Menger Manifolds and n-shape theories

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THESIS

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Contents

Introduction

The concept of shape was introduced by Borsuk [Bo_1] as a generalization of homotopy theory. In many studies, shape theory has been developed as a big branch of geometric topology (cf. [Bo_3], [Bo_4], [MS_2]). Using Hilbert cube manifolds (Q-manifolds) theory, Chapman [Cha_1] established the so-colled complement theorem, that is, *two Z-sets in Q have the same shape type if and only if their complements are homeomorphic.*

On the other hand, Menger manifold (μ^{n+1}-manifolds) were introduced and investigated by Bestvina [Be]. From many studies (cf. [Chi_4], [Chi_5], [Dr] etc.), it has become clear that μ^{n+1}-manifolds are "$(n+1)$-dimensional" analogues of Q-manifolds. In [Chi_1], Chigogidze introduced the concept of n-shape for compacta and proved the n-shape version of the complement theorem, that is, *two Z-sets in μ^{n+1} have the same n-shape type if and only if their complements are homeomorphic.* One can see a survey on μ^{n+1}-manifolds in [CKT]. The complement theorems are surveyed in [Sh_4] and [Sh_5].

In this paper, we introduce some variations of n-shape theory and investigate the relation between them. These results were obtained in [Aka_1], [Aka_2], [Aka_3], [Aka_4] and [AS].

In Chapter 1, we give terminology and notation, and present some basic properties which will be needed in the sequel. The n-homotopy extension theorem is strengthened for pairs.

We discuss μ^{n+1}-manifold pairs in Chapter 2 and adapt the Z-set approximation theorem and the Z-set unknotting theorem to pairs.

The concept of shape was extended to compact pairs by Borsuk [Bo_3] (cf. [Bo_4]) and by Mardešić and Segal [MS_1] (cf. [MS_2]), but their definitions do not coincide (see [Ma]). The shape of pairs in the sense of Mardešić-Segal is defined by using inverse systems, but it can be also defined using the Borsuk approach (cf. [KO]). In [Fe], Felt tried to generalize Chapman's complement theorem to compact pairs by using the definition of shape in the sense of Mardešić-Segal. However, his proof contained a gap, which was recovered in [Sa]. In Chapter 3, we give the definition of n-shape of compact pairs with dimension at most $n+1$ by Mardešić-Segal's method and generalize the Chigogidze's complement theorem for compact pairs.

Property UV^∞ arose in the study of cellularity and is connected with shape theory through the result that a compact metric space X has UV^∞ if and only if X has the shape of a singleton. Property SUV^∞ was introduced by Hartley [Ha] as a noncompact variant of property UV^∞. In [Sh_1], Sher defined property SUV^n as a finite-dimensional variant of property SUV^∞, and proved that if a closed connected subset X of a piecewise linear n-manifold has SUV^n, then X has SUV^∞. In Chapter 4, we define the notion of proper n-shape for locally compact spaces and proper n-shape category n-$\mathrm{SH}_p\mathcal{LK}$. Using this notion, we give the characterization of property SUV^n. As a corollary, strenthening Sher's result [Sh_1], if a locally compact connected space with dimension at most n has property SUV^n, then it has property SUV^∞.

In Chapter 5, we give the other approach to proper n-shape and proper n-shape category n-$\mathrm{SH}'_p\mathcal{LK}(n+1)$ whose objects are locally compact spaces with dimension at most $n+1$. Let n-$\mathrm{SH}_p\mathcal{LK}(n+1)$ be a full-subcategory of n-$\mathrm{SH}_p\mathcal{LK}$ whose objects are at most $n+1$ dimensional. We construct a categorical embedding from n-$\mathrm{SH}_p\mathcal{LK}(n+1)$ to n-$\mathrm{SH}'_p\mathcal{LK}(n+1)$. Moreover, strengthening n-shape of compact pairs, we introduce the notion of relative n-shape. Then we prove that if two locally compact spaces with dimension at most $n+1$, whose quasi-component spaces are compact, have the same proper

n-shape type, then their compact pairs of Freudenthal compactifications and the spaces of ends are relative n-shape equivalent. Using the result of Chapter 4, we conclude that if a locally compact space with dimension at most $n+1$ has SUV^n, then its Freudenthal compactification has UV^n.

In Chapter 6, applying proper n-shape, we consider Z-sets in μ^{n+1}-manifolds and prove a μ^{n+1}-manifold version of the result of [Sh_2], that is, if two Z-sets in μ^{n+1}-manifolds are proper n-shape equivalent, then they have arbitrarily small homeomorphic μ^{n+1}-manifold closed neighborhoods.

Chapter 1

Preliminaries

The purpose of this chapter is to introduce basic notation and terminology. In particular, we discuss the n-homotopy extension theorem for pairs.

1.1 The results of μ^{n+1}-manifold theory

All spaces in this paper are assumed to be separable and metrizable, and maps to be continuous. Let $\omega = \{0, 1, 2, \ldots\}$ denote the set of natural number. Let $Q = [0,1]^\omega$ be the Hilbert cube and μ^{n+1} the $(n+1)$-dimensional universal Menger compactum. A μ^{n+1}-*manifold* (resp. a Q-manifold) is a topological manifold modeled on μ^{n+1} (resp. Q). A closed set A in a space X is called a Z-*set* if the identity map $\mathrm{id}_X : X \to X$ is approximable arbitrary closely by maps $f : X \to X \setminus A$.

A pair (of space) (X, X_0) means that X is a space and X_0 is a closed set in X. Let $f, g : (X, X_0) \to (Y, Y_0)$ be (proper) maps of pairs. By $f \simeq g$ ($f \simeq_p g$), we means that f is (properly) homotopic to g. We say that $f, g : (X, X_0) \to (Y, Y_0)$ are (*properly*) n-*homotopic* and denote $f \simeq^n g$ ($f \simeq^n_p g$) if $f\alpha \simeq g\alpha$ ($f\alpha \simeq_p g\alpha$) for any (proper) map $\alpha : (Z, Z_0) \to (X, X_0)$ of a pair (Z, Z_0) with $\dim \leq n$.

Next theorems are obtained by Bestvina [Be].

Theorem 1.1.1 (The Z-set unknotting theorem: estimating version) *Let M be a μ^{n+1}-manifold. For each open cover $\mathcal{U}$ of M there exists an open refinement $\mathcal{V}$ satisfying the following. If $f : A \to B$ is a homeomorphism between Z-sets in M which is $\mathcal{V}$-close to id_A, then f extends to a homeomorphism $\bar{f} : M \to M$ such that $\bar{f}$ is $\mathcal{U}$-close to id_M.* $\square$

Theorem 1.1.2 (The Z-set unknotting theorem:n-homotopy version) *Let M be a μ^{n+1}-manifold and $f : A \to B$ be a homeomorphism between Z-sets in M. If $f \simeq^n_p \mathrm{id}_A$ in M, then f extends to a homeomorphism $\bar{f} : M \to M$ with $\bar{f} \simeq^n_p \mathrm{id}_M$.* $\square$

Theorem 1.1.3 (The Z-set approximation theorem) *Let M be a μ^{n+1}-manifold, X a locally compact space with $\dim \leq n+1$ and A a closed set in X. If $f : X \to M$ is a proper map which restricts to a Z-embedding on A, then f is approximable a Z-embedding $g : X \to M$ such that $f|_A = g|_A$.* $\square$

A map $f : X \to Y$ between spaces is called *n-invertible* if for any map $\alpha : Z \to Y$ from any space Z with $\dim Z \leq n$ to Y, there exists a map $\tilde{\alpha} : Z \to X$ such that $f\tilde{\alpha} = \alpha$. It is easy to observe that if f and α are proper, then $\tilde{\alpha}$ is also proper.

Theorem 1.1.4 ([Chi$_3$]) *There is an $(n+1)$-invertible UV^n-surjection $f : \mu^{n+1} \to Q$ such that the inverse image of a Z-set in Q is a Z-set in μ^{n+1} and the fibers are homeomorphic ($\approx$) to μ^{n+1}. Moreover, the inverse image of $Q_0 = \{(x_i) \in Q \mid x_1 = 0\}$ is homeomorphic to μ^{n+1}.* $\square$

Let μ_0^{n+1} be a Z-set in μ^{n+1} and homeomorphic to μ^{n+1} and $M = \mu^{n+1} \setminus \mu_0^{n+1}$. By the Z-set unknotting theorem, we may assume $f^{-1}(*) = \mu_0^{n+1}$ for some $* \in Q$ (a point is a Z-set in Q). The following follows from Theorem 1.1.4.

Proposition 1.1.5 *There is an $(n+1)$-invertible proper UV^n-surjection $f : M \to Q \setminus \{*\}$ such that the fibers are Z-sets and homeomorphic to μ^{n+1}.* $\square$

Theorem 1.1.6 *For each $n \geq 0$, there exists an n-invertible UV^{n-1}-surjection $f_n : (\mu^n, \mu_0^n) \to (Q, Q_0)$.* □

Remark. Let (X, X_0) be an arbitrary pair of spaces. We can assume that $X \subset Q$ and $X_0 = X \cap Q_0$. Then we have a pair $(Z, Z_0) = (f_n^{-1}(X), f_n^{-1}(X_0))$ of spaces with $\dim \leq n$ and an n-invertible proper UV^{n-1}-surjection $\alpha = f_n|_{(Z,Z_0)} : (Z, Z_0) \to (X, X_0)$. It is easy to see that $f \simeq^n g$ for two maps $f, g : (X, X_0) \to (Y, Y_0)$ if and only if $f\alpha \simeq g\alpha$.

1.2 The n-homotopy extension theorem

In this section, we treat pairs of spaces instead of spaces. Therefore, we need to extend some well-known results to pairs. A pair of LC^n spaces is simply called an *LC^n-pair.*

1.2.1 The extension theorem for pairs

Lemma 1.2.1 *Let (X, X_0) be a pair of spaces with $\dim \leq n+1$ and (Y, Y_0) an LC^n-pair. Let (A, A_0) be a closed pair in (X, X_0) with $A_0 = A \cap X_0$. Then every map $f : (A, A_0) \to (Y, Y_0)$ has an extension $F : (U, U_0) \to (Y, Y_0)$, where U is a neighborhood of A in X and $U_0 = U \cap X_0$. Moreover if (Y, Y_0) is a $C^n \cap LC^n$-pair, then f has an extension $F : (X, X_0) \to (Y, Y_0)$.*

Proof. Since $\dim X_0 \leq n+1$ and Y_0 is LC^n, the map $f|_{A_0} : A_0 \to Y_0$ extends to a map $g : U_0 \to Y_0$, where U_0 is a neighborhood of A_0 in X_0 [Hu]. Note that U_0 is closed in $(X \setminus X_0) \cup U_0$. Since Y is LC^n, the map $g \cup f : U_0 \cup A \to Y$ extends to a map $F : U \to Y$, where U is a neighborhood of $U_0 \cup A$ in $(X \setminus X_0) \cup U_0$. Then $U_0 = U \cap X_0$. Consequently, (U, U_0) is the desired neighborhood and F is the desired extension of f. The proof of the additional statement is similar. □

Lemma 1.2.2 *Let (Y, Y_0) be an LC^n-pair. Then every open cover $\mathcal{U}$ of Y has an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that if (X, X_0) is a pair of* $\dim \leq n$ *then any two $\mathcal{V}$-close maps $f, g : (X, X_0) \to (Y, Y_0)$ are $\mathcal{U}$-homotopic.*

Proof. Since Y is LC^n, every open cover $\mathcal{U}$ of Y has an open refinement $\mathcal{V}$ such that for any two $\mathcal{V}$-close maps $f, g : X \to Y$ of a space X with $\dim X \leq n$, any $\mathcal{V}$-homotopy $h : X_0 \times I \to Y$ from $f|_{X_0}$ to $g|_{X_0}$ extends to a $\mathcal{U}$-homotopy $H : X \times I \to Y$ from f to g [Hu, p.112]. Moreover, since Y_0 is also LC^n, there exists an open cover $\mathcal{V}_0$ of Y_0 such that if $f|_{X_0}, g|_{X_0} : X_0 \to Y_0$ are $\mathcal{V}_0$-close, then we can find a $\mathcal{V}$-homotopy $h : X_0 \times I \to Y_0$ connecting $f|_{X_0}$ to $g|_{X_0}$. Now let $f, g : (X, X_0) \to (Y, Y_0)$ be maps. If f and g are $\mathcal{V}$-close in Y and $f|_{X_0}$ and $g|_{X_0}$ are $\mathcal{V}_0$-close in Y_0, then there exists a $\mathcal{U}$-homotopy $H : (X, X_0) \times I \to (Y, Y_0)$ connecting f and g. For every $V \in \mathcal{V}_0$ there exists an open set V' in Y such that $V = V' \cap Y_0$. Put $\mathcal{V}_0' = \{V' \mid V \in \mathcal{V}_0\} \cup \{Y \setminus Y_0\}$. The common open refinement of $\mathcal{V}$ and $\mathcal{V}_0'$ is the desired cover of Y. □

Corollary 1.2.3 *Let (Y, Y_0) be an LC^n-pair. Then there exists an open cover $\mathcal{U}$ of Y such that any two $\mathcal{U}$-close maps defined on an arbitrary pair of spaces are n-homotopic.* □

The following is due to Chigogidze [Chi_1, Proposition 2.1]:

Proposition 1.2.4 *Let Y be LC^n. Then every open cover $\mathcal{U}$ of Y has an open refinement $\mathcal{V}$ satisfying the following condition:*

$(*)_n$ *For an arbitrary space X with* $\dim \leq n+1$ *and any two $\mathcal{V}$-close maps from a closed set in X, if one of them extends to a map from X, then the other also extends to a map from X, which is $\mathcal{U}$-close to the former extension.*

Remark on the proof. In [Chi_1], spaces are separable completely metrizable. However the proof of Proposition 2.1 in [Chi_1] is valid without the assumption of complete metrizability since every space can be embedded in a linear subspace of the Hilbert space ℓ_2 as a closed set.

As a corollary we have the following:

Lemma 1.2.5 *Every open cover $\mathcal{U}$ of an LC^n space Y has an open refinement $\mathcal{V}$ satisfying the following condition:*

$(\sharp)_n$ *Suppose that $f, g : X \to Y$ are $\mathcal{V}$-close maps, $\alpha : Z \to X$ is a map, $\dim Z \leq n$, A is closed in X, $\varphi : \alpha^{-1}(A) \times I \to Y$ is a $\mathcal{V}$-homotopy with $\varphi_0 = f\alpha|_{\alpha^{-1}(A)}$ and $\varphi_1 = g\alpha|_{\alpha^{-1}(A)}$. Then φ extends to a $\mathcal{U}$-homotopy $\tilde{\varphi} : Z \times I \to M$ with $\tilde{\varphi}_0 = f\alpha$ and $\tilde{\varphi}_1 = g\alpha$.*

Proof. Let $\mathcal{W}$ be an open star-refinement of $\mathcal{U}$. By Proposition 1.2.4, $\mathcal{W}$ has an open refinement $\mathcal{V}$ satisfying $(*)_n$. We extend φ to the map $\varphi' : \alpha^{-1}(A) \times I \cup Z \times \{0, 1\} \to M$ by $\varphi'(z, 0) = f\alpha(z)$ and $\varphi'(z, 1) = g\alpha(z)$ for each $z \in Z$. Since φ is a $\mathcal{V}$-homotopy and g is $\mathcal{V}$-close to f, φ' is $\mathcal{V}$-close to the restriction of the map $f\alpha\mathrm{pr}_Z : Z \times I \to Y$, where $\mathrm{pr}_Z : Z \times I \to Z$ is the projection. By $(*)_n$, φ' extends to a map $\tilde{\varphi} : Z \times I \to Y$ which is $\mathcal{W}$-close to $f\alpha\mathrm{pr}_Z$. It is easy to see that $\tilde{\varphi}$ is the desired homotopy. □

1.2.2 The n-homotopy extension theorem for pairs

The n-homotopy extension theorem was established by Chigogidze [Chi_1, Proposition 2.2]. We strengthen it by including a covering estimate, which might be useful in some applications. Although we have not used the covering estimate, we give a detailed proof, which will help those who have some difficulty concerning the first sentence in Chigogidze's proof.

Theorem 1.2.6 (The n-homotopy extension theorem for pairs) *Suppose that (Y, Y_0) is a locally compact LC^n-pair, (X, X_0) is a locally compact pair with $\dim \leq n+1$ and $f, g : (A, A_0) \to (Y, Y_0)$ are proper maps of a closed pair (A, A_0) in (X, X_0) with $A_0 = A \cap X_0$ which are properly n-homotopic. If f*

extends to a proper map $\tilde{f} : (X, X_0) \to (Y, Y_0)$*, then* g *also extends to a proper map* $\tilde{g} : (X, X_0) \to (Y, Y_0)$ *which is properly* n*-homotopic to* $\tilde{f}$*.*

To prove the n-homotopy extension theorem for pairs, we first generalize the standard trick of bridge maps (cf. [Hu]).

Lemma 1.2.7 *Let* $X_1, \cdots, X_k$ *be closed sets in* X *such that* $X_C = \bigcap_{i \in C} X_i$ *is* LC^n *for any* $C \subset \{1, \cdots, k\}$*, where* $X_\emptyset = X$*. Then each open cover* $\mathcal{U}$ *of* X *has an open refinement* $\mathcal{V}$ *satisfying the following condition:*

$(a)_n$ *Each* $V \in \mathcal{V}$ *is contained in some* $U \in \mathcal{U}$ *such that every map* $f : S^i \to V \cap X_C$ $(0 \leq i \leq n; C \subset \{1, \cdots, k\})$ *extends to a map* $\tilde{f} : B^{i+1} \to U \cap X_C$*.*

Proof. Let $\mathcal{W}$ be an open star-refinement of $\mathcal{U}$. For each $C \subset \{1, \ldots, k\}$, X_C is LC^n, hence we have an open cover $\mathcal{V}_C$ of X such that each $V \in \mathcal{V}_C$ is contained in some $W_C \in \mathcal{W}$ satisfying

$(a)_{nC}$ each map $f : S^i \to V \cap X_C$ $(i = 0, \cdots, n)$ extends to a map $\tilde{f} : B^{i+1} \to W_C \cap X_C$.

Let $\mathcal{V}$ be an open cover of X which refines all $\mathcal{V}_C$ $(C \subset \{1, \ldots, k\})$. Then $\mathcal{V}$ is the desired refinement of $\mathcal{U}$. In fact, let $V \in \mathcal{V}$. For each $C \subset \{1, \ldots, k\}$, V is contained in some $W_C \in \mathcal{W}$ satisfying $(a)_{nC}$. Since $\mathcal{W}$ is a star-refinement of $\mathcal{U}$, $\bigcup_{C \subset \{1, \cdots, k\}} W_C$ is contained in some $U \in \mathcal{U}$. Then $(a)_n$ is satisfied. □

Lemma 1.2.8 *Let* $X_1, \ldots, X_k$ *be closed in* X*. Then each open cover* $\mathcal{U}$ *of* X *has an open refinement* $\mathcal{V}$ *such that*

(b) $V \cap \bigcap \{X_i \mid X_i \cap V \neq \emptyset\} \neq \emptyset$ *for each* $V \in \mathcal{V}$ *with* $V \cap \bigcap_{i \in C} X_i \neq \emptyset$*,*

and $\mathcal{V}$ *is locally finite if so is* $\mathcal{U}$*.*

Proof. For each $C \subset \{1, \ldots, k\}$, let

$$\mathcal{V}_C = \{U \setminus \bigcup_{j \notin C} X_j \mid U \in \mathcal{U}, U \cap \bigcap_{i \in C} X_i \neq \emptyset\},$$

where $\mathcal{V}_\emptyset = \{U \in \mathcal{U} \mid U \cap \bigcup_{i=1}^k X_i = \emptyset\}$. Then $\mathcal{V} = \bigcup_{C \subset \{1,\cdots,k\}} \mathcal{V}_C$ is the desired refinement of $\mathcal{U}$. In fact, for each $V \in \mathcal{V}$, choose $C \subset \{1,\ldots,k\}$ so that $V \in \mathcal{V}_C$. In the case $C = \emptyset$, $V \cap \bigcup_{i=1}^k X_i = \emptyset$. In the case $C \neq \emptyset$, by the definition of $\mathcal{V}_C$, $V \cap \bigcap_{i \in C} X_i \neq \emptyset$ and $V \cap X_j = \emptyset$ if $j \notin C$, namely $C = \{i \mid X_i \cap V \neq \emptyset\}$. Thus (b) is satisfied. The additional statement is clear. □

Lemma 1.2.9 *Let* $\dim X \leq n+1$ *and* $X_1, \cdots, X_k$ *be closed in* X *such that* $X_C = \bigcap_{i \in C} X_i$ *is* LC^n *for any* $C \subset \{1,\ldots,k\}$, *where* $X_\emptyset = X$. *Then for each open cover* $\mathcal{U}$ *of* X, *there exist a locally finite simplicial complex* K *with* $\dim \leq n+1$, *subcomplexes* $K_1, \cdots, K_k$ *of* K, *maps* $p : X \to |K|$ *and* $q : |K| \to X$ *such that* $p(X_j) \subset |K_j|$, $q(|K_j|) \subset X_j$ *and* qp *is* $\mathcal{U}$*-close to* id_X.

Proof. By Lemma 1.2.7, we have a sequence of open covers of X as follows:

$$\mathcal{U} = \mathcal{U}_n \overset{(a)_n}{>} \mathcal{V}_n \overset{*}{>} \mathcal{U}_{n-1} \overset{(a)_{n-1}}{>} \cdots \overset{*}{>} \mathcal{U}_0 \overset{(a)_0}{>} \mathcal{V}_0,$$

where $\mathcal{U} \overset{(a)_j}{>} \mathcal{V}$ means that $\mathcal{V}$ is a refinement of $\mathcal{U}$ satisfying $(a)_j$ in Lemma 1.2.7 and $\mathcal{U} \overset{*}{>} \mathcal{V}$ means that $\mathcal{V}$ is a star-refinement of $\mathcal{U}$. Since $\dim X \leq n+1$, $\mathcal{V}_\prime$ has a locally finite open star-refinement $\mathcal{V}$ of $\operatorname{ord} \mathcal{V} \leq \mathrm{n}+2$ satisfying (b) by Lemma 1.2.8. Let $K = N(\mathcal{V})$ be the nerve of $\mathcal{V}$ and $K_j = N(\mathcal{V}_j)$ nerves of $\mathcal{V}_j = \{V \in \mathcal{V} \mid V \cap X_j \neq \emptyset\}$ $(j = 1, \cdots, k)$. Note that K is a locally finite simplicial complex with $\dim \leq n+1$ and K_j's are subcomplexes of K. Let $p : X \to |K|$ be a canonical map. Then $p(X_j) \subset |K_j|$ for each $j = 1, \cdots, k$.

We construct $q : |K| \to X$ by the skeleton-wise induction. For each $V \in \mathcal{V} = K^{(0)}$, let $X_V = \bigcap\{X_i \mid X_i \cap V \neq \emptyset\}$, where $X_V = X$ if $V \cap \bigcup_{i=1}^k X_i = \emptyset$. Then $V \cap X_V \neq \emptyset$ by (b). By choosing a point $q^{(0)}(V) \in V \cap X_V$ for each $V \in K^{(0)}$, we can define $q^{(0)} : |K^{(0)}| \to X$. For each $\tau \in K^{(1)}$, let $X_\tau = \bigcap\{X_i \mid q^{(0)}(\partial\tau) \subset X_i\}$, where $X_\tau = X$ if $q^{(0)}(\partial\tau) \not\subset X_i$ for any $i = 1, \cdots, k$. Then there exists $V \in \mathcal{V}_\prime$ such that $q^{(0)}(\partial\tau) \subset V \cap X_\tau$, By $(a)_0$, $q^{(0)}|_{\partial\tau}$ can be extended to a map $q_\tau : \tau \to U \cap X_\tau$ for some $U \in \mathcal{U}_\prime$. Then we can extend $q^{(0)}$ to a map

$q^{(1)} : |K^{(1)}| \to X$ such that $q^{(1)}(|K_j^{(1)}|) \subset X_j$ for each $j = 1, \cdots, k$ and $q^{(1)}$ maps each $\tau \in K^{(1)}$ into some $U \in \mathcal{U}_{\prime}$. Assume that $q^{(i)} : |K^{(i)}| \to X$ has been defined such that $q^{(i)}(|K_j^{(i)}|) \subset X_j$ for each $j = 1, \cdots, k$ and $q^{(i)}$ maps each $\tau \in K^{(i)}$ into some $U \in \mathcal{U}_{\rangle-\infty}$. For each $\tau \in K^{(i+1)}$, let $X_\tau = \bigcap\{X_i \mid q^{(i)}(\partial\tau) \subset X_i\}$, where $X_\tau = X$ if $q^{(0)}(\partial\tau) \not\subset X_i$ for any $i = 1, \cdots, k$. Since $\mathcal{U}_{i-1} \overset{*}{<} \mathcal{V}_i$, $q^{(i)}(\partial\tau) \subset V \cap X_\tau$ for some $V \in \mathcal{V}_i$. By $(a)_i$, $q^{(i)}|_{\partial\tau}$ can be extended to a map $q_\tau : \tau \to U \cap X_\tau$ for some $U \in \mathcal{U}_{\rangle}$. Then we can extend $q^{(i)}$ to a map $q^{(i+1)} : |K^{(i+1)}| \to X$ such that $q^{(i+1)}(|K_j^{(i+1)}|) \subset X_j$ for each $j = 1, \cdots, k$ and $q^{(i+1)}$ maps each $\tau \in K^{(i+1)}$ into some $U \in \mathcal{U}_{\rangle}$. Thus we can obtain the desired map $q = q^{(n+1)} : |K| \to X$. □

We use the following well-known fact:

Lemma 1.2.10 *Each k-dimensional metrizable space X can be embedded in a k-dimensional LC^{k-1} metrizable space $E(X)$ as a closed set, where if X is locally compact then $E(X)$ can be locally compact.*

Proof. For completeness, we give a proof. In the proof of [Ko, Theorem 1], Kodama constructed a $(k+1)$-dimensional ANR $M(X) = M \cup X$, where X is closed in $M(X)$ and M is a cell complex. Replacing M with the k-skeleton $M^{(k)}$, we obtain a k-dimensional subspace $E(X) = M^{(k)} \cup X$ of $M(X)$. Then $M^{(k)}$ is an ANR which is open in $E(X)$. To see that $E(X)$ is LC^{k-1} at each $x \in X$, let U be a neighborhood of x in $M(X)$. By the definition of the topology for $M(X)$, U contains a neighborhood W of x in $M(X)$ such that $\tau \cap W \neq \emptyset$ implies $\tau \subset U$ for any cell τ of M. Since $M(X)$ is locally contractible, W contains a neighborhood V of x in $M(X)$ which contracts in W. Then each map $f : S^i \to V \cap E(X)$ $(i = 1, \cdots, k-1)$ extends to a map $g : B^{i+1} \to W$. For each $(k+1)$-cell τ of M, choose a map $f_\tau : g^{-1}(\tau) \to \partial\tau$ so that $f_\tau|_{g^{-1}(\partial\tau)} = g|_{g^{-1}(\partial\tau)}$. We can define a map $\tilde{f} : B^{i+1} \to U \cap E(X)$ by $\tilde{f}|_{g^{-1}(\tau)} = f_\tau$ for each $(k+1)$-cell $\tau \in M$. Clearly $\tilde{f}$ is an extension of f. Thus $E(X)$ is LC^{k-1}. □

Now we prove the following n-homotopy extension theorem:

Theorem 1.2.11 *Suppose that Y is an LC^n space, $\mathcal{U}$ is an open cover of Y, X is a space with $\dim \leq n+1$, $\alpha : Z \to X$ is an n-invertible map from a space $\dim \leq n$, A is closed in X and $f, g : A \to Y$ are maps such that $f\alpha|_{\alpha^{-1}(A)}$ is $\mathcal{U}$-homotopic to $g\alpha|_{\alpha^{-1}(A)}$. If f extends to a map $\tilde{f} : X \to Y$, then g also extends to a map $\tilde{g} : X \to Y$ such that $\tilde{f}\alpha$ and $\tilde{g}\alpha$ are $\mathrm{st}^5\mathcal{U}$-homotopic.*

Proof. We divide the proof into three cases. In each case, let $\varphi : \alpha^{-1}(A) \times I \to Y$ be a $\mathcal{U}$-homotopy such that $\varphi_0 = f\alpha|_{\alpha^{-1}(A)}$ and $\varphi_1 = g\alpha|_{\alpha^{-1}(A)}$.

