Multi-Valued Function Spaces and Compactifications of Function Spaces

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THESIS

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Contents

Introduction

Given a space E, an *E-manifold* is a space locally homeomorphic to E, that is, each point has a neighborhood homeomorphic ($\approx$) to an open subset of E. Given a space E and a dense set F in E, an *(E,F)-manifold* is a pair (M,N) of spaces such that each $x \in M$ has a neighborhood U homeomorphic to an open set V of E with $(U, N \cap U) \approx (V, F \cap V)$. As a space E, we mainly deal with the Hilbert cube $Q = [-1,1]^\infty$, the pseudo-interior $s = (-1,1)^\infty$ of Q and the Hilbert space $\ell_2(A)$, where A is an infinite set. The earliest non-trivial result in Infinite-Dimensional Topology is the homogeneity of Q [Kel]. It is well-known that s is homeomorphic ($\approx$) to the Hilbert space $\ell_2 = \ell_2(\mathbb{N})$ [An_1].

Concerning Q-manifolds, many interested results have been obtained, e.g., the triangulation theorem [Ch_4], the factor theorem, the homeomorphisms approximation theorem [Ed], the classification theorem [Ch_3], the characterization of Q-manifolds [To_5], etc. It should be remarkable that the excellent applications of the Q-manifold theory to other branches has been obtained, e.g., the complement theorem in shape theory [Ch_2], the proof of the topological invariance of Whitehead torsion [Ch_5], the proof of the finiteness of the homotopy type of compact ANR's [We_2]. About $\ell_2(A)$-manifolds, the classification theorem [HS] (cf. [He_2]), the factor theorem [To_1], the stability theorem [AS], the open embedding theorem [He_1], the characterization of

$\ell_2(A)$-manifolds [To_6] and others were also established.

To characterize the pair (Q, s) or (Q, s)-manifold, the notion of cap-sets was introduced by Anderson [An_2] (cf. [Ch_1]). In our research, this notion plays an important role. On the other hand, this has been extended to the notion of absorbers by different authors (cf. [Ba], [BGM], [BM], [We_1], etc.). They enable us to study the topology of certain incomplete dense subsets of Q or $\ell_2(A)$. In Chapter 2, we modify the absorbers in the sense of Banakh [Ba] and extend some theorems on them.

In this thesis, we mainly study infinite-dimensional manifolds of functions or multi-valued functions. Let (X, d) and (Y, d') be metric spaces, $\mathbf{I} = [0, 1]$ the unit interval, $\mathbb{R}$ the real line, and $\overline{\mathbb{R}} = [-\infty, \infty]$ the extended real line. The product space $X \times Y$ admits the metric ρ defined as follows:

$$\rho((x, y), (x', y')) = \max\{d(x, x), d'(y, y')\}.$$

By 2^X, we denote the hyperspace of non-empty closed subsets of X endowed with the infinite-valued metric d_{H} defined by

$$d_{\mathrm{H}}(E, F) = \max\{\sup_{z \in E} d(z, F), \sup_{z \in F} d(z, E)\}.$$

By $\exp(X)$, we denote the subspace of 2^X consisting of compact subsets of X. Then d_{H} is a metric on $\exp(X)$. A multi-valued function $\varphi\colon X \to Y$ is said to be *upper semi-continuous* (u.s.c.) if the subset $\{x \in X \mid \varphi(x) \subset U\}$ is open for any open subset U of Y. Let $\mathrm{USC}(X, Y)$ be the space of u.s.c. multi-valued functions $\varphi\colon X \to Y$ such that each $\varphi(x)$ is compact, and let

$$\mathrm{USCC}(X, Y) = \{\varphi \in \mathrm{USC}(X, Y) \mid \text{ each } \varphi(x) \text{ is connected}\}.$$

By identifying each function $\varphi \in \mathrm{USC}(X, Y)$ with its graph, we regard $\mathrm{USC}(X, Y)$ as a subspace of $2^{X \times Y}$. In case X is non-compact, we consider the spaces

$$\mathrm{USC}_B(X, Y) = \{\varphi \in \mathrm{USC}(X, Y) \mid \textstyle\bigcup_{x \in X} \varphi(x) \text{ is bounded}\},$$

$$\mathrm{USCC}_B(X,Y) = \mathrm{USC}_B(X,Y) \cap \mathrm{USCC}(X,Y).$$

It should be noted that ρ_{H} is a metric on $\mathrm{USC}_B(X,Y)$. In case $Y = \mathbb{R}$, we denote simply $\mathrm{USC}_B(X) = \mathrm{USC}_B(X,\mathbb{R})$ and $\mathrm{USCC}_B(X) = \mathrm{USCC}_B(X,\mathbb{R})$.

Through Chapter 3 to 5, we consider compactifications of function spaces on a compactum X. The Banach space $\mathrm{C}(X)$ of real-valued continuous functions on X is homeomorphic to s by [An$_1$] and [Ka]. Since the Hilbert cube Q is a natural compactification of s, we can regard Q as a compactification of $\mathrm{C}(X)$. It is worthwhile considering the following:

Problem 1. *How can we obtain a natural compactification of* $\mathrm{C}(X)$ *homeomorphic to* Q*?*

On the other hand, Fedorchuk [Fe$_2$] showed that if X is a locally connected compactum without isolated points then the closure $\mathrm{C}_{\mathrm{H}}(X)$ of $\mathrm{C}(X)$ in $2^{X\times\mathbb{R}}$ coincides with $\mathrm{USCC}(X)$. He also proved the following:

Theorem 1 [Fe$_2$]. *If X is an infinite locally connected compact metric space, then* $\mathrm{C}_{\mathrm{H}}(X,\mathbf{I}) \approx Q$ *and* $\mathrm{C}_{\mathrm{H}}(X) \approx Q \setminus \{\mathrm{pt}\}$, *hence* $\alpha(\mathrm{C}_{\mathrm{H}}(X)) \approx Q$, *where* $\alpha(\mathrm{C}_{\mathrm{H}}(X))$ *is the Alexandroff one-point compactification of* $\mathrm{C}_{\mathrm{H}}(X)$.

And then, he posed the following question:

Problem 2 [Fe$_2$]. *For an infinite locally connected comactum X, are the pairs* $(\alpha(\mathrm{C}_{\mathrm{H}}(X,\mathbf{I})), \mathrm{C}(X,\mathbf{I}))$ *and* (Q,s) *homeomorphic ?*

These two probrems are studied in Chapter 3. Considering the closure $\overline{\mathrm{C}}(X)$ of $\mathrm{C}(X)$ in $2^{X\times\overline{\mathbb{R}}}$ as a compactification of $\mathrm{C}(X)$, we can answer to Probrem 1 as follows:

Theorem 2 [SU$_2$]. *If X is infinite, locally connected and compact then*

$$(\overline{\mathrm{C}}(X), \mathrm{C}(X)) \approx (Q,s).$$

By applying this theorem, we have the affirmative answer to Probrem 2. In the above, if X has no isolated points, then $\overline{\mathrm{C}}(X)$ coincides with $\mathrm{USCC}(X, \overline{\mathbb{R}})$

In case X is not locally connected, if X is a zero-dimensional compacum without isolated points, then the space $\mathrm{USC}(X, \overline{\mathbb{R}})$ is a compactification of $\mathrm{C}(X)$ [FK]. But, $\mathrm{USC}(X)$ is not an ANR in this case, though $\mathrm{USCC}(X)$ is an AR (this will be shown in Chapter 6). The space $\mathrm{USC}(X)$ is studied in Chapter 4. The following result is obtained:

Theorem 3. *The space* $\mathrm{USC}(X, Y)$ *is a* Q*-manifold if* X *and* Y *are locally connected compacta and* Y *has no isolated points. Furthermore, if* Y *is connected then* $\mathrm{USC}(X, Y) \approx Q$.

It is one of important probrems in Infinite-Dimensional Topology to determin whether or when the homeomorphism group $\mathrm{H}(M)$ of a compact n-manifold M is an ANR, and hence an ℓ_2-manifold. For $n = 1$ (or $n = 2$), it was proved that $\mathrm{H}(M)$ is an ℓ_2-manifold in [An$_3$] (or [LM]+[Ge$_1$]+[To$_1$]). The case $n > 2$, this probrem is still open. In the case M is a compact Q-manifold, it was shown that $\mathrm{H}(M)$ is an ℓ_2-manifold ([Fer], [To$_3$]). The following is related with this probrem:

Probrem 3 [We$_3$].

(1) *Is the closure* $\mathrm{clH}(M)$ *in* $\mathrm{C}(M, M)$ *an ANR?*

(2) *Is* $\mathrm{H}(M)$ *homotopy dense in* $\mathrm{clH}(M)$*?*

Let $\overline{\mathrm{H}}(M)$ be the closure of $\mathrm{H}(M)$ in $\exp(M^2)$. Then $\overline{\mathrm{H}}(M)$ is a compactification of $\mathrm{H}(M)$. We have the following version of Probrem 3:

Probrem 4. *Is the pair* $(\overline{\mathrm{H}}(M), \mathrm{H}(M))$ *a* (Q, s)*-manifold?*

On the other hand, the space $\mathrm{R}(M)$ of retractions of a compact n-manifold M has been studied. It was also proved in [BS] (or [Ca$_2$]) that $\mathrm{R}(M)$ is an ℓ_2-manifold for

$n = 1$ (or $n = 2$). The case $n > 2$ is unknown. In case M is a compact Q-manifold, it was proved that $\mathrm{R}(M)$ is an ℓ_2-manifold ([Ch_6]+[Sa_1]). Let $\overline{\mathrm{R}}(M)$ be the closure of $\mathrm{R}(M)$ in $\exp(M^2)$. Then $\overline{\mathrm{R}}(M)$ is a compactification of $\mathrm{R}(M)$. Similarly to Probrem 4, we have the following:

Probrem 5. *Is the pair* $(\overline{\mathrm{R}}(M), \mathrm{R}(M))$ *a* (Q, s)*-manifold?*

Probrems 4 and 5 above are considered in Chapter 5. In case $n = 1$, we have the following theorems:

Theorem 4 [SU_1]. *If* G *is a graph, then* $(\overline{\mathrm{H}}(G), \mathrm{H}(G))$ *is a* (Q, s)*-manifold.*

Theorem 5 [Ue_2]. $(\overline{\mathrm{R}}(\mathbf{I}), \mathrm{R}(\mathbf{I})) \approx (Q, s)$.

In the remaining part, we consider the space $\mathrm{USCC}_B(X)$ without assuming that X is compact. In Chapter 6, for a complete metric space X, we give necessary and sufficient conditions in order that $\mathrm{USCC}_B(X)$ is an AR [Ue_3]. As a corollary, we obtain that $\mathrm{USCC}_B(X)$ is an AR whenever X is compact [Ue_3]. We also give a necessary and sufficient condition for X in order that $\mathrm{USCC}_B(X)$ is closed in $2^{X \times \mathbb{R}}$ [Ue_3]. Furthermore, the converse of Theorem 2 can be proved, that is,

Theorem 6 [SU_4]. *If* $\mathrm{USCC}_B(X, \mathbf{I}) \approx Q$*, then* X *is infinite, locally connected and compact.*

Chapter 7 is devoted to the study of a compactification of the multi-valued function space $\mathrm{USCC}_B(X)$ for a certain non-compact totally bounded space X. A metric space $X = (X, d)$ (or a metric d) has *Property S* if X is covered by finitely many connected sets with arbitrarily small diameters. Curtis [Cu_1] proved that X admits a Peano compactification $\tilde{X}$ such that

$$(\exp(\tilde{X}), \exp(X)) \approx (Q, s)$$

if and only if X is connected, locally connected, completely metrizable, nowhere locally compact and admits an admissible metric d with Property S. We have the following version of Curtis' result:

Theorem 7 [SU$_4$]. *A metrizable space X has a metrizable compactification $\tilde{X}$ such that*

$$(\mathrm{USCC}(\tilde{X}, \mathbf{I}), e_{\tilde{X}}(\mathrm{USCC}(X, \mathbf{I}))) \approx (Q, s)$$

if and only if X is completely metrizable, non-compact and admits an admissible metric with Property S, where $e_{\tilde{X}} \colon \mathrm{USCC}(X) \to \mathrm{USCC}(\tilde{X})$ is a natural isometric embedding defined by $e_{\tilde{X}}(\varphi) = \mathrm{cl}_{\tilde{X}\times\mathbb{R}}\,\varphi$

In Chapter 8, we consider when the space $\mathrm{USCC}_B(X)$ is homeomorphic to Hilbert space. It is said that X is *uniformly locally connected* if for each $\varepsilon > 0$ there is $\delta > 0$ such that each pair of points $x, x' \in X$ with $d(x, x') < \delta$ are contained in some connected set in X with diameter $< \varepsilon$. We have the following:

Theorem 8 [SU$_3$]. *If X is a non-compact uniformly locally connected complete metric space, then $\mathrm{USCC}_B(X)$ is homeomorphic to a non-separable Hilbert space. In case X is separable, $\mathrm{USCC}_B(X) \approx \ell_2(2^{\mathbb{N}})$.*

CHAPTER 1

Preliminaries

In this chapter, we introduce some definitions, and we present some basic results.

§1.1. GENERL DEFINITIONS AND NOTATIONS

All spaces in this desertation are assumed to be a metric spaces. A continuous function is called a *map*. The standard sets and spaces are listed below:

(1) $\mathbb{N}$: the set of natural numbers,

(2) $\mathbb{R} = (-\infty, \infty)$: the real line with usual metric,

(3) $\overline{\mathbb{R}} = [-\infty, \infty]$: the extended real line,

(4) $\mathbf{I} = [0, 1]$: the unit closed interval,

(5) Δ^n : the standard n-simplex.

Let $X = (X, d)$ be a space, $A, B \subset X$ and $\varepsilon > 0$. We use the following notations:

(6) (A, B) : the pair of the spaces such that $B \subset A$,

(7) $\mathrm{cl}_X A$: the closure of A in X,

(8) $\mathrm{int}_X A$: the interior of A in X,

(9) $\partial_X A$: the boundary of A in X,

(10) $w(X)$: the weight of X,

(11) $\mathrm{card}(X)$: the cardinality of X,

(12) $\mathrm{diam}(X)$: the diameter of X,

(13) $\mathrm{mesh}(\mathcal{U}) = \sup\{\mathrm{diam}\, U \mid U \in \mathcal{U}\}$,

(14) $\mathrm{cov}(X)$: the collection of open covers of X,

(15) $\mathrm{dist}(A, B) = d(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}$,

(16) $B_d(A, \varepsilon) = \{x \in X \mid d(x, A) < \varepsilon\}$: the open ε-ball about A (we denote $B_d(A, \varepsilon) = B_d(a, \varepsilon)$ if $A = \{a\}$),

(17) $\overline{B}_d(A, \varepsilon) = \mathrm{cl}_X\, B_d(A, \varepsilon)$ (we denote $\overline{B}_d(A, \varepsilon) = \overline{B}_d(a, \varepsilon)$ if $A = \{a\}$).

It should be remarked that $\overline{B}_d(A, \varepsilon) \neq \{x \in X \mid d(x, A) \leq \varepsilon\}$ in general. Let K be a simplicial complex. We denote:

(18) $\mathrm{Sd}(K)$: the barycentric subdivision of a simplicial complex K,

(19) $|K|$: the geometric realization of K,

(20) $K^{(n)}$: the n-skeleton of K,

(21) $\sigma^{(n)}$: the n-face of $\sigma \in K$.

(22) $\sigma < \tau$ means that σ is a proper face of τ.

For $\mathcal{U} \in \mathrm{cov}(X)$, we denote:

(23) $\mathrm{N}(\mathcal{U})$: the nerve of $\mathcal{U} \in \mathrm{cov}(U)$,

(24) $\mathrm{St}(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U} \mid A \cap U \neq \emptyset\}$ (we denote $\mathrm{St}(A, \mathcal{U}) = \mathrm{St}(a, \mathcal{U})$ if $A = \{a\}$).

Let Y be a space, $C \subset Y$, $\mathcal{V} \in \mathrm{cov}(Y)$ and $f, g \colon X \to Y$ functions. Then we denote

(25) $f|A$: the restriction of f onto A,

(26) a continuous function f is called a *map*,

(27) id (or id_X) : the identity map on X,

(28) $\mathrm{pr}_X \colon X \times Y \to X$, $\mathrm{pr}_Y \colon X \times Y \to Y$: the projections.

(29) We say that f is *$\mathcal{V}$-close* (or *ε-close*) *to* g if $\{f(x), g(x)\}$ is contained in some member of $\mathcal{V}$ (or if $d_Y(f(x), g(x)) < \varepsilon$) for any $x \in X$.

(30) A map $h \colon X \times \mathbf{I} \to Y$ is called a *homotopy*. For $t \in \mathbf{I}$, we define the map

$h_t\colon X \to Y$ by $h_t(x) = h(x, y)$.

(31) By $X \cong Y$, we mean that X is homotopically equivalent to Y,

(32) $X \approx Y$ means that X is homeomorphic to Y. Correspondingly, $(X, A) \approx (Y, C)$ means that X is homeomorphic to Y by a homeomorphism $h\colon X \mapsto Y$ such that $h(A) = C$.

(33) $f \cong g$ means that f and g are homotopic, i.e., there is a homotopy $h\colon X \times \mathbf{I} \to Y$ such that $f = h_0$ and $g = h_1$,

(34) we say that f and g are $\mathcal{U}$- (or ε-) homotopic if there is a homotopy $h\colon X \times \mathbf{I} \to Y$ such that $\{h(\{x\} \times \mathbf{I}) \mid x \in X\}$ refines $\mathcal{U}$ (or $\operatorname{mesh}\{h(\{x\} \times \mathbf{I}) \mid x \in X\} < \varepsilon$) with $f = h_0$ and $g = h_1$.

Let $\varphi\colon X \to Y$ be a multi-valued function.

(35) $\varphi(X) = \bigcup_{x \in X} \varphi(x) \subset Y$,

(36) φ is called *bounded* if $\operatorname{diam} \varphi(X) < \infty$,

(37) φ is called *lower semi-continuous* (abbrev. l.s.c.) if $\{x \in X \mid \varphi(x) \cap U \neq \emptyset\}$ is open in X for any open set of Y,

(38) φ is called *upper semi-continuous* (abbrev. u.s.c.) if $\{x \in X \mid \varphi(x) \subset U\})$ is open in X for any open set of Y.

§1.2. Basic facts on ANR theory.

Let X be a space and $A \subset X$. A map $r\colon X \to A$ is called a *retraction* of X onto A if $r|A = \mathrm{id}$, namely r is an extension of the identity map of A. Then A is called a *retract* of X. A *neighborhood retract* of X is a closed set in X which is a retract of some neighborhood in X. A space X is called an *absolute retract* (or an *absolute neighborhood retract*) if X is a retract (or a neighborhood retract) of an arbitrary space which contains X as a closed set. We abbreviate an absolute retract (or an absolute neighborhood retract) to an AR (or an ANR).

In this section, we list some results of ANR theory which will be used in the sequel.

Theorem 1.2.1 (cf. [vM$_2$]). *A contractible ANR is an AR.*

A subset $A \subset X$ is called *homotopy negligible in* X, if there is a homotopy $h \colon X \times \mathbf{I} \to X$ such that $h_0 = \mathrm{id}$ and $h_t(X) \subset X \setminus A$ for each $t > 0$. Correspondingly, A is called *homotopy dense* in X if the complement $X \setminus A$ is homotopy negligible in X.

Theorem 1.2.2 [To$_4$]. *A homotopy dense subset of an AR (resp. ANR) is also an AR (resp. ANR).*

Theorem 1.2.3 [To$_4$]. *Let X be an ANR and $A \subset X$ a subset. Then, the following statements are equivalent;*

(1) *A is homotopy dense (resp. negligible) set in X,*

(2) *for every $\mathcal{U} \in \mathrm{cov}(X)$ and a map $f \colon \mathbf{I}^n \to X$ with $f(\partial\mathbf{I}^n) \subset A$ (resp. $f(\partial\mathbf{I}^n) \subset X \setminus A$) there exists a map $\tilde{f} \colon \mathbf{I}^n \to A$ (resp. $\tilde{f} \colon \mathbf{I}^n \to X \setminus A$) such that $\tilde{f}|\partial\mathbf{I}^n = f|\partial\mathbf{I}^n$ and $\tilde{f}$ is ε-close to f,*

(3) *for every open set U of X, the set $U \cap A$ is homotopy dense (resp. negligible) in U.*

Theorem 1.2.4 [Bsk]. *Let Y be an ANR, $\mathcal{U} \in \mathrm{cov}(Y)$ and $h \colon A \times \mathbf{I} \to Y$ a $\mathcal{U}$-homotopy of a closed set A in a space X. If h_0 extends to a map $f \colon X \to Y$, then h extends t a $\mathcal{U}$-homotopy $\tilde{h} \colon A \times \mathbf{I} \to Y$ such that $\tilde{h} = f$.*

Recall that a space X is *homotopy dominated by a space* Y if there are maps $f \colon X \to Y$ and $g \colon Y \to X$ such that $gf \cong \mathrm{id}_X$. In case gf is $\mathcal{U}$- (or ε-) homotopic to id for $\mathcal{U} \in \mathrm{cov}(X)$ (or $\varepsilon > 0$), we say that X is $\mathcal{U}$- (or ε-) *homotopy dominated by* Y.

Theorem 1.2.5 [Ha]. *For a metric space X, the following are equivalent:*

(1) *X is an ANR;*

(2) *For each $\mathcal{U} \in \mathrm{cov}(X)$, there is a simplicial complex K such that X is $\mathcal{U}$-homotopy dominated by $|K|$;*

(3) *For each $\varepsilon > 0$, X is ε-homotopy dominated by an ANR.*

A homotopy equivalence $f\colon X \to Y$ is said to be a *fine homotopy equivalence* if for each $\mathcal{U} \in \mathrm{cov}(Y)$, there exists a map $g\colon Y \to X$ such that fg and gf are $\mathcal{U}$- and $f^{-1}(\mathcal{U})$-homotopy equivalent to id_Y and id_X, respectively.

Theorem 1.2.6 [Ko]. *Let $f\colon X \to Y$ be a map from an ANR X to a space Y. If $f(X)$ is dense in Y and for each $\mathcal{U} \in \mathrm{cov}(Y)$ there exists a map $g\colon Y \to X$ such that gf is $f^{-1}(\mathcal{U})$-homotopy equivalent to id_X, then f is a fine homotopy equivalence and Y is an ANR.*

§1.3. Hilbert space manifolds and Hilbert cube manifolds.

Let E be a space. A space M is defined to be an *E-manifold* if each point $x \in M$ has a neighborhood homeomorphic to an open subset of E. Then, we say that M is modeled on E, and E is called a *model* space.

As a model space, we mainly deal with the *Hilbert cube*

$$Q = [-1, 1]^\infty$$

endowed with the metric d defined by

$$d(x, y) = \sum_{i \in \mathbb{N}} 2^{-i}|x_i - y_i|.$$

We will study also the following subspaces of Q:

- $s = (-1,1)^\infty$: the *pseudo-interior* of Q,
- $B(Q) = Q \setminus s$: the *pseudo-boundary* of Q,
- $\Sigma = \{(x_i) \in Q \mid \sup_i |x_i| < 1\}$: the *radial interior* of Q,
- $\sigma = \{(x_i) \in s \mid t_i \neq 0 \text{ for finitely many of } i\}$

  : the linear span of orthonormal basis of Q.

We introduce the following characterization of Q-manifolds that will be used quite often.

Theorem 1.3.1 [To$_5$]. *A space X is a Q-manifold (resp. $X \approx Q$) if and only if*

(1) *X is a locally compact ANR (resp. X is a compact AR),*

(2) *for any $\mathcal{U} \in \operatorname{cov}(X)$ and any map $f\colon Q \times \{0,1\} \to X$ there is a map $g\colon Q \times \{0,1\} \to X$ such that g is $\mathcal{U}$-close to f and $g(Q \times \{0\}) \cap g(Q \times \{1\}) = \emptyset$.*

In the above characterization, the condition (2) is equivalent to the following;

(2)$'$ for any $\varepsilon > 0$, any $n \in \mathbb{N}$ and any map $f\colon \mathbf{I}^n \times \{0,1\} \to X$ there is a map $g\colon \mathbf{I}^n \times \{0,1\} \to X$ such that g is ε-close to f and $g(\mathbf{I}^n \times \{0\}) \cap g(\mathbf{I}^n \times \{1\}) = \emptyset$.

The condition (2)' is called the *disjoint cells property* (abbrev. DCP).

A closed subset $A \subset X$ is called a (*strong*) *Z-set* in X if for every $\mathcal{U} \in \operatorname{cov}(X)$ there is a map $p\colon X \to X$ $\mathcal{U}$-close to id with $p(X) \cap A = \emptyset$ ($\operatorname{cl}_X p(X) \cap A = \emptyset$). The union of countably many Z-sets is called a *Z_σ-set.* An embedding $e\colon X \to Y$ is called a *Z-embedding* if $e(X)$ is a Z-set in Y. We also introduce the following characterization of the pair $(Q, B(Q))$:

Theorem 1.3.2 [An$_2$] **(cf.** [Ch$_1$]. *For a subset $M \subset Q$, $(Q, M) \approx (Q, B(Q))$ if and only if M is a Z_σ-set in Q and satisfies the following condition:*

($\flat$) *for any pair (A, B) of compacta in Q such that $B \subset M$ and for any $\varepsilon > 0$, there exists a closed embedding $h\colon A \to M$ such that $h|B = \mathrm{id}$ and h is ε-close* *to* id.

For an infinite set Γ, let $\mathcal{F}(\Gamma)$ be the set of all non-empty finite subsets of Γ. Then, $\mathcal{F}(\Gamma)$ is directed by $\subset$. Let $x \in \mathbb{R}^\Gamma$ be a real-valued function on Γ. By $\sum_{\gamma\in\Gamma} x(\gamma) < \infty$, we mean that $(\sum_{\gamma\in F} x(\gamma))_{F\in\mathcal{F}(\Gamma)}$ is convergent, whence we define $\sum_{\gamma\in\Gamma} x(\gamma) = \lim_{F\in\mathcal{F}(\Gamma)} \sum_{\gamma\in F} x(\gamma)$. By $\ell_2(\Gamma)$, we denote the Hilbert space defined as follows:

- $\ell_2(\Gamma) = \{x \in \mathbb{R}^\Gamma \mid \sum_{\gamma\in\Gamma} x(\gamma)^2 < \infty\}$

with the inner product $\langle x, y\rangle = \sum x(\gamma)y(\gamma)$.

In [To$_6$, Theorem 3.1] (cf. [To$_7$]), Toruńczyk characterized Hilbert space as the following:

Theorem 1.3.3 [To$_6$]. *Let A be a discrete space and $H = (H, d)$ a complete AR with weight $w(H) = \operatorname{card} A$. Then $H \approx \ell_2(A)$ if and only if the following two conditions are satisfied:*

(1) *For each $n \in \mathbb{N}$ and $\mathcal{U} \in \operatorname{cov}(H)$, each map $f\colon \mathbf{I}^n \times A \to H$ is $\mathcal{U}$-close to a map $g\colon \mathbf{I}^n \times A \to H$ such that $\{g(\mathbf{I}^n \times \{a\}) \mid a \in A\}$ is discrete in H.*

(2) *For any countable family $\{K_n \mid n \in \mathbb{N}\}$ of finite dimensional simplicial complexes with $\operatorname{card} K^{(0)} \leq \operatorname{card} A$ and for any $\mathcal{U} \in \operatorname{cov}(H)$, each map $f\colon \bigoplus_{n\in\mathbb{N}} |K_n| \to H$ is $\mathcal{U}$-close to a map $g\colon \bigoplus_{n\in\mathbb{N}} |K_n| \to H$ such that $\{g(|K_n|) \mid n \in \mathbb{N}\}$ is discrete in H,*

where $\bigoplus$ represents the topological sum.

§1.4. Hyperspaces

By 2^X, we denote the hyperspace of non-empty closed subsets of X endowed with the infinite-valued metric d_{H} defined by

$$d_{\mathrm{H}}(E,F) = \max\{\sup_{z\in E} d(z,F), \sup_{z\in F} d(z,E)\}.$$

This d_{H} is called *the Hausdorff metric induced by d*. By $\exp(X)$ (or $(2^X)_m$), we denote the subspace of 2^X consisting of compact (or bounded) subsets of X. Then d_{H} is a metric on $\exp(X)$, and on $(2^X)_m$ (cf. [Ku, p.214]). If X is complete, then $(2^X)_m$ is also complete [Ku, p.407]. In case X is compact, we can regard $(2^X)_m = \exp(X) = 2^X$. When X is unbounded, $2^X \neq (2^X)_m$ and d_{H} is not a metric on the whole 2^X (e.g., $X \notin (2^X)_m$ and $d_{\mathrm{H}}(\{x\},X) = \infty$ for any $x \in X$), but d_{H} induces the topology on 2^X. In fact, $A \in 2^X$ has a neighborhood base consisting of

$$\{B \in 2^X \mid d_{\mathrm{H}}(A,B) < \varepsilon\} \ \big(= \{B \in 2^X \mid A \subset N_d(B,\varepsilon),\ B \subset N_d(A,\varepsilon)\}\big).$$

§1.5. Spaces of multi-valued functions

The product space $X \times Y$ admits the metric ρ defined as follows:

$$\rho((x,y),(x',y')) = \max\{d(x,x), d'(y,y')\}.$$

Let $2^{X\times Y}$ admit the Hausdorff metric ρ_{H} induced by ρ. By $\mathrm{USC}(X,Y)$, we denote the space of u.s.c. multi-valued functions $\varphi\colon X \to Y$ such that each $\varphi(x)$ is compact. The following proposition unable us to identify each function $\varphi \in \mathrm{USC}(X,Y)$ with its graph $\mathrm{Gr}\,\varphi \in 2^{X\times Y}$:

Proposition 1.5.1 (cf. [FK]). *Let $\varphi\colon X \to Y$ be a multi-valued function with each $\varphi(x)$ is compact. Then,* $\mathrm{Gr}\,\varphi$ *is closed in $X \times Y$ if and only if $\varphi \in \mathrm{USC}(X,Y)$.*

By this proposition, we can regard $\mathrm{USC}(X,Y)$ as a subspace of $2^{X\times Y}$. We also consider the subspaces

- $\mathrm{USC}_B(X,Y) = \{\varphi \in \mathrm{USC}(X,Y) \mid \bigcup_{x\in X} \varphi(x) \text{ is bounded}\}$,
- $\mathrm{USCC}_B(X,Y) = \{\varphi \in \mathrm{USC}_B(X,Y) \mid \text{ each } \varphi(x) \text{ is connected}\}$.

It should be denoted that ρ_{H} is a metric on $\mathrm{USC}_B(X, Y)$. In case $Y = \mathbb{R}$, we denote simply $\mathrm{USC}_B(X) = \mathrm{USC}_B(X, \mathbb{R})$ and $\mathrm{USCC}_B(X) = \mathrm{USCC}_B(X, \mathbb{R})$.

Let $\mathrm{C}_B(X)$ be the subspace of $\mathrm{USCC}_B(X)$ consisting of bounded maps. By

$$||f|| = \sup\{|f(x)| \ : \ x \in X\},$$

we define the *sup-norm* on $\mathrm{C}_B(X)$. The space $\mathrm{C}_B(X)$ with the sup-norm is a Banach space. In general, the topology of $\mathrm{C}_B(X)$ (induced by ρ_{H}) is coarser than the one induced by the sup-norm, that is,

Proposition 1.5.2. *For any $f, g \in \mathrm{C}_B(X)$, we have $||f - g|| \leq \rho_{\mathrm{H}}(f, g)$.*

Proof. This easily follows by the definition. □

In case X is compact, since every map $f \colon X \to Y$ is bounded, we write simply $\mathrm{C}(X) = \mathrm{C}_B(X)$. In this case, we may assume that $\mathrm{C}(X)$ has the sup-norm, that is,

Proposition 1.5.3. *If X is compact, then the topology of $\mathrm{C}(X)$ induced by the sup-metric coincides with the one induced by ρ_{H}.*

Proof. This easily follows by the compactness. □

Chapter 2

Remarks on absorbers

By $\mathcal{M}_0$, $\mathcal{G}_\delta = \mathcal{M}_1$ and $\mathcal{F}_\sigma = \mathcal{A}_1$, we denote the class of compacta, the class of completely metrizable spaces and the class of σ-compacta, respectively. We say that a class $\mathcal{C}$ is *topological* if it contains all topological copies of every member of $\mathcal{C}$, and that $\mathcal{C}$ is *closed-hereditary* if it contains every closed subspace of any member of $\mathcal{C}$. By $\mathcal{C}_\sigma$ we denote the class whose elements are countable union of its closed sets belong to $\mathcal{C}$. A pair (X, Y) is said to be *strongly* $(\mathcal{M}_0, \mathcal{C})$*-universal* if for every $B \in \mathcal{M}_0$ and every subset $C \subseteq B$ with $C \in \mathcal{C}$, any map $f : B \to X$ which restricts to a Z-embedding on a closed subset $K \subseteq B$ can be approximated by a Z-embedding $g : B \to X$ such that $g|K = f|K$ and $g^{-1}(Y) \setminus K = C \setminus K$.

Many kinds of absorbers has been defined in various papers. We will pick two of them out, one in [BGM] and the other in [Ba], and give other names to them.

Definition 2.0.1 [BGM]. We call X a *strong* $(\mathcal{M}_0, \mathcal{C})$*-absorber in* M if;

(1) the pair (M, X) is strongly $(\mathcal{M}_0, \mathcal{C})$-universal,

(2) X is contained in a σ-compact Z_σ-set,

(3) $X \in \mathcal{C}$.

Definition 2.0.2 [Ba]. We call X a *weak* $(\mathcal{M}_0, \mathcal{C})$-*absorber in* M if;

(1) the pair (M, X) is strongly $(\mathcal{M}_0, \mathcal{C})$-universal,

(2) X is contained in a σ-compact Z_σ-set,

(3) $X \in \mathcal{C}_\sigma$.

In [BGM], the topological uniqueness for strong $(\mathcal{M}_0, \mathcal{C})$-absorbers in a Q-, s- or Σ-manifold M was proved, that is, every two strong $(\mathcal{M}_0, \mathcal{C})$-absorbers in M are homeomorphic by a homeomorphism of M onto itself. But T. Banakh mentioned in [Ba] that the topological uniqueness for weak $(\mathcal{M}_0, \mathcal{C})$-absorbers in Q- or s-manifolds is hold. However, weak $(\mathcal{M}_0, \mathcal{C})$-absorbers in a Σ-manifold are not topologically unique. To avoid this defect, we define here as follows;

Definition 2.0.3. We call X an $(\mathcal{M}_0, \mathcal{C})$-*absorber in* M if

(1) the pair (M, X) is strongly $(\mathcal{M}_0, \mathcal{C}_\sigma)$-universal,

(2) X is contained in a σ-compact Z_σ-set,

(3) $X \in \mathcal{C}_\sigma$.

It is easy to see that strong $(\mathcal{M}_0, \mathcal{C})$-absorbers are weak $(\mathcal{M}_0, \mathcal{C})$-absorbers and that $(\mathcal{M}_0, \mathcal{C})$-absorbers are weak $(\mathcal{M}_0, \mathcal{C})$-absorbers. In addition, one should remark that our $(\mathcal{M}_0, \mathcal{C})$-absorbers are nothing but strong $(\mathcal{M}_0, \mathcal{C}_\sigma)$-absorbers. Then we have the topological uniqueness of our absorbers from the result of [BGM].

Theorem 2.0.1. *Let $\mathcal{C}$ be a closed-hereditary topological class, M be a Q-, s- or Σ-manifold, X and Y be two $(\mathcal{M}_0, \mathcal{C})$-absorbers in M. Then for every open cover $\mathcal{U}$ of M there exists a homeomorphism $h : M \to M$ such that $h(X) = Y$ and h is $\mathcal{U}$-close to* id.

In §2.2, we prove that weak $(\mathcal{M}_0, \mathcal{C})$-absorbers in Q- or s-manifolds are $(\mathcal{M}_0, \mathcal{C})$-absorbers if $\mathcal{C}$ is a closed-hereditary topological class, and that the existence of an absorber in one of Q-, s- or Σ-manifold implies the existence of the absorbers in all Q-, s- and Σ-manifolds. Moreover a certain classification theorem is proved in §2.2. In §2.3, generalizing Theorem 1-1 in [Ba], we prove that if X is an $(\mathcal{M}_0, \mathcal{C})$-absorber in Q-, s- or Σ-manifold M for a certain class $\mathcal{C}$, then for every $Y \in \mathcal{A}_1$ and $Z \subset Y$ with $X \cup Z \in \mathcal{C}_\sigma$, the union $X \cup Z$ is also an $(\mathcal{M}_0, \mathcal{C})$-absorber in M. In our approach, the existence of absorbers in Σ-manifolds is essential. Throughout this chapter, we will make use of the following proposition due to [BGM].

Proposition 2.0.1. *Let M be an ANR, $X \subset M$ and $F \subset M$ be such that $M \setminus F$ is homotopy negligible in M. If $\mathcal{C}$ is a closed-hereditary class and the couple $(F, F \cap X)$ is strongly $(\mathcal{M}_0, \mathcal{C})$-universal then the couple (M, X) is also strongly $(\mathcal{M}_0, \mathcal{C})$-universal. Moreover if X is an $(\mathcal{M}_0, \mathcal{C})$-absorber in F, so is X in M.*

§2.1. Comparisons between absorbing pairs

The following theorem is mentioned in [Ba], and we will give a sketch of proof.

Theorem 2.1.1. *Let $\mathcal{C}$ be a closed-hereditary topological class, M be a Q-manifold or an s-manifold, X and Y be two weak $(\mathcal{M}_0, \mathcal{C})$-absorbers in M. Then for every open cover $\mathcal{U}$ of M there is a homeomorphism $h : M \to M$ such that $h(X) = Y$ and h is $\mathcal{U}$-close to* id.

