On the structure of Hopf modules

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THESIS

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CONTENTS

Introduction

The theory of comodule algebras is a new subject with roots in the invariant theory of affine algebraic groups and in the Hopf Galois theory of commutative, or non-commutative rings. The notion of Hopf modules is a useful tool in this subject. Therefore the theory and results on Hopf modules are surely of interest. This notes is a study to this material.

In chapter I we give a self-contained introduction to the homological theory of comodules over coalgebras and Hopf algebras. Section 1 is an exposition of basic concepts such as cotensor products, coflat comodules, injective comodules, projective comodules and change of coalgebras. Hopf algebraic proofs are obtained for some results of Cline-Parshall-Scott [3] on rational modules over affine algebraic groups. Section 2 deals with the representation theory of co-Frobenius coalgebras and coseparable coalgebras. Simplified proofs are obtained for some results of Larson [13] and Lin [14]. In particular, if C is a left co-Frobenius coalgebra then an injective cover of a finite dimensional right C-comodule is finite dimensional and every injective right C-comodule is C-projective. Also for a Hopf algebra H the following are equivalent: (1) H has a non-zero left (or right) integral. (2) H is left (or right) co-Frobenius. Section 3 contains some results on the cohomology of coalgebras. Let C be a coalgebra, N a C-C-bicomodule (i.e., a left $C^e = C \otimes C^{op}$-comodule) and X an injective resolution

$$0 \longrightarrow C \longrightarrow X_0 \longrightarrow X_1 \longrightarrow \cdots$$

of C as left C^e-comodule. The cohomology groups $H^n(N, C)$ [resp. $\mathrm{Hoch}^n(N, C)$] of C with coefficients in N are the cohomology groups of the complex $\mathrm{Com}_{C^e}(N, X)$ [resp. $N \square_{C^e} X$]. The map $\lambda: L \longrightarrow C$, $\lambda\omega(c \otimes d) = c\varepsilon(d) - \varepsilon(c)d$, ($\omega: C \otimes C \longrightarrow L$ the cokernel of the comultiplication Δ of C) is a universal coderivation and the following are equivalent for the coalgebra C: (1) C is coseparable. (2) $H^n(N, C) = (0)$ for all $n \geqq 1$ and all left C^e-comodules N. (3) Every coderivation from any left C^e-comodule to C is inner. (4) $\lambda: L \longrightarrow C$ is an inner coderivation. Also we show that extensions D of C with $D = C \wedge C$ correspond uniquely to elements of $H^2(D/C, C)$. These results are used to give a proof of the Sweedler-Sullivan splitting theorem [17] for coalgebras with coseparable coradical. We also derives some related results for augmented coalgebras and Hopf algebras.

In chapter II we introduce and study the concept of (A, B)-Hopf modules, where A is a Hopf algebra over a field k and B is a right A-comodule algebra. These are A-comodules which also have a B-module structure and where the comodule structures are compatible with this module structure. Our approach to the subject was inspired primarily by the paper of Takeuchi [25]. Our Hopf module is a generalization of his Hopf module. After the definition we prove that B is an injective A-comodule if and only if there is an A-comodule map $A \longrightarrow B$ preserving the identity, (it is called a total integral in chapter IV), and with these equivalent conditions every (A, B)-Hopf module is an injective (or equivalently a coflat) A-comodule. Using this we show that for any Hopf algebra A' and any quotient Hopf algebra A of A', if A' is a coflat (right or left) A-comodule, then it is faithfully coflat. In the latter part of this chapter we dualize the results and give counterparts for [C, A]-Hopf modules with a right A-module coalgebra C.

In chapter III we develope a Hopf module theory. For the best results it is frequently necessary to assume that B has a total integral , or even the stronger hypothesis that B is cleft (that is, there is an A-comodule map in the units of the convolution algebra Hom(A, B)). Section 1 summarizes some preliminary remarks on the category of Hopf modules. Section 2 studies (generalized) integrals and contains an application to questions of splittings in Hopf modules. In section 3 we prove the fundamental theorem about Hopf modules over cleft comodule algebras. Section 4 gives a construction of a smash product #(A,B), which plays an important role in the comodule algebra theory.

Throughout chapter I-III we work over a fixed field k. All algebras, coalgebras and so on are over k. But in the last chapter, all algebras and Hopf algebras will be over a fixed commutative ring R.

In the final chapter we are concerned with the relationship between a comodule algebra B over a fixed commutative ring R and the invariant subalgebra $C = \{ b \in B \mid \rho(b) = b \otimes 1 \}$. We give, in section 1, necessary and sufficient conditions for B to have a total integral. From this we deduce a version of Maschke's theorem for Hopf modules. When A is the dual Hopf algebra of the group algebra of a finite group which acts as automorphisms of B, the existence of a total integral is equivalent to that of an element b in B such that $\Sigma_{x \in G} x(b) = 1_B$. Section 2 is devoted to a new exposition of the foundation of the Hopf Galois theory. It turns out that the concepts of A-Galois extensions with normal bases (Kreimer-Takeuchi [12]) corresponds to our cleftness. We treat, in section 3, the case in which R is a field and A, B are commutative. In this case we prove that the following are equivalent: (1) The map $\beta: B \otimes_C B \longrightarrow B \otimes A$, $\beta(b \otimes_C b') = (b \otimes 1)\rho(b')$, is surjective and B has a total integral. (2) The category of (A, B)-Hopf modules is equivalent to the category of right C-modules. (3) β is bijective and B is a faithfully flat C-module. Using this we give an improvement of Takeuchi's theorem on the correspondence between the set of quotient Hopf algebras and the set of left coideal subalgebras.

We freely use the terminology and results of Sweedler [19].

Chapter I

Homological coalgebra

Throughout this chapter, the field k is fixed. Vector spaces over k are called k-spaces, and linear maps between k-spaces are called k-maps.

§1. Coalgebras and comodules

A coalgebra over k is a k-space C together with k-maps $\Delta\colon C \to C \otimes C$ and $\varepsilon\colon C \to k$ such that $(I \otimes \varepsilon)\Delta = (\varepsilon \otimes I)\Delta = I$. If C is a coalgebra, a left C-comodule is a k-space M together with a k-map $\rho\colon M \to C \otimes M$ such that $(I \otimes \rho)\rho = (\Delta \otimes I)\rho$ and $(\varepsilon \otimes I)\rho = I$. If M and N are left C-comodules, a comodule map from M to N is a k-map $f\colon M \to N$ such that $(I \otimes f)\rho_M = \rho_N f$. The k-space of all comodule maps from M to N is denoted by $\mathrm{Com}_C(M,N)$ and the category of left C-comodules is denoted by $\mathbb{M}^C$. Similarly, we define ${}^C\mathbb{M}$, the category of right C-comodules.

1.1. Cotensor products and injective comodules.

If M is a right C-comodule and N is a left C-comodule, the <u>cotensor product</u> $M \square_C N$ is the kernel of the k-map

$$\rho_M \otimes I - I \otimes \rho_N : M \otimes N \to M \otimes C \otimes N.$$

Given comodule maps $f: M \to M'$ and $g: N \to N'$, the k-map $f \otimes g : M \otimes N \to M' \otimes N'$ induces a k-map

$$f \square_C g : M \square_C N \to M' \square_C N'.$$

It is clear that $- \square_C -$ is an additive covariant bifunctor from $\mathbb{M}^C \times {}^C\mathbb{M}$ to $\mathbb{M}_k$, the category of k-spaces, and is left exact. The mapping $m \otimes c \to \varepsilon(c)m$ and $c \otimes n \to \varepsilon(c)n$ yield natural isomorphisms $M \square_C C \simeq M$ and $C \square_C N \simeq N$. We shall usually identify these isomorphic k-spaces.

Let C and D be two coalgebras. Suppose that M in addition to being a left C-comodule with structure map ρ^-: $M \to C \otimes M$, is also a right D-comodule with structure map $\rho^+: M \to M \otimes D$ and that $(I \otimes \rho^+)\rho^- = (\rho^- \otimes I)\rho^+$. We then say that M is a (C,D)-<u>bicomodule</u>. If N is a left D-comodule then the map

$$\rho^- \otimes I : M \otimes N \to C \otimes M \otimes N$$

gives a left C-comodule structure to $M \otimes N$ and $M \square_D N$ is a C-subcomodule of $M \otimes N$. Similarly if L is a right C-comodule then $L \square_C M$ becomes a right D-comodule. With the structure described above we have the associativity of cotensor product:

$$(L \square_C M) \square_D N \simeq L \square_C (M \square_D N).$$

If N is a left C-comodule which is finite dimensional as a k-space then the dual space N^* is a right C-comodule with structure map

$$N^* \to \mathrm{Hom}(N,C) \simeq N^* \otimes C, \quad n^* \to (I \otimes n^*)\rho_N.$$

If M is a right C-comodule we have canonically

$$M \square_C N \simeq \mathrm{Com}_C(N^*,M).$$

Since every comodule is the union of its finite dimensional subcomodules, this implies that the functor $M \square_C -$ from ${}^C\mathfrak{M}$ to $\mathfrak{M}_k$ is exact if and only if so is the functor $\mathrm{Com}_C(-,M)$ from $\mathfrak{M}^C$ to $\mathfrak{M}_k$ (cf. Takeuchi [24]). A right C-comodule M is called <u>injective</u> (or C-<u>injective</u>) if the functor $\mathrm{Com}_C(-,M)$ is exact, and is called <u>projective</u> (or C-<u>projective</u>) if the functor $\mathrm{Com}_C(M,-)$ is exact.

By the flatness of injective comodules and the associativity of cotensor products we have:

Proposition 1. <u>Let</u> L <u>be a right</u> C-<u>comodule and</u> M <u>be a</u> (C,D)-<u>bicomodule</u>. <u>If</u> L <u>is</u> C-<u>injective and</u> M <u>is</u> D-<u>injective then</u> $L \square_C M$ <u>is</u> D-<u>injective</u>.

We use the opposite coalgebra C^{op} to convert a left (or right) C-comodule V into a right (or left) C^{op}-comodule. Every (C,D)-bicomodule M becomes a left $C \otimes D^{op}$-comodule. Similarly M may be regarded as a left $D^{op} \otimes C$-comodule, a right $C^{op} \otimes D$-comodule and a right $D \otimes C^{op}$-comodule.

Let C, D and E be coalgebras. For a (D^{op},C)-bicomodule L, a (C,E)-bicomodule M and a (E,D^{op})-bicomodule N, we have a natural isomorphism

$$(L \square_C M) \square_{D\otimes E} N \simeq L \square_{C\otimes D} (M \square_E N).$$

Proposition 2. (1) <u>Let</u> L <u>be a</u> (D^{op},C)-<u>bicomodule</u>. <u>If</u> L <u>is injective as a right</u> $C \otimes D$-<u>comodule then it is injective as a right</u> D-<u>comodule</u>.

(2) <u>Let</u> L <u>be a right</u> D-<u>comodule and</u> M <u>be a right</u> E-<u>comodule</u>. <u>If</u> L <u>is</u> D-<u>injective and</u> M <u>is</u> E-<u>injective then</u> $M \otimes N$ <u>is</u>

injective as a right $D \otimes E$ _-comodule_.

Proof. (1) Setting $M = C$ and $E = k$ in the above isomorphism, yields the isomorphism for every left D-comodule N,

$$L \square_D N \simeq L \square_{C\otimes D} (C \otimes N).$$

This shows that the functor $L \square_D -$ is exact.

(2) Setting $C = k$, yields the isomorphism for every left $D \otimes E$ -comodule N,

$$(L \otimes M) \square_{D\otimes E} N \simeq L \square_D (M \square_E N).$$

This shows that the functor $(L \otimes M) \square_{D\otimes E} -$ is exact. Q.E.D.

Let W be a right C-comodule. For every k-space X, $X \otimes W$ is a right C-comodule with structure map

$$I \otimes \rho_W : X \otimes W \to X \otimes W \otimes C,$$

which we denote $(X) \otimes W$. $(X) \otimes W$ is a direct sum of copies of W. The next well-known result is fundamental.

Proposition 3. _Let_ V _be a right_ C-_comodule and_ X _be a_ k-_space_. _Then the map_

$$\phi: \mathrm{Com}_C(V, (X) \otimes C) \to \mathrm{Hom}(V, X)$$

given by $\phi(F) = (I \otimes \varepsilon)F$ _for each_ $F \in \mathrm{Com}_C(V, (X) \otimes C)$ _is a_ k-_isomorphism_. _The inverse_ ψ _of_ ϕ _is given by_ $\psi(f) = (f \otimes I)\rho_V$ _for each_ $f \in \mathrm{Hom}(V, X)$.

Proof. Straightfoward.

A right C-comodule M is called _free_ if there exists a k-space X such that $M \simeq (X) \otimes C$ as right C-comodules.

Corollary 1. _Every free comodule is injective_.

Note that an injective comodule need not be free. For example, take $C = C_1 \oplus C_2$, the direct sum of coalgebras C_1 and C_2. Then the C_1 is clearly not free, but is injective as a C-comodule. In [24], Takeuchi showed that if C is cocommutative and irreducible then every injective comodule is free.

Corollary 2. _Every comodule can be embedded in a free comodule_.

Proof. For every right C-comodule M, its structure map ρ_M is a C-comodule map from M to $(M) \otimes C$. Since $(I \otimes \varepsilon)\rho_M = I$, ρ_M is a monomorphism. Q.E.D.

We note that a C-comodule V is injective if and only if it is a direct summand of a free C-comodule.

If C is a coalgebra, then $C^* = \mathrm{Hom}(C, k)$ is an algebra, with multiplication defined by $\alpha\beta = (\alpha \otimes \beta)\Delta : C \to k$, where $\alpha, \beta \in C^*$. If V is a right C-comodule, then defining $c^* \rightharpoonup v = (I \otimes c^*)\rho_V(v)$ for $c^* \in C^*$, $v \in V$, makes V into a left C^*-module. In a similar fashion, left C-comodules have a right C^*-module structure. For right C-comodules M, N we have

$$\mathrm{Com}_C(M, N) = \mathrm{Mod}_{C^*}(M, N).$$

Thus M^C may be regarded as a full subcategory of the category of left C^*-modules. It follows that if a right C-comodule M is injective (resp. projective) as a left C^*-module then it is injective (resp. projective) as a right C-comodule.

Proposition 4. Let M be a finite dimensional right C-comodule. Then M is injective (resp. projective) as a left C^*-module if and only if it is injective (resp. projective) as a right C-comodule.

Proof. We need to show the " if " part. Suppose that M is C-injective. Then the map

$$0 \to M \xrightarrow{\rho} (M) \otimes C \simeq C \oplus \dots \oplus C \quad \text{(finite times)}$$

splits as right C-comodules, and so as left C^*-modules. Taking the dual, the map

$$C^* \oplus \dots \oplus C^* \to M^* \to 0$$

splits as right C^*-modules. This means that M^* is projective as a right C^*-module. Therefore $M = M^{**}$ is injective as a left C^*-module.

Next we show that if M is C-projective then it is projective as a left C^*-module. Since M^* is injective as a left C-comodule, it follows from the above that M^* is injective as a right C^*-module. Therefore $M = M^{**}$ is projective as a right C^*-module. This completes the proof.

1.2. Change of coalgebras.

We shall consider two coalgebras C and D, and a coalgebra map $\pi: C \to D$. Every right C-comodule V may be treated as a right D-comodule with structure map

$$(\pi \otimes I)\rho : V \to V \otimes C \to V \otimes D,$$

which we denote V_π. Similarly for left comodules. In particular C itself may be regarded as a left or a right D-comodule. Regarding C as a (D,C)-bicomodule, we form the right C-comodule

$$W^\pi = W \square_D C, \quad \text{where } W \text{ is a right } D\text{-comodule},$$

which we call the _induced_ comodule for W.

Proposition 5. _The following are equivalent_:

(i) _The functor_ $- \square_D C$ _from_ M^D _to_ M^C _is exact_.

(ii) C _is injective as a left_ D-_comodule_.

(iii) _Every injective left_ C-_comodule is injective as a left_ D-_comodule_.

Proof. The equivalence of (i) and (ii) has already been proved in 1.1. (ii) implies (iii) by Proposition 1, and (iii) implies (ii) since C is an injective left C-comodule. Q.E.D.

The next result is a generalization of Proposition 3.

Proposition 6. _Let_ V _be a right_ C-_comodule and_ W _be a right_ D-_comodule_. _Then the map_

$$\phi: \mathrm{Com}_C(V, W^\pi) \to \mathrm{Com}_D(V_\pi, W)$$

given by $\phi(F) = (I \square \pi)F$, _for each_ $F \in \mathrm{Com}_C(V, W^\pi)$, _is a_ k-_isomorphism_.

The inverse ψ _of_ ϕ _is given by_ $\psi(f) = (f \otimes I)\rho_V$ _for each_ $f \in \mathrm{Com}_D(V_\pi, W)$.

Proof. For $F \in \mathrm{Com}_C(V, W^\pi)$, the following diagram is commutative:

$$\begin{array}{ccccccc}
V & \xrightarrow{F} & W \,\square_D\, C & \xrightarrow{I \square \pi} & W \,\square_D\, D & \simeq & W \\
\downarrow \rho & & \downarrow I \otimes \Delta & & \downarrow I \otimes \Delta & & \downarrow \rho \\
V \otimes C & \xrightarrow{F \otimes I} & W \,\square_D\, C \otimes C & & & & \\
\downarrow I \otimes \pi & & \downarrow I \square I \otimes \pi & & & & \\
V \otimes D & \xrightarrow{F \otimes I} & W \,\square_D\, C \otimes D & \xrightarrow{I \square \pi \otimes I} & W \,\square_D\, D \otimes D & \simeq & W \otimes D.
\end{array}$$

This implies that $\phi(F) \in \mathrm{Com}_D(V_\pi, W)$.

Next we show that $\psi(f) \in \mathrm{Com}_C(V, W^\pi)$ for $f \in \mathrm{Com}_D(V_\pi, W)$. We have

$$\begin{aligned}
(\rho_W \otimes I)\psi(f) &= (\rho_W f \otimes I)\rho_V = ((f \otimes I)(I \otimes \pi)\rho_V \otimes I)\rho_V \\
&= (f \otimes I \otimes I)(I \otimes \pi \otimes I)(I \otimes \Delta)\rho_V \\
&= (I \otimes (\pi \otimes I)\Delta)(f \otimes I)\rho_V = (I \otimes (\pi \otimes I)\Delta)\psi(f).
\end{aligned}$$

This concludes that the image of the map $\psi(f)$ is contained in $W \,\square_D\, C$. So $\psi(f)$ is clearly a C-comodule map from V into W^π. It is easily checked that $\psi\phi = I$ and $\phi\psi = I$. Q.E.D.

Corollary. *If a right D-comodule W is injective then W^π is injective as a right C-comodule.*

A right C-comodule V is said to be π-*injective* if for every exact sequence of C-comodules

$$0 \to M' \to M \to M'' \to 0$$

which splits as D-comodules, the sequence

$$0 \to \mathrm{Com}_C(M'', V) \to \mathrm{Com}_C(M, V) \to \mathrm{Com}_C(M', V) \to 0$$

is also exact. Proposition 6 shows that W^π is π-injective for every right D-comodule W. Let $V = V_1 \oplus V_2$, where V_1 and V_2 are subcomodules of V. Then V is π-injective if and only if V_1 and V_2 are π-injective.

For every right C-comodule V, the structure map ρ may be regarded as a C-comodule map from V into $(V_\pi)^\pi = V \,\square_D\, C$. The composition

$$V \xrightarrow{\rho} V \square_D C \xrightarrow{I \square \pi} V \square_D D = V$$

is the identity, which shows that V map be treated as a direct summand of $(V \square_D C)_\pi$ as a D-comodule, since $I \square \pi$ is a D-comodule map. This observation leads us to the following result.

