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ON LINEAR SYSTEMS AND AUTOMORPHISMS OF AN ALGEBRAIC CURVE

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PREFACE

The aim of this article is to study on curves, particularly linear systems of a curve, Weierstrass points, automorphisms of a curve and relations among them.

This article consists of four parts. Throughout these parts, my idea is based upon detailed studies of linear systems of a curve.

In Part 1, I study Weierstrass points and automorphisms of special modular curves. Part 2 and 3 are devoted to investigation of linear systems themselves, or the homogeneous coordinate ring of a projective curve. The fourth part treats automorphisms of a curve from a projective point of view.

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CONTENTS

P A R T 1

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ON WEIERSTRASS POINTS OF RIEMANN SURFACES ASSOCIATED WITH $\Gamma(n, 2n)$

By
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Introduction. Let Γ be a congruence subgroup of $SL(2, \boldsymbol{R})$, and let $R(\Gamma)$ be the Riemann surface associated with Γ, i.e., $R(\Gamma)$ is the canonical compactification of $\Gamma\backslash H$, where H is the upper half plane of $\boldsymbol{C}$. If the genus of $R(\Gamma)$ is not less than two, then we can ask the following problems:

Problem 1. Is $R(\Gamma)$ hyperelliptic?

Problem 2. Are the cusps of $R(\Gamma)$ Weierstrass points?

Historically, Problems 1 and 2 are completely solved for $\Gamma=\Gamma(n)$ by H. Petersson [8] and by B. Schoeneberg [9] respectively. In the case of $\Gamma_0(n)$, partial solutions are given by J. Lehener and M. Newmann [6] and A.O.L. Atkin [2].

The purpose of this note is to answer both problems in the case of $\Gamma=\Gamma(n, 2n)$ (as for the definition of $\Gamma(n, 2n)$, see Definition 1). Our results are the following:

1. *$R(\Gamma(n, 2n))$ is non-hyperelliptic for any positive integer $n\geqq 4$* (see Theorem 4).

2. *Every cusp of $R(\Gamma(n, 2n))$ is a Weierstrass point for any even integer $n\geqq 4$. But there is an example, where the opposite situation may occur if n is odd* (see Theorem 6 and Remark 1).

Notation. $SL(2, \boldsymbol{R})$ (resp. $SL(2, \boldsymbol{Z})$) is the special linear group of degree two over the real number field $\boldsymbol{R}$ (resp. the rational integer ring $\boldsymbol{Z}$), and $PSL(2, \boldsymbol{R})$ is the projective special linear group of degree two over $\boldsymbol{R}$. When Δ is a subgroup of $SL(2, \boldsymbol{R})$, the image of Δ under the canonical homomorphism $SL(2, \boldsymbol{R})\longrightarrow PSL(2, \boldsymbol{R})$ is denoted by $\bar{\Delta}$. H^* means the disjoint union of the upper half plane H, the rational numbers $\boldsymbol{Q}$ and $\{\infty\}$.

Let Γ be a congruence subgroup of $SL(2, \boldsymbol{Z})$, and let $\sigma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an element of Γ (or $\bar{\Gamma}$). When z is a point on H^*, $\sigma(z)$ means $(az+b)/(cz+d)$, where $a/0$ $(a\neq 0)$ means ∞ and $(a\infty+b)/(c\infty+d)$ means a/c. For a point z on H^*, Γ_z (resp. $\bar{\Gamma}_z$) means the isotropy subgroup of Γ (resp. $\bar{\Gamma}$) at z. We denote the canonical projection $H^*\longrightarrow R(\Gamma)$ by π_Γ, and the genus of $R(\Gamma)$ by $g(\Gamma)$.

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§1. Genera of $R(\Gamma(n, 2n))$

We start with the definition of the congruence subgroup $\Gamma(n, 2n)$ for a positive integer n. These groups prove themselves useful when one discusses the theory of moduli of abelian varieties by means of the theory of theta functions in the case of general dimension. But we treat only the one dimensional case.

DEFINITION 1. Let n be a positive integer. We define,

$$\Gamma(n, 2n)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma(n) \mid ab\equiv cd\equiv 0 \mod. 2n\right\},$$

The following lemma is an easy consequence of the definition (cf. Igusa [3]).