Case 1: (X, A) is a polyhedral pair. Let K be a triangulation of X with L a subcomplex triangulating A such that $\tilde{f}$ maps each simplex of K in some member of $\mathcal{U}$, and g maps each simplex of L in some member of $\mathcal{U}$. Since $\dim Z \leq n$, we have a map $\alpha' : Z \to |K^{(n)}|$ which is contiguous to α, hence α and α' are homotopic by a homotopy whose each homotopy track is contained in some simplex of K. By the n-invertibility of α, we have a map $\gamma : |K^{(n)}| \to Z$ such that $\alpha\gamma = \mathrm{id}$. Then $\varphi' = \varphi(\gamma|_{\alpha^{-1}(A)} \times \mathrm{id}_I) : |L^{(n)}| \times I \to Y$ is a $\mathcal{U}$-homotopy such that $\varphi'_0 = f|_{|L^{(n)}|}$ and $\varphi'_1 = g|_{|L^{(n)}|}$. Since Y is LC^n, φ' extends to a $\mathcal{U}$-homotopy $\psi' : |K^{(n)}| \times I \to Y$ with $\psi'_0 = \tilde{f}|_{|K^{(n)}|}$. Thus, we have a $\mathcal{U}$-homotopy $\psi = \psi'(\alpha' \times \mathrm{id}_I) : Z \times I \to Y$ such that $\psi_0 = \tilde{f}\alpha'$ and $\psi_1 = \psi'_1\alpha'$. Now, we can extend g to a map $\tilde{g} : X \to Y$ as follows: $\tilde{g}|_{|K^{(n)}|} = \psi'_1$ and $\tilde{g}|_\sigma = (\psi' \cup f \times \{0\})h_\sigma$ for each n-simplex $\sigma \in K \setminus L$, where $h_\sigma : \sigma \to \partial\sigma \cup \sigma \times \{0\}$ is a homeomorphism such that $h(x) = (x, 1)$ for each $x \in \partial\sigma$. By the definition, $\tilde{g}$ maps each simplex of K into some member of $\mathrm{st}^2\mathcal{U}$, whence $\psi_1 = \tilde{g}\alpha'$ is $\mathrm{st}^2\mathcal{U}$-homotopic to $\tilde{g}\alpha$. On the other hand, since $\tilde{f}$ maps each simplex of K in some member of $\mathcal{U}$, $\psi_0 = \tilde{f}\alpha'$ is $\mathcal{U}$-homotopic to $\tilde{f}\alpha$. Therefore $\tilde{f}\alpha$ and $\tilde{g}\alpha$ are $\mathrm{st}^3\mathcal{U}$-homotopic.

Case 2: (X, A) is an LC^n-pair. Since Y is LC^n, $\mathcal{U}$ has an open refinement $\mathcal{W}$ such that any two $\mathcal{W}$-close maps from a space with $\dim \leq n$ are $\mathcal{U}$-homotopic. By Proposition 1.2.4, $\mathcal{W}$ has an open refinement $\mathcal{V}$ satisfying $(*)_n$. Using Lemma 1.2.9, we have a countable locally finite simplicial complex

K with $\dim \leq n+1$, a subcomplex L of K, and maps $p : (X, A) \to (|K|, |L|)$, $q : (|K|, |L|) \to (X, A)$ such that $gqp|_A$ is $\mathcal{V}$-close to g and $\tilde{f}qp\alpha$ is $\mathcal{U}$-homotopic to $\tilde{f}\alpha$. Let $\beta : Z' \to |K|$ be an n-invertible map, where $\dim Z' \leq n$. By the n-invertibility of β and α, we have maps $\tilde{\alpha} : Z \to Z'$ and $\tilde{\beta} : Z \to Z'$ such that $p\alpha = \beta\tilde{\alpha}$ and $q\beta = \alpha\tilde{\beta}$. Since $fq\beta|_{\beta^{-1}(|L|)} = f\alpha\tilde{\beta}|_{\beta^{-1}(|L|)}$ is $\mathcal{U}$-homotopic to $g\alpha\tilde{\beta}|_{\beta^{-1}(|L|)}$ and $fq|_{|L|}$ extends to the map $\tilde{f}q$, it follows from Step 1 that $gq|_{|L|}$ also extends to a map $g' : |K| \to Y$ such that $\tilde{f}q\beta$ is $\mathrm{st}^3\mathcal{U}$-homotopic to $g'\beta$. Then $\bar{g} = g'p : X \to Y$ is an extension of $gqp|_A : A \to Y$. Since $gqp|_A$ is $\mathcal{V}$-close to g, g also extends to a map $\tilde{g} : X \to Y$ such that $\tilde{g}\alpha$ is $\mathcal{U}$-homotopic to $\bar{g}\alpha$. On the other hand, $\tilde{f}\alpha$ is $\mathcal{U}$-homotopic to $\tilde{f}qp\alpha = \tilde{f}q\beta\tilde{\alpha}$, which is $\mathrm{st}^3\mathcal{U}$-homotopic to $g'\beta\tilde{\alpha} = g'p\alpha = \bar{g}\alpha$. Consequently, $\tilde{f}\alpha$ and $\tilde{g}\alpha$ are $\mathrm{st}^4\mathcal{U}$-homotopic.

General Case. We can embed (X, A) in an $(n+1)$-dimensional LC^n-pair (E, F) as a closed pair such that $A = X \cap F$. (In fact, first embed A in some F as a closed set and then $X \cup F$ in some E as a closed set.) Since Y is LC^n, we may assume that f and g are defined on an open neighborhood V of A in F and $\tilde{f}$ is defined on an open neighborhood W of X in E such that $W \cap F = V$. Let $\beta : Z' \to E$ be an n-invertible proper map, where $\dim Z' \leq n$. (The existence of such a map follows from results in [Dr]. Refer the proof of [Chi$_1$, Proposition 2.1] and Remark on the proof of Proposition 1.2.4.) Then we have maps $\tilde{\alpha} : Z \to Z'$ and $\tilde{\beta} : \beta^{-1}(X) \to Z$ such that $\alpha = \beta\tilde{\alpha}$ and $\beta|_{\beta^{-1}(X)} = \alpha\tilde{\beta}$. Since Y is LC^n and β is proper, the $\mathcal{U}$-homotopy $\psi = \varphi(\tilde{\beta} \times \mathrm{id}_I) : \beta^{-1}(A) \times I \to Y$ extends to a $\mathcal{U}$-homotopy extends to a $\mathcal{U}$-homotopy $\tilde{\varphi} : \beta^{-1}(A') \times I \to Y$ such that $\tilde{\varphi}_0 = f\beta|_{\beta^{-1}(A')}$ and $\tilde{\varphi}_1 = g\beta|_{\beta^{-1}(A')}$, where A' is an open neighborhood of A in V. Let $X' = W \setminus (F \setminus A')$. Then (X', A') is an LC^n-pair with $\dim \leq n+1$, f and g are defined on A', the extension $\tilde{f}$ of f is defined on X', $f\beta|_{\beta^{-1}(A')}$ is $\mathcal{U}$-homotopic to $g\beta|_{\beta^{-1}(A')}$. By Case 2, g extends to a map $\tilde{g} : X' \to Y$ such that $\tilde{f}\beta$ and $\tilde{g}\beta$ are $\mathrm{st}^5\mathcal{U}$-homotopic, whence $\tilde{f}\alpha = \tilde{f}\beta\tilde{\alpha}$ and $\tilde{g}\beta\tilde{\alpha} = \tilde{g}\alpha$ are also $\mathrm{st}^5\mathcal{U}$-homotopic. This completes the proof. □

Remark. In the above theorem, *if X and Y are locally compact, α, f, g and the homotopy from $f\alpha|_{\alpha^{-1}(A)}$ to $g\alpha|_{\alpha^{-1}(A)}$ are proper, then the homotopy from $\tilde{f}\alpha|_{\alpha^{-1}(A)}$ to $\tilde{g}\alpha|_{\alpha^{-1}(A)}$ can be proper.* In fact, subdividing K in Case 1, we may assume that for any two contiguous maps from a locally compact space to $|K|$, if one of them is proper then the other is also proper. Then all maps and homotopies are proper. In case 2, take an open refinement $\mathcal{U}'$ of $\mathcal{U}$ such that any map from a locally compact space to Y is proper if it is $\mathcal{U}'$-close to a proper map, and let $\mathcal{W}$ be an open refinement of $\mathcal{U}'$ such that any two $\mathcal{W}$-close maps from a space with $\dim \leq n$ are $\mathcal{U}'$-homotopic, whence if one of two $\mathcal{W}$-close maps is proper then the other and the homotopy are also proper. Hence all maps and homotopies can be proper. In General Case, embed (X, A) in an $(n+1)$-dimensional locally compact LC^n-pair (E, F) as a closed pair such that $A = X \cap F$, and $\beta : Z' \to E$ be proper. Then X' is locally compact and all maps and homotopies can be proper.

The n-homotopy extension theorem for pairs can be proved by replacing spaces with pairs in the proof of Theorem 1.2.11.

Theorem 1.2.12 *Let (Y, Y_0) be an LC^n-pair, $\mathcal{U}$ is an open cover of Y, (X, X_0) is a pair with* $\dim \leq n+1$, *$\alpha : (Z, Z_0) \to (X, X_0)$ is an n-invertible map from a pair with* $\dim \leq n$, *(A, A_0) is a closed pair in (X, X_0) with $A_0 = A \cap X_0$, and $f, g : (A, A_0) \to (Y, Y_0)$ are maps such that $f\alpha|_{(\alpha^{-1}(A), \alpha^{-1}(A_0))}$ is $\mathcal{U}$-homotopic to $g\alpha|_{(\alpha^{-1}(A), \alpha^{-1}(A_0))}$. If f extends to a map $\tilde{f} : (X, X_0) \to (Y, Y_0)$ then g also extends to a map $\tilde{g} : (X, X_0) \to (Y, Y_0)$ such that $\tilde{f}\alpha$ is* $\mathrm{st}^5\mathcal{U}$*-homotopic to $\tilde{g}\alpha$.*

Proof. When spaces are replaced by pairs, we verify that all maps and homotopies in each cases are relative.

In Case 1, K and L are also replaced by pairs, whence the maps α', γ and φ can be relative. Now remark that α and α' are relatively homotopic by the same homotopy. Since (Y, Y_0) is an LC^n-pair, we can extend φ' to a relative homotopy $\psi' : (|K^{(n)}|, |K_0^{(n)}|) \times I \to (Y, Y_0)$. Thus the homotopy ψ is also relative. By the definition, $\tilde{g}$ is naturally relative and $\tilde{f}\alpha'$ is relatively homotopic to $\tilde{g}\alpha'$. Since the homotopy from α to α' is relative, the homotopies from $\tilde{f}\alpha$ to $\tilde{f}\alpha'$ and from $\tilde{g}\alpha$ to $\tilde{g}\alpha'$ are also relative. Consequently, $\tilde{f}\alpha$ and $\tilde{g}\alpha'$ are also relatively homotopic.

In Case 2, let $\mathcal{W}$ be an open refinement of $\mathcal{U}$ satisfying the condition of Lemma 1.2.2, whence any two $\mathcal{W}$-close maps from a pair with dim $\leq n$ are relatively $\mathcal{U}$-homotopic. Using Proposition 1.2.4 twice, we have an open refinement $\mathcal{V}$ of $\mathcal{W}$ satisfying $(*)_n$ for pairs. Then it is easy to see that all maps and homotopies are relative when all spaces, K and L are replaced by pairs. Lemma 1.2.9 guarantees that all spaces, K and L can be replaced by pairs in this Case 2.

In General Case, first embed (A, A_0) in an $(n+1)$-dimensional LC^n-pair (F, F_0) as a closed pair such that $A_0 = A \cap F_0$, and then embed $(X \cup F, X_0 \cup F_0)$ in in an $(n+1)$-dimensional LC^n-pair (E, E_0) as a closed pair $X_0 \cup F_0 = (X \cup F) \cap E_0$, whence $X \cap F = A$ and $X_0 \cap F_0 = A_0$. Then similarly to Theorem 1.2.11, the general case can be reduced to Case 2. □

Remark. Similarly to Theorem 1.2.11, if (X, X_0) and (Y, Y_0) are locally compact, α, f, g and the homotopy from $f\alpha|_{(\alpha^{-1}(A),\alpha^{-1}(A_0))}$ to $g\alpha|_{(\alpha^{-1}(A),\alpha^{-1}(A_0))}$ are proper, then the homotopy from $\tilde{f}\alpha|_{(\alpha^{-1}(A),\alpha^{-1}(A_0))}$ to $\tilde{g}\alpha|_{(\alpha^{-1}(A),\alpha^{-1}(A_0))}$ can be proper. Thus we have Theorem 1.2.6.

Chapter 2

μ^{n+1}-manifold pairs

In this chapter, we study Z-pairs in μ^{n+1}-manifold pairs and adapt the Z-set approximation theorem and the Z-set unknotting theorem to Z-pairs.

2.1 Z-pairs in μ^{n+1}-manifold pair

Let μ_0^{n+1} be a Z-set in μ^{n+1} which is homeomorphic to μ^{n+1}. We call (A, A_0) a *Z-pair* in (X, X_0) if A and A_0 are Z-sets in X and X_0, respectively, and $A_0 = A \cap X_0$. For a pair (X, X_0) of spaces, X_0 is assumed to be a *closed* set in X.

A pair (M, M_0) is said to be a *μ^{n+1}-manifold pair* if M and M_0 are μ^{n+1}-manifolds and M_0 is a Z-set in M. For example, (μ^{n+1}, μ_0^{n+1}) is a μ^{n+1}-manifold pair. We say (A, A_0) *Z-pair* in a pair (X, X_0) if A and A_0 are Z-sets in X and X_0, respectively, and $A_0 = A \cap X_0$.

An embedding $f : (X, X_0) \to (M, M_0)$ is a Z-*embedding* if $(f(X), f(X_0))$ is a Z-pair in (M, M_0).

Lemma 2.1.1 *Let (A, A_0) and (B, B_0) be Z-pairs in (μ^{n+1}, μ_0^{n+1}) and let $f : (A, A_0) \to (B, B_0)$ be a homeomorphism. Then f extends to a homeomorphism $h : (\mu^{n+1}, \mu_0^{n+1}) \to (\mu^{n+1}, \mu_0^{n+1})$.*

Proof. By the Z-set unknotting theorem, $f|_{A_0} : A_0 \to B_0$ extends to a homeomorphism $h' : \mu_0^{n+1} \to \mu_0^{n+1}$. Then $f \cup h' : A \cup \mu_0^{n+1} \to B \cup \mu_0^{n+1}$ is a homeomorphism between two Z-sets. Again, by the Z-set unknotting theorem, $f \cup h'$ extends to a homeomorphism $h : (\mu^{n+1}, \mu_0^{n+1}) \to (\mu^{n+1}, \mu_0^{n+1})$. □

Proposition 2.1.2 *Let (M, M_0) be a μ^{n+1}-manifold pair and (X, X_0) a closed pair in (M, M_0) such that X is a Z-set in M. Then for any open cover $\mathcal{U}$ of M there exists a homeomorphism $h : M \to M$ such that h and id_M are $\mathcal{U}$-close, $h|_{X_0} = \mathrm{id}_{X_0}$ and $h(X) \cap M_0 = X_0$.*

Proof. By [AHW, Lemma 3], $M \setminus X_0$ has an open cover $\mathcal{U}'$ such that if a homeomorphism $h : M \setminus X_0 \to M \setminus X_0$ is $\mathcal{U}'$-close to $\mathrm{id}_{M\setminus X_0}$ then h can be extended to a homeomorphism $\bar{h} : M \to M$ by $\bar{h}|_{X_0} = \mathrm{id}_{X_0}$. Let $\mathcal{V}$ be a common open refinement of $\mathcal{U} \cap (M \setminus X_0)$ and $\mathcal{U}'$. By the Z-set unknotting theorem, $\mathcal{V}$ has an open refinement $\mathcal{W}$ such that if $g : Z_1 \to Z_2$ is a homeomorphism between two Z-sets in $M \setminus X_0$ and g is $\mathcal{W}$-close to id_{Z_1}, then g extends to a homeomorphism $\bar{g} : M \setminus X_0 \to M \setminus X_0$ which is $\mathcal{V}$-close to $\mathrm{id}_{M\setminus X_0}$. Since $M_0 \setminus X_0$ is a Z-set in $M \setminus X_0$, it follows from [Be, 2.3.6, 2.3.8] that there exists a Z-embedding $f_1 : X \setminus X_0 \to M \setminus X_0$ which is $\mathcal{W}$-close to $\mathrm{id}_{X\setminus X_0}$ and $f_1(X \setminus X_0) \cap (M_0 \setminus X_0) = \emptyset$. Then the homeomorphism $f_1 : X \setminus X_0 \to f_1(X \setminus X_0)$ extends to a homeomorphism $f_2 : M \setminus X_0 \to M \setminus X_0$ which is $\mathcal{V}$-close to $\mathrm{id}_{M\setminus X_0}$. By the property of $\mathcal{U}'$, there exists a homeomorphism $\bar{f}_2 : M \to M$ such that $\bar{f}_2|_{M\setminus X_0} = f_2$ and $\bar{f}_2|_{X_0} = \mathrm{id}_{X_0}$. Since $\bar{f}_2(X \setminus X_0) \cap (M_0 \setminus X_0) = \emptyset$, we have $\bar{f}_2(X) \cap M_0 = X_0$. Then $\bar{f}_2$ is the desired homeomorphism. □

By Proposition 2.1.2 and the Z-set approximation theorem, we have the following:

Corollary 2.1.3 *Every compact pair (X, X_0) of $\dim \leq n+1$ can be embedded in (μ^{n+1}, μ_0^{n+1}) as a Z-pair.* □

Proposition 2.1.4 *Let (X, X_0) be a Z-pair in (μ^{n+1}, μ_0^{n+1}). Then for every open neighborhood (U, U_0) of (X, X_0) with $U_0 = U \cap \mu_0^{n+1}$, there exists a compact μ^{n+1}-manifold pair (M, M_0) such that $X \subset M \subset U$, $X_0 \subset M_0 \subset U_0$ and $M_0 = M \cap \mu_0^{n+1}$.*

Proof. Since $\dim X \leq n+1$, it can be assumed that $(X, X_0) \subset (I^{2(n+1)+2}, I^{2(n+1)+1})$ and $X_0 = X \cap I^{2(n+1)+1}$. By Lemma 2.1.1, we can assume that (μ^{n+1}, μ_0^{n+1}) is obtained from $(I^{2(n+1)+2}, I^{2(n+1)+1})$ by the construction of [Be, Chap.2] so that $(X, X_0) \subset (\mu^{n+1}, \mu_0^{n+1})$. Take an open pair (V, V_0) in $(I^{2(n+1)+2}, I^{2(n+1)+1})$ such that $U = \mu^{n+1} \cap V$ and $U_0 = \mu_0^{n+1} \cap V_0$. Then there exists a regular neighborhood (R, R_0) of (X, X_0) in $(I^{2(n+1)+2}, I^{2(n+1)+1})$ such that $R \subset V$ and $R_0 = R \cap V_0$. Consequently, the desired (M, M_0) is obtained from (R, R_0) by the construction of [Be, Chap.2]. □

Proposition 2.1.5 *Let (X, X_0) be a compact pair with* dim $\leq n+1$ *and (M, M_0) a compact μ^{n+1}-manifold pair. Let (Y, Y_0) be a Z-pair in (M, M_0) and $f : (X, X_0) \to (M, M_0)$ a map. Then for any $\varepsilon > 0$ there exists a Z-embedding $h : (X, X_0) \to (M \setminus Y, M_0 \setminus Y_0)$ with $d(f, h) < \varepsilon$.*

Proof. By the Z-set unknotting theorem, there exists $\delta > 0$ ($\delta < \varepsilon/2$) such that if $\alpha : Z_1 \to Z_2$ is a homeomorphism between Z-sets in M with $d(\alpha, \mathrm{id}_{Z_1}) < \delta$, then α extends to a homeomorphism $\bar{\alpha} : M \to M$ with $d(\bar{\alpha}, \mathrm{id}_M) < \varepsilon/2$. Since (Y, Y_0) is a Z-pair in (M, M_0), there exist Z-embeddings $g_0 : X_0 \to M_0 \setminus Y_0 \subset M \setminus Y$ and $g : X \to M \setminus Y$ such that $d(g_0, f|_{X_0}) < \delta/2$ and $d(g, f) < \delta/2$ $(< \varepsilon/2)$, hence $d(g_0 g^{-1}|_{g(X_0)}, \mathrm{id}_{g(X_0)}) = d(g_0, g|_{X_0}) \leq d(g_0, f|_{X_0}) + d(f|_{X_0}, g|_{X_0}) < \delta$. Then the homeomorphism $\mathrm{id}_Y \cup g_0 g^{-1}|_{g(X_0)} : Y \cup g(X_0) \to Y \cup g_0(X_0)$ between Z-sets in M extends to a homeomorphism $h' : M \to M$ with $d(h', \mathrm{id}_M) < \varepsilon/2$. Observe that $h'g(X) \subset M \setminus Y$ and $h'g(X_0) = g_0(X_0) \subset M_0 \setminus Y_0$. By Proposition 2.1.2, there exists a homeomorphism $h'' : M \to M$ such that $d(h'', \mathrm{id}_M) < \varepsilon/4$, $h''|_{h'g(X_0)} = \mathrm{id}_{h'g(X_0)}$ and $h''(h'g(X)) \cap M_0 =$

$h''(h'g(X_0))$. We obtain the desired Z-embedding $h = h''h'g : (X, X_0) \to (M, M_0)$ because

$$d(h, f) \leq d(h''h'g, h'g) + d(h'g, g) + d(g, f) < \varepsilon/4 + \varepsilon/2 + \varepsilon/4 = \varepsilon. \quad \square$$

Proposition 2.1.6 *Let (Z, Z_0) be a compact pair with* $\dim \leq n$ *and (M, M_0) a compact μ^{n+1}-manifold pair. Let (Y, Y_0) be a Z-pair in (M, M_0) and $H : (Z, Z_0) \times I \to (M, M_0)$ a homotopy such that $H_0(Z) \cap Y = \emptyset$ and $H_1(Z) \cap Y = \emptyset$. Then for each $\varepsilon > 0$ there exists a homotopy $\tilde{H} : (Z, Z_0) \times I \to (M \setminus Y, M_0 \setminus Y_0)$ such that $H_0 = \tilde{H}_0$, $H_1 = \tilde{H}_1$ and $d(H, \tilde{H}) < \varepsilon$.*

Proof. Since $(M \setminus Y, M_0 \setminus Y_0)$ is LC^n-pair, by Lemma 1.2.2, there exists an open covering $\mathcal{V}$ of $M \setminus Y$ such that two $\mathcal{V}$-close maps from (Z, Z_0) to $(M \setminus Y, M_0 \setminus Y_0)$ are $\varepsilon/3$-homotopic. Since $H_0(Z) \cup H_1(Z)$ is compact in $M \setminus Y$, there exists a Lebesgue number $\delta > 0$ ($\delta < \varepsilon/3$) such that $d(x, y) < \delta$, where $x \in H_0(Z) \cup H_1(Z)$ and $y \in M \setminus Y$, implies that $x, y \in V$ for some $V \in \mathcal{V}$. By Proposition 2.1.5, there exists a Z-embedding $H' : (Z, Z_0) \times I \to (M \setminus Y, M_0 \setminus Y_0)$ which is δ-close to H. Then there exist $\varepsilon/3$-homotopies $F^{(i)} : (Z, Z_0) \times I \to (M \setminus Y, M_0 \setminus Y_0)$ ($i = 0, 1$) such that $F^{(0)}_0 = H_0$, $F^{(0)}_1 = H'_0$, $F^{(1)}_0 = H'_1$ and $F^{(1)}_1 = H_1$. Since Z is compact, there exists $t_0 > 0$ such that $\mathrm{diam}H(\{x\} \times [0, t_0]) < \varepsilon/3$, $\mathrm{diam}H(\{x\} \times [1 - t_0, 1]) < \varepsilon/3$, $\mathrm{diam}H'(\{x\} \times [0, t_0]) < \varepsilon/3$ and $\mathrm{diam}H'(\{x\} \times [1 - t_0, 1]) < \varepsilon/3$, for each $x \in Z$. We define the desired homotopy $\tilde{H}$ as follows:

$$\tilde{H}(x, t) = \begin{cases} F^{(0)}(x, \frac{2}{t_0}t) & (0 \leq t \leq \frac{t_0}{2}), \\ H'(x, 2t - t_0) & (\frac{t_0}{2} \leq t \leq t_0), \\ H'(x, t) & (t_0 \leq t \leq 1 - t_0), \\ H'(x, 2t + t_0 - 1) & (1 - t_0 \leq t \leq 1 - \frac{t_0}{2}), \\ F^{(1)}(x, \frac{2}{t_0}(t - 1) + 1) & (1 - \frac{t_0}{2} \leq t \leq 1). \end{cases}$$

To see that $d(H, \tilde{H}) < \varepsilon$, let $(x, t) \in Z \times I$. If $0 \leq t \leq t_0/2$, then

$$\begin{aligned} d(\tilde{H}(x, t), H(x, t)) &\leq d(F^{(0)}(x, \frac{2}{t_0}t), F^{(0)}(x, 0)) + d(H(x, 0), H(x, t)) \\ &< \varepsilon/3 + \varepsilon/3 < \varepsilon. \end{aligned}$$

If $t_0/2 \leq t \leq t_0$, then

$$\begin{aligned} d(\tilde{H}(x,t), H(x,t)) &\leq d(H'(x, 2t-t_0), H'(x,0)) \\ &\quad + d(F^{(0)}(x,1), F^{(0)}(x,0)) + d(H(x,0), H(x,t)) \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

If $1 - t_0 \leq t \leq 1$, we conclude that $d(\tilde{H}(x,t), H(x,t)) < \varepsilon$ by the same arguments. If $t_0 \leq t \leq 1 - t_0$, then

$$d(\tilde{H}(x,t), H(x,t)) = d(H'(x,t), H(x,t)) < \delta < \varepsilon/3 < \varepsilon. \quad \square$$

Corollary 2.1.7 *Let (X, X_0) be a compact pair and (M, M_0) a compact μ^{n+1}-manifold pair. Let (Y, Y_0) be a Z-pair in (M, M_0) and $f, g : (X, X_0) \to (M \setminus Y, M_0 \setminus Y_0)$ maps. Then $f \simeq^n g$ in (M, M_0) implies that $f \simeq^n g$ in $(M \setminus Y, M_0 \setminus Y_0)$.* $\square$

2.2 The Z-pair approximation theorem

In this section, we extend Theorem 1.1.3 to the Z-pair approximation theorem.

Theorem 2.2.1 *Let (M, M_0) be a μ^{n+1}-manifold pair, (X, X_0) a locally compact pair with* $\dim \leq n+1$, *(A, A_0) a closed pair in (X, X_0) with $A \cap X_0 = A_0$ and let f $(X, X_0) \to (M, M_0)$ be a proper map such that $f|_{(A,A_0)}$ is a Z-embedding. Then for any open cover $\mathcal{U}$ of M, f is $\mathcal{U}$-close to a Z-embedding g $(X, X_0) \to (M, M_0)$ such that $f|_{(A,A_0)} = g|_{(A,A_0)}$ and for any proper map α $(Z, Z_0) \to (X, X_0)$ of a pair with* $\dim \leq n$, *there is a proper $\mathcal{U}$-homotopy $\varphi : (Z, Z_0) \times I \to (M, M_0)$ such that $\varphi_0 = f\alpha$, $\varphi_1 = g\alpha$ and $\varphi_t|_{(\alpha^{-1}(A), \alpha^{-1}(A_0))} = f\alpha|_{(\alpha^{-1}(A), \alpha^{-1}(A))}$ for all $t \in I$, namely, $f\alpha \simeq_{\mathcal{U}} g\alpha$ rel. $(\alpha^{-1}(A), \alpha^{-1}(A_0))$. In particular, f is relatively properly n-homotopic to g.*

By Lemma 1.2.5, we can easily prove the absolute case of Theorem 2.2.1, which strengthens the Z-set approximation theorem.

