable ordinal number. If* ω_1 *has an order topology, then* ω_1 *has property* $\mathscr{D}$, *but does not have propertry* $\mathscr{B}$ *(strictly speaking,* ω_1 *does not have property* $\mathscr{B}(\omega_1)$*).*

proof *X has property* $\mathscr{D}$.

It is known that this space ω_1 is normal. In order to show by Proposition 1.2 that ω_1 has property $\mathscr{D}$, let κ be any infinite cardinal and $\mathscr{F} = \{F_\lambda \mid \lambda < \kappa\}$ any decreasing collection of closed subsets of X with $\bigcap_{\lambda<\kappa} F_\lambda = \phi$. We may assume each F_λ is not empty.

For each $\alpha \in \omega_1$, we let

$f(\alpha)$ = the first of $\{\lambda \mid \lambda < \kappa,\ [0, \alpha] \cap F_\lambda = \phi\}$.

Since $[0, \alpha]$ is compact and $\{F_\lambda \mid \lambda < \kappa\}$ is the decreasing collection with $\bigcap_{\lambda<\alpha} F_\lambda = \phi$, $f(\alpha)$ is well-defined.

We shall show that, if we let $\Lambda = \{f(\alpha) \mid \alpha < \omega_1\}$, then Λ is cofinal in κ. For this purpose, we assume that Λ is not cofinal in κ, that is, there exists some $\lambda_0 < \kappa$ such that $f(\alpha) \leq \lambda_0$ for any $\alpha < \omega_1$. This means that $[0, \alpha] \cap F_{\lambda_0} = \phi$ for any $\alpha < \omega_1$, that is, $F_{\lambda_0} = \phi$. This is contradictory.

Now, we select any point α_λ of $f^{-1}(\lambda)$ for each $\lambda \in \Lambda$. Then it is seen that $\{\alpha_\lambda \mid \lambda \in \Lambda\}$ is cofinal in ω_1. If we let

$$G_\lambda = (\alpha_\lambda, \omega_1) \qquad \text{if } \lambda \in \Lambda$$
$$= [0, \omega_1) \qquad \text{if } \lambda \in \kappa - \Lambda,$$

then $\{G_\lambda \mid \lambda \in \kappa\}$ is a collection of open subsets of ω_1 such that $F_\lambda \subset G_\lambda$ for each $\lambda < \kappa$, and furthermore $\bigcap_{\lambda\in\kappa} G_\lambda$ is empty by the cofinality of $\{\alpha_\lambda \mid \lambda \in \Lambda\}$. Hence X has property $\mathscr{D}$.

Space ω_1 does not have property $\mathscr{B}(\omega_1)$.

For each $\alpha < \omega_1$, we let $F_\alpha = [\alpha, \omega_1)$. Then $\{F_\alpha \mid \alpha < \omega_1\}$ is a decreasing collection of closed subsets of ω_1 with $\bigcap_{\alpha<\omega_1} F_\alpha = \phi$.

If ω_1 has property $\mathscr{B}(\omega_1)$, there exists a decreasing collection $\{G_\alpha \mid \alpha < \omega_1\}$ of open subsets of ω_1 by Proposition 1.1 such that

(1) $\quad F_\alpha \subset G_\alpha \quad$ for each $\alpha < \omega_1$

and

(2) $\quad \bigcap_{\alpha<\omega_1} \mathrm{cl}G_\alpha = \phi$.

For each $\alpha < \omega_1$, we let

$$f(\alpha) = \text{the first of } \{\beta < \omega_1 \mid (\beta, \omega_1) \subset G_\alpha\}.$$

Then f is a mapping from $[1, \omega_1)$ to ω_1 by (1) such that

(3) $f(\alpha) < \alpha$ for each α with $1 \leq \alpha < \omega_1$

and

(4) $f(\beta) \leq f(\alpha)$ for $\alpha, \beta < \omega_1$ with $\beta < \alpha$.

By definition of $f(\alpha)$, we have $[f(\alpha)+1, \omega_1) \subset G_\alpha$ for each $\alpha < \omega_1$ and hence $\bigcap_{\alpha<\omega_1} [f(\alpha)+1, \omega_1) \subset \bigcap_{\alpha<\omega_1} G_\alpha = \phi$. This means that

(5) $\{f(\alpha) \mid \alpha < \omega_1\}$ is cofinal in ω_1.

By pressing down lemma (see K. Kunen [1980]) and (3), there exists some $\alpha_0 < \omega_1$ (where we may assume that $\alpha_0 \geq 1$) such that

(6) $\{\alpha \mid \alpha < \omega_1, f(\alpha) \leq \alpha_0\}$ is cofinal in ω_1.

On the other hand, there exists some $\alpha_1 < \omega_1$ by (5) such that

(7) $\alpha_0 < f(\alpha_1)$.

For α_1, there exists some $\alpha_2 < \omega_1$ by (6) such that

(8) $\alpha_1 < \alpha_2$ and $f(\alpha_2) \leq \alpha_0$.

By (4), (7) and (8), $\alpha_0 < f(\alpha_1) \leq f(\alpha_2) \leq \alpha_0$. This is contradictory. □

Remarks

1. W. M. Fleishman ([1970]) proved the following theorem almost simultaneous with our Example 1.3: "Every open cover of a linealy ordered space is shrinkable." So it is seen that every open cover of ω_1 is shrinkable.

2. As in the proof of the above example, we can show that:

For any regular cardinal number κ, *there exists a normal space which has property* $\mathcal{D}$ *but does not have property* $\mathcal{B}(\kappa)$.

In fact, the space κ with order topology is such a space. Furthermore it is seen that *there is a normal space which has property* $\mathcal{B}(\lambda)$ *for every* $\lambda < \kappa$, *but does not have property* $\mathcal{B}(\kappa)$.

To discuss the gap between property $\mathcal{D}$ and countable paracompactness, let us first mention the following theorem due to M. E. Rudin.

1.4. **Theorem** (M. E. Rudin [1978])

For each infinite cardinal number κ, *there exists a normal space* X_κ *without property* $\mathcal{D}(\kappa)$.

A normal space without property $\mathcal{D}(\kappa)$ is generally called a κ-*Dowker space* which was previously defined by Rudin [1971] means a normal but not countably paracompact space. Thus, an ω-Dowker space is nothing but a Dowker space.

Let X_κ be the κ-Dowker space given in Theorem 1.4. Let $\mathfrak{U}$ be any open cover of X_κ with $|\mathfrak{U}| < cf(\kappa)$. Then $\mathfrak{U}$ is shown to be refined by a cover of mutually disjoint open sets (that is, X_κ is *ultra* λ-*paracompact* for any $\lambda < ck(\kappa)$). Therefore, taking $\kappa = \omega_1$, we have the following:

1.5. **Example** (M. E. Rudin [1983-a] and [1985])

There is a normal space X which is countably paracompact but does not have property $\mathcal{D}$.

Next we shall introduce a Navy's space which is repeatedly quoted in this paper. This space is normal and paraLindelöf but it is not paracompact (K. Navy [1981]), where a space X is called to be *paraLindelöf* if every open cover of X has a locally countable open refinement.

We present this space due to the definition which is appeared in Rudin's paper [1983-a].