LEMMA 1. (0) *$\Gamma(n, 2n)$ is a congruence subgroup of $SL(2, \mathbf{Z})$.*

(1) $\Gamma(n, 2n)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma(n) \mid ac\equiv bd\equiv 0 \mod. 2n\right\}.$

(2) *If n is an even integer, then*

$$\Gamma(n, 2n)=\left\{I_2+n\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma(n) \mid b\equiv c\equiv 0 \mod. 2\right\}.$$

(3) *If n is an odd integer, then*
$\Gamma(n, 2n)=\Gamma(n)\cap\Gamma(1, 2)$.

We continue to investigate the group $\Gamma(n, 2n)$.

LEMMA 2. (1) (Igusa [3]) *Assume that n is an even integer. Then,*

(a) *$\Gamma(n, 2n)$ is a normal subgroup of $\Gamma(n)$,*

(b) *$\Gamma(n)/\Gamma(n, 2n)\simeq(\mathbf{Z}/2\mathbf{Z})\oplus(\mathbf{Z}/2\mathbf{Z})$, and this isomorphism is induced by*

$$I_2+n\begin{pmatrix} a & b \\ c & d \end{pmatrix}\longmapsto(b \mod. 2,\ c \mod. 2),$$

(c) $$[\bar{\Gamma}(1):\ \bar{\Gamma}(n, 2n)]=\begin{cases} 24 & (\text{if } n=2) \\ 2n^3\prod_{p\mid n}\left(1-\frac{1}{p^2}\right) & (\text{if } n\geqq 4), \end{cases}$$

where $p|n$ means that p runs over all prime numbers dividing n.

(2) *Assume that n is an odd integer. Then,*

(a) *$\Gamma(n, 2n)$ is a normal subgroup of $\Gamma(1, 2)$, but it is not a normal subgroup of $\Gamma(n)$,*

(b) $\Gamma(n, 2n)=\Gamma(2n)+\Gamma(2n)\begin{pmatrix} 1-n & n \\ -n & 1+n \end{pmatrix}$, *where the right hand side of the equality means the coset decomposition of $\Gamma(n, 2n)$ mod. $\Gamma(2n)$,*

(c) $$[\bar{\Gamma}(1):\ \bar{\Gamma}(n, 2n)]=\begin{cases} 3 & (\text{if } n=1) \\ \frac{3}{2}n^3\prod_{p\mid n}\left(1-\frac{1}{p^2}\right) & (\text{if } n\geqq 3). \end{cases}$$

PROOF. (1) All of them are the special cases of results in Igusa's paper [3].

(2) (a) Since $\Gamma(n, 2n)=\Gamma(n)\cap\Gamma(1,2)$, $\Gamma(n, 2n)$ is a normal subgroup of $\Gamma(1,2)$. It can easily be verified that $\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}$ belongs to $\Gamma(n)$ and that $\begin{pmatrix}1-n & n\\ -n & 1+n\end{pmatrix}$ belongs to $\Gamma(n, 2n)$ but that

$$\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}\begin{pmatrix}1-n & n\\ -n & 1+n\end{pmatrix}\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}^{-1}=\begin{pmatrix}1-n-n^2 & n+2n^2+n^3\\ -n & 1+n+n^2\end{pmatrix}$$

does not belongs to $\Gamma(n, 2n)$, so $\Gamma(n, 2n)$ is not a normal subgroup of $\Gamma(n)$.

(b) Let G be the subgroup of $\Gamma(1)$ generated by $\Gamma(2n)$ and $\begin{pmatrix}1-n & n\\ -n & 1+n\end{pmatrix}$. Then we see that the coset decomposition of G mod. $\Gamma(2n)$ is given by $G=\Gamma(2n)+\Gamma(2n)\begin{pmatrix}1-n & n\\ -n & 1+n\end{pmatrix}$ and that the inclusion relations $\Gamma(n)\supsetneqq\Gamma(n, 2n)\supset G\supsetneqq\Gamma(2n)$ holds. Since $[\Gamma(n):\ \Gamma(2n)]=6$, we have $\Gamma(n, 2n)=G$.

(c) It is clear that $[\bar{\Gamma}(n, 2n):\ \bar{\Gamma}(2n)]=2$. Combining this equality with the known formula:

$$[\bar{\Gamma}(1):\ \bar{\Gamma}(n)]=\begin{cases}6 & (\textit{if } n=2)\\ \dfrac{1}{2}n^3\prod_{p|n}\left(1-\dfrac{1}{p^2}\right) & (\textit{if } n\geqq 3),\end{cases}$$

we obtain our formula. *Q.E.D.*

The following proposition is the main purpose in this section.