Proposition 7. _The following statements concerning a right C-comodule_ V _are equivalent_:

(i) V _is_ π-_injective_.

(ii) _Every exact sequence of_ C-_comodules_

$$0 \to V \to M \to N \to 0$$

which splits as D-_comodules, also splits as_ C-_comodules_.

(iii) _There exists a_ C-_comodule map_ $g: (V_\pi)^\pi \to V$ _such that_ $\rho g = I$, _that is_, V _is a direct summand of_ $(V_\pi)^\pi$ _as a_ C-_comodule_.

Proposition 8. _Let_ V _be a right_ C-_comodule_. _If_ V _is_ π-_injective and_ D-_injective, then it is_ C-_injective_.

Proof. By Corollary of Proposition 6, $V \square_D C$ is C-injective. Since V is π-injective, V is a direct summand of $V \square_D C$ as a C-comodule. Therefore V is C-injective. Q.E.D.

1.3. Comodules over Hopf algebras

A _Hopf algebra_ over k is a k-space H together with k-maps $\Delta: H \to H \otimes H$, $\varepsilon: H \to k$, $m: H \otimes H \to H$, $u: k \to H$ and $S: H \to H$ such that (H, Δ, ε) is a coalgebra over k, (H, m, u) is an algebra over k, m and u are coalgebra maps and $m(I \otimes S)\Delta = u\varepsilon = m(S \otimes I)\Delta$. The map S is called the _antipode_ of the Hopf algebra. Let V_i $(i = 1, 2)$ be right H-comodules with the structure map $\rho_i: V_i \to V_i \otimes H$ $(i = 1, 2)$. Then the composition

$$\rho = (I \otimes I \otimes m)(I \otimes t \otimes I)(\rho_1 \otimes \rho_2) : V_1 \otimes V_2 \to V_1 \otimes V_2 \otimes H$$

gives $V_1 \otimes V_2$ the structure of a right H-comodule, which we call the tensor product comodule of V_1 and V_2.

Now we shall consider two Hopf algebras H and L, and a Hopf algebra map $\pi: H \to L$ (i.e. π is both a coalgebra map and algebra map, and $\pi S_L = S_H \pi$). Using the fact that the antipode is an anti-coalgebra map and an anti-algebra map, we have the next result.

Proposition 9. *Let V be a right H-comodule and W be a right L-comodule. Then the map*

$$\phi: V \otimes W^{\pi} \to (V_{\pi} \otimes W)^{\pi}$$

given by $\phi(\Sigma\, v \otimes w \otimes h) = \Sigma\, v_{(0)} \otimes w \otimes v_{(1)}h$ (*we write* $\rho_V(v) = \Sigma\, v_{(0)} \otimes v_{(1)}$) *is an isomorphism of H-comodules. The inverse* ψ *of* ϕ *is given by* $\psi(\Sigma\, v \otimes w \otimes h) = \Sigma\, v_{(0)} \otimes w \otimes S(v_{(1)})h$.

Taking $L = k$, $\pi = \varepsilon_H$ and $W = k$, we have:

Corollary 1. *Let V be a right H-comodule. Then there exists an isomorphism*

$$V \otimes H \simeq (V) \otimes H$$

as H-comodules.

Corollary 2. *Let V be a right H-comodule and W be an injective right H-comodule. Then the tensor product comodule* $V \otimes W$ *is H-injective.*

Proof. Since W is injective, W is a direct summand of $(W_{\varepsilon})^{\varepsilon} = (W) \otimes H$. Hence $V \otimes W$ is a direct summand of $V \otimes (W_{\varepsilon})^{\varepsilon}$. By the above Proposition, $V \otimes (W_{\varepsilon})^{\varepsilon} \simeq (V_{\varepsilon} \otimes W_{\varepsilon})^{\varepsilon}$, and this implies that $V \otimes W$ is H-injective. Q.E.D.

An algebra map $\omega: L \to H$ is called a (right) cross-section of $\pi: H \to L$ if it is a right L-comodule map, that is, $(I \otimes \pi)\Delta_H \omega = (\omega \otimes I)\Delta_L$. Assume that there exists a cross-section. Then, defining $h \leftharpoonup \ell = h\omega(\ell)$ for $h \in H$, $\ell \in L$, H makes into a right L-module. We compute

$$(I \otimes \pi)\Delta(h \leftharpoonup \ell) = (I \otimes \pi)\Delta(h)\cdot(I \otimes \pi)\Delta(\omega(\ell))$$
$$= (\Sigma\, h_{(1)} \otimes \pi(h_{(2)}))\cdot(\Sigma\, \omega(\ell_{(1)}) \otimes \ell_{(2)})$$
$$= \Sigma\, h_{(1)} \leftharpoonup \ell_{(1)} \otimes \pi(h_{(2)})\ell_{(2)}.$$

This shows that H is a Hopf module. So we can apply the structure Theorem of Hopf modules (Sweedler [19], p. 84) to obtain an isomorphism of H to $(H') \otimes L$ as L-comodules, where $H' = \{h \in H \mid (I \otimes \pi)\Delta(h) = h \otimes 1\}$. Thus we have proved:

Proposition 10. *Let* $\pi: H \to L$ *be a Hopf algebra map. If there exists a right cross-section of* π, *then* H *is free as a right* L-*comodule*.

§2. A bilinear form for coalgebras.

2.1. Co-Frobenius coalgebras.

We shall consider a coalgebra C and a bilinear form $b: C \times C \to k$. Then b induces two k-maps $\tau: C \otimes C \to k$ and $\theta: C \to C^*$ by setting $\tau(c \otimes d) = b(c, d)$ and $\theta(d)(c) = b(c, d)$, for $c, d \in C$. The next Lemma is clear.

Lemma 1. *In the above situation, the following are equivalent:*

(i) $\Sigma\, c_{(1)}b(c_{(2)}, d) = \Sigma\, b(c, d_{(1)})d_{(2)}$, *for all* $c, d \in C$.

(ii) $b(c \leftharpoonup c^*, d) = b(c, c^* \rightharpoonup d)$, *for all* $c, d \in C$, $c^* \in C^*$.

(iii) $(I \otimes \tau)(\Delta \otimes I) = (\tau \otimes I)(I \otimes \Delta)$.

(iv) $\theta: C \to C^*$ *is a left* C^*-*module map*.

A bilinear form $b: C \times C \to k$ is called C-*balanced* if the above conditions hold.

Lemma 2. *Let* $b: C \times C \to k$ *be a* C-*balanced bilinear form and* X *be a subspace of* C. *Then we have:*

(1) *If* X *is a left coideal* (i.e. $\Delta(X) \subset C \otimes X$), *then* $X^{\perp} = \{d \in C \mid b(x, d) = 0$ *for all* $x \in X\}$ *is a right coideal*.

(2) *If* X *is a right coideal of* C, *then* ${}^{\perp}X = \{c \in C \mid b(c, x)$

$= 0$ for all $x \in X\}$ is a left coideal.

Proof. Let X be a left coideal. Note that $\Delta(X) \subset C \otimes X$ and $X \leftharpoonup C^* \subset X$ are equivalent. Now we have

$$b(X, C^* \rightharpoonup X^\perp) = b(X \leftharpoonup C^*, X^\perp) \subset b(X, X^\perp) = 0.$$

Hence $C^* \rightharpoonup X^\perp \subset X^\perp$, and so $X^\perp$ is a right coideal. This completes the proof of (1). In the similar way we have the proof of (2).

Q.E.D.

A bilinear form $b: C \times C \to k$ is called left non-degenerated if $C^\perp = \{0\}$, equivalently $\theta: C \to C^*$ is injective. A coalgebra C is called left co-Frobenius if there exists a bilinear form $b: C \times C \to k$ which is left non-degenerated and C-balanced, i.e. if there exists a left C^*-monomorphism from C to C^*. We note that if a coalgebra C is co-semi-simple then it is left (and right) co-Frobenius. For we let $C = \oplus_\lambda C_\lambda$, where C_λ are simple subcoalgebras of C. Since $A_\lambda = C_\lambda{}^*$ is a simple algebra, we have $A_\lambda \simeq A_\lambda{}^*$ as left A_λ-modules. Hence we have $C_\lambda \simeq C_\lambda{}^*$ as left $C_\lambda{}^*$-modules, and so as left C^*-modules. Thus we have

$$C = \oplus_\lambda C_\lambda \simeq \oplus_\lambda C_\lambda{}^* \hookrightarrow \Pi_\lambda C_\lambda{}^* = C^*$$

as left C^*-modules.

Theorem 1 (I-p. Lin). Let C be a left co-Frobenius coalgebra. Then we have:

(1) An injective cover of every finite dimensionl right C-comodule is finite dimensional.

(2) Every injective right C-comodule is C-projective.

Proof. (1) Let M be a finite dimensional right C-comodule and let $\sigma(M) = \oplus_{i=1}^n S_i$ be the socle of M (i.e. the sum of all simple right C-subcomodules of M). For the notion of socles and injective covers, we refer to Green[10]. It is easy to see that an injective cover $J(M)$ of M is isomorphic to $\oplus_{i=1}^{n} J(S_i)$, where $J(S_i)$ denotes an injective cover of S_i. Therefore in order to prove (1) it suffices to prove that $J(S)$ is finite dimensional for each simple right C-subcomodules S of M. We may assume that S is a minimal right coideal of C and $J(S) \subset C$. Let x be a non-zero element in S. Then we have $S = C^* \rightharpoonup x$. Since C is left co-Frobenius, there exists an element c in C such that $b(c, x) \neq 0$. We claim that $(c - C^*)^{\perp} \cap S = \{0\}$. Suppose that there exists a non-zero element y in S such that y lies in $(c - C^*)^{\perp}$. Since $S = C^* \rightharpoonup x = C^* \rightharpoonup y$ there exists an element c^* in C^* such that $c^* \rightharpoonup y = x$. Then

$$b(c \leftharpoonup c^*, y) = b(c, c^* \rightharpoonup y) = b(c, x) \neq 0.$$

But $y \in (c \leftharpoonup C^*)^{\perp}$ implies $b(c \leftharpoonup c^*, y) = 0$. This is a contradiction.

Since $c \leftharpoonup C^*$ is a left coideal, $(c \leftharpoonup C^*)^{\perp}$ is a right coideal, by Lemma 2. It follows that $(c \leftharpoonup C^*)^{\perp} \cap J(S) = \{0\}$. In generally, if X is a finite dimensional subspace of C, $X^{\perp}$ is cofinite dimensional since $X^{\perp}$ is the kernel of the map $C \to X^*$ defined by $c \to \theta(c)|X$. Thus we have that $(c \leftharpoonup C^*)^{\perp}$ is cofinite dimensional. It follows that $J(S)$ is finite dimensional. Thus (1) is proved.

(2) Let V be an injective right C-comodule and let $\sigma(V) = \oplus_\lambda S_\lambda$ be the socle of V. Then we have $V \simeq \oplus_\lambda J(S_\lambda)$. Since $J(S_\lambda)$ is finite dimensional it follows from Proposition 4 that $J(S_\lambda)$ is an injective left C^*-module. The embedding

$$J(S_\lambda) \subset C \overset{\theta}{\to} C^*$$

yields that $J(S_\lambda)$ is a direct summand of C^* as a left C^*-module. Therefore $J(S_\lambda)$ is a projective left C^*-module, and so is V. Thus V is a projective right C-comodule. This completes the proof.

Corollary 1. *If C is a left co-Frobenius coalgebra then C is projective as a right C-comodule.*

Corollary 2. *Let C be a left co-Frobenius coalgebra. Then the category of left C-comodules has enough projectives.*

Proof. We have to show that for each left C-comodule N there exists an epimorphism $P \to N \to 0$ with P projective. Without loss of generality, we may assume that N is finite dimensional. Then we consider a monomorphism of finite dimensional right C-comodules $0 \to N^* \to J(N^*)$. Taking the dual, we have an epimorphism of left C-comodules $J(N^*)^* \to N \to 0$. Q.E.D.

2.2. Integrals.

An *augmented coalgebra* is a coalgebra C together with a coalgebra map $u: k \to C$. Clearly $u(1)$ is a grouplike element of C. Using $u: k \to C$ we may convert any k-space X into a left (or right) C-comodule ${}_uX$ (or X_u) by setting $\rho(x) = u(1) \otimes x$ (or $\rho(x) = x \otimes u(1)$). In particular k has a left (or right) C-comodule structure.

$x \in C^*$ is called a *left integral* if x is a left C-comodule map from C to k, i.e. $\Sigma\, c_{(1)}\langle x, c_{(2)}\rangle = \langle x, c\rangle u(1)$ for all $c \in C$. We note that $x \in C^*$ is a left integral if and only if $c^* \cdot x = \langle c^*, u(1)\rangle x$ for all $c^* \in C^*$. An augmented coalgebra need not have a non-zero integral. However, if C is left co-Frobenius then C has a non-zero left integral. In fact, it is easily checked that $b(-, u(1)) = \theta(u(1))$ is a non-zero left integral.

Proposition 11. *Let C be an augmented coalgebra. If C is finite dimensional and left co-Frobenius then the k-space of left integrals is one dimensional.*

Proof. We have that $C \simeq C^*$ as right C-comodules. Therefore

$\mathrm{Com}_C(C, k) \simeq C^* \square_C k \simeq C \square_C k \simeq k.$ Q.E.D.

Lemma 3. *Let H be a Hopf algebra. If J is a non-zero right ideal and a right coideal, then J is equal to H.*

Proof. If $\varepsilon(J) = \{0\}$ then for all $h \in J$,

$h = \Sigma\, \varepsilon(h_{(1)})h_{(2)} = 0$ (since $\Delta(J) \subset J \otimes H$). Hence we must have $\varepsilon(J) \neq \{0\}$. Thus there exists an element h in J such that $\varepsilon(h) = 1$. Since $1 = \varepsilon(h) = \Sigma\, h_{(1)}S(h_{(2)})$ and $J \cdot H \subset J$, we have $1 \in J$. Q.E.D.

Theorem 2 (Lin-Larson-Sweedler-Sullivan). *The following statements concerning a Hopf algebra H are equevalent:*

(i) H *has a non-zero left integral.*

(ii) H *is left co-Frobenius.*

(iii) H *has a non-zero right integral.*

(iv) H *is right co-Frobenius.*

Proof. (i) => (ii). Let x be a non-zero left integral. We define a bilinear form $b: H \times H \to k$ as follows;

$b(c, d) = \langle x, cS(d)\rangle$, for all $c, d \in H$.

Then we compute

$$\begin{aligned}\Sigma\, b(c, d_{(1)})d_{(2)} &= \Sigma \langle x, cS(d_{(1)})\rangle d_{(2)} \\ &= \Sigma\, c_{(1)}S(d_{(2)})\langle x, c_{(2)}S(d_{(1)})\rangle d_{(3)} \\ &= \Sigma\, c_{(1)}\varepsilon(d_{(2)})\langle x, c_{(2)}S(d_{(1)})\rangle \\ &= \Sigma\, c_{(1)}\langle x, c_{(2)}S(d)\rangle = \Sigma\, c_{(1)}b(c_{(2)}, d).\end{aligned}$$

This shows that $b: H \times H \to k$ is C-balanced. Next we show that $H^{\perp}$ (= $\{d \in H \mid b(c, d) = 0$ for all $c \in H\}$) is zero. Let $d \in H^{\perp}$ and $h \in H$. For all $c \in H$, we have

$b(c, dh) = \langle x, cS(dh)\rangle = \langle x, cS(h)S(d)\rangle = b(cS(h), d) = 0.$

Hence $dh \in H$, so $H^{\perp}$ is a right ideal of H. Since $x \neq 0$, $H^{\perp}$ is a proper right ideal. Also $H^{\perp}$ is a right coideal, by Lemma 2. Therefore we have $H^{\perp} = \{0\}$, by Lemma 3.

(ii) => (iii). In the proof of Theorem 1, (1), we obtained that

H contains a proper right coideal of finite codimension. Therefore, by (2.14) in Sweedler [20], H has a non-zero right integral.

(iii) => (iv). The proof is the same as (i) => (ii).

(iv) => (i). The proof is the same as (ii) => (iii).

2.3. Coseparable coalgebras.

Let C be a coalgebra. For every right C-comodule V, we have $\mathrm{Com}_C(V, C) \simeq V^*$, by Proposition 3. If in addition V is a (C, C)-bicomodule then we have an isomorphism

$$\mathrm{Com}_{C,C}(V, C) \simeq \{\gamma \in V^* \mid (I \otimes \gamma)\rho^- = (\gamma \otimes I)\rho^+\},$$
$$\tau \longrightarrow \varepsilon\tau$$
$$(I \otimes \gamma)\rho^- \longleftarrow \gamma .$$

A coalgebra C is called <u>coseparable</u> if there exists a k-map $\tau\colon C \otimes C \to k$ such that $(I \otimes \tau)(\Delta \otimes I) = (\tau \otimes I)(I \otimes \Delta)$ and $\tau\Delta = \varepsilon$. We have immediately from the above isomorphism that C is coseparable if and only if there exists a (C, C)-bicomodule map $\pi\colon C \otimes C \to C$ such that $\pi\Delta = I$. We note that Δ may be viewed as a (C, C)-bicomodule map from C to $C \otimes C$. Thus we may conclude that C is coseparable if and only if C is injective as a $C \otimes C^{op}$-comodule.

Let C and D be coalgebras and let $\tau\colon C \otimes C \to k$ be a k-map such that $(I \otimes \tau)(\Delta \otimes I) = (\tau \otimes I)(I \otimes \Delta)$. For any (C, D)-bicomodule M, N and for each $f \in \mathrm{Com}_D(M, N)$, we define a k-map

$$f_C\colon M \to N$$

by setting $f_C = (\tau \otimes I)(I \otimes \rho_N^-)(I \otimes f)\rho_M^-$.

Lemma 4. <u>In the above situation,</u> f_C <u>is a</u> (C, D)-<u>bicomodule map</u>.

Proof. We can construct the following commutative diagram:

$$\begin{array}{ccccccccc}
M & \xrightarrow{\rho} & C\otimes M & \xrightarrow{I\otimes f} & C\otimes N & \xrightarrow{I\otimes \rho} & C\otimes C\otimes N & \xrightarrow{\tau\otimes I} & N \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle I\otimes\rho} & & \downarrow{\scriptstyle I\otimes\rho} & & \downarrow{\scriptstyle I\otimes I\otimes\rho} & & \downarrow{\scriptstyle \rho} \\
M\otimes D & \xrightarrow{\rho\otimes I} & C\otimes M\otimes D & \xrightarrow{I\otimes f\otimes I} & C\otimes N\otimes D & \xrightarrow{I\otimes\rho\otimes I} & C\otimes C\otimes N\otimes D & \xrightarrow{\tau\otimes I\otimes I} & N\otimes D
\end{array}$$

This shows that f_C is a right D-comodule map. We also have a commutative diagram:

$$\begin{array}{ccccccccc}
M & \xrightarrow{\rho} & C\otimes M & \xrightarrow{I\otimes f} & C\otimes N & \xrightarrow{I\otimes \rho} & C\otimes C\otimes N & \xrightarrow{\tau\otimes I} & N \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \Delta\otimes I} & & \downarrow{\scriptstyle \Delta\otimes I} & & & & \downarrow{\scriptstyle \rho} \\
C\otimes M & \xrightarrow{I\otimes\rho} & C\otimes C\otimes M & \xrightarrow{I\otimes I\otimes f} & C\otimes C\otimes N & \xrightarrow{I\otimes I\otimes\rho} & C\otimes C\otimes C\otimes N & \xrightarrow{I\otimes\tau\otimes I} & C\otimes N
\end{array}$$

This shows that f_C is a left C-comodule. Q.E.D.