Theorem 2.2.2 *Let M be a μ^{n+1}-manifold, X a locally compact space with* $\dim \leq n+1$*, A a closed set in X and let $f : X \to M$ be a proper map such that $f|_A$ is a Z-embedding. Then for any open cover $\mathcal{U}$ of M, f is $\mathcal{U}$-close to a Z-embedding $g : X \to M$ such that $f|_A = g|_A$ and if $\alpha : Z \to X$ is a proper map and* $\dim Z \leq n$ *then there is a proper $\mathcal{U}$-homotopy $\varphi : Z \times I \to M$ with $\varphi_0 = f\alpha$, $\varphi_1 = g\alpha$ and $\varphi_t|_{\alpha^{-1}(A)} = f\alpha|_{\alpha^{-1}(A)}$ for all $t \in I$, hence $f\alpha \simeq_{\mathcal{U}} g\alpha$ rel. $\alpha^{-1}(A)$ and $f\alpha \simeq_p g\alpha$ rel. $\alpha^{-1}(A)$.*

Proof. Replacing $\mathcal{U}$ by a refinement, we may assume that, if one of two $\mathcal{U}$-close maps from any locally compact space to M is proper, then the other one is also proper. By Lemma 1.2.5, $\mathcal{U}$ has an open refinement of $\mathcal{V}$ satisfying $(\sharp)_n$. By the Z-set approximation theorem, f is $\mathcal{V}$-close to a Z-embedding $g : X \to M$ such that $f|_A = g|_A$. Applying $(\sharp)_n$ to f, g and the constant homotopy $f\alpha\mathrm{pr}_{\alpha^{-1}(A)} : \alpha^{-1}(A) \times I \to M$, we can obtain the desired homotopy φ. $\square$

Now, we can prove Theorem 2.2.1.

Proof of Theorem 2.2.1. As in the proof of Theorem 2.2.2, we may assume that if one of two $\mathcal{U}$-close maps of a locally compact space to M is proper then the other is also proper. By Lemma 1.2.5, $\mathcal{U}$ has an open refinement $\mathcal{V}$ satisfying $(\sharp)_n$. Let $\mathcal{U}_1$ be an open star-refinement of $\mathcal{V}$. By Proposition 1.2.4, $\mathcal{U}_1$ has an open refinement $\mathcal{V}_1$ satisfying $(*)_n$. By Theorem 2.2.2, $f|_{X_0}$ is $\mathcal{V}_1$-close to a Z-embedding $g_0 : X_0 \to M_0$ such that $f|_{A_0} = g_0|_{A_0}$ and there is a $\mathcal{V}_1$-homotopy $\varphi : Z_0 \times I \to M_0$ such that $\varphi_0 = f\alpha|_{Z_0}$, $\varphi_1 = g_0\alpha|_{Z_0}$ and

$$\varphi_t|_{\alpha^{-1}(A_0)} = f\alpha|_{\alpha^{-1}(A_0)} \quad \text{for all } t \in I.$$

Then $g_0 \cup f|_A : X_0 \cup A \to M$ is a Z-embedding because $f(A_0) \subset g_0(X_0) \cap f(A) \subset f(A) \cap M_0 = f(A_0)$. Since $g_0 \cup f|_A$ is $\mathcal{V}_1$-close to $f|_{X_0 \cup A}$, $g_0 \cup f|_A$ extends to a map $f_1 : X \to M$ which is $\mathcal{U}_1$-close to f. By the Z-set approximation theorem, f_1 is $\mathcal{U}_1$-close to a Z-embedding $g_1 : X \to M$ such that $g_1|_{X_0 \cup A} = f_1|_{X_0 \cup A} = g_0 \cup f|_A$. It should be remarked that $g_1 : (X, X_0) \to (M, M_0)$ is an extension of f and both $g_1 : X \to M$ and $g_1|_{X_0} = g_0 : X_0 \to M_0$ are Z-embeddings but g_1 is not the desired one, because it can happen that $g_1(X) \cap M_0 \neq g_1(X_0)$.

By [AHW, Lemma 3], we have an open cover $\mathcal{W}$ of $M \setminus g_1(A)$ $(= M \setminus f(A))$ such that, if $h : M \setminus g_1(A) \to M \setminus g_1(A)$ is a homeomorphism $\mathcal{W}$-close to $\mathrm{id}_{M \setminus g_1(A)}$, then $h \cup \mathrm{id}_{g_1(A)} : M \to M$ is a homeomorphism. We may assume that $\mathcal{W}$ refines $\mathcal{U}_1$. By Proposition 2.1.2, we have a homeomorphism $h : M \setminus g_1(A) \to M \setminus g_1(A)$ $\mathcal{W}$-close to $\mathrm{id}_{M \setminus g_1(A)}$ such that $h|_{g_1(X_0)} = \mathrm{id}_{g_1(X_0)}$ and $h(g_1(X)) \cap M_0 = g_1(X_0)$. Let $g = \bar{h} g_1 : (X, X_0) \to (M, M_0)$, where $\bar{h} = h \cup \mathrm{id}_{g_1(A)} : M \to M$. Then g is the desired Z-embedding. In fact, g is $\mathcal{U}_1$-close to g_1. Recall that g_1 is $\mathcal{U}_1$-close to f_1 and f_1 is $\mathcal{U}_1$-close to f. Consequently, g and f are $\mathrm{st}\,\mathcal{U}_1$-close, hence they are $\mathcal{V}$-close. Since $\bar{h}|_{g(X_0 \cup A)} = \mathrm{id}$, we have $g|_{X_0 \cup A} = \bar{h} g_1|_{X_0 \cup A} = g_1|_{X_0 \cup A} = g_0 \cup f|_A$. On the other hand, the homotopy φ can be extended to a homotopy $\bar{\varphi} : (Z_0 \cup \alpha^{-1}(A)) \times I \to M$ by $\bar{\varphi}_t|_{\alpha^{-1}(A)} = f\alpha|_{\alpha^{-1}(A)}$ for all $t \in I$. Then $\bar{\varphi}$ is a $\mathcal{V}$-homotopy such that

$$\bar{\varphi}_0 = f\alpha|_{Z_0 \cup \alpha^{-1}(A)} \text{and} \bar{\varphi}_1 = (g_0 \cup f|_A)\alpha|_{Z_0 \cup \alpha^{-1}(A)} = g\alpha|_{Z_0 \cup \alpha^{-1}(A)}.$$

By using $(\sharp)_n$, we can extend $\bar{\varphi}$ to the desired homotopy $\tilde{\varphi}$. □

2.3 The Z-pair unknotting theorem

In this section, we prove the Z-pair unknotting theorem as an extension of Theorem 1.1.2.

Theorem 2.3.1 *Let (M, M_0) be a μ^{n+1}-manifold pair and $f : (A, A_0) \to (M, M_0)$ a Z-embedding of a Z-pair (A, A_0) in (M, M_0). If f is relatively properly n-homotopic to $\mathrm{id}_{(A,A_0)}$ in (M, M_0), then f extends to a homeomorphism $\tilde{f} : (M, M_0) \to (M, M_0)$ which is relatively properly n-homotopic to $\mathrm{id}_{(M,M_0)}$.*

Proof. Step 1. In this step, it is not asserted that $\tilde{f}$ is properly n-homotopic $(\simeq_p^n)$ to $\mathrm{id}_{(M,M_0)}$. Let $\alpha : (Z, Z_0) \to (M, M_0)$ be an n-invertible proper map of a pair with dim $\leq n$. By Theorem 1.2.6, f extends to a proper map $\bar{f} : (M, M_0) \to (M, M_0)$ which is properly n-homotopic to $\mathrm{id}_{(M,M_0)}$. By Theorem 2.2.2, there exist Z-embeddings $g_0, g_1 : M_0 \to M_0$ such that $g_0|_{A_0} = \mathrm{id}_{A_0}$, $g_1|_{A_0} = f$, $g_0\alpha|_{Z_0} \simeq_p \alpha|_{Z_0}$ rel. $\alpha^{-1}(A_0)$ and $g_1\alpha|_{Z_0} \simeq_p \bar{f}\alpha|_{Z_0}$ rel. $\alpha^{-1}(A_0)$. Since $g_0 \simeq_p^n \mathrm{id}_{M_0}$ in M_0 and $g_1 \simeq_p^n \bar{f}|_{M_0} \simeq_p^n \mathrm{id}_{M_0}$ in M_0, by using the Z-set unknotting theorem, we have a homeomorphism $h : M_0 \to M_0$ such that $hg_0 = g_1$. Then $h|_{A_0} = hg_0|_{A_0} = g_1|_{A_0} = f|_{A_0}$ and $h\alpha|_{Z_0} \simeq_p hg_0\alpha|_{Z_0} = g_1\alpha|_{Z_0}$ rel. $\alpha^{-1}(A_0)$. Since $h(M_0) \cap f(A) = f(A) \cap M_0 = f(A_0)$, $h \cup f : M_0 \cup A \to M$ is a Z-embedding. Observe that

$$(h \cup f)\alpha|_{Z_0 \cup \alpha^{-1}(A)} \simeq_p (g_1 \cup f)\alpha|_{Z_0 \cup \alpha^{-1}(A)} \simeq_p \bar{f}\alpha|_{Z_0 \cup \alpha^{-1}(A)} \quad \mathrm{rel.}\alpha^{-1}(A),$$

hence, $h \cup f \simeq_p^n \mathrm{id}_{M_0 \cup A}$. Using again the Z-set unknotting theorem, we have a homeomorphism $\tilde{f} : M \to M$ such that $\tilde{f}|_{M_0 \cup A} = h \cup f$, i.e., $\tilde{f}|_{M_0} = h$ and $\tilde{f}|_A = f$. Then $\tilde{f} : (M, M_0) \to (M, M_0)$ is the desired homeomorphism.

Step 2. By Theorem 1.2.6, f extends to a map $\bar{f} : (M, M_0) \to (M, M_0)$ which is properly n-homotopic to $\mathrm{id}_{(M,M_0)}$. By Theorem 2.2.1, we have Z-embeddings $f_0, f_1 : (M, M_0) \to (M, M_0)$ such that

$$f_0|_{(A,A_0)} = \mathrm{id}_{(A,A_0)},\ f_0 \simeq_p^n \mathrm{id}_{(M,M_0)},\ f_1|_{(A,A_0)} = f \text{ and } f_1 \simeq_p^n \bar{f} \simeq_p^n \mathrm{id}_{(M,M_0)}.$$

Applying Step 1 to the Z-embedding $f_1 f_0^{-1} : (f_0(M), f_0(M_0)) \to (M, M_1)$, we have a homeomorphism $h : (M, M_0) \to (M, M_0)$ such that $hf_0 = f_1$. Then h is the desired homeomorphism. In fact, $h|_{(A,A_0)} = hg_0|_{(A,A_0)} = g_1|_{(A,A_0)} = f$ and

$$h \simeq_p^n hf_0 = f_1 \simeq_p^n \mathrm{id}_{(M,M_0)} \quad \text{in } (M, M_0). \ \Box$$

Remark. In our approach as above, we can't give a covering estimate to Theorem 2.3.1. The problem remains to establish the Z-pair unknotting theorem with a covering estimate like in Theorem 2.2.1.

Chapter 3

The complement theorem in n-shape theory for compact pairs

The concept of shape was introduced by Borsuk [Bo_1]. In many studies, shape theory has been developed as a big branch of geometric topology (cf. [Bo_3], [MS_2]). Using Q-manifold theory, Chapman [Cha_1] established the so-called complement theorem, that is,

Theorem A *If X and Y are Z-sets in Q, then X and Y are shape equivalent if and only if $Q \setminus X$ and $Q \setminus Y$ are homeomorphic.* □

At the same time, he introduced the notion of weak proper homotopy and proved the following theorem:

Theorem B *Let $\mathcal{S}$ denote the category of Z-sets in Q and shape morphisms, and let $\mathcal{P}$ denote the category of complements of Z-sets in Q and weak proper homotopy classes of proper maps. Then there exists a category isomorphism T from $\mathcal{S}$ onto $\mathcal{P}$ such that $T(X) = Q \setminus X$ for each Z-set X in Q.* □

On the other hand, μ^{n+1}-manifolds were introduced and investigated by Bestvina [Be]. From many studies, it has become clear that μ^{n+1}-manifolds are "$(n+1)$-dimensional" analogues of Q-manifolds. In [Chi_1], Chigogidze introduced the concept of n-shape for compacta and proved the n-shape version of Theorem by using μ^{n+1} manifold theory.

The concept of shape was extended to compact pairs by Borsuk [Bo_3] (cf. [Bo_4]) and by Mardešić and Segal [MS_1] (cf. [MS_2]), but their definitions do not coincide (see [Ma]). The shape of pairs in the sense of Mardešić-Segal is defined by using inverse systems, but it can be also defined using the Borsuk approach (cf. [KO]). In [Fe], Felt tried to generalize Chapman's complement theorem to compact pairs using the definition of shape in the sense of Mardešić-Segal. However, his proof contained a gap, which was recovered by [Sa].

In this chapter, we introduce the notion of weak proper n-homotopy and prove the n-shape version of Theorem B. Moreover, we generalize Chigogidze's complement theorem to compact pairs.

Let μ_0^{n+1} be a Z-set in μ^{n+1} which is homeomorphic to μ^{n+1}. We call (A, A_0) a *Z-pair* in (X, X_0) if A and A_0 are Z-sets in X and X_0, respectively, and $A_0 = A \cap X_0$. For a pair (X, X_0) of spaces, X_0 is assumed to be a *closed* set in X. A pair (M, M_0) is said to be a *μ^{n+1}-manifold pair* if M and M_0 are μ^{n+1}-manifolds and M_0 is a Z-set in M. An embedding $f : (X, X_0) \to (M, M_0)$ is a *Z-embedding* if $(f(X), f(X_0))$ is a Z-pair in (M, M_0).

The following is our main theorem in this chapter.

Theorem C *There exists a categorical isomorphism Ψ from the n-shape category $\mathcal{S}^n_{\mathrm{pair}}$ of Z-pairs in (μ^{n+1}, μ_0^{n+1}) to the weak proper n-homotopy category $\mathcal{P}^n_{\mathrm{pair}}$ of complements of Z-pairs such that $\Psi(X, X_0) = (\mu^{n+1} \setminus X, \mu_0^{n+1} \setminus X_0)$, for each Z-pair (X, X_0) in (μ^{n+1}, μ_0^{n+1}).*

Theorem D *For any two Z-pairs (X, X_0) and (Y, Y_0) in (μ^{n+1}, μ_0^{n+1}), n-*

$\mathrm{Sh}(X, X_0) = n\text{-}\mathrm{Sh}(Y, Y_0)$ *if and only if*

$$(\mu^{n+1} \setminus X, \mu_0^{n+1} \setminus X_0) \approx (\mu^{n+1} \setminus Y, \mu_0^{n+1} \setminus Y_0).$$

3.1 The n-shape of compact pairs and weak proper n-homotopy

An inverse sequence of LC^n-pairs $(\mathbf{X}, \mathbf{X_0}) = \{(X_i, X_{0i}), p_i^{i+1}\}$ is called an LC^n-*sequence.* In case $\dim X_i \leq n+1$ for each i, $(\mathbf{X}, \mathbf{X_0})$ is called an $LC^n(n+1)$-*sequence.* We say that $(\mathbf{X}, \mathbf{X_0})$ is *associated* with (X, X_0) if $(X, X_0) = \lim_{\leftarrow}(\mathbf{X}, \mathbf{X_0})$. The following is an easy consequence of Proposition 2.1.4.

Proposition 3.1.1 *Let (X, X_0) be a Z-pair in (μ^{n+1}, μ_0^{n+1}). Then there exists an $LC^n(n+1)$-sequence $(\mathbf{X}, \mathbf{X_0}) = \{(X_i, X_{0i}), p_i^{i+1}\}$ associated with (X, X_0) such that each (X_i, X_{0i}) is a compact μ^{n+1}-manifold pair which is a neighborhood of (X, X_0) with $X_{i+1} \subset \mathrm{int} X_i$, $X_{0i} = X_i \cap \mu_0^{n+1}$ and each $p_i^{i+1} : (X_{i+1}, X_{0i+1}) \to (X_i, X_{0i})$ is the inclusion.* □

For every compact pair (X, X_0) with $\dim \leq n+1$, Corollary 2.1.3 and Proposition 3.1.1 guarantee the existence of an $LC^n(n+1)$-sequence $(\mathbf{X}, \mathbf{X_0}) = \{(X_i, X_{0i}), p_i^{i+1}\}$ associated with (X, X_0).

Let $(\mathbf{X}, \mathbf{X_0}) = \{(X_i, X_{0i}), p_i^{i+1}\}$ and $(\mathbf{Y}, \mathbf{Y_0}) = \{(Y_i, Y_{0i}), q_i^{i+1}\}$ be LC^n-sequences. An *n-morphism* $\mathbf{f} = (f, \{f_i\}) : (\mathbf{X}, \mathbf{X_0}) \to (\mathbf{Y}, \mathbf{Y_0})$ consists of an increasing function $f : \omega \to \omega$ and a collection $\{f_i\}$ of maps $f_i : (X_{f(i)}, X_{0f(i)}) \to (Y_i, Y_{0i})$ such that $f_i p_{f(i)}^{f(j)} \simeq^n q_i^j f_j$ in (Y_i, Y_{0i}), for each $i, j \in \omega$ with $j \geq i$. The identity map $\mathbf{id}_{(\mathbf{X},\mathbf{X_0})} : (\mathbf{X}, \mathbf{X_0}) \to (\mathbf{X}, \mathbf{X_0})$ is defined by $\mathrm{id}_\omega : \omega \to \omega$ and $\mathrm{id}_{(X_i, X_{0i})} : (X_i, X_{0i}) \to (X_i, X_{0i})$. The composition of two n-morphisms $\mathbf{f} = (f, \{f_i\}) : (\mathbf{X}, \mathbf{X_0}) \to (\mathbf{Y}, \mathbf{Y_0})$ and $\mathbf{g} = (g, \{g_i\}) : (\mathbf{Y}, \mathbf{Y_0}) \longrightarrow (\mathbf{Z}, \mathbf{Z_0}) = \{(Z_i, Z_{0i}), r_i^{i+1}\}$ are defined by $\mathbf{fg} = (fg, \{g_i f_{g(i)}\})$. Observe that $\mathbf{fg}$ forms an n-morphism.

Two n-morphisms $\mathbf{f} = (f, \{f_i\}), \mathbf{g} = (g, \{g_i\}) : (\mathbf{X}, \mathbf{X_0}) \to (\mathbf{Y}, \_fY_0)$ are said to be *n-homotopic* (notation: $\mathbf{f} \simeq^n \mathbf{g}$) if for each $i \in \omega$ there exists $j \geq f(i), g(i)$ such that $f_i p^j_{f(i)} \simeq^n g_i p^j_{g(i)}$ in (Y_i, Y_{0i}). It is clear that the relation $\simeq^n$ of n-morphisms is an equivalence relation. The equivalence class of $\mathbf{f}$ is denoted by $[\mathbf{f}]_n$.

By the same argument as in [MS$_1$], we have the following proposition which corresponds to theorem 12 in [MS$_1$].

Proposition 3.1.2 *Let* $(\mathbf{X}, \mathbf{X_0})$ *and* $(\mathbf{Y}, \mathbf{Y_0})$ *be* $LC^n(n+1)$*-sequences associated with* (X, X_0) *and* (Y, Y_0) *respectively. Then* $(X, X_0) \simeq^n (Y, Y_0)$ *implies* $(\mathbf{X}, \mathbf{X_0}) \simeq^n (\mathbf{Y}, \mathbf{Y_0})$. □

Corollary 3.1.3 *If* $(\mathbf{X}, \mathbf{X_0})$ *and* $(\mathbf{X'}, \mathbf{X'_0})$ *are* $LC^n(n+1)$*-sequences associated with the same compact pair* (X, X_0)*, then* $(\mathbf{X}, \mathbf{X_0}) \simeq^n (\mathbf{X'}, \mathbf{X'_0})$. □

Corollary 3.1.3 implies that every compact pair (X, X_0) of dim $\leq n+1$ determines an n-homotopy type of an $LC^n(n+1)$-sequence $(\mathbf{X}, \mathbf{X_0})$ associated with (X, X_0), which is called the *n-shape* of (X, X_0), and is denoted by n-$\mathrm{Sh}(X, X_0)$.

Let (X, X_0) and (Y, Y_0) be compact pairs with dim $\leq n+1$ and let $(\mathbf{X}, \mathbf{X_0})$ and $(\mathbf{Y}, \mathbf{Y_0})$ be $LC^n(n+1)$-sequences associated with (X, X_0) and (Y, Y_0), respectively. The *n-shape morphism* from (X, X_0) to (Y, Y_0) is represented by the n-homotopy class of n-morphisms from $(\mathbf{X}, \mathbf{X_0})$ to $(\mathbf{Y}, \mathbf{Y_0})$. For more information about n-shape, refer to [Chi$_2$].

Two proper maps $f, g : (X, X_0) \to (Y, Y_0)$ are said to be *weakly properly homotopic* (notation: $f \simeq_{wp} g$) if for any compactum D in Y there exists a homotopy $H : (X, X_0) \times I \to (Y, Y_0)$ connecting f and g and a compactum C in X such that $H(X \setminus C, X_0 \setminus C) \subset (Y \setminus D, Y_0 \setminus D)$. Two proper maps $f, g : (X, X_0) \to (Y, Y_0)$ are *weakly properly n-homotopic* (notation: $f \simeq^n_{wp} g$) if $f\alpha \simeq_{wp} g\alpha$ for any proper map $\alpha : (Z, Z_0) \to (X, X_0)$ of a pair (Z, Z_0) with dim $\leq n$. Observe that $f \simeq^n_{wp} g$ if and only if $f\alpha \simeq_{wp} g\alpha$ for some n-invertible proper map $\alpha : (Z, Z_0) \to (X, X_0)$ of a pair (Z, Z_0) with dim $\leq n$ (see the remark after Theorem 1.1.6). Moreover, we have the following:

Proposition 3.1.4 *Let $f, g : (X, X_0) \to (Y, Y_0)$ be proper maps. Then $f \simeq^n_{wp} g$ if and only if for each compactum D in Y there exists a compactum C in X such that $f|_{(X\setminus C, X_0\setminus C)} \simeq^n g|_{(X\setminus C, X_0\setminus C)}$ in $(Y \setminus D, Y_0 \setminus D)$.*

Proof. Let $\alpha : (Z, Z_0) \to (X, X_0)$ be an n-invertible proper map of a pair (Z, Z_0) with dim$\leq n$. First, assume $f \simeq^n_{wp} g$. Since $f\alpha \simeq_{wp} g\alpha$, for each compactum D in Y, there exists a compactum E in Z such that $f\alpha|_{(Z\setminus E, Z_0\setminus E)} \simeq g\alpha|_{(Z\setminus E, Z_0\setminus E)}$ in $(Y \setminus D, Y_0 \setminus D)$. Then $C = \alpha(E)$ is the desired compactum. Conversely, assume that for each compactum D in Y there is a compactum C in X satisfying the condition. Then $E = \alpha^{-1}(C)$ is a compactum in Z and $f\alpha|_{(Z\setminus E, Z_0\setminus E)} \simeq g\alpha|_{(Z\setminus E, Z_0\setminus E)}$ in $(Y \setminus D, Y_0 \setminus D)$. Thus, $f\alpha \simeq_{wp} g\alpha$, which implies $f \simeq^n_{wp} g$. □

By the definitions, we have $f \simeq^n_p g \Rightarrow f \simeq^n_{wp} g \Rightarrow f \simeq^n g$. Clearly the relation $\simeq^n_{wp}$ is an equivalent relation. The weak proper n-homotopy class of a proper map f is denoted by $\{f\}_n$.

3.2 A categorical isomorphism from $\mathcal{S}^n_{\text{pair}}$ to $\mathcal{P}^n_{\text{pair}}$

Let $\mathcal{S}^n_{\text{pair}}$ be the category of Z-pairs in (μ^{n+1}, μ^{n+1}_0) and n-shape morphisms. Let $\mathcal{P}^n_{\text{pair}}$ be the category of complements of Z-pairs in (μ^{n+1}, μ^{n+1}_0) and weak proper n-homotopy classes of proper maps. In this section, we will construct a categorical isomorphism $\Psi : \mathcal{S}^n_{\text{pair}} \to \mathcal{P}^n_{\text{pair}}$.

Let (X, X_0) and (Y, Y_0) be Z-pairs in (μ^{n+1}, μ^{n+1}_0) and let

$$(\mathbf{X}, \mathbf{X_0}) = \{(X_i, X_{0i}), p_i^{i+1}\} \text{ and } (\mathbf{Y}, \mathbf{Y_0}) = \{(Y_i, Y_{0i}), q_i^{i+1}\}$$

be nested sequences of compact μ^{n+1}-manifold neighborhoods of (X, X_0) and (Y, Y_0) in (μ^{n+1}, μ^{n+1}_0) respectively, which are obtained by Proposition 3.1.1.

An n-morphism $\mathbf{f} : (\mathbf{X}, \mathbf{X_0}) \to (\mathbf{Y}, \mathbf{Y_0})$ is called a *nice n-morphism* if $f_i(X_{f(i)}) \cap Y = \emptyset$, for each $i \in \omega$.

Lemma 3.2.1 *For each n-morphism $\mathbf{g} = (g, \{g_i\}) : (\mathbf{X}, \mathbf{X_0}) \to (\mathbf{Y}, \mathbf{Y_0})$, there exists a nice n-morphism $\mathbf{f} = (f, \{f_i\}) : (\mathbf{X}, \mathbf{X_0}) \to (\mathbf{Y}, \mathbf{Y_0})$ such that each $f_i : (X_{f(i)}, X_{0f(i)}) \to (Y_i, Y_{0i})$ is a Z-embedding, $f = g$ and $f_i \simeq^n g_i$ in (Y_i, Y_{0i}), hence $\mathbf{f} \simeq^n \mathbf{g}$.*

Proof. Recall that (X_i, X_{0i}) and (Y_i, Y_{0i}) are compact μ^{n+1}-manifold pairs. Take $\varepsilon_i > 0$ such that two ε_i-close maps into (Y_i, Y_{0i}) are n-homotopic. Let $i \in \omega$ be fixed. By Proposition 2.1.5, there exists a Z-embedding $f_i : (X_{g(i)}, X_{0g(i)}) \to (Y_i \backslash Y, Y_{0i} \backslash Y_0)$ with $d(f_i, g_i) < \varepsilon_i$. Then $\mathbf{f} = (g, \{f_i\})$ is the desired n-morphism. In fact, since $\mathbf{g}$ is an n-morphism and $g_i \simeq^n f_i$, we conclude that $f_i p_{g(i)}^{g(j)} \simeq^n q_i^j f_j$, for each $j \geq i$. □

For each nice n-morphism $\mathbf{f} : (\mathbf{X}, \mathbf{X_0}) \to (\mathbf{Y}, \mathbf{Y_0})$, we say that a proper map $\tilde{f} : (\mu^{n+1} \setminus X, \mu_0^{n+1} \setminus X_0) \to (\mu^{n+1} \setminus Y, \mu_0^{n+1} \setminus Y_0)$ is *associated* with $\mathbf{f}$ provided

$$\tilde{f}|_{(X_{f(i)} \backslash X, X_{0f(i)} \backslash X_0)} \simeq^n f_i|_{(X_{f(i)} \backslash X, X_{0f(i)} \backslash Y_0)} \text{ in } (Y_i \setminus Y, Y_{0i} \setminus Y_0), \text{ for each } i \in \omega.$$

By Lemma 3.2.1, we have the relative case of Lemma 4.2 in [Chi$_1$].