PROPOSITION 3. (1) *Let n be an even integer. Then $R(\Gamma(n, 2n))$ has no elliptic point. The number of cusps on $R(\Gamma(n, 2n))$ is equal to* 6 *if $n=2$, and is equal to $n^2\prod_{p|n}\left(1-\frac{1}{p^2}\right)$ if $n\geqq 4$. The genus of $R(\Gamma(n, 2n))$ is given by the following formula:*

$$g(\Gamma(n, 2n))=\begin{cases}0 & (\textit{if } n=2)\\ 1+\dfrac{1}{6}n^2(n-3)\prod_{p|n}\left(1-\dfrac{1}{p^2}\right) & (\textit{if } n\geqq 4).\end{cases}$$

(2) *Let n be an odd integer. Then $R(\Gamma(n, 2n))$ has no elliptic point if $n\geqq 3$, and has exactly one elliptic point, whose order is two, if $n=1$. In the case of $n=1$, $R(\Gamma(1,2))$ has exactly two cusps which are represented by ∞ and* 1. *In the case of $n\geqq 3$, $R(\Gamma(n, 2n))$ has $\frac{1}{2}n^2\prod_{p|n}\left(1-\frac{1}{p^2}\right)$ cusps lying over $\pi_{\Gamma(1,2)}(\infty)$ and $\pi_{\Gamma(1,2)}(1)$ respectively. Hence the number of the cusps on $R(\Gamma(n, 2n))$ is equal to $n^2\prod_{p|n}\left(1-\frac{1}{p^2}\right)$. The genus of $R(\Gamma(n, 2n))$ is given by the following formula:*

$$g(\Gamma(n, 2n))=\begin{cases}0 & (\textit{if } n=1)\\ 1+\dfrac{1}{8}n^2(n-4)\prod_{p|n}\left(1-\dfrac{1}{p^2}\right) & (\textit{if } n\geqq 3).\end{cases}$$

PROOF. (1) Since $\Gamma(n)$ $(n\geqq 2)$ has no elliptic element, $\Gamma(n, 2n)$ has no elliptic element. Since $[\bar{\Gamma}(1)_\infty : \bar{\Gamma}(n, 2n)_\infty]=2n$, the number of cusps on $R(\Gamma(n, 2n))$ is equal to $[\bar{\Gamma}(1): \bar{\Gamma}(n, 2n)]/2n$. Our formula comes from the genus formula of Riemann surfaces associated with modular groups (cf. G. Shimura [10] Proposition 1.40).

(2) Since $R(\Gamma(2))$ is a covering Riemann surface of genus 0 over $R(\Gamma(1, 2))$, $R(\Gamma(1, 2))$ is of genus 0. Suppose that $R(\Gamma(1, 2))$ has an elliptic point of order 3. Then $\Gamma(1, 2)$ must have an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ whose trace is 1 or -1. On the other hand, $ab\equiv cd\equiv 0$ mod. 2. These contradict to $ad-bc=1$, so $R(\Gamma(1, 2))$ has no elliptic point of order 3. By the similar argument, we see that ∞ and 1 on H^* are not $\Gamma(1, 2)$—equivalent. Let ν_∞ be the number of the cusps on $R(\Gamma(1, 2))$, and ν_2 the number of elliptic points of order 3. Then we have $\nu_\infty\geqq 2$ and $\nu_2\geqq 1$. On the other hand, by the genus formula, we get the equation $0=1+3/12-\nu_2/4-\nu_\infty/12$. Therefore we have $\nu_\infty=2$ and $\nu_2=1$.

Next, we shall calculate the number of cusps on $R(\Gamma(n, 2n))$ in the case of $n\geqq 3$. We put,

$$\nu_\infty^{(\infty)}=\text{the number of cusps on } R(\Gamma(n, 2n)) \text{ lying over } \pi_{\Gamma(1,2)}(\infty)$$
$$=[\bar{\Gamma}(1, 2): \bar{\Gamma}(n, 2n)]/[\bar{\Gamma}(1, 2)_\infty : \bar{\Gamma}(n, 2n)_\infty],$$
$$\nu_\infty^{(1)}=\text{the number of cusps on } R(\Gamma(n, 2n)) \text{ lying over } \pi_{\Gamma(1,2)}(1)$$
$$=[\bar{\Gamma}(1, 2): \bar{\Gamma}(n, 2n)]/[\bar{\Gamma}(1, 2)_1 : \bar{\Gamma}(n, 2n)_1].$$

Since $\bar{\Gamma}(n, 2n)_\infty=\left\{\overline{\begin{pmatrix} 1 & 2nm \\ 0 & 1 \end{pmatrix}}\,\middle|\, m\in \boldsymbol{Z}\right\}$, we have $\nu_\infty^{(\infty)}=\frac{1}{2}n^2\prod_{p|n}\left(1-\frac{1}{p^2}\right)$.