Lemma 5. _Let_ L, M, N _and_ P _be_ (C, D)-_bicomodules_. _For each_ $g \in Com_{C,D}(L, M)$, $f \in Com_D(M, N)$ _and_ $h \in Com_{C,D}(N, P)$, _we have_ $(hfg)_C = hf_Cg$.

Proof. $(hfg)_C = (\tau \otimes I)(I \otimes \rho_P^-)(I \otimes hfg)\rho_L^-$

$= (\tau \otimes I)(I \otimes I \otimes h)(I \otimes \rho_N^- f)(I \otimes g)\rho_L^-$

$= h(\tau \otimes I)(I \otimes \rho_N^- f)\rho_M^- g = hf_Cg.$ Q.E.D.

Lemma 6. _Let_ C _be a coseparable coalgebra_. _Let_ M _and_ N _be_ (C, D)-_bicomodules_. _If_ $f: M \to N$ _is a_ (C, D)-_bicomodule map_, _then_ $f_C = f$.

Proof. $f_C = (\tau \otimes I)(I \otimes \rho_N^- f)\rho_M^-$

$= (\tau \otimes I)(I \otimes I \otimes f)(I \otimes \rho_M^-)\rho_M^-$

$= f(\tau \otimes I)(\Delta \otimes I)\rho_M^-$

$= f(\varepsilon \otimes I)\rho_M^- = f.$ Q.E.D.

Proposition 12. _If_ C _is a coseparable coalgebra and_ D _is a co-semi-simple coalgebra then_ $C \otimes D$ _is a co-semi-simple coalgebra_.

Proof. It suffices to prove that every (C, D)-bicomodule M is completely reducible. Let N be a (C, D)-subcomodule of M. Since D is co-semi-simple, there exists a D-comodule map $f: M \to N$ such that $fi = I$, where $i: N \to M$ is the inclusion. We then have $f_Ci = I$,by Lemma 4 and 5. Since f_C is a (C, D)-

bicomodule map, it follows that N is a direct summand of M as a (C, D)-bicomodule. Q.E.D.

Corollary. _If a coalgebra C is coseparable then it is co-semi-simple_.

§3. Cohomology

Since ${}^{C}M$ is an abelian category and has enough injectives, we can define the functor $Ext_C^n(M, N)$ as the n-th right derived functor of the functor $Com_C(-, N)$. Explicitly, we take an injective resolution X of a left C-comodule N:

$$0 \to N \to X_0 \to X_1 \to X_2 \to \cdots$$

Then $Ext_C^n(M, N)$ is defined as the n-th cohomology group of the complex $Com_C(M, X)$.

3.1. Cohomology of coalgebras.

Let C be a coalgebra and N be a (C, C)-bicomodule. Let $C^e = C \otimes C^{op}$ be the enveloping coalgebra of C. Then we regard N as a left (or right) C^e-comodule. In particular we regard C as a left C^e-comodule. Now we define the n-th _cohomology group of_ C _with coefficients in_ N as

$$H^n(N, C) = Ext^n_{C^e}(N, C).$$

Thus we have $H^n(N, C) = H^n(Com_{C,C}(N, X))$, where X is an injective resolution of C as a left C^e-comodule. On the other hand, consider the complex $N \,\square_{C^e}\, X$ and we define another n-th cohomology group as

$$Hoch^n(N, C) = H^n(N \,\square_{C^e}\, X).$$

We note that if N is finite dimensional then $H^n(N, C) \simeq Hoch^n(N^*, C)$.

Next we shall describe a construction of a standard complex. For each integer $n \geq -1$, let $S^n(C)$ denote the (n+2)-fold tensor product of C. We convert $S^n(C)$ into a (C, C)-bicomodule by setting $\rho^-(c_0 \otimes c_1 \otimes \dots \otimes c_{n+1}) = \Delta(c_0) \otimes c_1 \otimes \dots \otimes c_{n+1}$ and $\rho^+(c_0 \otimes c_1 \otimes \dots \otimes c_{n+1}) = c_0 \otimes \dots \otimes c_n \otimes \Delta(c_{n+1})$. Clearly $S^n(C)$ is injective as a left C^e-comodule. We now define for each $n \geqq 0$ a C^e-comodule map

$$d^n\colon S^n(C) \to S^{n+1}(C)$$

by $d^n(c_0 \otimes c_1 \otimes \dots \otimes c_{n+1}) = \Sigma_{i=0}^{n+1}(-1)^i c_0 \otimes \dots \otimes \Delta(c_i) \otimes \dots \otimes c_{n+1}$.
We define for each $n \geqq 1$ a right C-comodule map

$$s^n\colon S^n(C) \to S^{n-1}(C)$$

by $s^n(c_0 \otimes c_1 \otimes \dots \otimes c_{n+1}) = \varepsilon(c_0)c_1 \otimes \dots \otimes c_{n+1}$.
One verifies directly that

$$s^{n+1}d^n + d^{n-1}s^n = I \quad (n \geqq 1).$$

This shows that

$$C = S^{-1}(C) \xrightarrow{\Delta} S^0(C) \xrightarrow{d^0} S^1(C) \xrightarrow{d^1} \dots$$

is an injective resolution of C as a left C^e-comodule. We observe that $S^0(C) = C \otimes C$ coincides with $C^e = C \otimes C^{op}$ as a C^e-comodule. More generally we have $S^n(C) \simeq C^e \otimes C^{[n]}$ as a C^e-comodule, where $C^{[n]}$ is the n-fold tensor product of C for each $n > 0$, and $C^{[0]} = k$.

In computing the cohomology groups we use the identifications:

$$\mathrm{Com}_{C^e}(N, S^n(C)) = \mathrm{Com}_{C^e}(N, C^e \otimes C^{[n]}) = \mathrm{Hom}(N, C^{[n]})$$

$$N \,\square_{C^e}\, S^n(C) = N \,\square_{C^e}\, (C^e \otimes C^{[n]}) = N \otimes C^{[n]}.$$

Thus $H^n(N, C)$ are the cohomology groups of the complex $\{\mathrm{Hom}(N, C^{[n]})\}_{n\geqq 0}$ with differentiation

$$\delta^n\colon \mathrm{Hom}(N, C^{[n]}) \to \mathrm{Hom}(N, C^{[n+1]})$$

by
$$\delta^n(f) = (I \otimes f)\rho_N^- - (\Delta \otimes I \otimes \dots)f + (I \otimes \Delta \otimes \dots)f$$
$$- \dots \pm (I \otimes \dots \otimes \Delta)f \mp (f \otimes I)\rho_N^+ .$$

And $\mathrm{Hoch}^n(N, C)$ are the cohomology groups of the comlex $\{N \otimes C^{[n]}\}_{n\geqq 0}$ with differentiation

$$D^n\colon N \otimes C^{[n]} \to N \otimes C^{[n+1]}$$

by $D^n(v \otimes c_1 \otimes \ldots c_n) = \rho^+(v) \otimes c_1 \otimes \ldots \otimes c_n$

$+ \Sigma_{i=1}^{n}(-1)^i v \otimes c_1 \otimes \ldots \Delta(c_i) \otimes \ldots \otimes c_n$

$+ (-1)^{n+1}\Sigma\, v_{(0)} \otimes c_1 \otimes \ldots \otimes c_n \otimes v_{(-1)}$,

where we write $\rho^-(v) = \Sigma\, v_{(-1)} \otimes v_{(0)} \in C \otimes N$.

We obtain that $H^0(N, C) = \{\gamma \in N^* \mid (I \otimes \gamma)\rho^- = (\gamma \otimes I)\rho^+\}$ $\cong \mathrm{Com}_{C,C}(N, C)$ and $\mathrm{Hoch}^0(N, C) = \{n \in N \mid t\rho^-(n) = \rho^+(n)\}$.

A k-map $f: N \to C$ from a (C, C)-bicomodule N into C with the property $\Delta f = (I \otimes f)\rho^- + (f \otimes I)\rho^+$ is called a <u>coderivation</u> from N into C. The coderivation f is called an <u>inner</u> coderivation provided that there exists a $\gamma \in N^*$ such that $f = (I \otimes \gamma)\rho^- - (\gamma \otimes I)\rho^+$. Thus we have an exact sequence

$$0 \to H^0(N, C) \to N^* \to \mathrm{Coder}(N, C) \to H^1(N, C) \to 0,$$

where $\mathrm{Coder}(N, C)$ denotes the k-space of all coderivations from N into C.

We now introduce an universal coderivation. Let L be the cokernel of $\Delta : C \to C \otimes C$. Then we have an exact sequence of (C, C)-bicomodules

$$0 \to C \xrightarrow{\Delta} C \otimes C \xrightarrow{\omega} L \to 0.$$

We denote $c \circ c' = \omega(c \otimes c')$ and we define a map

$$\lambda: L \to C$$

by $\lambda(c \circ c') = c\varepsilon(c') - \varepsilon(c)c'$. It is easily checked that λ is a coderivation from L into C. Moreover λ is an universal coderivation in the following sense:

Proposition 13. <u>For any</u> (C, C)-<u>bicomodule</u> N, <u>the map</u>

$$\mathrm{Com}_{C,C}(N, L) \to \mathrm{Coder}(N, L)$$

<u>sending</u> σ <u>to</u> $\lambda\sigma$, <u>is an</u> k-<u>isomorphism</u>.

Proof. Let $f \in \mathrm{Coder}(N, C)$. Then $\omega(f \otimes I)\rho_N^+ \in \mathrm{Com}_{C,C}(N, L)$. For any $n \in N$, We have

$\lambda\omega(f \otimes I)\rho_N^+(n) = \Sigma\, \lambda\omega(f(n_{(0)}) \otimes n_{(1)})$

$= \Sigma\, f(n_{(0)})\varepsilon(n_{(1)}) - \Sigma\, \varepsilon(f(n_{(0)}))n_{(1)} = f(n)$, since $\varepsilon f = 0$ for any coderivation f. Hence we have $\lambda\omega(f \otimes I)\rho_N^+ = f$.

Coversely, let $\sigma \in \mathrm{Com}_{C,C}(N, L)$. Then we have

$\omega(\lambda\sigma \otimes I)\rho_N^+ = \omega(\lambda \otimes I)\rho_L^+\sigma = \sigma$, since $\omega(\lambda \otimes I)\rho_L^+ = I$.

Thus the correspondence $\sigma \to \lambda\sigma$ gives a k-isomorphism, and this completes the proof.

Theorem 3. The following statements concerning a coalgebra C are equivalent:

(i) C is coseparable.

(ii) For every (C, C)-bicomodule N, we have $H^n(N, C) = \{0\}$ for all $n \geq 1$.

(iii) Every coderivation from any (C, C)-bicomodule into C is an inner coderivation.

(iv) $\lambda: L \to C$ is an inner coderivation.

Proof. (i) => (ii) is immediate from the fact that a coseparable coalgebra C is injective as a C^e-comodule. (ii) => (iii) and (iii) => (iv) are obvious. Now we prove (iv) => (i). Suppose that λ is inner. Then there exists a $\gamma \in L^*$ such that $\lambda = (I \otimes \gamma)\rho_L^- - (\gamma \otimes I)\rho_L^+$. We define a (C, C)-bicomodule map $\xi: L \to C \otimes C$ by $\xi = (I \otimes \gamma \otimes I)(\rho_L^- \otimes I)\rho_L^+$. Then we have

$$\begin{aligned}\xi &= ((\lambda + (\gamma \otimes I)\rho_L^+) \otimes I)\rho_L^+ \\ &= (\lambda \otimes I)\rho_L^+ + (\gamma \otimes I \otimes I)(I \otimes \Delta)\rho_L^+ \\ &= (\lambda \otimes I)\rho_L^+ + \Delta(\gamma \otimes I)\rho_L^+ .\end{aligned}$$

Hence we have $\omega\xi = \omega(\lambda \otimes I)\rho_L^+ = I$. This means that the exact sequence

$$0 \to C \xrightarrow{\Delta} C \otimes C \xrightarrow{\omega} L \to 0,$$

splits as a (C, C)-bicomodule. Therefore we have that C is coseparable, and the theorem is completely proved.

3.2. Extensions of coalgebras.

Let C be a coalgebra. An <u>extension</u> of C is any coalgebra D which contains C as a subcoalgebra.

Now we consider an extension D of C with $D = C \wedge C$, i.e. $\Delta(D) \subset D \otimes C + C \otimes D$ (see Sweedler [19], p.179). In this case we may regard the quotient space $\bar{D} = D/C$ as a (C, C)-bicomodule by

$$\rho^+ : D \xrightarrow{\Delta} D \otimes D \xrightarrow{p \otimes I} \bar{D} \otimes D$$

$$\rho^- : D \xrightarrow{\Delta} D \otimes D \xrightarrow{I \otimes p} D \otimes \bar{D}$$

where $p: D \longrightarrow \bar{D}$ denotes the natural projection, since we have $\operatorname{Im} \rho^+ \subset \bar{D} \otimes C$ and $\operatorname{Im} \rho^- \subset C \otimes \bar{D}$.

Let ψ be a k-map of $D \to C$ such that $\psi|C$ = identity. We then have that the following giagrams are commutative:

$$\begin{array}{ccc} D & \xrightarrow{\Delta} & D \otimes D \\ \downarrow p & \rho^+ & \downarrow p\otimes\psi \\ \bar{D} & \xrightarrow{\rho^+} & \bar{D} \otimes C \end{array} \qquad \begin{array}{ccc} D & \xrightarrow{\Delta} & D \otimes D \\ \downarrow p & \rho^- & \downarrow \psi\otimes p \\ \bar{D} & \xrightarrow{\rho^-} & C \otimes \bar{D} \end{array}$$

Define a map $f: D \to C \otimes C$ by setting

$$f = (\psi \otimes \psi)\Delta - \Delta\psi.$$

Then $f(C) = 0$ and thus f induces a k-map $\bar{f}: \bar{D} \to C \otimes C$ with $\bar{f}p = f$.

Lemma 6. $\bar{f}$ <u>is a</u> 2-<u>cocycle in</u> $\operatorname{Hom}(\bar{D}, C^{[2]})$.

Proof. We compute

$$\delta^2(\bar{f})p = (I \otimes \bar{f})\rho^- p - (\Delta \otimes I)\bar{f}p + (I \otimes \Delta)\bar{f}p - (\bar{f} \otimes I)\rho^+ p$$

$$= (\psi \otimes f)\Delta - (\Delta \otimes I)f + (I \otimes \Delta)f - (f \otimes \psi)\Delta$$

$$= \{(\psi \otimes \psi \otimes \psi)(I \otimes \Delta)\Delta - (\psi \otimes \Delta\psi)\Delta\} - \{(\Delta\psi \otimes \psi)\Delta - (\Delta \otimes I)\Delta\psi\}$$

$$+ \{(\psi \otimes \Delta\psi)\Delta - (I \otimes \Delta)\Delta\psi\} - \{(\psi \otimes \psi \otimes \psi)(\Delta \otimes I)\Delta + (\Delta\psi \otimes \psi)\Delta\}$$

$$= 0.$$

Since p is surjective we have $\delta^2(\bar{f}) = 0$. Q.E.D.

Let ψ_1 and ψ_2 be k-maps of $D \to C$ such that $\psi_1|C = \psi_2|C$ = identity. Construct the maps $\bar{f}_1$ and $\bar{f}_2$ as above.

Lemma 7. $\bar{f}_1$ <u>and</u> $\bar{f}_2$ <u>are cohomologous</u>.

Proof. Let $g = \psi_1 - \psi_2$. Since $g(C) = 0$, g induces a k-maps $\overline{g}: \overline{D} \to C$ with $\overline{g}p = g$. Then

$$\delta^1(\overline{g})p = (I \otimes \overline{g})\rho^- p - \Delta\overline{g}p + (\overline{g} \otimes I)\rho^+ p$$
$$= (\psi_1 \otimes g)\Delta - \Delta g + (g \otimes \psi_1)\Delta .$$

This implies that

$$\overline{f}_2 p = (\psi_2 \otimes \psi_2)\Delta - \Delta\psi_2$$
$$= ((\psi_1 - g) \otimes (\psi_1 - g))\Delta - \Delta(\psi_1 - g)$$
$$= \{(\psi_1 \otimes \psi_1)\Delta - \Delta\psi_1\} - \{(\psi_1 \otimes g) - \Delta g - (g \otimes \psi_1)\Delta\}$$
$$= \overline{f}_1 p - \delta^1(\overline{g})p.$$

Therefore $\overline{f}_2 = \overline{f}_1 - \delta^1(\overline{g})$, and this shows that $\overline{f}_1$ and $\overline{f}_2$ are cohomologous. Q.E.D.

Summarizing, we find that an extension D of C with $D = C \wedge C$ defines uniquely an element $[\overline{f}]$ = class of $\overline{f}$, in $H^2(\overline{D}, C)$.

Theorem 4. <u>Let</u> D <u>be an extension of a coalgebra</u> C <u>with</u> $D = C \wedge C$. <u>Then we have that</u> $[\overline{f}] = 0$ <u>in</u> $H^2(\overline{D}, C)$ <u>if and only if there exists a coalgebra map</u> $\psi: D \to C$ <u>such that</u> $\psi|C = I$.

Proof. Suppose that $[\overline{f}] = 0$. Let ψ be a k-map of $D \to C$ such that $\psi|C = I$. $\overline{f}$ can be viewed as the 2-cocycle associated with ψ. Since $[\overline{f}] = 0$ there exists a $\overline{g} \in \mathrm{Hom}(\overline{D}, C)$ such that $\overline{f} = \delta^1(\overline{g})$. Set $\psi' = \psi - \overline{g}p$. Then ψ' is a k-map of $D \to C$ such that $\psi'|C = I$. Let $\overline{f}'$ be the 2-cocycle associated with ψ'. The proof of Lemma 7 then implies that

$$\overline{f}' = \overline{f} - \delta^1(\overline{g}) = \overline{f} - \overline{f} = 0,$$

that is, ψ' is a coalgebra map.

The " if " part of the assertion is clear.

Q. E. D.

Remark. More generally we can show that the second cohomology group $H^2(M, C)$ for a (C, C)-bicomodule M is in one-to-one correspondence with the set of equivalence classes of extensions over C with cokernel M

$$C \xrightarrow{i} D \xrightarrow{p} M$$

(that is, D is a coalgebra, i is an injective coalgebra map, $i(C) \wedge i(C) = D$, p is a surjective k-map which induces $D/i(C) \simeq M$ as a (C, C)-bicomodule.) Two extensions $C \xrightarrow{i} D \xrightarrow{p} M$ and $C \xrightarrow{i'} D' \xrightarrow{p'} M$ over C with cokernel M are *equivalent* if there exists a coalgebra isomorphism $f : D \to D'$ such that the diagram

$$\begin{array}{ccccc} & & D & & \\ & \overset{i}{\nearrow} & \downarrow f & \overset{p}{\searrow} & \\ C & & & & M \\ & \underset{i'}{\searrow} & D' & \underset{p'}{\nearrow} & \end{array}$$

is commutative.

These results are used to give a proof of the next theorem for coalgebras with coseparable.

Theorem 5 (Sullivan [8]). *For C a coalgebra with coseparable coradical R, there exists a coalgebra map $\psi: C \to R$ such that $\psi|R = I$.*

Proof. C has a filtration by subcoalgebras $R = C_0 \subset C_1 \subset \cdots$ where $C_i = \wedge^{i+1}R$ $(i = 0,1,2,\ldots)$. Thus it is enough to construct a sequence $\psi_0, \psi_1, \ldots$ such that ψ_i is a coalgebra map of $C_i \to R$ and $\psi_i | C_{i-1} = \psi_{i-1}$, for all $i \geqq 1$. For since $C = \bigcup C_i$ there is a unique coalgebra map $\psi: C \to R$ which extends all the ψ_i. It is clear that $\psi | R = I$, therefore all is good.