Navy's space S Let F be the set of all functions from ω into ω_1. For n $(1 < n < \omega)$, let $\Sigma_n = \{ f|n \mid f \in F\}$, and P_n be the set of all subsets of Σ_n. Let $\Sigma = \bigcup_{n<\omega} \Sigma_n$ and P be the set of all finite subsets of $\bigcup_{n<\omega} P_n$.

Let $\Delta = \{(\sigma,\tau,\mathcal{A}) \mid$ (1) $\mathcal{A} \in P$, and σ, $\tau \in \Sigma_n$ for some n

(2) $A \in \mathcal{A} \cap P_m$ for some $m \leq n$, then $\sigma|m \in A$ iff $\tau|m \in A$

(3) $\sigma(0)<\tau(0)<\sigma(1)<\tau(1)< \ldots <\sigma(n-1)<\tau(n-1)$ $\}$.

For $\rho \in \Sigma$, and $\mathcal{B} \in P$, let

$B(\rho,\mathcal{B}) = \{f \in F \mid f \supset \rho\} \cup \{\langle\sigma,\tau,\mathcal{A}\rangle \in \Delta \mid$ (1) $\sigma \supset \rho$ or $\tau \supset \rho$

$(2)\ \mathscr{B} \subset \mathscr{A}$ $\Big\}$,

where $f \supset \rho$ means that f extends ρ.

Navy's space S is $F \cup \Delta$ topologized by having $\{B(\rho,\mathscr{B}) \mid \rho \in \Sigma$ and $\mathscr{B} \in P\} \cup \{\{(\sigma,\tau,\mathscr{A})\} \mid (\sigma,\tau,\mathscr{A}) \in \Delta\}$ as an open base.

1.6. **Example** (M. E. Rudin [1983-a] and [1985])

Navy's space S is a normal space with property $\mathscr{B}$ and every open cover of S is shrinkable, but is not paracompact.

If we apply our theorems of a later chapter, it is very easier than Rudin's one to prove that every open cover of S is shrinkable and S has property $\mathscr{B}$. So we shall show them.

proof of some part of Example 1.6

Since it is known that S is countably paracompact and paraLindelöf (K. Navy [1981]), S has property $\mathscr{B}$ by the below Corollary 3.4.

Next we can show that every open cover of S is shrinkable. By Fleissner [1984] the subspace F of S is ultra paracompact. So every open cover of F is shrinkable. On the other hand, Δ is a discrete subspace of S. Hence every open cover of S is shrinkable (by Proposition 4.1). □

In 1951, Bing introduced the concept of collectionwise normality and constructed the valuable example G.

Example G and its subspaces have proved to be rich source of examples among topological properties (ref. I. W. Lewis [1977]).

For completeness let us recall Bing's example G:

Let P be uncountable set and D two-point set $\{0, 1\}$. Furthermore let $\mathcal{P}$ be the power set of P and F the product space of $\mathcal{P}$-copies of D, that is, $F = \{f \mid f \text{ is a mapping from } \mathcal{P} \text{ to } D\}$.

For each $p \in P$, define $f_p \in F$ as follows:

$$f_p(Q) = 1 \quad \text{if } p \in Q$$
$$= 0 \quad \text{otherwise.}$$

Let F_0 as $\{f_p \mid p \in P\}$. For eah $\mathcal{R} \in [\mathcal{P}]$ and $p \in P$, let $U(p, \mathcal{R}) = \{f \in F \mid f(R) = f_p(R) \text{ for any } R \in \mathcal{R}\}$, where $[\mathcal{P}]$ denotes the set of all finite subsets of $\mathcal{P}$. Let us define a nbd base $\mathcal{U}(f)$ of $f \in F$ as follows:

$$\mathcal{U}(f) = \{\{f\}\} \quad \text{if } f \in F - F_0$$
$$= \{U(p, \mathcal{R}) \mid \mathcal{R} \in [\mathcal{P}]\} \quad \text{if } f = f_p \in F_0.$$

Therefore each point $f \in F - F_0$ is isolated in F and each point $f_p \in F_0$ is isolated in F_0 (but f_p is not isolated in F).

1.7. **Example** (K. Chiba [1984], or ref. Y. Yasui [1989])

Let F be a Bing's example G and X any subspace of F. Then every open cover of X is shrinkable, but F does not have property $\mathcal{B}$.

proof

Since the shrinkability of any open cover of X is due to Proposition 4.1, we shall sketch the outline that F does not have property $\mathscr{B}$.

Without loss of generality, we may assume that $P = \omega_1$. So we let $F_0 = \{f_\alpha \mid \alpha < \omega_1\}$. If we let $H_\alpha = \{f_\beta \mid \alpha \leq \beta < \omega_1\}$ for each $\alpha < \omega_1$, then $\{H_\alpha \mid \alpha < \omega_1\}$ is a decreasing collection of closed sets with $\bigcap_{\alpha<\omega_1} H_\alpha = \phi$. It is seen that if $\{U_\alpha \mid \alpha < \omega\}$ is a decreasing open collection such that $H_\alpha \subset U_\alpha$ for any α, then $\bigcap_{\alpha<\omega_1} \mathrm{cl} U_\alpha \neq \phi$. □

As the last example of this chapter, we shall consider that in the definition of property $\mathscr{B}(\omega)$ = property $\mathscr{D}(\omega)$ (Theorem 0.2), we cannot weaken to the following: Any countable increasing open cover $\{U_n \mid n < \omega\}$ of X has a countable increasing closed cover $\{F_n \mid n < \omega\}$ such that $F_n \subset U_n$ for each n.

1.8. **Example**

Let $X = \{(x, y) \mid x, y$ are real numbers with $y \geq 0\}$ with Niemytzki's tangent disc topology. Then every countable increasing open cover $\{U_n \mid n < \omega\}$ of X has an increasing closed cover $\{F_n \mid n < \omega\}$ such that $F_n \subset U_n$ for each $n < \omega$.

proof

We recall Niemytzki's tangent disc topology. Let $L = \{(x, 0) \mid x: \text{real}\}$ and τ be the Euclidean topology for X.

We generate a topology τ^* on X by adding to τ all sets of the form $\{p\} \cup D$, where $p \in L$ and D is an open disc in $X - L$ which is tangent to L at the point p. Then τ^* is called the *Niemytzki's tangent disc topology*.

Let $\{U_n \mid n < \omega\}$ be any countable increasing open cover of X. We select a countable open base $\{B_n \mid n < \omega\}$ for open half-plane $X - L$ such that $(cl_X B_n) \cap L = \phi$ and each $cl_X B_n$ is contained in some U_m. Let f be a mapping from ω to ω as follows:

$$f(m) = \min \{n \mid cl_X B_m \subset U_n\} \quad \text{for each } m < \omega.$$

If we let

$$F_n = \cup \{clB_m \mid f(m) \le n,\ m \le n\}$$

for each $n < \omega$, then it is seen that $\{F_n \mid n < \omega\}$ is an increasing closed cover of X such that $F_n \subset U_n$ for each $n < \omega$. □

CHAPTER 2

FUNDAMENTALS OF PROPERTIES $\mathscr{B}$ AND $\mathscr{D}$

In Chapter 2 we shall see the fundamentals of properties $\mathscr{B}$ and $\mathscr{D}$ and their applications.