Lemma 3.2.2 *Let (X, X_0) be a Z-pair in (μ^{n+1}, μ_0^{n+1}). Then there exists a nice n-morphism $\boldsymbol{\varphi}^X = (\mathrm{id}_\omega, \{\varphi_i^X\}) : (\mathbf{X}, \mathbf{X_0}) \to (\mathbf{X}, \mathbf{X_0})$ such that φ_i^X is a Z-embedding, $\varphi_i^X \simeq^n \mathrm{id}_{(X_i, X_{0i})}$ in (X_i, X_{0i}), hence, $\boldsymbol{\varphi}^X \simeq^n \mathbf{id}_{(\mathbf{X}, \mathbf{X_0})}$. Moreover, the identity map $\mathrm{id}_{(\mu^{n+1} \backslash X, \mu_0^{n+1} \backslash X_0)}$ is associated with $\boldsymbol{\varphi}^X$, that is,*

$$\varphi_i^X|_{(X_i \backslash X, X_{0i} \backslash X_0)} \simeq^n \mathrm{id}_{(X_i \backslash X, X_{0i} \backslash X_0)} \textit{ in } (X_i \setminus X, X_{0i} \setminus X_0), \textit{ for each } i \in \omega.$$

Proof. The first statement is a direct consequence of Lemma 3.2.1. To see the additional statement, let $\alpha : (Z, Z_0) \to (X_i, X_{0i})$ be an n-invertible proper map of a pair (Z, Z_0) with dim $\leq n$. Remark that Z is compact. Since

$\varphi_i^X \simeq^n \mathrm{id}_{(X_i,X_{0i})}$ in (X_i, X_{0i}), there exists a homotopy $H : (Z, Z_0) \times I \to (X_i, X_{0i})$ such that $H_1 = \varphi_i^X \alpha$ and $H_0 = \alpha$. Let $f_1 = \varphi_i^X \alpha$ and $j \geq 2$ fixed. Since Z is compact, there exists a sequence $1 > t_1 > t_2 > \ldots > 0$ such that $H(\{x\} \times [0, t_j]) < 2^{-j}$, for each $x \in Z$. By Lemma 1.2.2, there exists $\delta_j > 0$ ($\delta_j < 2^{-j-2}$) such that any two δ_j-close maps from (Z, Z_0) to (X_i, X_{0i}) are 2^{-j-2}-homotopic in (X_i, X_{0i}). By Proposition 2.1.5, there exists a Z-embedding $f_j : (Z, Z_0) \to (X_i \setminus X, X_{0i} \setminus X_0)$ such that $d(f_j, H_{t_j}) < \delta_j$. Then there exists a 2^{-j-2}-homotopy $G^j : (Z, Z_0) \times I \to (X_i, X_{0i})$ such that $G_0^j = H_{t_j}$ and $G_1^j = f_j$. We can define a homotopy $F^j : (Z, Z_0) \times I \to (X_i, X_{0i})$ connecting f_{j+1} and f_j as follows:

$$F^j(x,t) = \begin{cases} G^{j+1}(x, 1-3t) & (0 \leq t \leq 1/3), \\ H(x, 3(t_j - t_{j+1})t + 2t_{j+1} - t_j) & (1/3 \leq t \leq 2/3), \\ G^j(x, 3t-2) & (2/3 \leq t \leq 1). \end{cases}$$

By Proposition 2.1.6, there exists a homotopy $\tilde{F}^j : (Z, Z_0) \times I \to (X_i \setminus X, X_{0i} \setminus X_0)$ such that $\tilde{F}_0^j = F_0^j$, $\tilde{F}_1^j = F_1^j$ and $d(\tilde{F}^j, F^j) < 2^{-j-4}$. Note that $\tilde{F}^j$ is a 2^{-j+1}-homotopy. In fact,

$$\begin{aligned} \mathrm{diam}\tilde{F}^j(\{x\} \times I) &\leq 2 \cdot 2^{-j-4} + \mathrm{diam} F^j(\{x\} \times I) \\ &\leq 2^{-j-3} + 2 \cdot 2^{-j-3} + 2 \cdot 2^{-j-2} + \mathrm{diam} H(\{x\} \times [t_{j+1}, t_j]) \\ &\leq 2^{-j-3} + 2^{-j-2} + 2^{-j-1} + 2^{-j} \\ &< 2^{-j+1}. \end{aligned}$$

Again by Proposition 2.1.6, there exists $\tilde{F}^1 : (Z, Z_0) \times I \to (X_i \setminus X, X_{0i} \setminus X_0)$ such that $\tilde{F}_0^1 = f_2$ and $\tilde{F}_1^1 = f_1$. By using a linear homeomorphism $s_j : [t_{j+1}, t_j] \to I$, we can define a homotopy $\tilde{H} : (Z, Z_0) \times I \to (X_i, X_{0i})$ such that $\tilde{H}_0 = \alpha$ and $\tilde{H}|_{(Z,Z_0)\times(0,1]} = \bigcup_{j\in\omega} \tilde{F}^j \circ (\mathrm{id}_{(Z,Z_0)} \times s_j)$.

It is clear that $\tilde{H}$ is continuous on $Z \times (0,1]$. To prove the continuity of $\tilde{H}$ at $(x, 0) \in Z \times \{0\}$, let $\varepsilon > 0$. Then there exists $m \in \omega$ such that $2^{-m+2} < \varepsilon/2$. For each $x \in Z$, there exists a neighborhood U of x

in Z such that $d(\alpha(x), \alpha(x')) < \varepsilon/2$ for each $x' \in U$. For each $t \in (0, t_m]$, choose $j \geq m$ so that $t \in [t_{j+1}, t_j]$. Since $f_j \to \alpha$, $d(\tilde{H}(x', t), \alpha(x')) = d(\tilde{F}^j(x', s_j(t)), \alpha(x')) \leq d(\tilde{F}^j(x', s_j(t)), f_{j+1}(x')) + \sum_{i>j} d(f_i(x'), f_{i+1}(x')) \leq 2^{-m+1} + 2^{-m} + \ldots = 2^{-m+2} < \varepsilon/2$. Thus, for each $(x', t) \in U \times (0, t_m]$,

$$d(\tilde{H}(x', t), \tilde{H}(x, 0)) \leq d(\tilde{H}(x', t), \alpha(x')) + d(\alpha(x'), \alpha(x)) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Observe that $\tilde{H}(Z \times (0, 1]) \cap X = \emptyset$, which implies

$$\varphi_i^X \alpha|_{(Z \setminus \alpha^{-1}(X), Z_0 \setminus \alpha^{-1}(X_0))} \simeq \alpha|_{(Z \setminus \alpha^{-1}(X), Z_0 \setminus \alpha^{-1}(X_0))} \text{ in } (X_i \setminus X, X_{0i} \setminus X_0). \quad \square$$

Lemma 3.2.3 *For each nice n-morphism* $\mathbf{f} : (\mathbf{X}, \mathbf{X_0}) \to (\mathbf{Y}, \mathbf{Y_0})$*, there exists a proper map* $\tilde{f} : (\mu^{n+1} \setminus X, \mu_0^{n+1} \setminus X_0) \to (\mu^{n+1} \setminus Y, \mu_0^{n+1} \setminus Y_0)$ *associated with* $\mathbf{f}$.

Proof. Because of similarity and bothersome notation, we prove only the absolute case. Note that $f_i(X_{f(i)}) \subset Y_i \setminus Y$. Since $\mu^{n+1} \setminus Y$ is $C^n \cap LC^n$, by Lemma 1.2.1, the map $f_1 : X_{f(1)} \to Y_1 \setminus Y \subset \mu^{n+1} \setminus Y$ has an extension $h_1 : \mu^{n+1} \to \mu^{n+1} \setminus Y$.

Assume now that we have constructed $h_i : \mu^{n+1} \to \mu^{n+1} \setminus Y$ such that

1_i. $h_i|_{X_{f(i)}} = f_i$,

2_i. $h_i|_{\mu^{n+1} \setminus \operatorname{int} X_{f(i-1)}} = h_{i-1}|_{\mu^{n+1} \setminus \operatorname{int} X_{f(i-1)}}$,

3_i. $h_i|_{X_{f(i-1)}} \simeq^n f_{i-1}|_{X_{f(i-1)}}$ in $Y_{i-1} \setminus Y$.

We will construct $h_{i+1} : \mu^{n+1} \to \mu^{n+1} \setminus Y$. Consider the map $f_{i+1} \cup f_i|_{\operatorname{bd} X_{f(i)}} : X_{f(i+1)} \cup \operatorname{bd} X_{f(i)} \to Y_i \setminus Y$. Since $\mathbf{f}$ is an n-morphism and Y is a Z-set in Y_i, we have $f_i|_{X_{f(i+1)}} \cup f_i|_{\operatorname{bd} X_{f(i)}} \simeq^n f_{i+1} \cup f_i|_{\operatorname{bd} X_{f(i)}}$ in $Y_i \setminus Y$. By Lemma 1.2.6, there exists an extension $F_{i+1} : X_{f(i)} \to Y_i \setminus Y$ such that $F_{i+1}|_{X_{f(i+1)}} = f_{i+1}$, $F_{i+1}|_{\operatorname{bd} X_{f(i)}} = f_i|_{\operatorname{bd} X_{f(i)}}$ and $F_{i+1}|_{X_{f(i)}} \simeq^n f_i$ in $Y_i \setminus Y$. We can define the map $h_{i+1} : \mu^{n+1} \to \mu^{n+1} \setminus Y$ as follows:

$$h_{i+1}(x) = \begin{cases} h_i(x) & \text{for } x \in \mu^{n+1} \setminus \operatorname{int} X_{f(i)}, \\ F_{i+1}(x) & \text{for } x \in X_{f(i)}. \end{cases}$$

Observe that h_{i+1} is well defined and satisfies 1_{i+1}–3_{i+1}. By the induction, we have a sequence of maps $h_i : \mu^{n+1} \to \mu^{n+1} \setminus Y$.

Now, we define a map $\tilde{f} : \mu^{n+1} \setminus X \to \mu^{n+1} \setminus Y$ by $\tilde{f} = \lim_{i\to\infty} h_i$. By 2_i, $\tilde{f}$ is well defined. Indeed, $\tilde{f}|_{\mu^{n+1}\setminus \mathrm{int} X_{f(i)}} = h_i|_{\mu^{n+1}\setminus \mathrm{int} X_{f(i)}}$. For every compactum D in $\mu^{n+1} \setminus Y$, $\mu^{n+1} \setminus D$ is a neighborhood of Y. Then there exists Y_i such that $Y \subset Y_i \subset \mu^{n+1} \setminus D$. Let $C = \mu^{n+1} \setminus \mathrm{int}\, X_{f(i)}$. Since $(\mu^{n+1} \setminus X) \setminus C \subset X_{f(i)}$ and $\tilde{f}(X_{f(i)} \setminus X) \subset Y_i$, it follows that $\tilde{f}((\mu^{n+1} \setminus X) \setminus C) \subset Y_i$. Therefore $\tilde{f}((\mu^{n+1} \setminus X) \setminus C) \cap D = \emptyset$. Thus $\tilde{f}$ is a proper map. Moreover, by composing the n-homotopies of 3_i, it is easy to see that $\tilde{f}|_{X_{f(i)}\setminus X} \simeq^n f_i|_{X_{f(i)}\setminus X}$ in $Y_i \setminus Y$ for each $i \in \omega$. □

Lemma 3.2.4 *Let* $\tilde{f}, \tilde{g} : (\mu^{n+1} \setminus X, \mu_0^{n+1} \setminus X_0) \to (\mu^{n+1} \setminus Y, \mu_0^{n+1} \setminus Y_0)$ *be proper maps associated with nice n-morphisms* $\mathbf{f} = (f, \{f_i\}), \mathbf{g} = (g, \{g_i\}) : (\mathbf{X}, \mathbf{X_0}) \to (\mathbf{Y}, \mathbf{Y_0})$*, respectively. Then* $\mathbf{f} \simeq^n \mathbf{g}$ *if and only if* $\tilde{f} \simeq^n_{wp} \tilde{g}$.

Proof. By the same reason as in Lemma 3.2.3, we prove only the absolute case. First we prove the "only if" part. For each compactum E in $\mu^{n+1} \setminus Y$, there exists an $i \in \omega$ such that $E \cap Y_i = \emptyset$. Since $\mathbf{f} \simeq^n \mathbf{g}$, $\mathbf{f}$ and $\mathbf{g}$ are nice n-morphisms and Y is a Z-set, by Proposition 2.1.6, there exists $j \geq f(i), g(i)$ such that $f_i|_{X_j} \simeq^n g_i|_{X_j}$ in $Y_i \setminus Y$. Since $\tilde{f}$ and $\tilde{g}$ are associated with $\mathbf{f}$ and $\mathbf{g}$ respectively, we have $\tilde{f}|_{X_j\setminus X} \simeq^n f_i|_{X_j\setminus X} \simeq^n g_i|_{X_j\setminus X} \simeq^n \tilde{g}|_{X_j\setminus X}$ in $Y_i \setminus Y$. Note that $D = \mu^{n+1} \setminus \mathrm{int}\, X_j$ is a compactum in $\mu^{n+1} \setminus X$, $(\mu^{n+1} \setminus X) \setminus D \subset X_j \setminus X$ and $Y_i \setminus Y \subset (\mu^{n+1} \setminus Y) \setminus E$. Then it follows that $\tilde{f}|_{(\mu^{n+1}\setminus X)\setminus D} \simeq^n \tilde{g}|_{(\mu^{n+1}\setminus X)\setminus D}$ in $(\mu^{n+1} \setminus Y) \setminus E$. Thus we have $\tilde{f} \simeq^n_{wp} \tilde{g}$.

To prove the "if" part, assume $\tilde{f} \simeq^n_{wp} \tilde{g}$. For each $i \in \omega, E = \mu^{n+1} \setminus \mathrm{int}\, Y_i$ is a compactum in $\mu^{n+1} \setminus Y$. Then we have a compactum D in $\mu^{n+1} \setminus X$ such that $\tilde{f}|_{(\mu^{n+1}\setminus X)\setminus D} \simeq^n \tilde{g}|_{(\mu^{n+1}\setminus X)\setminus D}$ in $(\mu^{n+1} \setminus Y) \setminus E$. Take $j \geq f(i), g(i)$ such that $X_j \subset \mu^{n+1} \setminus D$. Let $\boldsymbol{\varphi}^X$ be a nice n-morphism obtained by Lemma 3.2.2. Then $\varphi_j^X(X_j) \subset X_j \setminus X$ and $\varphi_j^X \simeq^n \mathrm{id}_{X_j}$ in X_j. From the definition, it follows that

$$f_i|_{X_j} \simeq^n f_i\varphi_j^X|_{X_j} \simeq^n \tilde{f}\varphi_j^X|_{X_j} \simeq^n \tilde{g}\varphi_j^X|_{X_j} \simeq^n g_i\varphi_j^X|_{X_j} \simeq^n g_i|_{X_j} \text{ in } Y_i.$$

Consequently, $\mathbf{f} \simeq^n \mathbf{g}$. □

We define the functor $\Psi : \mathcal{S}^n_{\text{pair}} \to \mathcal{P}^n_{\text{pair}}$ by $\Psi(X, X_0) = (\mu^{n+1} \setminus X, \mu_0^{n+1} \setminus Y_0)$ and $\Psi([\mathbf{f}]_n) = \{\tilde{f}\}_n$, where $\tilde{f}$ is a proper map associated with a nice n-morphism which is n-homotopic to $\mathbf{f}$. By Lemmas 3.2.1, 3.2.3 and 3.2.4, Ψ is well-defined.

Now we prove Theorem C, that is,

Theorem 3.2.5 $\Psi : \mathcal{S}^n_{\text{pair}} \to \mathcal{P}^n_{\text{pair}}$ *is a categorical isomorphism.*

Proof. By the same reason as in Lemma 3.2.3, we prove only the absolute case.

Step 1 (Ψ is functorial). First, we prove that $\Psi([\mathbf{id_X}]_n) = \{\mathrm{id}_{\mu^{n+1}\setminus X}\}_n$. Let $\boldsymbol{\varphi}^X$ be a nice n-morphism obtained by Lemma 3.2.2. It suffices to prove that $\tilde{\varphi} \simeq^n_{wp} \mathrm{id}_{\mu^{n+1}\setminus X}$, for a proper map $\tilde{\varphi}$ associated with $\boldsymbol{\varphi}^X$. For each compactum D in $\mu^{n+1} \setminus X$, there exists $i \in \omega$ such that $X_i \cap D = \emptyset$. Since $\tilde{\varphi}$ is associated with $\boldsymbol{\varphi}^X$, $\tilde{\varphi}|_{X_i\setminus X} \simeq^n \varphi_i^X|_{X_i\setminus X}$ in $X_i \setminus X$. By Lemma 3.2.2, $\varphi_i^X|_{X_i\setminus X} \simeq^n \mathrm{id}_{X_i\setminus X}$ in $X_i \setminus X$. Let $E = \mu^{n+1} \setminus \mathrm{int}\, X_i$. Since $(\mu^{n+1} \setminus X) \setminus E \subset X_i \setminus X$ and $X_i \setminus X \subset (\mu^{n+1} \setminus X) \setminus D$, we have $\tilde{\varphi}|_{(\mu^{n+1}\setminus X)\setminus E} \simeq^n \mathrm{id}_{(\mu^{n+1}\setminus X)\setminus E}$ in $(\mu^{n+1} \setminus X) \setminus D$, which implies $\tilde{\varphi} \simeq^n_{wp} \mathrm{id}_{\mu^{n+1}\setminus X}$.

Next, let $\mathbf{f} = (f, \{f_i\}) : \mathbf{X} \to \mathbf{Y}$ and $\mathbf{g} = (g, \{g_i\}) : \mathbf{Y} \to \mathbf{Z} = \{Z_i, r_i^{i+1}\}$ be n-morphisms. We prove that $\Psi([\mathbf{f}]_n[\mathbf{g}]_n) = \Psi([\mathbf{f}]_n)\Psi([\mathbf{g}]_n)$. By Lemma 3.2.1, we can assume that $\mathbf{f}$ and $\mathbf{g}$ are nice n-morphisms. Then $\mathbf{gf} : \mathbf{X} \to \mathbf{Z}$ is also a nice n-morphism. Let $\tilde{f}, \tilde{g}$ and $\tilde{h}$ be proper maps associated with $\mathbf{f}, \mathbf{g}$ and $\mathbf{gf}$ respectively. For every compactum E in $\mu^{n+1} \setminus Z$ there exists $i \in \omega$ such that $Z_i \cap E = \emptyset$. Since Y and Z are Z-sets in μ^{n+1}, $g_i|_{Y_{g(j)}} \simeq^n g_j$ in $Z_i \setminus Z$ for all $j \geq i$, and $f_{g(i)}|_{X_{f(j)}} \simeq^n f_j$ in $Y_{g(i)} \setminus Y$ for all $j \geq g(i)$, by the definition of an n-morphism. It follows that

$$\tilde{h}|_{X_{fg(i)}\setminus X} \simeq^n g_i f_{g(i)}|_{X_{fg(i)}\setminus X} = g_i|_{Y_{g(i)}\setminus Y} f_{g(i)}|_{X_{fg(i)}\setminus X}$$

$$\simeq^n \tilde{g}|_{Y_{g(i)} \setminus Y} \tilde{f}|_{X_{fg(i)} \setminus X} \text{ in } Z_i \setminus Z.$$

Put $C = \mu^{n+1} \setminus \operatorname{int} X_{fg(i)}$. Then $\tilde{h}|_{(\mu^{n+1} \setminus X) \setminus C} \simeq^n \tilde{g}\tilde{f}|_{(\mu^{n+1} \setminus X) \setminus C}$ in $(\mu^{n+1} \setminus Z) \setminus E$, which implies $\tilde{g}\tilde{f} \simeq^n_{wp} \tilde{h}$.

Step 2 (Ψ is an isomorphism). By Lemma 3.2.4, Ψ is a monomorphism, so it suffices to see that Ψ is an epimorphism. To this end, let $\tilde{f} : \mu^{n+1} \setminus X \to \mu^{n+1} \setminus Y$ be a proper map. For any $i \in \omega$, $E = \mu^{n+1} \setminus \operatorname{int} Y_i$ is a compactum in $\mu^{n+1} \setminus Y$. Then $D = \tilde{f}^{-1}(E)$ is a compactum in $\mu^{n+1} \setminus X$. Thus we can take $n_i \in \omega$ such that $X_{n_i} \cap D = \emptyset$, hence $\tilde{f}(X_{n_i} \setminus X) \subset Y_i \setminus Y$. We have an increasing map $f : \omega \to \omega$ such that $f(i) \geq n_i$. Using the nice n-morphism $\boldsymbol{\varphi}^X$ in Lemma 3.2.2, we can define the map $f_i = \tilde{f}\varphi^X_{f(i)} : X_{f(i)} \to Y_i \setminus Y \subset Y_i$. Thus, we obtain the n-morphism $\mathbf{f} = (f, \{f_i\})$. In fact, for each $i, j \in \omega$ with $i \leq j$, $\varphi^X_{f(i)}|_{X_{f(j)}} \simeq^n \varphi^X_{f(j)}$ in $X_{f(i)} \setminus X$, because $\boldsymbol{\varphi}^X$ is an n-morphism and X is a Z-set in $X_{f(i)}$. Then we have $f_i p^{f(j)}_{f(i)} = \tilde{f}\varphi^X_{f(i)}|_{X_{f(j)}} \simeq^n \tilde{f}\varphi^X_{f(j)} = q^j_i f_j$ in Y_i. Thus, $\mathbf{f}$ is an n-morphism. By Lemma 3.2.2, $\tilde{f}|_{X_{f(i)} \setminus X} \simeq^n \tilde{f}\varphi^X_{f(i)}|_{X_{f(i)} \setminus X}$ in $Y_i \setminus Y$, which implies that $\tilde{f}$ is associated with $\mathbf{f}$. Consequently, we have $\Psi([\mathbf{f}]_n) = \{\tilde{f}\}_n$. □

3.3 The complement theorem for Z-pairs

Let (X, X_0) and (Y, Y_0) be Z-pairs in (μ^{n+1}, μ_0^{n+1}). By Proposition 3.1.1, we have $LC^n(n+1)$-sequences $(\mathbf{X}, \mathbf{X_0}) = \{(X_i, X_{0i}), p_i^{i+1}\}$ and $(\mathbf{Y}, \mathbf{Y_0}) = \{(Y_i, Y_{0i}), q_i^{i+1}\}$ associated with (X, X_0) and (Y, Y_0) (i.e., $(X, X_0) = \lim_{\leftarrow}(\mathbf{X}, \mathbf{X_0})$ and $(Y, Y_0) = \lim_{\leftarrow}(\mathbf{Y}, \mathbf{Y_0})$) such that all (X_i, X_{0i}) and (Y_i, Y_{0i}) are compact μ^{n+1}-manifold pairs which are neighborhoods of (X, X_0) and (Y_i, Y_{0i}) in (μ^{n+1}, μ_0^{n+1}) respectively, and all p_i^{i+1} and q_i^{i+1} are inclusion maps. By Proposition 1 in [Sh6], we may assume that all (X_i, X_{0i}) and (Y_i, Y_{0i}) have Z-pairs $(\delta X_i, \delta X_{0,i})$ and $(\delta Y_i, \delta Y_{0,i})$ respectively, such that

$$\mathrm{bd}_{\mu^{n+1}} X_i \subset \delta X_i, \ \mathrm{bd}_{\mu_0^{n+1}} X_{0i} \subset \delta X_{0i},$$

$$\mathrm{bd}_{\mu^{n+1}} Y_i \subset \delta Y_i \text{ and } \mathrm{bd}_{\mu_0^{n+1}} Y_{0i} \subset \delta Y_{0i}.$$

Throughout this section, we use these notations. The next lemma is the relative version of Lemma 4.1 in [Chi$_1$].

Lemma 3.3.1 *Let* $\mathbf{f} : (\mathbf{X}, \mathbf{X_0}) \to (\mathbf{Y}, \mathbf{Y_0})$ *and* $\mathbf{g} : (\mathbf{Y}, \mathbf{Y_0}) \to (\mathbf{X}, \mathbf{X_0})$ *be n-morphisms such that* $\mathbf{gf} \simeq^{\mathrm{n}} \mathbf{id}_{(\mathbf{X},\mathbf{X_0})}$. *Assume that* $h : (\mu^{n+1}, \mu_0^{n+1}) \to (\mu^{n+1}, \mu_0^{n+1})$ *is a homeomorphism and* $i_0 \in \omega$ *such that*

(1) $(\mathrm{int}_{\mu^{n+1}} h(X_{i_0})), \mathrm{int}_{\mu_0^{n+1}} h(X_{0i_0})) \supset (Y, Y_0)$ *and*

(2) $h^{-1}|_{(Y,Y_0)} \simeq^n g_{i_0}|_{(Y,Y_0)}$ *in* (X_{i_0}, X_{0i_0}).

Then there exists a homeomorphism $h' : (\mu^{n+1}, \mu_0^{n+1}) \to (\mu^{n+1}, \mu_0^{n+1})$ *with* $j_0 > i_0$ *such that*

(0)$'$ $h'|_{(\mu^{n+1}\setminus X_{i_0}, \mu_0^{n+1}\setminus X_{0i_0})} = h|_{(\mu^{n+1}\setminus X_{i_0}, \mu_0^{n+1}\setminus X_{0i_0})}$,

(1)$'$ $(\mathrm{int}_{\mu^{n+1}} h(X_{i_0}), \mathrm{int}_{\mu_0^{n+1}} h(X_{0i_0})) \supset (Y_{j_0}, Y_{0j_0}) \supset$

$(\mathrm{int}_{\mu^{n+1}} Y_{j_0}, \mathrm{int}_{\mu_0^{n+1}} Y_{0j_0}) \supset (h'(X), h'(X_0))$, *and*

(2)$'$ $h'|_{(X,X_0)} \simeq^n f_{j_0}|_{(X,X_0)}$ *in* (Y_{j_0}, Y_{0j_0}).

Proof. Let $\alpha : (Z, Z_0) \to (\mu^{n+1}, \mu_0^{n+1})$ be an n-invertible map of a compact pair with dim $\leq n$. By (2), there is a homotopy $\varphi : (\alpha^{-1}(Y), \alpha^{-1}(Y_0)) \times I \to (X_{i_0}, X_{0i_0})$ with $\varphi_0 = h^{-1}\alpha|_{(\alpha^{-1}(Y),\alpha^{-1}(Y_0))}$ and $\varphi_1 = g_{i_0}\alpha|_{(\alpha^{-1}(Y),\alpha^{-1}(Y_0))}$. By Lemma 1.2.1, we can find $i_1 > i_0$ such that

$$(Y_{g(i_1)}, Y_{0g(i_1)}) \subset (\mathrm{int}_{\mu^{n+1}} \mathrm{h}(\mathrm{X}_{\mathrm{i}_0}), \mathrm{int}_{\mu_0^{n+1}} \mathrm{h}(\mathrm{X}_{0\mathrm{i}_0}))$$

and φ extends to a homotopy $\tilde{\varphi} : (\alpha^{-1}(Y_{g(i_1)}), \alpha^{-1}(Y_{0g(i_1)})) \times I \to (X_{i_0}, X_{0i_0})$ with

$$\tilde{\varphi}_0 = h^{-1}\alpha|_{(\alpha^{-1}(Y_{g(i_1)}),\alpha^{-1}(Y_{0g(i_1)}))} \mathrm{and} \tilde{\varphi}_1 = \mathrm{g}_{\mathrm{i}_0}\alpha|_{(\alpha^{-1}(\mathrm{Y}_{\mathrm{g}(\mathrm{i}_1)}),\alpha^{-1}(\mathrm{Y}_{0\mathrm{g}(\mathrm{i}_1)}))}.$$

Hence, $h^{-1}|_{(Y_{g(i_1)},Y_{0g(i_1)})} \simeq^n g_{i_0}|_{(Y_{g(i_1)},Y_{0g(i_1)})}$ in (X_{i_0}, X_{0i_0}). Since $\mathbf{g}$ is an n-morphism, we have $h^{-1}|_{(Y_{g(i_1)},Y_{0g(i_1)})} \simeq^n g_{i_1}$ in (X_{i_0}, X_{0i_0}).

By Theorem 2.2.1, we have a Z-embedding $f' : (X, X_0) \to (Y_{g(i_1)}, Y_{0g(i_1)})$ which is n-homotopic to $f_{g(i_1)}|_{(X,X_0)}$ in $(Y_{g(i_1)}, Y_{0g(i_1)})$. Since $\mathbf{gf} \simeq^{\mathbf{n}} \mathbf{id}_{(\mathbf{X},\mathbf{X}_0)}$, it follows that

$$h^{-1}f' \simeq^n h^{-1}f_{g(i_1)}|_{(X,X_0)} = h^{-1}|_{(Y_{g(i_1)},Y_{0g(i_1)})}f_{g(i_1)}|_{(X,X_0)}$$
$$\simeq^n g_{i_1}f_{g(i_1)}|_{(X,X_0)} \simeq^n \mathrm{id}_{(X,X_0)} \text{ in } (\mathrm{X}_{\mathrm{i}_0}, \mathrm{X}_{0\mathrm{i}_0}),$$

which implies that $f'' = f'h^{-1}|_{(h(X),h(X_0))} \simeq^n \mathrm{id}_{(h(X),h(X_0))}$ in $(h(X_{i_0}), h(X_{0i_0}))$. Observe that

$$f'' \cup \mathrm{id}_{(h(\delta X_{i_0}),h(\delta X_{0i_0}))} : (h(X \cup \delta X_{i_0}), h(X_0 \cup \delta X_{0i_0})) \to (h(X_{i_0}), h(X_{0i_0}))$$

is a Z-embedding which is n-homotopic to id in $(h(X_{i_0}), h(X_{0i_0}))$. By Theorem 2.3.1, this extends to a homeomorphism $h'' : (h(X_{i_0}), h(X_{0i_0})) \to (h(X_{i_0}), h(X_{0i_0}))$. Since $h''|_{(h(\delta X_{i_0}),h(\delta X_{0i_0}))} = \mathrm{id}$, we have the homeomorphism

$$h''' = h'' \cup \mathrm{id}_{(\mu^{n+1}\setminus h(X_{i_0}),\mu_0^{n+1}\setminus h(X_{0i_0}))} : (\mu^{n+1}, \mu_0^{n+1}) \to (\mu^{n+1}, \mu_0^{n+1}).$$

Let $h' = h'''h$ and $j_0 = g(i_1) > i_0$. Observe that $h'|_{(X,X_0)} = h''h|_{(X,X_0)} = f''h|_{(X,X_0)} = f'$. Then it is easy to see that h' and j_0 satisfy the required conditions. □

Proof of Theorem D. Since the "only if" part follows from Theorem C, it remains to prove the "if" part.