To construct the sequence, assume inductively that we have $\psi_0, \psi_1, \ldots, \psi_n$ for some fixed $n \geqq 1$. Let J_n denote the kernel of ψ_n. C_{n+1}/J_n can be viewed as an extension coalgebra of C_n/J_n. Then it is easily checked that $C_{n+1}/J_n = C_n/J_n \wedge C_n/J_n$ and $C_n/J_n \simeq R$. It follows from Theorem 3 and Theorem 4 that there exists a coalgebra map $f: C_{n+1}/J_n \to C_n/J_n$ with $f|(C_n/J_n) = I$. Now we define a coalgebra map $\psi_{n+1}: C_{n+1} \to R$ by the composite

$$C_{n+1} \xrightarrow{\text{proj.}} C_{n+1}/J_n \xrightarrow{f} C_n/J_n \simeq R.$$

Then we have $\psi_{n+1} | C_n = \psi_n$, and this completes the proof.

Remark. Given two coalgebra maps ψ and ψ' with $\psi | R = \psi' | R$ = identity, we can find a relation between ψ and ψ'. In fact, C becomes a (R, R)-bicomodule by

$$\rho^-: C \overset{\Delta}{\to} C \otimes C \xrightarrow{\psi \otimes I} R \otimes C$$

$$\rho^+: C \overset{\Delta}{\to} C \otimes C \xrightarrow{I \otimes \psi'} C \otimes R.$$

Since R is a (R, R)-subcomodule of C, C/R is an (R, R)-bicomodule. Since $\psi | R = \psi' | R$, $\psi - \psi' : C \to R$ induces a k-map $\overline{\psi - \psi'}: C/R \to R$. Then it is easy to show that $\overline{\psi - \psi'}$ is a coderivation from a (R, R)-bicomodule C/R into R. It follows from Theorem 3 that there exists an element γ in $(C/R)^*$ such that $\delta^0(\gamma) = \overline{\psi - \psi'}$. Rewriting this equation, we have

$$(\psi \otimes \gamma p)\Delta - (\gamma p \otimes \psi')\Delta = \psi - \psi'$$

where $p: C \to C/R$ denotes the natural projection. Set $d^* = \varepsilon - \gamma p$ (in C^*). Then we obtain

$$\psi(d^* \rightharpoonup c) = \psi'(c \leftharpoonup d^*) \qquad \text{for all } c \in C.$$

3.3. Cohomology of augmented coalgebras.

Let (C, u) be an augmented coalgebra (see 2.2). Then k has a left C-comodule structure, and cohomology groups $\mathrm{Ext}^n_C(N, k)$ are defined for every left C-comodule N.

Theorem 6. <u>For every left</u> C-<u>comodule</u> N, <u>we have</u>
$\mathrm{Ext}^n_C(N, k) \cong H^n(N_u, C)$.

Proof. We apply Proposition 6 to obtain that for every (C, C)-bicomodule V,

$$\mathrm{Com}_C(N, V \square_C k) \cong \mathrm{Com}_{C,C}(N_u, V).$$

Therefore it suffices to show that the complex $\{X^n \square_C k\}$ is an injective resolution of k as a left C-comodule, for each injective resolution of C as a C^e-comodule;

$$C \to X^0 \xrightarrow{d^0} X^1 \xrightarrow{d^1} X^2 \xrightarrow{d^2} \cdots\cdots$$

Taking $V = X^n$ in the above isomorphism, we obtain that $X^n \square_C k$ is injective as a left C-comodule. Now let $Z^n = \mathrm{Ker}\, d^n = \mathrm{Im}\, d^{n-1}$ $(n \geqq 1)$. Then we have the exact sequences of (C, C)-bicomodules;

$$0 \to C \to X^0 \to Z^1 \to 0$$

$$0 \to Z^n \to X^n \to Z^{n+1} \to 0 \qquad (n \geqq 1).$$

Since C and X^0 are injective as a left C-comodule, so is clearly Z^1. It follows by induction that Z^n $(n \geqq 1)$ is injective as a left C-comodule, since from Proposition 2,(1) X^n $(n \geqq 0)$ is injective as a left C-comodule. Therefore we have the exact sequences ;

$$0 \to Z^n \square_C k \to X^n \square_C k \to Z^{n+1} \square_C k \to 0.$$

This shows that the comlex $\{X^n \square_C k\}$ is an injective resolution of k as a left C-comodule, and completes the proof.

Remark. Similarly, we can show for every right C-comodule M that $\mathrm{Hoch}^n({}_uM, C)$ coincides with the n-th cohomology group $H^n(M \square_C X)$, where X is an injective resolution of k as a left C-comodule, since we have that for every (C, C)-bicomodule V,

$$M \,\square_C\, (V \,\square_C\, k) \;\cong\; {}_u M \,\square_{C^e}\, V.$$

Now consider the particular case when C is a Hopf algebra. We define a k-map

$$\nabla\colon C^e = C \otimes C^{op} \to C$$

by setting $\nabla(c \otimes d^{op}) = cS(d)$, where S is the antipode of C.

Clearly, ∇ is a coalgebra map. Given a (C, C)-bicomodule N we shall denote by ${}_\nabla N$ (or N_∇) the k-space N regarded as a left (or right) C-comodule by means of the map ∇. In particular $(C^e)_\nabla$ is a (C^e, C)-bicomodule. Assume that C is involutory, i.e. S^2 = identity. Then the map

$$\alpha\colon (C^e)_\nabla \to C \otimes C$$

defined by setting $\alpha(c \otimes d^{op}) = \Sigma\, c_{(1)} \otimes c_{(2)}S(d)$ is a right C-comodule isomorphism, where $C \otimes C$ regarded as a right C-comodule by $\rho\colon C \otimes C \xrightarrow{I \otimes \Delta} C \otimes C \otimes C$. The inverse of α is given by $x \otimes y \to \Sigma\, x_{(1)} \otimes (S(y)x_{(2)})^{op}$. Therefore $(C^e)_\nabla$ is free as a right C-comodule. It follows that for each injective resolution of k as a left C-comodule, $k \to X^0 \to X^1 \to \ldots,$ we have an exact sequence

$$(C^e)_\nabla \,\square_C\, k \to (C^e)_\nabla \,\square_C\, X^0 \to \ldots . \qquad (*)$$

Moreover $(C^e)_\nabla \,\square_C\, X^n$ $(n \geqq 1)$ is injective as a left C^e-comodule, by Corollary of Proposition 5. Since $(C^e)_\nabla \,\square_C\, k \cong C$, it follows that the sequence $(*)$ is an injective resolution of C as a left C^e-comodule. Thus we have:

Theorem 7. _Let C be an involutory Hopf algebra. For every (C, C)-bicomodule N, we have_

$$\mathrm{Ext}^n_C({}_\nabla N, k) \cong H^n(N, C).$$

CHAPTER II

ON THE STRUCTURE OF RELATIVE HOPF MODULES

All vector spaces will be over a field k. Map always means k-linear map, and the unadorned tensor product $V \otimes W$ is understood to be $V \otimes_k W$. We use the sigma notation. Thus, if C is a coalgebra, we write $\Delta(c) = \Sigma\, c_{(1)} \otimes c_{(2)}$, for $c \in C$. If M is a right C-comodule with comodule structure map $\rho : M \longrightarrow M \otimes C$, we write for $m \in M$,

$$\rho(m) = \Sigma\, m_{(0)} \otimes m_{(1)}.$$

Throughout this chapter A is a Hopf algebra with antipode S.

Let B be an algebra and a right A-comodule. The comodule structure map will be denoted by $\rho_B : B \longrightarrow B \otimes A$ and for $\rho_B(b)$ we write $\Sigma\, b_{(0)} \otimes b_{(1)}$. B is called a right A-comodule algebra if ρ_B is an algebra map.

A is itself a right A-comodule algebra via $\Delta : A \longrightarrow A \otimes A$. More generally, if B is a subalgebra and a right coideal of A then B becomes a right A-comodule algebra. The ground field k has a trivial right A-comodule algebra structure given by

$$u_A : k \longrightarrow A \simeq k \otimes A.$$

Definition. Let B be a right A-comodule algebra. M is called a right (A, B)-Hopf module if M is a right A-comodule and a right B-module such that the following diagram commutes

$$\begin{array}{ccccc} M \otimes B & \xrightarrow{\ \omega\ } & M & \xrightarrow{\ \rho\ } & M \otimes A \\ \Big\downarrow {\scriptstyle \rho_M \otimes \rho_B} & & & & \Big\uparrow {\scriptstyle \omega_M \otimes M_A} \\ M \otimes A \otimes B \otimes A & & \xrightarrow{\ I \otimes T \otimes I\ } & & M \otimes B \otimes A \otimes A \end{array}$$

(ω_M is the B-module action of M, ρ_M is the A-comodule structure map of M, M_A is the multiplication in A, T is the twist map).

The diagram can be expressed as

$$\rho_M(mb) = \Sigma\, m_{(0)}b_{(0)} \otimes m_{(1)}b_{(1)}$$

for all $m \in M$, $b \in B$.

We note that B is itself a right (A, B)-Hopf module via ρ_B and $M_B : B \otimes B \longrightarrow B$.

Theorem 1. Let B be a right A-comodule algebra where there is a right A-comodule map $\phi : A \longrightarrow B$ with $\phi(1_A) = 1_B$. Then every right (A, B)-Hopf module is injective as an A-comodule.

Proof. Let M be a right (A, B)-Hopf module. If $M \otimes A$ has the right A-comodule structure given by $I \otimes \Delta : M \otimes A \longrightarrow (M \otimes A) \otimes A$ then the comodule structure map $\rho_M : M \longrightarrow M \otimes A$ is an A-comodule map. We show that there is an A-comodule map $\lambda : M \otimes A \longrightarrow M$ with $\lambda\rho_M = I$. Thus M is injective since it is isomorphic to a direct summand of $M \otimes A$, an injective A-comodule.

Define $\lambda : M \otimes A \longrightarrow M$ as the composite

$$M \otimes A \xrightarrow{\rho \otimes I} M \otimes A \otimes A \xrightarrow{I \otimes S \otimes I} M \otimes A \otimes A \xrightarrow{I \otimes M_A} M \otimes A$$

$$\xrightarrow{I \otimes \phi} M \otimes B \xrightarrow{\omega} M$$

so that $\lambda(m \otimes a) = \Sigma\, m_{(0)}\phi(S(m_{(1)})a)$ for $m \in M$, $a \in A$.

For any $m \in M$,

$$\lambda\rho_M(m) = \lambda(\Sigma\, m_{(0)} \otimes m_{(1)}) = \Sigma\, m_{(0)}\phi(S(m_{(1)})m_{(2)})$$

$$= \Sigma\, m_{(0)}\varepsilon(m_{(1)})\phi(1_A) = m$$

so that $\lambda\rho_M$ is the identity of M.

Next we claim λ is an A-comodule map.

$$\rho_M\lambda(m \otimes a) = \rho_M(\Sigma\, m_{(0)}\phi(S(m_{(1)})a))$$

$$= \Sigma\, m_{(0)}\phi(S(m_{(2)})a)_{(0)} \otimes m_{(1)}\phi(S(m_{(2)})a)_{(1)}$$

The condition that ϕ be a right A-comodule map is exactly $\rho_B\phi = (\phi \otimes I)\Delta_A$ or for $a \in A$, $\Sigma\, \phi(a)_{(0)} \otimes \phi(a)_{(1)} = \Sigma\phi(a_{(1)})\otimes a_{(2)}$. Since the antipode S is an anti-algebra map the above expression equals

$$\Sigma\, m_{(0)}\phi(S(m_{(3)})a_{(1)}) \otimes m_{(1)}S(m_{(2)})a_{(2)}$$

$$= \Sigma\, m_{(0)}\phi(S(m_{(2)})a_{(1)}) \otimes \varepsilon(m_{(1)})a_{(2)}$$

$$= \Sigma\, m_{(0)}\phi(S(m_{(1)})a_{(1)}) \otimes a_{(2)}$$

$$= (\lambda \otimes I)(I \otimes \Delta_A)(m \otimes a).$$

Thus λ is an A-comodule map. q. e. d.

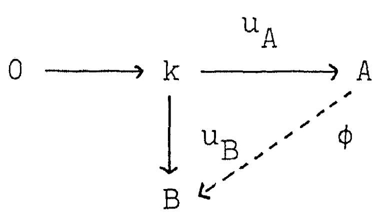
In case $B = k$, the above result reduces to [19. LEMMA 14.0.2].

<u>Corollary</u>. The following statements concerning a right A-comodule algebra B are equivalent:

(i) B is an injective A-comodule.

(ii) There is a right A-comodule map $\phi : A \longrightarrow B$ with $\phi(1_A) = 1_B$.

<u>Proof</u>. Consider the diagram of right A-comodules



If B is an injective A-comodule then the diagram can be completed by an A-comodule map ϕ to a commutative diagram. Thus we have that (i) implies (ii).

Since B may be regarded as a right (A, B)-Hopf module it follows from Theorem 1 that (ii) implies (i). q. e. d.

Let A, A' be Hopf algebras and $f : A' \longrightarrow A$ be a Hopf algebra map. Then A' becomes a right A-comodule algebra via

$$A' \xrightarrow{\ \Delta\ } A' \otimes A' \xrightarrow{\ I \otimes f\ } A' \otimes A.$$

Theorem 2. Let $f : A' \longrightarrow A$ be a surjective Hopf algebra map. If there is a right A-comodule map $\phi : A \longrightarrow A'$ with $\phi(1_A)=1_{A'}$ then we have:

(1) A' is injective as a right A-comodule.

(2) For any left A-comodule V, the canonical map

$$A' \square_A V \xrightarrow{f \otimes I} A \square_A V \simeq V$$

is surjective, where $\square_A$ denotes the cotensor product over A.

Proof. (1) is clear by Theorem 1 and thus we need only show (2). Since f is an A-comodule map, $\operatorname{Ker} f$ becomes a right (A, A')-Hopf module in a natural way. Thus we have from Theorem 1 that $\operatorname{Ker} f$ is an injective A-comodule. This implies that the sequence

$$0 \longrightarrow \operatorname{Ker} f \longrightarrow A' \xrightarrow{f} A \longrightarrow 0$$

is a split exact sequence of right A-comodules. Cotensoring over A by V yields the exact sequence

$$0 \longrightarrow (\operatorname{Ker} f) \square_A V \longrightarrow A' \square_A V \longrightarrow A \square_A V \longrightarrow 0.$$

q. e. d.

Remark. The above Theorem shows that for surjective Hopf algebra map $f : A' \longrightarrow A$, f is right coflat if and only if it is right faithfully coflat.

We return to the first setting where B is a right A-comodule algebra.

Define the A-invariant subspace of B to be the set

$$B_0 = \{ b \in B \mid \rho_B(b) = b \otimes 1_A \}.$$

It is clear that B_0 is a subalgebra of B.

Let V be a right B_0-module. Then $V \otimes_{B_0} B$ is a right B-module in the usual way. It is also a right A-comodule with comodule structure map $\rho : v \otimes_{B_0} b \longmapsto \Sigma\, v \otimes_{B_0} b_{(0)} \otimes b_{(1)}$ (this is well defined). One easily checks that $V \otimes_{B_0} B$ is a right (A, B)-Hopf module.

Let M be a right (A, B)-Hopf module. Define the set

$$M_0 = \{\, m \in M \mid \rho_M(m) = m \otimes 1_A \,\}.$$

For any $m \in M_0$ and $b \in B_0$ we have $mb \in M_0$ and thus M_0 is a right B_0-module. Define

$$\alpha : M_0 \otimes_{B_0} B \longrightarrow M$$

by $\alpha(m \otimes_{B_0} b) = mb$ for $m \in M_0$, $b \in B$. It is then an (A, B)-Hopf map, that is, an A-comodule map and a B-module map.

<u>Theorem 3</u>. Let B be a right A-comodule algebra. If there is a right A-comodule map $\phi : A \longrightarrow B$ which is an algebra map then for every right (A, B)-Hopf module M,

$$\alpha : M_0 \otimes_{B_0} B \longrightarrow M$$

is an isomorphism of (A, B)-Hopf modules.

<u>Proof</u>. Let $P : M \longrightarrow M$ be the composite

$$M \xrightarrow{\ \rho\ } M \otimes A \xrightarrow{\ I \otimes S\ } M \otimes A \xrightarrow{\ I \otimes \phi\ } M \otimes B \xrightarrow{\ \omega\ } M.$$

Explicitly $P(m) = \Sigma\, m_{(0)}\phi(S(m_{(1)}))$.

We claim $P(M) \subset M_0$:

$$\rho_M P(m) = \Sigma\, m_{(0)}\phi(S(m_{(2)}))_{(0)} \otimes m_{(1)}\phi(S(m_{(2)}))_{(1)}$$

$$= \Sigma\, m_{(0)}\phi(S(m_{(3)})) \otimes m_{(1)}S(m_{(2)})$$

$$= \Sigma\, m_{(0)}\phi(S(m_{(2)})) \otimes \varepsilon(m_{(1)})1_A$$

$$= \Sigma\, m_{(0)}\phi(S(m_{(1)})) \otimes 1_A = P(m) \otimes 1_A.$$

Thus P is in fact a map $M \longrightarrow M_0$.

Define $\beta : M \longrightarrow M_0 \otimes_{B_0} B$ by

$$\beta(m) = \Sigma\, P(m_{(0)}) \otimes_{B_0} \phi(m_{(1)}).$$

We will show $\alpha\beta = I$ and $\beta\alpha = I$:

$$\alpha\beta(m) = \alpha(\Sigma\, m_{(0)}\phi(S(m_{(1)})) \otimes_{B_0} \phi(m_{(2)}))$$

$$= \Sigma\, m_{(0)}\phi(S(m_{(1)}))\phi(m_{(2)})$$

$$= \Sigma\, m_{(0)}\phi(S(m_{(1)})m_{(2)}) = m.$$

For $m \in M_0$, $b \in B$,

$$\beta\alpha(m \otimes_{B_0} b) = \beta(mb) = \Sigma\, P(mb_{(0)}) \otimes_{B_0} \phi(b_{(1)})$$

$$= \Sigma\, mb_{(0)}\phi(S(b_{(1)})) \otimes_{B_0} \phi(b_{(2)})$$

since $\Sigma\, b_{(0)}\phi(S(b_{(1)})) \in B_0$ for any $b \in B$

$$= \Sigma\, m \otimes_{B_0} b_{(0)}\phi(S(b_{(1)}))\phi(b_{(2)})$$

$$= \Sigma\, m \otimes_{B_0} b_{(0)}\phi(\varepsilon(b_{(1)})1_B) = m \otimes_{B_0} b.$$ q. e. d.

In case $B = A$ and $\phi = I$, the above Theorem reduces to [19, THEOREM 4.1.1].

We dualize Theorem 1, 2 and 3.

Let C be a coalgebra which is a right A-module. C is a right A-<u>module</u> <u>coalgebra</u> if the following hold for all $c \in C$, $a \in A$:

(1) $\Delta(ca) = \Sigma\, c_{(1)}a_{(1)} \otimes c_{(2)}a_{(2)}$

(2) $\varepsilon(ca) = \varepsilon(c)\varepsilon(a)$.

A is itself a right A-module coalgebra via $M_A : A \otimes A \longrightarrow A$. The ground field k has a trivial right A-module coalgebra structure.

Let N be a right C-comodule and a right A-module. N is called a right $[C, A]$-Hopf module if the following folds for all $n \in N$, $a \in A$:

$$\rho(na) = \Sigma\, n_{(0)}a_{(1)} \otimes n_{(1)}a_{(2)}.$$

Suppose that there exists a right A-module map $\psi : C \longrightarrow A$ with $\varepsilon_A\psi = \varepsilon_C$. For any right $[C, A]$-Hopf module N, define $\lambda : N \longrightarrow N \otimes A$ as the composite

$$N \xrightarrow{\rho} N \otimes C \xrightarrow{I \otimes \psi} N \otimes A \xrightarrow{I \otimes \Delta} N \otimes A \otimes A$$

$$\xrightarrow{I \otimes S \otimes I} N \otimes A \otimes A \xrightarrow{\omega \otimes I} N \otimes A$$

so that $\lambda(n) = \Sigma\, n_{(0)}S(\psi(n_{(1)})_{(1)}) \otimes \psi(n_{(1)})_{(2)}$. If $N \otimes A$ has the right A-module structure given by $(N \otimes A) \otimes A \xrightarrow{I \otimes M} N \otimes A$ then λ is an A-module map with $\omega\lambda = I$. Thus N is a projective A-module since it is isomorphic to a direct summand of $N \otimes A$, a free A-module.