It is seen that properties $\mathscr{B}$ and $\mathscr{D}$ are closed hereditary, that is, every closed subspace of a space with property $\mathscr{B}$ (resp. property $\mathscr{D}$) has property $\mathscr{B}$ (resp. property $\mathscr{D}$), but it does not hold for open subspace.

We shall next consider the closed images of spaces with property $\mathscr{B}$ (resp. $\mathscr{D}$). It is known that the closed image of a paracompact space is necessarily paracompact, but M. E. Rudin [1983-a] and [1985] (resp. H. Ohta [1985]) showed that property $\mathscr{B}$ (resp. property $\mathscr{D}$) is not preserved under a closed mapping. As a matter of course, in a class of normal spaces, property $\mathscr{D}$ is preserved under closed mappings.

Really Rudin used Navy's space S for this example (see Example 1.6). Using the notation in the front of Example 1.6, for $\alpha < \omega_1$, let $F_\alpha = \{f \in F \mid f(0) = \alpha\}$. Let T be the quotient space gotten from S by idetifying the terms of F_α for each $\alpha < \omega_1$.

Furthermore f is a quotient mapping from S to T. Then Rudin showed that f is closed but T is a normal space which does not have property $\mathscr{B}$.

On the other hand, Ohta's example showed that the countable paracompactness is not preserved by closed continuous mapping. Such a space was assured by P. Zenor [1969], but the range space of Zenor's mapping is not regular. But Ohta showed that countable paracompactness is not an invariant of closed mappings in the realm of Tychonoff spaces.

As mentioned above, any closed continuous image of property $\mathscr{B}$ does not necessarily have property $\mathscr{B}$, but the following is seen:

2.1. **Proposition**

Every perfect image of a space with propertry $\mathscr{B}(\kappa)$ has also property $\mathscr{B}(\kappa)$ for any infinite cardinal number κ.

A mapping is called to be *perfect* if it is closed and every inverse image of any one-point set is compact.

From this fact, we have easily the following corollary with respect to the union of spaces with property $\mathscr{B}$:

2.2. **Corollary**

If a space X has a locally finite closed cover

$\mathcal{F} = \{F_\alpha \mid \alpha \in A\}$ *such that each* F_α *has property* $\mathcal{B}$, *then* X *has also property* $\mathcal{B}$.

2.3. **Corollary**

Let $\{U_n \mid n < \omega\}$ *be a countable open cover of a space* X. *If* $\mathrm{cl}U_n$ *has property* $\mathcal{B}$ *for any* $n < \omega$, *then* X *has property* $\mathcal{B}$.

proof

If we let $F_n = \mathrm{cl}U_n - \bigcup_{i<n} U_i$ for $n \geq 2$ and $F_1 = \mathrm{cl}U_1$, then $\{F_n \mid n < \omega\}$ is a locally finite closed cover of X each of which has property $\mathcal{B}$. So X has property $\mathcal{B}$ by Corollary 2.2. □

CHAPTER 3

CHARACTERIZATIONS OF PROPERTY $\mathscr{B}$

The purpose of Chapter 3 is to give characterizations of property $\mathscr{B}$ and to apply them and furthermore to clear up this concept.

1. Property $\mathscr{B}$ and certain open covers

Most covering properties are defined by refining an arbitrary open cover with an open cover having an appropriate local property such as paracompactness, metacompactness and subparacompactness (see Definitions 4.2 and 4.3).

One of the purposes of this section is to study the following problem: "For any increasing open cover of a space having $\mathscr{B}$-property, can we take its refinement with a 'nice' local property?"

3.1. **Theorem** (Y. Yasui [1986])

Let X be a space and κ an infinite cardinal number. Then the following conditions are equivalent:

(1) X has property $\mathscr{B}(\kappa)$.

(2) *Every increasing open cover* $\{U_\alpha \mid \alpha < \kappa\}$ *of* X *has an open cover* $\{V_\alpha \mid \alpha < \kappa\}$ *of* X *such that*

(2-1) $V_\alpha \subset U_\alpha$ *for any* $\alpha < \kappa$,

and

(2-2) *for each* $x \in X$, *there exist some open nbd* O_x *of* x *and some* $\alpha_x < \kappa$ *such that* $O_x \cap (\bigcup_{\alpha \geq \alpha_x} V_\alpha) = \phi$.

(3) *Every increasing open cover* $\{U_\alpha \mid \alpha < \kappa\}$ *of* X *has an open cover* $\{V_\alpha \mid \alpha < \kappa\}$ *of* X *such that*

(3-1) $\mathrm{cl} V_\alpha \subset U_\alpha$ *for any* $\alpha < \kappa$,

and

(3-2) *for each* $x \in X$, *there exist some open nbd* O_x *of* x *and some* $\alpha_x < \kappa$ *such that* $O_x \cap (\bigcup_{\alpha \geq \alpha_x} V_\alpha) = \phi$.

proof (1) → (3):

Let $\{U_\alpha \mid \alpha < \kappa\}$ be an increasing open cover of X. Then we have two increasing open covers $\{T_\alpha \mid \alpha < \kappa\}$ and $\{S_\alpha \mid \alpha < \kappa\}$ of X such that $\mathrm{cl} S_\alpha \subset T_\alpha \subset \mathrm{cl} T_\alpha \subset U_\alpha$ for each α.

Without loss of generality, we may assume that

(*) $T_\alpha = \cup \{T_\beta \mid \beta < \alpha\}$ for any limit ordinal $\alpha < \kappa$.

Let

$$V_\alpha = T_\alpha - \mathrm{cl}(S_{\alpha-1}) \quad \text{if } \alpha \text{ is not limit}$$
$$= \phi \quad \text{if } \alpha \text{ is limit.}$$

Then the collection $\{V_\alpha \mid \alpha < \kappa\}$ of open sets will be a cover of X. Let x be any point of X and α_0 the first of $\{\alpha \mid$

$x \in T_\alpha\}$, then α_0 is not limit by (*). So $x \notin cl(S_{\alpha_0-1})$.

Therefore we have $x \in V_{\alpha_0}$.

To see that $\{V_\alpha | \alpha < \kappa\}$ satisfies (3-2), let x be any point of X. Since $\{S_\alpha | \alpha < \kappa\}$ is a cover of X, there exists some $\alpha_x < \kappa$ with $x \in S_{\alpha_x}$. Then for any non-limit ordinal α with $\alpha_x < \alpha < \kappa$, we have $S_{\alpha_x} \cap V_\alpha \subset S_{\alpha_x} - cl(S_{\alpha_x-1}) \subset S_{\alpha_x} - cl(S_{\alpha_x}) = \phi$.

(3) → (2): clear

(2) → (1):

Let $\{U_\alpha | \alpha < \kappa\}$ be an increasing open cover of X and $\{V_\alpha | \alpha < \kappa\}$ as given in (2).

If we let

$$T_\alpha = \cup \{O | O \text{ is open and } O \cap (\bigcup_{\beta \geq \alpha} V_\beta) = \phi\}$$

for each $\alpha < \kappa$, then $\{T_\alpha | \alpha < \kappa\}$ is an increasing open cover of X. So we shall show that $clT_\alpha \subset U_\alpha$ for any α.

We have $T_\alpha \cap (\bigcup_{\beta \geq \alpha} V_\beta) = \phi$ and hence $(clT_\alpha) \cap (\bigcup_{\beta \geq \alpha} V_\beta) = \phi$ for each $\alpha < \kappa$.