Proof. Put $X = \bigcup_{i=1}^\infty X_i$, where X_i is closed in X and $X_i \in \mathcal{C}$ for every i. For every i, we can take closed subset E_i of M such that $E_i \cap X = X_i$. From the definition we also take compact Z-sets $\mathcal{A} = \{A_j \mid j \in \omega\}$ such that $X \subseteq \bigcup \mathcal{A}$. Without loss of generality, we may assume that $\mathcal{A} = \{E_i \cap A_j \mid i, j \in \omega\}$. Then, $A_i \cap X \in \mathcal{C}$ for every i. Similarly, we can take compact Z-sets $\mathcal{B} = \{B_i \mid i \in \omega\}$ such that $Y \subseteq \cup\mathcal{B}$ and

$B_i \cap Y \in \mathcal{C}$ for every i. Put $f_0 = \mathrm{id}$. By induction we shall construct sequences of homeomorphisms $f_i : M \to M$ and $g_i = f_i \circ \cdots \circ f_0$ with the properties:

(1) $X \cap \bigcup_{j=1}^{i} A_j = g_i^{-1}(Y) \cap \bigcup_{j=1}^{i} A_j$

(2) $g_i(X) \cap \bigcup_{j=1}^{i} B_j = Y \cap \bigcup_{j=1}^{i} B_j$

(3) $f_i | (\bigcup_{j=1}^{i-1} g_{i-1}(A_j) \cup B_j) = \mathrm{id}$

Assume that f_i has been constructed. We write $K = \bigcup_{j=1}^{i}(g_i(A_j) \cup B_j)$. Since the class $\mathcal{C}$ is topological, we have $C = g_i(A_{i+1} \cap X) \in \mathcal{C}$. Hence, the pair $(g_i(A_{i+1}) \cup K, C)$ belongs to $(\mathcal{M}_0, \mathcal{C})$. Since (M, Y) is strongly $(\mathcal{M}_0, \mathcal{C})$-universal, we can find a Z-embedding $\alpha : g_i(A_{i+1}) \cup K \to M$ that fixes K and has the property;

$$\alpha^{-1}(Y) \cap g_i(A_{i+1}) = C.$$

Let $\tilde{\alpha}$ be an extension of α to a homeomorphism of M. Put $K' = K \cup (\alpha \circ g_i(A_{i+1}))$. Since the pair $(M, \tilde{\alpha} \circ g_i(X))$ is strongly $(\mathcal{M}_0, \mathcal{C})$-universal we can find a Z-embedding $\beta : K' \cup B_{i+1} \to M$ that fixes K' such that

$$\beta^{-1}(\tilde{\alpha} \circ g_i(X)) \cap B_{i+1} = Y \cap B_{i+1}.$$

Let $\tilde{\beta}$ be an extension of β to a homeomorphism of M. If we put $f_{i+1} = \tilde{\beta}^{-1} \circ \tilde{\alpha}$ then one can easily verify the induction hypothesis for $i+1$. Since $\tilde{\alpha}$ and $\tilde{\beta}$ and hence f_{i+1} can be chosen arbitrarily close to the *id*, we may assume that $h = \lim_{i \to \infty} g_i$ is a homeomorphism of M. The function h maps X onto Y. □

Remark. Although weak $(\mathcal{M}_0, \mathcal{C})$-absorbers are not strong $(\mathcal{M}_0, \mathcal{C})$-absorbers in general, it follows from Theorem 2.1.1 that if there exists a strong $(\mathcal{M}_0, \mathcal{C})$-absorber in a Q- or s-manifold M then weak $(\mathcal{M}_0, \mathcal{C})$-absorbers in M are strong $(\mathcal{M}_0, \mathcal{C})$-absorbers. Theorem 2.1.1 does not hold for weak $(\mathcal{M}_0, \mathcal{C})$-absorbers in Σ-manifolds. For example, let $X = \Sigma \times s$ be a subset of $\Sigma \approx \Sigma \times Q$ and $Y = \Sigma \times B(Q) \times s$ be a subset of $\Sigma \approx \Sigma \times Q \times Q$. By Proposition 6.1 of [BGM], we can easily verify that both X

and Y are weak $(\mathcal{M}_0, \mathcal{G}_\delta)$-absorbers in Σ. Nevertheless, the pairs (Σ, X) and (Σ, Y) are not homeomorphic, because $\Sigma \setminus X$ is σ-compact but $\Sigma \setminus Y$ is not. If $\Sigma \setminus Y$ is σ-compact, so is $pr(\Sigma \setminus Y)$, where $pr : \Sigma \times Q \times Q \to Q \times Q$ is the projection. Hence, $B(Q) \times s = pr(Y) = (Q \times Q) \setminus pr(\Sigma \setminus Y)$ is a G_δ-subset of $Q \times Q$. This is a contradiction.

By the way, in Q- or s-manifolds there is no strong $(\mathcal{M}_0, \mathcal{G}_\delta)$-absorber by the Bair Category Theorem as mentioned in [BGM, p.173]. Does there a strong $(\mathcal{M}_0, \mathcal{G}_\delta)$-absorber exist in Σ-manifolds? This question is also answered in negative. Assume that there exists a strong $(\mathcal{M}_0, \mathcal{G}_\delta)$-absorber X in a Σ-manifold M^Σ. Then there exists s-manifold M^s in which M^Σ is an $(\mathcal{M}_0, \mathcal{F}_\sigma)$-absorber (cf. [$\mathrm{Ch}_1$]). By Proposition 2.0.1, X is a strong $(\mathcal{M}_0, \mathcal{G}_\delta)$-absorber in M^s. This is a contradiction.

We first prove the following:

Proposition 2.1.1. *Let $\mathcal{C}$ be a closed-hereditary topological class, M be a Q-, s- or Σ-manifold, X be its weak $(\mathcal{M}_0, \mathcal{C})$-absorber, K be a σ-compact Z-set in M and $C \in \mathcal{C}_\sigma$ be a subset of K. Then the set $Y = (X \setminus K) \cup C$ is a weak $(\mathcal{M}_0, \mathcal{C})$-absorber in M.*

Proof. The open set $U = M \setminus K$ can be written as $\bigcup_{i=1}^{\infty} F_i$, where F_i is closed in M for every i. Since X is a weak $(\mathcal{M}_0, \mathcal{C})$-absorber, we can put $X = \bigcup_{j=1}^{\infty} X_j$, where $X_j \in \mathcal{C}$ is closed in X for every j. Then, the set $F_i \cap X_j$ is closed in $F_i \cap X$ and belongs to $\mathcal{C}$ for every i and j by the condition of $\mathcal{C}$. Hence the sets $F_i \cap X = \cup_{j=1}^{\infty}(F_i \cap X_j)$ belong to $\mathcal{C}_\sigma$. In addition, it is easily seen that the sets $F_i \cap X$ are closed in Y. Therefore Y belongs to $\mathcal{C}_\sigma$. It is clear that Y is contained in a σ-compact Z_σ-set. Since the pair $(M \setminus K, Y \setminus K) = (M \setminus K, X \setminus K)$ is strongly $(\mathcal{M}_0, \mathcal{C})$-universal, the pair (M, Y) is strongly $(\mathcal{M}_0, \mathcal{C})$-universal by Proposition 2.0.1. Thus, Y is an weak $(\mathcal{M}_0, \mathcal{C})$-absorber in M. □

Using Theorem 2.1.1 and Proposition 2.1.1, we can prove the following:

Theorem 2.1.2. *Let $\mathcal{C}$ be a closed-hereditary topological class and M be a Q- or s-manifold. If X is a weak $(\mathcal{M}_0, \mathcal{C})$-absorber in M, then X is an $(\mathcal{M}_0, \mathcal{C})$-absorber in M.*

Proof. It suffices to prove that the pair (M, X) is strongly $(\mathcal{M}_0, \mathcal{C}_\sigma)$-universal. Let B be compact, K be closed in B, $C \in \mathcal{C}_\sigma$ be a subset of B, $f : B \to M$ be a map which restricts a Z-embedding on K and $\mathcal{U}$ be an open cover of M. By [vM$_2$], pick a Lebesgue number $\varepsilon > 0$ for $\mathcal{U}$ with respect to $f(B)$. We can approximate f to a Z-embedding $g : B \to M$ such that $g|K = f|K$, and $d(f,g) < \frac{\varepsilon}{2}$. By Proposition 2.1.1, we can find that $Y = (X \setminus g(B)) \cup g(C \setminus K)$ is a weak $(\mathcal{M}_0, \mathcal{C})$-absorber in $M \setminus g(K)$. Similarly observe that $X' = X \setminus g(K)$ is a weak $(\mathcal{M}_0, \mathcal{C})$-absorber in $M \setminus g(K)$. By Theorem 2.1.1, we can get a homeomorphism h of $M \setminus g(K)$ which is $\mathcal{V} \cap \mathcal{W}$-close to id such that $h(Y) = X'$, where $\mathcal{W}$ is an open cover of $M \setminus g(K)$ with mesh($\mathcal{W}$)$< \frac{\varepsilon}{2}$, and $\mathcal{V}$ is a Dugundji cover for $M \setminus g(K)$. By [AHW], we can extend h to a homeomorphism $h' : M \to M$ with $h'|g(K) = \mathrm{id}$. Put $p = h' \circ g$. Then, p is a Z-embedding such that $p^{-1}(X) \setminus K = C \setminus K$, $p = f$ on K and $d(p, f) < \varepsilon$. □

Remark. For Σ-manifolds, Theorem 2.1.2 does not hold. In the example of the Remark of Theorem 2.1.1, $\Sigma \times s$ is a weak $(\mathcal{M}_0, \mathcal{G}_\delta)$-absorber in $\Sigma \times Q$. However, it is not an $(\mathcal{M}_0, \mathcal{G}_\delta)$-absorber. If it were, there exists an embedding $f : Q \to \Sigma \times Q$ such that $f^{-1}(\Sigma \times s) = \Sigma$. Then, $p_\Sigma \circ f(Q)$ is compact, where $p_\Sigma : \Sigma \times Q \to \Sigma$ is the projection. Since $(p_\Sigma \circ f(Q)) \times s$ is complete metrizable, so is $f(Q) \cap ((p_\Sigma \circ f(Q)) \times s) = f(\Sigma)$. But Σ is not complete metrizable. This is a contradiction.

The following is a generalization of [Ba, Proposition 2.1].

Theorem 2.1.3. *Let $\mathcal{C}$ be a closed-hereditary topological class. Suppose that for some $A \in \{Q, s, \Sigma\}$, there exists an A-manifold M^A and a weak $(\mathcal{M}_0, \mathcal{C})$-absorber X in M^A. Then for every $B \in \{Q, s, \Sigma\}$ and any B-manifold N^B there exists an $(\mathcal{M}_0, \mathcal{C})$-absorber Y in N^B.*

To prove Theorem 2.1.3, we need the following three lemmas.

Lemma 2.2.1. *Let $A \in \{Q, s, \Sigma\}$. If there exists an A-manifold M and a (resp. weak) $(\mathcal{M}_0, \mathcal{C})$-absorber X in M, then for every open subset U of A there exists an (resp. weak) $(\mathcal{M}_0, \mathcal{C})$-absorber Y in U. In particular, A has an (resp. weak) $(\mathcal{M}_0, \mathcal{C})$-absorber.*

Proof. We consider the case of $A = Q$. By [vM$_2$], there exists a basic open subset W of M which can be written as $Q \times [0,1)$. We will prove that $X' = X \cap W$ is an (weak) $(\mathcal{M}_0, \mathcal{C})$-absorber in W. Since W is an open subset of M, the pair (W, X') is strongly $(\mathcal{M}_0, \mathcal{C})$-universal ([BGM]), and X' is contained in a σ-compact Z_σ-set in W by [vM$_2$]. As same as the proof of Proposition 2.1.1, we can find that $Y \in \mathcal{C}_\sigma$. Hence X' is an (weak) $(\mathcal{M}_0, \mathcal{C})$-absorber in $W \approx Q \times [0,1)$. Then, X' is an (weak) $(\mathcal{M}_0, \mathcal{C})$-absorber in $Q \times [0,1] \approx Q$ by Proposition 2.0.1. Similarly as above, it follows that the subset $Y = X' \cap U$ is an (weak) $(\mathcal{M}_0, \mathcal{C})$-absorber in U for any open subset U of Q. On the other hand, in the case of $A = s$ or Σ, the open embedding theorem (cf. [Ch$_1$]) and the above argument lead to the proof. □

Lemma 2.1.2. *Let $A \in \{Q, s, \Sigma\}$. If there exists an open subset of A, and a (resp. weak) $(\mathcal{M}_0, \mathcal{C})$-absorber in it, then for every A-manifold M^A there exists a (resp. weak) $(\mathcal{M}_0, \mathcal{C})$-absorber in M^A.*

Proof. We will prove the version of weak $(\mathcal{M}_0, \mathcal{C})$-absorbers only.

In case $A = s$ or Σ, M^A can be embedded into A as an open set by the open embedding theorem for s or Σ (see [Ch$_1$]). Then, M^A has a weak $(\mathcal{M}_0, \mathcal{C})$-absorber by Lemma 2.1.1.

We next consider the case of $A = Q$. Since M^Q has an open cover by the sets which are homeomorphic to open subsets of Q, by Lemma 2.1.1 we can take the family $\mathcal{U}$ of open subsets of M^A satisfying the property that for every $U \in \mathcal{U}$ and every open subset V of U, V has a weak $(\mathcal{M}_0, \mathcal{C})$-absorber. It is obvious that $\mathcal{U}$ is open-hereditary and the discrete sum of elements of $\mathcal{U}$ belongs to $\mathcal{U}$. If $V_1 \cup V_2$ belongs to $\mathcal{U}$ for every $V_1, V_2 \in \mathcal{U}$, by virtue of so-called the localization principle, our assertion is completely proved (see [BP]).

Thus, let V_1 and V_2 be elements of $\mathcal{U}$. Then V_1 and V_2 have weak $(\mathcal{M}_0, \mathcal{C})$-absorbers X_1 and X_2 respectively. We show that $Y = X_1 \cup (X_2 \setminus V_1)$ is a weak $(\mathcal{M}_0, \mathcal{C})$-absorber in $V_1 \cup V_2$. It is sufficient to verify that the pair $(V_1 \cup V_2\ , Y)$ is strongly $(\mathcal{M}_0, \mathcal{C})$-universal. Now let B be compact, K be closed in B, C be a subset of B which belongs to $\mathcal{C}$, $f : B \to V_1 \cup V_2$ be a map which restrict to a Z-embedding on K, and $\mathcal{U}$ be an open cover of $V_1 \cup V_2$. There exists a Lebesgue number $\varepsilon > 0$ of $\mathcal{U}$ with respect to $f(B)$. Without loss of generality, we may assume that f is a Z-embedding. Put $f_0 = f$, $K_0 = \emptyset$, $K' = K \cup f^{-1}(V_1 \setminus V_2)$, $B \setminus K' = \bigcup_{i=0}^{\infty} K_i$ where K_i is compact and $K_i \subset \operatorname{int}(K_{i+1})$ for every i, and ε_i=min$\{2^{-i-1} \cdot \varepsilon, d(f(K_i), f(K) \cup (V_1 \setminus V_2))\}$. By induction, we can construct a map $f_i : B \to V_1 \cup V_2$ for every i such that

(1) $f_i^{-1}(X_2) \cap K_i = C \cap K_i$,

(2) $f_i|K_i$ is a Z-embedding,

(3) $f_i|K' \cup K_{i-1} = f_{i-1}|K' \cup K_{i-1}$,

(4) $d(f_i, f_{i-1}) < \varepsilon_i$.

Assume that f_{n-1} has been constructed. Since the pair (V_2, X_2) is strongly $(\mathcal{M}_0, \mathcal{C})$-universal, $f_{i-1}|K_i$ can be approximated by a Z-embedding p such that $p^{-1}(X_2) = C \cap K_i$, $p|K_{i-1} = f_{i-1}|K_{i-1}$, and p is close enough to $f_{i-1}|K_i$ in order to have a extension satisfying the inductive condition. The inductive step is over.

Observe that $g = \lim_{i \to \infty} f_i : B \to V_1 \cup V_2$ is a Z-embedding such that $g^{-1}(X_2) \setminus K' = g^{-1}(V_2) \cap C \setminus K'$, $\hat{d}(f, g) < \frac{\varepsilon}{2}$, and $g|K' = f|K'$. Similarly, by the universality of (V_1, X_1), we can approximate g to a map $h : B \to V_1 \cup V_2$ such that $h^{-1}(X_1) \setminus K = C \cap h^{-1}(V_1) \setminus K$, h is equal to g on $g^{-1}(V_2 \setminus V_1) \cup K$, and $\hat{d}(g, h) < \frac{\varepsilon}{2}$. Then, g is a

Z-embedding such that $g^{-1}(Y) \setminus K = C \setminus K$, $g|K = f|K$, and $\hat{d}(f, g) < \varepsilon$. $\square$

Lemma 2.1.3. *If any one of Q, s and Σ, has a weak $(\mathcal{M}_0, \mathcal{C})$-absorber, then all of Q, s and Σ have $(\mathcal{M}_0, \mathcal{C})$-absorbers.*

Proof. It suffices to show that the following statements are equivalent:

(1) Q has an $(\mathcal{M}_0, \mathcal{C})$-absorber,

(2) s has an $(\mathcal{M}_0, \mathcal{C})$-absorber,

(3) Σ has a weak $(\mathcal{M}_0, \mathcal{C})$-absorber,

(4) Σ has an $(\mathcal{M}_0, \mathcal{C})$-absorber.

The implications (4) $\Rightarrow$ (3), (3) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) follow from Proposition 2.0.1 and Theorem 2.1.2. To complete the proof, we will prove (1) $\Rightarrow$ (4).

Write $\Sigma = \bigcup_{i=1}^{\infty} Q_n$, where $Q_n \approx Q$, each Q_n is a Z-set in Q_{n+1}, and the tower $(Q_n)_{n \in \mathbb{N}}$ has the compact absorption property in the sense of [Ch$_1$] in Q. By Lemma 2.1.1 and (1), the Q-manifold $Q_n \setminus Q_{n-1}$ has an $(\mathcal{M}_0, \mathcal{C})$-absorber X_n for every $n \geq 1$. Then $X = \bigcup_{n=1}^{\infty} X_n$, is an $(\mathcal{M}_0, \mathcal{C})$-absorber in Σ. To prove this, we shall show that the pair (Σ, X) is strongly $(\mathcal{M}_0, \mathcal{C}_\sigma)$-universal.

Let B be compact, K be closed in B, C be a subset of B which belongs to $\mathcal{C}_\sigma$, $f : B \to \Sigma$ be a map which restricts a Z-embedding on K, and $\mathcal{U}$ be an open cover of Σ. Notice that every compact subset of Σ is Z-set in Σ. Without loss of generality, we may assume that f is a Z-embedding. There exists a Lebesgue number $\varepsilon > 0$ for $\mathcal{U}$ with respect to $f(B)$. Write $B \backslash K = \bigcup_{n=0}^{\infty} B_n$, where $B_0 = \emptyset$, B_n is compact and $B_n \subset \operatorname{int} B_{n+1}$ for every $n \geq 1$. For every $n \geq 1$, put $\varepsilon_n = \min\{2^{-i} \cdot \varepsilon\ ,\ \frac{1}{2} d(f(K), f(B_n))\}$ and $f_0 = f$. By induction, we will construct maps $(f_n)_{n=1}^{\infty} : B \to \Sigma$ and $i_n \in \mathbb{N}$ such that

(1) $f_n^{-1}(X) \cap B_n = C \cap B_n$,

(2) $f_n|B_n$ is a Z-embedding,

(3) $f_n|B_{n-1} = f_{n-1}|B_{n-1}$,

(4) $f_n|B \setminus \text{int}B_{n+1} = f|B \setminus \text{int}B_{n+1}$,

(5) $f_n(B_n) \subseteq Q_{i_n}$,

(6) $i_n \geq i_{n-1}$,

(7) $\hat{d}(f_n, f_{n-1}) < \varepsilon_n$.

Assume that i_{n-1} and f_{n-1} have been constructed. From the compact absorption property, there exists a number i_n and an embedding $p : B_n \to Q_{i_n}$ such that p is sufficiently close to f_{n-1} and restricts to f_{n-1} on B_{n-1}. Since the set $\cup_{j=1}^{i_n} X_j$ is an $(\mathcal{M}_0, \mathcal{C}_\sigma)$-absorber in $Q' = \cup_{j=1}^{i_n} Q_j$ by Proposition 2.1.1, we can approximate p to a Z-embedding $q : B_n \to Q'$ such that $q^{-1}(X) = C \cap B_n$, $q|B_{n-1} = f_{n-1}|B_{n-1}$ and q is close enough to p to have an extension f_n satisfying the inductive condition.

Observe that the map $\lim_{n\to\infty} f_n$ is the required Z-embedding to show the universality of the pair (Σ, X). □

Proof of Theorem 2.1.3. Let $A, B \in \{Q, s, \Sigma\}$, M^A and N^B are A-manifold and B-manifold, respectively. Assume that M^A has a weak $(\mathcal{M}_0, \mathcal{C})$-absorber. Then A has a weak $(\mathcal{M}_0, \mathcal{C})$-absorber by Lemma 2.1.1. Hence B has an $(\mathcal{M}_0, \mathcal{C})$-absorber by Lemma 2.1.3. Therefore N^B has an $(\mathcal{M}_0, \mathcal{C})$-absorber by Lemma 2.2.2. Thus, Theorem 2.1.2 is proved. □

As a corollary of Theorem 2.1.2, we have the following:

Corollary 2.1.1. *Let $\mathcal{B}$ be an absolute Borelian class of spaces, and $A \in \{Q, s, \Sigma\}$. Then every A-manifold M has an $(\mathcal{M}_0, \mathcal{B})$-absorber X. Moreover if $\mathcal{B}$ is other than the class $\mathcal{M}_1$, X is a strong $(\mathcal{M}_0, \mathcal{B})$-absorber.*

Proof. It is easy to see that $s \times \Sigma$ is an $(\mathcal{M}_0, \mathcal{M}_1)$-absorber in $Q \times Q$. On the other hand, there is strong $(\mathcal{M}_0, \mathcal{B})$-absorber in Q by [BGM, p.174] if $\mathcal{B}$ is a Borelian class other than $\mathcal{M}_1$. Then, M has an $(\mathcal{M}_0, \mathcal{B})$-absorber X by Theorem 2.1.3. Moreover if

$\mathcal{B}$ is an additive Borelian class, the $(\mathcal{M}_0, \mathcal{B})$-absorber X is a strong $(\mathcal{M}_0, \mathcal{B})$-absorber because of $(\mathcal{B})_\sigma = \mathcal{B}$. On the other hand, let $\mathcal{B}$ be a multiplicative Borelian class other than $\mathcal{M}_1$ and Ω be a strong $(\mathcal{M}_0, \mathcal{B})$-absorber in Q. Since $\Omega \times \Sigma$ belongs to $\mathcal{B}$, $\Omega \times \Sigma$ is a strong $(\mathcal{M}_0, \mathcal{B})$-absorber in $Q \times A$. By the triangulation theorem (see [Ch_1]), there exists a locally finite simplicial complex K such that $A \times K \approx M$. Since $\Sigma \times K$ is σ-compact, $\Omega \times \Sigma \times K$ is also a strong $(\mathcal{M}_0, \mathcal{B})$-absorber in $Q \times A \times K \approx M$. Therefore, M has a strong $(\mathcal{M}_0, \mathcal{B})$-absorber. □

§2.2. The classification of absorbers

The following easy observation will be used frequently in our argument.

Proposition 2.2.1. *Suppose that $A \in \{Q, s, \Sigma\}$. Let $\mathcal{C}$ be a closed-hereditary topological class, M be an A-manifold and X be an $(\mathcal{M}_0, \mathcal{C})$-absorber in M. Then there exists an $(\mathcal{M}_0, \mathcal{F}_\sigma)$-absorber Y in M such that X is an $(\mathcal{M}_0, \mathcal{C})$-absorber in Y.*

Proof. By Corollary 2.1.1, there exists an $(\mathcal{M}_0, \mathcal{F}_\sigma)$-absorber G in M. Since G is a Σ-manifold, G has an $(\mathcal{M}_0, \mathcal{C})$-absorber X' by Theorem 2.1.3. By Proposition 2.0.1, X' is an $(\mathcal{M}_0, \mathcal{C})$-absorber in M. Theorems 2.0.1 and 2.1.1 give a homeomorphism $h : M \to M$ such that $h(X') = X$. Then, X is an $(\mathcal{M}_0, \mathcal{C})$-absorber in $Y = h(G)$. □

For Borelian classes (see [BM]), we have the following:

Corollary 2.2.1. *Let $\mathcal{A}_\alpha$ (resp. $\mathcal{M}_\alpha$) be an additive (multiplicative) Borelian classes for a countable ordinal α, M be a Q-, s- or Σ-manifold and X be an $(\mathcal{M}_0, \mathcal{A}_\alpha)$-absorber (an $(\mathcal{M}_0, \mathcal{M}_\alpha)$-absorber) in M. Then there exists an $(\mathcal{M}_0, \mathcal{M}_\alpha)$-absorber (an $(\mathcal{M}_0, \mathcal{A}_\alpha)$-absorber) Y in M such that $X \cup Y$ is an $(\mathcal{M}_0, \mathcal{F}_\sigma)$-absorber in M.*

Proof. We first assume that X is an $(\mathcal{M}_0, \mathcal{A}_\alpha)$-absorber. By Proposition 2.2.1 and Corollary 2.1.1, M has an $(\mathcal{M}_0, \mathcal{F}_\sigma)$-absorber G in which X is a strong $(\mathcal{M}_0, \mathcal{A}_\alpha)$-absorber. Observe that G is a Σ-manifold. Then the pair $(G, G \setminus X)$ is strongly $(\mathcal{M}_0, \mathcal{M}_\alpha)$-universal since the pair (G, X) is strongly $(\mathcal{M}_0, \mathcal{A}_\alpha)$-universal. The pair $(M, G \setminus X)$ is strongly $(\mathcal{M}_0, \mathcal{M}_\alpha)$-universal by Proposition 2.0.1 because $M \setminus G$ is homotopy negligible in M. In addition, $G \setminus X$ belongs to $(\mathcal{M}_\alpha)_\sigma$ since G is σ-compact. Hence $Y = G \setminus X$ is an $(\mathcal{M}_0, \mathcal{C})$-absorber in M. Similarly we can prove the case that X is an $(\mathcal{M}_0, \mathcal{M}_\alpha)$-absorber for $\alpha > 1$.

We next assume X is an $(\mathcal{M}_0, \mathcal{M}_1)$-absorber. We can find that every Σ-manifold F has a weak $(\mathcal{M}_0, \mathcal{G}_\delta)$-absorber as a $\mathcal{G}_\delta$-subset as follows. There exists a simplicial complex K such that $F \cong Q \times \Sigma \times K$ by the triangulation theorem and the stability theorem (see [Ch$_1$]). Then, it is easy to see that the product $s \times \Sigma \times K$ is such an absorber in $Q \times \Sigma \times K \cong F$. Hence, as same as we saw in Proposition 2.2.1, M has an $(\mathcal{M}_0, \mathcal{F}_\sigma)$-absorber G in which X is a weak $(\mathcal{M}_0, \mathcal{M}_1)$-absorber and $\mathcal{G}_\delta$. Therefore, the complement $Y = G \setminus X$ belongs to $\mathcal{A}_1$. Thus, Y is an $(\mathcal{M}_0, \mathcal{A}_1)$-absorber. $\square$

Now we prove the following:

Theorem 2.2.1. *Let $A, B \in \{Q, s, \Sigma\}$, M^A be an A-manifold, M^B be a B-manifold, and X,Y be weak $(\mathcal{M}_0, \mathcal{C})$-absorber in M^A,M^B respectively. Then, X is homeomorphic to Y if M^A is homotopy equivalent to M^B.*

Proof. We first consider the case of $A = B = \Sigma$. By [Ch$_1$], there exist s-manifolds S^A and S^B in which M^A and M^B are $(\mathcal{M}_0, \mathcal{F}_\sigma)$-absorbers respectively. Since $S^A \setminus M^A$ is homotopy negligible in S^A, by Proposition 2.0.1 X is a weak $(\mathcal{M}_0, \mathcal{C})$-absorber in S^A, and similarly so is Y in S^B. In addition, we can find that $S^A \cong M^A \cong M^B \cong S^B$ by [To$_4$]. Then, it follows from [HS] that $S^A \approx S^B$. Thus, $X \approx Y$ follows from Theorem 2.1.1.

On the other hand, let $A, B \in \{Q, s, \Sigma\}$. Since any $(\mathcal{M}_0, \mathcal{F}_\sigma)$-absorber in these manifolds is Σ-manifold, by Proposition 2.2.1 there exist Σ-manifolds T^A and T^B in which X and Y are weak $(\mathcal{M}_0, \mathcal{C})$-absorbers such that $T^A \cong M^A$ and $T^B \cong M^B$, respectively. Therefore it follows from the above case that $X \approx Y$. $\square$

As a corollary of Theorem 2.2.1, we have the following classification theorem for absorber.

Corollary 2.2.1. *Let $\mathcal{C}$ be a closed-hereditary topological class which contains the class of all finite-dimensional compactum, $A, B \in \{Q, s, \Sigma\}$, M^A and M^B are A-manifold and B-manifold, and X and Y are weak $(\mathcal{M}_0, \mathcal{C})$-absorbers in M^A and M^B respectively. Then, following statements are all equivalent:*

(1) $X \approx Y$,

(2) $X \cong Y$,

(3) $M^A \cong M^B$.

Proof. The implication (1) $\Rightarrow$ (2) is clear, (2) $\Rightarrow$ (3) follows from [Ba] and (3) $\Rightarrow$ (1) follows from Theorem 2.2.1. $\square$

Remark. In the implication (1) $\Rightarrow$ (3) or (2) $\Rightarrow$ (3) of Corollary 2.2.1, we need some conditions of the class $\mathcal{C}$ like that $\mathcal{C}$ contains the class of all finite-dimensional compactum. For example, let $\mathcal{C}$ be a class whose elements consist of at most one point. Observe that a countable dense subset D of Q is an $(\mathcal{M}_0, \mathcal{C})$-absorber in it. Let $p : Q \to I_1$ be a projection onto the first coordinate of Q. We can easily pick a point $x \in \operatorname{int} I_1$ such that $p^{-1}(x) \cap D = \emptyset$. Then D is an $(\mathcal{M}_0, \mathcal{C})$-absorber also in $Q \setminus p^{-1}(x)$. But $Q \setminus p^{-1}(x)$ is not homotopy equivalent to Q.

§2.3. Enlarging absorbers.

The following is a generalization of [Ba].

Theorem 2.3.1. *Let $\mathcal{C}$ be a closed-hereditary topological class such that $C \times Q \in \mathcal{C}$ and $A \cup C \in \mathcal{C}_\sigma$ for every $C \in \mathcal{C}$ and $A \in \mathcal{A}_1$. Assume that M is a Q-, Σ- or s-manifold, and X is its $(\mathcal{M}_0, \mathcal{C})$-absorber. Then for every σ-compactum $Y \subset M \setminus X$ and every $Z \subset Y$ with $X \cup Z \in \mathcal{C}_\sigma$, the union $X \cup Z$ is an $(\mathcal{M}_0, \mathcal{C})$-absorber in M.*

Proof. We first assume that M is a Σ-manifold. By using Theorem 2.0.1 and σ-compactness of M, we can prove this case in the same way as the proof of the case M is a Q-manifold ([Ba]). We next assume M is an s-manifold. By Proposition 2.2.1, M has an $(\mathcal{M}_0, \mathcal{F}_\sigma)$-absorber and hence Σ-manifold G in which X is an $(\mathcal{M}_0, \mathcal{C})$-absorber. By the first case, $X \cup (Z \cap G)$ is an $(\mathcal{M}_0, \mathcal{C})$-absorber in G because $Z \cap G$ is σ-compact. Since $M \setminus G$ is homotopy negligible in M, the pair $(M, X \cup Z)$ is strongly $(\mathcal{M}_0, \mathcal{C})$-universal by Proposition 2.0.1. Hence, $X \cup Z$ is an $(\mathcal{M}_0, \mathcal{C})$-absorber in M. □

CHAPTER 3

A Hilbert cube compactification of the Banach space of real-valued continuous functions

Let X be a compact space. Recall that the Banach space $\mathrm{C}(X)$ of continuous real-valued functions of X is homeomorphic to the pseudo-interior $s = (-1,1)^\omega$ of the Hilbert cube $Q = [-1,1]^\omega$ by the Kadec-Anderson Theorem ([Ka] and [An$_1$]). Thus, Q is a compactification of $\mathrm{C}(X)$. In this chapter, we consider such a **natural** compactification of the Banach space $\mathrm{C}(X)$.

Identifying a map $f: X \to \mathbb{R} \subset \overline{\mathbb{R}}$ with the graph of $f \subset X \times \overline{\mathbb{R}}$, we can regard $\mathrm{C}(X) \subset \exp(X \times \overline{\mathbb{R}})$. By Proposition 1.6.3, the topology of $\mathrm{C}(X)$ inherited from $\exp(X \times \overline{\mathbb{R}})$ is the same as the one induced from the sup-norm. Then the closure of $\mathrm{C}(X)$ in $\exp(X \times \overline{\mathbb{R}})$ is a compactification of $\mathrm{C}(X)$, which we denote by $\overline{\mathrm{C}}(X)$. The following is our main result in this chapter:

Main Theorem. *For a locally connected infinite compactum X,*

$$(\overline{\mathrm{C}}(X), \mathrm{C}(X)) \approx (Q, s).$$

After the notation of [Fe$_1$], the closure of $\mathrm{C}(X)$ in $\exp(X \times \mathbb{R})$ is denoted by $\mathrm{C_H}(X)$ $(= \mathrm{C_H}(X, \mathbb{R}))$. One should note that $\overline{\mathrm{C}}(X) \neq \mathrm{C_H}(X)$ and our $\overline{\mathrm{C}}(X)$ is equal to $\mathrm{C_H}(X,\overline{\mathbb{R}})$. Fedorchuk [Fe$_2$] (cf. [Fe$_1$]) proved that if X is locally connected and has no isolated points then $\mathrm{C_H}(X)$ coincides with $\mathrm{USCC}(X)$, and $\mathrm{C_H}(X,[-1,1]) = \mathrm{USCC}(X,[-1,1])$ (this is also valid in case X is non-metrizable [Fe$_2$, Theorem 1.9]). He also proved that if X is infinite and locally connected then $\mathrm{C_H}(X,[-1,1]) \approx Q$ and $\mathrm{C_H}(X) \approx Q \times [0,1) \approx Q \setminus \{\mathrm{pt}\}$, hence $\alpha(\mathrm{C_H}(X)) \approx Q$, where $\alpha(\mathrm{C_H}(X))$ is the Alexandroff one-point compactification of $\mathrm{C_H}(X)$. However, in his proof of [Fe$_2$, Proposition 2.3] (i.e., $\mathrm{C_H}(X,[-1,1]) \approx Q$), Fedorchuk slipped up on the proof of (a version of) Toruńczyk's disjoint n-cells property (see §3.3.). In his proof of the fact that $\mathrm{C_H}(X,[-1,1])$ is an AR, there is no problem (cf. [Fe$_2$ Proposition 2.1]). However, reproving this fact on the way to prove the main theorem, we will give an alternative proof. Thus one can read this paper without Fedorchuk's paper [Fe$_2$].

As a corollary of Main Theorem, we can prove the following:

Corollary 1. *For a locally connected infinite compactum X,*

$$(\mathrm{C_H}(X), \mathrm{C}(X)) \approx (Q \times [0,1), s \times [0,1)).$$

By this corollary and [Ch$_1$, Theorems 6.4(1), 6.6 and 6.2], we have

$$(\alpha(\mathrm{C_H}(X)), \alpha(\mathrm{C_H}(X)) \setminus \mathrm{C}(X)) \approx (Q, Q \setminus s),$$

that is, $(\alpha(\mathrm{C_H}(X)), \mathrm{C}(X)) \approx (Q, s)$. Thus we have the affirmative answer to [Fe$_2$, Question 2.6], that is,

Corollary 2. *For a locally connected infinite compactum X,*

$$(\alpha(\mathrm{C_H}(X)), \mathrm{C}(X)) \approx (Q, s) \qquad \square$$

A map $f\colon X \to \mathbb{R}$ is *Lipschitz* if there is some $k > 0$ such that $|f(x) - f(y)| \leqslant k{\cdot}d(x,y)$ for each $x, y \in X$. Here the metric d for X can be replaced by any admissible metric. By $\mathrm{LIP}(X)$, we denote the subspace of $\mathrm{C}(X)$ consisting of Lipschitz maps. Then $(\mathrm{C}(X), \mathrm{LIP}(X)) \approx (s, \Sigma)$ for any infinite compactum X [SW$_1$]. The following follows from Main Theorem and [Ch$_1$, Lemma 4.3]:

Corollary 3. *For a locally connected infinite compactum X,*

$$(\overline{\mathrm{C}}(X), \mathrm{C}(X), \mathrm{LIP}(X)) \approx (Q, s, \Sigma). \qquad \square$$

In case X is a Euclidean polyhedron, let $\mathrm{PL}(X)$ be the subspace of $\mathrm{C}(X)$ consisting of PL maps. Then $(\mathrm{C}(X), \mathrm{PL}(X)) \approx (s, \sigma)$ for any compact Euclidean polyhedron X with $\dim X \neq 0$ [Ge$_2$]. Furthermore, we have

$$(\mathrm{C}(X), \mathrm{LIP}(X), \mathrm{PL}(X)) \approx (s, \Sigma, \sigma),$$

where X admits the metric inherited from Euclidean space [Sa$_3$]. By Corollary 3 and [CDM, Theorem 2.4], this can be extended as follows:

Corollary 4. *For a compact Euclidean polyhedron X with* $\dim X \neq 0$,

$$(\overline{\mathrm{C}}(X), \mathrm{C}(X), \mathrm{LIP}(X), \mathrm{PL}(X)) \approx (Q, s, \Sigma, \sigma). \qquad \square$$

In our results, the local connectedness of X is essential. For instance, as will be seen, $\overline{\mathrm{C}}(X)$ is not locally path-connected in case X is the comb space (see §3.2.).

§3.1. The pairs $(\overline{\mathrm{C}}(X), \mathrm{C}(X))$ and (Q, s) are homeomorphic

To prove Main Theorem, we use the characterization of the pseudo-boundary $B(Q) = Q \setminus s$ of Q (Theorem 1.3.2).

Since X is compact and locally connected, it has only finitely many components $X_1, \cdots, X_n$. Then

$$(\overline{\mathrm{C}}(X), C_{\mathrm{H}}(X), \mathrm{C}(X)) \approx \left(\prod_{i=1}^n \overline{\mathrm{C}}(X_i), \prod_{i=1}^n \mathrm{C}_{\mathrm{H}}(X_i), \prod_{i=1}^n \mathrm{C}(X_i)\right).$$

Since X is infinite, some X_i is non-degenerate. By using the following easy or well-known facts, we can easily reduce Main Theorem and Corollary 1 to the case X is a Peano continuum (= a locally connected continuum):

(1) in case X is a singleton, $\mathrm{C_H}(X) = \mathrm{C}(X)$ and $(\overline{\mathrm{C}}(X), \mathrm{C}(X)) \approx (\overline{\mathbb{R}}, \mathbb{R})$;

(2) $[0,1) \times [0,1) \approx [0,1) \times (0,1) \approx [0,1) \times [0,1]$;

(3) $(Q \times [0,1], s \times (0,1)) \approx (Q \times [0,1], s \times [0,1]) \approx (Q, s)$.