We may assume that $(X_1, X_{01}) = (\mu^{n+1}, \mu_0^{n+1})$. Using the same argument of [Chi$_1$], we construct homeomorphisms $h_i : (\mu^{n+1}, \mu_0^{n+1}) \to (\mu^{n+1}, \mu_0^{n+1})$ $(i \in \omega)$ by induction. Since $g_1|_{(Y,Y_0)} \simeq^n \mathrm{id}_{(Y,Y_0)}$ in (μ^{n+1}, μ_0^{n+1}), we apply Lemma 3.3.1 to obtain a homeomorphism $h_1 : (\mu^{n+1}, \mu_0^{n+1}) \to (\mu^{n+1}, \mu_0^{n+1})$ with $j_1 \in \omega$ such that

(1) $(\mathrm{int}_{\mu^{n+1}}\mathrm{Y}_{\mathrm{j}_1}, \mathrm{int}_{\mu_0^{n+1}}\mathrm{Y}_{0\mathrm{j}_1}) \supset (\mathrm{h}_1(\mathrm{X}), \mathrm{h}_1(\mathrm{X}_0))$ and

(2) $h_1|_{(X,X_0)} \simeq^n f_{j_1}|_{(X,X_0)}$ in (Y_1, Y_{01}).

By Lemma 3.3.1, there exists a homeomorphism $h' : (\mu^{n+1}, \mu_0^{n+1}) \to (\mu^{n+1}, \mu_0^{n+1})$ with $j_2 > j_1$ such that

(0)′ $h'|_{(\mu^{n+1}\backslash Y_{j_1},\mu_0^{n+1}\backslash Y_{0j_1})} = h_1^{-1}|_{(\mu^{n+1}\backslash Y_{j_1},\mu_0^{n+1}\backslash Y_{0j_1})}$,

(1)′ $(h_1^{-1}(\mathrm{int}_{\mu^{n+1}} Y_{j_1}), h_1^{-1}(\mathrm{int}_{\mu_0^{n+1}} Y_{0j_1})) \supset (X_{j_2}, X_{0j_2}) \supset$

$(\mathrm{int}_{\mu^{n+1}} X_{j_2}, \mathrm{int}_{\mu_0^{n+1}} X_{0j_2}) \supset (h'(Y), h'(Y_0))$, and

(2)′ " $h'|_{(Y,Y_0)} \simeq^n g_{j_2}|_{(Y,Y_0)}$ in (X_{j_2}, X_{0j_2}).

Let $h_2 = h'^{-1}$. Then

(0) $h_2^{-1}|_{(\mu^{n+1}\backslash Y_{j_1},\mu_0^{n+1}\backslash Y_{0j_1})} = h_1^{-1}|_{(\mu^{n+1}\backslash Y_{j_1},\mu_0^{n+1}\backslash Y_{0j_1})}$,

(1) $(\mathrm{int}_{\mu^{n+1}} Y_{j_1}, \mathrm{int}_{\mu_0^{n+1}} Y_{0j_1}) \supset (h_2(X_{j_2}), h_2(X_{0j_2})) \supset$

$(\mathrm{int}_{\mu^{n+1}} h_2(X_{j_2}), \mathrm{int}_{\mu_0^{n+1}} h_2(X_{0j_2})) \supset (Y, Y_0)$, and

(2) $h_2^{-1}|_{(Y,Y_0)} \simeq^n g_{j_2}|_{(Y,Y_0)}$ in (X_{j_2}, X_{0j_2}).

Again by Lemma 3.3.1, we have a homeomorphism $h_3 : (\mu^{n+1}, \mu_0^{n+1}) \to (\mu^{n+1}, \mu_0^{n+1})$ with $j_3 > j_2$ such that

(0) $h_3|_{(\mu^{n+1}\backslash X_{j_2},\mu_0^{n+1}\backslash X_{0j_2})} = h_2|_{(\mu^{n+1}\backslash X_{j_2},\mu_0^{n+1}\backslash X_{0j_2})}$,

(1) $(h_2(\mathrm{int}_{\mu^{n+1}} X_{j_2}), h_2(\mathrm{int}_{\mu_0^{n+1}} X_{0j_2})) \supset (Y_{j_3}, Y_{0j_3}) \supset$

$(\mathrm{int}_{\mu^{n+1}} Y_{j_3}, \mathrm{int}_{\mu_0^{n+1}} Y_{0j_3}) \supset (h_3(X), h_3(X_0))$, and

(2) $h_3|_{(X,X_0)} \simeq^n f_{j_3}|_{(X,X_0)}$ in (Y_{j_3}, Y_{0j_3}).

Similarly to h_2, we can obtain a homeomorphism $h_4 : (\mu^{n+1}, \mu_0^{n+1}) \to (\mu^{n+1}, \mu_0^{n+1})$ with $j_4 > j_3$ such that

(0) $h_4^{-1}|_{(\mu^{n+1}\backslash Y_{j_3},\mu_0^{n+1}\backslash Y_{0j_3})} = h_3^{-1}|_{(\mu^{n+1}\backslash Y_{j_3},\mu_0^{n+1}\backslash Y_{0j_3})}$,

(1) $(\mathrm{int}_{\mu^{n+1}} Y_{j_3}, \mathrm{int}_{\mu_0^{n+1}} Y_{0j_3}) \supset (h_4(X_{j_4}), h_4(X_{0j_4})) \supset$

$(\mathrm{int}_{\mu^{n+1}} h_4(X_{j_4}), \mathrm{int}_{\mu_0^{n+1}} h_4(X_{0j_4})) \supset (Y, Y_0)$, and

(2) $h_4^{-1}|_{(Y,Y_0)} \simeq^n g_{j_4}|_{(Y,Y_0)}$ in (X_{j_4}, X_{0j_4}).

Thus, we define inductively homeomorphisms $h_i : (\mu^{n+1}, \mu_0^{n+1}) \to (\mu^{n+1}, \mu_0^{n+1})$ with j_i. Similarly to [Chi$_1$], we can define a homeomorphism $h : (\mu^{n+1}, \mu_0^{n+1}) \to (\mu^{n+1}, \mu_0^{n+1})$ by

$$h|_{(\mu^{n+1}\setminus X_{j_{2i}}, \mu_0^{n+1}\setminus X_{0j_{2i}})} = h_{2i}|_{(\mu^{n+1}\setminus X_{j_{2i}}, \mu_0^{n+1}\setminus X_{0j_{2i}})}.$$

Hence the inverse of h is given by

$$h^{-1}|_{(\mu^{n+1}\setminus Y_{j_{2i-1}}, \mu_0^{n+1}\setminus Y_{0j_{2i-1}})} = h_{2i-1}^{-1}|_{(\mu^{n+1}\setminus Y_{j_{2i-1}}, \mu_0^{n+1}\setminus Y_{0j_{2i-1}})}. \quad \square$$

Chapter 4

Proper n-shape and property SUV^n

Property UV^∞ arose in the study of cellularity and is connected with shape theory through the result that a compact metric space X has property UV^∞ if and only if X has the shape of a singleton. Property SUV^∞ was introduced by Hartley [Ha] as a noncompact variant of property UV^∞. Generalizing shape theory to locally compact spaces, Ball and Sher introduced the notion of proper shape [BS]. In [Sh_1], Sher proved that a locally compact metric space X has property SUV^∞ if and only if X has the proper shape of a tree (i.e., a locally finite, connected and simply connected simplicial 1-complex). He also defined property SUV^n as a noncompact variant of property UV^n and proved that, if $X \in SUV^n$ is a closed connected subset in a piecewise linear n-manifold, then $X \in SUV^\infty$.

On the other hand, the concept of n-shape was introduced by Chigogidze [Chi_1]. He discussed the relation of n-shape and property UV^n. In particular, a compact metric space X has property UV^n if and only if X has the n-shape of a singleton. In this chapter, we introduce the notion of proper n-shape and give a characterization of property SUV^n, that is, X *has property* SUV^n *if and* *only if X has the proper n-shape of a tree* (Theorem 4.2.3). As a corollary, the above result of Sher can be strengthened as follows: *if $X \in SUV^n$ is connected and* $\dim X \leq n$, *then* $X \in SUV^\infty$ (Corollary 4.2.4).

4.1 Definition of proper n-shape

Let X and Y be locally compact spaces. Recall that a proper map $\alpha : X \to Y$ is said to be *properly n-invertible* if for any locally compact space Z with $\dim Z \leq n$ and any proper map $\beta : Z \to Y$ there is a proper map $\varphi : Z \to X$ such that $\alpha\varphi = \beta$. By Theorem 1.1.4 and considering one-point compactification, for any locally compact space X, there exists a proper n-invertible surjection $\alpha : Z \to X$ with $\dim Z \leq n$.

Two proper maps $f, g : X \to Y$ are *properly n-homotopic* (written by $f \simeq_p^n g$) if, for any proper map $\alpha : Z \to X$ from a locally compact space Z with $\dim Z \leq n$ into X, the compositions $f\alpha$ and $g\alpha$ are properly homotopic in the usual sense ($f\alpha \simeq_p g\alpha$). Note that we may only prove when α is a properly n-invertible.

If there exist proper maps $f : X \to Y$ and $g : Y \to X$ such that $fg \simeq_p^n \mathrm{id}_Y$ and $gf \simeq_p^n \mathrm{id}_X$, then X and Y are said to be *properly n-homotopically equivalent* (written by $X \simeq_p^n Y$). If only the first relation is valid, then it is said that X *properly n-homotopically dominates* Y, or Y is *properly n-homotopically dominated by* X (written by $X \geq_p^n Y$, or $Y \leq_p^n X$).

Suppose that X and Y are closed sets in locally compact AR's M and N, respectively. Let $\Lambda = (\Lambda, \leq)$ be a directed set. A net $\mathbf{f} = \{f_\lambda \mid \lambda \in \Lambda\}$ of maps $f_\lambda : M \to N$ is called *a proper n-fundamental net* from X to Y in M and N if, for every closed neighborhood V of Y in N, there exist a closed neighborhood U of X in M and an index $\lambda_0 \in \Lambda$ such that

$$f_\lambda|_U \simeq_p^n f_{\lambda_0}|_U \text{ in } V \text{ for all } \lambda \geq \lambda_0.$$

One should remark that each f_λ need not be proper but $f_\lambda|_U$ is proper for some closed neighborhood U of X in M (cf. [BS, Lemma 3.2]). We denote that $\mathbf{f}: X \to Y$ in (M, N).

The identity n-fundamental net is defined by $\mathbf{i}_X = \{\mathrm{id}_M\} : X \to X$ in (M, M).

Let $\mathbf{g} = \{g_\delta \mid \delta \in \Delta\}$ be a proper n-fundamental net from Y to Z in N and P, where Δ is a directed set and P is a locally compact AR containing Z as a closed set. Then the composition of $\mathbf{f}$ and $\mathbf{g}$ is defined by $\mathbf{gf} = \{g_\delta f_\lambda \mid (\lambda, \delta) \in \Lambda \times \Delta\} : X \to Z$ in (M, P), where $\Lambda \times \Delta$ is the directed set with the order $(\lambda, \delta) \geq (\lambda_0, \delta_0) \Leftrightarrow \lambda \geq \lambda_0, \delta \geq \delta_0$.

Let $\mathbf{f} = \{f_\lambda \mid \lambda \in \Lambda\}, \mathbf{g} = \{g_\delta \mid \delta \in \Delta\} : X \to Y$ in (M, N) be proper n-fundamental nets. We say that $\mathbf{f}$ and $\mathbf{g}$ are *properly n-homotopic* (written by $\mathbf{f} \simeq_p^n \mathbf{g}$) if for each closed neighborhood V of Y there exist a closed neighborhood U of X and $(\lambda_0, \delta_0) \in \Lambda \times \Delta$ such that

$$f_\lambda|_U \simeq_p^n g_\delta|_U \text{ in } V \text{ for all } (\lambda, \delta) \geq (\lambda_0, \delta_0).$$

If there exist proper n-fundamental nets $\mathbf{f} : X \to Y$ in (M, N) and $\mathbf{g} : Y \to X$ in (N, M) such that $\mathbf{gf} \simeq_p^n \mathbf{i}_X$ and $\mathbf{fg} \simeq_p^n \mathbf{i}_Y$, then X and Y are said to be *properly n-fundamentally equivalent* in M and N. In this case we write $X \simeq_{pF}^n Y$ in (M, N). If only the relation $\mathbf{fg} \simeq_p^n \mathbf{i}_Y$ is valid, then we say that X *properly n-fundamentally dominate* Y in (M, N) or Y is *properly n-fundamentally dominated by* X in (N, M), and write $X \geq_{pF}^n Y$ in (M, N) or $Y \leq_{pF}^n X$ in (N, M). The following is easy to see.

Proposition 4.1.1 *Let X, Y and Z be closed subsets of locally compact AR's M, N and P, respectively. If $X \geq_{pF}^n Y$ in (M, N) and $Y \geq_{pF}^n Z$ in (N, P), then $X \geq_{pF}^n Z$ in (M, P). If $X \simeq_{pF}^n Y$ in (M, N) and $Y \simeq_{pF}^n Z$ in (N, P), then $X \simeq_{pF}^n Z$ in (M, P).* □

Let $f : X \to Y$ be a proper map and $\mathbf{f} : X \to Y$ in (M, N) a proper n-fundamental net. We say that $\mathbf{f}$ is *generated by f* (or f *generates* $\mathbf{f}$) provided

$f = f_\lambda|_X$ for all $\lambda \in \Lambda$. Since N is an AR, each proper map f has an extension $\tilde{f} : M \to N$. By [BS, Lemma 3.2], we have the following.

Lemma 4.1.2 *Each proper map $f : X \to Y$ generates a proper n-fundamental net $\mathbf{f} = \{\tilde{f}\} : X \to Y$ in (M, N).* $\square$

Two proper maps $f, g : X \to Y \subset N$ are said to be *weakly properly n-homotopic in N* if they are properly n-homotopic in each closed neighborhood of Y in N. It is clear that any two properly n-homotopic maps from X to Y are weakly properly n-homotopic in N.

Lemma 4.1.3 *Let $\mathbf{f}, \mathbf{g}$ be two proper n-fundamental nets from X to Y in M and N generated by $f, g : X \to Y$, respectively. Then f and g are weakly properly n-homotopic in N if and only if $\mathbf{f} \simeq_p^n \mathbf{g}$.*

Proof. Suppose that $\mathbf{f} \simeq_p^n \mathbf{g}$. Then for each closed neighborhood V of Y there exist a closed neighborhood U of X and $(\lambda_0, \delta_0) \in \Lambda \times \Delta$ such that

$$f_\lambda|_U \simeq_p^n g_\delta|_U \text{ in } V \text{ for all } (\lambda, \delta) \geq (\lambda_0, \delta_0).$$

Thus, $f = f_\lambda|_X \simeq_p^n g_\delta|_X = g$ in V.

Conversely, suppose that f and g are weakly properly n-homotopic in N. Let V and V_1 be closed neighborhoods of Y in N such that $V_1 \subset \text{int}V$. Since $\mathbf{f}$ and $\mathbf{g}$ are properly n-fundamental nets, there exist a closed neighborhood U_1 of X and $(\lambda_0, \delta_0) \in \Lambda \times \Delta$ such that

$$f_\lambda|_{U_1} \simeq_p^n f_{\lambda_0}|_{U_1} \text{ and } g_\delta|_{U_1} \simeq_p^n g_{\delta_0}|_{U_1} \text{ in } V_1 \text{ for all } (\lambda, \delta) \geq (\lambda_0, \delta_0).$$

It follows from the assumption that

$$f_{\lambda_0}|_X = f \simeq_p^n g = g_{\delta_0}|_X \text{ in } V_1.$$

Let $\alpha : Z \to M$ be an n-invertible proper map with $\dim Z \leq n$. Then there exists a proper homotopy $H : \alpha^{-1}(X) \times I \to V_1$ such that $H_0 = f_{\lambda_0}\alpha|_{\alpha^{-1}(X)}$

and $H_1 = g_{\delta_0}\alpha|_{\alpha^{-1}(X)}$. Let $\hat{f} = f_{\lambda_0}\alpha$ and $\hat{g} = g_{\delta_0}\alpha$. By [BS, Lemma 3.4], there exists a closed neighborhood W of $\alpha^{-1}(X)$ in Z such that $\hat{f}|_W \simeq_p \hat{g}|_W$ in V. Since α is proper, we can find a closed neighborhood U of X in M such that $U \subset U_1$ and $\alpha^{-1}(U) \subset W$. Hence, $\hat{f}|_{\alpha^{-1}(U)} \simeq_p \hat{g}|_{\alpha^{-1}(U)}$ in V, which implies that

$$f_\lambda|_U \simeq^n_p f_{\lambda_0}|_U \simeq^n_p g_{\delta_0}|_U \simeq^n_p g_\delta|_U \text{ in } V. \quad \square$$

Observe that $X \leq^n_p Y$ (or $X \simeq^n_p Y$) implies $X \leq^n_{pF} Y$ (or $X \simeq^n_{pF} Y$) in (M, N) for any locally compact *AR*'s M and N containing X and Y as closed sets, respectively. In fact, then we have two maps $f : X \to Y$ and $g : Y \to X$ such that $gf \simeq^n_p \mathrm{id}_X$. By Lemma 4.1.2, f and g generate proper n-fundamental nets $\mathbf{f} : X \to Y$ in (M, N) and $\mathbf{g} : Y \to X$ in (N, M), respectively. Then $\mathbf{gf}$ is a proper n-fundamental net generated by the composition gf. Since $gf \simeq^n_p \mathrm{id}_X$, we have $\mathbf{gf} \simeq^n_p \mathbf{i}_X$ by Lemma 4.1.3. (Similarly, $\mathbf{fg} \simeq^n_p \mathbf{i}_Y$.) In particular, if X is homeomorphic ($\approx$) to X', then $X \simeq^n_{pF} X'$ in (M, M') for any locally compact *AR*'s M and M' containing X and X' as closed sets, respectively. Then the following holds from Proposition 4.1.1.

Proposition 4.1.4 *Let X, X', Y and Y' be closed sets in locally compact AR's M, M', N and N', respectively. Suppose that $X \approx X'$ and $Y \approx Y'$. Then $X \geq^n_{pF} Y$ in (M, N) if and only if $X' \geq^n_{pF} Y'$ in (M', N'); and $X \simeq^n_{pF} Y$ in (M, N) if and only if $X' \simeq^n_{pF} Y'$ in (M', N').* $\square$

This proposition shows that the relations of proper n-fundamental dominance and proper n-fundamental equivalence do not depend on the choice of ambient *AR*'s. We can define *the proper n-shape* as follows: locally compact spaces X and Y are *the same proper n-shape* (written by $n\text{-}\mathrm{Sh}_p(X) = n\text{-}\mathrm{Sh}_p(Y)$) if and only if $X \simeq^n_{pF} Y$ in (M, N) for some locally compact *AR*'s M and N. Analogously, X is said to *properly n-shape dominate* Y, or Y is *properly n-shape dominated by* X (written by $n\text{-}\mathrm{Sh}_p(X) \geq n\text{-}\mathrm{Sh}_p(Y)$, or $n\text{-}\mathrm{Sh}_p(Y) \leq n$-

$\mathrm{Sh}_p(X)$) if and only if $X \geq^n_{pF} Y$ in (M, N) for some locally compact *AR*'s M and N.

Let $\mathcal{LK}$ be the class of locally compact spaces. We define *the proper n-shape category*, n-$\mathrm{SH}_p\mathcal{LK}$, whose objects are in $\mathcal{LK}$ and whose morphisms are the proper n-homotopy classes of proper n-fundamental nets.

Suppose that X is rim-compact (i.e., any point has arbitrary small neighborhoods with compact boundaries). Then *the Freudenthal compactification* of X, here denoted by FX, is defined as the least upper bound of all compactifications Y of X such that $\mathrm{ind}(Y \setminus X) = 0$. We call $EX = FX \setminus X$ *the space of ends* of X. It is known that FX is metrizable if and only if the space QX of quasi-components of X is compact, where EX is homeomorphic to a closed set of the Cantor set.

Let $\Sigma = \{X \mid X$ is locally compact and QX is compact $\}$. Suppose that $X, Y \in \Sigma$ and that $f : X \to Y$ is a proper map. Then f has the unique extension $Ff : (FX, EX) \to (FY, EY)$. If $g : X \to Y$ is a proper map and $f \simeq_p g$, then $Ff|_{EX} = Fg|_{EX}$. Also, the assignment $f \to Ff$ is functorial; that is, $F(\mathrm{id}_X) = \mathrm{id}_{FX}$ and $F(fg) = (Ff)(Fg)$ for proper maps $f : X \to Y$ and $g : Y \to Z$. For details, refer to [BS].

Lemma 4.1.5 *Let X be a closed set in a locally compact ANR M with* $\dim X \leq$ *n and $f, g : M \to Z$ be proper maps from M to any space Z. If $f \simeq^n_p g$, then there exists a closed neighborhood E of X in M such that $f|_E \simeq_p g|_E$.*

Proof. Since M is a locally compact *ANR*, M has an open covering $\mathcal{U}$ such that any two $\mathcal{U}$-close proper maps of an arbitrary space to M are properly homotopic ([Mi, Theorem 5.1.1] and [Cha_2, Theorem 4.1(2)]). By [Mi, Theorem 5.1.4], there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that for every locally finite simplicial complex $\mathcal{T}$ and every subcomplex $\mathcal{S}$ of $\mathcal{T}$ containing all vertices of $\mathcal{T}$, every partial realization of $\mathcal{T}$ in M relative to $(\mathcal{S}, \mathcal{V})$ can be extended to a full realization of $\mathcal{T}$ in M relative to $\mathcal{U}$. Let $\mathcal{V}'$ be an open star-refinement of

$\mathcal{V}$. Since $\dim X \leq n$, there exists an open covering $\mathcal{W}$ of X in M such that $\mathcal{W}$ refines $\mathcal{V}'$ and the order of $\mathcal{W}$ is at most n (cf. [Mi, Theorem 4.3.5]). Let $E \subset \bigcup \mathcal{W}$ be a closed neighborhood of X in M.

Let $\eta : E \to K$ be a canonical map from E to the nerve K of $\mathcal{W} \cap E$ and $\zeta^{(0)} : K^{(0)} \to M$ be a map defined by $\zeta^{(0)}(x(W)) \in W$ for each $W \in \mathcal{W}$. Since $\zeta^{(0)}$ is a partial realization of K relative to $(K^{(0)}, \mathcal{V})$, there exists a full realization $\zeta : K \to M$ relative to $\mathcal{U}$. By the proof of [Mi, Theorem 5.1.8], we conclude that id_E and $\zeta\eta$ are $\mathcal{U}$-close, which implies that $\zeta\eta$ is proper homotopic to id_E. Since $\dim K \leq n$ and $f \simeq_p^n g$, we have

$$f|_E \simeq_p f\zeta\eta \simeq_p g\zeta\eta \simeq_p g|_E. \quad \square$$

Remark. Let $\mu^n = M_n^{2n+1} \subset I^{2n+1}$ be the n-dimensional Menger compactum constructed in the $(2n+1)$-dimensional cube I^{2n+1} [CKT]. Since $\dim FX \leq n$ and by the Z-set unknotting theorem [Be], we can assume that FX is a closed set in μ^n and $EX = FX \cap I^{2n} \times \{1\}$, which implies that $X \subset \mu^n \setminus EX \subset I^{2n} \times [0,1) = M$. Let $f, g : M \to Z$ be proper maps from M to any space Z and $f|_V \simeq_p^n g|_V$ for some closed neighborhood of X in M. Then there exists a closed neighborhood $U = \bigcup_{i\in\omega} B_i$ of X in V, where B_i are similar figures to $B = \{(x_1, \cdots, x_{2n+1}) \in I^{2n+1} \mid$ there exists $(y_1, \cdots, y_{2n+1}) \in P$ such that $|x_i - y_i| \leq 1/3$ for some $i \in \{1, \cdots, 2n+1\}\}$ and $P = \{(y_1, \cdots, y_{2n+1}) \in I^{2n+1} \mid y_{i_1}, \cdots, y_{i_{n+1}} \in \{0,1\}, 1 \leq i_1 < \cdots < i_{n+1} \leq 2n+1\}$. There exists a deformation retraction from B to the n-dimensional subcomplex P of B, so we obtain a proper deformation retract $\beta : U \to L$ from U to n-dimensional subcomplex L of U. Let $\gamma : L \to U$ be an inclusion. Then $\mathrm{id}_U \simeq_p \gamma\beta$ in U, which implies that $f|_U \simeq_p f\gamma\beta \simeq_p g\gamma\beta \simeq_p g|_U$.

Proposition 4.1.6 *If* $\mathrm{Sh}_p(X) = \mathrm{Sh}_p(Y)$*, then* n-$\mathrm{Sh}_p(X) = n$-$\mathrm{Sh}_p(Y)$*; the converse holds in case* $\dim X, \dim Y \leq n$.

Proof. First statement is obvious since a proper homotopy is a proper n-homotopy.

In case $\dim X, \dim Y \leq n$, suppose that X and Y are closed sets in locally compact *ANR*'s M and N, respectively. If $n\text{-Sh}_p(X) = n\text{-Sh}_p(Y)$, then there exist proper n-fundamental nets $\mathbf{f} : X \to Y$ in (M, N) and $\mathbf{g} : Y \to X$ in (N, M) such that $\mathbf{gf} \simeq_p^n \mathbf{i}_X$ and $\mathbf{fg} \simeq_p^n \mathbf{i}_Y$. Let V be a closed neighborhood of Y in N. Then there exist a closed neighborhood U of X in M and $\lambda_0 \in \Lambda$ such that $f_\lambda|_U \simeq_p^n f_{\lambda_0}|_U$ in V for all $\lambda \geq \lambda_0$. Note that int$U$ is a locally compact *ANR*. By Lemma 4.1.5, there exists a closed neighborhood W of X in U such that $f_\lambda|_W \simeq_p f_{\lambda_0}|_W$ in V for all $\lambda \geq \lambda_0$. Thus, we conclude that $\mathbf{f}$ is a proper fundamental net. Similarly, $\mathbf{g}$ is a proper fundamental net, $\mathbf{gf} \simeq_p \mathbf{i}_X$ and $\mathbf{fg} \simeq_p \mathbf{i}_Y$. The proposition is proved. □

4.2 Proper n-shape and property SUV^n

Let X be a closed set in a locally compact *ANR* M, and $n \in \omega$. We say that X has *property SUV^n in M* [Sh1] if, for each closed neighborhood U of X in M, there exists a closed neighborhood V of X in U such that the following holds;

> For any proper map $h : S^i \times \omega \to V$ $(1 \leq i \leq n)$ there exists a proper map $\bar{h} : B^{i+1} \times \omega \to U$ such that $\bar{h}|_{S^i \times \omega} = h$.

One should note that $i \neq 0$ in the above condition. (If $i = 0$, then the above condition implies $EX =$ one-point.) It is easy to see that property SUV^n is independent of the choice of M. We write $X \in SUV^n$ if X embeds into some locally compact *ANR* as a closed SUV^n subset.

Suppose $X \in \Sigma$. A sequence α of points of X is said to be *admissible in X* [Ba] if (1) no subsequence of α converges to a point of X and (2) no compact subset of X separates in X two infinite subsequences of α. Two admissible sequences α and β in X are *equivalent* if no compact subset of X separates an infinite subsequence of α from an infinite subsequence of β. This relation is an equivalence relation of the set of all admissible sequences in X. The equivalent class of α is denoted by $[\alpha]$.

Lemma 4.2.1 *Let T be a tree and $X \in \Sigma$. For proper maps $f, g : X \to T$, the following are equivalent:*

(i) $Ff|_{EX} = Fg|_{EX}$;

(ii) $f \simeq_p g$;

(iii) $f \simeq_p^n g$ *for all* $n \in \omega$;

(iv) $f \simeq_p^0 g$.

Proof. The implication (i)⇒(ii) follows from [Sh$_1$, Theorem 2.3] and the implications (ii)⇒(iii)⇒(iv) are trivial.