Therefore

$$clT_\alpha \subset X - (\bigcup_{\beta \geq \alpha} V_\beta) \subset \bigcup_{\beta < \alpha} V_\beta \subset \bigcup_{\beta < \alpha} U_\beta \subset U_\alpha. \quad \square$$

Almost all the covering properties are defined by terms that an arbitrary open cover has a refinement with some property (see Definitions 4.2 and 4.3). If it is possible to have such characterizations of property $\mathscr{B}$, its utility will be exploited in more areas.

The second purpose of this section is to have such characterizations:

3.2. Theorem (Y. Yasui [1986], [1987] and [1989])

Let X be a space and κ an infinite cardinal number. Then the following conditions are equivalent:

(1) X has property $\mathcal{B}(\kappa)$.

(2) Every open cover $\{U_\alpha \mid \alpha < \kappa\}$ of X has an open cover $\{U_{\alpha\beta} \mid \beta < \alpha;\ \alpha < \kappa\}$ of X such that

(2-1) $V_{\alpha\beta} \subset U_\beta$ for any β, α with $\beta < \alpha$,

and

(2-2) each $x \in X$ has some nbd O and some ordinal $\alpha_x < \kappa$ such that $O \cap (\cup \{V_{\alpha\beta} \mid \beta < \alpha;\ \alpha \geq \alpha_x\}) = \phi$.

(3) Every open cover $\mathcal{U}$ of X with cardinality κ has an open refinement $\mathcal{V}$ which satisfies the following:

(*) Each $x \in X$ has some nbd O such that the cardinality of $\{V \in \mathcal{V} \mid O \cap V \neq \phi\}$ is less than κ.

proof (1) → (2):

Let $\mathcal{U} = \{U_\alpha \mid \alpha < \kappa\}$ be an open cover of X. If we let $W_\alpha = \bigcup_{\beta<\alpha} U_\beta$ for each $\alpha < \kappa$, then $\{W_\alpha \mid \alpha < \kappa\}$ is an increasing open cover of X such that $W_\alpha = \bigcup_{\beta<\alpha} W_\beta$ for each limit ordinal $\alpha < \kappa$.

Since X has property $\mathscr{B}(\kappa)$, there exists an increasing open cover $\mathscr{V} = \{V_\alpha \mid \alpha < \kappa\}$ of X such that $\mathrm{cl}V_\alpha \subset W_\alpha$ for each α. We may assume that $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$ for each limit α.

For each α, $\beta < \kappa$ with $\beta < \alpha$, let

$$\begin{aligned} V_{\alpha\beta} &= U_\beta - \mathrm{cl}(V_{\alpha-1}) \quad \text{if } \alpha \text{ is not limit and } \beta < \alpha \\ &= \phi \quad \text{otherwise.} \end{aligned}$$

Then it is clear that $V_{\alpha\beta} \subset U_\beta$ for each α, β with $\beta < \alpha$.

To see that $\{V_{\alpha\beta} \mid \beta < \alpha\}$ is a cover of X, let x be any point of X. If α_x is the first of $\{\alpha \mid \alpha < \kappa,\ x \in W_\alpha\}$, then α_x is not limit and $x \notin W_{\alpha_x - 1}$. Hence $x \notin \mathrm{cl}(V_{\alpha_x - 1})$.

Since $x \in W_{\alpha_x} = \bigcup_{\beta<\alpha_x} U_\beta$, there is some $\beta < \alpha_x$ with $x \in U_\beta$, and hence $x_\beta \in U - \mathrm{cl}(V_{\alpha_x - 1}) = V_{\alpha_x\beta}$. This means that $\{V_{\alpha\beta} \mid \beta < \alpha\}$ is a cover of X.

Since $V_{\alpha\beta} \subset U_\beta$ for any β, α with $\beta < \alpha$, it is sufficient to show that $\{V_{\alpha\beta} \mid \beta < \alpha\}$ satisfies the condition (2-2).

Let $x \in X$ and $\alpha_0 < \kappa$ with $x \in V_{\alpha_0}$. We have $V_{\alpha_0} \cap (X - \mathrm{cl}V_\alpha)$ for any α with $\alpha_0 < \alpha < \kappa$, because $\{V_\alpha \mid \alpha\}$ is increasing. Then, for any non-limit ordinal α with $\alpha > \alpha_0+1$ and any ordinal β with $\beta < \alpha$, it follows that

$$V_{\alpha_0} \cap V_{\alpha\beta} \subset V_{\alpha_0} \cap (X - \mathrm{cl}V_{\alpha-1}) = \phi.$$

(2) → (3):

Let $\mathscr{U}$ be an open cover of X with cardinality κ and $\mathscr{U}$ express as $\{U_\alpha \mid \alpha < \kappa\}$.

By (2), there exists an open cover $\{V_{\alpha\beta} \mid \beta < \alpha;\ \alpha < \kappa\}$ of X that

(2-1) $V_{\alpha\beta} \subset U_\beta$ for any $\beta < \alpha$

and

(2-2) each $x \in X$ has some open nbd O_x and some ordinal α_x such that $O_x \cap (\cup \{V_{\alpha\beta} \mid \beta < \alpha;\ \alpha \geq \alpha_x\}) = \phi$.

Then the cover $\{V_{\alpha\beta} \mid \beta < \alpha\}$ is a refinement of $\mathfrak{U}$ by (2-1).

On the other hand, let x be any point of X and O_x, α_x be as given in (2-2). Then the cardinality of $\{V_{\alpha\beta} \mid O_x \cap V_{\alpha\beta} \neq \phi\}$ is less than or equal to the cardinality of $\{(\alpha, \beta) \mid \beta < \alpha;\ \alpha \leq \alpha_x\}$. Then the cardinality of $\{V_{\alpha\beta} \mid O_x \cap V_{\alpha\beta} \neq \phi\}$ is less than $\kappa = |\mathfrak{U}|$.

(3) → (1): It is clear by Theorem 3.1.

These complete the proof. □

By Theorem 3.2, we have the following:

3.3. Theorem (Y. Yasui [1986], [1987] and [1989])

The following conditions are equivalent for a space X:

(1) X has property $\mathfrak{B}$.

(2) Every infinite open cover $\{U_\alpha \mid \alpha < \tau\}$ *of X has an open cover* $\{V_{\alpha\beta} \mid \beta < \alpha;\ \alpha < \tau\}$ *of X such that*

(2-1) $V_{\alpha\beta} \subset U_\beta$ *for any* β, α *with* $\beta < \alpha$,

and

(2-2) *each* $x \in X$ *has some nbd* O *and some ordinal* $\alpha_x < \tau$ *such that* $O \cap (\cup \{V_{\alpha\beta} \mid \beta < \alpha;\ \alpha \geq \alpha_x\}) = \phi$.

(3) *Every infinite open cover* $\mathfrak{U}$ *of* X *has an open refinement* $\mathcal{V}$ *which satisfies the following:*

(*) *Each* $x \in X$ *has a nbd* O *such that the cardinality of* $\{V \in \mathcal{V} \mid O \cap V \neq \phi\}$ *is less than* $|\mathfrak{U}|$.

In Chapter 0, we recalled a Navy's space. A value for the existence of the space is to show that paracompactness is stronger than paraLindelöfness in normal spaces.