In the rest of this section, we assume that

($\natural$) *X is a non-degenerate Peano continuum.*

Observe that $(\overline{\mathbb{R}}, \mathbb{R}) \approx ([-1,1], (-1,1))$ and $\mathrm{C_H}(X, [-1,1])$ is not only the closure of $\mathrm{C}(X, [-1,1])$ but also the closure of $\mathrm{C}(X, (-1,1))$ in $\exp(X \times [-1,1])$. (One should note that $\mathrm{C_H}(X, (-1,1))$ is the closure of $\mathrm{C}(X, (-1,1))$ in $\exp(X \times (-1,1))$.) Then we can replace $(\overline{\mathrm{C}}(X), \mathrm{C}(X))$ by

$$(\mathrm{C_H}(X, [-1,1]), \mathrm{C}(X, (-1,1))).$$

Since X is a Peano continuum, X has an admissible *convex* metric d ([Bi], [Mo]), whence *each two points $x, x' \in X$ can be joined by an arc in X isometric to the segment $[0, d(x,x')]$ in $\mathbb{R}$.* Recall that ρ is the metric on $X \times [-1,1]$ defined by

$$\rho((x,t),(x',t')) = \max\{d(x,x'), |t - t'|\}$$

and ρ_{H} is the Hausdorff metric on $\exp(X \times [-1,1])$ induced by ρ. For $F \subset X \times [-1,1]$, we denote

$$N_\rho(F, \varepsilon) = \{z \in X \times [-1,1] \mid \rho(z, F) < \varepsilon\} \subset X \times [-1,1]$$

Recall that $\rho_{\mathrm{H}}(G,H) < \varepsilon$ if and only if $G \subset N_\rho(F,\varepsilon)$ and $F \subset N_\rho(G,\varepsilon)$ because

$$\rho_{\mathrm{H}}(G,H) = \inf\{\varepsilon > 0 \mid G \subset N_\rho(F,\varepsilon) \text{ and } F \subset N_\rho(G,\varepsilon)\}.$$

By $p\colon X\times[-1,1]\to X$ and $q\colon X\times[-1,1]\to[-1,1]$, we denote the projections of $X\times[-1,1]$ onto X and $[-1,1]$, respectively. Recall a function (or a multi-valued function) $\varphi\colon X\to[-1,1]$ is identified with the graph of φ, whence $\varphi(x)=q(\varphi\cap p^{-1}(x))$ for each $x\in X$ and $\varphi|A=\varphi\cap p^{-1}(A)$ for $A\subset X$. As is shown in the proof of [Fe$_2$, Theorem 1.9],

($\sharp$) *$N_\rho(f,\varepsilon)\cap p^{-1}(x)$ is connected for each $f\in \mathrm{C}(X,[-1,1])$, $\varepsilon>0$ and $x\in X$.*

Since $\mathrm{C}(X,(-1,1))$ is dense in $\mathrm{C_H}(X,[-1,1])$ and it is an AR, one might thought it is an easy observation to absorb compact subsets of $C_{\mathrm{H}}(X,[-1,1])$ into $C(X,(-1,1))$. However, one should keep in his mind the following example: Let U and V be nonempty disjoint open sets in X such that $\operatorname{diam}(U\cup V)<\varepsilon$. Take $x\in U$ and $y\in V$ and let $f,g\colon X\to[0,1]$ be Urysohn maps such that $f(x)=1$, $f(X\setminus U)=0$, $g(y)=1$ and $g(X\setminus V)=0$. Then $\rho_{\mathrm{H}}(f,g)<\varepsilon$ but $\rho_{\mathrm{H}}(f,\frac{1}{2}f+\frac{1}{2}g)>\frac{1}{2}$, hence the path from f to g in $C(X)$ defined by $(1-t)f+tg$ has the diameter $\geqslant\frac{1}{2}$. This example helps one to understand the proof below.

Lemma 3.1.1. *Let $f\colon K^{(0)}\to \mathrm{C}(X,(-1,1))$ be a map of the 0-skeleton of a locally finite simplicial complex K. Then f extends to a map $h\colon|K|\to\mathrm{C}(X,(-1,1))$ with*

$$(*)\qquad \operatorname{diam}_{\rho_{\mathrm{H}}}h(\sigma)\leqslant 4\operatorname{diam}_{\rho_{\mathrm{H}}}f(\sigma^{(0)})\quad \textit{for every } \sigma\in K,$$

where $\sigma^{(0)}=\sigma\cap K^{(0)}$.

Proof. For each $\sigma\in K\setminus K^{(0)}$, choose $\varepsilon_\sigma>0$ so that

$$\operatorname{diam}_{\rho_{\mathrm{H}}}f(\sigma^{(0)})<\varepsilon_\sigma<\tfrac{3}{2}\operatorname{diam}_{\rho_{\mathrm{H}}}f(\sigma^{(0)})$$

and $\varepsilon_{\sigma'}\leqslant\varepsilon_\sigma$ if $\sigma'\prec\sigma$. For each $v\in K^{(0)}$, let

$$\varepsilon_v=\min\left\{\operatorname{diam}_{\rho_{\mathrm{H}}}f(\sigma^{(0)})\;\middle|\;\sigma\in\operatorname{St}(v,K)\right\}>0,$$

where $\mathrm{St}(v, K)$ is the star at v in K. Since X has no isolated points and K is locally finite, we can choose a finite subset A_v of X and an open set U_v in X for each $v \in K^{(0)}$ so that $f(v) \subset N_\rho(f(v) \cap p^{-1}(A_v), \varepsilon_v)$, $A_v \subset U_v$ and $U_v \cap U_{v'} = \emptyset$ if $v \neq v' \in \sigma^{(0)}$ and $\sigma \in K$. For each $v \in K^{(0)}$, let $r_v : X \to [0,1]$ be a Urysohn map such that $r_v(A_v) = 1$ and $r_v(X \setminus U_v) = 0$.

Take the barycentric subdivision $\mathrm{Sd}\, K$ of K. The barycenter of $\sigma \in K$ is denoted by $b(\sigma)$. We first extend f to a map $g \colon |\mathrm{Sd}\, K^{(0)}| \to \mathrm{C}(X, (-1,1))$ as follows: for every $\sigma \in K$,

$$g(b(\sigma))(x)$$

$$= \begin{cases} (\dim \sigma + 1)^{-1} \sum_{w \in \sigma^{(0)}} f(w)(x) & \text{if } x \in X \setminus \bigcup_{v \in \sigma^{(0)}} U_v, \\ \dfrac{1 - r_v(x)}{\dim \sigma + 1} \displaystyle\sum_{w \in \sigma^{(0)}} f(w)(x) + r_v(x) f(v)(x) & \text{if } x \in U_v \text{ and } v \in \sigma^{(0)}. \end{cases}$$

Then observe that $g(b(\sigma))(a) = f(v)(a)$ if $\sigma \in K, v \in \sigma^{(0)}$ and $a \in A_v$. Since $f(w) \subset N_\rho(f(v), \varepsilon_\sigma)$ for $v, w \in \sigma^{(0)}$, it follows from ($\sharp$) that $g(b(\sigma)) \cap p^{-1}(x) \subset N_\rho(f(v), \varepsilon_\sigma) \cap p^{-1}(x)$ for each $x \in X$, hence $g(b(\sigma)) \subset N_\rho(f(v), \varepsilon_\sigma)$ for $v \in \sigma^{(0)}$.

Next we extend g to a map $h \colon |K| = |\mathrm{Sd}\, K| \to C(X)$ as follows:

$$h\left(\textstyle\sum_{i=0}^n t_i \cdot b(\sigma_i)\right) = \sum_{i=0}^n t_i \cdot g(b(\sigma_i))$$

for each $\sigma_0 \precneqq \cdots \precneqq \sigma_n \in K$ and $t_i \geqslant 0$ such that $\sum_{i=0}^n t_i = 1$. Then

$$\rho_{\mathrm{H}}(h(z), f(v)) < \varepsilon_\sigma \quad \text{if } z \in |\mathrm{St}(v, \mathrm{Sd}\, K)| \cap \sigma.$$

In fact, we can write $z = \sum_{i=0}^n t_i \cdot b(\sigma_i)$, where $v = \sigma_0 \precneqq \sigma_1 \precneqq \cdots \precneqq \sigma_n \prec \sigma$, $t_i \geqslant 0$ and $\sum_{i=0}^n t_i = 1$. For every $a \in A_v$, we have $h(z)(a) = f(v)(a)$ because $g(b(\sigma_i))(a) = f(v)(a)$ for each $i = 1, \cdots, n$. Then $f(v) \cap p^{-1}(A_v) \subset h(z)$, which implies $f(v) \subset N_\rho(h(z), \varepsilon_v) \subset N_\rho(h(z), \varepsilon_\sigma)$. On the other hand, $g(b(\sigma_i)) \subset$

$N_\rho(f(v),\varepsilon_{\sigma_i}) \subset N_\rho(f(v),\varepsilon_\sigma)$ for each $i = 1,\cdots,n$. Then it follows from ($\sharp$) that $h(z)\cap p^{-1}(x) \subset N_\rho(f(v),\varepsilon_\sigma)\cap p^{-1}(x)$ for each $x \in X$, hence $h(z) \subset N_\rho(f(v),\varepsilon_\sigma)$.

To see ($*$), for each $z, z' \in \sigma \in K$, choose $v, v' \in \sigma^{(0)}$ so that $z \in |\mathrm{St}(v, \mathrm{Sd}\,K)|$ and $z' \in |\mathrm{St}(v', \mathrm{Sd}\,K)|$. Then we have

$$\begin{aligned}\rho_{\mathrm{H}}(h(z),h(z')) &\leqslant \rho_{\mathrm{H}}(h(z),f(v)) + \rho_{\mathrm{H}}(f(v),f(v')) + \rho_{\mathrm{H}}(h(z'),f(v')) \\ &< \varepsilon_\sigma + \mathrm{diam}_{\rho_{\mathrm{H}}} f(\sigma^{(0)}) + \varepsilon_\sigma \\ &\leqslant \tfrac{3}{2}\mathrm{diam}_{\rho_{\mathrm{H}}} f(\sigma^{(0)}) + \mathrm{diam}_{\rho_{\mathrm{H}}} f(\sigma^{(0)}) + \tfrac{3}{2}\mathrm{diam}_{\rho_{\mathrm{H}}} f(\sigma^{(0)}) \\ &= 4\mathrm{diam}_{\rho_{\mathrm{H}}} f(\sigma^{(0)}).\end{aligned}$$

Therefore ($*$) holds. □

Lemma 3.1.2. *There exists a homotopy*

$$\gamma\colon \mathrm{C_H}(X,[-1,1]) \times [0,1] \to \mathrm{C_H}(X,[-1,1])$$

such that $\gamma_0 = \mathrm{id}$ *and* $\gamma_t(\mathrm{C_H}(X,[-1,1])) \subset \mathrm{C}(X,(-1,1))$ *for* $t > 0$, *namely the complement* $\mathrm{C_H}(X,[-1,1]) \setminus \mathrm{C}(X,(-1,1))$ *is locally homotopy negligible (l.h.n.) in* $\mathrm{C_H}(X,[-1,1])$.

Proof. For each $n \in \mathbb{N}$, let $\mathcal{U}_n$ be a finite open cover of $\mathrm{C_H}(X,[-1,1])$ such that $\mathrm{diam}_{\rho_{\mathrm{H}}} U < (n+1)^{-1}$ for each $U \in \mathcal{U}_n$. We define

$$\mathcal{W}_1 = \{U \times (2^{-1},1] \mid U \in \mathcal{U}_1\} \quad \text{and}$$
$$\mathcal{W}_n = \{U \times ((n+1)^{-1},(n-1)^{-1}) \mid U \in \mathcal{U}_n\} \text{ for } n > 1.$$

Then $\mathcal{W} = \bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ is a locally finite open cover of $\mathrm{C_H}(X,[-1,1]) \times (0,1]$ and $\mathcal{U}_n = \{p(W) \mid W \in \mathcal{W}_n\}$ for each $n \in \mathbb{N}$. Let K be the nerve of $\mathcal{W}$ and $g\colon \mathrm{C_H}(X,[-1,1]) \times (0,1] \to |K|$ a canonical map, that is, each $g(\varphi,t)$ is contained in the simplex spanned by all vertices $W \in \mathcal{W}$ containing (φ,t). For each $n \in \mathbb{N}$, let K_n be the nerve of

$\mathcal{W}_n \cup \mathcal{W}_{n+1}$. Then each K_n is a finite subcomplex of K and $K = \bigcup_{n\in\mathbb{N}} K_n$. By choosing $f(W) \in p(W) \cap C(X, (-1,1))$ for each $W \in K^{(0)} = \mathcal{W}$, we have a map $f\colon K^{(0)} \to C(X,(-1,1))$ such that $\mathrm{diam}_{\rho_{\mathrm{H}}} f(\sigma^{(0)}) < 2(n+1)^{-1}$ for each $\sigma \in K_n$. By using Lemma 3.1.1, we can extend f to a map $h\colon |K| \to \mathrm{C}(X,(-1,1))$ such that $\mathrm{diam}_{\rho_{\mathrm{H}}} h(\sigma) < 4\mathrm{diam}_{\rho_{\mathrm{H}}} f(\sigma^{(0)})$. Thus we obtain the map

$$hg\colon \mathrm{C_H}(X,[-1,1]) \times (0,1] \to \mathrm{C}(X,(-1,1)) \subset \mathrm{C_H}(X,[-1,1]).$$

For each $(\varphi,t) \in \mathrm{C_H}(X,[-1,1]) \times (0,1]$, choose $n \in \mathbb{N}$ and $W \in \mathcal{W}_n$ so that $(n+1)^{-1} < t \leqslant n^{-1}$ and $(\varphi,t) \in W$. Then we have $\sigma \in K_n$ such that $g(\varphi,t) \in \sigma$ and $W \in \sigma^{(0)}$. Since $h(W) = f(W) \in p(W) \cap \mathrm{C}(X,(-1,1))$ and $\varphi \in p(W) \in \mathcal{U}_n$, we have $\rho_{\mathrm{H}}(h(W),\varphi) \leqslant \mathrm{diam}_{\rho_{\mathrm{H}}} p(W) < (n+1)^{-1}$. Since $h(W), hg(\varphi,t) \in h(\sigma)$ and $\mathrm{diam}_{\rho_{\mathrm{H}}} h(\sigma) < 4\mathrm{diam}_{\rho_{\mathrm{H}}} f(\sigma^{(0)}) < 8(n+1)^{-1}$, it follows that

$$\begin{aligned}\rho_{\mathrm{H}}(hg(\varphi,t),\varphi) &\leqslant \rho_{\mathrm{H}}(hg(\varphi,t),h(W)) + \rho_{\mathrm{H}}(h(W),\varphi)\\ &< 8(n+1)^{-1} + (n+1)^{-1} = 9(n+1)^{-1} < 9t.\end{aligned}$$

Then hg can be extended to the desired homotopy γ by $\gamma_0 = \mathrm{id}$. □

Using Lemma 3.1.2, we can easily prove the following Fedorchuk's result [$\mathrm{Fe}_{1,2}$]:

Proposition 3.1.1. *For a locally connected infinite compactum X,*

$$\mathrm{C_H}(X,[-1,1]) \approx Q.$$

Proof. Since $\mathrm{C}(X,(-1,1))$ homotopy dense in $\mathrm{C_H}(X,[-1,1])$ by Lemma 3.1.2 and $\mathrm{C}(X,(-1,1)) \approx s$, we can easily verify that $\mathrm{C_H}(X,[-1,1])$ is an AR and has the disjoint cells property, hence $\mathrm{C_H}(X,[-1,1]) \approx Q$ by Toruńczyk's characterization of Q (Theorem 1.3.1). □

Proof of Main Theorem. Since $\mathrm{C}(X,(-1,1))$ is completely metrizable, the complement

$$M = \mathrm{C_H}(X,[-1,1]) \setminus \mathrm{C}(X,(-1,1)) \subset \mathrm{C_H}(X,[-1,1])$$

is an F_σ-set, hence a Z_σ-set by Lemma 3.1.2. It remains to prove that M satisfies the condition ($\flat$) in $\mathrm{C_H}(X,[-1,1])$, whence the result follows from Theorem 1.3.2.

Let (A,B) be a pair of compacta in $\mathrm{C_H}(X,[-1,1])$ such that $B \subset M$ (i.e., $B \cap \mathrm{C}(X,(-1,1)) = \emptyset$), and let $\varepsilon > 0$. Define a map $\alpha\colon A \to [0,1]$ by

$$\alpha(\varphi) = \tfrac{1}{3}\min\{1, \varepsilon, \rho_{\mathrm{H}}(\varphi, B)\}.$$

By Lemma 3.1.2, we can find a map $f\colon A \to \mathrm{C_H}(X,[-1,1])$ such that $f(A\setminus B) \subset \mathrm{C}(X,(-1,1))$, $f|B = \mathrm{id}$ and $\rho_{\mathrm{H}}(f(\varphi),\varphi) < \alpha(\varphi)$ for each $\varphi \in A\setminus B$. Since $A\setminus B$ is completely metrizable and $\mathrm{C}(X,(-1,1)) \approx s$, we have a closed embedding $g\colon A\setminus B \to \mathrm{C}(X,(-1,1))$ such that $\rho_{\mathrm{H}}(g(\varphi), f(\varphi)) < \alpha(\varphi)$ for each $\varphi \in A\setminus B$. Let $x_0 \in X$ and define $h\colon A\setminus B \to M$ as follows:

$$h(\varphi)(x) = \begin{cases} [g(\varphi)(x), \min\{1, g(\varphi)(x) + \alpha(\varphi)\}] & \text{if } x = x_0, \\ g(\varphi)(x) & \text{otherwise.} \end{cases}$$

As is easily observed, h is continuous and injective. For each $\varphi \in A\setminus B$,

$$\begin{aligned}\rho_{\mathrm{H}}(h(\varphi),\varphi) &\leqslant \rho_{\mathrm{H}}(h(\varphi), g(\varphi)) + \rho_{\mathrm{H}}(g(\varphi), f(\varphi)) + \rho_{\mathrm{H}}(f(\varphi),\varphi) \\ &< \alpha(\varphi) + \alpha(\varphi) + \alpha(\varphi) = 3\alpha(\varphi) = \min\{\varepsilon, \rho_{\mathrm{H}}(\varphi, B)\}.\end{aligned}$$

Hence we can extend h to the map $\tilde{h}\colon A \to M$ by $\tilde{h}|B = \mathrm{id}$. Then $\tilde{h}$ is ε-close to id. Since $\rho_{\mathrm{H}}(\varphi, h(\varphi)) < \rho_{\mathrm{H}}(\varphi, B)$ for each $\varphi \in A\setminus B$, $\tilde{h}(A\setminus B) = h(A\setminus B)$ does not meet $h(B)$. Then it follows that $\tilde{h}$ is injective, whence it is an embedding since A is compact. $\square$

Remark. By only a couple of adjustments in the proof of Lemma 3.1.1, we can prove the following:

Lemma 3.1.1' (due to the referee in [SU$_2$]). *Let* $M = \mathrm{C_H}(X,[-1,1]) \setminus \mathrm{C}(X,(-1,1))$ *be the complement of* $\mathrm{C}(X,(-1,1))$ *in* $\mathrm{C_H}(X,[-1,1])$. *Then each map* $f\colon \mathbf{S}^n \to M$ *can be extended to a map* $h\colon \mathbf{B}^{n+1} \to M$ *so that* $\mathrm{diam}_{\rho_H} h(\mathbf{B}^{n+1}) \leqslant 4\mathrm{diam}_{\rho_H} f(\mathbf{S}^n)$.

Sketch of Proof. For every $s \in \mathbf{S}^n$ and every $\varepsilon > 0$, there exist an open neighborhood G_s of s in $\mathbf{S}^n$ and a finite subset A_s of X so that $f(s') \subset N_\rho(f(s) \cap p^{-1}(A_s), \varepsilon)$ for every $s' \in G_s$. Now, we can find finitely many $s_1, s_2, \cdots, s_p \in \mathbf{S}^n$ so that open sets $G_i = G_{s_i}$ cover $\mathbf{S}^n$. Then, given $s \in \mathbf{S}^n$, there exists i such that $f(s) \subset N_\rho(f(s_i) \cap p^{-1}(A_{s_i}), \varepsilon)$. We can make $A_{s_i} = A_i$ pairwise disjoint. As in the proof of Lemma 3.1.1, enlarge the sets A_i to open sets $U_i \subset X$ which are also pairwise disjoint. Use these A_i and U_i (replacing the vertices v by the points s_i, and $\dim \sigma + 1$ by p) to define $h(0)$ in the way that we have defined $g(b(\sigma))$, where 0 is the center of the ball $\mathbf{B}^{n+1}$. Write each $z \in \mathbf{B}^{n+1}$ in the form ts $(s \in \mathbf{S}^n,\ t \in [0,1])$ and define $h(z) = tf(s) + (1-t)h(0)$. Then, $h\colon \mathbf{B}^{n+1} \to M$ is the desired one. □

Since $\mathrm{C_H}(X, [-1,1])$ is an AR (which is proved by Fedorchuk in [Fe_2]), according to [To_4, Corollary 3.3], the above lemma shows that the complement $M = \mathrm{C_H}(X, [-1,1]) \setminus \mathrm{C}(X, (-1,1))$ is l.h.n., hence it is a Z_σ-set in $\mathrm{C_H}(X, [-1,1])$. Thus we don't need Lemma 3.3.2 to prove Main Theorem. But it follows from Lemma 3.1.2 that $\mathrm{C_H}(X, [-1,1])$ is an AR. Namely, Lemma 3.1.2 gives us an alternative proof of this result.

Proof of Corollary 1 (cf. Proof of [Fe_2, Proposition 2.4]). By Main Theorem, $(\mathrm{C_H}(X, [-1,1]), \mathrm{C}(X, (-1,1)) \approx (Q, s)$, that is,

$$(\mathrm{C_H}(X, [-1,1]), \mathrm{C_H}(X, [-1,1]) \setminus \mathrm{C}(X, (-1,1)) \approx (Q, Q \setminus s).$$

Let

$$D = \mathrm{C_H}(X, [-1,1]) \setminus \mathrm{C_H}(X, (-1,1)) \subset \mathrm{C_H}(X, [-1,1]) \setminus \mathrm{C}(X, (-1,1)),$$

where $\mathrm{C_H}(X, (-1,1))$ is the closure of $\mathrm{C}(X, (-1,1))$ in $\exp(X \times (-1,1))$. Then D is a Z-set in $\mathrm{C_H}(X, [-1,1]) \approx Q$. For each $\varphi \in \mathrm{C_H}(X, [-1,1])$, $\varphi \in D$ if and only if

$\varphi \not\subset X \times (-1,1)$, i.e., $\max \varphi(x) = 1$ or $\min \varphi(x) = -1$ for some $x \in X$. We can define a homotopy $\eta \colon D \times [0,1] \to D$ as follows:

$$\begin{aligned}\eta_t(\varphi)(x) &= (1-t)\varphi(x) + t[-1,1] \\ &= [(1-t)\min\varphi(x) - t, (1-t)\max\varphi(x) + t] \quad \text{for each } \ x \in X.\end{aligned}$$

Then $\eta_0 = \mathrm{id}$ and $\eta_1(\varphi)(x) = [-1,1]$ for every $\varphi \in D$, i.e., $\eta_1(D) = \{X \times [-1,1]\}$. Therefore, D is contractible. By Chapman's Complement Theorem [Ch$_2$],

$$\mathrm{C_H}(X,(-1,1)) = \mathrm{C_H}(X,[-1,1]) \setminus D \approx Q \setminus \{\mathrm{pt}\} \approx Q \times [0,1).$$

It follows from [Ch$_1$, Lemma 5.4 and Theorem 6.2] that

$$\begin{aligned}&(\mathrm{C_H}(X,(-1,1)), \mathrm{C_H}(X,(-1,1)) \setminus \mathrm{C}(X,(-1,1)) \\ &\qquad\qquad \approx (Q \times [0,1), Q \times [0,1) \setminus s \times [0,1)).\end{aligned}$$

Thus we have

$$\begin{aligned}(\mathrm{C_H}(X), \mathrm{C}(X)) &\approx (\mathrm{C_H}(X,(-1,1)), \mathrm{C}(X,(-1,1)) \\ &\approx (Q \times [0,1), s \times [0,1)). \qquad \square\end{aligned}$$

§3.2. Remarks and counter-examples

In our results, the local connectedness of X is essential.

Example 3.2.1. *There exists a continuum X such that $\mathrm{C_H}(X,[-1,1])$ is not locally path-connected, that is, neither $\mathrm{C}(X,[-1,1])$ nor $\mathrm{C}(X,(-1,1))$ has the l.h.n. complement in $\mathrm{C_H}(X,[-1,1])$.*

Proof. Let X be the comb space, that is,

$$X = (\{n^{-1} \mid n \in \mathbb{N}\} \cup \{0\}) \times [0,1] \cup [0,1] \times \{0\} \subset \mathbb{R}^2$$

and let

$$\varphi_0 = X \times \{1\} \cup \{0\} \times [0,1] \times \{-1,1\} \cup \{(0,0)\} \times [-1,1] \in \exp(X \times [-1,1]).$$

For each $n \in \mathbb{N}$, we define $f_n, g_n \in \mathrm{C}(X, [-1,1])$ by

$$f_n(x,y) = \begin{cases} -1 & \text{if } x = n^{-1}, \\ 1 & \text{if } x \leqslant (n+1)^{-1} \text{ or } x \geqslant (n-1)^{-1}, \\ -2n(n+1)x + 2n + 1 & \text{if } (n+1)^{-1} \leqslant x \leqslant n^{-1}, \\ 2n(n-1)x - 2n + 1 & \text{if } n^{-1} \leqslant x \leqslant (n-1)^{-1}, \end{cases}$$

and

$$g_n(x,y) = \begin{cases} -1 & \text{if } x \leqslant (n+1)^{-1}, \\ 1 & \text{if } x \geqslant n^{-1}, \\ 2n(n+1)x - 2n - 1 & \text{if } (n+1)^{-1} \leqslant x \leqslant n^{-1}. \end{cases}$$

Then $\rho_{\mathrm{H}}(f_n, \varphi_0) = \rho_{\mathrm{H}}(g_n, \varphi_0) = n^{-1}$. Hence $\varphi_0 \in \mathrm{C}_{\mathrm{H}}(X, [-1,1])$. Let

$$W = \{\varphi \in \mathrm{C}_{\mathrm{H}}(X, [-1,1]) \mid \varphi \cap [0,1] \times [1/2, 1] \times [-1/2, 1/2] = \emptyset\}.$$

Then W is an open neighborhood of φ_0 in $\mathrm{C}_{\mathrm{H}}(X, [-1,1])$. For each neighborhood V of φ_0 in W, choose $n \in \mathbb{N}$ so large that $f_n, g_n \in V$. By the same way as above, we can show that there is no path in W connecting f_n and g_n. Thus $\mathrm{C}_{\mathrm{H}}(X, [-1,1])$ is not locally path-connected at φ_0. □

Remark. Since $\mathrm{C}(X, [0,1]) \approx \mathrm{C}(X, [0,1)) \approx s$ (cf. [Sa_2]), we can show similarly to Main Theorem and Corollary 1 that if X is locally connected then

$$(\mathrm{C}_{\mathrm{H}}(X, [0,1]), \mathrm{C}(X, [0,1])) \approx (Q, s) \quad \text{and}$$

$$(\mathrm{C}_{\mathrm{H}}(X, [0,1)), \mathrm{C}(X, [0,1))) \approx (Q \times [0,1), s \times [0,1)).$$

We can summarize them as follows:

Corollary 3.2.1. *Let X be a locally connected infinite compactum and $Y \subset \mathbb{R}$ a non-degenerate interval. Then, $(\mathrm{C_H}(X,Y), \mathrm{C}(X,Y))$ is a (Q,s)-manifold.*

Note that $\mathrm{C}(X,Y)$ is an s-manifold even if Y is an arbitrary separable completely metrizable ANR without isolated points [Sa$_2$]. However, in the above corollary, Y cannot be replaced by the unit circle $\mathbf{S}^1 \subset \mathbb{R}^2$ even though $X = [0,1]$, that is,

Example 3.2.2. *The pair $(\mathrm{C_H}([0,1],\mathbf{S}^1), \mathrm{C}([0,1],\mathbf{S}^1))$ is not a (Q,s)-manifold pair.*

Proof. Let ρ be the metric on $[0,1] \times \mathbf{S}^1$ defined by

$$\rho((x,y),(x',y')) = \max\{|x-x'|, d(y,y')\},$$

where d is the arc-length metric on $\mathbf{S}^1$ (whence the diameter of $\mathbf{S}^1$ is equal to π), and let ρ_H be the Hausdorff metric on $\exp([0,1]\times\mathbf{S}^1)$ induced by ρ. First, we show that $\mathrm{C_H}([0,1],\mathbf{S}^1) = \mathrm{USCC}([0,1],\mathbf{S}^1)$, where $\mathrm{USCC}([0,1],\mathbf{S}^1)$ is the space of of u.s.c. multi-valued functions $\varphi\colon [0,1] \to \mathbf{S}^1$ such that each $\varphi(x)$ is a continuum (= an arc or a point in this case).

To see the inclusion "$\subset$", let $\varphi \in \mathrm{C_H}([0,1],\mathbf{S}^1)$. Then φ is u.s.c. because φ is closed in $[0,1]\times\mathbf{S}^1$. Assume that $\varphi(x_0)$ is disconnected for some $x_0 \in [0,1]$. Then there are disjoint open sets $V_1, V_2 \subset \mathbf{S}^1$ such that $\varphi(x_0) \subset V_1 \cup V_2$ and $V_i \cap \varphi(x_0) \neq \emptyset$, that is, there are $y_i \in V_i \cap \varphi(x_0)$, $i=1,2$. Since φ is u.s.c., x_0 has an open neighborhood U in $[0,1]$ such that $\varphi(U) = \bigcup_{x\in U}\varphi(x) \subset V_1 \cup V_2$. Choose $\varepsilon > 0$ so that

$$N_\rho(\varphi(x_0),\varepsilon) \subset U \times (V_1 \cup V_2),$$

and $f \in \mathrm{C}([0,1],\mathbf{S}^1)$ so that $\rho_\mathrm{H}(f,\varphi) < \varepsilon$. Then there are $x_1, x_2 \in [0,1]$ such that $|x_i - x_0| < \varepsilon$ and $d(f(x_i), y_i) < \varepsilon$, $i=1,2$. Since the ε-neighborhood of y_i is connected, it is contained in V_i, hence $f(x_i) \in V_i$, $i=1,2$. We may assume that $x_1 < x_2$. Note that $x_1 \leqslant x \leqslant x_2$ implies $|x - x_0| < \varepsilon$, so $x \in U$. Hence $f([x_1,x_2]) \subset V_1 \cup V_2$, which contradicts to the connectedness of $f([x_1,x_2])$. Consequently, $\varphi(x)$ is connected for each $x \in [0,1]$.

To see the inclusion "$\supset$", let $\varphi \in \mathrm{USCC}([0,1],\mathbf{S}^1)$ and $\varepsilon > 0$. Choose

$$F = \left\{(x_i, y_j^{(i)}) \mid i = 1, \cdots, n;\ j = 1, \cdots, k(i)\right\} \subset \varphi,$$

so that $\rho_{\mathrm{H}}(F,\varphi) < \varepsilon$, where $0 = x_1 < x_2 < \cdots < x_n < 1$ and $y_1^{(i)}, \cdots, y_{k(i)}^{(i)} \in \varphi(x_i)$ $(i = 1, \cdots, n)$. Note that $x_{i+1} - x_i < \varepsilon$, where $x_{n+1} = 1$. For each $i = 1, \cdots, n$, choose

$$x_i = x_1^{(i)} < x_2^{(i)} < \cdots < x_{k(i)}^{(i)} < x_{i+1}.$$

Since each $\varphi([x_i, x_{i+1}]) = \bigcup_{x \in [x_i, x_{i+1}]} \varphi(x)$ is connected, we can construct a map $f\colon [0,1] \to \mathbf{S}^1$ so that $f(x_j^{(i)}) = y_j^{(i)}$, $f([x_1^{(i)}, x_{k(i)}^{(i)}]) \subset \varphi(x_i)$ and $f([x_{k(i)}^{(i)}, x_{i+1}]) \subset \varphi([x_i, x_{i+1}])$. Observe that $\rho_{\mathrm{H}}(f,\varphi) < \varepsilon$. Thus, $\varphi \in C_{\mathrm{H}}([0,1],\mathbf{S}^1)$.

Next, we show that $\mathrm{C_H}([0,1],\mathbf{S}^1) = \mathrm{USCC}([0,1],\mathbf{S}^1)$ is contractible. Let

$$\gamma\colon \mathrm{USCC}([0,1],\mathbf{S}^1) \times [0,1] \to \mathrm{USCC}([0,1],\mathbf{S}^1)$$

be a homotopy defined by $\gamma_0 = \mathrm{id}$ and $\gamma_t(\varphi)(x) = \mathrm{cl}_{\mathbf{S}^1} N_d(\varphi(x), t\pi)$ for $t > 0$, whence

$$\gamma_1(\varphi) = [0,1] \times \mathbf{S}^1 \in \mathrm{USCC}([0,1],\mathbf{S}^1).$$

Since each $\varphi(x)$ is connected and d is the arc-length metric on $\mathbf{S}^1$, each $\gamma_t(\varphi)(x)$ is connected. To see the upper semi-continuity of $\gamma_t(\varphi)$ at $x_0 \in [0,1]$, let V be an open neighborhood of $\gamma_t(\varphi)(x_0)$ in $\mathbf{S}^1$. Choose $\delta > 0$ so that $N_d(\gamma_t(\varphi)(x_0), \delta) \subset V$. Since φ is u.s.c., we have a neighborhood U of x_0 in $[0,1]$ such that $\varphi(V) = \bigcup_{x \in U} \varphi(x) \subset N_d(\varphi(x_0), \delta/2)$. For each $x \in U$ and $y \in \gamma_t(\varphi)(x)$, there is some $y' \in \varphi(x)$ such that $d(y,y') < t\pi + \delta/2$. Since $d(y', \varphi(x_0)) < \delta/2$, we have $d(y, \varphi(x_0)) < t\pi + \delta$. In case $d(y, \varphi(x_0)) \leqslant t\pi$, $y \in \gamma_t(\varphi)(x_0) \subset V$. Otherwise, we have $y'' \in \mathbf{S}^1$ such that $d(y, y'') < \delta$ and $d(y'', \varphi(x_0)) = t\pi$ because d is the arc-length metric on $\mathbf{S}^1$. Then

$y \in N_d(\gamma_t(\varphi)(x_0), \delta) \subset V$. Therefore each $\gamma_t(\varphi)$ is u.s.c. Thus γ is well-defined. The continuity of γ is easily checked.

On the other hand, $\mathrm{C}([0,1], \mathbf{S}^1)$ has the homotopy type of $\mathbf{S}^1$. Hence $\mathrm{C}([0,1], \mathbf{S}^1)$ has not the l.h.n. complement in $\mathrm{C_H}([0,1], \mathbf{S}^1)$. $\square$

Related with our results, there is the following general problem:

Problem. *Under a sufficiently general condition, find a natural local compactification* $\overline{\mathrm{C}(X,Y)}$ *of* $\mathrm{C}(X,Y)$ *so that* $(\overline{\mathrm{C}(X,Y)}, \mathrm{C}(X,Y))$ *is a* (Q,s)*-manifold pair.*

Note that $C(X,Y)$ is an s-manifold if X is an infinite compactum and Y is an separable completely metrizable ANR without isolated points [Sa$_2$]. In Chapter 7, this Probrem under the condition $Y = \mathbb{R}$ will be soleved, that is, we will prove the converse of Main Theorem.

APPENDIX

In [Fe$_2$], Fedorchuk proved that $\mathrm{C_H}(X, [-1,1]) \approx Q$ and $\mathrm{C_H}(X) \approx Q \times [0,1)$ for a locally connected infinite compact metric space. This result is reduced to the case X is a non-degenerate Peano continuum [Fe$_2$, Propositions 2.3 and 2.4]. The proof of [Fe$_2$, Proposition 2.3] contains a gap, where the map d_ϵ is defined but it is not continuous even if $X = [0,1]$. Before giving a counter-example, we recall the definition of d_ϵ. Given $x_0 \in X$ and $0 < \epsilon < 1$, $d_\epsilon \colon C_\mathrm{H}(X, [-1,1]) \to C_\mathrm{H}(X, [-1,1])$ is defined as follows:

$$d_\epsilon(\varphi) = (1-\epsilon)\varphi \cup \{x_0\} \times [\min(1-\epsilon)\varphi(x_0), \max(1-\epsilon)\varphi(x_0) + \epsilon].$$

In other words,

$$d_\epsilon(\varphi)(x) = \begin{cases} (1-\epsilon)\varphi(x) & \text{if } x \neq x_0, \\ [\min(1-\epsilon)\varphi(x_0), \max(1-\epsilon)\varphi(x_0) + \epsilon] & \text{if } x = x_0. \end{cases}$$

Counter-example. Let $X = [0,1]$ and $x_0 = 0$. For each $n \in \mathbb{N} \cup \{0\}$, we define $\varphi_n \in C_{\mathrm{H}}(X, [-1,1])$ as follows:

$$\varphi_0 = X \times \{0\} \cup \{0\} \times [0, \tfrac{1}{2}] \quad \text{and}$$
$$\varphi_n = X \times \{0\} \cup \{1/n\} \times [0, \tfrac{1}{2}] \quad \text{for } n > 0.$$

Then φ_n converges to φ_0. As is easily observed, we have

$$d_\epsilon(\varphi_0) = X \times \{0\} \cup \{0\} \times [0, \tfrac{1}{2}(1+\epsilon)] \quad \text{and}$$
$$d_\epsilon(\varphi_n) = X \times \{0\} \cup \{0\} \times [0, \epsilon] \cup \{1/n\} \times [0, \tfrac{1}{2}(1-\epsilon)] \quad \text{for } n > 0.$$

It follows that

$$\rho_{\mathrm{H}}(d_\epsilon(\varphi_n), d_\epsilon(\varphi_0)) \geqslant \min\{\tfrac{1}{2}(1-\epsilon), \epsilon\} \quad \text{for sufficiently large } n,$$

which means that $d_\epsilon(\varphi_n)$ does not converge to $d_\epsilon(\varphi_0)$.

One should remark that d_ϵ is always continuous on $C(X, [-1,1])$ because the evaluation $f \mapsto f(x_0)$ is continuous. We have used such a map in our proof of the main theorem.

Chapter 4

Spaces of multi-valued functions with compact fibers

Let X and Y be locally connected compacta. Recall that $\mathrm{USC}(X,Y)$ is the space of u.s.c. multi-valued functions $\varphi\colon X \to Y$ such that each $\varphi(x)$ is compact, whence $\mathrm{USC}(X,Y)$ is endowed with the Hausdorff metric ρ_H. In this chapter, it is proved that $\mathrm{USC}(X,Y)$ is a Q-manifold provided that Y has no isolated points.