We summarize this in the following theorem :

Theorem 4. Let C be a right A-module coalgebra where there is a right A-module map $\psi : C \longrightarrow A$ with $\varepsilon\psi = \varepsilon$. Then every right $[C, A]$-Hopf module is a projective A-module.

Remarks. If C is finite dimensional then $\psi(C)$ is a non-zero finite dimensional right ideal of A so that A must be finite dimensional (19 , p.107). In case $C = k$, the above Theorem reduces to [19, THEOREM 5.1.8].

We state without proof the dual of Corollary of Theorem 1 and

Theorem 2 :

Corollary 1. Let C be a right A-module coalgebra. The following are equivalent :

(i) C is a projective A-module.

(ii) There is a right A-module map $\psi : C \longrightarrow A$ with $\varepsilon\psi = \varepsilon$.

Corollary 2. Let H be a Hopf algebra and A a Hopf subalgebra. If there is a right A-module map $\psi : H \longrightarrow A$ with $\varepsilon\psi = \varepsilon$ then we have :

(1) H is a projective A-module.

(2) For any left A-module V, the canonical map

$$V \simeq A \otimes_A V \longrightarrow H \otimes_A V$$

is injective.

Let C be a right A-module coalgebra. If A^+ denotes the kernel of $\varepsilon : A \longrightarrow k$ then CA^+ is a coideal of C. Hence $\bar{C} = C/CA^+$ has a unique coalgebra structure such that the projection $p : C \longrightarrow \bar{C}$ is a coalgebra map.

Let N be a right $[C, A]$-Hopf module. Then the map p induces the right $\bar{C}$-comodule structure of N

$$N \xrightarrow{\ \rho\ } N \otimes C \xrightarrow{\ I \otimes p\ } N \otimes \bar{C}.$$

NA^+ is then a $\bar{C}$-subcomodule of N. Thus $\bar{N} = N/NA^+$ has a unique comodule structure $\bar{\rho} : \bar{N} \longrightarrow \bar{N} \otimes \bar{C}$ making the projection $\pi : N \longrightarrow \bar{N}$ a $\bar{C}$-comodule map, that is, $\bar{\rho}\pi = (\pi \otimes p)\rho_N$.

Note that we have $\pi(na) = \pi(n)\varepsilon(a)$, for $n \in N$, $a \in A$.

Since C has the left $\bar{C}$-comodule structure induced by

$$C \xrightarrow{\ \Delta\ } C \otimes C \xrightarrow{\ p \otimes I\ } \bar{C} \otimes C$$

for any right $\overline{C}$-comodule W, $W \square_{\overline{C}} C$ is defined and it is a right $[C, A]$-Hopf module via

$$W \square_{\overline{C}} C \xrightarrow{I \otimes \Delta} W \square_{\overline{C}} C \otimes C$$

$$W \square_{\overline{C}} C \otimes A \xrightarrow{I \otimes \omega} W \square_{\overline{C}} C.$$

For any right $[C, A]$-Hopf module N, define $\alpha : N \longrightarrow \overline{N} \otimes C$ be the composite

$$N \xrightarrow{\rho} N \otimes C \xrightarrow{\pi \otimes I} \overline{N} \otimes C.$$

It is easy to see that $\alpha(N) \subset \overline{N} \square_{\overline{C}} C$. Thus α is in fact a map $N \longrightarrow \overline{N} \square_{\overline{C}} C$. α is then a $[C, A]$-Hopf module map.

In these terms Theorem 3 can be dualized as follows :

<u>Theorem 5</u>. Let C be a right A-module coalgebra. If there is a right A-module map $\psi : C \longrightarrow A$ which is a coalgebra map then for every right $[C, A]$-Hopf module N,

$$\alpha : N \longrightarrow \overline{N} \square_{\overline{C}} C$$

is an isomorphism of $[C, A]$-Hopf modules.

Since the proof of Theorem 3 is not so easily dualized we include a proof of Theorem 5.

<u>Proof</u>. Let $Q : N \longrightarrow N$ denote the composite

$$N \xrightarrow{\rho} N \otimes C \xrightarrow{I \otimes \psi} N \otimes A \xrightarrow{I \otimes S} N \otimes A \xrightarrow{\omega} N$$

so that $Q(n) = \Sigma\, n_{(0)} S(\psi(n_{(1)}))$ for $n \in N$.

For $n \in N$ and $a \in A$,

$$\begin{aligned} Q(na) &= \Sigma\, n_{(0)} a_{(1)} S(\psi(n_{(1)} a_{(2)})) \\ &= \Sigma\, n_{(0)} a_{(1)} S(\psi(n_{(1)}) a_{(2)}) \\ &= \Sigma\, n_{(0)} a_{(1)} S(a_{(2)}) S(\psi(n_{(1)})) \\ &= \Sigma\, n_{(0)} \varepsilon(a) S(\psi(n_{(1)})) = Q(n\varepsilon(a)). \end{aligned}$$

Hence Q vanishes on NA^{+}. Thus there is a map $\overline{Q}$ making

$$\begin{array}{ccc} N & \xrightarrow{\;Q\;} & N \\ & {\scriptstyle\pi}\searrow \quad \nearrow{\scriptstyle\overline{Q}} & \\ & \overline{N} & \end{array}$$

commute. In particular, if we define $Q_0 : C \longrightarrow C$ by $Q_0(c) = \Sigma\, c_{(1)}S(\psi(c_{(2)}))$ then Q_0 factors through $\overline{C}$, that is, there is a map $\overline{Q}_0 : \overline{C} \longrightarrow C$ with $Q_0 = \overline{Q}_0 p$. Note that we have

$$\omega_C(Q_0 \otimes \psi)\Delta_C = I_C.$$

Let $\beta : \overline{N} \,\square_{\overline{C}}\, C \longrightarrow N$ denote the composite

$$\overline{N} \,\square_{\overline{C}}\, C \xrightarrow{\text{inclusion}} \overline{N} \otimes C \xrightarrow{\overline{Q} \otimes \psi} N \otimes A \xrightarrow{\;\omega\;} N.$$

For any $n \in N$

$$\begin{aligned} \beta\alpha(n) &= \beta(\Sigma\, \pi(n_{(0)}) \otimes n_{(1)}) \\ &= \Sigma\, n_{(0)}S(\psi(n_{(1)}))\psi(n_{(2)}) \\ &= \Sigma\, n_{(0)}S(\psi(n_{(1)})_{(1)})\psi(n_{(1)})_{(2)} \\ &= \Sigma\, n_{(0)}\varepsilon\psi(n_{(1)}) = \Sigma\, n_{(0)}\varepsilon(n_{(1)}) = n. \end{aligned}$$

For any $\pi(n) \in \overline{N}$, $c \in C$

$$\begin{aligned} &\alpha\omega_N(\overline{Q} \otimes \psi)(\pi(n) \otimes c) \\ &= (\pi \otimes I)\rho(Q(n)\psi(c)) \\ &= (\pi \otimes I)(\Sigma\, Q(n)_{(0)}\psi(c)_{(1)} \otimes Q(n)_{(1)}\psi(c)_{(2)}) \\ &= (\pi \otimes I)(\Sigma\, n_{(0)}S(\psi(n_{(3)}))\psi(c_{(1)}) \otimes n_{(1)}S(\psi(n_{(2)}))\psi(c_{(2)})) \\ &= \Sigma\, \pi(n_{(0)})\varepsilon(n_{(3)})\varepsilon(c_{(1)}) \otimes n_{(1)}S(\psi(n_{(2)}))\psi(c_{(2)}) \\ &= \Sigma\, \pi(n_{(0)}) \otimes n_{(1)}S(\psi(n_{(2)}))\psi(c) \\ &= \Sigma\, \pi(n_{(0)}) \otimes Q_0(n_{(1)})\psi(c). \end{aligned}$$

Let $\Sigma\, \pi(n) \otimes c \in \overline{N} \,\square_{\overline{C}}\, C$, thus

$$(\bar{\rho} \otimes I)(\Sigma\ \pi(n) \otimes c) = (I \otimes p \otimes I)(I \otimes \Delta)(\Sigma\ \pi(n) \otimes c).$$

Since $\bar{\rho}\pi = (\pi \otimes p)\rho_N$ we have

$$(\pi \otimes p \otimes I)(\rho_N \otimes I)(\Sigma\ n \otimes c) = (I \otimes p \otimes I)(I \otimes \Delta)(\Sigma\ \pi(n) \otimes c).$$

Applying $(I \otimes \omega_C)(I \otimes \bar{Q}_0 \otimes \psi)$ to this, we have

$$\Sigma\ \pi(n_{(0)}) \otimes Q_0(n_{(1)})\psi(c) = \Sigma\ \pi(n) \otimes c.$$

Thus we have shown that $\alpha\beta$ is the identity on $\bar{N} \square_{\bar{C}} C$. q.e.d.

CHAPTER III

CLEFT COMODULE ALGEBRAS AND HOPF MODULES

In this chapter we develope our Hopf module theory over comodule algebras. We work a fixed field k. All algebras, coalgebras, and so on are over k.

§1. The category M_B^A.

We begin by establishing our notation and summarizing background results; the facts here stated can be found in [7]. Throughout, A denotes a Hopf algebra over a field k and B denotes a right A-comodule algebra. A right (A,B)-Hopf module is a k-module M which is also a right A-comodule and a right B-module such that $\rho_M(mb) = \Sigma\, m_{(0)}b_{(0)} \otimes m_{(1)}b_{(1)}$ for all $m \in M$ and $b \in B$, where $\rho_M\colon M \longrightarrow M \otimes A$ denotes the comodule structure map on M and we write $\rho_M(m) = \Sigma\, m_{(0)} \otimes m_{(1)}$. If M_1, M_2 are right (A,B)-Hopf modules and $f \in \mathrm{Hom}(M_1,M_2)$, then f is said to be a <u>Hopf module map</u> if it is also an A-comodule map and a B-module map. We denote by M_B^A the category whose objects are right (A,B)-Hopf modules, and whose morphisms are Hopf module maps. M_B^A is an abelian category.

<u>Example 1.1.</u> For a right B-module V, we have $V \otimes A \in M_B^A$ via $\rho(v \otimes a) = \Sigma\, v \otimes a_{(1)} \otimes a_{(2)}$ and $(v \otimes a)b = \Sigma\, vb_{(0)} \otimes ab_{(1)}$. In particular we have $B \otimes A \in M_B^A$. Note that for $M \in M_B^A$ the comodule structure map $\rho_M\colon M \longrightarrow M \otimes A$ is a Hopf module map.

<u>Example 1.2.</u> For a right A-comodule W, we have $W \otimes B \in M_B^A$ via $\rho(w \otimes b) = \Sigma\, w_{(0)} \otimes b_{(0)} \otimes w_{(1)}b_{(1)}$ and $(w \otimes b)b' = w \otimes bb'$. Note that for $M \in M_B^A$ the B-action $\omega_M\colon M \otimes B \longrightarrow M$ is a Hopf module map.

Let $B_0 = \{b \in B \mid \rho_B(b) = b \otimes 1_A\}$ and $M_0 = \{m \in M \mid \rho_M(m) = m \otimes 1_A\}$ for $M \in \mathcal{M}_B^A$. It is easy to see that B_0 is a subalgebra of B and M_0 is a right B_0-module. Let $\mathcal{M}_{B_0}$ denote the category of right B_0-modules. Then

$$R: \mathcal{M}_B^A \longrightarrow \mathcal{M}_{B_0}, \qquad M \longmapsto M_0,$$

is obviously a covariant functor. The functor R has a left adjoint L defined by $L(V) = V \otimes_{B_0} B$ for $V \in \mathcal{M}_{B_0}$ where the right A-comodule structure is given by $\rho(v \otimes_{B_0} b) = \Sigma\, v \otimes_{B_0} b_{(0)} \otimes b_{(1)}$; $V \otimes_{B_0} B$ is a right B-module in the usual way and in fact $L(V)$ is a right (A,B)-Hopf module. The adjoint situation is as follows;

$$\mathcal{M}_B^A(L(V),M) \underset{\psi_{V,M}}{\overset{\phi_{V,M}}{\rightleftarrows}} \mathcal{M}_{B_0}(V,R(M))$$

(for $g: L(V) \longrightarrow M$ put $\phi_{V,M}g(v) = g(v \otimes_{B_0} 1)$ and for $f: V \longrightarrow R(M)$ take $\psi_{V,M}f(v \otimes_{B_0} b) = f(v)b$.)

$$\Phi_V = \phi_{V,L(V)}(1_{L(V)}): V \longrightarrow RL(V) = (V \otimes_{B_0} B)_0, \quad v \longmapsto v \otimes_{B_0} 1,$$

$$\Psi_M = \psi_{R(M),M}(1_{R(M)}): LR(M) = M_0 \otimes_{B_0} B \longrightarrow M, \quad m \otimes_{B_0} b \longmapsto mb.$$

<u>Example 1.3.</u> We consider $B \otimes A \in \mathcal{M}_B^A$ as in example 1.1. Since $\{a \in A \mid \Delta(a) = a \otimes 1\} \simeq k$ it follows that $(B \otimes A)_0 \simeq B$. Identifying these isomorphic k-spaces it is easy to see that the map $\Psi_{B \otimes A}: LR(B \otimes A) \longrightarrow B \otimes A$ is equal to

$$\beta: B \otimes_{B_0} B \longrightarrow B \otimes A, \quad b \otimes_{B_0} c \longmapsto \Sigma\, bc_{(0)} \otimes c_{(1)}.$$

§2. Generalized integrals.

Let A be a Hopf algebra and B a right A-comodule algebra. The element $\phi \in \mathrm{Hom}(A,B)$ is called a <u>generalized integral</u> for B if $\rho_B\phi = (\phi \otimes I)\Delta_A$. Equivalently, ϕ is a right A-comodule map. The ground field k has a right A-comodule algebra structure given by $u_A: k \longrightarrow A \simeq k \otimes A$, and an element $x \in A^* = \mathrm{Hom}(A, k)$ is a generalized integral for k if and only if $yx = y(1)x$ for all $y \in A^*$. The set of all generalized integrals for B will be denoted by $\mathrm{Com}(A,B)$.

The following result is essentially theorem 1 and its corollary in [7], however we include it for the convenience of reader.

<u>Theorem 2.1.</u> The follwing are equivalent;

(1) all right (A,B)-Hopf modules are injective as A-comodules,

(2) B is an injective A-comodule,

(3) there is $\phi \in \mathrm{Com}(A,B)$ where $\phi(1_A) = 1_B$,

(4) there is $\phi \in \mathrm{Com}(A,B)$ where $\phi(I_A)$ is invertible in B.

<u>Proof.</u> Since $B \in M_B^A$, (1) $\Rightarrow$ (2) is clear. For (2) $\Rightarrow$ (3), we consider the following diagram of right A-comodules

$$\begin{array}{ccccc} 0 & \longrightarrow & k & \xrightarrow{u_A} & A \\ & & \downarrow{\scriptstyle u_B} & & \\ & & B. & & \end{array}$$

If B is an injective A-comodule then there is $\phi \in \mathrm{Com}(A,B)$

such that $\phi u_A = u_B$.

(3) $\Rightarrow$ (4) is trivial. Now suppose $\phi \in \mathrm{Com}(A,B)$ where $\phi(1_A)$ is invertible in B. Replacing ϕ by the map $a \longmapsto \phi(1)^{-1}\phi(a)$ we have that $\phi(1_A) = 1_B$ and $\phi \in \mathrm{Com}(A,B)$. Thus (4) $\Rightarrow$ (3) holds. Finally we prove (3) $\Rightarrow$ (1): From example 1.1 it is enough to show that for $M \in \mathcal{M}_B^A$, $\rho_M: M \longrightarrow M \otimes A$ splits as A-comodules, since $M \otimes A$ is an injective A-comodule.

We define the map $\lambda_M: M \otimes A \longrightarrow M$ as the composite

$$\lambda_M: M \otimes A \xrightarrow{\rho \otimes I} M \otimes A \otimes A \xrightarrow{I \otimes S \otimes I} M \otimes A \otimes A$$
$$\xrightarrow{I \otimes m_A} M \otimes A \xrightarrow{I \otimes \phi} M \otimes B \xrightarrow{\omega_B} M$$

(where S denotes the antipode of A and m_A denotes the multiplication of A.) Thus we have for $m \in M$ and $a \in A$,

$$\lambda_M(m \otimes a) = \Sigma\, m_{(0)}\phi(S(m_{(1)})a).$$

It is straightforward to show that λ_M is an A-comodule map by $\phi \in \mathrm{Com}(A,B)$. Moreover $\lambda_M\rho_M = I$, by $\phi(1_A) = 1_B$ (see [7]). Thus M is an injective A-comodule. q.e.d.

Let M be a right (A,B)-Hopf module and N a right A-comodule. For $\phi \in \mathrm{Com}(A,B)$ and $f \in \mathrm{Hom}(N,M)$, we define $\phi \rightharpoonup f \in \mathrm{Hom}(N,M)$ by

$$\phi \rightharpoonup f: N \xrightarrow{\rho_N} N \otimes A \xrightarrow{f \otimes I} M \otimes A \xrightarrow{\lambda_M} M.$$

Thus we have for $n \in N$,

$$(\phi \rightharpoonup f)(n) = \Sigma\, f(n_{(0)})_{(0)}\phi(S(f(n_{(0)})_{(1)})n_{(1)}).$$

<u>Lemma 2.2.</u> Let $\phi \in \mathrm{Com}(A,B)$. Let M, P be right (A,B)-Hopf modules and N a right A-comodule, $f \in \mathrm{Hom}(N,M)$, $g \in \mathrm{Hom}(M,P)$.

(a) $\phi \rightharpoonup f$ is an A-comodule map.

(b) If $\phi(1_A) = 1_B$ and f is an A-comodule map then $\phi \rightharpoonup f = f$.

(c) If f is an A-comodule map then $\phi \rightharpoonup (gf) = (\phi \rightharpoonup g)f$.

(d) If g is a Hopf module map then $\phi \rightharpoonup (gf) = g(\phi \rightharpoonup f)$.

(e) If A and B are commutative, and if g is a B-module map, then $\phi \rightharpoonup g$ is a Hopf module map.

<u>Proof.</u> (a) It follows from the following commutative diagram

$$\begin{array}{ccccccc}
N & \xrightarrow{\rho} & N \otimes A & \xrightarrow{f \otimes I} & M \otimes A & \xrightarrow{\lambda_M} & M \\
\downarrow \rho & & \downarrow I\otimes\Delta & & \downarrow I\otimes\Delta & & \downarrow \rho \\
N \otimes A & \xrightarrow{\rho \otimes I} & N \otimes A \otimes A & \xrightarrow{f\otimes I\otimes I} & M \otimes A \otimes A & \xrightarrow{\lambda_M \otimes I} & M \otimes A.
\end{array}$$

(b) $\phi \rightharpoonup f = \lambda_M(f \otimes I)\rho_N = \lambda_M \rho_M f = f$.

(c) $\phi \rightharpoonup (gf) = \lambda_P(g \otimes I)(f \otimes I)\rho_N = \lambda_P(g \otimes I)\rho_M f = (\phi \rightharpoonup g)f$.