Furthermore such a space is only one as far as I know, and M. E. Rudin showed that this space has also property $\mathcal{B}$ ([1983-a] and [1985]).

If we show the following as a corollary of Theorem 3.3, our proof that Navy's space has property $\mathcal{B}$ is very simpler than Rudin's one.

3.4. **Corollary**

Every countably paracompact and paraLindelöf space has property $\mathcal{B}$.

proof

Let $\mathfrak{U}$ be an infinite open cover of X and $\mathfrak{U}$ express as $\{U_\alpha \mid \alpha < \tau\}$ for some τ, where τ is the minimal ordinal whose cardinality is equal to $|\mathfrak{U}|$.

case 1 Assume cof(τ) (= the cofinality of τ) is countable.

Let $\{\alpha_n \mid n < \omega\}$ be an increasing sequence of ordinals which converges to τ. Since $\{W_{\alpha_n} \mid n < \omega\}$ is a countable open cover of X, where $W_{\alpha_n} = \bigcup_{\alpha < \alpha_n} U_\alpha$, there exists a locally finite open cover $\{V_n \mid n < \omega\}$ of X such that $V_n \subset W_{\alpha_n}$ for each n.

Then each $x \in X$ has an open nbd O_x such that the cardinality of $\{W_{\alpha_n} \cap U_\alpha \mid \alpha < \alpha_n$ and $O_x \cap (W_{\alpha_n} \cap U_\alpha) \neq \phi\}$ is less than $|\mathfrak{U}|$.

case 2 Assume cof(τ) is not countable.

By paraLindelöf property of X, $\mathfrak{U}$ has a locally countable open refinement $\mathcal{V}$. Hence each $x \in X$ has an open nbd O_x whcih intersects V for at most countably many $V \in \mathcal{V}$, this means that the cardinality of $\{V \in \mathcal{V} \mid O_x \cap V \neq \phi\}$ is less than $|\mathfrak{U}|$. □

As a corollary of the above theorem, the following will be seen.

3.5. Corollary (T. Tani and Y. Yasui [1972])

A space X has property $\mathcal{B}$ if and only if every increasing open cover of X has a cushioned open refinement.

proof "only if" part:

Let $\mathcal{U} = \{U_\alpha \mid \alpha < \tau\}$ be an increasing open cover of *X*.

By Theorem 3.1, there exists an open cover $\mathcal{V} = \{V_\alpha \mid \alpha < \tau\}$ of X such that

(1) $V_\alpha \subset U_\alpha$ for any $\alpha < \tau$

and

(2) each $x \in X$ has an open nbd O_x and some $\alpha_x < \tau$ such that

$O_x \cap (\bigcup_{\alpha \geq \alpha_x} U_\alpha) = \phi$.

We let $W_\alpha = \cup \{O_x \mid \alpha_x = \alpha\}$ for each α. To see that a cover $\mathcal{W} = \{W_\alpha \mid \alpha\}$ is cushioned in $\mathcal{U}$, let A be any subset of τ, where we may assume that A is not cofinal in τ, and let α_0 be the sup of A (and hence $\alpha_0 < \tau$).

We have $(\bigcup_{\alpha \in A} W_\alpha) \cap (\bigcup_{\alpha \geq \alpha_0} U_\alpha) = \phi$, and

$cl(\bigcup_{\alpha \in A} W_\alpha) \subset \bigcup_{\alpha < \alpha_0} U_\alpha \subset \bigcup_{\alpha \in A} U_\alpha$.

This means $\mathcal{W}$ is cushioned in $\mathcal{U}$.

"if" part:

Let $\mathcal{U} = \{U_\alpha \mid \alpha < \tau\}$ be an increasing open cover of X. Then we have an open cover $\mathcal{V} = \{V_\alpha \mid \alpha < \tau\}$ which is cushioned in

$\mathcal{U}$.

Let $W_\alpha = \bigcup_{\beta<\alpha} V_\beta$ for each α, then it is seen that the collection $\{W_\alpha \mid \alpha < \tau\}$ is an increasing open cover of X such that $\mathrm{cl}W_\alpha \subset U_\alpha$ for any α. □

Remark

It is well-known that a regular space X is paracompact if and only if every open cover of X has a cushioned open refinement (E. Michael [1959]). Since paracompactness is not equivalent to property $\mathscr{B}$, we cannot exchange "every increasing open cover" for "every open cover" in Corollary 3.5.

2. Property $\mathscr{B}$ and the normality of product spaces

Some interesting and useful characterizations of separations and covering properties of a space X are given in terms of the normality of the product spaces X×Y with all members Y of some class of spaces. In other wards, this means the existences of test spaces of covering properties.

We know them for the classes of paracompact spaces, κ-paracompact spaces and collectionwise normal countably paracompact spaces (Theorem 0.1, K. Morita [1961], O. T. Alas

[1971] and H. Tamano [1960]).

So we shall find a test space of normal spaces with property $\mathscr{B}(\kappa)$ in this section.

Before we give such a characterization of property $\mathscr{B}$, we shall define some terminology.

For an infinite cardinal number κ, we let a *space* I_κ as follows:

(1) I_κ is the set of all ordinals $\leq \kappa$.

(2) I_κ has an open base $\{\{\alpha\} \mid \alpha < \kappa\} \cup \{(\alpha, \kappa] \mid \alpha < \kappa\}$.

Therefore each $\alpha < \kappa$ is isolated in I_κ and the point κ has a usual order nbd base in I_κ.

3.6. **Theorem** (Y. Yasui [1983])

Let X be a normal space and κ *an infinite cardinal number. Then X has property* $\mathscr{B}(\kappa)$ *if and only if* $X \times I_\kappa$ *is normal.*

proof "if" part:

To see that X has property $\mathscr{B}(\kappa)$, let $\{U_\alpha \mid \alpha < \kappa\}$ be an increasing open cover of X.

Let $H = \cup \{(X - U_\alpha) \times \{\alpha\} \mid \alpha < \kappa\}$ and $K = X \times \{\kappa\}$.

Since H and K are disjoint closed sets of $X \times I_\kappa$, there are disjoint open sets W and V of $X \times I_\kappa$ such that $H \subset W$ and $K \subset V$.

For each $\alpha < \kappa$, let

$V_\alpha = \{x \in X \mid O \times (\alpha, \kappa] \subset V$ for some nbd O of $x\}$.

Then it is seen that $\{V_\alpha \mid \alpha < \kappa\}$ is an increasing open cover of X which satisfies $\mathrm{cl}V_\alpha \subset U_\alpha$ for $\alpha < \kappa$. Hence X has property $\mathscr{B}$.

"only if" part:

Let H and K be any disjoint closed sets of $X \times I_\kappa$.

Since $H \cap (X \times \{\kappa\})$ and $K \cap (X \times \{\kappa\})$ are closed and disjoint in $X \times \{\kappa\}$, there is an open set O of X such that

(a) $H \cap (X \times \{\kappa\}) \subset O \times \{\kappa\}$

and

(b) $((\mathrm{cl}O) \times \{\kappa\}) \cap (K \cap (X \times \{\kappa\})) = \phi$.

For each $\alpha < \kappa$, we let

(c) $U_\alpha = (X - \mathrm{cl}O) \cup (\cup\{P \mid P\text{: open, } P \times [\alpha, \kappa] \cap K = \phi\})$.