§4.1. The space $\mathrm{USC}(X,Y)$ for locally connected compacta X and Y

The following is our main result:

Theorem 4.1.1. *Let X and Y be locally connected compacta. Suppose that Y has no isolated point. Then,* $\mathrm{USC}(X,Y)$ *is a compact Q-manifold.*

Since X is locally connected and compact, it has only finitely many components $X_1,\ldots,X_n$. Then, we can write

$$\mathrm{USC}(X,Y) \approx \prod_{i=1}^{n} \mathrm{USC}(X_i,Y).$$

Hence, the theorem above can be deduced to the case that X is a Peano continuum. We may assume that the metric ρ_H is induced by the convex metric d for X (cf. [Bi], [Mo]).

We prove that $\mathrm{USC}(X,Y)$ is an ANR. For every finite collection $\mathcal{A}$ of subsets of $X \times Y$, put

$$\langle \mathcal{A} \rangle = \{\varphi \in \mathrm{USC}(X,Y) \mid \varphi \subset \cup\mathcal{A} \text{ and } \varphi \cap A \neq \emptyset \text{ for every } A \in \mathcal{A}\}.$$

Then, the collection

$$\mathcal{B} = \{\langle \mathcal{A} \rangle \mid \mathcal{F} \text{ is a finite family of open subsets of } X \times Y\}$$

is an open base for $\mathrm{USC}(X,Y)$.

Lemma 4.1.1. *If X and Y are locally connected compacta, then* $\mathrm{USC}(X,Y)$ *is locally connected.*

Proof. Take $\varphi \in \mathrm{USC}(X,Y)$ and let $\mathcal{U}$ be a neighborhood of A in $\mathrm{USC}(X,Y)$. There exists a finite family $\mathcal{V}$ of open subsets of $X \times Y$ such that $A \in \langle \mathcal{V} \rangle \subseteq \mathcal{U}$. Since $X \times Y$ is compact and locally connected, there exists a finite family $\mathcal{E}$ of connected compact subsets of $X \times Y$ such that $A \in \langle \mathcal{E}' \rangle \subseteq \langle \mathcal{E} \rangle \subseteq \langle \mathcal{V} \rangle$, where $\mathcal{E}' = \{\mathrm{int} E \mid E \in \mathcal{E}\}$. Since $X \times Y$ is locally connected compact, we may assume that $\bigcup_{E \in \mathcal{E}} E$ is the union of finitely many disjoint Peano continua $F_1, \ldots, F_m$.

We claim that $\langle \mathcal{E} \rangle$ is contractible. For each $i \leq m$, there exists a contraction $r^i \colon 2^{F_i} \times [0,1] \to 2^{F_i}$ such that $r^i_s(A) \subseteq r^i_t(A)$ for $0 \leq s \leq t \leq 1$, $r^i_0 = \mathrm{id}$ and $r^i_1(2^{F_i}) = \{F_i\}$. For any $\varphi \in \langle \mathcal{E} \rangle$ and $i \leq m$, put $\varphi_i = \varphi \cap F_i$. We define the map $r \colon \langle \mathcal{E} \rangle \times [0,1] \to \langle \mathcal{E} \rangle$ by

$$r_t(\varphi) = \bigcup_{i=1}^{m} r^i_t(\varphi_i).$$

It is easily seen that r is well-defined, continuous, $r_0 = \mathrm{id}$ and $r_1(\langle \mathcal{E} \rangle) = \{\cup\mathcal{E}\}$, whence r is a contraction. Therefore, $\langle \mathcal{E} \rangle$ is contractible. Thus, $\mathrm{USC}(X,Y)$ is locally connected. □

Proposition 4.1.1. *The space* $\mathrm{USC}(X,Y)$ *is an ANR if* X *and* Y *are locally connected compacta.*

Proof. To prove that $\mathrm{USC}(X,Y)$ is an ANR, we aim at applying Theorem 5.2.12 in [vM$_2$]. Since $\mathcal{B}$ is closed under finite intersections, it suffices to prove that every component of an element of $\mathcal{B}$ is homotopically trivial. To this end, let $U_1, \ldots, U_n \subset X \times Y$ be nonempty and open, and consider a component C of $B = \langle\{U_1, \ldots, U_n\}\rangle$. Since $\mathrm{USC}(X,Y)$ is compact and locally connected, C is path-connected by Theorem 5.3.13 in [vM$_2$]. For the verification that C is homotopically trivial, it suffices to consider a continuous function $g\colon \mathbf{S}^n \to C$, where $n \geq 1$. The function $\overline{g}\colon 2^{\mathbf{S}^n} \to \mathrm{USC}(X,Y)$ defined by $\overline{g}(A) = \bigcup_{x \in A} g(x)$ is well-defined. By Corollary 5.3.7 in [vM$_2$], $\overline{g}$ is continuous. It is easily seen that $\overline{g}$ is an extension of g, and $\overline{g}(A) \in C$ for every $A \in 2^{\mathbf{S}^n}$. By Proposition 5.3.11 in [vM$_2$], there exists a continuous function $f_n\colon \mathbf{B}^{n+1} \to 2^{\mathbf{S}^n}$ such that $f_n(x) = \{x\}$ for every $x \in \mathbf{S}^n$. The composition $\overline{g} \circ f_n\colon \mathbf{B}^{n+1} \to C$ is required extension of f. Therefore, $\mathrm{USC}(X,Y)$ is an ANR. □

Recall that the convex metric d is defined by

$$d(x,y) = \inf\{\operatorname{diam} C \mid C \text{ is connected and contains } \{x,y\}\}.$$

Lemma 4.1.2. *Let* (X,d) *be a compactum. If* d *is a convex metric, then the map* $b\colon X \times \mathbb{R} \to 2^X$ *defined by* $b(x,t) = \overline{B}_d(x,t)$ *is continuous.*

Proof. Take $\varepsilon > 0$, $x \in X$ and $r \in \mathbb{R}$. Let $x' \in X$ and $r' \in \mathbb{R}$ be points such that $d(x,x') < \varepsilon$ and $|r - r'| < \varepsilon$. It is easily obtained that

$$\overline{B}_d(x,r) \subset B_d(x', r+\varepsilon) \subset B_d(x', r'+2\varepsilon).$$

Recall that the metric for 2^X is induced by the convex metric d for X, which has the property that each two points $x, x' \in X$ can be joined by an arc in X isometric to the segment $[0, d(x, x')]$ in $\mathbb{R}$. Then, we have

$$B_d(x', r' + 2\varepsilon) \subset B_d(\overline{B}_d(x', r'), 2\varepsilon).$$

Hence, $\overline{B}_d(x, r) \subset B_d(\overline{B}_d(x', r'), 2\varepsilon)$. Conversely, $\overline{B}_d(x', r') \subset B_d(\overline{B}_d(x, r), 2\varepsilon)$. It follows that $d_H(\overline{B}_d(x, r), \overline{B}_d(x', r')) < 2\varepsilon$. Therefore, b is continuous. □

Put

$$\mathrm{USC}_F(X, Y) = \{\varphi \in \mathrm{USC}(X, Y) \mid \text{ each } \varphi(x) \text{ is finite }\}.$$

Proposition 4.1.2. $\mathrm{USC}_F(X, Y)$ *is homotopy dense in* $\mathrm{USC}(X, Y)$.

Proof. We consider the homotopy dense subset $\mathcal{F}_\infty = \{F \in 2^{X \times Y} \mid \mathrm{card} F < \infty\}$ in $2^{X \times Y}$ ([vM2]). There exists a homotopy $h\colon 2^{X \times Y} \times [0, 1] \to 2^{X \times Y}$ such that $h_0 = \mathrm{id}$ and $h(2^{X \times Y} \times (0, 1]) \subset \mathcal{F}_\infty$. Define the homotopy

$$H\colon \mathrm{USC}(X, Y) \times [0, 1] \to \mathrm{USC}(X, Y)$$

by

$$H(\varphi, t) = \bigcup_{(x,y) \in h(\varphi,t)} \overline{B}(x, \rho_H(\varphi, h(\varphi, t))) \times \{y\}.$$

Then, H is well-defined. To the contrary, we assume that $p(H(\varphi, t)) \neq X$ for some $(\varphi, t) \in \mathrm{USC}(X, Y) \times [0, 1]$, where $p\colon X \times Y \to Y$is the projection. Put $\varepsilon = \rho_H(\varphi, h(\varphi, t)) + \rho_H(X, H(\varphi, t))/2$. But $B_d(h(\varphi, t), \varepsilon)$ doesn't contain φ. This is a contradiction. It follows by Lemma 4.1.2. that H is continuous. It is easy to see that $H_0 = \mathrm{id}$ and $H(\mathrm{USC}(X, Y) \times (0, 1]) \subset \mathrm{USC}_F(X, Y)$, as required. □

Proposition 4.1.3. *Let X and Y be locally connected compacta. If Y has no isolated point, then the complement* $\mathrm{USC}(X, Y) \setminus \mathrm{USC}_F(X, Y)$ *is homotopy dense in* $\mathrm{USC}(X, Y)$.

Proof. Put $\varepsilon = \mathrm{diam} Y$. Since Y is locally connected, we can define the admissible metric d_Y for Y by

$$d_Y(y, y') = \inf\{\varepsilon + 1, \mathrm{diam} C \mid C \text{ is continuum containing } y \text{ and } y'\}.$$

We may assume that ρ_H is induced by d_Y. Define the homotopy

$$H\colon \mathrm{USC}(X,Y) \times [0,1] \to \mathrm{USC}(X,Y)$$

by

$$H(\varphi,t)(x) = \overline{B}_{d_Y}(\varphi(x),t).$$

Since Y has no isolated point, $H(\mathrm{USC}(X,Y) \times (0,1]) \cap \mathrm{USC}_F(X,Y) = \emptyset$. Hence, H is required one. $\square$

Proof of Theorem 4.1.1. By using Propositions 4.1.1, 4.1.2 and 4.1.3, we apply Theorem 1.3.1 to obtain Theorem 4.1.1. $\square$

In case Y is connected, we have the following:

Corollary 4.1.1. *Let X be a locally connected compactum and Y be a nondegenerate Peano continuum. Then,* $\mathrm{USC}(X,Y)$ *is homeomorphic to the Hilbert cube.*

Proof. Let d_Y be a convex metric for Y. We may assume that $\operatorname{diam} Y \leq 1$. The homotopy $H\colon \mathrm{USC}(X,Y) \times [0,1] \to \mathrm{USC}(X,Y)$ defined by

$$H(\varphi,t)(x) = \overline{B}_{d_Y}(\varphi(x),t)$$

is a contraction. Hence, $\mathrm{USC}(X,Y)$ is homeomorphic to the Hilbert cube by Theorem 1.3.1. $\square$

Chapter 5

Hilbert cube manifold compactifications of some function spaces

Let G be a finite graph (= 1-dimensional compact polyhedron) in Euclidean space $\mathbb{R}^n$. Anderson [An_3] proved that the homeomorphism group $\mathrm{H}(G)$ of G with the topology induced by the sup-metric is an s-manifold and Sakai [Sa_4] showed that the triple $(\mathrm{H}(G), \mathrm{H}^{\mathrm{LIP}}(G), \mathrm{H}^{\mathrm{PL}}(G))$ is locally homeomorphic to (s, Σ, σ) (i.e., it is an (s, Σ, σ)-manifold triple), where $\mathrm{H}^{\mathrm{LIP}}(G)$ and $\mathrm{H}^{\mathrm{PL}}(G)$ are subgroups of $\mathrm{H}(G)$ consisting of Lipschitz homeomorphisms and PL homeomorphisms, respectively. By identifying each $h \in \mathrm{H}(G)$ with the graph of h, which is a compact set in $G \times G$, we can regard $\mathrm{H}(G)$ as a subspace of the hyperspace $\exp(G \times G)$. By $\overline{\mathrm{H}}(G)$, we denote the closure of $\mathrm{H}(G)$ in $\exp(G \times G)$. Then $\overline{\mathrm{H}}(G)$ is a compactification of $\mathrm{H}(G)$. In §5.1, we show that $\overline{\mathrm{H}}(G)$ is a Q-manifold. In fact, we can prove that the quadruple $(\overline{\mathrm{H}}(G), \mathrm{H}(G), \mathrm{H}^{\mathrm{LIP}}(G), \mathrm{H}^{\mathrm{PL}}(G))$ is locally homeomorphic to (Q, s, Σ, σ) (i.e., it is a (Q, s, Σ, σ)-manifold quadruple) for any finite graph $G \subset \mathbb{R}^n$.

Let $\mathrm{E}(\mathbf{I}, G)$ be the space of embeddings of the unit interval $\mathbf{I} = [0,1]$ into a graph $G \subset \mathbb{R}^n$, where $\mathrm{E}(\mathbf{I}, G)$ has the topology induced by the sup-metric. In [Sa_4], it was also shown that the triple $(\mathrm{E}(\mathbf{I}, G), \mathrm{E}^{\mathrm{LIP}}(\mathbf{I}, G), \mathrm{E}^{\mathrm{PL}}(\mathbf{I}, G))$ is locally homeomorphic to (s, Σ, σ) (i.e., it is an (s, Σ, σ)-manifold triple), where $\mathrm{E}^{\mathrm{LIP}}(\mathbf{I}, G)$ and $\mathrm{E}^{\mathrm{PL}}(\mathbf{I}, G)$ are the subspaces of $\mathrm{E}(\mathbf{I}, G)$ consisting of Lipschitz embeddings and PL embeddings, respectively. Similarly as above, consider $\mathrm{E}(\mathbf{I}, G)$ as a subset of the hyperspace $\exp(\mathbf{I} \times G)$ of nonempty compact sets in $\mathbf{I} \times G$. Now we don't assume that G is compact, so $\exp(\mathbf{I} \times G)$ is not compact in general but it is locally compact. Then the closure $\overline{\mathrm{E}}(\mathbf{I}, G)$ of $\mathrm{E}(\mathbf{I}, G)$ in $\exp(\mathbf{I} \times G)$ is a local compactification of $\mathrm{E}(\mathbf{I}, G)$. In §5.2, we can also prove that the quadruple $(\overline{\mathrm{E}}(\mathbf{I}, G), \mathrm{E}(\mathbf{I}, G), \mathrm{E}^{\mathrm{LIP}}(\mathbf{I}, G), \mathrm{E}^{\mathrm{PL}}(\mathbf{I}, G))$ is locally homeomorphic to (Q, s, Σ, σ) (i.e., it is a (Q, s, Σ, σ)-manifold quadruple) for any graph $G \subset \mathbb{R}^n$.

On the other hand, it was proved in [BS] that $\mathrm{R}(\mathbf{I})$ is homeomorphic to s, where $\mathrm{R}(\mathbf{I})$ be the space of retractions $f : \mathbf{I} \to \mathbf{I}$ endowed with the sup-metric. We also give a natural compactification $\overline{\mathrm{R}}(\mathbf{I})$ of $\mathrm{R}(\mathbf{I})$ as a closure of $\mathrm{R}(\mathbf{I})$ in the hyperspace $\exp(\mathbf{I} \times \mathbf{I})$. In §5.3, we will show that $(\overline{\mathrm{R}}(\mathbf{I}), \mathrm{R}(\mathbf{I})) \approx (Q, s)$.

§5.1. A Hilbert cube compactification of $\mathrm{H}_\partial(\mathbf{I})$

We denote $\mathrm{H}_\partial(\mathbf{I}) = \{h \in \mathrm{H}(\mathbf{I}) \mid h|\partial\mathbf{I} = \mathrm{id}\}$, $\mathrm{H}_\partial^{\mathrm{LIP}}(\mathbf{I}) = \mathrm{H}^{\mathrm{LIP}}(\mathbf{I}) \cap \mathrm{H}_\partial(\mathbf{I})$, $\mathrm{H}_\partial^{\mathrm{PL}}(\mathbf{I}) = \mathrm{H}^{\mathrm{PL}}(\mathbf{I}) \cap \mathrm{H}_\partial(\mathbf{I})$, and $\overline{\mathrm{H}}_\partial(\mathbf{I})$ is the closure of $\mathrm{H}_\partial(\mathbf{I})$. In this section, we prove the following:

Theorem 5.1.1. *The quadruple $(\overline{\mathrm{H}}(G), \mathrm{H}(G), \mathrm{H}^{\mathrm{LIP}}(G), \mathrm{H}^{\mathrm{PL}}(G))$ is locally homeomorphic to (Q, s, Σ, σ) (i.e., it is a (Q, s, Σ, σ)-manifold quadruple) for any finite graph $G \subset \mathbb{R}^n$.*

By the arguments in [An$_3$] (cf. [Sa$_4$]), for any finite graph G, there exists a compact polyhedron K such that $\mathrm{H}(G) \approx \mathrm{H}_\partial(\mathbf{I}) \times K$. By the same homeomorphism, we have $\mathrm{H}^{\mathrm{LIP}}(G) \approx \mathrm{H}_\partial^{\mathrm{LIP}}(\mathbf{I}) \times K$ and $\mathrm{H}^{\mathrm{PL}}(G) \approx \mathrm{H}_\partial^{\mathrm{PL}}(\mathbf{I}) \times K$. As is easily observed, this homeomorphism can be extend to $\overline{\mathrm{H}}(G) \approx \overline{\mathrm{H}}_\partial(\mathbf{I}) \times K$. Consequently,

$$(\overline{\mathrm{H}}(G), \mathrm{H}(G), \mathrm{H}^{\mathrm{LIP}}(G), \mathrm{H}^{\mathrm{PL}}(G)) \approx$$
$$(\overline{\mathrm{H}}_\partial(\mathbf{I}) \times K, \mathrm{H}_\partial(\mathbf{I}) \times K, \mathrm{H}_\partial^{\mathrm{LIP}}(\mathbf{I}) \times K, \mathrm{H}_\partial^{\mathrm{PL}}(\mathbf{I}) \times K).$$

Since $(Q \times L, s \times L, \Sigma \times L, \sigma \times L) \approx (Q, s, \Sigma, \sigma)$ for any compact contractible polyhedron L, Theorem 5.1.1 is reduced to the following theorem:

Theorem 5.1.2. $(\overline{\mathrm{H}}_\partial(\mathbf{I}), \mathrm{H}_\partial(\mathbf{I}), \mathrm{H}_\partial^{\mathrm{LIP}}(\mathbf{I}), \mathrm{H}_\partial^{\mathrm{PL}}(\mathbf{I})) \approx (Q, s, \Sigma, \sigma)$.

Moreover, this can be reduced to the following:

Theorem 5.1.3. $(\overline{\mathrm{H}}_\partial(\mathbf{I}), \mathrm{H}_\partial(\mathbf{I})) \approx (Q, s)$.

In fact, $(\mathrm{H}_\partial(\mathbf{I}), \mathrm{H}_\partial^{\mathrm{LIP}}(\mathbf{I}), \mathrm{H}_\partial^{\mathrm{PL}}(\mathbf{I})) \approx (s, \Sigma, \sigma)$ by [Sa$_4$, Theorem 3]. On the other hand, $(\overline{\mathrm{H}}_\partial(\mathbf{I}), \mathrm{H}_\partial(\mathbf{I}), \mathrm{H}_\partial^{\mathrm{LIP}}(\mathbf{I})) \approx (Q, s, \Sigma)$ by [Ch$_1$, Lemma 4.3] and Theorem 5.1.3. Then we can apply [CDM, Theorem 2.4] to obtain Theorem 5.1.2.

Thus, to obtain Theorem 5.1.1, it suffices to prove Theorem 5.1.3.

Put $\mathrm{USC}(\mathbf{I}) = \mathrm{USC}(\mathbf{I}, \mathbf{I})$ and $\mathrm{USCC}(\mathbf{I}) = \mathrm{USCC}(\mathbf{I}, \mathbf{I})$. It follows that $\overline{\mathrm{H}}_\partial(\mathbf{I}) \subset \mathrm{USCC}(\mathbf{I})$. Identifying each $\varphi \in \mathrm{USC}(\mathbf{I})$ with the graph of φ, we regard $\mathrm{USC}(\mathbf{I})$ as the subspace of $\exp(\mathbf{I}^2)$ as follows:

$$\mathrm{USC}(\mathbf{I}) = \{\varphi \in \exp(\mathbf{I}^2) \mid p_1(\varphi) = \mathbf{I}\},$$

where $p_1 : \mathbf{I}^2 \to \mathbf{I}$ is the projection onto the first factor.

Lemma 5.1.1. *Let* $\varphi \in \overline{\mathrm{H}}_\partial(\mathbf{I})$ *and* (x_1, y_1), $(x_2, y_2) \in \varphi \subset \mathbf{I}^2$. *Then* $x_1 < x_2$ *implies* $y_1 \leqslant y_2$. *In other words,* $\max \varphi(x_1) \leqslant \min \varphi(x_2)$ *for* $x_1 < x_2$.

Proof. We have a sequence $h_n \in \mathrm{H}_\partial(\mathbf{I})$, $n \in \mathbb{N}$, which converges to φ in $\exp(\mathbf{I}^2)$. Then we can choose $(x_i^n, y_i^n) \in h_n$ so that $\lim_{n\to\infty}(x_i^n, y_i^n) = (x_i, y_i)$, $i = 1, 2$. Assume that $x_1 < x_2$. Then $x_1^n < x_2^n$ for sufficiently large $n \in \mathbb{N}$, whence $y_1^n = h_n(x_1^n) < h_n(x_2^n) = y_2^n$ since h_n is increasing. Therefore $y_1 = \lim_{n\to\infty} y_1^n \leqslant \lim_{n\to\infty} y_2^n = y_2$. $\square$

Lemma 5.1.2. *There exists a homotopy* $\gamma\colon \overline{\mathrm{H}}_\partial(\mathbf{I}) \times [0,1] \to \overline{\mathrm{H}}_\partial(\mathbf{I})$ *such that* $\gamma_0 = \mathrm{id}$ *and* $\gamma_t(\overline{\mathrm{H}}_\partial(\mathbf{I})) \subset \mathrm{H}_\partial(\mathbf{I})$ *for* $t > 0$, *namely* $\overline{\mathrm{H}}_\partial(\mathbf{I}) \setminus \mathrm{H}_\partial(\mathbf{I})$ *is locally homotopy negligible in* $\overline{\mathrm{H}}_\partial(\mathbf{I})$.

Proof. We define the homotopy $\gamma\colon \mathbf{I}^2 \times [0,1] \to \mathbf{I}^2$ as follows:

$$\gamma_t(x,y) = (1-\tfrac{t}{2})(x,y) + \tfrac{t}{2}(y,x) = ((1-\tfrac{t}{2})x + \tfrac{t}{2}y, (1-\tfrac{t}{2})y + \tfrac{t}{2}x).$$

By the same notation γ, we denote the homotopy $\gamma\colon \exp(\mathbf{I}^2)\times[0,1] \to \exp(\mathbf{I}^2)$ induced by the above γ, that is, $\gamma_t(\varphi) = \{\gamma_t(x,y) \mid (x,y) \in \varphi\}$. For each $\varphi \in \overline{\mathrm{H}}_\partial(\mathbf{I})$, φ is a compact and connected set in $\mathbf{I}^2$ which contains the points $(0,0),(1,1)$, hence $\gamma_t(\varphi)$ is also such a set in $\mathbf{I}^2$. Therefore $\gamma_t(\varphi)\colon \mathbf{I} \to \mathbf{I}$ is a u.s.c. multi-valued function. Since $\gamma_0 = \mathrm{id}_{\exp(\mathbf{I}^2)}$, the restriction of γ would be the desired homotopy if we could prove that $\gamma_t(\varphi)\colon \mathbf{I} \to \mathbf{I}$ is a bijective single-valued function for $t > 0$ and $\varphi \in \overline{\mathrm{H}}_\partial(\mathbf{I})$. In fact, this means $\gamma_t(\varphi) \in \mathrm{H}_\partial(\mathbf{I})$ since the continuity of as a single-valued function comes from the upper semi-continuity of as a multi-valued function.

Now fix $0 < t \leqslant 1$ and $\varphi \in \overline{\mathrm{H}}_\partial(\mathbf{I})$. Let (x_1,y_1), $(x_2,y_2) \in \gamma_t(\varphi) \subset \mathbf{I}^2$. It suffices to show that $x_1 = x_2$ if and only if $y_1 = y_2$. We can write

$$x_1 = (1-\tfrac{t}{2})a_1 + \tfrac{t}{2}b_1,\ y_1 = (1-\tfrac{t}{2})b_1 + \tfrac{t}{2}a_1;$$
$$x_2 = (1-\tfrac{t}{2})a_2 + \tfrac{t}{2}b_2,\ y_2 = (1-\tfrac{t}{2})b_2 + \tfrac{t}{2}a_2,$$

where $(a_i,b_i) \in \varphi$, $i = 1,2$. Assume $x_1 = x_2$, that is,

$$(1-\tfrac{t}{2})a_1 + \tfrac{t}{2}b_1 = (1-\tfrac{t}{2})a_2 + \tfrac{t}{2}b_2.$$

Since $1 - \frac{t}{2} > 0$ and $\frac{t}{2} > 0$, if $a_1 < a_2$ then $b_1 > b_2$ and if $a_1 > a_2$ then $b_1 < b_2$, which are contradictions by Lemma 5.1.1. Thus we have $a_1 = a_2$. Then it follows that $(a_1,b_1) = (a_2,b_2)$, hence $y_1 = y_2$. Similarly we can see the converse, that is, $y_1 = y_2$ implies $x_1 = x_2$. □

Remark. In the above proof, observe that $\gamma_t(\varphi) \in \mathrm{H}_\partial(\mathbf{I})$ if $t > 0$ and $\varphi \in \mathrm{USCC}(\mathbf{I})$ satisfies the condition:

($\natural$) $0 \in \varphi(0)$, $1 \in \varphi(1)$, $\max\varphi(x_1) \leqslant \min\varphi(x_2)$ if $x_1 < x_2$.

Then it follows that

$$\overline{\mathrm{H}}_\partial(\mathbf{I}) = \{\varphi \in \mathrm{USCC}(\mathbf{I}) \mid \varphi \text{ satisfies } (\natural)\}.$$

One should also note that $\gamma_1(\varphi) = \mathrm{id}_{\mathbf{I}}$ for all $\varphi \in \overline{\mathrm{H}}_\partial(\mathbf{I})$, hence $\overline{\mathrm{H}}_\partial(\mathbf{I})$ is contractible. Indeed, since $\mathrm{H}_\partial(\mathbf{I}) \approx s$, it is easy to verify that $\overline{\mathrm{H}}_\partial(\mathbf{I})$ is an AR and has the disjoint cells property, hence $\overline{\mathrm{H}}_\partial(\mathbf{I}) \approx Q$ by Toruńczyk's characterization of Q (Theorem 1.3.1).

Now, we shall prove Theorem 5.1.3.

Proof of Theorem 5.1.3. As observed in the above, first note that $\overline{\mathrm{H}}_\partial(\mathbf{I}) \approx Q$. Since $\mathrm{H}_\partial(\mathbf{I})$ is completely metrizable, $\overline{\mathrm{H}}_\partial(\mathbf{I}) \setminus \mathrm{H}_\partial(\mathbf{I})$ is an F_σ-set, hence a Z_σ-set in $\overline{\mathrm{H}}_\partial(\mathbf{I})$ by Lemma 5.1.2. Thus it remains to prove that $\overline{\mathrm{H}}_\partial(\mathbf{I}) \setminus \mathrm{H}_\partial(\mathbf{I})$ satisfies the condition ($\flat$) of Theorem 1.3.1 in $\overline{\mathrm{H}}_\partial(\mathbf{I})$, whence we have the result by Theorem 1.3.1.

Let (A, B) be a pair of compacta in $\overline{\mathrm{H}}_\partial(\mathbf{I})$ such that $B \cap \mathrm{H}_\partial(\mathbf{I}) = \emptyset$, and $\varepsilon > 0$. Define a map $\alpha\colon A \to [0,1]$ by $\alpha(\varphi) = \frac{1}{3}\min\{1, \varepsilon, \rho_{\mathrm{H}}(\varphi, B)\}$. By using Lemma 5.1.1, we can define a map $f\colon A \to \overline{\mathrm{H}}_\partial(\mathbf{I})$ such that $f(A \setminus B) \subset \mathrm{H}_\partial(\mathbf{I})$, $f|B = \mathrm{id}$ and $\rho_{\mathrm{H}}(f(\varphi), \varphi) < \alpha(\varphi)$ for each $\varphi \in A \setminus B$. Since $\mathrm{H}_\partial(\mathbf{I}) \approx s$ [An$_3$] and $A \setminus B$ is completely metrizable, we have a closed embedding $g\colon A \setminus B \to \mathrm{H}_\partial(\mathbf{I})$ such that $\rho_{\mathrm{H}}(g(\varphi), f(\varphi)) < \alpha(\varphi)$ for each $\varphi \in A \setminus B$. Now we define $h\colon A \setminus B \to \overline{\mathrm{H}}_\partial(\mathbf{I}) \setminus \mathrm{H}_\partial(\mathbf{I})$ as follows:

$$h(\varphi)(x) = \begin{cases} [1 - \alpha(\varphi), 1] & \text{if } x = 1, \\ (1 - \alpha(\varphi))g(\varphi)(x) & \text{otherwise.} \end{cases}$$

As is easily observed, h is continuous and injective. For each $\varphi \in A \setminus B$,

$$\begin{aligned}\rho_{\mathrm{H}}(h(\varphi), \varphi) &\leqslant \rho_{\mathrm{H}}(h(\varphi), g(\varphi)) + \rho_{\mathrm{H}}(g(\varphi), f(\varphi)) + \rho_{\mathrm{H}}(f(\varphi), \varphi) \\ &< \alpha(\varphi) + \alpha(\varphi) + \alpha(\varphi) = 3\alpha(\varphi) = \min\{\varepsilon, \rho_{\mathrm{H}}(\varphi, B)\}.\end{aligned}$$

Hence we can extend h to the map $\tilde{h}\colon A \to \overline{\mathrm{H}}_\partial(\mathbf{I})$ by $\tilde{h}|B = \mathrm{id}$, which is ε-close to id. Since $\rho_{\mathrm{H}}(\varphi, h(\varphi)) < \rho_{\mathrm{H}}(\varphi, B)$ for each $\varphi \in A \setminus B$, $\tilde{h}(A \setminus B) = h(A \setminus B)$ does not meet $h(B)$, which implies that $\tilde{h}$ is injective. Then $\tilde{h}$ is an embedding since A is compact. □

Remark. Let $C(\mathbf{I}, \mathbf{I})$ be the space of (continuous) maps of $\mathbf{I}$ to itself and $M_0(\mathbf{I})$ be the space $M_0(\mathbf{I})$ of monotone maps $f\colon \mathbf{I} \to \mathbf{I}$ with $f(0) = 0$ and $f(1) = 1$. The closure of $\mathrm{H}_\partial(\mathbf{I})$ in $C(\mathbf{I}, \mathbf{I})$ is the space $M_0(\mathbf{I})$, that is, $M_0(\mathbf{I}) = \overline{\mathrm{H}}_\partial(\mathbf{I}) \cap C(\mathbf{I}, \mathbf{I})$. It is shown that $M_0(\mathbf{I}) \approx s$ [Ge1]. In the above proof, observe that $h(A \setminus B) \subset \overline{\mathrm{H}}_\partial(\mathbf{I}) \setminus M_0(\mathbf{I})$. For any pair (A, B) of compacta $\overline{\mathrm{H}}_\partial(\mathbf{I})$ such that $B \cap M_0(\mathbf{I}) = \emptyset$, and for any $\varepsilon > 0$, we have an embedding $\tilde{h}\colon A \to M_0(\mathbf{I})$ such that $\tilde{h}$ is ε-close to id and $\tilde{h}|B = \mathrm{id}$. Thus we obtain $(\overline{\mathrm{H}}_\partial(\mathbf{I}), M_0(\mathbf{I})) \approx (Q, s)$.

§5.2. A Q-MANIFOLD LOCAL COMPACTIFICATION OF $\mathrm{E}(\mathbf{I}, X)$

In this section, we show the following:

Theorem 5.2.1. *The quadruple* $(\overline{\mathrm{E}}(\mathbf{I}, G), \mathrm{E}(\mathbf{I}, G), \mathrm{E}^{\mathrm{LIP}}(\mathbf{I}, G), \mathrm{E}^{\mathrm{PL}}(\mathbf{I}, G))$ *is locally homeomorphic to* (Q, s, Σ, σ) *(i.e., it is a* (Q, s, Σ, σ)*-manifold quadruple) for any graph* $G \subset \mathbb{R}^n$.

This is a corollary of the following:

Theorem 5.2.2. *Let X be a locally compact 1-dimensional ANR with no isolated points and a locally convex metric d, that is, each point of X has a neighborhood U satisfying the following condition:*

(∗) *for each $x, y \in U$, there is $z \in U$ such that $d(x, z) = d(y, z) = d(x, y)/2$.*

Then the quadruple $(\overline{\mathrm{E}}(\mathbf{I},X),\mathrm{E}(\mathbf{I},X),\mathrm{E}^{\mathrm{LIP}}(\mathbf{I},X),\mathrm{E}^{\mathrm{PL}}(\mathbf{I},X))$ *is locally homeomorphic to* (Q,s,Σ,σ) *(i.e., it is a* (Q,s,Σ,σ)*-manifold quadruple).*

Here it is naturally defined that a map $f\colon \mathbf{I}\to X$ is PL (see [Sa$_4$, p.1173]).

Proof. Similarly to [Sa$_4$, Theorem 4], Theorem 5.2.2 can be reduced to the case X is a dendrite (= 1-dimensional compact AR) with a convex metric d. Let $b(X) = X^2\setminus \Delta X$, where ΔX is the diagonal of X^2, and let $\beta\colon \mathrm{E}(\mathbf{I},X)\to b(X)$ be the map defined by $\beta(h) = (h(0),h(1))$. In the proof of [Sa$_4$, Theorem 5], we constructed a homeomorphism $\varphi\colon \mathrm{E}(\mathbf{I},X)\to b(X)\times \mathrm{H}_\partial(\mathbf{I})$ such that $\mathrm{pr}_{b(X)}\circ\varphi=\beta$, $\varphi(\mathrm{E}^{\mathrm{LIP}}(\mathbf{I},X)) = b(X)\times \mathrm{H}_\partial^{\mathrm{LIP}}(\mathbf{I})$ and $\varphi(\mathrm{E}^{\mathrm{PL}}(\mathbf{I},X)) = b(X)\times \mathrm{H}_\partial^{\mathrm{PL}}(\mathbf{I})$, It is easy to see that φ can be extend to the homeomorphism $\bar{\varphi}\colon \overline{\mathrm{E}}(\mathbf{I},X)\to b(X)\times \overline{\mathrm{H}}_\partial(\mathbf{I})$. Therefore we have

$$\begin{aligned}&(\overline{\mathrm{E}}(\mathbf{I},X),\mathrm{E}(\mathbf{I},X),\mathrm{E}^{\mathrm{LIP}}(\mathbf{I},X),\mathrm{E}^{\mathrm{PL}}(\mathbf{I},X))\approx\\ &\qquad(b(X)\times\overline{\mathrm{H}}_\partial(\mathbf{I}),b(X)\times\mathrm{H}_\partial(\mathbf{I}),b(X)\times\mathrm{H}_\partial^{\mathrm{LIP}}(\mathbf{I}),b(X)\times\mathrm{H}_\partial^{\mathrm{PL}}(\mathbf{I})).\end{aligned}$$

Since $b(X)$ is a locally compact ANR, the above quadruple is locally homeomorphic to (Q,s,Σ,σ). □

Remark. It is known that $\overline{\mathrm{H}}(P)\approx Q$ for the pseudo-arc P [Kaw]. However $\mathrm{H}(P)\not\approx s$. In fact, $\mathrm{H}(P)$ contains no non-degenerate continua [Le].

The corresponding result for the space $C(X,\mathbf{I})$ of a locally connected infinite compactum X to the interval $\mathbf{I}$ has been obtained in Chapter 3. However, as is shown in Example 3.2.2, the interval $\mathbf{I}$ in this result cannot be replaced by the circle $\mathbf{S}^1$, that is, the corresponding result for the space $C(X,\mathbf{S}^1)$ does not hold even if $X=\mathbf{I}$. One should note that Theorem 5.2.1 is the result containing the space $\mathrm{E}(\mathbf{I},\mathbf{S}^1)$.

§5.3. A HILBERT CUBE COMPACTIFICATION OF R(I)

In this section, we prove the following:

Theorem 5.3.1. $(\overline{\mathrm{R}}(\mathbf{I}), \mathrm{R}(\mathbf{I})) \approx (Q, s)$.

We identify each $\varphi \in \mathrm{USCC}(\mathbf{I}, \mathbf{I})$ with its graph $\mathrm{Gr}(\varphi) \subset \mathbf{I} \times \mathbf{I}$, and assume that the first and second factor of the product $\mathbf{I} \times \mathbf{I}$ to be the domain and the range of φ, respectively. Let p_1 and $p_2 \colon \mathbf{I} \times \mathbf{I} \to \mathbf{I}$ be the projection onto the first and second factor, respectively. Define maps $a, b : \exp(\mathbf{I}^2) \to [0,1]$ by

$$a(\varphi) = \min p_2(\varphi) \quad , \quad b(\varphi) = \max p_2(\varphi).$$

Observe that $\varphi|_{[a(\varphi), b(\varphi)]} = \mathrm{id}$ for every $\varphi \in \mathrm{R}(\mathbf{I})$. Moreover, put

$$P = \{\varphi \in \mathrm{USCC}(\mathbf{I}, \mathbf{I}) \;\mid\; a(\varphi) \neq b(\varphi) \Rightarrow \varphi|(a(\varphi), b(\varphi)) = \mathrm{id}\}.$$

Then, note that $\overline{\mathrm{R}}(\mathbf{I}) \subset P$. In fact, P is closed in $\mathrm{USCC}(\mathbf{I}, \mathbf{I})$, $\mathrm{R}(\mathbf{I}) \subset P$ and $\mathrm{USCC}(\mathbf{I}, \mathbf{I})$ is closed in $\exp(\mathbf{I}^2)$.

For $i \in \{0, 1\}$, put

$$\mathrm{USCC}^i(\mathbf{I}, \mathbf{I}) = \{\varphi \in \mathrm{USCC}(\mathbf{I}, \mathbf{I}) \mid i \in \varphi(1 - i)\},$$

$$C^i(\mathbf{I}, \mathbf{I}) = C(\mathbf{I}, \mathbf{I}) \cap \mathrm{USCC}^i(\mathbf{I}, \mathbf{I}).$$

For any $\varphi \in \mathrm{USCC}^i(\mathbf{I}, \mathbf{I})$ and $\varepsilon > 0$, similarly to Theorem 1.9 in [Fe_2], we can take a map $f \in C^i(\mathbf{I}, \mathbf{I})$ such that $d_H(\varphi, f) < \varepsilon$ and $f(1) = 0$, that is, the subset $C^i(\mathbf{I}, \mathbf{I})$ is dense in $\mathrm{USCC}^i(\mathbf{I}, \mathbf{I})$. By the same method as Lemma 3.1.1 and 3.1.2, we can construct a homotopy $G^i \colon \mathrm{USCC}^i(\mathbf{I}, \mathbf{I}) \times [0,1] \to \mathrm{USCC}^i(\mathbf{I}, \mathbf{I})$ such that $G^i_0 = \mathrm{id}$ and $G^i_t(\mathrm{USCC}^i(\mathbf{I}, \mathbf{I})) \subset C^i(\mathbf{I}, \mathbf{I})$ for each $t > 0$.