(d) We show that $\lambda_P(g \otimes I_A) = g\lambda_M$. In fact, for $m \in M$, $a \in A$,

$\lambda_P(g \otimes I)(m \otimes a) = \lambda_P(g(m) \otimes a) = \Sigma\, g(m)_{(0)}\phi(S(g(m)_{(1)})a)$

$= \Sigma\, g(m_{(0)})\phi(S(m_{(1)})a) = g(\Sigma\, m_{(0)}\phi(S(m_{(1)})a) = g\lambda_M(m \otimes a)$.

Now we have

$\phi \rightharpoonup (gf) = \lambda_P(g \otimes I)(f \otimes I)\rho_N = g\lambda_M(f \otimes I)\rho_N = g(\phi \rightharpoonup f)$.

(e) $(\phi \rightharpoonup g)(mb) = \lambda_P(g \otimes I)(\Sigma\, m_{(0)}b_{(0)} \otimes m_{(1)}b_{(1)})$

$= \lambda_P(\Sigma\, g(m_{(0)})b_{(0)} \otimes m_{(1)}b_{(1)})$

$= \Sigma\, g(m_{(0)})_{(0)}b_{(0)}\phi(S(g(m_{(0)})_{(1)}b_{(1)})m_{(1)}b_{(2)})$

$$= \Sigma\, \xi(m_{(0)})_{(0)} b_{(0)} \phi(S(b_{(1)}) S(\xi(m_{(0)})_{(1)}) m_{(1)} b_{(2)})$$

$$= \Sigma\, \xi(m_{(0)})_{(0)} b_{(0)} \phi(S(\xi(m_{(0)})_{(1)}) m_{(1)} \varepsilon(b_{(1)}))$$

(since A is commutative)

$$= \Sigma\, \xi(m_{(0)})_{(0)} \phi(S(\xi(m_{(0)})_{(1)}) m_{(1)}) b \quad \text{(since } B \text{ is commutative)}$$

$$= (\phi \rightharpoonup \xi)(m) b.$$

q.e.d.

Proposition 2.3. Assume that B is injective as an A-comodule, and A, B are commutative. Let M, P be right (A,B)-Hopf modules and $g: M \longrightarrow P$ a Hopf module map and $f: P \longrightarrow M$ a B-module map such that $gf = I_P$. Then there is a Hopf module map $h: P \longrightarrow M$ such that $gh = I_P$.

Proof. From theorem 2.1 there is $\phi \in Com(A,B)$ where $\phi(1) = 1$. If we put $h = \phi \rightharpoonup f$ then by lemma 2.2, (e), h is a Hopf module map. Moreover, $gh = g(\phi \rightharpoonup f) = \phi \rightharpoonup (gf) = \phi \rightharpoonup I_P = I_P$. q.e.d.

The next result is a generalization of [15, Cor. 4].

Theorem 2.4. Let A, B be as in above proposition and P a right (A,B)-Hopf module which is projective as a B-module. If $\Psi_P: P_0 \otimes_{B_0} B \longrightarrow P$, $p_0 \otimes_{B_0} b \longmapsto p_0 b$, is surjective then it is an isomorphism.

Proof. We consider $P_0 \otimes B$ as a right (A,B)-Hopf module by $\rho(p_0 \otimes b) = \Sigma\, p_0 \otimes b_{(0)} \otimes b_{(1)}$ and $(p_0 \otimes b)b' = p_0 \otimes bb'$. Thus the map $P_0 \otimes B \longrightarrow P$, $p_0 \otimes b \longmapsto p_0 b$ becomes a Hopf module map and is also surjective. Since P is B-projective this map splits in M_B^A by proposition 2.3.

Since $(P_0 \otimes B)_0 \otimes_{B_0} B \simeq P_0 \otimes B_0 \otimes_{B_0} B \simeq P_0 \otimes B$ we have that the adjunction Ψ_M: $M_0 \otimes_{B_0} B \longrightarrow M$ is an isomorphism when $M = P_0 \otimes B$, hence for $M = P$. q.e.d.

§3. Cleft comodule algebras.

If C is a coalgebra and B an algebra then $\mathrm{Hom}(C,B)$ has an algebra structure by convolution $*$. For $f, g \in \mathrm{Hom}(C,B)$ the product $f*g$ is $m_B(f \otimes g)\Delta_C$. The unit of $\mathrm{Hom}(C,B)$ is $u_B\varepsilon_C$. $\mathrm{Reg}(C,B)$ denotes the multiplicative group of all invertible elements of $\mathrm{Hom}(C,B)$.

Definition. Let A be a Hopf algebra and B a right A-comodule algebra. B is called cleft if there is $\phi \in \mathrm{Com}(A,B) \cap \mathrm{Reg}(A,B)$.

Notes. (a) A may be viewed as a right A-comodule algebra via Δ_A. I_A: $A \longrightarrow A$ is a A-comodule map which is invertible since A has an antipode. Thus A is a cleft A-comodule algebra.

(b) If B is a cleft A-comodule algebra then it satisfies the condition (4) Of theorem 2.1, so B is an injective A-comodule.

Proposition 3.1. Let A be an irreducible Hopf algebra and B a right A-comodule algebra. Then B is cleft if and only if it is injective as an A-comodule.

Proof. LEMMA 9.2.3 of [19] and theorem 2.1 implies the result.

q.e.d.

For $\phi \in \mathrm{Reg}(A,B)$ ϕ^{-1} denotes the inverse of ϕ with respect to convolution *. Thus we have

$$m_B(\phi \otimes \phi^{-1})\Delta_A = u_B\varepsilon_A = m_B(\phi^{-1} \otimes \phi)\Delta_A.$$

Lemma 3.2. If $\phi \in \mathrm{Com}(A,B) \cap \mathrm{Reg}(A,B)$ then the following diagram is commutative;

$$\begin{array}{ccccc} A & \xrightarrow{\phi^{-1}} & B & \xrightarrow{\rho_B} & B \otimes A \\ \big\downarrow{\scriptstyle\Delta} & & & & \big\uparrow{\scriptstyle\phi^{-1}\otimes S} \\ A \otimes A & & \xrightarrow{\quad T \quad} & & A \otimes A \end{array}$$

(where T denotes the twist map)

Thus for $a \in A$, $\rho_B\phi^{-1}(a) = \Sigma\, \phi^{-1}(a_{(2)}) \otimes S(a_{(1)})$.

Proof. In the convolution algebra $\mathrm{Hom}(A, B \otimes A)$, we have

$$\begin{aligned} \rho_B\phi^{-1} &= (\rho_B\phi)^{-1} && \text{since } \rho_B \text{ is an algebra map} \\ &= ((\phi \otimes I)\Delta_A)^{-1} && \text{since } \phi \in \mathrm{Com}(A;B) \\ &= (\phi^{-1} \otimes S)T\Delta_A && \text{by a direct calculation.} \end{aligned}$$

q.e.d.

The above lemma is similar in sprit to [18, Lemma 8.2 (b)].

The next result is a generalization of [7, Theorem 3].

Theorem 3.3. Let A be a Hopf algebra and B a right A-comodule algebra. If B is cleft then for every right (A,B)-Hopf module M,

$$\Psi_M: M_0 \otimes_{B_0} B \longrightarrow M, \quad m \otimes_{B_0} b \longmapsto mb,$$

is an isomorphism of Hopf modules.

Proof. Let $\phi \in \mathrm{Com}(A,B) \cap \mathrm{Reg}(A,B)$. We define a map $p: M \longrightarrow M$ by $p(m) = \Sigma\, m_{(0)}\phi^{-1}(m_{(1)})$ for $m \in M$. Then

$$\rho_M(p(m)) = \Sigma\, m_{(0)}\phi^{-1}(m_{(3)}) \otimes m_{(1)}S(m_{(2)}) \qquad \text{by lemma 3.2}$$

$$= \Sigma\, m_{(0)}\phi^{-1}(m_{(2)}) \otimes \varepsilon(m_{(1)})1_A = p(m) \otimes 1_A.$$

Thus p is in fact a map $M \longrightarrow M_0$. Now define a map

$$\chi_M: M \longrightarrow M_0 \otimes_{B_0} B$$

by $\chi_M(m) = \Sigma\, p(m_{(0)}) \otimes_{B_0} \phi(m_{(1)})$. Then for $m \in M$

$$\Psi_M\chi_M(m) = \Sigma\, m_{(0)}\phi^{-1}(m_{(1)})\phi(m_{(2)}) = \Sigma\, m_{(0)}\varepsilon(m_{(1)}) = m.$$

And for $m \in M_0$ and $b \in B$

$$\chi_M\Psi_M(m \otimes_{B_0} b) = \Sigma\, p(mb_{(0)}) \otimes_{B_0} \phi(b_{(1)})$$

$$= \Sigma\, mb_{(0)}\phi^{-1}(b_{(1)}) \otimes_{B_0} \phi(b_{(2)})$$

$$= \Sigma\, m \otimes_{B_0} b_{(0)}\phi^{-1}(b_{(1)})\phi(b_{(2)}) \quad \text{since } \Sigma\, b_{(0)}\phi^{-1}(b_{(1)}) \in B_0$$

$$= m \otimes_{B_0} b.$$

Thus we have shown that χ_M is the inverse of Ψ_M. q.e.d.

We deduce several consequences.

Corollary 3.4. If B is cleft then the map as in example 1.3

$$\beta: B \otimes_{B_0} B \longrightarrow B \otimes A, \quad b \otimes_{B_0} c \longmapsto \Sigma\, bc_{(0)} \otimes c_{(1)},$$

is an isomorphism of Hopf modules.

Corollary 3.5. If B is cleft then every right (A,B)-Hopf module M is a free A-comodule; this means that the map $M_0 \otimes A \longrightarrow M$ given by $m \otimes a \longmapsto m\phi(a)$ is an isomorphism of A-comodules. The inverse map is given by $m \longmapsto \Sigma\, m_{(0)}\phi^{-1}(m_{(1)}) \otimes m_{(2)}$.

Corollary 3.6. If B is cleft , then it is faithfully flat as a left B_0-module and the functor

$$R: M_B^A \longrightarrow M_{B_0}$$

is an equivalence of categories.

Remark. We can dualize the above theorem in the following;
Let A be a Hopf algebra and C a right A-module coalgebra. If there is an A-module map in $\mathrm{Reg}(C,A)$ then we have for every right $[C,A]$-Hopf module N an isomorphism

$$N \simeq N/NA^+ \,\square_{C/CA^+}\, C.$$

For the notation for dualization, see [7].

§4. Smash products.

Let B be a right A-comodule algebra. We define the algebra $\#(A,B)$ to be $\mathrm{Hom}(A,B)$ as a vector space. Multiplication $\#$ is defined by setting, for $f, g \in \mathrm{Hom}(A,B)$ and $a \in A$,

$$(f\#g)(a) = \Sigma\, f(g(a_{(2)})_{(1)}a_{(1)})g(a_{(2)})_{(0)}.$$

A calculation shows that $\#(A,B)$ is an associative algebra with unit $u_B\varepsilon_A$. $\#(A,B)$ is called the <u>smash product</u> of B by A.

We note that if the comodule structure of B is trivial (that is, $\rho_B(b) = b \otimes 1_A$ for all $b \in B$), then the multiplication $f\#g$ is equal to the convolution product $f*g$.

If A is finite dimensional then B has a left A^*-module algebra structure in the natural way and the canonical linear isomorphism $\mathrm{Hom}(A,B) \simeq B \otimes A^*$ induces an isomorphism of algebras

$$\#(A,B) \simeq B\#A^*.$$

We list below some basic properties.

<u>4.1.</u> B and A^* can be embedded in $\#(A,B)$ as subalgebras via $b \longmapsto (a \longmapsto \varepsilon(a)b)$ and $x\ (\in A^*) \longmapsto (a \longmapsto x(a)1_B)$.

<u>4.2.</u> $\#(A,B)$ is a left A-module via $(a.f)(d) = f(da)$ for $a, d \in A$ and $f \in \#(A,B)$. In fact $\#(A,B)$ is a left A-module algebra, i. e., A easures $\#(A,B)$ to $\#(A,B)$. Moreover,

$$\{f \in \#(A,B) \mid a.f = \varepsilon(a)f, \text{ for all } a \in A\} = B.$$

4.3. B is a left $\#(A,B)$-module via $f \rightharpoonup b = \Sigma f(b_{(1)})b_{(0)}$.
Thus this induces the algebra map

$$\pi\colon \#(A,B) \longrightarrow \mathrm{End}^r_{B_0}(B), \qquad \pi(f) = f \rightharpoonup b,$$

where $\mathrm{End}^r_{B_0}(B)$ denotes the subalgebra of all right B_0-endomorphisms of B.

4.4. If J is a subspace of B, then J is a $\#(A,B)$-submodule of B if and only if J is a left ideal of B and $\rho_B(J) \subset J\otimes A$.

Theorem 4.5. Let A be a Hopf algebra with $S^2 = I$. Let B be a cleft right A-comodule algebra. Then the map as in 4.3 is an isomorphism of algebras.

Proof. The proof is similar in spirit to [26, Theorem 1.1.].
Define a map $\beta'\colon B \otimes_{B_0} B \longrightarrow B \otimes A$ by $\beta'(b \otimes_{B_0} c) = \Sigma\, b_{(0)}c \otimes b_{(1)}$. Then β' is a right B-module map where the B-module structure on $B \otimes A$ is defined by $(b \otimes a)b' = bb' \otimes a$.
Now we construct the following commutative diagram

$$\begin{array}{ccc}
\#(A,B) & \xrightarrow{\qquad\quad \pi \quad\qquad} & \mathrm{End}^r_{B_0}(B) \\
\downarrow\wr & & \downarrow\wr \\
\mathrm{Hom}^r_B(B \otimes A,\ B) & \xrightarrow{\quad \beta'^* \quad} & \mathrm{Hom}^r_B(B \otimes_{B_0} B,\ B).
\end{array}$$

Thus we are done when we show that β' is a linear isomorphism.
This follows from the next lemma, since β as in example 1.3

is a linear isomorphism by corollary 3.4.

Lemma 4.6. Let A be a Hopf algebra with $S^2 = I$. Then

(a) The map $\theta: B \otimes A \longrightarrow B \otimes A$ given by $\theta(b \otimes a) = \Sigma\, b_{(0)} \otimes b_{(1)}a$ is an linear isomorphism. The inverse of θ is given by $\theta^{-1}(b \otimes a) = \Sigma\, b_{(0)} \otimes S(b_{(1)})a$.

(b) $\beta' = \theta(I \otimes S)\beta$.

Proof. Straightforward.

CHAPTER IV

ALGEBRAS WITH TOTAL INTEGRALS

We freely use the sigma notation of Sweedler for coalgebras and comodules. For a coalgebra the diagonal map is denoted by Δ and the counit map by ε. For a comodule the structure map is usually denoted by ρ. For a Hopf algebra the antipode is denoted by S and the composite-inverse to S is denoted by $\overline{S}$ if it exists.

Throughout R will be a fixed commutative ring , A will be a given Hopf algebra over R and B will be an A-comodule algebras.

1. Total integrals

(1.1) By M_B^A we denote the category of right (A, B)-Hopf modules (see [7] and [8]); thus the objects are right B-modules M which are right A-comodules so that $\rho_M(mb) = \Sigma\, m_{(0)}b_{(0)} \otimes m_{(1)}b_{(1)}$ for m in M and b in B. Morphisms are right B-module maps which are A-comodule maps. Similarly we can define the category ${}_B\mathcal{M}^A$ of left (A, B)-Hopf modules; the objects are left B-modules N which are right A-comodules so that $\rho_N(bn) = \Sigma\, b_{(0)}n_{(0)} \otimes b_{(1)}n_{(1)}$ for n in N and b in B. Morphisms are left B-module maps which are A-comodule maps. Note that nowhere in the above definitions has the antipode S appeared. But the antipode plays an important role in this paper.

(1.2) The antipode S is not necessarily bijective. When S is bijective, we denote by $\overline{S}$ the composite-inverse to S. Since S is an anti-algebra, anti-coalgebra map, so is $\overline{S}$. Also we have for a in A, $\Sigma\, a_{(2)}\overline{S}(a_{(1)}) = \varepsilon(a)1_A = \Sigma\, \overline{S}(a_{(2)})a_{(1)}$.

Let B^{op} denote the opposite algebra of B. To the Hopf algebra A is associated a Hopf algebra A^{op} as follows. As a coalgebra A^{op} is A and as an algebra A^{op} is the opposite algebra of A and the antipode for A^{op} is $\overline{S}$. Then the structure map ρ_B induces a right A^{op}-comodule structure on B^{op} which is also an algebra map of B^{op} to $B^{op} \otimes A^{op}$. Thus B^{op} is an A^{op}-comodule algebra. We note that ${}_B\mathcal{M}^A = \mathcal{M}_{B^{op}}^{A^{op}}$.

Remarks. If A is a finitely generated, projective R-module, then the antipode is always bijective ([12], Prop.(1.1)). When A is either commutative or cocommutative, the usual proof in ([19], Prop.4.0.1.) shows that SS = I. This means that S is bijective and $S = \overline{S}$.

(1.3) Assume that the Hopf algebra A is finitely generated, projective as an R-module. Then $A^* = \text{Hom}(A, R)$ has a natural Hopf algebra structure. An A-comodule algebra B is a left A^*-module algebra by the rule $a^*.b = \Sigma\, a^*(b_{(1)})b_{(0)}$ for a^* in A^* and b in B. We denote by $B \# A^*$ the smash product of B with A^*; thus $B \# A^*$ is $B \otimes A^*$ as an R-module, elements $b \otimes a^*$ will be written $b \# a^*$, and the multiplication in $B \# A^*$ is given by $(b \# a^*)(c \# d^*) = \Sigma\, b(a^*_{(1)}.c) \# a^*_{(2)}d^*$. It is easy to see that each N in ${}_B M^A$ is a left $B \# A^*$-module by the rule $(b \# a^*).n = \Sigma\, a^*(n_{(1)})bn_{(0)}$ for n in N. Conversely any left $B \# A^*$-module is a left (A, B)-Hopf module in a natural way. Thus we have ${}_B M^A = {}_{B\#A^*}M$, the category of left $B \# A^*$-modules.

(1.4) Let D be a coalgebra over R. A right D-comodule W is called a <u>relative injective</u> if, for every D-comodule map $i: U \longrightarrow V$ for which there exists an R-module map $p: V \longrightarrow U$ with $pi = I_U$, and for every D-comodule map $f: U \longrightarrow W$, there exists a D-comodule map $g: V \longrightarrow W$ with $gi = f$. <u>This is equivalent to the existence of a D-comodule map $\lambda: W \otimes D \longrightarrow W$ with $\lambda\rho_W = I_W$, where the D-comodule structure on $W \otimes D$ is given by $I \otimes \Delta$.</u> In fact, since $\rho_W: W \longrightarrow W \otimes D$ is a D-comodule map and $(I \otimes \varepsilon)\rho = I$, it follows that if W is relative injective then there exists a D-comodule map $\lambda: W \otimes D \longrightarrow W$ with $\lambda\rho = I$.

Conversely, now suppose there exists such a map λ. Let $U \xrightarrow{i} V \xrightarrow{p} U$ and $U \xrightarrow{f} W$ be as above. Define g by $\lambda(fp \otimes I)\rho_V: V \longrightarrow W$. Then $gi = \lambda(fp \otimes I)\rho_V i = \lambda(fp\otimes I)(i\otimes I)\rho_U$

$= \lambda(f \otimes I)\rho_U = \lambda\rho_W f = f$. We compute $(g \otimes I)\rho_V =$
$(\lambda \otimes I)(fp \otimes I \otimes I)(\rho_V \otimes I)\rho_V = (\lambda \otimes I)(fp \otimes I \otimes I)(I \otimes \Delta)\rho_V$
$= (\lambda \otimes I)(I \otimes \Delta)(fp \otimes I)\rho_V = \rho_W\lambda(fp \otimes I)\rho_V = \rho_W g$. Therefore, g is a comodule map. This completes the proof.