Since $\{U_\alpha \mid \alpha < \kappa\}$ is an increasing cover of X, we have an increasing open cover $\{V_\alpha \mid \alpha < \kappa\}$ of X such that

(d) $\mathrm{cl}V_\alpha \subset U_\alpha$ for each α.

If we let $P = \cup \{(V_\alpha \cap O) \times (\alpha, \kappa] \mid \alpha < \kappa\}$, then P is an open set of $X \times I_\kappa$ such that

(e) $(X \times \{\kappa\}) \cap H \subset P$ (by (c))

and

(f) $(\mathrm{cl}_{X \times I_\kappa} P) \cap K = \phi$ (by (d)).

Quite similarly we have an open set Q of $X \times I_\kappa$ such that

(e)′ $(X \times \{\kappa\}) \cap K \subset Q$

and

(f)′ $(\mathrm{cl}_{X\times I_\kappa} Q) \cap H = \phi$.

On the other hand, there is an open set P_α of $X\times\{\alpha\}$ for each α such that

(g) $H \cap (X\times\{\alpha\}) \subset P_\alpha$

and

(h) $(\mathrm{cl}_{X\times\{\alpha\}} P_\alpha) \cap (K \cap (X\times\{\alpha\})) = \phi$,

because $H \cap (X\times\{\alpha\})$ and $K \cap (X\times\{\alpha\})$ are disjoint closed sets of $X\times\{\alpha\}$ which is normal.

By the topology of I_κ, each P_α is open in $X\times I_\kappa$ and $\mathrm{cl}_{X\times\{\alpha\}} P_\alpha = \mathrm{cl}_{X\times I_\kappa} P_\alpha$. Hence if we let $U = P \cup (\bigcup_{\alpha<\kappa} P_\alpha) - \mathrm{cl}Q$, then U is an open set of $X\times I_\kappa$ which contains H (by (e), (f)′ and (g)), and clU is disjoint from K (by (e)′, (f) and (h)). □

By the above theorem, we have a class of test spaces of property $\mathscr{B}$ as follows:

3.7. **Theorem** (Y. Yasui [1983])

Let X be a normal space. Then X has property $\mathscr{B}$ if and only if $X\times I_\kappa$ is normal for any infinite cardinal number κ.

Remarks

1. On "only if" part of the above Theorem 3.7, it is

sufficient to show only the case $K = X\times\{\kappa\}$ by M. Starbird ([1974]). But in the above proof we did not use his theorem because we wanted to explain the characteristic property of space I_κ.

2. When we characterize the countable paracompactness and paracompactness etc. by using the terms of normality of some product spaces, all of the spaces which are test spaces are compact. For countable paracompactness, the test spaces are $\omega+1$, I and all compact metric spaces (see Theorems 0.1) and for paracompactness, their test spaces are Stoně-Cech compactification and all compact spaces (see H. Tamano [1960]).

But our space I_κ which is used in Theorems 3.6 and 3.7 is not compact for any $\kappa > \omega$. Therefore the following question will be raise: "Is there any class $\mathcal{P}$ of compact spaces such that a normal space X has property $\mathcal{B}$ if and only if $X\times P$ is normal for any $P \in \mathcal{P}$?" But this question answered negatively by M. E. Rudin under some set-axiom ([1983-a] and [1985]).

In fact, let S be Navy's space and T some quotient space of S which is described in explanation of Chapter 2. Since T is the closed continuous image of S, $T\times C$ is normal for compact C whenever $S\times C$ is normal (M. E. Rudin [1975]). As mentioned above, S has property $\mathcal{B}$ but T does not have property $\mathcal{B}$.

CHAPTER 4
SHRINKING PROPERTY

1. Spaces having shrinking property

In 1984, K. Chiba proved that every open cover of a normal subparacompact or perfectly normal space is shrinkable.

One of the purposes of this section is to generalize the above theorem.

At first we shall study the shrinkability of open covers of specific spaces which are useful to study some examples:

4.1. Proposition (ref. Y. Yasui [1985])

Let X be a normal space having a disjoint cover $\{A, B\}$. If A is a discrete subset of X and every open cover of B is shrinkable, then every open cover of X is shrinkable.

proof

Let $\{U_\lambda | \lambda \in \Lambda\}$ be an open cover of X. Since $\{U_\lambda \cap B | \lambda \in \Lambda\}$ is an open cover of the subspace B, there exists an open cover

$\{V_\lambda \mid \lambda \in \Lambda\}$ of B such that $cl_B V_\lambda \subset U_\lambda$ for any λ. Since $cl_B V_\lambda = cl_X V_\lambda \subset U_\lambda$, we have some open set W_λ of X such that

$$cl_X V_\lambda \subset W_\lambda \subset cl_X W_\lambda \subset U_\lambda \quad (\text{for } \lambda \in \Lambda).$$

If we let $W = \bigcup_{\lambda\in\Lambda} W_\lambda$ and $O_\lambda = W_\lambda \cup ((X - W) \cap U_\lambda)$, then $\{O_\lambda \mid \lambda \in \Lambda\}$ is an open cover of X. Since W is an open set containing B and each point of X - B is isolated in X, we have;

$$cl_X O_\lambda = cl_X W_\lambda \cup cl_X((X - W) \cap U_\lambda) \subset U_\lambda \cup ((X - W) \cap U_\lambda)$$

Hence $cl_X O_\lambda \subset U_\lambda$. This completes the proof. □

Secondly we shall study a class of spaces whose open covers are shrinkable.

To begin with, we shall define the terminologies:

4.2. Definition

A space X is *subparacompact* if every open cover of X has a σ-discrete closed refinement.

4.3. Definition

A space X is *submetacompact* if every open cover of X has a sequence $\{\mathfrak{U}_n \mid n < \omega\}$ of open refinements such that for each $x \in X$, there is some n such that $ord(x, \mathfrak{U}_n)$ is finite, where *ord*$(x, \mathfrak{U}_n)$ denotes the cardinality of $\{U \mid x \in U, U \in \mathfrak{U}_n\}$.

Furthermore the above sequence $\{\mathfrak{U}_n \mid n < \omega\}$ is called *θ-sequence* or *θ-refinement* of $\mathfrak{U}$.

Subparacompactness was introduced by McAuley who called it as *F_σ-screenable* ([1958]). He showed that every collectionwise normal F_σ-screenable space is paracompact.

A submetacompact space was introduced by Worrell and Wicke who called it as *θ-refinable* space ([1965]). After then Junnila called it to be 'submetacompact' ([1978]). Junnila showed that a space X is subparacompact if and only if every open cover of X has a sequence $\{\mathfrak{U}_n \mid n < \omega\}$ of open refinements such that for each $x \in X$, there is some n with $\mathrm{ord}(x, \mathfrak{U}_n) = 1$.

Therefore it is seen that every subparacompact space is submetacompact.

Next, we recall the following:

4.4. **Definition**

A space X is said to be *perfectly normal* if X is normal and every open subset of X is a union of countably many closed subsets of X, that is, every open subset is a F_σ-set.

Whenever we discuss the shrinkage of all open covers of a space, the space must be normal.