Each $\varphi \in \mathrm{USCC}([a, b], [c, d])$ is linearly transferred to the element of $\mathrm{USCC}(\mathbf{I}, \mathbf{I})$ by the function $\mathrm{T}^{[c,d]}_{[a,b]} : \mathrm{USCC}([a, b], [c, d]) \to \mathrm{USCC}(\mathbf{I}, \mathbf{I})$, that is,

$$\mathrm{T}^{[c,d]}_{[a,b]}(\varphi) = \big\{ (\frac{x - a}{b - a}, \frac{y - c}{d - c}) \mid (x, y) \in \varphi \big\}.$$

The inverse of $\mathrm{T}_{[a,b]}^{[c,d]}$ is denoted by ${\mathrm{T}^{-1}}_{[a,b]}^{[c,d]} \colon \mathrm{USCC}(\mathbf{I},\mathbf{I}) \to \mathrm{USCC}([a,b],[c,d])$.

These functions will be used in the following lemma.

Lemma 5.3.1. *There exists a homotopy $F : P \times [0,1] \to P$ such that $F_0 = id$ and $F_t(P) \subset \mathrm{R}(\mathbf{I})$ for $t > 0$.*

Proof. First, we will define a homotopy $H\colon P \times [0,1] \to P$ such that $H_0 = \mathrm{id}$ and

$$H_t(P) \subset \{\varphi \in P \mid p_2(\varphi) \cap \{0,1\} = \emptyset \text{ or } a(\varphi) = b(\varphi)\}.$$

For each $\varphi \in P$ and each $t \in [0,1]$, put numbers

$$\begin{aligned} a_t(\varphi) &= (1 - \frac{t}{2})a(\varphi) + \frac{t}{2}b(\varphi), \\ b_t(\varphi) &= (1 - \frac{t}{2})b(\varphi) + \frac{t}{2}a(\varphi), \end{aligned}$$

and define a retraction $r_{(\varphi,t)} : \mathbf{I} \to \mathbf{I}$ by

$$r_{(\varphi,t)}(y) = \begin{cases} a_t(\varphi) & \text{if } y \in [0, a_t(\varphi)], \\ y & \text{if } y \in [a_t(\varphi), b_t(\varphi)], \\ b_t(\varphi) & \text{if } y \in [b_t(\varphi), 1]. \end{cases}$$

for every $y \in \mathbf{I}$. The homotopy $H\colon P \times [0,1] \to P$ is defined by

$$H_t(\varphi)(x) = r_{(\varphi,t)}(\varphi(x)) \subset \mathbf{I}$$

for every $\varphi \in P$, $t \in [0,1]$ and $x \in \mathbf{I}$. Observe that $H_t(\varphi) = \varphi$ for each $t \in [0,1]$ and each $\varphi \in P$ such that $a(\varphi) = b(\varphi)$.

Next, by using the homotopy G^i, we approximate multi-valued functions $H_t(\varphi)$ by (single-valued) continuous maps. Let

$$\begin{aligned} L_t(\varphi) &= \big({\mathrm{T}^{-1}}_{[0,a_t(\varphi)]}^{[a_t(\varphi),b_t(\varphi)]} \circ G_t^0 \circ \mathrm{T}_{[0,a_t(\varphi)]}^{[a_t(\varphi),b_t(\varphi)]}\big)(H_t(\varphi)|[0,a_t(\varphi)]) \subset \mathbf{I}^2 \quad \text{and} \\ \mathrm{R}_t(\varphi) &= \big({\mathrm{T}^{-1}}_{[b_t(\varphi),1]}^{[a_t(\varphi),b_t(\varphi)]} \circ G_t^1 \circ \mathrm{T}_{[b_t(\varphi),1]}^{[a_t(\varphi),b_t(\varphi)]}\big)(H_t(\varphi)|[b_t(\varphi),1]) \subset \mathbf{I}^2 \end{aligned}$$

for every $t \in (0,1]$ and for every $\varphi \in P$ such that $a(\varphi) \neq b(\varphi)$. Now we define the desired homotopy $F : P \times [0,1] \to P$ as follows:

$$F_t(\varphi) = \begin{cases} \varphi & \text{if } t = 0 \text{ or } a(\varphi) = b(\varphi), \\ L_t(\varphi) \cup \mathrm{id}|[a_t(\varphi), b_t(\varphi)] \cup \mathrm{R}_t(\varphi) & \text{otherwise.} \quad \square \end{cases}$$

Proof of Theorem 5.3.1. Since P is closed in $\exp(\mathbf{I}^2)$, it follows from Lemma 5.3.1 that $\overline{\mathrm{R}}(\mathbf{I}) = P$. Because $\mathrm{R}(\mathbf{I})$ is homotopy co-negligible in $\overline{\mathrm{R}}(\mathbf{I})$ (Lemma 5.3.1) and $\mathrm{R}(\mathbf{I}) \approx s$ ([BS]), we can easily verify that $\overline{\mathrm{R}}(\mathbf{I})$ is an AR and has the disjoint cells property, hence $\overline{\mathrm{R}}(\mathbf{I}) \approx Q$ by Toruńczyk's characterization of Q (Theorem 1.3.1). For convenience sake, we identify $\overline{\mathrm{R}}(\mathbf{I})$ with Q, and assume $\mathrm{R}(\mathbf{I}) \subset Q$. It is easily seen by Lemma 5.3.1 that $Q \setminus \mathrm{R}(\mathbf{I})$ is a Z_σ-set in Q.

We will prove that $\mathrm{R}(\mathbf{I})$ satisfies the condition $(*)$. Let $\alpha\colon A \to [0,1]$ be the map defined by $\alpha(\varphi) = \frac{1}{3}\min\{\varepsilon, d_{\mathrm{H}}(\varphi, B)\}$. By using Lemma 5.3.1, we can define a map $f\colon A \to Q$ such that $f(A \setminus B) \subset \mathrm{R}(\mathbf{I})$, $f|B = \mathrm{id}$ and $d_{\mathrm{H}}(f(\varphi), \varphi) < \alpha(\varphi)$ for each $\varphi \in A \setminus B$. Since $\mathrm{R}(\mathbf{I}) \approx s$ and $A \setminus B$ is completely metrizable, we have a closed embedding $g\colon A \setminus B \to \mathrm{R}(\mathbf{I})$ such that $d_{\mathrm{H}}(g(\varphi), f(\varphi)) < \alpha(\varphi)$ for each $\varphi \in A \setminus B$. We may assume that $g(A)$ doesn't intersect the subset $\mathrm{R}_c(\mathbf{I})$ consisting of all constant maps because $\mathrm{R}_c(\mathbf{I}) \approx \mathbf{I}$ is a compact subset of $\mathrm{R}(\mathbf{I}) \approx s$, whence it is a Z-set in $\mathrm{R}(\mathbf{I})$. Now we define $h\colon A \setminus B \to Q \setminus \mathrm{R}(\mathbf{I})$ as follows:

$$h(\varphi)(x) = \begin{cases} [g(\varphi)(x), \min\{b(\varphi), g(\varphi)(x) + \alpha(\varphi)\}] & \text{if } x = a(\varphi), \\ g(\varphi)(x) & \text{otherwise} \end{cases}$$

(recall that $a(\varphi) = \min \bigcup_{x \in \mathbf{I}} \varphi(x)$ and $b(\varphi) = \max \bigcup_{x \in \mathbf{I}} \varphi(x)$). As is easily observed, h is continuous and injective. For each $\varphi \in A \setminus B$,

$$\begin{aligned} d_{\mathrm{H}}(h(\varphi), \varphi) &\leqslant d_{\mathrm{H}}(h(\varphi), g(\varphi)) d_{\mathrm{H}}(g(\varphi), f(\varphi)) + d_{\mathrm{H}}(f(\varphi), \varphi) \\ &< \alpha(\varphi) + \alpha(\varphi) + \alpha(\varphi) = 3\alpha(\varphi) \leqslant d_{\mathrm{H}}(\varphi, B). \end{aligned}$$

Hence we can extend h to the map $\tilde{h}\colon A \to Q$ by $\tilde{h}|B = \mathrm{id}$. Since $d_{\mathrm{H}}(\varphi, h(\varphi)) < d_{\mathrm{H}}(\varphi, B)$ for each $\varphi \in A \setminus B$, $\tilde{h}(A \setminus B) = h(A \setminus B)$ does not meet $h(B)$. Then it follows that $\tilde{h}$ is injective, whence it is an embedding since A is compact. Thus we have the desired embedding $\tilde{h}$. By Theorem 1.3.2, we have the result. $\square$

CHAPTER 6

Spaces of multi-valued functions on metric spaces

This chapter is devoted to study the space $\mathrm{USCC}_B(X)$ not only in the case X is compact but also in the case X is non-compact.

Fedorchuk [$\mathrm{Fe}_{1,2}$] proved that if X is an infinite locally connected compact metric space then $\mathrm{USCC}(X, \mathbf{I})$ is homeomorphic to the Hilbert cube Q and $\mathrm{USCC}(X) \approx Q \setminus \{0\}$ ($\approx Q \times [0, 1)$) (cf. Chapter 3). In §6.1, we prove the converse of this result. In particular, under the condition X is compact, $\mathrm{USCC}(X)$ is closed in $2^{X \times \mathbb{R}}$ if and only if X is locally connected. It is worthwhile considering whether the condition X is compact can be deleted. But, this is not the case. In §6.2, we give a necessary and sufficient condition for a space X in order that $\mathrm{USCC}_B(X)$ is closed set of $2^{X \times \mathbb{R}}$. We also give a necessary and sufficient condition in order that $\mathrm{USCC}_B(X)$ is an AR (§6.3).

§6.1. THE CONVERSE OF FEDORCHUK'S RESULT

The following is our main result in this section:

Theorem 6.1.1. *For a metric space X, the following are equivalent:*

(a) $\mathrm{USCC}(X, \mathbf{I}) \approx Q$;

(b) $\mathrm{USCC}_B(X) \approx Q \setminus \{0\}$ $(\approx Q \times [0,1))$;

(c) *X is infinite, locally connected and compact.*

To prove Theorem 6.1.1, we show the following:

Proposition 6.1.1. *For a locally compact metric space X, $\mathrm{USCC}_B(X, \mathbf{I})$ is closed in $2^{X\times\mathbf{I}}$ if and only if X is locally connected.*

Proof. The "if" part is proved in [Fe$_2$].

To see the "only if" part, assume that X is not locally connected. Then some $x_0 \in X$ has a compact neighborhood B_0 such that any neighborhood of x_0 contained in B_0 is not connected. Let $\delta = d(x_0, X \setminus B_0) > 0$. Then we have disjoint non-empty closed sets A_1 and B_1 in X such that $B_0 = A_1 \cup B_1$, $d(x_0, A_1) < 2^{-1}\delta$ and $x_0 \in B_1$. In fact, since B_0 is compact, the intersection of clopen sets in B_0 containing x_0 is the component of B_0, which is not a neighborhood of x_0. Then we have a clopen set B_1 in B_0 and $x_1 \in B_0 \setminus B_1$ with $d(x_0, x_1) < 2^{-1}\delta$, whence $A_1 = B_0 \setminus B_1$ and B_1 satisfy the condition. Using the same argument inductively, we have disjoint non-empty closed sets A_n and B_n in X, $n \in \mathbb{N}$, such that $B_{n-1} = A_n \cup B_n$, $d(x_0, A_n) < 2^{-n}\delta$ and $x_0 \in B_n$. For each $n \in \mathbb{N}$, let

$$\varphi_n = \bigcup_{i=1}^{n} A_i \times \{0\} \cup B_n \times \{1\} \cup (X \setminus \mathrm{int}_X B_0) \times \mathbf{I} \in \mathrm{USCC}_B(X, \mathbf{I}).$$

Note that $\varphi_n(\mathrm{int}_X B_0) = \{0, 1\}$. Since $2^{B_0\times\mathbf{I}} = \exp(B_0 \times \mathbf{I})$ is compact, $(\varphi_n|B_0)_{n\in\mathbb{N}}$ has a subsequence $(\varphi_{n_i}|B_0)_{i\in\mathbb{N}}$ converging to some $\varphi' \in 2^{B_0\times\mathbf{I}}$. Then $(\varphi_{n_i})_{i\in\mathbb{N}}$ converges to $\varphi = \varphi' \cup (X \setminus \mathrm{int}_X B_0) \times \mathbf{I}$ in $2^{X\times\mathbf{I}}$. Since $(x_0, 0) \in \varphi_n$ for all $n \in \mathbb{N}$, we have $(x_0, 0) \in \varphi$. For each $n \in \mathbb{N}$, choose $x_n \in A_n$ so that $d(x_n, x_0) < 2^{-n}\delta$. Since $\rho((x_0, 1), (x_n, 1)) < 2^{-n}\delta$ and $(x_n, 1) \in \varphi_n$, we have $(x_0, 1) \in \varphi$. However $(x_0, \frac{1}{2}) \notin \varphi$

because $B_0 \times (0,1) \cap \varphi_n = \emptyset$ for any $n \in \mathbb{N}$. This means that $\varphi \cap \{x_0\} \times \mathbf{I}$ (i.e., $\varphi(x_0)$) is not connected, hence $\varphi \notin \mathrm{USCC}_B(X, \mathbf{I})$. This is a contradiction. □

For a space X, there exists the natural closed embedding $i_X : X \to \mathrm{USCC}_B(X, \mathbf{I})$ defined as follows:

$$i_X(x) = X \times \{0\} \cup \{x\} \times \mathbf{I} \subset X \times \mathbf{I} \quad \text{for each } x \in X,$$

whence each $i_X(x) \in \mathrm{USCC}_B(X, \mathbf{I})$ is defined by

$$i_X(x)(y) = \begin{cases} \{0\} & \text{if } y \neq x, \\ \mathbf{I} & \text{if } y = x. \end{cases}$$

Observe that $\rho_{\mathrm{H}}(i_X(x), i_X(x')) = d(x, x')$ if $d(x, x') < 1$, hence i_X is locally isometric. It is easy to see that $i_X(X)$ is closed in $\mathrm{USCC}_B(X, \mathbf{I})$.

Proof of Theorem 6.1.1. The implications (c) ⇒ (a) and (c) ⇒ (b) are the Fedorchuk's result [$\mathrm{Fe}_{1,2}$] (cf. Appendix in Chapter 3).

(a) ⇒ (c): By using the embedding i_X above, X can be embedded in $\mathrm{USCC}_B(X, \mathbf{I})$ as a closed set, hence X is compact. By Proposition 6.1.1 above, X is locally connected. If X is a singleton, the space $\mathrm{USCC}_B(X, \mathbf{I})$ is homeomorphic to the hyperspace of subcontinua (i.e., closed subintervals) of $\mathbf{I}$, so $\mathrm{USCC}_B(X, \mathbf{I}) \approx \mathbf{I}^2$ (cf. [Du, §3]). Hence, if X is finite then $\mathrm{USCC}_B(X, \mathbf{I}) \approx \mathbf{I}^{2n}$, where n is the number of points of X. Therefore, X must be infinite.

(b) ⇒ (c): Since $\mathrm{USCC}_B(X)$ is locally compact, $\varphi_0 = X \times \{0\} \in \mathrm{USCC}_B(X)$ has a compact neighborhood N in $\mathrm{USCC}_B(X)$. Choose $\delta > 0$ so that every $\varphi \in \mathrm{USCC}_B(X)$ with $\rho_{\mathrm{H}}(\varphi, \varphi_0) < \delta$ belongs to N. Then, $\mathrm{USCC}(X, [0, \delta]) \subset N$ and $\mathrm{USCC}(X, [0, \delta])$ is closed in $\mathrm{USCC}_B(X)$. Hence, $\mathrm{USCC}(X, \mathbf{I}) \approx \mathrm{USCC}(X, [0, \delta])$ is compact. As seen in the above, it follows that X is compact and locally connected. Since

$$\mathrm{USCC}_B(X) = \mathrm{USCC}(X) \approx \mathrm{USCC}(X, (0,1)) \subset \mathrm{USCC}(X, \mathbf{I}),$$

$\mathrm{USCC}(X, \mathbf{I})$ is infinite-dimensional, which implies that X is infinite. $\square$

By $\mathrm{C}_B(X)$, we denote the Banach space of bounded continuous real-valued functions of X with the sup-norm and let $\mathrm{C}(X, \mathbf{I}) = \{f \in \mathrm{C}_B(X) \mid f(X) \subset \mathbf{I}\}$. Although $\mathrm{C}_B(X) \subset \mathrm{USCC}_B(X)$ as sets, the Banach space $\mathrm{C}_B(X)$ is not a subspace of $\mathrm{USCC}_B(X)$ in case X is non-compact (cf. [FK, Remark 3.6] and Supplement). In Chapter 8, it is also shown that if X is locally connected and has no isolated points then the closures of $\mathrm{C}(X, \mathbf{I})$ and $\mathrm{C}_B(X)$ in $2^{X \times \mathbf{I}}$ are $\mathrm{USCC}(X, \mathbf{I})$ and $\mathrm{USCC}_B(X)$, respectively. In case X is locally compact, the converse also holds by Proposition 6.1.1 above.

Corollary 6.1.1. *For a locally compact metric space X,*

$$\mathrm{cl}_{2^{X \times \mathbf{I}}} \mathrm{C}(X, \mathbf{I}) = \mathrm{USCC}(X, \mathbf{I}) \quad \textit{and/or} \quad \mathrm{cl}_{2^{X \times \mathbb{R}}} \mathrm{C}_B(X) = \mathrm{USCC}_B(X)$$

if and only if X is locally connected and has no isolated point. $\square$

§6.2. The closedness of $\mathrm{USCC}(X)$ in $2^{X \times \mathbb{R}}$

Let $X = (X, d)$ be a space. Throughout the chapter, we use the notation

$$B_d^C(E, \varepsilon) = X \setminus B_d(E, \varepsilon)$$

for $E \subset X$. In case $E = \{x\}$, we use $B_d^C(x, \varepsilon) = B_d^C(E, \varepsilon)$. The main result in this section is the following:

Proposition 6.2.1. *The following statements are equivalent.*

(1) *$\mathrm{USCC}(X)$ is closed in $2^{X \times \mathbb{R}}$.*

(2) *For any $\varepsilon > 0$, any $x \in X$ and any sequence $\{E_n\}_{n=1}^{\infty}$ of clopen subsets in $\overline{B}_d(x, \varepsilon)$ with $d(E_n, x) \to 0$ and $x \notin E_n$ for all n, the sequence $\{B_d^C(x, \varepsilon) \cup E_n\}_{n=1}^{\infty}$ or $\{B_d^C(x, \varepsilon) \cup (B_d^C(x, \varepsilon) \setminus E_n)\}_{n=1}^{\infty}$ doesn't converge in 2^X.*

Proof. To prove (1) $\Rightarrow$ (2), assume that there exist $\varepsilon > 0$, $x_0 \in X$ and a sequence $\{E_n\}_{n=1}^{\infty}$ consisting of clopen subsets of $\overline{B}_d(x_0, \varepsilon)$ such that $d(E_n, x_0) \to 0$, $x_0 \notin E_n$ for all n, $B_d^C(x_0, \varepsilon) \cup E_n \to E$ in 2^X and $B_d^C(x_0, \varepsilon) \cup (\overline{B}_d(x_0, \varepsilon) \setminus E_n) \to F$ in 2^X. Observe that$x_0 \in E \cap F$ and $E \cup F = X$. For each $n \in \mathbb{N}$, define $\varphi_n \in \mathrm{USCC}(X)$ by

$$\varphi_n(x) = \begin{cases} [0,1] & \text{if } x \notin B_d(x_0, \varepsilon), \\ 0 & \text{if } x \in B_X(x_0, \varepsilon) \setminus E_n, \\ 1 & \text{if } x \in B_X(x_0, \varepsilon) \cap E_n. \end{cases}$$

Then, $\varphi_n \to \varphi \in \mathrm{USC}(X)$, where

$$\varphi(x) = \begin{cases} [0,1] & \text{if } x \notin B_X(x_0, \varepsilon), \\ 0 & \text{if } x \in F \setminus E, \\ 1 & \text{if } x \in E \setminus F, \\ \{0,1\} & \text{if } x \in E \cap F \cap B_X(x_0, \varepsilon). \end{cases}$$

Then, $\varphi \notin \mathrm{USCC}(X)$. This contradicts to (1).

For (2) $\Rightarrow$ (1), suppose that $\mathrm{USCC}(X)$ is not closed in $2^{X\times\mathbb{R}}$. There exist $\varphi \in 2^{X\times\mathbb{R}} \setminus \mathrm{USCC}(X)$ and $\{\varphi_n\}_{n=1}^{\infty} \subset \mathrm{USCC}(X)$ such that $\varphi_n \to \varphi$. We may assume that $\varphi \in \mathrm{USC}(X) \setminus \mathrm{USCC}(X)$ since $\mathrm{USC}(X)$ is closed in $2^{X\times\mathbb{R}}$. We can take $x \in X$ such that $\varphi(x)$ is not connected, whence we can take a non-degenerate closed interval $[s,t] \subset \mathbb{R}$ and $\varepsilon > 0$ such that $(\overline{B}_d(x, \varepsilon) \times [s,t]) \cap \varphi = \emptyset$, $(-\infty, s] \cap \varphi(x) \neq \emptyset$ and $[t, \infty) \cap \varphi(x) \neq \emptyset$. Without loss of generality, we may assume that $\varphi_n(x) \subset [t, \infty)$ and

$$\varphi_n(\overline{B}_d(x, \varepsilon)) \cap (\overline{B}_d(x, \varepsilon) \times [s,t]) = \emptyset \quad \text{for all } n.$$

Put

$$E = p(\overline{B}_d(x, \varepsilon) \times (-\infty, s] \cap \varphi),$$

$$F = p(\overline{B}_d(x, \varepsilon) \times [t, \infty) \cap \varphi),$$

$$E_n = p(\overline{B}_d(x, \varepsilon) \times (-\infty, s] \cap \varphi_n),$$

$$F_n = p(\overline{B}_d(x, \varepsilon) \times [t, \infty) \cap \varphi_n),$$

where $p \colon X \times \mathbb{R} \to X$ is the projection. Observe that E_n and F_n are disjoint clopen subsets in $\overline{B}_d(x,\varepsilon)$, and that $x \notin E_n$, $E_n \cup F_n = \overline{B}_d(x,\varepsilon)$ for each n. Then, we have $B_d^C(x,\varepsilon) \cup E_n \to B_d^C(x,\varepsilon) \cup E$ and $B_d^C(x,\varepsilon) \cup F_n \to B_d^C(x,\varepsilon) \cup F$. This is a contradiction. □

In case X is locally connected, X satisfies the condition (2). In fact, if $x \in X$ has a connected neighborhood contained in $\overline{B}_X(x,\varepsilon)$, there is no sequence $\{E_n\}$ as in (2). Hence, we have the following:

Corollary 6.2.1. *If X is locally connected, then* USCC(X) *is closed in* $2^{X\times\mathbb{R}}$. □

But the converse of Corollary 2 is not true. For example, let

$$I_n = \{(t, t/n) \in \mathbb{R}^2 \mid t \in [1/n, 1]\} \quad n \in \mathbb{N}.$$

The set $X_0 = \{(0,0)\} \cup \bigcup_{n=1}^{\infty} I_n$ is endowed with the metric defined by

$$d((x,y),(x',y')) = \begin{cases} |x - x'| & \text{if } \{(x,y),(x',y')\} \subset I_n \text{ for some n,} \\ x + x' & \text{if otherwise.} \end{cases}$$

Then, X_0 is not locally connected. However, since X_0 satisfies the condition (2) of Proposition 6.2.1, USCC(X_0) is closed in $2^{X_0\times\mathbb{R}}$.

§6.3. The condition that USCC$_B(X)$ is an AR

Let $X = (X,d)$ be a complete metric space. In this section, as we saw in the above, we will give a necessary and sufficient condition in order that USCC$_B(X)$ is an AR.

For a finite subset $\mathcal{F} \subset$ USCC(X), the multi-valued function conv$(\mathcal{F}) \colon X \to \mathbb{R}$ is defined by

$$\mathrm{conv}(\mathcal{F})(x) = \left[\min \bigcup_{\varphi\in\mathcal{F}} \varphi(x) \ , \ \max \bigcup_{\varphi\in\mathcal{F}} \varphi(x)\right]$$

for each $x \in X$. Then, it is easy to see that $\mathrm{conv}(\mathcal{F}) \in \mathrm{USCC}(X)$.

We consider the following condition:

($*$) for every $\varepsilon > 0$ and every $\varphi \in \mathrm{USCC}(X)$ there exists $\delta(\varepsilon, \varphi) > 0$ such that $\rho_H(f, \varphi) < \delta(\varepsilon, \varphi)$ and $\rho_H(g, \varphi) < \delta(\varepsilon, \varphi)$ imply $\mathrm{conv}(f, g) \subset B_\rho(\varphi, \varepsilon)$ for any $f, g \in \mathrm{USCC}(X)$.

It is easily seen that the condition ($*$) is equivalent to the following:

($*$)' for every $\varepsilon > 0$ there exists a locally finite open cover $\mathcal{U}(\varepsilon)$ of $\mathrm{USCC}(X)$ such that $\mathrm{conv}(f, g) \subset B_\rho(f, \varepsilon)$ for every $\varphi \in \mathrm{USCC}(X)$ and $f, g \in \mathrm{St}(\varphi, \mathcal{U}(\varepsilon))$.

In the above statements, observe that $\mathrm{conv}(f, g) \subset B_\rho(f, \varepsilon)$ if and only if

$$\rho_H(\mathrm{conv}(f, g), f) < \varepsilon.$$

For every $\varphi, \psi \in \mathrm{USCC}_B(X)$, $t \in [0,1]$, define the multi-valued functions $t\varphi$ and $\varphi + \psi$ as follows:

$$t\varphi(x) = \{tr \in \mathbb{R} \mid r \in \varphi(x)\},$$

$$(\varphi + \psi)(x) = \{r + s \in \mathbb{R} \mid r \in \varphi(x), s \in \psi(x)\}.$$

It is easy to see that $t\varphi, \varphi + \psi \in \mathrm{USCC}_B(X)$. One should remark that the map $\mathrm{USCC}_B(X) \times \mathrm{USCC}_B(X) \ni (\varphi, \psi) \mapsto \varphi + \psi \in \mathrm{USCC}_B(X)$ is not continuous

Lemma 6.3.1. *For each $n \in \mathbb{N}$ and each subset $\{\varphi_1, \cdots, \varphi_n\} \subset \mathrm{USCC}_B(X)$, the map $h\colon [0,1]^n \to \mathrm{USCC}_B(X)$ defined by $(t_1, \cdots, t_n) \longmapsto t_1\varphi_1 + \cdots + t_n\varphi_n$ is continuous.*

Proof. By $|\cdot|_H$, we denote the Hausdorff metric on $2^{\mathbb{R}}$ induced by the usual metric.

First we prove the case $n = 1$. Take any $\varphi \in \mathrm{USCC}_B(X)$. Since φ is bounded, there exists $M > 0$ such that $\sup_{x \in X} \max\{|r| > 0 \mid r \in \varphi(x)\} < M$. Choose any $\varepsilon > 0$ and any convergent sequence $\{a^i\}_{i=1}^{\infty} \subset [0,1]$. Put $t = \lim_{i \to \infty} a^i$. For each

$x \in X$, we can write $a^i\varphi(x) = \{(a^i - t)r + tr \in \mathbb{R} \mid r \in \varphi(x)\}$. Hence,

$$\begin{aligned}\rho_H\big(h(t), h(a^i)\big) &= \rho_H\big(t\varphi, a^i\varphi\big) \\ &\leqslant \sup_{x\in X} |\cdot|_H\big(t\varphi(x), a^i\varphi(x)\big) \\ &\leqslant \sup_{x\in X}\{|(a^i - t)r| \geqslant 0 \mid r \in \varphi(x)\} \\ &\leqslant |a^i - t|M \to 0.\end{aligned}$$

Next assume that the lemma holds for $1, \cdots, n$. Take any $\varepsilon > 0$, any subset $\{\varphi_1, \cdots, \varphi_{n+1}\} \subset \mathrm{USCC}_B(X)$ and any sequence $\{a^i\}_{i=1}^{\infty} \subset [0,1]^{n+1}$ with $a^i \to t \in [0,1]^{n+1}$. Then,

$$\begin{aligned}\rho_H\big(h(t), h(a^i)\big) &= \rho_H\big(t_1\varphi_1 + \cdots + t_{n+1}\varphi_{n+1}\,,\, a_1^i\varphi_1 + \cdots + a_{n+1}^i\varphi_{n+1}\big) \\ &\leqslant \rho_H\big(t_1\varphi_1 + \cdots + t_{n+1}\varphi_{n+1}\,,\, t_1\varphi_1 + \cdots + t_n\varphi_n + a_{n+1}^i\varphi_{n+1}\big) \\ &\quad + \rho_H\big(t_1\varphi_1 + \cdots + t_n\varphi_n + a_{n+1}^i\varphi_{n+1}\,,\, a_1^i\varphi_1 + \cdots + a_{n+1}^i\varphi_{n+1}\big).\end{aligned}$$

Since $a_j^i \to t_j$ $(i \to \infty)$ for each $j \leqslant n+1$, it is easily seen by the inductive hypothesis that $\rho_H\big(h(t), h(a^i)\big) \to 0$. Therefore, the lemma follows by induction. □

Lemma 6.3.2. *Let (X,d) be complete, and let $f\colon K^{(0)} \to \mathrm{USCC}_B(X)$ be a map of the 0-skeleton of a locally finite simplicial complex K. Then f extends to a map $h\colon |K| \to \mathrm{USCC}_B(X)$ such that*

(♭) $\operatorname{diam}_{\rho_H} h(|\sigma|) \leqslant 3\max\{\rho_H(\operatorname{conv}(\varphi,\psi)\ ,\ \varphi) \geqslant 0 \mid \varphi, \psi \in f(\sigma^{(0)})\}$

for every $\sigma \in K$, where $\sigma^{(0)} = \sigma \cap K^{(0)}$.

Proof. Take the barycentric subdivision $\operatorname{Sd} K$ of K. The barycenter of $\sigma \in K$ is denoted by $b(\sigma)$. We first extend f to a map $g\colon \operatorname{Sd} K^{(0)} \to \mathrm{USCC}_B(X)$ as

$$g(b(\sigma)) = \operatorname{conv}(f(\sigma^{(0)})).$$

Observe that $f(v) \subset g(b(\sigma))$ and $g(b(\sigma)) \subset B_\rho(f(v), \varepsilon)$ for each $\sigma \in K$, $v \in \sigma^{(0)}$, where $\varepsilon = \max\{\rho_H(\operatorname{conv}(f(v), \psi), f(v)) \geqslant 0 \mid \psi \in f(\sigma^{(0)})\}$.

Next we extend g to a map $h\colon |K| = |\mathrm{Sd}\,K| \to \mathrm{USCC}_B(X)$ as follows:

$$h\Big(\sum_{i=0}^{n} t_i v_i\Big)(x) = \Big[\sum_{i=0}^{n} t_i \cdot \min g(v_i)(x)\ ,\ \sum_{i=0}^{n} t_i \cdot \max g(v_i)(x)\Big]$$

for each $x \in X$, $\{v_0, \cdots, v_n\} = \sigma \in \mathrm{Sd}\,K$ and $\sum_{i=0}^{n} t_i = 1$ with $t_i \geqslant 0$. Then, observe that $h(\sum_{i=0}^{n} t_i v_i) = t_0 g(v_0) + \cdots + t_n g(v_n)$. Hence, the map h is continuous by Lemma 6.3.1.

Finally we will verify the condition (b). Take any $\sigma \in K$ and $r, s \in |\sigma|$. Put $\varepsilon = \max\{\rho_H(\mathrm{conv}(\varphi, \psi)\ ,\ \varphi) \geqslant 0 \mid \varphi, \psi \in f(\sigma^{(0)})\} \geqslant 0$. Choose $v, w \in \sigma^{(0)}$ and $\mu, \tau \in \mathrm{Sd}\,K$ such that $\{r, v\} \subset |\mu| \subset |\sigma|$ and $\{s, w\} \subset |\tau| \subset |\sigma|$. It follows from the construction of h that $h(v) \subset h(r)$ and $h(w) \subset h(s)$. Similarly, we have $h(r) \subset h(b(\sigma)) \subset B_\rho(h(v), \varepsilon)$ and $h(s) \subset h(b(\sigma)) \subset B_\rho(h(w), \varepsilon)$, whence $\rho_H(h(a), h(v)) \leqslant \varepsilon$ and $\rho_H(h(b), h(w)) \leqslant \varepsilon$. It is easily seen that $\rho_H(h(v), h(w)) \leqslant \varepsilon$. Therefore, $\rho_H(h(a), h(b)) \leqslant 3\varepsilon$. □

Lemma 6.3.3. *If (X, d) is a complete metric space satisfying the condition $(*)$, then $\mathrm{USCC}_B(X)$ is an AR.*

Proof. We have the homotopy $H\colon \mathrm{USCC}_B(X) \times [0,1] \to \mathrm{USCC}_B(X)$ defined by

$$H_t(\varphi)(x) = [(1-t)\min \varphi(x), (1-t)\max \varphi(x)],$$

hence $\mathrm{USCC}_B(X)$ is contractible. To complete the proof, we will show the Hanner's condition for a space to be an ANR (Theorem 1.2.5), that is, for any $\varepsilon > 0$ there exist a locally finite simplicial complex K, maps $p\colon \mathrm{USCC}_B(X) \to |K|$ and $q\colon |K| \to \mathrm{USCC}_B(X)$ such that $q \circ p$ is ε-homotopic to the identity of $\mathrm{USCC}_B(X)$. We will verify this condition similarly to the proof of Lemma 3.1.2. For each $n \in \mathbb{N}$, take a locally finite open cover $\mathcal{U}(\frac{\varepsilon}{3\cdot 2^{n+1}})$ of $\mathrm{USCC}_B(X)$ as in the condition $(*)'$. Put

$\mathcal{U}_n = \mathcal{U}(\frac{\varepsilon}{3\cdot 2^{n+1}})$. We define

$$\mathcal{W}_1 = \{U \times (\frac{1}{2}, 1] \mid U \in \mathcal{U}_1\} \quad \text{and}$$
$$\mathcal{W}_n = \{U \times (\frac{1}{n+1}, \frac{1}{n-1}) \mid U \in \mathcal{U}_n\} \text{ for } n > 1.$$

Then $\mathcal{W} = \bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ is a locally finite open cover of $\mathrm{USCC}_B(X) \times (0,1]$ and $\mathcal{U}_n = \{pr(W) \mid W \in \mathcal{W}_n\}$ for each $n \in \mathbb{N}$, where $pr\colon \mathrm{USCC}_B(X) \times [0,1] \to \mathrm{USCC}_B(X)$ is the projection. Let K be the nerve of $\mathcal{W}$ and $g\colon \mathrm{USCC}_B(X) \times (0,1] \to |K|$ a canonical map, that is, each $g(\varphi, t)$ is contained in the simplex spanned by all vertices $W \in \mathcal{W}$ containing (φ, t). For each $n \in \mathbb{N}$, let K_n be the nerve of $\mathcal{W}_n \cup \mathcal{W}_{n+1}$. Then each K_n is a subcomplex of K and $K = \bigcup_{n\in\mathbb{N}} K_n$. Choosing $f(W) \in pr(W)$ for each $W \in K^{(0)} = \mathcal{W}$, we have a map $f\colon K^{(0)} \to \mathrm{USCC}_B(X)$ such that

$$\max\{\rho_H(\mathrm{conv}(\varphi, \psi)\ ,\ \varphi) \geqslant 0 \mid \varphi, \psi \in f(\sigma^{(0)})\} < \frac{\varepsilon}{3\cdot 2^{n+1}}$$

for each $\sigma \in K_n$. By Lemma 6.3.2, we can extend f to a map $q\colon |K| \to \mathrm{USCC}_B(X)$ such that

$$\mathrm{diam}_{\rho_H} q(\sigma) < 3\frac{2\varepsilon}{3\cdot 2^{n+1}} = \varepsilon/2^n$$

for each $\sigma \in K_n$. As same as we saw in Lemma 3.1.2, $q \circ g$ can be extended to the homotopy $H\colon \mathrm{USCC}_B(X) \times [0,1] \to \mathrm{USCC}_B(X)$ such that $H|\mathrm{USCC}(X) \times (0,1] = q \circ g$ and $H_0 = \mathrm{id}$. For every $\varphi \in \mathrm{USCC}_B(X)$, there exists a family of simplices $\{\sigma_n \in K_n \mid n \in \mathbb{N}\}$ such that $g(\{\varphi\} \times (0,1]) \subset \bigcup_{n=1}^{\infty} \sigma_n$. Hence,

$$H(\{\varphi\} \times [0,1]) \subset \{\varphi\} \cup q \circ g(\{\varphi\} \times (0,1]) \subset \{\varphi\} \cup \bigcup_{n=1}^{\infty} q(\sigma_n).$$

Therefore,

$$\mathrm{diam}\, H(\{\varphi\} \times [0,1]) \leqslant \mathrm{diam} \bigcup_{n=1}^{\infty} q(\sigma_n) \leqslant \sum_{n=1}^{\infty} \mathrm{diam}\, q(\sigma_n) \leqslant \sum_{n=1}^{\infty} \varepsilon/2^n \leqslant \varepsilon.$$

Thus, q and $p = g|\mathrm{USCC}(X) \times \{1\}$ are desired maps. □

We call a subset $E \subset X$ *ε-discrete* if we have $d(x, y) \geqslant \varepsilon$ for any distinct elements $x \neq y \in X$.

Theorem 6.3.1. *Let (X, d) be a complete metric space. Then the followings are equivalent:*

(1) $\mathrm{USCC}_B(X)$ *is an AR,*

(2) $\mathrm{USCC}_B(X)$ *is locally path-connected,*

(3) *for every $\varepsilon > 0$ and every 2ε-discrete sequence $\{x_n\}_{n=1}^\infty \subset X$, if $D_n = B_d(x_n, \varepsilon)$ is written as the union $D_n = E_n \cup F_n \cup G_n \cup G'_n$ of disjoint clopen subsets of D_n such that $x_n \in E_n$, then the numbers*

$$\alpha_n = \min\{\ d_H(E_n \cup G_n \cup D_n^C\ ,\ F_n \cup G_n \cup D_n^C)\ ,\qquad d_H(E_n \cup G'_n \cup D_n^C\ ,\ F_n \cup G'_n \cup D_n^C)\ \}$$

doesn't converge to 0, where $D_n^C = B_d^C(x_n, \varepsilon)$,

(4) *X satisfies the condition $(*)$.*

Proof. The implication (1) $\Rightarrow$ (2) is clear. The implication (4) $\Rightarrow$ (1) follows by Lemma 6.3.3.