To see the implication (iv)⇒(i), suppose that $f \simeq_p^0 g$. By [Ba, Theorem 2.5], an end $e \in EX$ may be considered to be the equivalence class $[\beta]$ of an admissible sequence β in X. Consider β as a proper map from ω to X. Then $f\beta \simeq_p g\beta$ by (iv), which implies $Ff(e) = [f\beta] = [g\beta] = Fg(e)$. Thus, we conclude that $Ff|_{EX} = Fg|_{EX}$. □

Lemma 4.2.2 *Suppose $X \in SUV^1$ and X is a closed set in $Q \times [0, 1)$. For each closed neighborhood U of X there exists a closed neighborhood V of X such that two proper maps $f, g : T \to V$ of a tree T are properly homotopic in U if $Ff|_{ET} = Fg|_{ET}$.*

Proof. Since $X \in SUV^1$, for each closed neighborhood U of X there exists a closed neighborhood V of X such that any proper map $h : S^1 \times \omega \to V$ has a proper extension $\bar{h} : B^2 \times \omega \to U$. We can assume that V is a Q-manifold. Let $V = \bigcup_{i=0}^{\infty} C_i$, where C_i is compact and $\emptyset = C_0 \subset C_1 \subset C_2 \subset \cdots$. Since f and g are proper, there exists a sequence $T = T_1 \supset T_2 \supset T_3 \supset \cdots$ of subcomplexes of T such that $ET_i = ET$ for each $i \in \omega$, $\bigcap_{i=1}^{\infty} T_i = \emptyset$ and $f(T_i) \cup g(T_i) \subset V \setminus C_{i-1}$.

For each $v \in T^{(0)}$, let $i(v) = \max\{i \in \omega \mid v \in T_i\}$. Take an admissible sequence $(x_j^v)_{j \in \omega}$ which is in the same component of v in $T_{i(v)}$. Since $Ff|_{ET} = Fg|_{ET}$, $(f(x_j^v))_{j\in\omega}$ and $(g(x_j^v))_{j\in\omega}$ are equivalence, there exists an index $k(v)$ such that $f(x_{k(v)}^v)$ and $g(x_{k(v)}^v)$ are in the same component D_v of $V \setminus C_{i(v)}$. Since V is locally path-connected, D_v is a path-component. Hence, there exists a path $\gamma : I \to D_v$ connecting $f(x_{k(v)}^v)$ and $g(x_{k(v)}^v)$. Let $\beta : I \to T_{i(v)}$ be a path such that $\beta(0) = v$ and $\beta(1) = x_{k(v)}^v$, and define a path $\alpha_v : I \to V \setminus C_{i(v)}$ by

$$\alpha_v(t) = \begin{cases} f\beta(3t) & \text{for } 0 \leq t \leq 1/3, \\ \gamma(3t-1) & \text{for } 1/3 \leq t \leq 2/3, \\ g\beta(3-3t) & \text{for } 2/3 \leq t \leq 1. \end{cases}$$

Then $\alpha_v(0) = f(v)$ and $\alpha_v(1) = g(v)$ for each $v \in T^{(0)}$. We define a map $H : T \times \{0\} \cup T^{(0)} \times I \cup T \times \{1\} \to V$ by

$$H(x,t) = \begin{cases} f(x) & \text{for } t = 0, \\ \alpha_x(t) & \text{for } x \in T^{(0)}, \\ g(x) & \text{for } t = 1. \end{cases}$$

Observe that H is a proper map. Let $\{\tau_i \mid i \in \omega\}$ be the set of all 1-simplices of T and $\phi_i : B^2 \times \{i\} \to \tau_i \times I$ $(i \in \omega)$ be homeomorphisms. Then the proper map $h : S^1 \times \omega \to V$ defined by $h|_{S^1 \times \{i\}} = H\phi_i|_{S^1 \times \{i\}}$ extends to a proper map $\bar{h} : B^2 \times \omega \to U$, which induces the proper extension $\tilde{H} : T \times I \to U$ of H such that $\bar{h}|_{B^2 \times \{i\}} = \tilde{H}\phi_i$. Since $\tilde{H}$ is a proper homotopy from f to g in U, we have the lemma. □

Theorem 4.2.3 *Let $X \in \Sigma$ be connected. Then the following are equivalent:*

(i) $n\text{-Sh}_p(X) = n\text{-Sh}_p(T)$ *for any tree T such that $EX \approx ET$;*

(ii) $n\text{-Sh}_p(X) = n\text{-Sh}_p(T)$ *for some tree T;*

(iii) $n\text{-Sh}_p(X) \leq n\text{-Sh}_p(T)$ *for some tree T;*

(iv) $X \in SUV^n$.

Proof. Since (i)⇒(ii)⇒(iii) are trivial, we show the implications (iii)⇒(iv)⇒(i).

(iii)⇒(iv): Let M be a locally compact AR with $X \subset M$ as a closed set. Suppose that there exist proper n-fundamental nets $\mathbf{f} = \{f_\lambda \mid \lambda \in \Lambda\} : X \to T$ in (M, T) and $\mathbf{g} = \{g_\delta \mid \delta \in \Delta\} : T \to X$ in (T, M) such that $\mathbf{gf} \simeq_p^n \mathbf{i}_X$. Then for each closed neighborhood U of X there exist a closed neighborhood V of X and $(\lambda_0, \delta_0) \in \Lambda \times \Delta$ such that

$$g_\delta f_\lambda|_V \simeq_p^n \mathrm{id}_V \text{ in } U \text{ for all } (\lambda, \delta) \geq (\lambda_0, \delta_0).$$

Let $h : S^i \times \omega \to V$ $(1 \leq i \leq n)$ be a proper map. Then $T_j = f_{\lambda_0} h(S^i \times \{j\})$ $(j \in \omega)$ are compact subtrees of T and $\{T_j \mid j \in \omega\}$ is locally finite in T. Since each T_j is an AR, we have an extension $h_j : B^{i+1} \times \{j\} \to T_j$ of $f_{\lambda_0} h|_{S^i \times \{j\}}$. Define a proper map $\bar{h} : B^{i+1} \times \omega \to T$ by $\bar{h}|_{B^{i+1} \times \{j\}} = h_j$. Then $g_{\delta_0} \bar{h}|_{S^i \times \omega} = g_{\delta_0} f_{\lambda_0} h \simeq_p h$ in U since $\dim S^i \times \omega \leq n$. By Borsuk's homotopy extension theorem for a proper maps (cf. [Sh_3, Theorem 3.1]), h has a proper extension $\tilde{h} : B^{i+1} \times \omega \to U$.

(iv)⇒(i): We can assume that $X \subset Q \times [0, 1) = M$ is a closed set such that $\mathrm{cl}_{Q \times [0,1]} X = FX$. Let T be a tree such that there exists a homeomorphism $f : EX \to ET$. By [Sh_1, Theorem 2.3], there exists a proper map $\bar{f} : X \to T$ such that $F\bar{f}|_{EX} = f$. Then $\bar{f}$ generates a proper n-fundamental net $\mathbf{f} = \{\bar{f}\} : X \to T$ in (M, T) by Lemma 4.1.2.

Let $\Delta = \{V \subset M \mid V$ is a closed connected neighborhood of X in M which is a Q-manifold and $EV = EX\}$. Note that Δ is the directed set with the order $V \geq V' \Leftrightarrow V \subset V'$. Let $U \in \Delta$. Since $X \in SUV^n$, we have a sequence $U = V_{n+1} \supset V_n \supset \cdots \supset V_1 = V_U$ in Δ satisfying the following;

(∗) Any proper map $h : S^i \times \omega \to V_i$ $(1 \leq i \leq n)$ has a proper extension $\bar{h} : B^{i+1} \times \omega \to V_{i+1}$.

Observe that $V_2 \supset V_1$ satisfy the condision of Lemma 4.2.2.

By the Triangulation Theorem for Q-manifolds [Cha$_2$], we can regard $V_U = K\times Q$, where K is a locally finite simplicial complex. Let $\eta : V_U = K\times Q \to K$ be the projection and $\zeta : K \to K \times \{0\} \subset K \times Q$ the canonical injection. Let T'_U be a maximal tree of K such that $EK = ET'_U$ and let $T_U = \zeta(T'_U)$ be a closed tree in V_U. Note that $ET_U = EV_U = EX$. By [Sh$_1$, Theorem 2.3], there exist proper maps $k_U : V_U \to T_U \subset V_U$ and $g_U : T \to T_U \subset V_U$ such that $Fk_U|_{EV_U} = \mathrm{id}_{EV_U}$ and $Fg_U|_{ET} = f^{-1} : ET \to EX = EV_U = ET_U$. Then $\mathbf{g} = \{g_U \mid U \in \Delta\}$ is a proper n-fundamental net because if $U' \geq V_U$, then $Fg_U|_{ET} = f^{-1} = Fg_{U'}|_{ET}$, hence $g_{U'} \simeq_p g_U$ in U by Lemma 4.2.2. Since $F(\tilde{f}g_U)|_{ET} = (F\tilde{f})(Fg_U)|_{ET} = ff^{-1} = \mathrm{id}_{ET}$, $\tilde{f}g_U \simeq^n_p \mathrm{id}_T$ in T for all $U \in \Delta$ by Lemma 4.2.1, which implies that $\mathbf{fg} \simeq^n_p \mathbf{i}_T$.

Since V_U is path-connected and $F(k_U\zeta)|_{ET'_U} = (Fk_U)(F\zeta)|_{ET'_U} = F\zeta|_{ET'_U}$, by the same argument of the proof of Lemma 4.2.2, we have a proper homotopy $H^{(1)} : K^{(1)} \times I \to V_2$ such that $H^{(1)}_0 = \zeta|_{K^{(1)}}$ and $H^{(1)}_1 = k_U\zeta|_{K^{(1)}}$.

Assume that a proper map $H^{(i-1)} : K^{(i-1)} \times I \to V_i$ $(2 \leq i \leq n)$ has been constructed so that $H^{(i-1)}_0 = \zeta|_{K^{(i-1)}}$ and $H^{(i-1)}_1 = k_U\zeta|_{K^{(i-1)}}$. By using $(*)$, we can extend $H^{(i-1)}$ to a proper map $H^{(i)} : K^{(i)} \times I \to V_{i+1}$ such that $H^{(i)}_0 = \zeta|_{K^{(i)}}$ and $H^{(i)}_1 = k_U\zeta|_{K^{(i)}}$. We have inductively a proper map $H^{(n)} : K^{(n)} \times I \to V_{n+1} = U$ such that $H^{(n)}_0 = \zeta|_{K^{(n)}}$ and $H^{(n)}_1 = k_U\zeta|_{K^{(n)}}$. Let $\beta : Z \to V_U$ be a proper map with $\dim Z \leq n$. By the standard trick of bridge maps, there exists a proper map $\gamma : Z \to K^{(n)}$ such that $\eta\beta \simeq_p \gamma$ in K.[1] Since $\beta \simeq_p \zeta\eta\beta \simeq_p \zeta\gamma$ in V_U, we have

$$\mathrm{id}_{V_U}\beta \simeq_p \zeta\gamma \overset{H^{(n)}}{\simeq_p} k_U\zeta\gamma \simeq_p k_U\beta \text{ in } U,$$

which implies that $\mathrm{id}_{V_U} \simeq^n_p k_U$ in U. Note that $\mathrm{id}_{V_{U'}} \simeq^n_p k_{U'}$ in U for all $U' \geq V_U$.

[1]Note that γ is contiguous to $\eta\beta$.

Since $F(g_U\tilde{f})|_{EV_U} = (Fg_U)(F\tilde{f})|_{EV_U} = f^{-1}f = \mathrm{id}_{EV_U} = Fk_U|_{EV_U}$, we have $g_U\tilde{f}|_{V_U} \simeq_p^n k_U$ in T_U by Lemma 4.2.1. Then

$$\mathrm{id}_{V_{U'}} \simeq_p^n k_{U'} \simeq_p^n g_{U'}\tilde{f}|_{V_{U'}} \text{ in } U \text{ for all } U' \geq V_U,$$

which implies that $\mathbf{gf} \simeq_p^n \mathbf{i}_X$. The theorem is proved. □

Now we can strengthen Sher's result [Sh_1, Theorem 4.1] as follows:

Corollary 4.2.4 *Let $X \in \Sigma$ be connected. If $X \in SUV^n$ and $\dim X \leq n$, then $X \in SUV^\infty$.*

Proof. By Theorem 4.2.3, there exists a tree T such that $n\text{-}\mathrm{Sh}_p(X) = n\text{-}\mathrm{Sh}_p(T)$. It implies that $\mathrm{Sh}_p(X) = \mathrm{Sh}_p(T)$ by Proposition 4.1.6. By [Sh_1, Corollary 3.5], we conclude that $X \in SUV^\infty$. □

The following table shows the relations of UV-properties and shapes.

X : compact;

$$\begin{array}{rcl} X \in UV^\infty & \Leftrightarrow & \mathrm{Sh}(X) = \mathrm{Sh}(1) \text{ [Bo}_2\text{, Theorem 9.1]} \\ & & \Updownarrow \ \dim X \leq n \text{ [Chi}_2\text{, Corollary 2.17]} \\ X \in UV^n & \Leftrightarrow & n\text{-}\mathrm{Sh}(X) = n\text{-}\mathrm{Sh}(1) \text{ [Chi}_2\text{, Proposition 3.1].} \end{array}$$

X : locally compact;

$$\begin{array}{rcl} X \in SUV^\infty & \Leftrightarrow & \mathrm{Sh}_p(X) = \mathrm{Sh}_p(T),\ \exists T : \text{tree [Sh}_1\text{, Theorem 3.1]} \\ & & \Updownarrow \ \dim X \leq n \text{ (Corollary 4.2.4)} \\ X \in SUV^n & \Leftrightarrow & n\text{-}\mathrm{Sh}_p(X) = n\text{-}\mathrm{Sh}_p(T'),\ \exists T' : \text{tree (Theorem 4.2.3).} \end{array}$$

Proposition 4.2.5 *For any two trees T_1 and T_2,*

(1) *$T_1 \simeq_p^0 T_2$ if and only if $ET_1 \approx ET_2$;*

(2) *$T_1 \geq_p^0 T_2$ if and only if ET_2 embeds in ET_1.*

Proof. We only prove (1). Note that (2) follows from the proof of (1).

First, we prove the "only if" part. Suppose that there exist two proper maps $f : T_1 \to T_2$ and $g : T_2 \to T_1$ such that $gf \simeq_p^0 \mathrm{id}_{T_1}$ and $fg \simeq_p^0 \mathrm{id}_{T_2}$. By Lemma

4.2.1, $(Fg)(Ff)|_{ET_1} = F(gf)|_{ET_1} = F(\mathrm{id}_{T_1})|_{ET_1} = \mathrm{id}_{ET_1}$ and $(Ff)(Fg)|_{ET_2} = F(fg)|_{ET_2} = F(\mathrm{id}_{T_2})|_{ET_2} = \mathrm{id}_{ET_2}$, which implies that $ET_1 \approx ET_2$.

Next, we prove the "if" part. Let $h : ET_1 \to ET_2$ be a homeomorphism. By [Sh_1, Theorem 2.3], there exist two proper maps $k : T_1 \to T_2$ and $l : T_2 \to T_1$ such that $h = Fk|_{ET_1}$ and $h^{-1} = Fl|_{ET_2}$. Then $F(lk)|_{ET_1} = (Fl)(Fk)|_{ET_1} = h^{-1}h = \mathrm{id}_{ET_1}$ and $F(kl)|_{ET_2} = (Fk)(Fl)|_{ET_2} = hh^{-1} = \mathrm{id}_{ET_2}$. By Lemma 4.2.1, we conclude that $lk \simeq_p^0 \mathrm{id}_{T_1}$ and $kl \simeq_p^0 \mathrm{id}_{T_2}$. □

By [Sh_1, Corollary 2.6] and Lemma 4.2.1, we have the following.

Corollary 4.2.6 *Let T_1 and T_2 be trees. Then $T_1 \geq_p^0 T_2$ or $T_2 \geq_p^0 T_1$.* □

Corollary 4.2.7 *Suppose $X_1, X_2 \in SUV^n$. Then*

(1) *n-$\mathrm{Sh}_p(X_1) = n$-$\mathrm{Sh}_p(X_2)$ if and only if $EX_1 \approx EX_2$;*

(2) *n-$\mathrm{Sh}_p(X_1) \geq n$-$\mathrm{Sh}_p(X_2)$ if and only if EX_2 embeds in EX_1.*

Proof. We only prove (1). note that (2) follows from the proof of (1).

Let T_1 and T_2 be trees such that $EX_1 \approx ET_1$ and $EX_2 \approx ET_2$. Then n-$\mathrm{Sh}_p(X_1) = n$-$\mathrm{Sh}_p(T_1)$ and n-$\mathrm{Sh}_p(X_2) = n$-$\mathrm{Sh}_p(T_2)$ by Theorem 4.2.3. If n-$\mathrm{Sh}_p(X_1) = n$-$\mathrm{Sh}_p(X_2)$, then n-$\mathrm{Sh}_p(T_1) = n$-$\mathrm{Sh}_p(T_2)$. Since T_1 and T_2 are AR's, this implies that there exist two proper maps $f : T_1 \to T_2$ and $g : T_2 \to T_1$ such that $gf \simeq_p^n \mathrm{id}_{T_1}$ and $fg \simeq_p^n \mathrm{id}_{T_2}$. By Proposition 4.2.5, we have $ET_1 \approx ET_2$, hence $EX_1 \approx EX_2$.

Conversely, if $EX_1 \approx EX_2$, then $ET_1 \approx ET_2$. By Proposition 4.2.5 and Lemma 4.2.1, we have $T_1 \simeq_p^n T_2$, hence n-$\mathrm{Sh}_p(X_1) = n$-$\mathrm{Sh}_p(T_1) = n$-$\mathrm{Sh}_p(T_2) = n$-$\mathrm{Sh}_p(X_2)$. □

Corollary 4.2.8 *Suppose $X_1, X_2 \in SUV^n$. Then n-$\mathrm{Sh}_p(X_1) \geq n$-$\mathrm{Sh}_p(X_2)$ or n-$\mathrm{Sh}_p(X_2) \geq n$-$\mathrm{Sh}_p(X_1)$.* □

Chapter 5

Proper n-shape and Freudenthal compactification

In this chapter, we give another approach to proper n-shape and proper n-shape category $n\text{-SH}'_p\mathcal{LK}(n+1)$.

For a class $\mathcal{M}$ of spaces, $\mathcal{M}(n)$ denotes the subclass of $\mathcal{M}$ consisting of spaces with $\dim \leq n$. Let $\Sigma = \{X \mid X \text{ is locally compact and } QX \text{ is compact }\}$.

In [BS], Ball and Sher studied the relation of proper maps and the Freudenthal compactifications, defined the notion of proper shape and proved that for $X, Y \in \Sigma$, if $\text{Sh}_p(X) = \text{Sh}_p(Y)$ then $\text{Sh}(FX, EX) = \text{Sh}(FY, EY)$ rel. (EX, EY) [BS, Corollary 4.8].

By $n\text{-SH}_p\mathcal{LK}(n+1)$ ($n\text{-SH}_p\Sigma(n+1)$), we denote the full-subcategory of $n\text{-SH}_p\mathcal{LK}$ whose objects are in $\mathcal{LK}(n+1)$ ($\Sigma(n+1)$). The proper n-shape of locally compact spaces with $\dim \leq n+1$ is also defined by using embeddings of them into locally compact $(n+1)$-dimensional $LC^n \cap C^n$-spaces. We denote such a proper n-shape category by $n\text{-SH}'_p\mathcal{LK}(n+1)$. It is not clear that $n\text{-SH}_p\mathcal{LK}(n+1)$ and $n\text{-SH}'_p\mathcal{LK}(n+1)$ are categorical isomorphic. However we can prove the following:

Theorem A *There is a categorical embedding* $\Phi : n\text{-SH}_p\mathcal{LK}(n+1) \to n$-

$\mathrm{SH}'_p\mathcal{LK}(n+1)$ *such that* $\Phi(X) = X$ *for each* $X \in \mathcal{LK}(n+1)$.

Let $\mathcal{K}^2(n+1)$ be the class of compact pairs with dim $\leq n+1$ and n-$\mathrm{SH}'_p\Sigma(n+1)$ be the full-subcategory of n-$\mathrm{SH}'_p\mathcal{LK}(n+1)$ whose objects are in $\Sigma(n+1)$. We also define the relative n-shape category n-$\mathrm{SH}_{rel}\mathcal{K}^2(n+1)$, strengthening n-shape category of pairs, and prove the following.

Theorem B *There is a functor* $\Psi : n\text{-}\mathrm{SH}'_p\Sigma(n+1) \to n\text{-}\mathrm{SH}_{rel}\mathcal{K}^2(n+1)$ *such that* $\Psi(X) = (FX, EX)$ *for each* $X \in \Sigma(n+1)$.

So we conclude the following corollaries.

Corollary 1 *There is a functor* $\Theta : n\text{-}\mathrm{SH}_p\Sigma(n+1) \to n\text{-}\mathrm{SH}_{rel}\mathcal{K}^2(n+1)$ *such that* $\Theta(X) = (FX, EX)$ *for each* $X \in \Sigma(n+1)$.

Corollary 2 *For* $X, Y \in \Sigma(n+1)$, *if* $n\text{-}\mathrm{Sh}_p(X) = n\text{-}\mathrm{Sh}_p(Y)$ *then*

$$n\text{-}\mathrm{Sh}(FX, EX) = n\text{-}\mathrm{Sh}(FY, EY) \text{ rel. } (EX, EY).$$

Corollary 3 *If* X *is connected,* SUV^n *and* $\dim X \leq n+1$, *then* $FX \in UV^n$.

For each $X \in \mathcal{LK}$, let $CX = X \cup \{\infty\}$ be the one-point compactification of X. Considering $(CX, \{\infty\})$ instead of (FX, EX), we have the following similarly to Theorem 2.

Theorem C *There is a functor* $\Psi' : n\text{-}\mathrm{SH}'_p\mathcal{LK}(n+1) \to n\text{-}\mathrm{SH}_{rel}\mathcal{K}^2(n+1)$ *such that* $\Psi(X) = (CX, \{\infty\})$ *for each* $X \in \mathcal{LK}(n+1)$.

As a corollary, we have the following.

Corollary 4 *Let* $\mu^{n+1}_\infty = \mu^{n+1} \setminus \{*\}$, *where* $* \in \mu^{n+1}$, *and let* $X, Y \subset \mu^{n+1}_\infty$ *be* Z*-sets. If* $n\text{-}\mathrm{Sh}_p(X) = n\text{-}\mathrm{Sh}_p(Y)$ *then* $\mu^{n+1}_\infty \setminus X \approx \mu^{n+1}_\infty \setminus Y$.

5.1 The proper n-shape theory by using $AE(n)$'s

Suppose that X and Y are closed sets in $(n+1)$-dimensional locally compact $LC^n \cap C^n$-spaces M and N, respectively. Let $\Lambda = (\Lambda, \leq)$ and $\Delta = (\Delta, \leq)$ be directed sets. A net $\mathbf{f} = \{f_\lambda \mid \lambda \in \Lambda\}$ of maps $f_\lambda : M \to N$ is called *a proper n-fundamental net* from X to Y in M and N if, for every closed neighborhood V of Y in N, there exist a closed neighborhood U of X in M and an index $\lambda_0 \in \Lambda$ such that

$$f_\lambda|_U \simeq_p^n f_{\lambda_0}|_U \text{ in } V \text{ for all } \lambda \geq \lambda_0.$$

One should remark that each f_λ need not be proper but $f_\lambda|_U$ is proper for some closed neighborhood U of X in M (cf. [BS, Lemma 3.2]). We denote that $\mathbf{f} : X \to Y$ in (M, N).

Let $\mathbf{f} = \{f_\lambda \mid \lambda \in \Lambda\}, \mathbf{g} = \{g_\delta \mid \delta \in \Delta\} : X \to Y$ in (M, N) be proper n-fundamental nets. We say that $\mathbf{f}$ and $\mathbf{g}$ are *properly n-homotopic* (written by $\mathbf{f} \simeq_p^n \mathbf{g}$) if for each closed neighborhood V of Y there exist a closed neighborhood U of X and $\lambda_0 \in \Lambda$, $\delta_0 \in \Delta$ such that

$$f_\lambda|_U \simeq_p^n g_\delta|_U \text{ in } V \text{ for all } \lambda \geq \lambda_0 \text{ and } \delta \geq \delta_0.$$

The proper n-homotopy class of $\mathbf{f}$ is denoted by $[\mathbf{f}]_p^n$.

By the same argument of previous chapter, we can define the notion of proper n-shape for $(n+1)$-dimensional locally compact spaces by using embeddings of them into $(n+1)$-dimensional locally compact $LC^n \cap C^n$-spaces. *The proper n-shape category with* $\dim \leq n+1$, $n\text{-SH}'_p\mathcal{LK}(n+1)$, is the category whose objects are in $\mathcal{LK}(n+1)$ and whose morphisms are the proper n-homotopy classes of proper n-fundamental nets. If $X, Y \in \mathcal{LK}(n+1)$ are isomorphic in $n\text{-SH}'_p\mathcal{LK}(n+1)$, then we denote $n\text{-Sh}'_p(X) = n\text{-Sh}'_p(Y)$.

Proposition 5.1.1 *Let $X, Y \in \mathcal{LK}$, X be an LC^n-space, $f : X \to Y$ be a proper UV^n-surjection and $\mathcal{U}$ be an open covering of Y. Suppose that two proper maps $\phi : W_0 \to X$ and $\psi : W \to Y$ such that $f\phi = \psi|_{W_0}$, where*

$W \in \mathcal{LK}(n+1)$ and W_0 is a closed subset of W. Then there exists a proper map $\gamma : W \to X$ such that $\gamma|_{W_0} = \phi$ and $f\gamma$ is $\mathcal{U}$-close to ψ.

Proof. By [Da, Theorem 16.11], Y is LC^n. Let $\mathcal{U}_1$ be a double star-refinement of $\mathcal{U}$. By [Chi$_1$, Proposition 2.1], there is an open refinement $\mathcal{U}'$ of $\mathcal{U}_1$ satisfying the following; for any two $\mathcal{U}'$-close proper maps $g, h : A \to Y$ from a closed subset A of a space B with dim $\leq n+1$ such that g has a proper extension $G : B \to Y$ it follows that h also has a proper extension $H : B \to Y$ which is $\mathcal{U}_1$-close to G (cf. [Cha$_2$, Theorem 4.2(2)]). By [Hu, p.156], there exists an open refinement $\mathcal{V}$ of $\mathcal{U}_1$ such that, for any simplicial polytope K with dim $\leq n+1$, every partial realization of K in Y relative to $\mathcal{V}$ extends to a full realization of K in Y relative to $\mathcal{U}_1$. Let $\mathcal{W}$ be a canonical cover (cf. [Hu, p.51]) of $W \setminus W_0$ with order $\leq n+1$ such that $\psi(\mathcal{W})$ is a refinement of $\mathcal{V}$. By the nerve replacement trick [Hu, p.53] and the definition of $\mathcal{V}$, we have a proper map $\psi' : W^* \to Y$ such that $\psi' p$ is $\mathcal{U}_1$-close to ψ and $\psi'|_{W_0} = \psi|_{W_0}$, where $W^* = N(\mathcal{W}) \cup W_0$ and $p : W \to W^*$ is a canonical map with $p|_{W_0} = \mathrm{id}_{W_0}$. Since $\psi'|_{W_0} = f\phi$ and X is LC^n, ϕ extends to a proper map $\tilde{\phi} : W' \to X$, where W' is a closed neighborhood of W_0 in W^* and $W' \setminus W_0$ is a subpolyhedron of $N(\mathcal{W})$, such that $f\tilde{\phi}$ is $\mathcal{U}'$-close to $\psi'|_{W'}$. Then $f\tilde{\phi}$ has a proper extension $\tilde{\psi}' : W^* \to Y$ which is $\mathcal{U}_1$-close to ψ'. Note that $f\tilde{\phi}|_{W' \setminus W_0} = \tilde{\psi}'|_{W' \setminus W_0}$. By the lifting property [La, Lemma A] (cf. [Be, Proposition 2.1.3]), we have a proper map $\gamma' : N(\mathcal{W}) \to X$ such that $\gamma'|_{W' \setminus W_0} = \tilde{\phi}|_{W' \setminus W_0}$ and $f\gamma'$ is $\mathcal{U}_1$-close to $\tilde{\psi}'|_{N(\mathcal{W})}$. Then $\gamma = (\gamma' \cup \phi)p$ is the desired proper map. □

By Proposition 1.1.5, we may assume that each $X \in \mathcal{LK}(n+1)$ is embedded as a closed set into an AR-space $M_X \in \mathcal{LK}$ and an $LC^n \cap C^n$-space $M'_X \in \mathcal{LK}(n+1)$, and there is an $(n+1)$-invertible proper UV^n-surjection $\alpha_X : M'_X \to M_X$ such that $\alpha_X|_X = \mathrm{id}_X$.