(1.5) Let A be a Hopf algebra over R and B an A-comodule algebra. By $\mathrm{Com}(A, B)$ we denote the R-module of right A-comodule maps from A into B; thus

$\mathrm{Com}(A, B) = \{ \phi \in \mathrm{Hom}(A, B) \mid \rho(\phi(a)) = \Sigma\, \phi(a_{(1)}) \otimes a_{(2)} \}$.

The elements of $\mathrm{Com}(A, B)$ are called <u>integrals</u>. An integral ϕ is called <u>total</u> if $\phi(1_A) = 1_B$ (see [13]).

Let consider $B \otimes A$. We can view $B \otimes A$ as a right (or left) (A, B)-Hopf module by $b \otimes a \longmapsto b \otimes \Delta(a)$ and

$(b \otimes a)\, b' = \Sigma\, bb'_{(0)} \otimes ab'_{(1)}$ or $b'(b \otimes a) = \Sigma\, b'_{(0)}b \otimes b'_{(1)}a$.

Note that the structure map $\rho\colon B \longrightarrow B \otimes A$ is a morphism in M_B^A (or ${}_BM^A$).

<u>Lemma</u>. For any A-comodule algebra B, the map

$\mathrm{Com}(A, B) \longrightarrow {}_BM^A(B \otimes A, B)$, $\phi \longmapsto (b\otimes a \longmapsto \Sigma b_{(0)}\phi(S(b_{(1)})a))$

is an isomorphism of R-modules. The inverse is given by

$F \longmapsto (a \longmapsto F(1_B \otimes a))$.

If, in addition, the antipode S is bijective then the map

$\mathrm{Com}(A, B) \longrightarrow M_B^A(B \otimes A, B)$, $\phi \longmapsto (b\otimes a \longmapsto \Sigma\, \phi(a\overline{S}(b_{(1)}))b_{(0)})$

is an isomorphism of R-modules.

The proof is easy, hence omitted.

(1.6) Theorem. Let B be an A-comodule algebra. Then the following are equivalent:

(1) There exists a total integral $\phi: A \longrightarrow B$.

(2) There exists an integral $\phi: A \longrightarrow B$ where $\phi(1_A)$ is invertible in B.

(3) B is a relative injective A-comodule.

(4) Any M in M_B^A is a relative injective A-comodule.

(5) There exists a map $\theta: B \otimes A \longrightarrow B$ in ${}_BM^A$ with $\theta\rho = I_B$.

If, in addition, the antipode S is bijective, these are equivalent to

(6) Any N in ${}_BM^A$ is a relative injective A-comodule.

(7) There exists a map $\theta': B \otimes A \longrightarrow B$ in M_B^A with $\theta'\rho = I_B$.

Proof. (1) => (2); trivial. (2) => (1); put $x = \phi(1)$. Then $\rho(x^{-1}) = x^{-1} \otimes 1$. Replacing ϕ by the map $a \longmapsto x^{-1}\phi(a)$, it follows that ϕ is a total integral. (1) => (5), (7); by Lemma. (5) => (1) and (7) => (1); by definition (1.4). (1) => (4), (6); suppose that ϕ is a total integral. Let $M \in M_B^A$ and $N \in {}_BM^A$. Define $\lambda_M: M \otimes A \longrightarrow M$ and $\lambda_N: N \otimes A \longrightarrow N$ by

$$\lambda_M(m \otimes a) = \Sigma\, m_{(0)}\phi(S(m_{(1)})a), \qquad \lambda_N(n \otimes a) = \Sigma\, \phi(a\bar{S}(n_{(1)}))n_{(0)}.$$

Then it is easy to check that $\lambda_M\rho_M = I_M$, $\lambda_N\rho_N = I_N$ and λ_M, λ_N are A-comodule maps. Hence M is a relative injective A-comodule. (4) => (3); by $B \in M_B^A$. (6) => (3); by $B \in {}_BM^A$. (3) => (1); now suppose that B is a relative injective A-comodule. Consider the unit map $u_A: R \longrightarrow A$ and $u_B: R \longrightarrow B$. Note that these maps are right A-comodule maps. The counit map $\varepsilon_A: A \longrightarrow R$ satisfies $\varepsilon_A u_A = I_R$. Since B is a relative injective A-comodule, it follows that there exists a right A-comodule map $\phi: A \longrightarrow B$ with $\phi u_A = u_B$. Therefore, ϕ is a total integral.

This completes the proof of the theorem.

Remark. The above map λ_M (λ_N) is not necessarily a B-module map.

(1.7) Theorem. Let A be a commutative Hopf algebra and B an A-comodule algebra. Assume that there exists a total integral $\phi: A \longrightarrow B$ with $\phi(A) \subset \mathrm{cent}(B)$, the centre of B. Then:

(1) For every M in $\mathcal{M}_B^A$, the map $\lambda_M: M \otimes A \longrightarrow M$ given by $\lambda_M(m \otimes a) = \Sigma\, m_{(0)}\phi(S(m_{(1)})a)$ is a right B-module map, where B-action on $M \otimes A$ is $(m \otimes a)b = \Sigma\, mb_{(0)} \otimes ab_{(1)}$. Consequently λ_M is a retract in $\mathcal{M}_B^A$ of $\rho_M: M \longrightarrow M \otimes A$. A similar result holds for the category ${}_B\mathcal{M}^A$.

(2) For every map $i: M' \longrightarrow M$ in $\mathcal{M}_B^A$ for which there exists a B-module map $p: M \longrightarrow M'$ such that $pi = I$, there exists a map $q: M \longrightarrow M'$ in $\mathcal{M}_B^A$ with $qi = I$. A similar result holds for the category ${}_B\mathcal{M}^A$.

Proof. (1) $\lambda_M((m \otimes a)b) = \lambda_M(\Sigma\, mb_{(0)} \otimes ab_{(1)})$

$= \Sigma\, m_{(0)}b_{(0)}\phi(S(m_{(1)}b_{(1)})ab_{(2)}) = \Sigma\, m_{(0)}b_{(0)}\phi(S(b_{(1)})S(m_{(1)})ab_{(2)})$

$= \Sigma\, m_{(0)}b\phi(S(m_{(1)})a)$ (since A is commutative)

$= \lambda_M(m \otimes a)b$ (since $\phi(A) \subset \mathrm{cent}(B)$).

(2) Define $q = \lambda_{M'}(p \otimes I)\rho_M: M \longrightarrow M'$. It is easy to check that q is an A-comodule map and $qi = I$. It remains to show that q is a B-module map. We compute $q(mb) = \lambda_{M'}(\Sigma\, p(m_{(0)})b_{(0)} \otimes m_{(1)}b_{(1)})$

$= \lambda_{M'}((\Sigma\, p(m_{(0)}) \otimes m_{(1)})b) = \lambda_{M'}(\Sigma\, p(m_{(0)}) \otimes m_{(1)})b$ (since $\lambda_{M'}$ is a B-module map by (1)) $= q(m)b$. q.e.d.

Remark. We obtain, under the assumptions of the above theorem, that $B \# A^*$ is semi-simple if A is a finitely generated, projective R-module and B is semi-simple.

(1.8) Example 1. Let G be a finite group with identity element denoted by e. Let A be the free R-module with basis G. Setting $(\sum_{x\in G} r_x x)(\sum_{x\in G} s_x x) = \sum_{x\in G} r_x s_x x$, $\varepsilon(\sum_{x\in G} r_x x) = r_e$, and for $x\in G$,

$\Delta(x) = \sum_{x=yz} y \otimes z = \sum_{z\in G} xz^{-1} \otimes z$, $S(x) = x^{-1}$, A is a commutative Hopf algebra. The identity element in A is $\sum_{x\in G} x$. Note that A is the dual Hopf algebra of the group algebra $R[G]$. Let B be an R-algebra and ρ an R-module map of B to $B \otimes A$ where we write $\rho(b) = \sum_{x\in G} x(b) \otimes x$. Then B is an A-comodule algebra with respect to ρ if and only if G acts as a group of automorphisms of the R-algebra B. Moreover, a left (A, B)-Hopf module N is just a left B-module and a left G-module such that $x(bn) = x(b)x(n)$ for all x in G, b in B and n in N.

Now let ϕ be in $\mathrm{Hom}(A, B) = \mathrm{Map}(G, B)$. It is easy to check that $\phi \in \mathrm{Com}(A, B)$ if and only if $\phi(xz^{-1}) = z(\phi(x))$ for $x, z\in G$, equivalently, $\phi(x) = x^{-1}(\phi(e))$ for $x \in G$. This shows that the map of $\mathrm{Com}(A, B)$ to B given by $\phi \longrightarrow \phi(e)$ is bijective.
A total integral ϕ corresponds to an element b in B such that $\sum_{x\in G} x(b) = 1_B$. Moreover, $\phi(A) \subset \mathrm{cent}(B) \iff b \in \mathrm{cent}(B)$. In

particular, if $|G|^{-1} \in B$, there exists a total integral ϕ with $\phi(A) \subset \mathrm{cent}(B)$ (say $b = |G|^{-1}$). Thus theorem (1.7) is a version of Maschke's theorem for Hopf modules.

Example 2. Suppose R is a ring of prime characteristic p, and $q = p^n$ for some positive integer n. Let A be the free R-module on basis $\{ c_0, c_1, \ldots, c_{q-1} \}$. Define

$$c_i c_j = \binom{i+j}{i} c_{i+j} \quad \text{if } i + j < q, \qquad c_i c_j = 0 \quad \text{if } i + j \geqq q$$

$$c_0 = 1_A, \quad \Delta(c_r) = \sum_{i=0}^{r} c_i \otimes c_{r-i}, \quad \varepsilon(c_i) = \delta_{i,0}, \quad S(c_i) = (-1)^i c_i.$$

Then A is a commutative and cocommutative Hopf algebra. Let B be an R-algebra and ρ an R- module map of B to $B \otimes A$ where we write $\rho(b) = \Sigma_{i=0}^{q-1} d_i(b) \otimes c_i$. If B is an A-comodule algebra with respect to ρ then $d_0 = I_B$, d_1 is a derivation of B with $d_1{}^q = 0$ and $d_i = d_1{}^i$ for $2 \leqq i < q$. Conversely, if d is a derivation of B with $d^q = 0$, then B becomes an A-comodule algebra as follows; $\rho(b) = \Sigma_{i=0}^{q-1} d^i(b) \otimes c_i$ for b in B.

A left (A,B)-Hopf module N is just a left B-module with an R-module map $\delta: N \longrightarrow N$ such that $\delta^q = 0$ and for all b in B, n in N, $\delta(bn) = b\delta(n) + d(b)n$.

Now let ϕ be in $\mathrm{Hom}(A, B)$ where we denote $\phi(c_i) = b_i$, $i=0,\ldots,q-1$. Then it is easy to compute that $\phi \in \mathrm{Com}(A, B)$ if and only if $d(b_i) = b_{i-1}$ ($1 \leqq i < q$). Thus the existence of a total integral ϕ is equivalent to the existence of an element b in B such that $d^{q-1}(b) = 1$ (say $b = \phi(c_{q-1})$). Moreover, $\phi(A) \subset \mathrm{cent}(B)$ if and only if the corresponding b is in $\mathrm{cent}(B)$.

Example 3. Again suppose R is of prime characteristic p and $q = p^n$ for some positive integer n. Now let A be the Hopf algebra $R[X]/(X^q) = R[x]$, where x is the residue class of X and $\Delta(x) = x \otimes 1 + 1 \otimes x$, $\varepsilon(x) = 0$ and $S(x) = -x$. This is the dual of the Hopf algebra in the previous example. In this case an A-comodule algebra is such an R-algebra B with $\{D_0=I, D_1, \ldots, D_{q-1}\}$, a set of iterative higher derivations of B, by the rule $\rho(b) = \Sigma_{i=0}^{q-1} D_i(b) \otimes x^i$. Where "iterative higher derivations" means that $D_r(bc) = \Sigma_{i=0}^{r} D_i(b) D_{r-i}(c)$ for all $1 \leqq r < q$, and $D_i D_j = \binom{i+j}{i} D_{i+j}$ if $i + j < q$, $D_i D_j = 0$ if $i + j \geqq q$.

Let ϕ be in $\mathrm{Hom}(A, B)$ where we denote $\phi(x^i) = b_i$ $(0 \leqq i < q)$. Then ϕ is an integral if and only if

$$D_i(b_j) = \binom{j}{i} b_{j-i} \quad \text{if } i \leqq j, \qquad D_i(b_j) = 0 \quad \text{if } i > j.$$

It follows that if there exists an element b in B such that $D_{q-1}(b) = 1$ then B has a total integral. In fact, putting $t = -D_{q-2}(b)$ we have $D_1(t) = 1$ and $D_i(t) = 0$ for $2 \leqq i < q$. Therefore we obtain (by induction) that

$$D_i(t^j) = \binom{j}{i} t^{j-i} \quad \text{for } i \leqq j \text{ and } D_i(t^j) = 0 \quad \text{for } i > j.$$

Thus the R-module map $\phi\colon A \longrightarrow B$ given by the rule $\phi(x^j) = t^j$ for $0 \leqq j < q$ is a total integral.

2. Invariant subalgebras and the map β

(2.1) Let A be a Hopf algebra over R and W a right A-comodule. By W_0 we denote the kernel of $W \underset{i}{\overset{\rho}{\rightrightarrows}} W \otimes A$, where ρ is the structure map of W and $i(w) = w \otimes 1$ for all $w \in W$.

Thus $W_0 = \{ w \in W \mid \rho(w) = w \otimes 1 \}$. We say that W_0 is the <u>invariant subspace</u> of W.

For an R-module V, $V \otimes A$ has a right A-comodule structure via $v \otimes a \longrightarrow v \otimes \Delta(a)$. Let $\Sigma v_i \otimes a_i$ be an element in the invariant subspace of $V \otimes A$. Hence $\Sigma v_i \otimes \Delta(a_i) = \Sigma v_i \otimes a_i \otimes 1$. Applying to $I \otimes \varepsilon \otimes I$ to both sides of this equation, we conclude that $\Sigma v_i \otimes a_i = \Sigma \varepsilon(a_i)v_i \otimes 1$. This shows that $(V \otimes A)_0 \simeq V$. In particular, $A_0 \simeq R$.

If V is a flat R-module, then we have for any right A-comodule W that $(V \otimes W)_0 \simeq V \otimes W_0$.

Note that for a family of A-comodules, $\{ W_i \}_{i \in I}$,

$$(\bigoplus_{i \in I} W_i)_0 \simeq \bigoplus_{i \in I} (W_i)_0 .$$

(2.2) Let B be an A-comodule algebra. By C we denote the invariant subspace of B. C is a subalgebra of B, which we call <u>the invariant subalgebra</u> of B. In the examples of (1.8), $C = \{ c \in B \mid x(c) = c$ for all $x \in G \}$, $\{ c \in B \mid d(c) = 0 \}$ and $\{ c \in B \mid D_i(c) = 0$ for $1 \leqq i < q \}$ respectively.

Let $M \in \mathcal{M}_B^A$, $m \in M_0$ and $c \in C$. Then $\rho_M(mc) = mc \otimes 1$. This shows that M_0 is a right C-module. Similarly, N_0 is a left C-module for any $N \in {}_B\mathcal{M}^A$. Thus $(-)_0 \colon \mathcal{M}_B^A \longrightarrow \mathcal{M}_C$ is a covariant functor, where $\mathcal{M}_C$ denotes the category of right C-modules. It has the left adjoint $L \colon \mathcal{M}_C \longrightarrow \mathcal{M}_B^A$ defined by $L(V) = V \otimes_C B$, where $(v \otimes_C b)b' = v \otimes_C bb'$, $v \otimes_C b \longrightarrow \Sigma v \otimes_C b_{(0)} \otimes b_{(1)}$. The adjunctions are as follows; for $V \in \mathcal{M}_C$ and $M \in \mathcal{M}_B^A$,

$$\Phi_V \colon V \longrightarrow (V \otimes_C B)_0, \qquad v \longmapsto v \otimes_C 1$$

Ψ_M: $M_0 \otimes_C B \longrightarrow M$, $\quad m \otimes_C b \longmapsto mb$.

A similar adjoint situation holds for (${}_B\mathcal{M}^A$, ${}_C\mathcal{M}$); for $U \in {}_C\mathcal{M}$, $N \in {}_B\mathcal{M}^A$

Φ'_U: $U \longrightarrow (B \otimes_C U)_0$, $\quad u \longmapsto 1 \otimes_C u$

Ψ'_N: $B \otimes_C N_0 \longrightarrow N$, $\quad b \otimes_C n \longmapsto bn$.

(2.3) Recall that $B \otimes A \in \mathcal{M}_B^A$ and $\in {}_B\mathcal{M}^A$ (see (1.5)). We denote by β (resp. β') the adjunction $\Psi_{B\otimes A}$ (resp. $\Psi'_{B\otimes A}$). Identifying $(B \otimes A)_0$ with B, we get

$$\beta: B \otimes_C B \longrightarrow B \otimes A, \quad b \otimes_C b' \longmapsto \Sigma\, bb'_{(0)} \otimes b'_{(1)}$$

$$\beta': B \otimes_C B \longrightarrow B \otimes A, \quad b \otimes_C b' \longmapsto \Sigma\, b_{(0)}b' \otimes b_{(1)}.$$

Note that β (resp. β') is not only a map in $\mathcal{M}_B^A$ (resp. in ${}_B\mathcal{M}^A$) but also a left (resp. right) B-module map in the natural way.

<u>Proposition</u>. If the antipode S of A is bijective, then the map $\alpha: B \otimes A \longrightarrow B \otimes A$ given by $\alpha(b \otimes a) = \Sigma\, b_{(0)} \otimes b_{(1)}S(a)$ is an R-module isomorphism. The inverse of α is given by $\alpha^{-1}(b \otimes a) = \Sigma\, b_{(0)} \otimes \overline{S}(a)b_{(1)}$. Moreover $\beta' = \alpha\beta$ holds. In particular β is injective, surjective, or bijective, respectively, if and only if β' is injective, surjective, or bijective.

The proof is easy, hence omitted.

<u>Remark</u>. For any subalgebra D of C we can consider the map β_D: $B \otimes_D B \longrightarrow B \otimes A$, $\quad b \otimes_D b' \longmapsto \Sigma\, bb'_{(0)} \otimes b'_{(1)}$. It is verified in the same way as ([23], 4.2 Lemma) that if β_D is bi-

jective and B is a faithfully flat right D-module then $C = D$.

(2.4) For an A-comodule algebra B, define the maps π, π', ω and ω' as follows:

$$\pi\colon \operatorname{Hom}(A, B) \longrightarrow \operatorname{End}_{C-}(B), \quad \pi(f)(b) = \Sigma\, b_{(0)} f(b_{(1)})$$

$$\pi'\colon \operatorname{Hom}(A, B) \longrightarrow \operatorname{End}_{-C}(B), \quad \pi'(f)(b) = \Sigma\, f(b_{(1)}) b_{(0)}$$

$$\omega\colon \operatorname{Com}(A, B) \longrightarrow \operatorname{Hom}_{-C}(B, C), \quad \omega(\phi)(b) = \Sigma\, \phi(\overline{S}(b_{(1)})) b_{(0)}$$

$$\omega'\colon \operatorname{Com}(A, B) \longrightarrow \operatorname{Hom}_{C-}(B, C), \quad \omega'(\phi)(b) = \Sigma\, b_{(0)} \phi(S(b_{(1)})).$$

Observe that the map π obtains as the composite

$$\operatorname{Hom}(A, B) \simeq \operatorname{Hom}_{B-}(B \otimes A, B) \xrightarrow{\beta^*} \operatorname{Hom}_{B-}(B \otimes_C B, B) \simeq \operatorname{End}_{C-}(B)$$

and π' obtains as the composite

$$\operatorname{Hom}(A, B) \simeq \operatorname{Hom}_{-B}(B \otimes A, B) \xrightarrow{\beta'^*} \operatorname{Hom}_{-B}(B \otimes_C B, B) \simeq \operatorname{End}_{-C}(B).$$

Also the map ω obtains by lemma (1.5) as the composite

$$\operatorname{Com}(A, B) \simeq M_B^A(B \otimes A, B) \xrightarrow{\beta^*} M_B^A(B \otimes_C B, B) \xrightarrow[\sim]{\text{adjoint.}} \operatorname{Hom}_{-C}(B, C)$$

and the map ω' obtains as the composite

$$\operatorname{Com}(A, B) \simeq {}_B M^A(B \otimes A, B) \xrightarrow{\beta'^*} {}_B M^A(B \otimes_C B, B) \xrightarrow{\sim} \operatorname{Hom}_{C-}(B, C).$$

Proposition. Let A be a Hopf algebra with bijective antipode and B an A-comodule algebra. Assume that the map β is bijective. Then: (1) The above π, π', ω and ω' are bijective.