To see (2) $\Rightarrow$ (3), assume that there exist $\varepsilon > 0$ and a 2ε-discrete sequence $\{x_n\}_{n=1}^\infty \subset X$ such that each $D_n = B_d(x_n, \varepsilon)$ can be written as $D_n = E_n \cup F_n \cup G_n \cup G'_n$ similarly to (3) but $\alpha_n \to 0$. We can define $\varphi_n \in \mathrm{USCC}_B(X)$ by

$$\varphi(x) = \begin{cases} [0,2] & \text{if } x \notin \bigcup_{n\in\mathrm{N}} D_n, \\ 0 & \text{if } x \in \bigcup_{n\in\mathrm{N}} E_n, \\ 2 & \text{if } x \in \bigcup_{n\in\mathrm{N}} F_n, \\ 0 & \text{if } x \in \bigcup_{n\in\mathrm{N}} G_n, \\ 2 & \text{if } x \in \bigcup_{n\in\mathrm{N}} G'_n. \end{cases}$$

Let $\varepsilon_0 = \min\{1, \varepsilon\}$. Take any $\delta > 0$ with $\delta < \varepsilon_0$. Since $\alpha_n \to 0$, there exists a number $n_0 \in \mathrm{N}$ such that $\alpha_{n_0} < \delta$. Define $\psi \in \mathrm{USCC}_B(X)$ by

$$\psi(x) = \begin{cases} 2 & \text{if } x \in E_{n_0}, \\ 0 & \text{if } x \in F_{n_0}, \\ \varphi(x) & \text{if otherwise.} \end{cases}$$

Observe that that $\rho_H(\varphi,\psi) < \alpha_{n_0} < \delta$. Let $P\colon [0,1] \to \mathrm{USCC}_B(X) \subset 2^{X\times\mathbb{R}}$ be a path from φ to ψ. The union $\bigcup_{t\in[0,1]} P(t) \subset X\times\mathbb{R}$ cannot be contained in $B_\rho(\varphi,\varepsilon_0)$. To the contrary, if $\bigcup_{t\in[0,1]} \subset B_\rho(\varphi,\varepsilon_0)$, then there exists $t_0 \in (0,1)$ such that $P(t_0)\cap(D_{n_0}\times(-\infty,1))$ is a non-empty clopen subset of $D_{n_0}\times(-\infty,1)$, and $P(t_0)\cap(\overline{B}_d(x_0,\varepsilon_0/2)\times(-\infty,1)) = \emptyset$. This implies that $P|[0,t_0]$ is not continuous. Hence, $\mathrm{USCC}_B(X)$ is not locally path-connected at φ.

Finally, we will prove (3) $\Rightarrow$ (4). Assume that there exist $\delta > 0$, $\varphi \in \mathrm{USCC}_B(X)$, $f_n, g_n \in \mathrm{USCC}_B(X)$ and $x_n \in X$ $(n \in \mathbb{N})$ such that $\rho_H(f_n,\varphi) < 1/2n$, $\rho_H(g_n,\varphi) < 1/2n$,

$$\mathrm{conv}(f_n, g_n)|\{x_n\} \not\subset B_\rho(\varphi,\delta).$$

Then, the sequence $\{x_n\}_{n=1}^\infty$ has no convergent subsequence. If fact, if there exists $\{x_{n_i}\}_{i=1}^\infty \subset \{x_n\}_{n=1}^\infty$ such that $x_{n_i} \to x_0$ for some $x_0 \in X$, then $f_{n_i}(x_{n_i})$ and $g_{n_i}(x_{n_i})$ are contained in $[\min\varphi(x_0)-\delta, \max\varphi(x_0)+\delta]$ for all i but finitely many i, whence $\mathrm{conv}(f_{n_i}, g_{n_i})|\{x_{n_i}\} \subset B_\rho(\varphi,\varepsilon)$. This is a contradiction. Therefore, by the completeness of X, $\{x_n \mid n \in \mathbb{N}\}$ has an infinite 2ε-discrete subset for some $0 < \varepsilon < \delta$. Thus, without loss of generality, we may assume that $\{x_n\}_{n=1}^\infty$ is 2ε-discrete, $\rho(f_n,\varphi) < \varepsilon$ and $\rho(g_n,\varphi) < \varepsilon$.

For every $n \in \mathbb{N}$, since

$$f_n \cup g_n \subset B_\rho(\varphi,\varepsilon) \ \text{ but } \ \mathrm{conv}(f_n,g_n)|\{x_n\} \not\subset B_\rho(\varphi,\varepsilon),$$

there exists a closed interval $[l_n, u_n] \subset \mathbb{R}$ such that

$$(\overline{B}_d(x_n,\varepsilon)\times[l_n,u_n]) \cap f_n = \emptyset,$$

$$(\overline{B}_d(x_n,\varepsilon)\times[l_n,u_n]) \cap g_n = \emptyset,$$

$$(\overline{B}_d(x_n,\varepsilon)\times(-\infty,l_n]) \cap f_n \neq \emptyset \text{ and } (\overline{B}_d(x_n,\varepsilon)\times[u_n,\infty)) \cap g_n \neq \emptyset$$

$$\left(\text{or, } (\overline{B}_d(x_n,\varepsilon)\times(-\infty,l_n]) \cap g_n \neq \emptyset \text{ and } (\overline{B}_d(x_n,\varepsilon)\times[u_n,\infty)) \cap f_n \neq \emptyset\right).$$

Let $p\colon X \times \mathbb{R} \to X$ be the projection. Put

$$E_n = p(f_n \cap (D_n \times (-\infty, l_n])) \cap p(g_n \cap (D_n \times [u_n, \infty))),$$

$$F_n = p(g_n \cap (D_n \times (-\infty, l_n])) \cap p(f_n \cap (D_n \times [u_n, \infty))),$$

$$G_n = p(f_n \cap (D_n \times (-\infty, l_n])) \setminus E_n,$$

$$G'_n = p(g_n \cap (D_n \times [u_n, \infty))) \setminus F_n,$$

where $D_n = \overline{B}_d(x_n, \varepsilon)$. It is clear that E_n, F_n, G_n and G'_n are disjoint clopen sets in D_n and $D_n = E_n \cup F_n \cup G_n \cup G'_n$ for each n. Exchanging E_n and F_n if necessary, we may assume that $x_n \in E_n$ for each n. Note that $d_H(p(A), p(B)) < d_H(A, B)$ for any $A, B \subset X \times \mathbb{R}$. Hence, for sufficiently large n, we have

$$d_H(E_n \cup G_n \cup D_n^C\ ,\ F_n \cup G_n \cup D_n^C) \leqslant \rho_H(f_n, g_n) < 1/n,$$

$$d_H(E_n \cup G'_n \cup D_n^C\ ,\ F_n \cup G'_n \cup D_n^C) \leqslant \rho_H(f_n, g_n) < 1/n.$$

Therefore, $\alpha_n \to 0$. This is a contradiction. □

Remark. In the proof of Theorem 6.3.1, we need the completeness of X for (3) ⇒ (4) only. But the implications (4) ⇒ (1) and (2) ⇒ (3) are valid for any metric space X.

By the statement (3) of Theorem 6.3.1, we have following:

Corollary 6.3.1. *If X is compact, then* $\mathrm{USCC}_B(X)$ *is an AR.* □

Chapter 7

A compactification of multi-valued function spaces

Let X be a dense subset of a metric space Y, we have the natural isometric embedding $e_Y\colon \mathrm{USC}_B(X) \to \mathrm{USC}_B(Y)$ defined by

$$e_Y(\varphi) = \mathrm{cl}_{Y\times\mathbb{R}}\,\varphi.$$

Then $e_Y(\mathrm{USC}(X,\mathbf{I})) \subset \mathrm{USC}(Y,\mathbf{I})$. But, in general,

$$e_Y(\mathrm{USCC}_B(X)) \not\subset \mathrm{USCC}_B(Y) \quad \text{nor} \quad e_Y(\mathrm{USCC}(X,\mathbf{I})) \not\subset \mathrm{USCC}(Y,\mathbf{I}).$$

For example, let $Y = \mathbf{S}^1$ be the unit circle of Euclidean plane $\mathbb{R}^2$ with the usual metric, $X = \mathbf{S}^1 \setminus \{(1,0)\}$, and $f\colon X \to \mathbb{R}$ be the map defined by $f(x,y) = y$ if $x \leqslant 0$ and $f(x,y) = y/|y|$ if $x > 0$. Then $e_Y(f)(1,0) = \{-1,1\}$ is not connected. In case Y is locally connected, it will be shown that

$$e_Y(\mathrm{USCC}_B(X)) \subset \mathrm{USCC}_B(Y) \quad \text{and/or} \quad e_Y(\mathrm{USCC}(X,\mathbf{I})) \subset \mathrm{USCC}(Y,\mathbf{I})$$

if and only if the complement $Y \setminus X$ is *locally non-separating* in Y, that is, $U \cap X \neq \emptyset$ is connected for each non-empty connected open set U in Y. We generalize Theorem 6.1.1 to pairs as follows:

Theorem A. *Let X be a dense subset of a locally connected metric space Y with the locally non-separating complement in Y. Then the following are equivalent:*

(a) $(\mathrm{USCC}(Y,\mathbf{I}), e_Y(\mathrm{USCC}(X,\mathbf{I}))) \approx (Q,s)$;

(b) $(\mathrm{USCC}_B(Y), e_Y(\mathrm{USCC}_B(X))) \approx (Q \times [0,1), s \times [0,1))$;

(c) $X \neq Y$, *X is G_δ in Y and Y is compact.*

In the above, it should be observed that if Y is locally connected and $Y \setminus X$ is locally non-separating in Y then X is dense in Y.

A metric space $X = (X,d)$ (or a metric d) has *Property S* if X is covered by finitely many connected sets with arbitrarily small diameters. It should be remarked that a metric space with Property S is totally bounded, hence *a complete metric space with Property S is compact.* The subspace of 2^X consisting of compacta is denoted by $\exp(X)$. In case X is compact, $\exp(X) = 2^X$. In [Cu₂], Curtis proved that X admits a Peano compactification $\tilde{X}$ such that $(\exp(\tilde{X}), \exp(X)) \approx (Q,s)$ if and only if X is connected, locally connected, completely metrizable, nowhere locally compact and admits an admissible metric d with Property S. We have the following version of this Curtis' result:

Theorem B. *A metrizable space X has a metrizable compactification $\tilde{X}$ such that*

$$(\mathrm{USCC}(\tilde{X},\mathbf{I}), e_{\tilde{X}}(\mathrm{USCC}(X,\mathbf{I}))) \approx (Q,s)$$

if and only if X is completely metrizable, non-compact and admits an admissible metric with Property S.

One should note that some admissible metric d for X cannot be extended to $\tilde{X}$ even if d has Property S. For example, let $X = (0,1)$ and $\tilde{X} = [0,1]$. Then, $X \approx \mathbf{S}^1 \setminus \{(1,0)\}$. The metric on X inherited from $\mathbf{S}^1$ has Property S but cannot be extended to $\tilde{X}$. The following is a direct consequence of Theorems A and B:

Corollary. *Let X be completely metrizable, non-compact and admits an admissible metric with Property S. Then X has a metric which induces the topology on* $\mathrm{USCC}_B(X)$ *such that* $\mathrm{USCC}(X, \mathbf{I}) \approx \mathrm{USCC}_B(X) \approx \ell_2$. $\square$

In the above, the topology of $\mathrm{USCC}(X, \mathbf{I})$ is not defined by using a complete metric on X. In Chapter 8, it will be proved that the spaces $\mathrm{USCC}_B(X)$ and $\mathrm{USCC}(X, \mathbf{I})$ are homeomorphic to a *non-separable* Hilbert space for a uniformly locally connected, non-compact and complete metric space X (*even if X is separable*). One should observe that $\mathrm{USCC}_B(\mathbb{R})$ is non-separable but $\mathrm{USCC}_B((0,1))$ is separable, where $\mathbb{R}$ and $(0,1)$ have the usual metrics.

§7.1. A compactification of multi-valued function spaces on a totally bounded spaces

First, we show the following:

Proposition 7.1.1. *Let X be a dense subset of a locally connected metric space Y. Then, the following are equivalent:*

(a) $e_Y(\mathrm{USCC}(X, \mathbf{I})) \subset \mathrm{USCC}(Y, \mathbf{I})$;

(b) $e_Y(\mathrm{USCC}_B(X)) \subset \mathrm{USCC}_B(Y)$;

(c) $Y \setminus X$ *is locally non-separating in* Y.

Proof. (c) $\Rightarrow$ (b): Suppose $e_Y(\mathrm{USCC}_B(X)) \not\subset \mathrm{USCC}_B(Y)$, that is, there exists $\varphi \in \mathrm{USCC}_B(X)$ such that $e_Y(\varphi) \notin \mathrm{USCC}_B(Y)$. Then $e_Y(\varphi)(y)$ is not connected for some $y \in Y \setminus X$, whence we have $t_1 < t < t_2$ such that $t_1, t_2 \in e_Y(\varphi)(y)$ but $t \notin e_Y(\varphi)$. Since $e_Y(\varphi)$ is closed in $Y \times \mathbf{I}$ and Y is locally connected, we have a connected open neighborhood U in y in Y and $\delta > 0$ such that

$$U \times (t-\delta, t+\delta) \cap e_Y(\varphi) = \emptyset,$$

whence $t \notin \varphi(x)$ for all $x \in U \cap X$, $t_1 < t - \delta$ and $t_2 > t + \delta$. By the definition of $e_Y(\varphi)$, we have $x_i \in U \cap X$ and $s_i \in \varphi(x_i)$, $i = 1, 2$, such that $|s_1 - t_i| < \delta$, whence $t \notin \varphi(x_i)$ and $s_1 < t < s_2$. Since $\varphi(x_i)$ is connected, $\varphi(x_1) \subset (-\infty, t)$ and $\varphi(x_2) \subset (t, \infty)$. Since φ is u.s.c.,

$$U_1 = \{x \in U \mid \varphi(x) \subset (-\infty, t)\} \quad \text{and} \quad U_2 = \{x \in U \mid \varphi(x) \subset (t, \infty)\}$$

are open in U. It follows that $U = U_1 \cup U_2$, $U_1 \cap U_2 = \emptyset$ and $x_i \in U_i \cap X$, $i = 1, 2$. Hence, $U \cap X$ is not connected, which means that $Y \setminus X$ is not locally non-separating in Y.

(b) $\Rightarrow$ (a): This is observed as follows:

$$\begin{aligned} e_Y(\mathrm{USCC}(X, \mathbf{I})) &= e_Y(\mathrm{USCC}_B(X)) \cap \mathrm{USC}(Y, \mathbf{I}) \\ &\subset \mathrm{USCC}_B(Y) \cap \mathrm{USC}(Y, \mathbf{I}) = \mathrm{USCC}(Y, \mathbf{I}). \end{aligned}$$

(a) $\Rightarrow$ (c): First, note that X is dense in Y. Otherwise, $e_Y(\varphi)(y) = \emptyset$ for each $\varphi \in \mathrm{USCC}(X, \mathbf{I})$ and $y \in Y \setminus \mathrm{cl}\, X$. Now, suppose that $Y \setminus X$ is not locally non-separating in Y, that is, there exists a connected open set U in Y such that $U \cap X$ is not connected. (Note that $U \cap X \neq \emptyset$ because X is dense in Y.) Let $U \cap X = U_1 \cap U_2$, where U_1 and U_2 are disjoint non-empty open sets in X. Note that $\mathrm{cl}_X U_1 \cap \mathrm{cl}_X U_2 \subset X \setminus U$. Let

$$\varphi = (X \setminus U) \times \mathbf{I} \cup U_1 \times \{0\} \cup U_2 \times \{1\} \in \mathrm{USCC}(X, \mathbf{I}).$$

Since $U \neq U \cap X$, we have $y \in U \setminus X$. Then $y \in \mathrm{cl}_Y U_1 \cap \mathrm{cl}_Y U_2$ because X is dense in Y. It follows that $e_Y(\varphi)(y) = \{0, 1\}$. Thus $e_Y(\varphi) \notin \mathrm{USCC}(Y, \mathbf{I})$, which contradicts to $e_Y(\mathrm{USCC}(X, \mathbf{I})) \subset \mathrm{USCC}(Y, \mathbf{I})$. Therefore, $Y \setminus X$ is locally non-separating in Y. □

Proposition 7.1.2. *Let X be a dense subset of a locally connected compact metric space Y with the locally non-separating complement $Y \setminus X$ in Y. Then X is G_δ in Y if and only if $e_Y(\mathrm{USCC}_B(X))$ is G_δ in* $\mathrm{USCC}(Y)$.

Proof. The "only if" part follows from

$$i_Y(X) = i_Y(Y) \cap e_Y(\mathrm{USCC}_B(X)),$$

where $i_Y : Y \longrightarrow \mathrm{USCC}(Y, \mathbf{I}) \subset \mathrm{USCC}_B(Y)$ is the natural closed embedding.

To see the "if" part, let $X = \bigcap_{n\in\mathbb{N}} U_n$, where each U_n is open in Y. For each $m, n \in \mathbb{N}$, let

$$G_{m,n} = \{\varphi \in \mathrm{USCC}_B(Y) \mid \rho_{\mathrm{H}}(\varphi, e_Y(\varphi|U_n)) < 1/m\}.$$

Since $e_Y(\mathrm{USCC}_B(X)) = \bigcap_{m,n\in\mathbb{N}} G_{m,n}$, it suffices to show that each $G_{m,n}$ is open in $\mathrm{USCC}_B(Y)$, or each $F_{m,n} = \mathrm{USCC}_B(Y) \setminus G_{m,n}$ is closed in $\mathrm{USCC}_B(Y)$.

Assume that a sequence $\varphi_i \in F_{m,n}$, $i \in \mathbb{N}$, which converges to $\varphi \in \mathrm{USCC}_B(Y)$. Since φ is bounded, $\varphi \subset Y \times [-a, a]$ for some $a > 0$. Then, we may assume that $\varphi_i \subset Y \times [-a, a]$ for all $i \in \mathbb{N}$. Since each φ_i is compact, we can choose $(x_i, t_i) \in \varphi_i$ so that

$$\rho((x_i, t_i), e_Y(\varphi_i|U_n)) = \rho_{\mathrm{H}}(\varphi_i, e_Y(\varphi_i|U_n)) \geqslant 1/m.$$

Since $Y \times [-a, a]$ is compact, we may assume that (x_i, t_i) converges to $(x_0, t_0) \in Y \times [-a, a]$, whence $(x_0, t_0) \in \varphi$. We show that $\rho((x_0, t_0), e_Y(\varphi|U_n)) \geqslant 1/m$, which means that $\varphi \in F_{m,n}$. Then, $F_{m,n}$ would be closed in $\mathrm{USCC}(Y, [-a, a])$.

Now, assume that $\rho((x_0, t_0), e_Y(\varphi|U_n)) < 1/m$. Then, we have $(y_0, s_0) \in \varphi|U_n$ such that $\rho((x_0, t_0), (y_0, s_0)) < 1/m$. Let

$$\delta = \min\{d(y_0, Y \setminus U_n),\ \tfrac{1}{2}(1/m - \rho((x_0, t_0), (y_0, s_0)))\} > 0.$$

Choose i so large that $\rho_{\mathrm{H}}(\varphi_i, \varphi) < \delta$ and $\rho((x_i, t_i), (x_0, t_0)) < \delta$. Then, we have $(y_i, s_i) \in \varphi_i$ such that $\rho((y_0, s_0), (y_i, s_i)) < \delta$. Since $d(y_0, y_i) < d(y_0, Y \setminus U_n)$, it follows that $y_i \in U_n$, hence $(y_i, s_i) \in \varphi_i|U_n$. Therefore,

$$\rho((x_i, t_i), (y_i, s_i)) \geqslant \rho((x_i, t_i), \mathrm{cl}_{Y\times\mathbf{I}}\, \varphi_i|U_n) \geqslant 1/m.$$

On the other hand,

$$\begin{aligned}\rho((x_i,t_i),(y_i,s_i)) &\leqslant \rho((x_i,t_i),(x_0,t_0)) + \rho((x_0,t_0),(y_0,s_0)) + \rho((y_0,s_0),(y_i,s_i)) \\ &< 2\delta + \rho((x_0,t_0),(y_0,s_0)) < 1/m,\end{aligned}$$

which is a contradiction. The proof is completed. $\square$

Now, we prove Theorems A and B.

Proof of Theorem A. (a) $\Rightarrow$ (b): As saw in the proof of [Fe$_2$, Proposition 2.4], $D = \mathrm{USCC}(Y, \mathbf{I}) \setminus \mathrm{USCC}(Y, (0,1))$ is a contractible Z-set in $\mathrm{USCC}(Y, \mathbf{I})$ and then

$$\mathrm{USCC}(Y,(0,1)) \approx \mathrm{USCC}(Y,\mathbf{I}) \setminus D \approx Q \times [0,1).$$

It follows from [Ch$_1$, Theorem 6.6] that

$$\big(\mathrm{USCC}(Y,(0,1)),\ e_Y(\mathrm{USCC}(X,\mathbf{I})) \setminus D\big) \approx \big(Q \times [0,1),\ s \times [0,1)\big),$$

where it should be noted that $e_Y(\mathrm{USCC}(X,\mathbf{I})) \setminus D \neq e_Y(\mathrm{USCC}(X,(0,1))$ but

$$e_Y(\mathrm{USCC}(X,\mathbf{I})) \setminus D = \big\{e_Y(\varphi) \bigm| \varphi \in \mathrm{USCC}(X,(a,b)) \text{ for some } 0 < a < b < 1\big\}.$$

By Theorem 6.1.1, Y is compact, whence $\mathrm{USCC}_B(Y) = \mathrm{USCC}(Y)$ and there exists a homeomorphism $h\colon \mathrm{USCC}(Y) \to \mathrm{USCC}(Y,(0,1))$ such that

$$h(e_Y(\mathrm{USCC}_B(X))) = \big\{e_Y(\varphi) \bigm| \varphi \in \mathrm{USCC}(X,(a,b)) \text{ for some } 0 < a < b < 1\big\}.$$

Consequently, we have

$$\begin{aligned}\big(\mathrm{USCC}_B(Y),\ e_Y(\mathrm{USCC}_B(X))\big) &\approx \big(\mathrm{USCC}(Y,(0,1)),\ e_Y(\mathrm{USCC}(X,\mathbf{I})) \setminus D\big) \\ &\approx \big(Q \times [0,1),\ s \times [0,1)\big).\end{aligned}$$

(b) $\Rightarrow$ (c): By Theorem 6.1.1, the condition (b) implies that $X \neq Y$ and Y is compact and locally connected by Theorem 6.1.1. Moreover, $Y \setminus X$ is locally non-separating in Y by Proposition 7.1.1 and X is G_δ in Y by Proposition 7.1.2.

(c) $\Rightarrow$ (a): We first consider the case that Y is connected, hence it is a Peano continuum. In this case, $\mathrm{USCC}(Y, \mathbf{I})$ is the closure of $\mathrm{C}(Y, \mathbf{I})$ in $\exp(Y \times \mathbf{I}) = 2^{Y \times \mathbf{I}}$ [Fe$_2$, Theorem 1.10]. Since $(\mathrm{USCC}(Y, \mathbf{I}), \mathrm{C}(Y, \mathbf{I})) \approx (Q, s)$ (Theorem 3.1.1), the complement $\mathrm{USCC}(Y, \mathbf{I}) \setminus \mathrm{C}(Y, \mathbf{I})$ is a Z_σ-set in $\mathrm{USCC}(Y, \mathbf{I})$. By Proposition 7.1.2, $e_Y(\mathrm{USCC}_B(X))$ is G_δ in $\mathrm{USCC}_B(Y)$, whence

$$e_Y(\mathrm{USCC}(X, \mathbf{I})) = e_Y(\mathrm{USCC}_B(X)) \cap \mathrm{USCC}(Y, \mathbf{I})$$

is also G_δ in $\mathrm{USCC}(Y, \mathbf{I})$. Then, the complement

$$M = \mathrm{USCC}(Y, \mathbf{I}) \setminus e_Y(\mathrm{USCC}(X, \mathbf{I}))$$

is F_σ in $\mathrm{USCC}(Y, \mathbf{I})$ and $M \subset \mathrm{USCC}(Y, \mathbf{I}) \setminus \mathrm{C}(Y, \mathbf{I})$, hence M is a Z_σ-set in $\mathrm{USCC}(Y, \mathbf{I})$. Let (A, B) be a pair of compacta in $\mathrm{USCC}(Y, \mathbf{I})$ such that $B \subset M$ and $\varepsilon > 0$. By all the same way as the proof of Theorem 3.1.1, but using a point $x_0 \in Y \setminus X$, we can define an embedding $h \colon A \to M$ such that $h|B = \mathrm{id}$ and h is ε-close to id. Applying the characterization of $B(Q) = Q \setminus s$ [An$_2$] (cf. [Ch$_1$, Lemma 8.1]), we have $(\mathrm{USCC}(Y, \mathbf{I}), M) \approx (Q, B(Q))$, hence

$$(\mathrm{USCC}(Y, \mathbf{I}), e_Y(\mathrm{USCC}(X, \mathbf{I}))) \approx (Q, s).$$

In the general case, we write $Y = \bigcup_{i=1}^n Y_i$, where each Y_i is a component of Y, which is closed and open in Y because of locally connectedness of Y. Since $Y \setminus X$ is locally non-separating in Y, each $X_i = X \cap Y_i$ is a component of X. Then

$$(\mathrm{USCC}(Y, \mathbf{I}), e_Y(\mathrm{USCC}(X, \mathbf{I}))) \approx \left(\prod_{i=1}^n \mathrm{USCC}(Y_i, \mathbf{I}), \prod_{i=1}^n e_{Y_i}(\mathrm{USCC}(X_i, \mathbf{I}))\right).$$

In case Y_i is a singleton, $X_i = Y_i$ and $\mathrm{USCC}(Y_i, \mathbf{I})$ is homeomorphic to the hyperspace of subcontinua of $\mathbf{I}$, hence $\mathrm{USCC}(Y_i, \mathbf{I}) \approx \mathbf{I}^2$ (cf. [Du, §3]). Hence the general case can be obtained the connected case. □

Proof of Theorem B. First, assume that X is completely metrizable and has an admissible metric with Property S. Then, X has only finitely many components, which are closed and open in X. Replacing the metric, we may assume that the distance between any two components of X is positive. Thus, as in the proof of Theorem A, it suffices to treat the case X is connected. In this case, X has a Peano compactification $\tilde{X}$ with a locally non-separating remainder $\tilde{X} \setminus X$ by [Cu$_2$, Proposition 2.4]. By complete metrizability, X is G_δ in $\tilde{X}$. Then, the "if" part follows from Theorem A.

Conversely, assume that X has a compactification $\tilde{X}$ such that

$$(\mathrm{USCC}(\tilde{X}, \mathbf{I}), e_{\tilde{X}}(\mathrm{USCC}(X, \mathbf{I}))) \approx (Q, s).$$

By Theorem A, $X \neq \tilde{X}$, X is G_δ in $\tilde{X}$, $\tilde{X}$ is locally connected and the remainder $\tilde{X} \setminus X$ is locally non-separating in $\tilde{X}$. Then X is completely metrizable and, as is easily observed, each component of $\tilde{X}$ is a Peano compactification of a component of X with locally non-separating remainder. By [Cu$_2$, Proposition 2.4], X admits an admissible metric d with Property S. Thus we have the "only if" part. □

§7.2. Supplement

As mentioned before, the Banach space $\mathrm{C}_B(X)$ is not a subspace of $\mathrm{USCC}_B(X)$ in case X is non-compact (cf. [FK, Remark 3.6]). Here we show the following:

Proposition 7.2.1. *In the following cases, the topology for* $\mathrm{C}(X, \mathbf{I})$ *induced by the sup-norm is different from the one induced by the Hausdorff metric* ρ_{H}*:*

(1) *X is non-complete;*

(2) *X has a non-totally bounded component;*

(3) *X has infinitely many components X_i, $i \in \mathbb{N}$, such that* $\inf_{i\in\mathbb{N}} \operatorname{diam} X_i > 0$ *and* $\inf_{i\neq j} \operatorname{dist}(X_i, X_j) > 0$.

Proof. (1) In this case, X has a non-convergent Cauchy sequence $(x_i)_{i\in\mathbb{N}}$. Then, for each $n \in \mathbb{N}$, we have $m > n$ such that $d(x_i, x_j) < \frac{1}{3}d(x_n, x_m)$ for all $i, j \geqslant m$. In fact, x_n is not an accumulation point of $(x_i)_{i\in\mathbb{N}}$, whence there is some $\delta > 0$ such that $d(x_n, x_i) > \delta$ for almost all $i \in \mathbb{N}$. Since $(x_i)_{i\in\mathbb{N}}$ is Cauchy, we can choose $m > n$ such that $d(x_n, x_m) > \delta$ and $d(x_i, x_j) < \frac{1}{3}\delta$ if $i, j \geqslant m$, whence $d(x_i, x_j) < \frac{1}{3}d(x_n, x_m)$ for all $i, j \geqslant m$. Therefore, by taking a subsequence, we can assume that $d(x_i, x_j) < \frac{1}{3}d(x_n, x_{n+1})$ for all $i, j > n$. For each $n \in \mathbb{N}$, let $\varepsilon_n = \frac{1}{3}d(x_n, x_{n+1})$. Then the collection $\{B(x_n, \varepsilon_n) \mid n \in \mathbb{N}\}$ is discrete in X and

$$\bigcup_{i>n} B(x_i, \varepsilon_i) \subset B(x_{n+1}, 2\varepsilon_n) \subset X \setminus \bigcup_{j\leqslant n} B(x_j, \varepsilon_j).$$

We define a map $f \in \mathrm{C}(X, \mathbf{I})$ as follows:

$$f(x) = \begin{cases} 1 - \varepsilon_{2n-1}^{-1} d(x, x_{2n-1}) & \text{if } x \in B(x_{2n-1}, \varepsilon_{2n-1}),\ n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $f(x_{2n-1}) = 1$ and $f(x_{2n}) = 0$ for each $n \in \mathbb{N}$. Then, any map $g \in \mathrm{C}(X, \mathbf{I})$ with $\sup_{x\in X} |f(x) - g(x)| = \delta < \frac{1}{2}$ is not uniformly continuous, because $\lim_{n\to\infty} d(x_{2n-1}, x_{2n}) = 0$ but

$$\lim_{n\to\infty} |g(x_{2n-1}) - g(x_{2n})| \geqslant (1 - \delta) - \delta = 1 - 2\delta > 0.$$

However, for each $\varepsilon > 0$, there exists a uniformly continuous map $h \in \mathrm{C}(X, \mathbf{I})$ with $\rho_{\mathrm{H}}(f, h) < \varepsilon$. In fact, choose $n \in \mathbb{N}$ so that $\varepsilon_{2n} < \varepsilon$ and define a map $h \in \mathrm{C}(X, \mathbf{I})$ as follows:

$$h(x) = \begin{cases} 1 - \varepsilon_n^{-1} d(x, x_{2n+1}) & \text{if } x \in B(x_{2n+1}, 2\varepsilon_n), \\ f(x) & \text{otherwise.} \end{cases}$$

(2) Let X_0 be a non-totally bounded component of X. Then we have $\delta > 0$ and $x_i \in X_0$, $i \in \mathbb{N}$, such that $d(x_i, x_j) > \delta$ if $i \neq j$. For each $i \in \mathbb{N}$, let $\delta_i = \min\{i^{-1},\ \frac{1}{3}\delta\} > 0$. We define a map $f \in \mathrm{C}(X, \mathbf{I})$ as follows:

$$f(x) = \begin{cases} 1 - \delta_i^{-1} d(x, x_i) & \text{if } x \in B(x_i, \delta_i),\ i \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Since X_0 is connected and $\operatorname{diam} X_0 > \delta$, there are $y_i \in X_0$, $i \in \mathbb{N}$, such that $d(x_i, y_i) = \delta_i$, whence $f(y_i) = 0$. By the same reason as the case (1), any map $g \in \mathrm{C}(X, \mathbf{I})$ with $\sup_{x \in X} |f(x) - g(x)| = \delta < \frac{1}{2}$ is not uniformly continuous. However, for each $\varepsilon > 0$, choose $n \in \mathbb{N}$ so that $n^{-1} < \varepsilon$ and define a uniformly continuous map $h \in \mathrm{C}(X, \mathbf{I})$ defined by

$$h(x) = \begin{cases} 1 - \min\{\varepsilon, \delta\}^{-1} d(x, x_i) & \text{if } x \in B(x_i, \min\{\varepsilon, \delta\}),\ i \geqslant n, \\ f(x) & \text{otherwise.} \end{cases}$$

Similarly as the above, there are $z_i \in X_0$, $i \in \mathbb{N}$, such that $d(x_i, z_i) = \min\{\varepsilon, \delta\}$, whence $h(z_i) = 0$. Then it is easy to see that $\rho_{\mathrm{H}}(f, h) < \varepsilon$.

(3) For each $i \in \mathbb{N}$, take $x_i \in X_i$. Choose $\delta > 0$ so that $\delta < \inf_{i \in \mathbb{N}} \operatorname{diam} X_i$ and $\delta < \inf_{i \neq j} \operatorname{dist}(X_i, X_j)$. Then, replacing X_0 by X_i in the proof of the case (2), we have the proof of this case. $\square$

CHAPTER 8

Multi-valued function spaces homeomorphic to the Hilbert spaces

In this chapter, we consider the case X is non-compact but complete. It is said that X is *uniformly* (or *d-uniformly*) *locally connected* if, for each $\varepsilon > 0$, there is $\delta > 0$ such that each pair of points $x, x' \in X$ with $d(x, x') < \delta$ are contained in some connected set in X with diameter $< \varepsilon$. Let m (or ℓ_∞) be the Banach space of bounded sequences in $\mathbb{R}$ with the sup-norm. Note that m is non-separable. Indeed, $m \approx \ell_2(2^{\mathrm{N}})$ [BP, Ch.VII, Theorem 6.1]. By applying Toruńczyk's characterization of Hilbert spaces [To_6] (cf. Theorem 1.3.3), we prove the following:

Main Theorem. *If $X = (X, d)$ is a uniformly locally connected, non-compact and complete metric space, then* $\mathrm{USCC}(X, \mathbf{I})$ *and* $\mathrm{USCC}_B(X)$ *are homeomorphic to a non-separable Hilbert space. In case X is separable,*

$$\mathrm{USCC}(X, \mathbf{I}) \approx \mathrm{USCC}_B(X) \approx m \approx \ell_2(2^{\mathrm{N}}).$$

In the above, the word "uniformly" cannot be removed, that is, the Main Theorem is not valid for a locally connected complete metric space X with no isolated points.

Example. *The following closed subspace X of Euclidean plane $\mathbb{R}^2$ is locally path-connected and has no isolated points, but* $\mathrm{USCC}(X,\mathbf{I})$ *and* $\mathrm{USCC}_B(X)$ *are not locally connected, hence they are not ANR's:*

$$X = \mathbb{R} \times \{0\} \cup \bigcup_{n\in\mathbb{N}} \{n, n+2^{-n}\} \times \mathbf{I} \subset \mathbb{R}^2.$$

Proof. We define a map $f\colon X \to \mathbf{I}$ by

$$f(s,t) = \begin{cases} 2t & \text{if } s \in \mathbb{N} \text{ and } 0 \leqslant t \leqslant \frac{1}{2}; \\ 1 & \text{if } s \in \mathbb{N} \text{ and } \frac{1}{2} \leqslant t \leqslant 1; \\ 0 & \text{otherwise.} \end{cases}$$

For each $\varepsilon > 0$, choose $n_0 \in \mathbb{N}$ so that $2^{-n_0} < \varepsilon$, and define $g\colon X \to \mathbf{I}$ by

$$g(s,t) = \begin{cases} 0 & \text{if } s = n_0; \\ 2t & \text{if } s = n_0 + 2^{-n_0} \text{ and } 0 \leqslant t \leqslant \frac{1}{2}; \\ 1 & \text{if } s = n_0 + 2^{-n_0} \text{ and } \frac{1}{2} \leqslant t \leqslant 1; \\ f(s,t) & \text{otherwise.} \end{cases}$$

Then $\rho_{\mathrm{H}}(f,g) = 2^{-n_0} < \varepsilon$ but g cannot be connected with f by any path in $\mathrm{USCC}_B(X)$ with diameter $< \frac{1}{2}$. □

In the above, $X \approx Y = \mathbb{R} \times \{0\} \cup \mathbb{N} \times \mathbf{I} \subset \mathbb{R}^2$, but $\mathrm{USCC}(X,\mathbf{I}) \not\approx \mathrm{USCC}(Y,\mathbf{I})$ because $\mathrm{USCC}(Y,\mathbf{I}) \approx \ell_2(2^{\mathbb{N}})$ by the Main Theorem.

Throughout the chapter, the open ε-ball in $X = (X,d)$ centered at $x \in X$ is denoted by $B(x,\varepsilon)$ (or $B_d(X,\varepsilon)$) and the closure of $B(x,\varepsilon)$ in X by $\overline{B}(x,\varepsilon)$. On the other hand, to avoid confusion, the ε-neighborhood of a subset $F \subset X$ in X is denoted by $N(F,\varepsilon)$ (or $N_d(F,\varepsilon)$), that is,

$$N(F,\varepsilon) = \bigcup_{x\in F} B(x,\varepsilon) = \{y \in X \mid d(y,F) < \varepsilon\} \subset X.$$

For $F \subset X \times \mathbb{R}$ and $A \subset X$, we denote $F|A = F \cap \mathrm{pr}_X^{-1}(A) = F \cap A \times \mathbb{R}$ and $F(A) = \mathrm{pr}_{\mathbb{R}}(F|A)$, where $\mathrm{pr}_X : X \times \mathbb{R} \to X$ and $\mathrm{pr}_{\mathbb{R}} \colon X \times \mathbb{R} \to \mathbb{R}$ are the projections. In case $A = \{x\}$, we denote $F|\{x\} = F|x$ and $F(\{x\}) = F(x)$.

§8.1. Relations among $C_B(X)$, $\mathrm{USCC}_B(X)$ and $2^{X\times\mathbb{R}}$

Let $X = (X, d)$ be a metric space. Recall that $(2^X)_m$ is the hyperspace of non-empty bounded closed subsets of X with the Hausdorff metric d_{H} defined by d. If X is complete, then $(2^X)_m$ is also complete [Ku, p.407]. In case X is compact, $(2^X)_m$ is the hyperspace 2^X of non-empty closed subsets of X. When X is unbounded, $2^X \neq (2^X)_m$ and d_{H} is not a metric on the whole 2^X (see Preliminaries §1.4).