Before we shall have a generalization of Chiba's theorem, we shall show some lemmas:

4.5. **Lemma** (Y. Yasui [1984-b])

Let $\mathfrak{U} = \{U_\alpha \mid \alpha \in A\}$ be an open cover of a normal space X and $X_f = \{x \in X \mid ord(x, \mathfrak{U})$ is finite$\}$.

Then for each $\alpha \in A$, there exists a sequence $\{U_{\alpha n} \mid n < \omega\}$ of open subsets such that

(1) $clU_{\alpha n} \subset U_\alpha$ *for any n,*

and

(2) $X_f \subset \cup \{U_{\alpha n} \mid \alpha \in A,\ n < \omega\}$.

proof

Let $X_n = \{x \in X \mid ord(x, \mathfrak{U}) \leq n\}$ for each $n < \omega$. Then X_n is closed in X and $X_f = \bigcup_n X_n$.

For each $\alpha \in A$, we let $F_{\alpha 1} = U_\alpha - \cup \{U_\beta \mid \beta \neq \alpha\}$. Then $F_{\alpha 1}$ is closed and is contained in U_α.

Since X is normal, we have an open set $U_{\alpha 1}$ of X such that $F_{\alpha 1} \subset U_{\alpha 1} \subset clU_{\alpha 1} \subset U_\alpha$. Then $\{U_{\alpha 1} \mid \alpha \in A\}$ is a cover of X_1.

We assume that for some n, there exist open sets $H_{\alpha i}$ of X for $\alpha \in A$ and i=1,2,..., n-1 such that $clU_{\alpha i} \subset U_\alpha$ for each α and each $i \leq n-1$, and $X_{n-1} \subset \cup \{U_{\alpha i} \mid \alpha \in A,\ i \leq n-1\}$.

Let for each $\alpha \in A$,

$$F_{\alpha n} = U_\alpha \cap X_n - \cup \{U_{\beta i} \mid \beta \in A,\ i \leq n-1\}.$$

Since $F_{\alpha n}$ is a closed subset of X which is contained in U_α, there exists an open set $U_{\alpha n}$ such that $F_{\alpha n} \subset U_{\alpha n} \subset clU_{\alpha n} \subset U_\alpha$.

Then $\{U_{\alpha i} \mid \alpha \in A,\ i \leq n\}$ is an open cover of X_n. By induction on n, we complete the proof of Lemma. □

4.6. **Lemma** (ref. A. Bešlagić [1986])

Assume that every open cover $\{U_\alpha \mid \alpha \in A\}$ *of a space X has an open cover* $\{V_{\alpha n} \mid \alpha \in A,\ n < \omega\}$ *of X such that* $\mathrm{cl}V_{\alpha n} \subset U_\alpha$ *for any* $\alpha \in A$, *and any* $n < \omega$. *Then every open cover of X is shrinkable.*

Every normal subparacompact or perfectly normal space is submetacompact. We shall prove the following theorem by using the above Lemmas:

4.7. **Theorem** (Y. Yasui [1984-b])

Every open cover of a normal submetacompact space is shrinkable.

proof

Let X be a normal submetacompact space and $\mathfrak{U} = \{U_\alpha \mid \alpha \in A\}$ an open cover of X.

Then there exists a sequence $\{\mathcal{V}_n \mid n < \omega\}$ of open refinements of $\mathfrak{U}$ satisfying that, for each $x \in X$, there is some n such that $\mathrm{ord}(x, \mathcal{V}_n)$ is finite. Since each $\mathcal{V}_n$ is a refinement of $\mathfrak{U}$, we may assume that $\mathcal{V}_n = \{V_{\alpha n} \mid \alpha \in A\}$ and $V_{\alpha n} \subset U_\alpha$ for any α and any n.

We let, for each n

$$X_n = \{x \in X \mid \operatorname{ord}(x, \mathscr{V}_n) \text{ is finite}\}.$$

Then by Lemma 4.6, there exist open subsets $H_{\alpha ni}$ $(i=1,2,..)$ of X such that $\mathrm{cl}H_{\alpha ni} \subset V_{\alpha n}$ for any α and any i, and

$$X_n \subset \cup \{H_{\alpha ni} \mid \alpha \in A,\ i < \omega\}.$$

Then $\{H_{\alpha ni} \mid \alpha \in A;\ n,\ i < \omega\}$ is a cover of X which satisfies the condition of Lemma 4.6, and hence $\mathscr{U}$ is shrinkable. □

By the above theorem, every normal submetacompact space has property $\mathscr{D}$, and so we have the following question:

"*Does every normal submetacompact space have property $\mathscr{B}$?*"
But this does not hold (see Example 1.7).

2. Gruenhage and Michael's problem

On the shrinkability of certain open cover each of which has some property, G. Gruenhage and E. Michael ([1983]) raised some question:

"*Is every cover of a regular space by open subsets with metrizable closures shrinkable ?*"

We shall give an affirmative answer to the above question as a corollary of the following:

4.8. **Theorem** (Y. Yasui [1984-a])

Every cover of a space by open subsets with perfectly normal closures is shrinkable.

proof

Let $\mathcal{U} = \{U_\alpha \mid \alpha \in A\}$ be an open cover of a space X such that $\mathrm{cl}U_\alpha$ is perfectly normal for any $\alpha \in A$.

Since $\mathrm{cl}U_\alpha$ is perfectly normal, there exists a collection $\{U_{\alpha n} \mid n < \omega\}$ of open subsets of X (for each α) such that

(1) $\mathrm{cl}_X U_{\alpha n} \subset U_{\alpha n+1}$ for $n < \omega$,

(2) $U_\alpha = \cup \{U_{\alpha n} \mid n < \omega\}$

and

(3) $U_{\alpha 0} = \phi$.

For each $\alpha \in A$ and each $n = 1,2,\dots$, we let

$$F_{\alpha n} = U_{\alpha n} - \{U_{\beta n-1} \mid \beta \in A\}.$$

Furthermore, let for each $\alpha \in A$,

$$F_\alpha = \mathrm{cl}_X(\cup \{F_{\alpha n} \mid n = 1,2,\dots\}).$$

Then we shall show that the collection $\mathcal{F} = \{F_\alpha \mid \alpha \in A\}$ of closed subsets of X satisfies the following claims:

claim 1 $F_\alpha \subset U_\alpha$ *for any* $\alpha \in A$.

Let $x \in X - U_\alpha$. Since $\{U_\beta \mid \beta \in A\}$ is a cover of X, there is some $\beta_0 \in A$ with $x \in U_{\beta_0}$ (hence $\alpha \neq \beta_0$).

By (2) there is some $n_0 \in \{1,2,\dots\}$ with $x \in U_{\beta_0 n_0}$, and by (1) and the definition of $F_{\alpha n}$, it is seen that

(4) $U_{\beta_0 n_0} \cap F_{\alpha m} = \phi$ for any $m > n_0$.

For n_0, we have $x \in X - cl_X U_{\alpha n_0}$ (by (2)) and

(5) $(X - cl_X U_{\alpha n_0}) \cap F_{\alpha m} = \phi$ for any $m \leq n_0$

by (1).

By (4) and (5), $U_{\beta_0 n_0} \cap (X - cl_X U_{\alpha n_0})$ is an open nbd of x which does not intersect with $\bigcup_m F_{\alpha m}$.

This shows $x \notin cl_X(\bigcup_m F_{\alpha m})$, that is, $x \notin F_\alpha$.

claim 2 $\mathcal{F}$ is a closed cover of X.