(2) The following are equivalent;

(i) there exists a total integral $\phi\colon A \longrightarrow B$,

(ii) C is a direct summand of the right C-module B,

(iii) C is a direct summand of the left C-module B.

(3) There exists a total integral ϕ with $\phi(A) \subset Z_B(C)$, the centralizer of C in B, if and only if C is a direct summand

of the (C, C)-bimodule B.

Proof. (1) follows from the above observation. (2) and (3) follow from (1).

(2.5) Theorem. Let A be a commutative Hopf algebra which is a projective R-module. Let B be an A-comodule algebra. Assume that there exists a total integral $\phi: A \longrightarrow B$ with $\phi(A) \subset \text{cent}(B)$ and the map $\beta: B \otimes_C B \longrightarrow B \otimes A$ is surjective. If either B is a flat R-module, or B is a flat right C-module, then β is bijective and the adjunction $\Psi_M: M_0 \otimes_C B \longrightarrow M$ is an isomorphism for all M in M_B^A.

Proof. We first show that β is bijective. The proof is similar in spirit to ([8], Theorem 2.4). Suppose that B is flat as a right C-module (or as an R-module). Noting $(B \otimes_C B)_0 \simeq B$ (or $(B \otimes B)_0 \simeq B$), it is easily verified that the adjunction Ψ_M is an isomorphism for $M = B \otimes_C B$ (or $M = B \otimes B$ where the right (A, B)-Hopf module structure on $B \otimes B$ is $b \otimes b' \longrightarrow b \otimes \rho_B(b')$ and $(b \otimes b')b'' = b \otimes b'b''$). We claim that $B \otimes A$ is projective as a right B-module. This will complete the proof because $B \otimes A$ is isomorphic to a M_B^A-direct summand of $B \otimes_C B$ (or $B \otimes B$) by theorem (1.7), (2) and $\Psi_{B \otimes A} = \beta$. Recall that the right B-module structure of $B \otimes A$ is $(b \otimes a)b' = \Sigma\, bb'_{(0)} \otimes ab'_{(1)}$. But $B \otimes A$ also has a right B-module structure via the first facter, and it is B-projective (since A is R-projective). It is easily checked that the two B-module structures of $B \otimes A$ are isomorphic by the correspondence $\tau(b \otimes a) = \Sigma\, b_{(0)} \otimes S(b_{(1)})a$ and $\tau^{-1}(b \otimes a) = \Sigma\, b_{(0)} \otimes b_{(1)}a$ (since A is commutative). This completes the proof that β is bijective.

Now let M be any object in M_B^A. By theorem (1.7), (1), M is a M_B^A-direct summand of $M \otimes A$. The adjunction $\Psi_{M\otimes A}$ factors as follows: $(M \otimes A)_0 \otimes_C B \simeq M \otimes_C B \simeq M \otimes_B (B \otimes_C B)$

$$\xrightarrow{\ I \otimes \beta\ } M \otimes_B (B \otimes A) \simeq M \otimes A.$$

Therefore, since β is bijective, $\Psi_{M\otimes A}$ is an isomorphism. Thus Ψ_M is an isomorphism for all M in M_B^A. q.e.d.

(2.6) Let A be a Hopf algebra and B an A-comodule algebra. For f, g $\in$ Hom(A, B), define $f*g = m_B(f \otimes g)\Delta_A$ where m_B is the multiplication map $B \otimes B \longrightarrow B$. Then this product * makes Hom(A, B) into an associative algebra with identity $u_B\varepsilon_A$. The set of *-invertible elements is denoted by Reg(A, B). Note that Alg(A, B) $\subset$ Reg(A, B).

We say B is <u>cleft</u> if there exists an integral in Reg(A, B). Let $\phi: A \longrightarrow B$ be an integral in Reg(A, B). If $\psi: A \longrightarrow B$ denotes the *-inverse of ϕ then $\Sigma\, \phi(a_{(1)})\psi(a_{(2)}) = \varepsilon(a)1_B = \Sigma\, \psi(a_{(1)})\phi(a_{(2)})$ for all a in A. In particular $\phi(1_A)\psi(1_A) = 1_B = \psi(1_A)\phi(1_A)$. Now define $\tilde{\phi}, \tilde{\psi} : A \longrightarrow B$ by $\tilde{\phi}(a) = \psi(1)\phi(a)$ and $\tilde{\psi}(a) = \psi(a)\phi(1)$. Then $\tilde{\phi}$ is a total integral and $\tilde{\phi}*\tilde{\psi} = u_B\varepsilon_A = \tilde{\psi}*\tilde{\phi}$. Thus we obtain that B is cleft if and only if there exists a total integral in Reg(A, B).

Consider the example 2 mentioned in (1.8) and let f, g be in Hom(A, B). Since $(f*g)(c_r) = \Sigma_{i=0}^{r} f(c_i)g(c_{r-i})$ and $(u_B\varepsilon_A)(c_r) = 0$ for $r \geqq 0$, $(u_B\varepsilon_A)(c_0) = 1_B$, it follows that $f \in \mathrm{Reg}(A, B)$ if and only if $f(c_0) \in U(B)$, the set of units of B. Therefore, in this case we have that if B has an total integral then B is cleft.

We note that if A is a finitely generated, projective R-module then the algebra Hom(A, B) is isomorphic to $B \otimes A^*$ and this induces $\mathrm{Reg}(A, B) \simeq U(B \otimes A^*)$. It follows that B as in the example 1 of (1.8) is cleft if and only if there exists an element b in B such that $\sum_{x \in G} x^{-1}(b)x$ is an unit in the group algebra B[G].

(2.7) <u>Lemma</u>. Let $\phi: A \longrightarrow B$ be an integral in Reg(A, B) and we denote by ψ the *-inverse of ϕ. Then:

(1) $\rho_B\psi = (\psi \otimes S)T\Delta$ holds, where $T: A \otimes A \longrightarrow A \otimes A$ is the twist map.

(2) If the antipode S is bijective then the map $\psi\overline{S}: A^{op} \longrightarrow B^{op}$ is an integral for the A^{op}-comodule algebra B^{op} and $\psi\overline{S}$ is in $\mathrm{Reg}(A^{op}, B^{op})$.

<u>Proof</u>. The proof of (1) is found in ([8], Lemma 3.2).

(2): We have $\rho_B(\psi\overline{S}) = (\psi \otimes S)T\Delta\overline{S}$ by (1)

$= (\psi \otimes S)(\overline{S} \otimes \overline{S})\Delta$ since $\overline{S}$ is an anti-coalgebra map

$= (\psi\overline{S} \otimes I)\Delta$.

This shows that $\psi\overline{S}$ is an integral for B and so for the A^{op}-comodule algebra B^{op}. We also compute

$$m_BT_B(\psi\overline{S} \otimes \phi\overline{S})\Delta = m_B(\phi\overline{S} \otimes \psi\overline{S})T_A\Delta = m_B(\phi \otimes \psi)\Delta\overline{S} = (\phi*\psi)\overline{S} = u_B\varepsilon_A.$$

Similarly, we have $m_B T_B(\phi\bar{S} \otimes \psi\bar{S})\Delta = u_B\varepsilon_A$. This shows that $\phi\bar{S}$ is the inverse of $\psi\bar{S}$ in $\mathrm{Reg}(A^{op}, B^{op})$ q.e.d.

(2.8) Theorem. For the next statements concerning an A-comodule algebra B, (1) $\Rightarrow$ (2) $\Rightarrow$ (3) holds:

(1) B is cleft.

(2) (i) For every M in M_B^A, $\Psi_M: M_0 \otimes_C B \simeq M$.

(ii) There exists a left C-module, right A-comodule isomorphism between $C \otimes A$ and B.

(3) (i) $\beta: B \otimes_C B \simeq B \otimes A$.

(ii) as (2)(ii).

If, in addition, the antipode S is bijective, then (3) $\Rightarrow$ (1) holds and these are equivalent to

(4) (i) For every N in ${}_BM^A$, $\Psi'_N: B \otimes_C N_0 \simeq N$.

(ii) There exists a right C-module, right A-comodule isomorphism between $C \otimes A$ and B.

(5) (i) $\beta': B \otimes_C B \simeq B \otimes A$.

(ii) as (4)(ii).

Proof. (1) $\Rightarrow$ (2)(i) may be verified in the same way as [8], Th.3.3. (1) $\Rightarrow$ (2)(ii). Now suppose that $\phi: A \longrightarrow B$ is an integral in $\mathrm{Reg}(A, B)$. It is easy to check that the map $F: C \otimes A \longrightarrow B$ given by $F(c \otimes a) = c\phi(a)$ is a right A-comodule map which is a left C-module map. Define the map $G: B \longrightarrow C \otimes A$ by $G(b) = \Sigma\, b_{(0)}\psi(b_{(1)}) \otimes b_{(2)}$, where ψ denotes the *-inverse of ϕ. Since $\Sigma\, b_{(0)}\psi(b_{(1)}) \in C$ by lemma (2.7)(1), G is in fact a map

of B into $C \otimes A$. It is easy to show that $GF = I$ and $FG = I$.
(2) => (3) is clear since $\beta = \Psi_{B \otimes A}$.
(3) => (1). Suppose that β is bijective and there exists a left C-module isomorphism $F: C \otimes A \longrightarrow B$ which is also an A-comodule map. Define $\phi: A \longrightarrow B$ by $\phi(a) = F(1_B \otimes a)$. Then ϕ is an integral and $F(c \otimes a) = c\phi(a)$ for all c in C, a in A. We will show that $\phi \in \mathrm{Reg}(A, B)$. Denote the inverse map of F by G and put $g = (I \otimes \varepsilon_A)G$. Since $G: B \longrightarrow C \otimes A$ is a left C-module, right A-comodule map, we have that $g \in \mathrm{Hom}_{C-}(B, C)$ and $G = (g \otimes I)\rho_B$. Since $\omega': \mathrm{Com}(A, B) \simeq \mathrm{Hom}_{C-}(B, C)$ (see (2.4)), there exists an integral $\eta: A \longrightarrow B$ such that

$$g(b) = \Sigma\, b_{(0)}\eta(S(b_{(1)})) \quad \text{for all } b \text{ in } B.$$

Then

$$1_C \otimes a = GF(1_C \otimes a) = G(\phi(a)) = \Sigma\, g(\phi(a_{(1)}) \otimes a_{(2)}$$
$$= \Sigma\, \phi(a_{(1)})\eta(S(a_{(2)})) \otimes a_{(3)}.$$

Applying $I \otimes \varepsilon$ we have that for all a in A,

$$\varepsilon(a)1_B = \Sigma\, \phi(a_{(1)})\eta(S(a_{(2)})).$$

This shows that $\phi * \eta S = u_B\varepsilon_A$.
It remains to show that $\eta S * \phi = u_B\varepsilon_A$. Using $FG = I_B$,

$$b = \Sigma\, g(b_{(0)})\phi(b_{(1)}) = \Sigma\, b_{(0)}\eta(S(b_{(1)}))\phi(b_{(2)}).$$

This shows that $\pi(\eta S * \phi) = I_B$ where π is as in (2.4). Since $\pi(u_B\varepsilon_A) = I_B$ and π is bijective it follows that $\eta S * \phi = u_B\varepsilon_A$.
This completes the proof of (3) => (1).

We note by lemma(2.7) (2) that B is cleft if and only if B^{op} is cleft as an A^{op}-comodule algebra. Also ${}_B\mathcal{M}^A$ is naturally isomorphic to $\mathcal{M}^{A^{op}}_{B^{op}}$. Therefore (1) <=> (4) <=> (5) are obtained by applying the above proof to B^{op}. q.e.d.

Remark. The condition (3) is nothing but the definition of A-Galois extensions with normal bases ([12]).

When A is projective as an R-module, a cleft comodule algebra B is a projective left C-module by (2)(ii) and is then a faithfully flat left C-module by lemma (2.4)(2). It follows that the adjunction $\Phi_V: V \longrightarrow (V \otimes_C B)_0$ is an isomorphism for any right C-module V (consider $\Phi_V \otimes_C B$ and apply (2)(i)). Thus in this case the functor $(-)_0: M_B^A \longrightarrow M_C$ is an equivalence of categories.

3. When R is a field

Throughout this section it is assumed that R is a field.

(3.1) Proposition. Let A be a Hopf algebra over a field R and B an A-comodule algebra with invariant subalgebra C. If B has a total integral then for any right C-module,

$$\Phi_V: V \longrightarrow (V \otimes_C B)_0, \quad v \longmapsto v \otimes_C 1$$

is an isomorphism.

Proof. Let $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ be an exact sequence in M_B^A. Since R is a field it is split as an A-comodule sequence by theorem (1.6). It follows that the sequence $0 \longrightarrow M'_0 \longrightarrow M_0 \longrightarrow M''_0 \longrightarrow 0$ is also exact. Thus the functor $(-)_0: M_B^A \longrightarrow M_C$ is exact. (This means that B is a projective object in M_B^A.)

Now let V be any right C-module. Take an free presentation $C^{(J)} \longrightarrow C^{(I)} \longrightarrow V \longrightarrow 0$. By the right exactness of tensor product and the exactness of $(-)_0$, we get a commutative diagram

$$\begin{array}{ccccccc} C^{(J)} & \longrightarrow & C^{(I)} & \longrightarrow & V & \longrightarrow & 0 \quad :\text{exact} \\ \downarrow\wr & & \downarrow\wr & & \downarrow \Phi_V & & \\ (C^{(J)} \otimes_C B)_0 & \longrightarrow & (C^{(I)} \otimes_C B)_0 & \longrightarrow & (V \otimes_C B)_0 & \longrightarrow & 0 \quad :\text{exact} \end{array}$$

This shows that Φ_V is an isomorphism by the Five-Lemma. q.e.d.

(3.2) Theorem. Let A be a commutative Hopf algebra over a field. Let B be an A-comodule algebra which is commutative as an algebra. Then the following are equivalent:

(1) (i) $\beta: B \otimes_C B \longrightarrow B \otimes A$ is surjective.

(ii) B is an injective A-comodule.

(2) The functor $(-)_0: M_B^A \longrightarrow M_C$ is an equivalence of categories.

(3) (i) β is bijective.

(ii) B is a faithfully flat C-module.

Proof. (1) $\Rightarrow$ (2): Immediate from theorem (2.5), proposition (3.1).
(2) $\Rightarrow$ (3): Clear.
(3) $\Rightarrow$ (1): It suffices, by [24], A.2.1., to prove that the functor $B \,\square_A\, -$ of the category of left A-comodules to the category of R-vecter spaces is exact. For any left A-comodule U,

$$B \otimes_C (B \,\square_A\, U) \simeq (B \otimes_C B) \,\square_A\, U \simeq (B \otimes A) \,\square_A\, U$$
$$\simeq B \otimes (A \,\square_A\, U) \simeq B \otimes U.$$

Since B is a faithfully flat C-module this shows that the functor $B \,\square_A\, -$ is exact. This proof is due to Takeuchi [22].

q.e.d.

Remark. This gives a purely Hopf-algebraic proof of (Oberst [16], Satz A).

(3.3) When B is a commutative Hopf algebra and A is a quotient Hopf algebra of B

Let B be a commutative Hopf algebra over a field R. Let $f: B \longrightarrow A$ be a given surjective Hopf algebra map and make B an A-comodule algebra by defining $\rho_B = (I \otimes f)\Delta_B$. Then $C = \{ b \in B \mid \Sigma\, b_{(1)} \otimes f(b_{(2)}) = b \otimes 1 \}$. C is a left coideal of B, that is, $\Delta_B(C) \subset B \otimes C$. We note that the map

$$\beta: B \otimes_C B \longrightarrow B \otimes A, \quad b \otimes_C b' \longmapsto \Sigma\, bb'_{(1)} \otimes f(b'_{(2)})$$

is surjective. In fact, it is enough to show that the composite

$$B \otimes B \xrightarrow{\ \xi\ } B \otimes B \xrightarrow{\ I \otimes f\ } B \otimes A$$

is surjective, where $\xi(b \otimes b') = \Sigma\, bb'_{(1)} \otimes b'_{(2)}$. But ξ is an automorphism ($\xi^{-1}(b \otimes b') = \Sigma\, bS(b'_{(1)}) \otimes b'_{(2)}$). Hence, the above composite is surjective because f is surjective.

Now assume that B is an injective A-comodule. Then B is a faithfully flat C-module by theorem (3.2). Viewing A as a right (A, B)-Hopf module by $a \longrightarrow \Delta_A(a)$ and $a.b = af(b)$, we get, by theorem (3.2), (2), that $R \otimes_C B \simeq A$. Applying $- \otimes_C B$ to the exact sequence $0 \longrightarrow C^+ \longrightarrow C \xrightarrow{\ \varepsilon\ } R \longrightarrow 0$, we have an exact sequence $C^+ \otimes_C B \longrightarrow C \otimes_C B \longrightarrow R \otimes_C B \longrightarrow 0$. Thus $R \otimes_C B \simeq B/C^+B$, and hence $A \simeq B/C^+B$ (as Hopf algebras).

Conversely, let D be a left coideal subalgebra of B. Since

D^+B is a Hopf ideal of B, B/D^+B is a Hopf algebra, where $D^+ = \{d \in D |\ \varepsilon_B(d) = 0\}$. The automorphism $\xi: B \otimes B \longrightarrow B \otimes B$ induces

$$B \otimes_D B \simeq B \otimes B/D^+B, \quad b \otimes_D b' \longrightarrow \Sigma\, bb'_{(1)} \otimes \overline{b'_{(2)}}.$$

Noting that D is contained in the invariant subalgebra of the B/D^+B -comodule algebra B, it follows from the remark in (2.3) and theorem (3.2) that if B is a faithfully flat D-module then B is an injective B/D^+B -comodule with invariant subalgebra D. Thus we have proved:

Theorem (Takeuchi). Let B be a commutative Hopf algebra over a field R. Then there is a bijective correspondence between the set of quotient Hopf algebras $B \longrightarrow A$ where B is an injective right A-comodule and the set of left coideal subalgebras $D \subset B$ over which B is a faithfully flat module.

Remarks. (1) It is known in [7], Theorem 2 that for any Hopf quotient $B \longrightarrow A$, B is injective as a right A-comodule if and only if it is faithfully coflat. Hence the above theorem is essencially the same to ([25], Theorem 3.).

(2) If D is a Hopf subalgebra of B then the D-module B is always faithfully flat ([21], THEOREM 3.1.). Hence B is an injective B/D^+B -comodule. If, moreover, B/D^+B is irreducible as a coalgebra (this is equivalent to " coradical(B) $\subset$ D ", see [25], Lemma 4), then B is cleft as a B/D^+B -comodule algebra by [8], Prop.3.1. In particular B is a free D-module.

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