The spaces $\mathrm{USCC}(X, \mathbf{I}) \subset \mathrm{USCC}_B(X)$ are regarded as subspaces of the hyperspace $2^{X\times\mathbb{R}}$. It should be noted that $\mathrm{USCC}(X, \mathbf{I}) \not\subset (2^{X\times\mathbb{R}})_m$ if X is unbounded, and that ρ_{H} is not a metric on $2^{X\times\mathbb{R}}$ but it is a metric on $\mathrm{USCC}_B(X)$.

One should remark that a different metric d' on X defines not only a different space $(2^X)_m$ but also a different topology of 2^X even if d' induces the same topology of X as d. However, if d' is uniformly equivalent to d, then d'_{H} induces the same topology on 2^X as d_{H}. Let d^* be the bounded metric on X defined by $d^*(x, y) = \min\{1, d(x, y)\}$. Note that every closed subset of X is bounded with respect to d^*. Since d^*_{H} is a metric on the whole 2^X, the space 2^X is metrizable. Moreover, if d is complete, then so is d^*, hence d^*_{H} is also complete (cf. [Ku, p.407]).

The following is elementary, but we give a proof for completeness.

Lemma 8.1.1. *Let $\varphi \in \mathrm{USCC}_B(X)$ and $A \subset X$. If A is connected, then the image $\varphi(A)$ is also connected.*

Proof. Assume that $\varphi(A)$ is disconnected. Then we have $t \in \mathbb{R} \setminus \varphi(A)$ such that $(-\infty, t) \cap \varphi(A) \neq \emptyset$ and $(t, \infty) \cap \varphi(A) \neq \emptyset$, whence $\varphi(x) \subset (-\infty, t)$ or $\varphi(x) \subset (t, \infty)$ for each $x \in A$ because of connectedness of $\varphi(x)$. Let $U = \{x \in X \mid \varphi(x) \subset (-\infty, t)\}$ and $V = \{x \in X \mid \varphi(x) \subset (t, \infty)\}$. Then $U \cap V = \emptyset$, $A \subset U \cap V$, $A \cap U \neq \emptyset$ and $A \cap V \neq \emptyset$. Since φ is u.s.c., these U and V are open sets in X. This contradicts the connectedness of A. Hence $\varphi(A)$ is connected. □

Without any completeness condition, the following can be proved (cf. [FK, Theorem 3.3(a)]).

Proposition 8.1.2. *If X is locally connected, then* $\mathrm{USCC}_B(X)$ *is closed in* $2^{X\times\mathbb{R}}$*, hence* $\mathrm{USCC}(X,\mathbf{I})$ *is closed in* $2^{X\times\mathbf{I}}$.

Proof. Let $\varphi \in \mathrm{cl}_{2^{X\times\mathbb{R}}} \mathrm{USCC}_B(X)$. Then, as is easily observed, $\varphi \subset X \times [-a, a]$ for some $a > 0$. If $\varphi(x) = \emptyset$ (i.e., $\varphi \cap \{x\} \times \mathbb{R} = \emptyset$), then $B(x, \varepsilon) \times \mathbb{R} \cap \varphi = \emptyset$ for some $\varepsilon > 0$. For any $\psi \in \mathrm{USCC}_B(X)$, since $\psi(x) \neq \emptyset$, $\rho_{\mathrm{H}}(\psi, \varphi) \geqslant \varepsilon$, which is a contradiction. Therefore, $\varphi(x) \neq \emptyset$ for every $x \in X$. Since φ is closed in $X \times \mathbb{R}$, it follows that $\varphi\colon X \to \mathbb{R}$ is u.s.c. We show that each $\varphi(x)$ is connected, which implies that $\varphi \in \mathrm{USCC}_B(X)$.

Assume that some $\varphi(x_0)$ is not connected. Then we can find some $t_1 < t_0 < t_2$ such that $t_1, t_2 \in \varphi(x_0)$ and $t_0 \notin \varphi(x_0)$. Choose $\varepsilon > 0$ so that

$$B(x_0, 2\varepsilon) \times (t_0 - \varepsilon, t_0 + \varepsilon) \cap \varphi = \emptyset,$$

whence $\rho((x, t_0), \varphi) \geqslant \varepsilon$ for each $x \in B(x_0, \varepsilon)$. Since X is locally connected, x_0 has a connected neighborhood $U \subset B(x_0, \varepsilon)$. Then U contains some $B(x_0, \delta) \subset U$, whence $\delta \leqslant \varepsilon$. For each $\psi \in \mathrm{USCC}_B(X)$ with $\rho_{\mathrm{H}}(\psi, \varphi) < \delta$, we have some $(x_i, s_i) \in \psi$, $i = 1, 2$, such that $d(x_i, x_0) < \delta$ and $|t_i - s_i| < \delta$, whence $x_1, x_2 \in U$, $s_1 < t_0$ and $s_2 > t_0$. Since $\psi(U)$ is connected by Lemma 8.1.1, it follows that $t_0 \in [s_1, s_2] \subset \psi(U)$, that is, $t_0 = \psi(x)$ for some $x \in U \subset B(x_0, \varepsilon)$. Then, $\rho_{\mathrm{H}}(\psi, \varphi) \geqslant \rho((x, t_0), \varphi) \geqslant \varepsilon$, which is a contradiction. Therefore, every $\varphi(x)$ is connected. Thus we have $\varphi \in \mathrm{USCC}_B(X)$. □

By the remark in the beginning of this section, the following easily follows from Proposition 8.1.2.

Corollary 8.1.3. *If X is complete and locally connected, then* $\mathrm{USCC}_B(X)$ *is complete, hence so is* $\mathrm{USCC}(X,\mathbf{I})$. □

Let $C_B(X)$ be the Banach space of bounded continuous real-valued functions of X with the sup-norm[1] and let $C(X, \mathbf{I}) = \{f \in C_B(X) \mid f(X) \subset \mathbf{I}\}$. In case X is compact, every continuous real-valued function of X is bounded, whence we denote $C_B(X) = C(X)$. For a compact space X, Fedorchuk [$\mathrm{Fe}_{1,2}$] proved that if X is locally connected and has no isolated points then $C(X)$ and $C(X, \mathbf{I})$ are dense in $\mathrm{USCC}(X)$ and $\mathrm{USCC}(X, \mathbf{I})$, respectively. This has been generalized in [FK] to non-compact spaces with some completeness condition. Here is given a proof without local connectedness nor any completeness condition.

Lemma 8.1.4. *For each $\varphi \in \mathrm{USCC}(X, \mathbf{I})$ and $\varepsilon > 0$, there exists a lower semi-continuous (l.s.c.) multi-valued function $\varphi_\varepsilon \colon X \to \mathbf{I}$ such that each $\varphi_\varepsilon(x)$ is a closed interval, $\varphi \subset \varphi_\varepsilon$ and $\rho_{\mathrm{H}}(\varphi, \mathrm{cl}_{X \times \mathbf{I}}\, \varphi_\varepsilon) \leqslant \varepsilon$.*

Proof. For each $x \in X$, let

$$V_x = (\min \varphi(x) - \varepsilon,\ \max \varphi(x) + \varepsilon) \cap \mathbf{I}.$$

Since φ is u.s.c., we can choose $\delta_x > 0$ so that $\delta_x \leqslant \varepsilon$ and $\varphi(x') \subset V_x$ if $x' \in B(x, \delta_x)$ (i.e., $d(x, x') < \delta_x$). Let $\psi \colon X \to \mathbf{I}$ be the multi-valued function defined by

$$\psi(x) = \bigcup \{V_y \mid d(x, y) < \delta_y\} \quad \text{for each } x \in X.$$

We define the multi-valued function $\varphi_\varepsilon \colon X \to \mathbf{I}$ by $\varphi_\varepsilon(x) = \mathrm{cl}_{\mathbf{I}}\, \psi(x)$. Then $\varphi \subset \psi \subset \varphi_\varepsilon$. As is easily observed, $\rho_{\mathrm{H}}(\varphi, \mathrm{cl}_{X \times \mathbf{I}}\, \psi) \leqslant \varepsilon$. Since $\mathrm{cl}_{X \times \mathbf{I}}\, \varphi_\varepsilon = \mathrm{cl}_{X \times \mathbf{I}}\, \psi$, we have $\rho_{\mathrm{H}}(\varphi, \mathrm{cl}_{X \times \mathbf{I}}\, \varphi_\varepsilon) \leqslant \varepsilon$. If $d(x, y) < \delta_y$ then $\varphi(x) \subset V_y$. Since $\varphi(x)$ and V_y are connected, each $\psi(x)$ is connected, hence so is $\varphi_\varepsilon(x)$.

To see that φ_ε is l.s.c., let V be an open set in $\mathbf{I}$ and $x \in X$ such that $\varphi_\varepsilon(x) \cap V \neq \emptyset$. Then we have $t \in \psi(x) \cap V$. By the definition of ψ, we can find $y \in X$ such that

[1] As in [FK, Remark 3.6], although $C_B(X) \subset \mathrm{USCC}_B(X)$, the Banach space $C_B(X)$ is not a subspace of $\mathrm{USCC}_B(X)$ in case X is non-compact.

$d(x,y) < \delta_y$ and $t \in V_y$. If $d(x,x') < \delta_y - d(x,y)$ then $d(x',y) < \delta_y$, hence $V_y \subset \psi(x') \subset \varphi_\varepsilon(x')$ by the definition. Thus we have $t \in \varphi_\varepsilon(x') \cap V$. Therefore, $\varphi_\varepsilon \colon X \to \mathbf{I}$ is l.s.c. $\square$

Remark. In the above, $\varphi_\varepsilon \neq \mathrm{cl}_{X\times\mathbf{I}}\,\psi$. For example, let $\varphi = \mathbf{I} \times \{0\} \cup [\frac{1}{2}, 1] \times \mathbf{I} \in \mathrm{USCC}(\mathbf{I}, \mathbf{I})$ and $\varepsilon = \frac{1}{2}$. Then $V_x = [0, \frac{1}{2})$ for $x < \frac{1}{2}$ and $V_x = \mathbf{I}$ for $x \geqslant \frac{1}{2}$. Define ψ as above by using

$$\delta_x = \begin{cases} \frac{1}{2} - x & \text{if } x < \frac{1}{2}; \\ \frac{1}{2} & \text{if } x \geqslant \frac{1}{2}. \end{cases}$$

Observe that $d(0,y) < \delta_y$ implies $y < \frac{1}{2}$, and that if $x \neq 0$ then $d(x, \frac{1}{2}) < \delta_{\frac{1}{2}} = \frac{1}{2}$. Therefore, $\psi = \{0\} \times [0, \frac{1}{2}) \cup (0,1] \times \mathbf{I} = \mathbf{I}^2 \setminus \{0\} \times [\frac{1}{2}, 1]$, hence $\mathrm{cl}_{X\times\mathbf{I}}\,\psi = \mathbf{I}^2$. On the other hand, $\varphi_{\frac{1}{2}} = \{0\} \times [0, \frac{1}{2}] \cup (0,1] \times \mathbf{I}$ because $\varphi_{\frac{1}{2}}(x) = \mathrm{cl}_{\mathbf{I}}\,\psi(x)$ for each $x \in \mathbf{I}$.

Theorem 8.1.5. *The following are equivalent for any metric space $X = (X, d)$:*

(a) $\mathrm{C}(X, \mathbf{I})$ *is dense in* $\mathrm{USCC}(X, \mathbf{I})$*;*

(b) $\mathrm{C}_B(X)$ *is dense in* $\mathrm{USCC}_B(X)$*;*

(c) *X has no isolated points.*

Proof. (a) $\Rightarrow$ (b): This follows from the fact that each $\varphi \in \mathrm{USCC}_B(X)$ is contained in some $\mathrm{USCC}_B(X, [-a, a])$.

(b) $\Rightarrow$ (c): When X has an isolated point x_0, let $\varphi = X \times \{0\} \cup \{x_0\} \times \mathbf{I} \in \mathrm{USCC}_B(X)$. Then, as is easily observed,

$$\rho_{\mathrm{H}}(\varphi, f) \geqslant \min\left\{\tfrac{1}{2}, d(x_0, X \setminus \{x_0\})\right\} > 0 \quad \text{for any } f \in \mathrm{C}_B(X),$$

which implies that $\mathrm{C}_B(X)$ is not dense in $\mathrm{USCC}_B(X)$.

(c) $\Rightarrow$ (a): For each $\varphi \in \mathrm{USCC}(X, \mathbf{I})$ and $\varepsilon > 0$, let $\varphi_\varepsilon \colon X \to \mathbf{I}$ be the l.s.c. multi-valued function obtained by Lemma 8.1.4. Choose a discrete closed subset D of φ so that $\rho((x,t), D) < \varepsilon/2$ for any $(x,t) \in \varphi$, whence $\rho_{\mathrm{H}}(\varphi, D) < \varepsilon/2$. Note that $\mathrm{pr}_X | D$

is finite-to-one and $\mathrm{pr}_X(D)$ is discrete in X. Since φ_ε is l.s.c. and X has no isolated points, for each $(x,t) \in \varphi_\varepsilon$, there are infinitely many $y \in X$ such that

$$d(x,y) < \varepsilon/2 \quad \text{and} \quad \varphi_\varepsilon(y) \cap (t-\varepsilon/2, t+\varepsilon/2) \neq \emptyset.$$

Then we can construct a discrete closed subset f of φ_ε such that $\mathrm{pr}_X|f$ is injective and $\rho_{\mathrm{H}}(D,f) < \varepsilon/2$, hence $\rho_{\mathrm{H}}(\varphi,f) < \varepsilon$. Then $A = \mathrm{pr}_X(f)$ is discrete in X and $f\colon A \to \mathbf{I}$ is a map[2] which is a selection for $\varphi_\varepsilon|A$ (i.e., $f(x) \in \varphi_\varepsilon(x)$ for each $x \in A$). By the Michael's Selection Theorem [Mi], we can extend f to $\tilde{f} \in \mathrm{C}(X,\mathbf{I})$ which is a selection for φ_ε. For any $(x,t) \in \varphi$, $\rho((x,t),\tilde{f}) \leqslant \rho((x,t),f) \leqslant \rho_{\mathrm{H}}(\varphi,f) < \varepsilon$. Since $\tilde{f} \subset \mathrm{cl}_{X\times\mathbf{I}}\,\varphi_\varepsilon$ and $\rho_{\mathrm{H}}(\varphi, \mathrm{cl}_{X\times\mathbf{I}}\,\varphi_\varepsilon) \leqslant \varepsilon$, it follows that $\rho((x,t),\varphi) \leqslant \varepsilon$ for any $(x,t) \in \tilde{f}$. Thus we have $\rho_{\mathrm{H}}(\tilde{f},\varphi) \leqslant \varepsilon$. Consequently, $\varphi \in \mathrm{cl}_{2^{X\times\mathbf{I}}}\,\mathrm{C}(X,\mathbf{I})$. □

Combining Theorem 8.1.5 with Proposition 8.1.2, we have the following corollary:

Corollary 8.1.6. *For any locally connected metric space X with no isolated points, $\mathrm{USCC}_B(X)$ (resp. $\mathrm{USCC}(X,\mathbf{I})$) is the closure of $\mathrm{C}_B(X)$ (resp. $\mathrm{C}(X,\mathbf{I})$) in $2^{X\times\mathbb{R}}$ (resp. $2^{X\times\mathbf{I}}$).* □

One should notice that any completeness is not assumed in the above (cf. [FK, Theorem 3.3(a)]).

§8.2. The AR-property of $\mathrm{USCC}_B(X)$ and $\mathrm{USCC}(X,\mathbf{I})$

In this section, using Borges' characterization of AR's in [Bo], we prove that $\mathrm{USCC}_B(X)$ and $\mathrm{USCC}(X,\mathbf{I})$ are AR's if $X = (X,d)$ is uniformly locally connected.

Now, we define a new metric d_c on X as follows:

$$d_c(x,x') = \begin{cases} \inf\{\, \mathrm{diam}_d\, C \mid C \in \mathcal{C}(x,x')\} & \text{if } \mathcal{C}(x,x') \neq \emptyset; \\ 1 & \text{otherwise,} \end{cases}$$

[2] Recall a map is identified with its graph.

where

$$\mathcal{C}(x,x') = \{C \subset X \mid C \text{ is connected, } x,x' \in C \text{ and } \operatorname{diam} C < 1\}.$$

As is easily observed, if X is uniformly locally connected, then d_c is uniformly equivalent to d, hence d_c induces the same topology of $2^{X\times\mathbb{R}}$ as d. Then, by replacing d by d_c, we can assume that

($\star$) *each pair of points $x, x' \in X$ with $d(x,x') < \varepsilon < 1$ are contained in a connected set C in X with* $\operatorname{diam} C < \varepsilon$.

Lemma 8.2.1. *The condition ($\star$) implies the following condition:*

($\sharp$) *$N_\rho(\varphi,\varepsilon)(x)$ is connected for each $\varphi \in \mathrm{USCC}_B(X)$, $0 < \varepsilon < 1$ and $x \in X$.*

Proof. Let $t_1 < t_2 \in N_\rho(\varphi,\varepsilon)(x)$ and $t_1 < t < t_2$. Then we have $x_1, x_2 \in X$ and $s_i \in \varphi(x_i)$ $(i = 1,2)$ such that $d(x_i,x) < \varepsilon$ and $|s_i - t_i| < \varepsilon$. Let

$$s = \frac{t_2 - t}{t_2 - t_1} s_1 + \frac{t - t_1}{t_2 - t_1} s_2.$$

By ($\star$), X has connected sets C_1 and C_2 such that $x_i, x \in C_i$ and $\operatorname{diam} C_i < \varepsilon$. Since $C = C_1 \cup C_2$ is connected, $s \in \varphi(x_0)$ for some $x_0 \in C$ by Lemma 8.1.1. Then $d(x_0,x) < \varepsilon$. Observe

$$t = \frac{t_2 - t}{t_2 - t_1} t_1 + \frac{t - t_1}{t_2 - t_1} t_2.$$

Then, it follows that

$$|s - t| \leqslant \frac{t_2 - t}{t_2 - t_1}|s_1 - t_1| + \frac{t - t_1}{t_2 - t_1}|s_2 - t_2| < \varepsilon.$$

Hence, $(x,t) \in N_\rho(\varphi,\varepsilon)$, i.e., $t \in N_\rho(\varphi,\varepsilon)(x)$. Therefore, $N_\rho(\varphi,\varepsilon)(x)$ is connected. □

By Δ^{n-1}, we denote the standard $(n-1)$-simplex in $\mathbb{R}^n$, that is,

$$\Delta^{n-1} = \{(t_1,\dots,t_n) \in \mathbb{R}^n \mid t_i \geqslant 0,\ \textstyle\sum_{i=1}^n t_i = 1\}.$$

It is said that a space Y is *hyper-connected* if there are functions $h_n\colon Y^n \times \Delta^{n-1} \to Y$ ($n \in \mathbb{N}$) which satisfy the following conditions:

(i) if $t_i = 0$ then

$$h_n(y_1, \ldots, y_n; t_1, \ldots, t_n)$$
$$= h_{n-1}(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n; t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n);$$

(ii) $\Delta^{n-1} \ni (t_1, \ldots, t_n) \mapsto h_n(y_1, \ldots, y_n; t_1, \ldots, t_n) \in Y$ is continuous for each $(y_1, \ldots, y_n) \in Y^n$;

(iii) each neighborhood U of $y \in Y$ contains a neighborhood V of y such that $h_n(V^n \times \Delta^{n-1}) \subset U$ for every $n \in \mathbb{N}$.

It should be noticed that each h_n need not be continuous. It is proved by C.R. Borges [Bo] that a metrizable space X is an AR if and only if X is hyper-connected.[3] We apply this characterization to prove the following:

Theorem 8.2.2. *For any uniformly locally connected metric space $X = (X, d)$, $\mathrm{USCC}_B(X)$ and $\mathrm{USCC}(X, \mathbf{I})$ are AR's.*

Proof. Since $\mathrm{USCC}(X, \mathbf{I})$ is a retract of $\mathrm{USCC}_B(X)$, it is sufficient to prove that $\mathrm{USCC}_B(X)$ is an AR.

By replacing the metric d by d_c, we can assume the condition ($\star$). Each point of $\Delta^{n-1} \setminus \{b_{n-1}\}$ can be uniquely represented as follows:

$$(1-t)b_{n-1} + z,\ z \in \partial\Delta^{n-1},\ 0 < t \leqslant 1,$$

where b_{n-1} is the barycenter of Δ^{n-1} and $\partial\Delta^{n-1}$ is the boundary of Δ^{n-1}. We shall inductively define $h_n\colon \mathrm{USCC}_B(X)^n \times \Delta^{n-1} \to \mathrm{USCC}_B(X)$ ($n \in \mathbb{N}$). First, let

[3] R. Cauty [Ca] introduced the local hyper-connectedness deferent from the one of [Bo] and showed that a metrizable space X is an ANR if and only if X is locally hyper-connected. The results of [Bo] and [Ca] hold for stratifiable spaces.

$h_1(\varphi, 1) = \varphi$ for every $\varphi \in \mathrm{USCC}_B(X)$. Assume that $h_1, \ldots, h_{n-1}$ have been defined, and define h_n as follows:

$$h_n(\varphi_1, \ldots, \varphi_n; b_{n-1})(x) = \left[\min \bigcup_{i=1}^n \varphi_i(x),\ \max \bigcup_{i=1}^n \varphi_i(x)\right]$$

and, for $z \in \partial\Delta^{n-1}$ and $0 < t \leqslant 1$,

$$\begin{aligned} &h_n(\varphi_1, \ldots, \varphi_n; (1-t)b_{n-1} + tz)(x) \\ &\qquad = (1-t)h_n(\varphi_1, \ldots, \varphi_n; b_{n-1})(x) + th_n(\varphi_1, \ldots, \varphi_n; z)(x), \end{aligned}$$

where $h_n(\varphi_1, \ldots, \varphi_n; z)$ is defined by the condition (i). Then the conditions (i) is clearly satisfied. Similarly to Lemma 6.3.1, the condition (ii) follows. We show that

$$h_n\left(B_{\rho_{\mathrm{H}}}(\varphi, \varepsilon)^n \times \Delta^{n-1}\right) \subset B_{\rho_{\mathrm{H}}}(\varphi, \varepsilon)$$

for each $\varphi \in \mathrm{USCC}_B(X)$ and $0 < \varepsilon < 1$. For $\varphi_1, \ldots, \varphi_n \in B_{\rho_{\mathrm{H}}}(\varphi, \varepsilon)$ and $z \in \Delta^{n-1}$, since $\varphi_1, \ldots, \varphi_n \subset N_\rho(\varphi, \varepsilon)$, it follows from Lemma 8.2.1 and the definition of h_n that

$$h_n(\varphi_1, \ldots, \varphi_n; z) \subset h_n(\varphi_1, \ldots, \varphi_n; b_{n-1}) \subset N_\rho(\varphi, \varepsilon).$$

On the other hand, since $h_n(\varphi_1, \ldots, \varphi_n; z)$ contains some φ_i and $\varphi \subset N_\rho(\varphi_i, \varepsilon)$, we have $\varphi \subset N_\rho(h_n(\varphi_1, \ldots, \varphi_n; z), \varepsilon)$. Therefore,

$$\rho_{\mathrm{H}}(h_n(\varphi_1, \ldots, \varphi_n; z), \varphi) < \varepsilon \quad \left(\text{i.e., } h_n(\varphi_1, \ldots, \varphi_n; z) \in B_{\rho_{\mathrm{H}}}(\varphi, \varepsilon)\right).$$

Thus the condition (iii) also holds. Consequently, $\mathrm{USCC}_B(X)$ is hyper-connected, hence it is an AR. □

§8.3. Multi-valued function spaces homeomorphic to the Hilbert spaces

To prove the Main Theorem, we use the following variant of Toruńczyk's characterization of Hilbert space [To$_6$] (Theorem 1.3.3):

Lemma 8.3.1. *Let A be a discrete space and $H = (H, d)$ a complete AR with weight $w(H) = \operatorname{card} A$. Then $H \approx \ell_2(A)$ if and only if the following condition is satisfied:*

(∗) *for any open cover $\mathcal{U}$ of H, there exists a map $f\colon H \times A \to H$ such that $\{f_a(H) \mid a \in A\}$ is discrete in H and each f_a is $\mathcal{U}$-close to* id,

where $f_a\colon H \to H$ is defined $f_a(x) = f(x, a)$.

Proof. Obviously, the condition (∗) implies the conditions (1) and (2) in Theorem 1.3.3, hence we have the "if" part. The "only if" part easily follows from the fact that the projection $\mathrm{pr}_1\colon H \times H \to H$ onto the first factor is a near homeomorphism (cf. [Sc]). □

Lemma 8.3.2. *Assume the condition ($\star$) in §8.2 and that X has no isolated points and there exist $D \subset X$ and $\delta, \varepsilon \in (0,1)$ such that $d(a, a') \geqslant \varepsilon$ for $a \neq a' \in D$ and each $a \in D$ has a connected neighborhood with diameter $> \delta$. Then, for any open cover $\mathcal{U}$ of $\mathrm{USCC}_B(X)$, there exists a map $h\colon \mathrm{USCC}_B(X) \times 2^D \to \mathrm{USCC}_B(X)$ such that $\{h_F(\mathrm{USCC}_B(X)) \mid F \in 2^D\}$ is discrete in $\mathrm{USCC}_B(X)$ and each h_F is $\mathcal{U}$-close to* id, *where $h_F\colon \mathrm{USCC}_B(X) \to \mathrm{USCC}_B(X)$ is defined by $h_F(\varphi) = h(\varphi, F)$.*

Proof. Let $\mathcal{V}$ be an open star-refinement of $\mathcal{U}$. Since $\mathrm{USCC}_B(X)$ is an AR (Theorem 8.2.2), we have a simplicial complex K with maps

$$p\colon \mathrm{USCC}_B(X) \to |K| \quad \text{and} \quad q\colon |K| \to \mathrm{USCC}_B(X)$$

such that qp is $\mathcal{V}$-close to id. Let $\alpha\colon \mathrm{USCC}_B(X) \to (0,1)$ be a map such that $\alpha(\varphi) <$

$\min\{\delta, \varepsilon\}$ for each $\varphi \in \mathrm{USCC}_B(X)$ and

$$\{\overline{B}_{\rho_{\mathrm{H}}}(\varphi, 2\alpha(\varphi)) \mid \varphi \in \mathrm{USCC}_B(X)\} \prec \mathcal{V}.$$

By subdividing K, we can assume the following two conditions:

(1) $\operatorname{diam}_{\rho_{\mathrm{H}}} q(\sigma) < \frac{1}{8}\alpha q(y)$ if $y \in \sigma \in K$;

(2) $\alpha q(y) < 2\alpha q(y')$ if $y, y' \in \sigma \in K$.

In fact, for each $\varphi \in \mathrm{USCC}_B(X)$, let

$$W_\varphi = B_{\rho_{\mathrm{H}}}(\varphi, \tfrac{1}{24}\alpha(\varphi)) \cap \{\psi \in \mathrm{USCC}_B(X) \mid \tfrac{2}{3}\alpha(\varphi) < \alpha(\psi) < \tfrac{4}{3}\alpha(\varphi)\},$$

and subdivide K so that each simplex is contained in some $q^{-1}(W_\varphi)$.

For each $v \in K^{(0)}$, we define $f(v) \in \mathrm{USCC}_B(X)$ as follows:

$$f(v) = q(v) \cup \bigcup_{a \in D} \overline{B}(a, \tfrac{1}{8}\alpha q(v)) \times [b(v,a), t(v,a)],$$

where

$$t(v,a) = \sup q(v)(\overline{B}(a, \tfrac{1}{8}\alpha q(v)));$$
$$b(v,a) = \inf q(v)(\overline{B}(a, \tfrac{1}{8}\alpha q(v))).$$

Obviously $\rho_{\mathrm{H}}(f(v), q(v)) \leqslant \frac{1}{8}\alpha q(v)$. If u and v are vertices of the same simplex of K, then

$$\begin{aligned}\rho_{\mathrm{H}}(f(u), f(v)) &\leqslant \rho_{\mathrm{H}}(f(u), q(u)) + \rho_{\mathrm{H}}(q(u), q(v)) + \rho_{\mathrm{H}}(f(v), q(v)) \\ &< \tfrac{1}{8}\alpha q(u) + \tfrac{1}{8}\alpha q(v) + \tfrac{1}{8}\alpha q(v) \\ &< \tfrac{1}{4}\alpha q(v) + \tfrac{1}{4}\alpha q(v) = \tfrac{1}{2}\alpha q(v).\end{aligned}$$

For the varycenter $\hat{\sigma}$ of each $\sigma \in K$, we define $f(\hat{\sigma}) \in \mathrm{USCC}_B(X)$ by

$$f(\hat{\sigma})(x) = \big[\min \textstyle\bigcup_{v \in \sigma^{(0)}} f(v)(x),\ \max \bigcup_{v \in \sigma^{(0)}} f(v)(x)\big].$$

Then, by Lemma 8.2.1, $f(\hat{\sigma}) \subset N_\rho(f(v), \frac{1}{2}\alpha q(v))$ for each $v \in \sigma^{(0)}$. Observe that if $0 < r \leqslant \min_{v\in\sigma^{(0)}} \frac{1}{8}\alpha q(v)$, then

$$f(\hat{\sigma})|\overline{B}(a,r) = \overline{B}(a,r) \times [b(\hat{\sigma},a), t(\hat{\sigma},a)] \quad \text{for each } \ a \in D,$$

where $b(\hat{\sigma},a) = \min_{v\in\sigma^{(0)}} b(v,a)$ and $t(\hat{\sigma},a) = \max_{v\in\sigma^{(0)}} t(v,a)$.

We define a map $f\colon |K| \to \mathrm{USCC}_B(X)$ as follows:

$$\begin{aligned} f(y)(x) &= \sum_{i=1}^{k} s_i f(\hat{\sigma}_i)(x) \\ &= \big[\textstyle\sum_{i=1}^k s_i \min f(\hat{\sigma}_i)(x),\ \sum_{i=1}^k s_i \max f(\hat{\sigma}_i)(x)\big], \end{aligned}$$

where $y = \sum_{i=1}^k s_i\hat{\sigma}_i$, $\sigma_1 < \cdots < \sigma_k \in K$, $s_i \leqslant 0$ and $\sum_{i=1}^k s_i = 1$. In the above, note that $\frac{1}{2}\alpha q(y) < \alpha q(v)$ for each $v \in \sigma_k^{(0)}$. Then, for each $a \in D$,

$$f(y)|\overline{B}(a, \tfrac{1}{16}\alpha q(y)) = \overline{B}(a, \tfrac{1}{16}\alpha q(y)) \times [\min f(y)(a), \max f(y)(a)].$$

For each $y \in |K|$, choose $v \in \sigma^{(0)}$ so that $y \in |\operatorname{St}(v, \operatorname{Sd} K)|$. Since $f(v) \subset f(y) \subset f(\hat{\sigma}) \subset N_\rho(f(v), \frac{1}{2}\alpha q(v))$, we have $\rho_{\mathrm{H}}(f(y), f(v)) < \frac{1}{2}\alpha q(v)$, hence

$$\begin{aligned} \rho_{\mathrm{H}}(f(y), q(y)) &\leqslant \rho_{\mathrm{H}}(f(y), f(v)) + \rho_{\mathrm{H}}(f(v), q(v)) + \rho_{\mathrm{H}}(q(v), q(y)) \\ &< \tfrac{1}{2}\alpha q(v) + \tfrac{1}{8}\alpha q(v) + \tfrac{1}{8}\alpha q(v) \\ &< \tfrac{3}{4}\alpha q(v) < \tfrac{3}{2}\alpha q(y). \end{aligned}$$

Now, for any $F \in 2^D$, we define $h_F\colon \mathrm{USCC}_B(X) \to \mathrm{USCC}_B(X)$ by

$$h_F(\varphi) = fp(\varphi) \cup \bigcup_{a\in F} \{a\} \times [\max fp(\varphi)(a), \max fp(\varphi)(a) + \tfrac{1}{2}\alpha qp(\varphi)].$$

Then h_F is $\mathcal{U}$-close to id. In fact, h_F is $\mathcal{V}$-close to qp because

$$\begin{aligned} \rho_{\mathrm{H}}(h_F(\varphi), qp(\varphi)) &\leqslant \rho_{\mathrm{H}}(h_F(\varphi), fp(\varphi)) + \rho_{\mathrm{H}}(fp(\varphi), qp(\varphi)) \\ &< \tfrac{1}{2}\alpha qp(\varphi) + \tfrac{3}{2}\alpha qp(\varphi) = 2\alpha qp(\varphi). \end{aligned}$$

To see the continuity of h_F, let $\varphi_n \to \varphi$ in $\mathrm{USCC}_B(X)$ as $n \to \infty$. Since fp and αqp are continuous, $fp(\varphi_n) \to fp(\varphi)$ and $\alpha qp(\varphi_n) \to \alpha qp(\varphi)$. Let $0 < r < \frac{1}{16}\alpha qp(\varphi)$. Then, $r < \frac{1}{16}\alpha qp(\varphi_n)$ for sufficiently large n, whence for each $a \in F$,

$$fp(\varphi_n)|\overline{B}(a,r) = \overline{B}(a,r) \times [\min fp(\varphi_n)(a), \max fp(\varphi_n)(a)] \quad \text{and}$$
$$fp(\varphi)|\overline{B}(a,r) = \overline{B}(a,r) \times [\min fp(\varphi)(a), \max fp(\varphi)(a)].$$

Hence, $\max fp(\varphi_n)(a) \to \max fp(\varphi)(a)$ for each $a \in F$. By the definition, it follows that $h_F(\varphi_n) \to h_F(\varphi)$.

We show that $\{h_F(\mathrm{USCC}_B(X)) \mid F \in 2^D\}$ is discrete in $\mathrm{USCC}_B(X)$. Suppose contrary that there exist $\varphi, \varphi_i \in \mathrm{USCC}_B(X)$ and $F_i \in 2^D$ $(i \in \mathbb{N})$ such that $h_{F_i}(\varphi_i) \to \varphi$ as $i \to \infty$ and $F_i \neq F_j$ if $i \neq j$. Then $\inf_{i\in\mathbb{N}} \alpha qp(\varphi_i) > 0$. Otherwise, $\lim_{n\to\infty} \alpha qp(\varphi_{i_n}) \to 0$ for some $i_1 < i_2 < \cdots$. As saw in the above, $\rho_{\mathrm{H}}(h_{F_{i_n}}(\varphi_{i_n}), qp(\varphi_{i_n})) < 2\alpha qp(\varphi_{i_n})$. Then it follows that $qp(\varphi_{i_n}) \to \varphi$, hence $\alpha qp(\varphi) = \lim_{n\to\infty} \alpha qp(\varphi_{i_n}) = 0$, which is a contradiction.

Let $\varepsilon_0 = \inf_{i\in\mathbb{N}} \frac{1}{16}\alpha qp(\varphi_i) > 0$. For any two distinct $i \neq j \in \mathbb{N}$, there exists $a \in D$ such that $a \in F_i \setminus F_j$ or $a \in F_j \setminus F_i$. Without loss of generality, we may assume that $a \in F_j \setminus F_i$. For simplicity, we denote $b_i = b(p(\varphi_i), a)$, $t_i = t(p(\varphi_i), a)$, $b_j = b(p(\varphi_j), a)$ and $t_j = t(p(\varphi_j), a)$. Then,

$$h_{F_i}(\varphi_i)|\overline{B}(a,\varepsilon_0) = \overline{B}(a,\varepsilon_0) \times [b_i, t_i] \quad \text{and}$$
$$h_{F_j}(\varphi_j)|\overline{B}(a,\varepsilon_0) = \overline{B}(a,\varepsilon_0) \times [b_j, t_j] \cup \{a\} \times [t_j, t_j + \alpha qp(\varphi_j)].$$

In case $t_i \leqslant t_j + \frac{1}{2}\alpha qp(\varphi_j)$, we have

$$\rho_{\mathrm{H}}(h_{F_i}, h_{F_j}) \geqslant \rho((a, t_j + \alpha qp(\varphi_j)), h_{F_i}) \geqslant \min\{\varepsilon_0, \tfrac{1}{2}\alpha qp(\varphi_j)\} = \varepsilon_0.$$

Recall that a has a connected neighborhood with diameter $> \delta$. Since $\varepsilon_0 < \frac{1}{16}\delta$, we have $c \in X$ so that $d(a,c) = \varepsilon_0/2$. In case $t_i \geqslant t_j + \frac{1}{2}\alpha qp(\varphi_j)$, it follows that

$$\rho_{\mathrm{H}}(h_{F_i}, h_{F_j}) \geqslant \rho((c, t_i), h_{F_j}) \geqslant \min\{\varepsilon_0/2, \tfrac{1}{2}\alpha qp(\varphi_j)\} = \varepsilon_0/2.$$

Consequently, $\rho_{\mathrm{H}}(h_{F_i}(\varphi_i), h_{F_j}(\varphi_j)) \geqslant \varepsilon_0/2$ if $i \neq j$, whence $h_{F_i}(\varphi_i)$ is not convergent. This is a contradiction. The proof is completed. □

Lemma 8.3.3. *Assume that X is not totally bounded. For each $n \in \mathbb{N}$, let D_n be a maximal subset of X such that $d(x,y) \geqslant 2^{-n}$ for any distinct points $x, y \in D_n$.*[4] *Then, $w(\mathrm{USCC}_B(X)) = \sup_{n\in\mathbb{N}} 2^{\operatorname{card} D_n}$. In case X is separable, $w(\mathrm{USCC}_B(X)) = 2^{\aleph_0}$.*[5]

Proof. For each $n \in \mathbb{N}$, let $\mathbb{Q}_n = \{2^{-n}m \mid m \in \mathbb{N}\} \subset \mathbb{R}$. Then $D_n \times \mathbb{Q}_n$ is discrete in $X \times \mathbb{R}$. Since X is not totally bounded, each D_n is infinite, hence $\operatorname{card}(D_n \times \mathbb{Q}_n) = \operatorname{card} D_n$. By the maximality, $d(x, D_n) < 2^{-n}$ for every $x \in X$, hence $\rho(z, D_n \times \mathbb{Q}_n) < 2^{-n}$ for every $z \in X \times \mathbb{R}$. For each $E \in 2^{X\times\mathbb{R}}$ and $n \in \mathbb{N}$, let

$$F = \left\{z \in D_n \times \mathbb{Q}_n \mid \rho(z, E) < 2^{-n}\right\} \in 2^{D_n \times \mathbb{Q}_n} \subset 2^{X\times\mathbb{R}}.$$

Then $\rho_{\mathrm{H}}(E, F) \leqslant 2^{-n}$. Hence, $\bigcup_{n\in\mathbb{N}} 2^{D_n\times\mathbb{Q}_n}$ is dense in $2^{X\times\mathbb{R}}$. Since the weight $w(2^{X\times\mathbb{R}})$ is equal to the density of $2^{X\times\mathbb{R}}$, it follows that

$$\begin{aligned} w(2^{X\times\mathbb{R}}) &\leqslant \operatorname{card} \textstyle\bigcup_{n\in\mathbb{N}} 2^{D_n\times\mathbb{Q}_n} \\ &\leqslant \sup_{n\in\mathbb{N}} \operatorname{card} 2^{D_n\times\mathbb{Q}_n} = \sup_{n\in\mathbb{N}} 2^{\operatorname{card}(D_n\times\mathbb{Q}_n)} = \sup_{n\in\mathbb{N}} 2^{\operatorname{card} D_n}, \end{aligned}$$

which implies $w(\mathrm{USCC}_B(X)) \leqslant \sup_{n\in\mathbb{N}} 2^{\operatorname{card} D_n}$.