Lemma 5.1.2 *Let $X, Y \in \mathcal{LK}(n+1)$. Then any proper n-fundamental net $\mathbf{f} = \{f_\lambda \mid \lambda \in \Lambda\} : X \to Y$ in (M_X, M_Y) in the sense of Chapter 4 induces a* *proper n-fundamental net* $\mathbf{f}' = \{f'_\lambda \mid \lambda \in \Lambda\} : X \to Y$ *in* (M'_X, M'_Y) *such that* $f_\lambda \alpha_X = \alpha_Y f'_\lambda$ *for each* $\lambda \in \Lambda$.

Proof. Since α_Y is $(n+1)$-invertible, for each map $f_\lambda : M_X \to M_Y$ there is a map $f'_\lambda : M'_X \to M'_Y$ such that $f_\lambda \alpha_X = \alpha_Y f'_\lambda$. We show that $\mathbf{f}' = \{f'_\lambda \mid \lambda \in \Lambda\} : X \to Y$ in (M'_X, M'_Y) is a proper n-fundamental net. Since α_Y is proper, for each closed neighborhood V' of Y in M'_Y there is a closed neighborhood V_1 of Y in M_Y such that $\alpha_Y^{-1}(\operatorname{int} V_1) \subset V'$. Note that $\alpha_Y^{-1}(\operatorname{int} V_1)$ is LC^n. Let $V \subset \operatorname{int} V_1$ be a closed neighborhood of Y in M_Y. Then there are a closed neighborhood U of X in M_X and an index $\lambda_0 \in \Lambda$ such that

$$f_\lambda|_U \simeq_p^n f_{\lambda_0}|_U \text{ in } V \text{ for all } \lambda \geq \lambda_0.$$

Let $U' = \alpha_X^{-1}(U)$ and fix $\lambda \geq \lambda_0$. Since $\alpha_Y f'_\lambda|_{U'} = f_\lambda \alpha_X|_{U'}$, we have $f'_\lambda(U') \subset \alpha_Y^{-1}(V) \subset \alpha_Y^{-1}(\operatorname{int} V_1) \subset V'$. Let $\beta' : Z \to U'$ be a proper map from $Z \in \mathcal{LK}(n)$ to U'. Since $f_\lambda|_U \simeq_p^n f_{\lambda_0}|_U$ in V,

$$f_\lambda|_U \alpha_X \beta' \simeq_p f_{\lambda_0}|_U \alpha_X \beta' \text{ in } V,$$

i.e., there is a proper homotopy $H : Z \times I \to \operatorname{int} V_1$ such that $H_0 = f_{\lambda_0}|_U \alpha_X \beta'$ and $H_1 = f_\lambda|_U \alpha_X \beta'$. Let $h : Z \times \{0,1\} \to \alpha_Y^{-1}(\operatorname{int} V_1)$ be the map defined by $h|_{Z\times\{0\}} = f'_{\lambda_0}|_{U'}\beta'$ and $h|_{Z\times\{1\}} = f'_\lambda|_{U'}\beta'$. Since $H|_{Z\times\{0,1\}} = \alpha_Y h$, by Proposition 5.1.1, there is a proper map $\tilde{h} : Z \times I \to \alpha_Y^{-1}(\operatorname{int} V)$ which is an extension of h. We conclude that $f'_\lambda|_{U'} \simeq_p^n f'_{\lambda_0}|_{U'}$ in $\alpha_Y^{-1}(\operatorname{int} V_1) \subset V'$. □

Lemma 5.1.3 *Let* $\mathbf{f} = \{f_\lambda \mid \lambda \in \Lambda\}, \mathbf{g} = \{g_\delta \mid \delta \in \Delta\} : X \to Y$ *in* (M_X, M_Y) *be proper n-fundamental nets and suppose* $\mathbf{f}, \mathbf{g}$ *induce proper n-fundamental nets* $\mathbf{f}' = \{f'_\lambda \mid \lambda \in \Lambda\}, \mathbf{g}' = \{g'_\delta \mid \delta \in \Delta\} : X \to Y$ *in* (M'_X, M'_Y) *such that* $f_\lambda \alpha_X = \alpha_Y f'_\lambda$ *and* $g_\delta \alpha_X = \alpha_Y g'_\delta$ *for each* $\lambda \in \Lambda$ *and* $\delta \in \Delta$. *Then* $\mathbf{f} \simeq_p^n \mathbf{g}$ *if and only if* $\mathbf{f}' \simeq_p^n \mathbf{g}'$.

Proof. Suppose that $\mathbf{f} \simeq_p^n \mathbf{g}$. Since α_Y is proper, for each closed neighborhood V' of Y in M'_Y there is a closed neighborhood V of Y in M_Y such that $\alpha_Y^{-1}(V) \subset$

V'. By the argument of Lemma 5.1.2, we may assume that $\alpha_Y^{-1}(V)$ is LC^n. Since $\mathbf{f} \simeq_p^n \mathbf{g}$, there are a closed neighborhood U of X in M_X and indices $\lambda_0 \in \Lambda$ and $\delta_0 \in \Delta$ such that

$$f_\lambda|_U \simeq_p^n g_\delta|_U \text{ in } V \text{ for all } \lambda \geq \lambda_0 \text{ and } \delta \geq \delta_0.$$

By replacing f_λ and f_{λ_0} to f_λ and g_δ in Lemma 5.1.2, we can conclude $\mathbf{f}' \simeq_p^n \mathbf{g}'$.

To prove the contrary, suppose that $\mathbf{f}' \simeq_p^n \mathbf{g}'$. Let V be a closed neighborhood of Y in M_Y. Since $\mathbf{f}' \simeq_p^n \mathbf{g}'$, for $V' = \alpha_Y^{-1}(V)$ there exist a closed neighborhood U' of X in M'_X and indices $\lambda_0 \in \Lambda$, $\delta_0 \in \Delta$ such that

$$f'_\lambda|_{U'} \simeq_p^n g'_\delta|_{U'} \text{ in } V' \text{ for all } \lambda \geq \lambda_0 \text{ and } \delta \geq \delta_0.$$

Since α_X is proper, there is a closed neighborhood U of X in M_X such that $\alpha_X^{-1}(U) \subset U'$. Let $\beta : Z \to U$ be a proper map from $Z \in \mathcal{LK}(n)$ to U. By the invertiblity of α_X, there is a proper map $\beta' : Z \to \alpha_X^{-1}(U)$ such that $\beta = \alpha_X \beta'$. Then

$$f_\lambda|_U \beta = f_\lambda|_U \alpha_X \beta' = \alpha_Y f'_\lambda|_{\alpha_X^{-1}(U)} \beta' \simeq_p \alpha_Y g'_\delta|_{\alpha_X^{-1}(U)} \beta' = g_\delta|_U \alpha_X \beta' = g_\delta|_U \beta \text{ in } V$$

for all $\lambda \geq \lambda_0$ and $\delta \geq \delta_0$, which implies $\mathbf{f} \simeq_p^n \mathbf{g}$. □

Theorem 5.1.4 *There is a categorical embedding* $\Phi : n\text{-SH}_p\mathcal{LK}(n+1) \to n\text{-SH}'_p\mathcal{LK}(n+1)$ *such that* $\Phi(X) = X$ *for each* $X \in \mathcal{LK}(n+1)$.

Proof. For a proper n-fundamental net $\mathbf{f} : X \to Y$ in (M_X, M_Y), we define $\Phi([\mathbf{f}]_p^n) = [\mathbf{f}']_p^n$, where $\mathbf{f}' : X \to Y$ in (M'_X, M'_Y) is induced in Lemma 5.1.2. By Lemmas 5.1.2 and 5.1.3, we may only prove that Φ is functorial, that is, $\Phi([\mathbf{g}]_p^n[\mathbf{f}]_p^n) = \Phi([\mathbf{f}]_p^n)\Phi([\mathbf{g}]_p^n)$ for each proper n-fundamental nets $\mathbf{f} : X \to Y$ in (M_X, M_Y) and $\mathbf{g} : Y \to Z$ in (M_Y, M_Z). Let $\mathbf{f}' : X \to Y$ in (M'_X, M'_Y) and $\mathbf{g}' : Y \to Z$ in (M'_Y, M'_Z) be proper n-fundamental nets induced from $\mathbf{f}$ and $\mathbf{g}$. For each $\lambda \in \Lambda$ and $\delta \in \Delta$, since $f_\lambda \alpha_X = \alpha_Y f'_\lambda$ and $g_\delta \alpha_Y = \alpha_Z g'_\delta$,

$$g_\delta f_\lambda \alpha_X = g_\delta \alpha_Y f'_\lambda = \alpha_Z g'_\delta f'_\lambda,$$

which means that $\mathbf{g}'\mathbf{f}' = \{g'_\delta f'_\lambda \mid (\lambda,\delta) \in \Lambda \times \Delta\}$ is induced from $\mathbf{gf} = \{g_\delta f_\lambda \mid (\lambda,\delta) \in \Lambda \times \Delta\}$. Therefore,

$$\Phi([\mathbf{g}]_p^n[\mathbf{f}]_p^n) = \Phi([\mathbf{gf}]_p^n) = [\mathbf{g}'\mathbf{f}']_p^n = [\mathbf{g}']_p^n[\mathbf{f}']_p^n = \Phi([\mathbf{g}]_p^n)\Phi([\mathbf{f}]_p^n). \ \square$$

5.2 The Freudenthal compactification and compact pairs

In this section, we recall the Freudenthal compactification and study the relation between proper maps and compact pairs.

Suppose that X is rim-compact (i.e., any point has arbitrary small neighborhoods with compact boundaries). *The Freudenthal compactification* of X, here denoted by FX, is defined as the least upper bound of all compactifications Y of X such that $\operatorname{ind}(Y \setminus X) = 0$. We call $EX = FX \setminus X$ *the space of ends* of X. It is known that FX is metrizable if and only if the space QX of quasi-components of X is compact, whence EX is homeomorphic to a closed set of the Cantor set.

Let $X, Y \in \Sigma$. Then each proper map $f : X \to Y$ has the unique extension $Ff : (FX, EX) \to (FY, EY)$. If $g : X \to Y$ is a proper map and $f \simeq_p^0 g$, then $Ff|_{EX} = Fg|_{EX}$ (cf. [Ba, Lemmas 2.3 and 2.7]). Also, the assignment $f \to Ff$ is functorial, that is, $F(\mathrm{id}_X) = \mathrm{id}_{FX}$ and $F(fg) = (Ff)(Fg)$ for proper maps $f : X \to Y$ and $g : Y \to Z$. For details, refer to [BS] and [DM].

Lemma 5.2.1 *Let $f, g : Z \to FY$ be maps from a compact space Z to FY and C a closed set in Z. Suppose that $f(Z \setminus C), g(Z \setminus C) \subset Y$ and $f(z) = g(z) \in EY$ for each $z \in C$. If $f|_{Z\setminus C} \simeq_p g|_{Z\setminus C}$ in Y, then $f \simeq g$ rel. C in FY.*

Proof. Let $H : (Z \setminus C) \times I \to Y$ be a proper homotopy such that $H_0 = f|_{Z\setminus C}$

and $H_1 = g|_{Z\setminus C}$. Define the homotopy $H' : Z \times I \to FY$ by

$$H'(z,t) = \begin{cases} H(z,t) & \text{for } z \in Z \setminus C, \\ f(z) = g(z) & \text{for } z \in C. \end{cases}$$

We prove that H' is continuous. Let $\{(z_i, t_i)\}_{i\in\omega}$ be a sequence in $(Z\setminus C)\times I$ such that $(z_i, t_i) \to (z_0, t_0) \in C \times I$ as $i \to \infty$. Let V be a neighborhood of $f(z_0) = g(z_0)$ in FY. Since $\dim EY = 0$, there exist open sets V_0, V_1 in FY such that $V_0 \subset V$, $EY \subset V_0 \cup V_1$ and $V_0 \cap V_1 = \emptyset$. Then $L = Y \setminus (V_0 \cup V_1) = FY \setminus (V_0 \cup V_1)$ is compact in Y, so $H^{-1}(L)$ is compact in $(Z \setminus C) \times I$. Let $U = Z \setminus pH^{-1}(L)$, where $p : Z \times I \to Z$ is a projection. Then U is a neighborhood of C and $H(U \times I) \subset V_0 \cup V_1$. Since $(z_i, t_i) \to (z_0, t_0)$, there exists $m_0 \in \omega$ such that $\{z_m\} \times I \subset U \times I$ for each $m \geq m_0$. Note that $H(\{z_m\} \times I) \subset H(U \times I) \subset V_0 \cup V_1$. By the continuity of f, $H(\{z_m\} \times 0) = f(z_m) \in V_0$. Then $H(\{z_m\} \times I) \subset V_0 \subset V$ since $V_0 \cap V_1 = \emptyset$. In particular, $H(z_m, t_m) \in V$ for each $m \geq m_0$, which implies that H' is continuous. □

Let $f, g : X \to Y$ be maps and $A \subset X$ a closed set. We denote $f \simeq^n g$ rel. A if $f\alpha \simeq g\alpha$ rel. $\alpha^{-1}(A)$ for any map $\alpha : Z \to X$ with $\dim Z \leq n$. To see $f \simeq^n g$ rel. A, it suffices to verify the condision in case that α is an n-invertible surjection. By using this notation, the following holds from Lemma 5.2.1.

Corollary 5.2.2 *Let $f, g : X \to Y$ be proper maps. Then $f \simeq^n_p g$ implies $Ff \simeq^n Fg$ rel. EX.* □

Remark. As is easily observed, the above is valid for maps between pairs.

We call (M, M_0) a *μ^{n+1}-manifold pair* if M and M_0 are μ^{n+1}-manifolds and M_0 is Z-set in M.

Lemma 5.2.3 *Let $f, g : U \to V$ be proper maps such that $f \simeq^n_p g$. Suppose that (FU, EU) and (FV, EV) are embedded in compact μ^{n+1}-manifold pairs (M, M_0) and (N, N_0), respectively, such that $EU = FU \cap M_0$ and $EV =$*

$FV \cap M_0$. If f has an extension $\tilde{f} : (M, M_0) \to (N, N_0)$ with $\tilde{f}^{-1}(EV) = EU$, then g has also an extension $\tilde{g} : (M, M_0) \to (N, N_0)$ such that $\tilde{g}^{-1}(EV) = EU$ and $\tilde{f} \simeq^n \tilde{g}$ rel. EU as maps between pairs.

Proof. By Theorem 1.2.6 and its remark, we can extend g to a proper map $\bar{g} : (M \setminus EU, M_0 \setminus EU) \to (N \setminus EV, N_0 \setminus EV)$ such that $\bar{g} \simeq^n_p \tilde{f}|_{(M\setminus EU, M_0 \setminus EU)}$. Since M_0 is Z-set in M, $F(M \setminus EU) = M$ by [No, Corollary 1]. Then we have $\tilde{g} = F(\bar{g}) : (M, M_0) \to (N, N_0)$ which has the desired property by Corollary 5.2.2. □

Remark. In the above, $g \cup \tilde{f}|_{M_0} : U \cup M_0 \to V \cup N_0$ is a map which is n-homotopic to $\tilde{f}|_{U \cup M_0}$ rel. M_0. Then we can obtain $\tilde{g}$ satisfying $\tilde{g}|_{M_0} = \tilde{f}|_{M_0}$.

Lemma 5.2.4 *Let $f, g : (M, M_0) \to (Y, Y_0)$ be maps from a compact pair (M, M_0) to an LC^n-pair (Y, Y_0) and $A \subset X$ be closed sets in M and $X_0 = X \cap M_0$. If $f|_{(X,X_0)} \simeq^n g|_{(X,X_0)}$ rel. A, then there exists a neighborhood pair (U, U_0) of (X, X_0) in (M, M_0) such that $f|_{(U,U_0)} \simeq^n g|_{(U,U_0)}$ rel. A.*

Proof. Let $\alpha : (Z, Z_0) \to (M, M_0)$ be an n-invertible UV^n-surjection from an n-dimensional compact pair (Z, Z_0) to (M, M_0). By the assumption, there exists a homotopy $H : (\alpha^{-1}(X), \alpha^{-1}(X_0)) \times I \to (Y, Y_0)$ such that $H_0 = f\alpha|_{(\alpha^{-1}(X), \alpha^{-1}(X_0))}$, $H_1 = g\alpha|_{(\alpha^{-1}(X), \alpha^{-1}(X_0))}$ and $H_t|_{\alpha^{-1}(A)} = f\alpha|_{\alpha^{-1}(A)}$ for all $t \in I$. By 1.2.2, there exists a neighborhood pair (W, W_0) of $(Z, Z_0) \times \{0, 1\} \cup (\alpha^{-1}(X), \alpha^{-1}(X_0)) \times I$ in $(Z, Z_0) \times I$ and an extension $H' : (W, W_0) \to (Y, Y_0)$ of H. Since α is proper, we can find a neighborhood pair (U, U_0) of (X, X_0) such that $(\alpha^{-1}(U), \alpha^{-1}(U_0)) \times I \subset (W, W_0)$, which implies that $f|_{(U,U_0)} \simeq^n g|_{(U,U_0)}$ rel. A. □

5.3 Proper n-shape and relative n-shape for compact pairs

Let $\mathcal{K}^2$ be the class of compact pairs. In this section, we define the relative n-shape category for compact pairs with dim $\leq n+1$, n-$\mathrm{SH}_{rel}\mathcal{K}^2(n+1)$, which is different from Chapter 3 and we construct a functor $\Psi : n\text{-}\mathrm{SH}'_p\Sigma(n+1) \to n\text{-}\mathrm{SH}_{rel}\mathcal{K}^2(n+1)$.

Let (X, X_0) be a Z-pair in (μ^{n+1}, μ_0^{n+1}) and let $(\mathbf{X}, \mathbf{X_0}) = \{(X_i, X_{0i}), p_i^{i+1}\}$ be an $LC^n(n+1)$-sequence associated with (X, X_0), where each (X_i, X_{0i}) is a compact μ^{n+1}-manifold pair which is a closed neighborhood of (X, X_0), and bonding maps are inclusions (Proposition 3.1.1). For convenience sake, we assume that $(X_1, X_{01}) = (\mu^{n+1}, \mu_0^{n+1})$. We call such an $LC^n(n+1)$-sequence *an inclusion sequence* associated with (X, X_0). Let (Y, Y_0) be a Z-pair in (μ^{n+1}, μ_0^{n+1}) and $(\mathbf{Y}, \mathbf{Y_0}) = \{(Y_i, Y_{0i}), q_i^{i+1}\}$ be an inclusion sequence associated with (Y, Y_0). An n-morphism $\mathbf{h} = (h, \{h_i\}) : (\mathbf{X}, \mathbf{X_0}) \to (\mathbf{Y}, \mathbf{Y_0})$ is said to be *a relative n-morphism* if $h_i p_{h(i)}^{h(j)} \simeq^n q_i^j h_j$ rel. X_0 for any i, j with $j \geq i$, and we denote $\mathbf{h} : (\mathbf{X}, \mathbf{X_0}) \to (\mathbf{Y}, \mathbf{Y_0})$ rel. X_0. Two relative n-morphisms $\mathbf{g}, \mathbf{h} : (\mathbf{X}, \mathbf{X_0}) \to (\mathbf{Y}, \mathbf{Y_0})$ rel. X_0 are *relative n-homotopic* ($\mathbf{g} \simeq^n \mathbf{h}$ rel. X_0) if for each $i \in \omega$ there is an index $j \geq g(i), h(i)$ such that $g_i|_{(X_j, X_{0j})} \simeq^n h_i|_{(X_j, X_{0j})}$ rel. X_0. By Lemma 5.2.4, $\mathbf{g} \simeq^n \mathbf{h}$ rel. X_0 if and only if $g_i|_X \simeq^n h_i|_X$ rel. X_0. The class of relative n-homotopy of the relative n-morphism $\mathbf{h}$ is denoted by $[\mathbf{h}]_{rel}^n$. *The relative n-shape category for compact pairs* n-$\mathrm{SH}_{rel}\mathcal{K}^2(n+1)$ is defined as a category whose objects are in $\mathcal{K}^2(n+1)$ and whose morphisms are the relative n-homotopy classes of relative n-morphisms. If there exist two relative n-morphisms $\mathbf{f} : (\mathbf{X}, \mathbf{X_0}) \to (\mathbf{Y}, \mathbf{Y_0})$ rel. X_0 and $\mathbf{g} : (\mathbf{Y}, \mathbf{Y_0}) \to (\mathbf{X}, \mathbf{X_0})$ rel. Y_0 such that $\mathbf{gf} \simeq^n \mathbf{i}_{(\mathbf{X},\mathbf{X_0})}$ rel. X_0 and $\mathbf{fg} \simeq^n \mathbf{i}_{(\mathbf{Y},\mathbf{Y_0})}$ rel. Y_0, we denote n-$\mathrm{Sh}(X, X_0) = n$-$\mathrm{Sh}(Y, Y_0)$ rel. (X_0, Y_0). The relative n-shape for compact pairs is stronger that the n-shape for compact pairs in the sense of Chapter 3, that is, n-$\mathrm{Sh}(X, X_0) = n$-$\mathrm{Sh}(Y, Y_0)$ rel. (X_0, Y_0) implies n-$\mathrm{Sh}(X, X_0) = n$-$\mathrm{Sh}(Y, Y_0)$.

Suppose that FX and FY are Z-sets in μ^{n+1}, $EX = FX \cap \mu_0^{n+1}$, $EY = FY \cap \mu_0^{n+1}$ and $M = \mu^{n+1} \setminus \mu_0^{n+1}$, $(\mathbf{X}, \mathbf{X_0}) = \{(X_i, X_{0i}), p_i^{i+1}\}$ and $(\mathbf{Y}, \mathbf{Y_0}) = \{(Y_i, Y_{0i}), q_i^{i+1}\}$ are inclusion sequences associated with (FX, EX) and (FY, EY), respectively. For each proper n-fundamental net $\mathbf{f} = \{f_\lambda \mid \lambda \in \Lambda\} : X \to Y$ in (M, M), we construct a relative n-morphism $\hat{\mathbf{f}} = (f, \{\hat{f}\}) : (\mathbf{X}, \mathbf{X_0}) \to (\mathbf{Y}, \mathbf{Y_0})$ rel. EX as follows:

Let $V_1 \supset V_2 \supset \cdots$ be closed neighborhoods of Y in M such that $Y = \bigcap_{i \in \omega} V_i$, $EV_i = EY$ and $FV_i \subset \operatorname{int} Y_i$. Choose closed neighborhoods $U_1^f \supset U_2^f \supset \cdots$ of X and indices $\lambda_1 \leq \lambda_2 \leq \cdots$ such that $X = \bigcap_{i \in \omega} U_i^f$, $EU_i^f = EX$ and

$$f_\lambda|_{U_i^f} \simeq_p^n f_{\lambda_i}|_{U_i^f} \text{ in } V_i \text{ for all } \lambda \geq \lambda_i.$$

Since (Y_1, Y_{01}) is an $LC^n \cap C^n$-pair, $F(f_{\lambda_1}|_{U_1^f}) : (FU_1^f, EU_1^f) \to (FV_1, EV_1) \subset (Y_1, Y_{01})$ has an extension $\hat{f}_1 : (X_1, X_{01}) \to (Y_1, Y_{01})$ by Lemma 1.2.1. Since $f_{\lambda_2}|_{U_2^f} \simeq_p^n f_{\lambda_1}|_{U_2^f}$ in V_1 and $f_{\lambda_1}|_{U_2^f}$ has an extension $\hat{f}_1$ with $EU_1^f = \hat{f}_1^{-1}(EV_1)$, by Lemma 5.2.3, $f_{\lambda_2}|_{U_2^f}$ has an extension $\tilde{f}_2 : (X_1, X_{01}) \to (Y_1, Y_{01})$ such that $\tilde{f}_2 \simeq^n \hat{f}_1$ in (V_1, V_{01}) rel. $EU_2^f = EX$. Note that $\tilde{f}_2(FU_2^f) \subset FV_2 \subset \operatorname{int} Y_2$. Then there exists a μ^{n+1}-manifold neighborhood pair $(\bar{U}_2^f, \bar{U}_{02}^f)$ of (FU_2^f, EU_2^f) such that $\tilde{f}_2(\bar{U}_2^f, \bar{U}_{02}^f) \subset (Y_2, Y_{02})$. Take $f(2) \in \omega$ such that $(X_{f(2)}, X_{0f(2)}) \subset (\bar{U}_2^f, \bar{U}_{02}^f)$ and let $\hat{f}_2 = \tilde{f}_2|_{(X_{f(2)}, X_{0f(2)})} : (X_{f(2)}, X_{0f(2)}) \to (Y_2, Y_{02})$. Observe that $\hat{f}_2 \simeq^n \hat{f}_1|_{(X_{f(2)}, X_{0f(2)})}$ in (Y_1, Y_{01}) rel. EX. Assume that we obtained an extension $\tilde{f}_i : (\bar{U}_i^f, \bar{U}_{0i}^f) \to (Y_i, Y_{0i})$ of $f_{\lambda_i}|_{U_i^f}$ for $i \geq 2$ such that $(\bar{U}_i^f, \bar{U}_{0i}^f)$ is a μ^{n+1}-manifold neighborhood pair of (FU_i^f, EU_i^f) and $\tilde{f}_i \simeq \tilde{f}_{i-1}|_{(\bar{U}_i^f, \bar{U}_{0i}^f)}$ in (Y_i, Y_{0i}) rel. EX. Since $f_{\lambda_{i+1}}|_{U_{i+1}^f} \simeq_p^n f_{\lambda_i}|_{U_{i+1}^f}$ in V_i and $f_{\lambda_i}|_{U_{i+1}^f}$ has an extension $\tilde{f}_i$ with $EU_i^f = \tilde{f}_i^{-1}(EV_i)$, by Lemma 5.2.3, $f_{\lambda_{i+1}}|_{U_{i+1}^f}$ has an extension $\tilde{f}_{i+1} : (\bar{U}_i^f, \bar{U}_{0i}^f) \to (Y_i, Y_{0i})$ such that $\tilde{f}_{i+1} \simeq^n \tilde{f}_i$ in (V_i, V_{0i}) rel. EX. Note that $\tilde{f}_{i+1}(FU_{i+1}^f) \subset FV_{i+1} \subset \operatorname{int} Y_{i+1}$. Then there exists a μ^{n+1}-manifold neighborhood pair $(\bar{U}_{i+1}^f, \bar{U}_{0i+1}^f)$ of (FU_{i+1}^f, EU_{i+1}^f) such that $\tilde{f}_{i+1}(\bar{U}_{i+1}^f, \bar{U}_{0i+1}^f) \subset (Y_{i+1}, Y_{0i+1})$. Take $f(i+1) \in \omega$ such that $(X_{f(i+1)}, X_{0f(i+1)}) \subset (\bar{U}_{i+1}^f, \bar{U}_{0i+1}^f)$ and let $\hat{f}_{i+1} = \tilde{f}_{i+1}|_{(X_{i+1}, X_{0i+1})} : (X_{f(i+1)}, X_{0f(i+1)}) \to (Y_{i+1}, Y_{0i+1})$. Observe that $\hat{f}_{i+1} \simeq^n \hat{f}_i|_{(X_{f(i+1)}, X_{0f(i+1)})}$ in (Y_i, Y_{0i}) rel. EX. By the induction, we have a sequence $\hat{\mathbf{f}} = (f, \{\hat{f}_i\}) : (\mathbf{X}, \mathbf{X_0}) \to (\mathbf{Y}, \mathbf{Y_0})$. It is easy to see that $\hat{\mathbf{f}}$ is a relative n-morphism. In fact, for each $j \geq i$,

$$\begin{aligned} q_i^j \hat{f}_j = \hat{f}_j \quad &\simeq^n \quad \hat{f}_{j-1}|_{(X_{f(j)}, X_{0f(j)})} \text{ in } (Y_{j-1}, Y_{0j-1}) \text{ rel. } EX \\ &\simeq^n \quad \hat{f}_{j-2}|_{(X_{f(j)}, X_{0f(j)})} \text{ in } (Y_{j-2}, Y_{0j-2}) \text{ rel. } EX \\ &\dots \\ &\simeq^n \quad \hat{f}_i|_{(X_{f(j)}, X_{0f(j)})} = \hat{f}_{f(j)} p_{f(i)}^{f(j)} \text{ in } (Y_i, Y_{0i}) \text{ rel. } EX \end{aligned}$$

The above construction of $\hat{\mathbf{f}}$ is denoted by $\mathbf{f} \Rightarrow \{V_i, U_i^f, \lambda_i, \tilde{f}_i, \bar{U}_i^f, f(i)\}(\hat{\mathbf{f}})$.