Since the closedness of F_α is clear, it sufficies to show that $\mathcal{F}$ is a cover of X. We let $x \in X$ and n_x be the first number of $\{n \mid x \in \bigcup_\alpha U_{\alpha n}\}$, and α_x any point of A with $x \in U_{\alpha n_x}$.

Then we have $x \in F_{\alpha_x n_x} \subset F_{\alpha_x}$. Hence $\mathcal{F}$ is a cover of X.

Lastly, since $cl_X U_\alpha$ is normal, we can find an open set V_α in $cl_X U_\alpha$ such that

(6) $F_\alpha \subset V_\alpha \subset cl_\alpha V_\alpha \subset U_\alpha$,

where $cl_\alpha V_\alpha$ denotes the closure of V_α in $cl_X U_\alpha$. It is seen that V_α is open in X and $cl_\alpha V_\alpha = cl_X V_\alpha$.

Therefore by (6) and claims 1 and 2, $\{V_\alpha \mid \alpha \in A\}$ is an open cover of X such that $cl_X V_\alpha \subset U_\alpha$ for each $\alpha \in A$. Then $\mathfrak{U}$ is shrinkable. □

Remark

There are several results for shrinkage of open covers in Σ-products. Let us recall a Σ-product which was introduced by H. H. Corson [1959]. Let $\{X_\lambda \mid \lambda \in \Lambda\}$ be a collection of spaces and $X = \Pi X_\lambda$ the product space of $\{X_\lambda \mid \lambda \in \Lambda\}$ and $f = (f_\lambda)_\lambda \in X$. If we let $\Sigma = \{x \in X \mid (\lambda \mid x_\lambda \neq f_\lambda)$ is at most countable$\}$, then the subspace Σ of X is called *Σ-product* of spaces $\{X_\lambda \mid \lambda\}$ with the *base point* $f \in X$. On the shrinkage of open covers of Σ-produts, there are many results (M. E. Rudin [1983-b], A. L. Donne [1985] and Y. Yajima [1986]).

As characterizations of many covering properties, it is enough to show that every increasing open cover has a corresponding refinement. For example (J. Mack [1967]):

"A space X is paracompact if and only if every increasing open cover of X has a locally finite open refinement."

For submetacompactness (resp. metacompactness), such a theorem was proved by H. J. K. Junnila [1978] (resp. W. B. Sconyers [1970]).

So we have the following question:

"If any increasing open cover of X is shrinkable, then is any open cover of X shrinkable ? "

This question means:

"If X has property $\mathscr{D}$, then is any open cover of X shrinkable ?"

This question was answered negatively under some set axiom:

4.10. **Example** ($\diamondsuit^{++}(E)$) (*A.* Bešlagić and *M. E.* Rudin [1985])

Let κ *be an infinite regular cardinal number.*
Then, there is a space Δ *such that* Δ *is ultra* κ*-paracompact and collectionwise normal, and every increasing open cover* $\{U_\alpha \mid \alpha \in A\}$ *of* Δ *has a refinement* $\{V_\alpha \mid \alpha \in A\}$ *consisting of open and closed sets, but there is an open cover* $\{W_\lambda \mid \lambda \in \Lambda\}$ *which has no closed refinement* $\{F_\lambda \mid \lambda \in \Lambda\}$ *with* $F_\lambda \subset W_\lambda$.

3. Countably many product spaces

It is seen that two many product space of the spaces with property $\mathscr{B}$ (resp. $\mathscr{D}$) does not have property $\mathscr{B}$ (resp. $\mathscr{D}$), because the square of Sorgenfrey line which is paracompact is not countabky paracompact.

Hence we must add some conditions whenever we discuss property $\mathscr{B}$ of product spaces.

In this section we shall comment about the countably many product spaces of property $\mathscr{B}$.

Let $X = \Pi \{X_n | n < \omega\}$ be a product space. As is well-known:

(*) "If X_n has some property P for each n, then X also has one."

does not hold for almost all the covering properties P. But:

(**) "$\prod_{i \leq n} X_i$ has property P and is perfectly normal for any n, then X has property P."

holds for many properties (A. Okuyama [1968] for paracompactness, E. Michael [1971] for Lindelöf property and T. C. Przymusinski [1984] for countable paracompactness).

For shrinkage of open covers, the following theorem is known:

4.11. **Theorem** (A. Bešlagić [1986])

Let X be a normal product space of $\{X_n | n < \omega\}$.
Then every open cover of X is shrinkable if and only if every open cover of $\Pi \{X_i | i \leq n\}$ *is shrinkable for all* $n < \omega$.

We shall replace 'is shrinkable' with 'has property $\mathscr{B}$'. Though its proof is the almost same way but the last part, some characterization of property $\mathscr{B}$ is useful:

4.12. **Theorem**

Let X be a normal product space of $\{X_n \mid n < \omega\}$. Then X has property $\mathscr{B}$ if and only if $\Pi\{X_i \mid i \leq n\}$ has property $\mathscr{B}$ for all $n < \omega$.

proof

Let $\{U_\alpha \mid \alpha < \tau\}$ be an increasing open cover of X. If we let for each $\alpha < \tau$ and each $n < \omega$,

$$U_{\alpha n} = \cup \{O \mid O \text{ is open in } \prod_{i \leq n} X_i \text{ and } O \times \prod_{i > n} X_i \subset U_\alpha\},$$

then $\{U_{\alpha n} \mid \alpha\}$ is an increasing collection of open sets of $\prod_{i \leq n} X_i$ for each n. Furthermore we let $O_n = (\bigcup_{\alpha < \tau} U_{\alpha n}) \times \prod_{i > n} X_i$ for each n. Then $\{O_n \mid n < \omega\}$ is an increasing open cover of X. Since X is countably paracompact (T. C. Przymusinski [1984]), there is an increasing open cover $\{S_n \mid n < \omega\}$ of X such that $\mathrm{cl} S_n \subset O_n$ for any n (Theorem 0.2). Let p_n be the projection from X to $\prod_{i \leq n} X_i$ and $T_n = \prod_{i \leq n} X_n - p_n(X - \mathrm{cl} S_n)$ for any $n < \omega$, then T_n is a closed subset of $\prod_{i \leq n} X_i$ and $T_n \subset \bigcup_\alpha U_{\alpha n}$.

Since T_n has property $\mathscr{B}$, there is an increasing open cover $\{V_{\alpha n} \mid \alpha < \tau\}$ of T_n such that $\mathrm{cl}_{T_n}(V_{\alpha n}) \subset U_{\alpha n}$ for each α (where the closure of $V_{\alpha n}$ in T_n = the closure of it in $\prod_{i \leq n} X_n$).

We let for each n and each α,

$$W_{\alpha n} = (V_{\alpha n} \cap \mathrm{Int}(T_n)) \times \prod_{i > n} X_i .$$

Then $\{W_{\alpha n} \mid \alpha\}$ is an increasing collection of open subsets of X such that $\mathrm{cl} W_{\alpha n} \subset U_\alpha$ for any α and any n. Since it is seen that $\{W_{\alpha n} \mid \alpha < \tau,\ n < \omega\}$ is a cover of X, X has property $\mathscr{B}$ (T. Tani and Y. Yasui [1972]). □

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