On the other hand, for each $n \in \mathbb{N}$ and $F \in 2^{D_n}$, let

$$\varphi_F = F \times \mathbf{I} \cup X \times \{0\} \in \mathrm{USCC}_B(X).$$

Since $\rho_{\mathrm{H}}(\varphi_F, \varphi_{F'}) \geqslant 2^{-n}$ for each $F \neq F' \in 2^{D_n}$, $\{B_{\rho_{\mathrm{H}}}(\varphi_F, 2^{-n-1}) \mid F \in 2^{D_n}\}$ is pairwise disjoint. Therefore,

$$w(\mathrm{USCC}_B(X)) \geqslant \operatorname{card} 2^{D_n} = 2^{\operatorname{card} D_n},$$

[4]The existence of such $D_n \subset X$ is guaranteed by Zorn's Lemma.
[5]In general, $\sup_{n\in\mathbb{N}} 2^{\operatorname{card} D_n} \neq 2^{\sup_{n\in\mathbb{N}} \operatorname{card} D_n} = 2^{w(X)}$.

hence $w(\mathrm{USCC}_B(X)) \geqslant \sup_{n\in\mathbb{N}} 2^{\mathrm{card}\, D_n}$. $\square$

Proof of Main Theorem. We apply Lemma 8.3.1 to show that $\mathrm{USCC}_B(X) \approx \ell_2(A)$, where $\mathrm{card}\, A = w(\mathrm{USCC}_B(X))$. We have proved that $\mathrm{USCC}_B(X)$ is a completely metrizable AR (Corollary 8.1.3 and Theorem 8.2.2). It remains to construct a map $f\colon \mathrm{USCC}_B(X) \times A \to \mathrm{USCC}_B(X)$ such as in Lemma 8.3.1. Let $\mathfrak{C}$ be the collection of all components of X and take D_n ($n \in \mathbb{N}$) as in Lemma 8.3.3. Then, observe that

$$\mathrm{card}\, \mathfrak{C} \leqslant w(X) = \mathrm{card} \textstyle\bigcup_{n\in\mathbb{N}} D_n = \sup_{n\in\mathbb{N}} \mathrm{card}\, D_n.$$

(1) *The case* $\mathrm{card}\, \mathfrak{C} = w(X)$: Since X is uniformly locally connected, $\mathrm{card}\, D_n \geqslant \mathrm{card}\, \mathfrak{C} = w(X)$ for sufficiently large $n \in \mathbb{N}$. On the other hand, $\mathrm{card}\, D_n \leqslant w(X)$ for all $n \in \mathbb{N}$ by the definition. Then we have $\sup_{n\in\mathbb{N}} 2^{\mathrm{card}\, D_n} = 2^{w(X)}$, hence $\mathrm{card}\, A = w(\mathrm{USCC}_B(X)) = 2^{w(X)}$ by Lemma 8.3.3.

We can write $\mathfrak{C} = \bigcup_{i\in\mathbb{N}} \mathfrak{C}_i$, where $\mathfrak{C}_i \cap \mathfrak{C}_j = \emptyset$ if $i \neq j$ and $\mathrm{card}\, \mathfrak{C}_i = w(X)$ for each $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, let $r_i\colon \mathrm{USCC}_B(X) \to m(\mathfrak{C}_i)$ be the map defined by $r_i(\varphi)(C) = \sup \varphi(C)$ ($\leqslant \varphi(X)$) for each $C \in \mathfrak{C}_i$. Since $m(\mathfrak{C}_i) \approx \ell_2(2^{\mathfrak{C}_i})$ ([BP, Ch.VII, Theorem 6.1]) and $w(\mathrm{USCC}_B(X) \times A) = 2^{w(X)} = \mathrm{card}\, 2^{\mathfrak{C}_i}$, we have a closed embedding $g_i\colon \mathrm{USCC}_B(X) \times A \to m(\mathfrak{C}_i)$ such that $\|g_i(\varphi, a) - r_i(\varphi)\| < 2^{-i}$ for each $(\varphi, a) \in \mathrm{USCC}_B(X) \times A$. It should be noted that $\{(g_i)_a(\mathrm{USCC}_B(X)) \mid a \in A\}$ is discrete in $\mathrm{USCC}_B(X)$.

For any open cover $\mathcal{U}$ of $\mathrm{USCC}_B(X)$, let $\alpha\colon \mathrm{USCC}_B(X) \to (0,1)$ be a map such that $\{B_{\rho_{\mathrm{H}}}(\varphi, \alpha(\varphi)) \mid \varphi \in \mathrm{USCC}_B(X)\} \prec \mathcal{U}$. Now, we define a map $f\colon \mathrm{USCC}_B(X) \times A \to \mathrm{USCC}_B(X)$ as follows:

$$f(\varphi,a)(x) = \begin{cases} \varphi(x) + g_i(\varphi,a)(C) - r_i(\varphi)(C) & \\ \qquad \text{for } x \in C \in \mathfrak{C}_i \text{ and } 2^{-i+1} < \alpha(\varphi); & \\ \varphi(x) + 2^i(\alpha(\varphi) - 2^{-i})(g_i(\varphi,a)(C) - r_i(\varphi)(C)) & \\ \qquad \text{for } x \in C \in \mathfrak{C}_i \text{ and } 2^{-i} \leqslant \alpha(\varphi) \leqslant 2^{-i+1}; & \\ \varphi(x) \quad \text{otherwise.} & \end{cases}$$

Then f_a is $\mathcal{U}$-close to id. In fact, for every $C \in \mathfrak{C}_i$,

$$|g_i(\varphi, a)(C) - r_i(\varphi)(C)| \leqslant \|g_i(\varphi, a) - r_i(\varphi)\| < 2^{-i},$$

hence $\rho_{\mathrm{H}}(f_a(\varphi), \varphi) < \alpha(\varphi)$.

We prove that $\{f_a(\mathrm{USCC}_B(X)) \mid a \in A\}$ is discrete in $\mathrm{USCC}_B(X)$. On the contrary, assume that there is a sequence $(\varphi_k, a_k) \in \mathrm{USCC}_B(X) \times A$ $(k \in \mathbb{N})$ such that $a_k \neq a_{k'}$ if $k \neq k'$, and $f_{a_k}(\varphi_k)$ converges to some $\varphi_0 \in \mathrm{USCC}_B(X)$. Then there is some $i_0 \in \mathbb{N}$ such that $2^{-i_0+1} < \alpha(\varphi_k)$ for all $k \in \mathbb{N}$. Otherwise, $\lim_{j\to\infty} \alpha(\varphi_{k(j)}) = 0$ for some $k(1) < k(2) < \cdots$, whence $\lim_{j\to\infty} \rho_{\mathrm{H}}(f_{a_{k(j)}}(\varphi_{k(j)}), \varphi_{k(j)}) = 0$. It follows that $\varphi_{k(j)}$ converges to φ_0, so $\alpha(\varphi_0) = \lim_{j\to\infty} \alpha(\varphi_{k(j)}) = 0$, which is a contradiction. For each $C \in \mathfrak{C}_{i_0}$,

$$\begin{aligned} r_{i_0}(f_{a_k}(\varphi_k))(C) &= \sup f(\varphi_k, a_k)(C) \\ &= \sup \varphi_k(C) + g_{i_0}(\varphi_k, a_k)(C) - r_{i_0}(\varphi_k)(C) \\ &= g_{i_0}(\varphi_k, a_k)(C) = (g_{i_0})_{a_k}(\varphi_k). \end{aligned}$$

Since r_{i_0} is continuous, $(g_{i_0})_{a_k}(\varphi_k) = r_{i_0}(f_{a_k}(\varphi_k))$ converges to $r_{i_0}(\varphi_0)$, which contradicts to the fact that $\{(g_{i_0})_a(\mathrm{USCC}_B(X)) \mid a \in A\}$ is discrete in $\mathrm{USCC}_B(X)$. Therefore, $\{f_a(\mathrm{USCC}_B(X)) \mid a \in A\}$ is discrete in $\mathrm{USCC}_B(X)$.

(2) *The case* $\operatorname{card} \mathfrak{C} < w(X)$: Since X is uniformly locally connected, we may assume the condition $(\star)$ in §8.2. Let X_0 be the set of isolated points of X. Then $d(x, X \setminus \{x\}) \geqslant 1$ for every $x \in X_0$ by $(\star)$. As is easily seen,

$$\mathrm{USCC}_B(X) \approx \mathrm{USCC}_B(X_0) \times \mathrm{USCC}(X \setminus X_0).$$

For each $n \in \mathbb{N}$, let $D'_n = D_n \setminus X_0$. Since $\operatorname{card} X_0 \leqslant \operatorname{card} \mathfrak{C} < w(X) = \sup_{n\in\mathbb{N}} \operatorname{card} D_n$, we have $\operatorname{card} X_0 < \operatorname{card} D_n$ for sufficiently large $n \in \mathbb{N}$, whence $\operatorname{card} D'_n = \operatorname{card} D_n$. By Lemma 8.3.3, we have

$$w(\mathrm{USCC}_B(X \setminus X_0)) = \sup_{n\in\mathbb{N}} 2^{\operatorname{card} D'_n} = \sup_{n\in\mathbb{N}} 2^{\operatorname{card} D_n} = w(\mathrm{USCC}_B(X)).$$

In the case (1) above, we have shown that $\mathrm{USCC}_B(X_0)$ is homeomorphic to a Hilbert space, hence it is a completely metrizable AR with $w(\mathrm{USCC}_B(X_0)) \leqslant w(\mathrm{USCC}_B(X))$. By [To$_2$, Theorem 3.1], it suffices to show that $\mathrm{USCC}_B(X \setminus X_0)$ is homeomorphic to a Hilbert space with the same weight. Thus we can assume that X has no isolated points.

For each $\delta > 0$, let $\mathfrak{C}(\delta) = \{C \in \mathfrak{C} \mid \operatorname{diam} C < \delta\}$. Let

$$D_n^1 = D_n \setminus \bigcup \mathfrak{C}(2^{-n}) \quad \text{for each } n \in \mathbb{N}.$$

It should be noted that each point of D_n^1 has a connected neighborhood in X with $\operatorname{diam} \geqslant 2^{-n}$ because it is contained in a component of X with $\operatorname{diam} \geqslant 2^{-n}$. Each member of $\mathfrak{C}(2^{-n})$ contains at most one point of D_n. Recall that $\operatorname{card} \mathfrak{C} < w(X) = \sup_{n \in \mathbb{N}} \operatorname{card} D_n$. Then, for sufficiently large $n \in \mathbb{N}$,

$$\operatorname{card}\left(D_n \cap \bigcup \mathfrak{C}(2^{-n})\right) \leqslant \operatorname{card} \mathfrak{C}(2^{-n}) \leqslant \operatorname{card} \mathfrak{C} < \operatorname{card} D_n,$$

whence we have $\operatorname{card} D_n = \operatorname{card} D_n^1$. Therefore, it follows from Lemma 8.3.3 that

$$\begin{aligned} \operatorname{card}\left(\textstyle\bigcup_{n \in \mathbb{N}} \{n\} \times 2^{D_n^1}\right) &= \sup_{n \in \mathbb{N}} \operatorname{card} 2^{D_n^1} = \sup_{n \in \mathbb{N}} 2^{\operatorname{card} D_n^1} \\ &= \sup_{n \in \mathbb{N}} 2^{\operatorname{card} D_n} = w(\mathrm{USCC}_B(X)). \end{aligned}$$

Thus we may assume that

$$A = \bigcup_{n \in \mathbb{N}} \{n\} \times 2^{D_n^1}.$$

For any open cover $\mathcal{U}$ of $\mathrm{USCC}_B(X)$, let $\mathcal{V}$ be an open star-refinement of $\mathcal{U}$. Since X is not totally bounded, we can apply Lemma 8.3.2 to obtain a map $g\colon \mathrm{USCC}_B(X) \times \mathbb{N} \to \mathrm{USCC}_B(X)$ such that $\{g_n(\mathrm{USCC}_B(X)) \mid n \in \mathbb{N}\}$ is discrete in $\mathrm{USCC}_B(X)$ and each g_n is $\mathcal{V}$-close to id. Choose an open refinement $\mathcal{W}$ of $\mathcal{V}$ so that the star $\operatorname{st}(W, \mathcal{W})$ of each $W \in \mathcal{W}$ meets at most one of $g_n(\mathrm{USCC}_B(X))$. Applying Lemma 8.3.2 again, we obtain maps $h_n\colon \mathrm{USCC}_B(X) \times 2^{D_n^1} \to \mathrm{USCC}_B(X)$ $(n \in \mathbb{N})$ such that

$\{(h_n)_F(\mathrm{USCC}_B(X)) \mid F \in 2^{D^1_n}\}$ is discrete in $\mathrm{USCC}_B(X)$ and each $(h_n)_F$ is $\mathcal{W}$-close to id. Then we define a maps $f\colon \mathrm{USCC}_B(X) \times A \to \mathrm{USCC}_B(X)$ by

$$f(\varphi,(n,F)) = h_n(g(\varphi,n),F) \quad \big(\text{i.e., } f_{(n,F)}(\varphi) = (h_n)_F \circ g_n(\varphi)\big).$$

Each $f_{(n,F)}$ is $\mathcal{U}$-close to id because it is $\mathcal{W}$-close to g_n.

We show that the collection $\{f_{(n,F)}(\mathrm{USCC}_B(X)) \mid (n,F) \in A\}$ is discrete in $\mathrm{USCC}_B(X)$. Each $\varphi \in \mathrm{USCC}_B(X)$ is contained in some $W \in \mathcal{W}$. Then this W meets at most one member of

$$\big\{f(\mathrm{USCC}_B(X) \times \{n\} \times 2^{D^1_n}) \bigm| n \in \mathbb{N}\big\}.$$

In fact, if $f_{(n,F)}(\psi), f_{(n',F')}(\psi') \in W$ for some $\psi,\psi' \in \mathrm{USCC}_B(X)$, $n \neq n' \in \mathbb{N}$, $F \in 2^{D^1_n}$ and $F' \in 2^{D^1_{n'}}$, then $g_n(\psi), g_{n'}(\psi') \in \mathrm{st}(W,\mathcal{W})$, which is a contradiction. In case

$$W \cap f(\mathrm{USCC}_B(X) \times \{n\} \times 2^{D^1_n}) \neq \emptyset,$$

we can choose a neighborhood W' of φ so that $W' \subset W$ and W' meets at most one of $(h_n)_F(\mathrm{USCC}_B(X))$. Since

$$f_{(n,F)}(\mathrm{USCC}_B(X)) = (h_n)_F \circ g_n(\mathrm{USCC}_B(X)) \subset (h_n)_F(\mathrm{USCC}_B(X)),$$

W' meets at most one of $f_{(n,F)}(\mathrm{USCC}_B(X))$. Thus $\{f_{(n,F)}(\mathrm{USCC}_B(X)) \mid (n,F) \in A\}$ is discrete in $\mathrm{USCC}_B(X)$.

Finally, we show that $\mathrm{USCC}(X,[-1,1]) \approx \ell_2(A)$, (i.e., $\mathrm{USCC}(X,\mathbf{I}) \approx \ell_2(A)$). Let

$$B = \big\{\varphi \in \mathrm{USCC}(X,[-1,1]) \bigm| \inf\varphi(X) = -1 \text{ or } \sup\varphi(X) = 1\big\}.$$

Then B is clearly closed in $\mathrm{USCC}(X,[-1,1])$ and

$$\mathrm{USCC}(X,[-1,1]) \setminus B \approx \mathrm{USCC}_B(X) \approx \ell_2(A).$$

We show that B is a strong Z-set in $\mathrm{USCC}(X,[-1,1])$, whence $\mathrm{USCC}(X,[-1,1]) \approx \ell_2(A)$ by [To$_7$, Theorem B1] (cf. [To$_2$]). For any map $\alpha\colon \mathrm{USCC}(X,[-1,1]) \to (0,1)$, we define a map $h\colon \mathrm{USCC}(X,[-1,1]) \to \mathrm{USCC}(X,[-1,1])$ by

$$h(\varphi)(x) = (1 - \frac{1}{2}\alpha(\varphi))\cdot\varphi(x).$$

Then we have $\rho_{\mathrm{H}}(h(\varphi),\varphi) < \alpha(\varphi)$ for each $\varphi \in \mathrm{USCC}(X,[-1,1])$. For each element φ_0 of $\mathrm{cl}\, h(\mathrm{USCC}(X,[-1,1]))$, there is a sequence $\varphi_k \in \mathrm{USCC}(X,\mathbf{I})$ ($k \in \mathbb{N}$) such that $h(\varphi_k) \to \varphi_0$. Then $b = \inf_{k\in\mathbb{N}} \alpha(\varphi_k) > 0$. Otherwise, $\lim_{j\to\infty} \alpha(\varphi_{k_j}) = 0$ for some $k_1 < k_2 < \cdots$, then φ_{k_j} converges to φ_0, so $\alpha(\varphi_0) = \lim_{j\to\infty} \alpha(\varphi_{k_j}) = 0$, which is a contradiction. For each $k \in \mathbb{N}$,

$$\sup\bigcup_{x\in X} h(\varphi_k)(x) = (1 - \tfrac{1}{2}\alpha(\varphi_k))\cdot \sup\bigcup_{x\in X} \varphi_k(x) \leqslant 1 - \tfrac{1}{2}b,$$

hence $\sup\bigcup_{x\in X} \varphi_0(x) \leqslant 1 - \frac{1}{2}b < 1$. Similarly, we have $\inf\bigcup_{x\in X} \varphi_0(x) \geqslant -1 + \frac{1}{2}b > -1$. Therefore, $\varphi_0 \notin B$. This means that

$$B \cap \mathrm{cl}\, h(\mathrm{USCC}(X,[-1,1])) = \emptyset.$$

Thus B is a strong Z-set in $\mathrm{USCC}(X,[-1,1])$. The proof is completed. □

Remark. Let P be the convex set in the Banach space $\mathrm{C}_B(X)^2 = \mathrm{C}_B(X) \times \mathrm{C}_B(X)$ defined as follows:

$$P = \{(f,g) \in \mathrm{C}_B(X)^2 \mid g(x) \geqslant 0 \ \text{ for all } \ x \in X\}.$$

Then it is easy to see that if $X = (X,d)$ is a discrete metric space (i.e., $\inf\{d(x,y) \mid x \neq y\} > 0$), then $\mathrm{USCC}_B(X) \approx P$. In fact, for each $\varphi \in \mathrm{USCC}_B(X)$, we define $m_\varphi, r_\varphi \in \mathrm{C}_B(X)$ by

$$m_\varphi(x) = \tfrac{1}{2}(\min\varphi(x) + \max\varphi(x));$$

$$r_\varphi(x) = \tfrac{1}{2}(\max\varphi(x) - \min\varphi(x)).$$

Then the desired homeomorphism $\xi\colon \mathrm{USCC}_B(X) \to P$ can be defined by $\xi(\varphi) = (m_\varphi, r_\varphi)$.

§8.4. Remarks on topologies for $\mathrm{C}_B(X)$ and $\mathrm{C}(X, \mathbf{I})$

Although the spaces $\mathrm{C}_B(X)$ and $\mathrm{C}(X, \mathbf{I})$ with the sup-metric are AR's for an arbitrary metric space X, the example of Introduction also shows that the spaces $\mathrm{C}_B(X)$ and $\mathrm{C}(X, \mathbf{I})$ with the Hausdorff metric ρ_{H} are not ANR's even if X is locally connected. One should also remark that $\mathrm{C}_B(X)$ is not a topological linear space in this topology. In fact, it can be easily derived from [FK, Remark 3.6] that the addition $\mathrm{C}_B(\mathbb{R})^2 \to \mathrm{C}_B(\mathbb{R})$ $((f, g) \mapsto f + g)$ is not continuous with respect to the Hausdorff metric. However, we can prove the following:

Theorem 8.4.1. *For any uniformly locally connected metric space $X = (X, d)$, the spaces $\mathrm{C}_B(X)$ and $\mathrm{C}(X, \mathbf{I})$ with the Hausdorff metric are AR's.*

A subset Z of a space Y is said to be *homotopy dense* in Y if there exists a homotopy $h\colon Y \times \mathbf{I} \to Y$ such that $h_0 = \mathrm{id}$ and $h_t(Y) \subset Z$ for $t > 0$. As is easily observed, a homotopy dense subset of an AR (resp. ANR) is also an AR (resp. ANR). By Theorem 8.2.2, in the case X has no isolated points, Theorem 8.4.1 is deduced from the following:

Theorem 8.4.2. *For any uniformly locally connected metric space $X = (X, d)$ with no isolated points, $\mathrm{C}_B(X)$ (resp. $\mathrm{C}(X, \mathbf{I})$) is homotopy dense in $\mathrm{USCC}_B(X)$ (resp. $\mathrm{USCC}(X, \mathbf{I})$).*

As a corollary of Theorem 8.4.2, we also have the following:

Corollary 8.4.3. *Let $X = (X, d)$ be an infinite σ-compact complete metric space, which is assumed to be uniformly locally connected in case X is non-compact. Then $\mathrm{C}_B(X)$ and $\mathrm{C}(X, \mathbf{I})$ with the Hausdorff metric are homeomorphic to a Hilbert space.*

To prove Theorem 8.4.2, we need the following non-compact version of Lemma 3.1.1:

Lemma 8.4.4. *Assume the condition $(\star)$ in §8.2 and that X has no isolated points. Let $f_0\colon K^{(0)} \to \mathrm{C}_B(X)$ be a map of the 0-skeleton of a locally finite simplicial complex K such that $\mathrm{diam}_{\rho_{\mathrm{H}}}\{f_0(\sigma^{(0)}) < 1$ for every $\sigma \in K$, where $\sigma^{(0)} = \sigma \cap K^{(0)}$. Then f_0 extends to a map $f\colon |K| \to \mathrm{C}_B(X)$ such that*

$$\mathrm{diam}_{\rho_{\mathrm{H}}} f(\sigma) \leqslant 4\,\mathrm{diam}_{\rho_{\mathrm{H}}} f_0(\sigma^{(0)}) \quad \textit{for every} \quad \sigma \in K,$$

where $\mathrm{C}_B(X)$ has the topology induced by ρ_{H}.

Sketch of Proof. By Lemma 8.2.1, we have the property $(\sharp)$ Then the proof is all the same proof as Lemma 3.1.1, where $\mathrm{C}(X, (-1, 1))$ is replaced by $\mathrm{C}_B(X)$. Now, since X is not compact, we cannot take $A_v \subset X$ as a finite set in the proof, but since K is locally finite and X has no isolated points, we can take $A_v \subset X$ as a discrete set with the same property, that is, $f(v) \subset N_\rho(f(v)|A_v, \varepsilon_v)$ (in other word, $f(v)|A_v = f(v) \cap p^{-1}(A_v)$ is ε_v-dense in $f(v)$), and each A_v has an open neighborhood U_v in X such that $U_v \cap U_{v'} = \emptyset$ if $v \neq v' \in \sigma^{(0)}$ and $\sigma \in K$. Any other change is not necessary. □

Remark. In the above, if $\mathrm{card}\,\mathrm{St}(v_0, K) > \mathrm{card}\,X$ at some vertex $v_0 \in K^{(0)}$, it is impossible to obtain discrete sets $A_v \subset X$, $v \in K^{(0)}$, such that $A_v \cap A_{v_0} = \emptyset$ for every $v \in \mathrm{St}(v_0, K)^{(0)}$. Then the local finiteness of K is assumed.

We can apply Lemma 8.4.4 to prove the following lemma by the same way as Lemma 3.1.2

Lemma 8.4.5. *Let $X = (X, d)$ be a uniformly locally connected metric space with no isolated points and $f\colon Y \to \mathrm{USCC}_B(X)$ a map of a separable metrizable space Y. Then there exists a homotopy $h\colon Y \times \mathbf{I} \to \mathrm{USCC}_B(X)$ such that $h_0 = f$ and $h_t(Y) \subset \mathrm{C}_B(X)$ for $t > 0$.*

Proof. By replacing the metric d by d_c, we can assume the condition $(\star)$ in §8.2. For each $n \in \mathbb{N}$, let $\mathcal{U}_n$ be an open cover of $\mathrm{USCC}_B(X)$ with $\operatorname{mesh}_{\rho_{\mathrm{H}}} \mathcal{U}_n < (n+1)^{-1}$. Since Y is separable metrizable, the open cover $f^{-1}(\mathcal{U}_n)$ of Y has a countable star-finite open refinement $\mathcal{V}_n$, whence the nerve of $\mathcal{V}_n$ is locally finite. We define

$$\mathcal{W}_1 = \{U \times (2^{-1}, 1] \mid U \in \mathcal{U}_1\} \quad \text{and}$$
$$\mathcal{W}_n = \{U \times ((n+1)^{-1}, (n-1)^{-1}) \mid U \in \mathcal{U}_n\} \quad \text{for } n > 1.$$

Thus we have a star-finite open cover $\mathcal{W} = \bigcup_{n\in\mathbb{N}} \mathcal{W}_n$ of $Y \times (0, 1]$. Let K be the nerve of $\mathcal{W}$ and $g\colon Y \times (0, 1] \to |K|$ a canonical map, that is, each $g(y, t)$ is contained in the simplex spanned by all vertices $W \in \mathcal{W}$ containing (y, t). Then K is locally finite. For each $n \in \mathbb{N}$, let K_n be the nerve of $\mathcal{W}_n \cup \mathcal{W}_{n+1}$. Then each K_n is a subcomplex of K and $K = \bigcup_{n\in\mathbb{N}} K_n$. Note that $K^{(0)} = \bigcup_{n\in\mathbb{N}} \mathcal{W}_n$. For each $W \in \mathcal{W}_n$, since $\mathrm{pr}_Y(W) \in \mathcal{V}_n \prec f^{-1}(\mathcal{U}_n)$, we can choose $\pi(W) \in \mathcal{U}_n$ so that $f\,\mathrm{pr}_Y(W) \subset \pi(W)$. Since $\mathrm{C}_B(X)$ is dense in $\mathrm{USCC}_B(X)$ by Theorem 8.1.5, we can also choose $k_0(W) \in \pi(W) \cap \mathrm{C}_B(X)$, whence $\rho_{\mathrm{H}}(k_0(W), f(y)) \leqslant \operatorname{mesh}_{\rho_{\mathrm{H}}} \mathcal{U}_n < (n+1)^{-1}$ for any $y \in \mathrm{pr}_Y(W)$. Thus we have a map $k_0\colon K^{(0)} \to \mathrm{C}_B(X)$ such that $\rho_{\mathrm{H}}(k_0(W), f(y)) < (n+1)^{-1}$ for any $W \in K_n^{(0)} = \mathcal{W}_n$ and $y \in \mathrm{pr}_Y(W)$, hence $\operatorname{diam}_{\rho_{\mathrm{H}}} k_0(\sigma^{(0)}) < 2(n+1)^{-1}$ for each $\sigma \in K_n$. By using Lemma 8.4.4, we can extend k_0 to a map $k\colon |K| \to \mathrm{C}_B(X)$ such that $\operatorname{diam}_{\rho_{\mathrm{H}}} k(\sigma) < 4 \operatorname{diam}_{\rho_{\mathrm{H}}} k_0(\sigma^{(0)})$. Thus we obtain the map

$$kg\colon Y \times (0, 1] \to \mathrm{C}_B(X) \subset \mathrm{USCC}_B(X).$$

For each $(y, t) \in Y \times (0, 1]$, choose $n \in \mathbb{N}$ and $W \in \mathcal{W}_n$ so that $(n+1)^{-1} < t \leqslant n^{-1}$ and $(y, t) \in W$. Then we have $\sigma \in K_n$ such that $g(y, t) \in \sigma$ and $W \in \sigma^{(0)}$. Since

$k(W), kg(y,t) \in k(\sigma)$ and $\operatorname{diam}_{\rho_{\mathrm{H}}} k(\sigma) < 4 \operatorname{diam}_{\rho_{\mathrm{H}}} k(\sigma^{(0)}) < 8(n+1)^{-1}$, it follows that

$$\begin{aligned}\rho_{\mathrm{H}}(kg(y,t), f(y)) &\leqslant \rho_{\mathrm{H}}(kg(y,t), k(W)) + \rho_{\mathrm{H}}(k(W), f(y)) \\ &< 8(n+1)^{-1} + (n+1)^{-1} = 9(n+1)^{-1} < 9t.\end{aligned}$$

Then kg can be extended to the desired homotopy h by $h_0 = f$. □

Remark. In the above lemma, the separability of Y is necessary because the local finiteness of K is assumed in Lemma 8.4.4. Note that $\mathrm{USCC}_B(X)$ is non-separable.

It is said that $Z \subset Y$ is *locally homotopy negligible* in Y if every neighborhood U of each point $x \in X$ contains a neighborhood V of x such that each map $f\colon (I^n, \partial I^n) \to (V, V \setminus Z)$, $n \in \mathbb{N}$ is homotopic in $(U, U \setminus Z)$ to a map g with $g(I^n) \subset U \setminus Z$ [To$_4$]. By using Lemma 8.4.5, it is easy to see the following:

Corollary 8.4.6. *For any uniformly locally connected space $X = (X,d)$ with no isolated points,* $\mathrm{USCC}_B(X) \setminus \mathrm{C}_B(X)$ *is locally homotopy negligible in* $\mathrm{USCC}_B(X)$. □

Proof of Theorem 8.4.2. Since $\mathrm{USCC}_B(X)$ is an AR by Theorem 8.2.2, according to Theorem 1.2.3, Corollary 8.4.6 implies that $\mathrm{C}_B(X)$ is homotopy dense in $\mathrm{USCC}_B(X)$.

By small adjustments, we can see that Lemmas 8.4.4 and 8.4.5 are valid for $\mathrm{USCC}(X, \mathbf{I})$. Then, it follows that $\mathrm{C}(X, \mathbf{I})$ is homotopy dense in $\mathrm{USCC}(X, \mathbf{I})$ for any uniformly locally connected metric space $X = (X,d)$ with no isolated points. □

Proof of Theorem 8.4.1. Let X_0 be the set of all isolated points of X. Since X is uniformly locally connected, there is $\delta > 0$ such that $d(a, X \setminus \{a\}) > \delta$ for every $a \in X_0$. It is easy to see that

$$\mathrm{C}_B(X) \approx \mathrm{C}_B(X_0) \times \mathrm{C}_B(X \setminus X_0),$$

where the topology of each spaces is induced the Hausdorff metric ρ_{H}. By Theorems 8.2.2 and 8.4.2, $\mathrm{C}_B(X \setminus X_0)$ with the Hausdorff metric is an AR. On the other hand,

$C_B(X_0)$ with the Hausdorff metric is also an AR because the Hausdorff metric on $C_B(X_0)$ induces the same topology as the sup-norm. Therefore, $C_B(X)$ with the Hausdorff metric is an AR. Moreover, $C(X, \mathbf{I})$ with the Hausdorff metric is also an AR because it is an retract of $C_B(X)$ with the Hausdorff metric. $\square$

Proof of Corollary 8.4.3. In case X is compact and infinite, the Hausdorff metric induces the same topology as the sup-metric. The separable Banach space $C(X) = C_B(X)$ is homeomorphic to the separable Hilbert space ℓ_2 [BP, Ch.VI, Theorem 5.1]. The space $C(X, \mathbf{I})$ is homeomorphic to the closed unit ball $C(X, [-1,1])$ of $C(X)$, hence $C(X, \mathbf{I}) \approx \ell_2$ [BP, Ch.VI, Theorem 5.1].

In case X is non-compact, as Theorem 8.4.1, it reduces the case X has no isolated points. Then it suffices to show that $\mathrm{USCC}_B(X) \setminus C_B(X)$ is an F_σ-set in $\mathrm{USCC}_B(X)$. In fact, $\mathrm{USCC}_B(X) \setminus C_B(X)$ would be Z_σ-set in $\mathrm{USCC}_B(X)$ by Theorem 8.4.2, hence $\mathrm{USCC}_B(X) \approx C_B(X)$ by [Cu_3, Corollary 1]. Moreover, since $\mathrm{USCC}(X, \mathbf{I}) \setminus C(X, \mathbf{I})$ would be also an F_σ-set in $\mathrm{USCC}(X, \mathbf{I})$, it would similarly follow that $\mathrm{USCC}(X, \mathbf{I}) \approx C(X, \mathbf{I})$.

Since X is σ-compact, X has compact subsets $X_1 \subset X_2 \subset \cdots$ with $X = \bigcup_{n \in \mathbb{N}} X_n$. For each $n \in \mathbb{N}$, let

$$F_n = \{\varphi \in \mathrm{USCC}_B(X) \mid \exists x \in X_n \text{ such that } \operatorname{diam} \varphi(x) \geqslant 1/n\}.$$

Then $\mathrm{USCC}_B(X) \setminus C_B(X) = \bigcup_{n \in \mathbb{N}} F_n$. To see that each F_n is closed in $\mathrm{USCC}_B(X)$, let $\varphi_i \in F_n$, $i \in \mathbb{N}$, and assume $\varphi_i \to \varphi \in \mathrm{USCC}_B(X)$ as $i \to \infty$. Then all φ_i and φ are contained in some $X \times [-r, r]$. For each $i \in \mathbb{N}$, we have $x_i \in X_n$ such that $\operatorname{diam} \varphi(x_i) \geqslant 1/n$, whence there are $s_i < t_i \in \varphi(x_i)$ with $t_i - s_i \geqslant 1/n$. Since X_n and $[-r, r]$ are compact, we may assume that $x_i \to x$ in X_n, $s_i \to s$ and $t_i \to t$ in $[-r, r]$. Then $t - s \geqslant 1/n$ and $s, t \in \varphi(x)$. Thus we have $\operatorname{diam} \varphi(x) \geqslant 1/n$, hence $\varphi \in F_n$. Therefore, $\mathrm{USCC}_B(X) \setminus C_B(X)$ is an F_σ-set in $\mathrm{USCC}_B(X)$. $\square$

Let $\mathrm{C}_B^U(X)$ be the subspace of the Banach space $\mathrm{C}_B(X)$ consisting of uniformly continuous functions and $\mathrm{C}^U(X,\mathbf{I}) = \mathrm{C}(X,\mathbf{I}) \cap \mathrm{C}_B^U(X)$. In case X is compact, $\mathrm{C}_B^U(X) = \mathrm{C}(X)$ and $\mathrm{C}^U(X,\mathbf{I}) = \mathrm{C}(X,\mathbf{I})$. As just have seen, the Banach space $\mathrm{C}_B(X)$ is not a subspace of $\mathrm{USCC}_B(X)$, but $\mathrm{C}_B^U(X)$ can be regarded as a subspace of $\mathrm{USCC}_B(X)$, that is,

Proposition 8.4.7. *The topology of $\mathrm{C}_B^U(X)$ induced by the sup-norm $\|\cdot\|$ coincides with the one induced by the Hausdorff metric ρ_{H}.*

Proof. Let $f \in \mathrm{C}_B^U(X)$. By the uniform continuity of f, for each $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x,y) < \delta$ implies $|f(x) - f(y)| < \varepsilon/2$. Let $g \in \mathrm{C}_B(X)$ such that $\rho_{\mathrm{H}}(f,g) < \min\{\varepsilon/2, \delta\}$. For each $x \in X$, since $\rho((x,g(x)),f) < \min\{\varepsilon/2,\delta\}$, we can choose $y \in X$ so that

$$\rho((x,g(x)),(y,f(y))) = \max\big\{d(x,y),\ |g(x) - f(y)|\big\} < \min\{\varepsilon/2,\delta\}.$$

Since $d(x,y) < \delta$, we have $|f(x) - f(y)| < \varepsilon/2$. Hence, it follows that

$$|f(x) - g(x)| \leqslant |f(x) - f(y)| + |f(y) - g(x)| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

Therefore, $\|f - g\| < \varepsilon$. Conversely, if $\|f - g\| < \varepsilon$ then

$$\begin{aligned}\rho_{\mathrm{H}}(f,g) &= \max\big\{\sup\nolimits_{x\in X} \rho((x,f(x)),g),\ \sup\nolimits_{x \in X} \rho((x,g(x)),f)\big\} \\ &\leqslant \sup_{x\in X} |f(x) - g(x)| = \|f - g\| < \varepsilon. \qquad \square\end{aligned}$$

Comparing with the result in Chapter 3, one may want to replace $\mathrm{C}(X,\mathbf{I})$ and $\mathrm{C}_B(X)$ in Theorem 8.1.5 (or Corollary 8.1.6) by $\mathrm{C}^U(X,\mathbf{I})$ and $\mathrm{C}_B^U(X)$, respectively, since they are subspaces of $\mathrm{USCC}(X,\mathbf{I})$ and $\mathrm{USCC}_B(X)$, respectively. However, $\mathrm{C}^U(X,\mathbf{I})$ is not dense in $\mathrm{USCC}(X,\mathbf{I})$ even if X is locally connected and has no isolated point. In fact, let $X = \bigcup_{n\in\mathbb{N}}[n - n^{-1}, n] \subset \mathbb{R}$ and define $f \in \mathrm{C}(X,\mathbf{I}) \subset \mathrm{USCC}(X,\mathbf{I})$

by $f(n-t) = nt$ if $0 \leqslant t \leqslant n^{-1}$. Then any $g \in \mathrm{C}(X, \mathbf{I})$ with $\rho_{\mathrm{H}}(f, g) < \frac{1}{4}$ is not uniformly continuous because $\frac{1}{4}, \frac{3}{4} \in g([n - n^{-1}, n])$ for all $n \in \mathbb{N}$. As an another example, let $X = \mathbb{R} \setminus \{0\}$ and and define $f \in \mathrm{C}(X, \mathbf{I}) \subset \mathrm{USCC}(X, \mathbf{I})$ by $f(x) = 0$ for $x < 0$ and $f(x) = 1$ for $x > 0$. Then any $g \in \mathrm{C}(X, \mathbf{I})$ with $\rho_{\mathrm{H}}(f, g) < \frac{1}{4}$ is not uniformly continuous because $g(x) < \frac{1}{4}$ if $x < 0$ and $g(x) > \frac{3}{4}$ if $x > 0$.

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