Theorem 5.3.1 *There is a functor* $\Psi : n\text{-SH}'_p\Sigma(n+1) \to n\text{-SH}_{rel}\mathcal{K}^2(n+1)$ *such that* $\Psi(X) = (FX, EX)$ *for each* $X \in \Sigma(n+1)$.

Proof. Let $\mathbf{f} = \{f_\lambda \mid \lambda \in \Lambda\} : X \to Y$ in (M, M) be a proper n-fundamental net. By the construction $\mathbf{f} \Rightarrow \{V_i, U_i^f, \lambda_i, \tilde{f}_i, \bar{U}_i^f, f(i)\}(\hat{\mathbf{f}})$, we have a relative n-morphism $\hat{\mathbf{f}} : (\mathbf{X}, \mathbf{X_0}) \to (\mathbf{Y}, \mathbf{Y_0})$ rel. EX. We define $\Psi([\mathbf{f}]_p^n) = [\hat{\mathbf{f}}]_{rel}^n$.

First, we prove that Ψ is well-defined, that is, if $\mathbf{f} \simeq_p^n \mathbf{g}$ for another proper n-fundamental net $\mathbf{g} = \{g_\delta \mid \delta \in \Delta\} : X \to Y$ in (M, M), then $\hat{\mathbf{f}} \simeq^n \hat{\mathbf{g}}$ rel. EX. By the construction $\mathbf{g} \Rightarrow \{V_i', U_i^g, \delta_i, \tilde{g}_i, \bar{U}_i^g, g(i)\}(\hat{\mathbf{g}})$, we have a relative n-morphism $\hat{\mathbf{g}} = (g, \{\hat{g}_i\}) : (\mathbf{X}, \mathbf{X_0}) \to (\mathbf{Y}, \mathbf{Y_0})$ rel. EX such that $\hat{g}_i = \tilde{g}_i|_{(X_{g(i)}, X_{0g(i)})}$ and $\tilde{g}_i|_{U_i^g} = g_{\delta_i}|_{U_i^g}$. Since $\mathbf{f} \simeq_p^n \mathbf{g}$, there exist a closed neighborhood W of X in M, $\lambda_0 \in \Lambda$ and $\delta_0 \in \Delta$ such that

$$f_\lambda|_W \simeq_p^n g_\delta|_W \text{ in } V_i \cup V_i' \text{ for all } \lambda \geq \lambda_0 \text{ and } \delta \geq \delta_0.$$

Let $W' = W \cap U_i^f \cap U_i^g$. For each $\lambda \geq \lambda_0, \lambda_i$ and $\delta \geq \delta_0, \delta_i$, since

$$f_\lambda|_{W'} \simeq_p^n f_{\lambda_i}|_{W'} \text{ in } V_i \text{ and } g_\delta|_{W'} \simeq_p^n g_{\delta_i}|_{W'} \text{ in } V_i',$$

we have

$$f_{\lambda_i}|_{W'} \simeq_p^n f_\lambda|_{W'} \simeq_p^n g_\delta|_{W'} \simeq_p^n g_{\delta_i}|_{W'} \text{ in } V_i \cup V_i'.$$

Since $\hat{f}_i$ and $\hat{g}_i$ are extensions of $f_{\lambda_i}|_{W'}$ and $g_{\delta_i}|_{W'}$, by Corollary 5.2.2,

$$\hat{f}_i|_{FX} \simeq^n \hat{g}_i|_{FX} \text{ rel. } EX \text{ in } Y_i,$$

which implies $\hat{\mathbf{f}} \simeq^n \hat{\mathbf{g}}$ rel. EX. Therfore, Ψ is well-defined.

Next, we show that Ψ is functorial. Let $\mathbf{f} : X \to Y$ in (M, M) and $\mathbf{g} : Y \to Z$ in (M, M) be proper n-fundamental nets and $(\mathbf{Z}, \mathbf{Z_0}) = \{(Z_i, Z_{0i}), r_i^{i+1}\}$ be an inclusion sequence associated with (FZ, EZ). By the constructions $\mathbf{f} \Rightarrow \{V_i, U_i^f, \lambda_i, \tilde{f}_i, \bar{U}_i^f, f(i)\}(\hat{\mathbf{f}})$ and $\mathbf{g} \Rightarrow \{W_i, V_i^f, \delta_i, \tilde{g}_i, \bar{V}_i^g, g(i)\}(\hat{\mathbf{g}})$, we have relative n-morphisms $\hat{\mathbf{f}} : (\mathbf{X}, \mathbf{X_0}) \to (\mathbf{Y}, \mathbf{Y_0})$ rel. EX and $\hat{\mathbf{g}} : (\mathbf{Y}, \mathbf{Y_0}) \to (\mathbf{Z}, \mathbf{Z_0})$ rel. EY. Observe that $V_{g(i)} \subset Y_{g(i)} \subset \bar{V}_i^g$. Since $\mathbf{gf} = \{g_\delta f_\lambda \mid (\lambda, \delta) \in \Lambda \times \Delta\} : X \to Z$ in (M, M) and $\hat{\mathbf{g}}\hat{\mathbf{f}} = (fg, \{\hat{g}_i \hat{f}_{g(i)}\}) : (\mathbf{X}, \mathbf{X_0}) \to (\mathbf{Z}, \mathbf{Z_0})$, it is easy to verify $\mathbf{gf} \Rightarrow \{W_i, U_{g(i)}^f, (\delta_i, \lambda_{g(i)}), \tilde{g}_i \tilde{f}_{g(i)}, \bar{U}_{g(i)}^f, fg(i)\}(\hat{\mathbf{g}}\hat{\mathbf{f}})$, hence

$$\Psi([\mathbf{gf}]_p^n) = [\hat{\mathbf{g}}\hat{\mathbf{f}}]_{rel}^n = [\hat{\mathbf{g}}]_{rel}^n [\hat{\mathbf{f}}]_{rel}^n = \Psi([\mathbf{g}]_p^n)\Psi([\mathbf{f}]_p^n). \ \square$$

Combining Theorems 5.1.4 and 5.3.1, we have the following.

Corollary 5.3.2 *There is a functor* $\Theta : n\text{-SH}_p\Sigma(n+1) \to n\text{-SH}_{rel}\mathcal{K}^2(n+1)$ *such that* $\Theta(X) = (FX, EX)$ *for each* $X \in \Sigma(n+1)$. □

As a direct consequence of the above, we have

Corollary 5.3.3 *For* $X, Y \in \Sigma(n+1)$, *if* $n\text{-Sh}_p(X) = n\text{-Sh}_p(Y)$ *then* $n\text{-Sh}(FX, EX) = n\text{-Sh}(FY, EY)$ *rel.* (EX, EY), *hence* $n\text{-Sh}(FX, EX) = n\text{-Sh}(FY, EY)$ *in the sense of Chapter* 3 *and* $EX \approx EY$. □

Corollary 5.3.4 *If* X *is connected* SUV^n *and* $\dim X \leq n+1$, *then* $FX \in UV^n$.

Proof. By Theorem 4.2.1, there exists a tree T such that $n\text{-Sh}_p(X) = n\text{-Sh}_p(T)$. By Corollary 5.3.3, $n\text{-Sh}(FX, EX) = n\text{-Sh}(FT, ET)$ rel. (EX, ET). In particular, $n\text{-Sh}(FX) = n\text{-Sh}(FT)$. Since FT is contractible, $n\text{-Sh}(FX) = n\text{-Sh}(1)$, that is, $FX \in UV^n$ (cf. [Chi$_1$, Proposition 3.1]). □

5.4 Proper n-shape and the one-point compactification

In this section, we consider the relation of proper n-shape and the one-point compactification.

Let $CX = X \cup \{\infty\}$ be *the one-point compactification* of $X \in \mathcal{LK}$. It is known that CX is metrizable. For a proper map $f : X \to Y$ between $X, Y \in \mathcal{LK}$, there is an unique extension $Cf : (CX, \{\infty\}) \to (CY, \{\infty\})$. If $f, g : X \to Y$ are proper maps and $f \simeq_p g$, then $Cf \simeq Cg$ rel. $\{\infty\}$. It is easy to see that the arguments of Section 5.3 are valid for the pairs $(CX, \{\infty\})$ and the maps Cf. By replacing FX by CX and EX by $\{\infty\}$ in the proof of Theorem 5.3.1, we can obtain the following.

Theorem 5.4.1 *There is a functor* $\Psi' : n\text{-SH}'_p\mathcal{LK}(n+1) \to n\text{-SH}_{rel}\mathcal{K}^2(n+1)$ *such that* $\Psi'(X) = (CX, \{\infty\})$ *for each* $X \in \mathcal{LK}(n+1)$. □

Combining Theorems 5.1.4 and 5.4.1, we have the following.

Corollary 5.4.2 *There is a functor* $\Theta' : n\text{-SH}_p\mathcal{LK}(n+1) \to n\text{-SH}_{rel}\mathcal{K}^2(n+1)$ *such that* $\Theta'(X) = (CX, \{\infty\})$ *for each* $X \in \mathcal{LK}(n+1)$. □

As a direct consequence of the above, we have

Corollary 5.4.3 *For* $X, Y \in \mathcal{LK}(n+1)$, *if* $n\text{-Sh}_p(X) = n\text{-Sh}_p(Y)$ *then* $n\text{-Sh}(CX, \{\infty\}) = n\text{-Sh}(CY, \{\infty\})$ *rel.* $(\{\infty\}, \{\infty\})$, *hence pointed compacta* $(CX, \{\infty\})$ *and* $(CY, \{\infty\})$ *have the same pointed* n*-shape.* □

Corollary 5.4.4 *Let* $\mu^{n+1}_\infty = \mu^{n+1} \setminus \{*\}$, *where* $* \in \mu^{n+1}$ *and let* $X, Y \subset \mu^{n+1}_\infty$ *be* Z*-sets. If* $n\text{-Sh}_p(X) = n\text{-Sh}_p(Y)$ *then* $\mu^{n+1}_\infty \setminus X \approx \mu^{n+1}_\infty \setminus Y$.

Proof. We can assume that $CX, CY \subset \mu^{n+1}$ as Z-sets with $\{*\} = \{\infty\}$. By Corollary 5.4.3, we have $n\text{-Sh}(CX, \{\infty\}) = n\text{-Sh}(CY, \{\infty\})$ rel. $(\{\infty\}, \{\infty\})$. In particular, $n\text{-Sh}(CX) = n\text{-Sh}(CY)$. By the complement theorem, $\mu^{n+1}_\infty \setminus X = \mu^{n+1} \setminus CX \approx \mu^{n+1} \setminus CY = \mu^{n+1}_\infty \setminus Y$. □

Chapter 6

Homeomorphic neighborhoods in μ^{n+1}-manifolds

In this chapter, we prove a μ^{n+1}-manifold version of the result of [Sh_2], that is, *if X and Y are Z-sets in μ^{n+1}-manifolds M and N respectively, and n-$\mathrm{Sh}'_p(X) = n$-$\mathrm{Sh}'_p(Y)$, then X and Y have arbitrarily small homeomorphic μ^{n+1}-manifold closed neighborhoods.* Here, we use the notion of proper n-shape for $(n+1)$-dimensional locally compact spaces by using embeddings of them into $(n+1)$-dimensional locally compact $LC^n \cap C^n$-spaces (see Section 5.1). As a corollary, if X is a connected Z-set in a μ^{n+1}-manifold and $X \in SUV^n$, then there exists a tree T such that X has arbitrarily small closed neighborhoods homeomorphic to the Δ_{n+1}-product $T\Delta_{n+1}\mu^{n+1}$ of T and μ^{n+1}. Here, the Δ_{n+1}-product is defined in [Iwa] which plays the rule of the Cartesian product in the category of μ^{n+1}-manifolds. For a locally finite polyhedron P, $P\Delta_{n+1}\mu^{n+1}$ is the μ^{n+1}-manifold having the same proper n-homotopy type of P.

6.1 Homeomorphic neighborhoods

We denote $\mu^{n+1}_\infty = \mu^{n+1} \setminus \{*\}$, where $* \in \mu^{n+1}$. Recall that two proper maps $f, g : X \to Y$ are *properly n-homotopic* (written by $f \simeq^n_p g$) if, for any proper map $\alpha : Z \to X$ from a space Z with $\dim Z \leq n$ into X, the compositions $f\alpha$ and $g\alpha$ are properly homotopic in the usual sense ($f\alpha \simeq_p g\alpha$). A μ^{n+1}-manifold M lying in a μ^{n+1}-manifold N is said to be *n-clean* in N [Chi$_5$] if M is closed in N and there exists a closed μ^{n+1}-manifold $\delta(M)$ in M such that

(i) $(N \setminus M) \cup \delta(M)$ is a μ^{n+1}-manifold;

(ii) $\delta(M)$ is a Z-set in both M and $(N \setminus M) \cup \delta(M)$; and

(iii) $M \setminus \delta(M)$ is open in N.

Remark 6.1.1 Let P be a PL-manifold and L a submanifold in P such that $\mathrm{Bd}\, L = \mathrm{Bd}\,(P \setminus L)$. By [Chi$_4$, Theorem 1.6], there exists a proper UV^n-surjection $f : N \to P$ from a μ^{n+1}-manifold N satisfying the following conditions:

(a) $f^{-1}(K)$ is a μ^{n+1}-manifold for any closed subpolyhedron K of P; and

(b) $f^{-1}(Z)$ is a Z-set in $f^{-1}(K)$ for any Z-set Z in a closed subpolyhedron K of P.

Then it is easy to see that $M = f^{-1}(L)$ is an n-clean submanifold of N with $\delta(M) = f^{-1}(\mathrm{Bd}\, L)$.

Lemma 6.1.2 *Let Y be closed in a locally compact $C^n \cap LC^n$-space N. Assume that $r : V_0 \to Y$ is a proper retraction of a closed neighborhood V_0 of Y in N. Then for each closed neighborhood V of Y in N there exists a closed neighborhood V' of Y in N such that $V' \subset V \cap V_0$ and $\mathrm{id}_{V'} \simeq^n_p r|_{V'}$ in V.*

Proof. Let $\mathcal{W}$ be an open cover of $V \cap V_0$ such that if one of any two $\mathcal{W}$-close maps from arbitrary locally compact space is proper, then the other is also proper. Since $\mathrm{int}\,(V \cap V_0)$ is LC^n, there exists an open cover $\mathcal{U}$ of $\mathrm{int}\,(V \cap V_0)$ such that any two $\mathcal{U}$-close maps from a space with $\dim \leq n$ to $\mathrm{int}\,(V \cap V_0)$ are $\mathcal{W}$-homotopic. By the continuity of r, for any $U \in \mathcal{U} \cap Y = \{U \in \mathcal{U} \mid U \cap Y \neq$

$\emptyset\}$ and $x \in U \cap Y$ there exists a closed neighborhood $\bar{V}_x$ of x in N such that $\bar{V}_x \subset U$ and $r(\bar{V}_x) \subset U \cap Y$. Since Y is locally compact, $\{\bar{V}_x \mid x \in Y\}$ has a locally finite refinement $\mathcal{V}'$. Then $V' = \bigcup \mathcal{V}'$ is the desired neighborhood. □

Let X and Y be closed sets in locally compact $C^n \cap LC^n$-spaces M and N respectively. Recall that a proper n-fundamental net $\mathbf{f} = \{f_\lambda \mid \lambda \in \Lambda\} : X \to Y$ in (M, N) is *generated by* a proper map $f : X \to Y$ (or f *generates* $\mathbf{f}$) provided $f = f_\lambda|_X$ for all $\lambda \in \Lambda$. The proper n-homotopy class $[\mathbf{f}]$ of $\mathbf{f}$ is *generated by* f if f generates some $\mathbf{f}' \in [\mathbf{f}]$.

Proposition 6.1.3 *If Y is a locally compact LC^n-space, then the proper n-homotopy class of any proper n-fundamental net $\mathbf{f} : X \to Y$ in (M, N) is generated by a proper map $f : X \to Y$.*

Proof. Since Y is LC^n, there exist a closed neighborhood V_0 of Y in N and a proper retraction $r : V_0 \to Y$ by [BS, Lemma 3.2]. By Lemma 6.1.2, for each closed neighborhood V of Y in N there exists a closed neighborhood V' of Y in N such that $r|_{V'} \simeq^n_p \mathrm{id}_{V'}$ in V. Then there exist a closed neighborhood U' of X in M and $\lambda_0 \in \Lambda$ such that $f_\lambda|_{U'} \simeq^n_p f_{\lambda_0}|_{U'}$ in V' for all $\lambda \geq \lambda_0$. Let $r' : N \to N$ be an extension of r and $f'_\lambda = r' f_{\lambda_0}$. Note that $\mathbf{f}' = \{f'_\lambda\}$ is generated by $f = r f_{\lambda_0}|_X$, i.e., $f'_\lambda|_X = f$. Since $f'_\lambda|_{U'} = r' f_{\lambda_0}|_{U'} \simeq^n_p f_{\lambda_0}|_{U'} \simeq^n_p f_\lambda|_{U'}$ in V for all $\lambda \geq \lambda_0$, $\mathbf{f}' \simeq^n_p \mathbf{f}$. □

Lemma 6.1.4 *Let X be a Z-set in a μ^{n+1}-manifold M. Then there exists a closed embedding $F : M \to \mu^{n+1}_\infty$ such that $F(M)$ is a neighborhood of $F(X)$ in μ^{n+1}_∞ and $F(M)$ is n-clean in μ^{n+1}_∞ with $\delta(F(M)) \approx M$.*

Proof. By [Chi$_1$, Theorem 9], there exists a proper $(n+1)$-invertible UV^n-surjection $f : M \to P$ from M to a locally finite $(n+1)$-dimensional polyhedron P. We can assume that P is a closed subpolyhedron in $(I^{2(n+1)+1} \times \{0\}) \setminus \{*\}$, where $* = (0, \cdots, 0) \in I^{2(n+1)+2}$. Then there exists a proper $(n+1)$-invertible UV^n-surjection $f : N \to I^{2(n+1)+2} \setminus \{*\}$ from a μ^{n+1}-manifold N as same as

Remark 6.1.1. Since $I^{2(n+1)+2} \setminus \{*\} \simeq_p^n \mu_\infty^{n+1}$ and f is proper UV^n, we have $N \approx \mu_\infty^{n+1}$ (cf. [Chi$_4$, Theorem 1.3]). Let $N(P)$ be a regular neighborhood of P. By Remark 6.1.1, $M' = f^{-1}(N(P))$ and $M'' = f^{-1}(\mathrm{Bd}\, N(P))$ are μ^{n+1}-manifolds in N and M' is n-clean with $\delta(M') = M''$. Since P is a Z-set in $N(P)$, it follows that $N(P) \simeq_p^n \mathrm{Bd}N(P)$. By [Chi$_4$, Theorem 1.4], there exist homeomorphisms $g : M \to M'$ and $g' : M \to M''$. By the Z-set unknotting theorem, there exists a homeomorphism $h : M' \to M'$ such that $h(g(X)) \cap M'' = \emptyset$. Then $F = hg$ is the desired closed embedding. □

Theorem 6.1.5 *Let X and Y are Z-sets in μ^{n+1}-manifolds M and N respectively, such that $n\text{-}\mathrm{Sh}_p'(X) \leq n\text{-}\mathrm{Sh}_p'(Y)$. Then, for each neighborhood U of X in M and each neighborhood V of Y in N, there exists an open neighborhood V' of Y such that for every μ^{n+1}-manifold closed neighborhood S of Y in $V' \cap V$, there exists a μ^{n+1}-manifold closed neighborhood R of X in U which is homeomorphic to S.*

Proof. By Lemma 6.1.4, we can assume that $M = N = \mu_\infty^{n+1}$ and U is n-clean. Let $\mathbf{f} : X \to Y$ in $(\mu_\infty^{n+1}, \mu_\infty^{n+1})$ and $\mathbf{g} = \{g_\delta \mid \delta \in \Delta\} : Y \to X$ in $(\mu_\infty^{n+1}, \mu_\infty^{n+1})$ be proper n-fundamental nets such that $\mathbf{gf} \simeq_p^n \mathbf{i}_X$. Let U' be a closed neighborhood of X such that $U' \subset \mathrm{int}\, U$. Then there exist $\delta_0 \in \Delta$, $\lambda_0 \in \Lambda$ and a closed neighborhood W of Y with $W \subset V$ such that $g_\delta f_\lambda|_X \simeq_p^n \mathrm{id}_X$ in U', $g_\delta|_W \simeq_p^n g_{\delta_0}|_W$ in U' for each $\delta \geq \delta_0, \lambda \geq \lambda_0$. By the Z-set approximation theorem, there exists a Z-embedding $g'_{\delta_0} : W \to \mathrm{int}\, U$ approximating g_{δ_0}. Note that $g'_{\delta_0}|_Y$ is properly n-homotopic to the inclusion in μ_∞^{n+1}. By the Z-set unknotting theorem, there exists a homeomorphism $h : \mu_\infty^{n+1} \to \mu_\infty^{n+1}$ such that $hg'_{\delta_0}|_Y = \mathrm{id}_Y$. Let $V' = h(\mathrm{int}\, U)$ and $S \subset V \cap V'$ be a closed μ^{n+1}-manifold neighborhood of Y. Then $S' = h^{-1}(S)$ is a μ^{n+1}-manifold closed neighborhood of $g'_{\delta_0}(Y)$ lying in $\mathrm{int}\, U$. Let W' be a closed neighborhood of Y lying in $\mathrm{int}\, S$ so that $g'_{\delta_0}(W') \subset \mathrm{int}\, S'$. Then there exists $\lambda \geq \lambda_0$ such that $f_\lambda(X) \subset W'$. By the Z-set approximation theorem, we can assume that f_λ is a Z-embedding.

Note that $g'_{\delta_0} f_\lambda(X) \subset g'_{\delta_0}(W') \subset \operatorname{int} S'$ and $g'_{\delta_0} f_\lambda|_X \simeq^n_p g_{\delta_0} f_\lambda|_X \simeq^n_p \mathrm{id}_X$ in $U' \subset \operatorname{int} U$. By the Z-set unknotting theorem, there exists a homeomorphism $h' : \mu^{n+1}_\infty \to \mu^{n+1}_\infty$ such that $h' g'_{\delta_0} f_\lambda|_X = \mathrm{id}_X$ and $h'|_{\mu^{n+1}_\infty \setminus \operatorname{int} U} = \mathrm{id}_{\mu^{n+1}_\infty \setminus \operatorname{int} U}$. Then $R = h'(S')$ is the desired neighborhood. □

For Δ_{n+1}-product, refer to [Iwa].

Lemma 6.1.6 *Let P be a locally finite polyhedron embedded in μ^{n+1}_∞ as a closed set. Every neighborhood U of P in μ^{n+1}_∞ contains a μ^{n+1}-manifold closed neighborhood V of P such that V is n-clean in μ^{n+1}_∞ and $V \approx \delta(V) \approx P\Delta_{n+1}\mu^{n+1}$.*

Proof. We can assume that $P \subset (I^{2(n+1)+1} \times \{0\}) \setminus \{*\} \subset I^{2(n+1)+2} \setminus \{*\} = M$ as a closed subpolyhedron and μ^{n+1}_∞ is obtained from M by the Lefschetz construction [CKT, 2.1, II]. Let $\mathcal{L}$ be a combinatorial triangulation of M and $\tilde{U}$ be a neighborhood of P in M such that $U = \mu^{n+1}_\infty \cap \tilde{U}$. By Whitehead's theorem [Wh], there exists a subdivision $\mathcal{L}'$ of $\mathcal{L}$ such that $\mathcal{L}'$ refines $\{M \setminus P\} \cup \{\tilde{U}\}$. Let $N(P, \operatorname{sd}\mathcal{L}')$ be a regular neighborhood of P obtained from the barycentric subdivision $\operatorname{sd}\mathcal{L}'$ of $\mathcal{L}'$ and let V be a μ^{n+1}-manifold obtained from $N(P, \operatorname{sd}\mathcal{L}')$ by the Lefschetz construction. Then V is n-clean in μ^{n+1}_∞ such that $\delta(V) = \mu^{n+1}_\infty \cap \operatorname{Bd} N(P, \operatorname{sd}\mathcal{L}')$ and $V \setminus \delta(V) = \mu^{n+1}_\infty \cap \operatorname{int} N(P, \operatorname{sd}\mathcal{L}')$. Now there exists a proper deformation retraction $r : N(P, \operatorname{sd}\mathcal{L}') \to P$, we have a proper UV^n-retraction $r|_V : V \to P$. Since there exists a proper UV^n-surjection $P\Delta_{n+1}\mu^{n+1} \to P$ [Iwa], and by [Chi$_4$, Theorem 1.4], V and $P\Delta_{n+1}\mu^{n+1}$ are homeomorphic. Since P is a Z-set in $N(P, \operatorname{sd}\mathcal{L}')$, $N(P, \operatorname{sd}\mathcal{L}') \simeq^n_p \operatorname{Bd} N(P, \operatorname{sd}\mathcal{L}')$, which implies $\delta(V) \approx V$ by [Chi$_4$, Theorem 1.4]. □

Theorem 6.1.7 *Let X be a Z-set in a μ^{n+1}-manifold M and P an $(n+1)$-dimensional locally finite polyhedron such that $n\text{-}\mathrm{Sh}'_p(X) \leq n\text{-}\mathrm{Sh}'_p(P)$. Then X has arbitrarily small closed neighborhoods U_α, $\alpha \in A$, such that*

(1) *each U_α is n-clean in M,*

(2) $U_\alpha \approx \delta(U_\alpha) \approx P\Delta_{n+1}\mu^{n+1}$ *and*

(3) *for each $\alpha, \beta \in A$ there exists a homeomorphism $h : U_\alpha \to U_\beta$ fixing X.*

Proof. By Lemma 6.1.4, we can assume that X and P are closed sets in μ_∞^{n+1}. Let $\mathbf{f} : X \to P$ and $\mathbf{g} : P \to X$ be proper n-fundamental nets in $(\mu_\infty^{n+1}, \mu_\infty^{n+1})$ such that $\mathbf{gf} \simeq_p^n \mathbf{i}_X$. By Proposition 6.1.3, $\mathbf{f}$ is generated by a proper map $f : X \to P$. Let $A = \{\alpha \mid \alpha \text{ is a closed neighborhood of } X \text{ in } \mu_\infty^{n+1}\}$. For each $\alpha \in A$, there exist $\delta_\alpha \in \Delta$ and a closed neighborhood W of P which is homeomorphic to $P\Delta_{n+1}\mu^{n+1}$, such that $g_\delta|_W \simeq_p^n g_{\delta_\alpha}|_W$ and $g_\delta f \simeq_p^n \mathrm{id}_X$ in α for all $\delta \geq \delta_\alpha$ by Lemma 6.1.6. By the same argument of Theorem 6.1.5, we may assume that $g_{\delta_\alpha}|_W$ is a Z-embedding of $P\Delta_{n+1}\mu^{n+1}$ into α and $X \subset g_{\delta_\alpha}(W)$. Then $g_{\delta_\alpha}^{-1}|_X \simeq_p^n g_{\delta_\alpha}^{-1} g_{\delta_\alpha} f|_X \simeq_p^n f$ in $P\Delta_{n+1}\mu^{n+1}$.

Let $\alpha, \beta \in A$. Since $g_{\delta_\alpha}^{-1}|_X \simeq_p^n f \simeq_p^n g_{\delta_\beta}^{-1}|_X$ in $P\Delta_{n+1}\mu^{n+1}$, by the Z-set unknotting theorem, there exists a homeomorphism $G : P\Delta_{n+1}\mu^{n+1} \to P\Delta_{n+1}\mu^{n+1}$ such that $Gg_{\delta_\alpha}^{-1}|_X = g_{\delta_\beta}^{-1}|_X$. Then $h = g_{\delta_\beta} G g_{\delta_\alpha}^{-1}$ is the desired homeomorphism. □

It is proved (Theorem 4.2.3) that if X is connected, then $X \in SUV^n$ if and only if $n\text{-Sh}_p(X) = n\text{-Sh}_p(T)$ for some tree T. By Theorem 5.1.4, we have the following:

Corollary 6.1.8 *Let X be a connected Z-set in a μ^{n+1}-manifold and $X \in SUV^n$. Then X has an arbitrarily small closed μ^{n+1}-manifold neighborhood V such that $V \approx T\Delta_{n+1}\mu^{n+1}$ for some tree T.* □

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