Let $\sigma=\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. Then $\sigma(1)=\infty$, so $\sigma(\Gamma(n, 2n)_1)\sigma^{-1}=(\sigma\Gamma(n, 2n)\sigma^{-1})_\infty$.

By Lemma 2, for any odd integer n, we see that

$$\sigma\Gamma(n, 2n)\sigma^{-1}=\sigma(\Gamma(2n)+\Gamma(2n)\begin{pmatrix} 1-n & n \\ -n & 1+n \end{pmatrix})\sigma^{-1}$$
$$=\Gamma(2n)+\Gamma(2n)\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

Hence $\overline{(\sigma\Gamma(n, 2n)\sigma^{-1})_\infty}=\left\{\overline{\begin{pmatrix} 1 & nm \\ 0 & 1 \end{pmatrix}}\,\middle|\, m\in \boldsymbol{Z}\right\}$, and also

$$[\bar{\Gamma}(1, 2)_1 : \bar{\Gamma}(n, 2n)_1]=[\overline{(\sigma\Gamma(1, 2)\sigma^{-1})_\infty} : \overline{(\sigma\Gamma(n, 2n)\sigma^{-1})_\infty}]=n.$$

Therefore $\nu_\infty^{(1)}=\frac{1}{2}n^2\prod_{p|n}\left(1-\frac{1}{p^2}\right)$. Finally, by the genus formula we get our formula.

Q.E.D.

§ 2. The answer to the first problem

Let R be a Riemann surface of genus g, and let P be a point on R. We say that a positive integer m is a gap at P, if there exists no meromorphic function on R with a pole of order m at P and holomorphic at any other point. It is known that for any P there are exactly g gaps and that except for finitely many points the gap sequence coincides with $\{1, 2, \cdots, g\}$. A point on R whose gap sequence differs from $\{1, 2, \cdots, g\}$ is called a Weierstrass point.

If the genus g is greater than or equal to two, then at least $2g+2$ Weierstrass points exist, the number of the Weierstrass points on R, say w, satisfies the following inequalities:

$$2g+2 \leqq w \leqq (g-1)g(g+1),$$

and w is equal to $2g+2$ if and only if R is hyperelliptic.

The purpose in this section is to prove the following theorem. Our proof gose in the same way as Peterson's proof for the principal congruence case [8].

THEOREM 4. *$R(\Gamma(n, 2n))$ is a non-hyperelliptic Riemann surface for any integer $n \geqq 4$.*

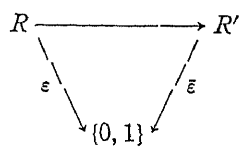
Before proving our theorem, we state some remarks. In the case of $n \geqq 4$, there exists a Weierstrass points on $R(\Gamma(n, 2n))$ because their genera are greater than two.

DEFINITION 2. Let R be a Riemann surface of genus $g \geqq 2$.

$\varepsilon = \varepsilon_R$: $R \longrightarrow \{0, 1\}$ is defined by the following:

$$\varepsilon(P) = \begin{cases} 0 & \text{if } P \text{ is not a Weierstrass point,} \\ 1 & \text{if } P \text{ is a Weierstrass point.} \end{cases}$$

Let G be an automorphism group of R, and let $R \longrightarrow R' = G \backslash R$ be the canonical covering map. Then there exists a function $\bar{\varepsilon} = \bar{\varepsilon}_{R,G}$: $R' \longrightarrow \{0, 1\}$ such that the following diagram is commute:



PROOF OF THEOREM 4. The first, we assume that n is an odd integer. We put,

$$\pi = \pi_{\Gamma(1,2)},$$

$$\mu = \mu(n) = [\bar{\Gamma}(1, 2): \ \bar{\Gamma}(n, 2n)] = \frac{1}{2} n^3 \prod_{p \mid n} \left(1 - \frac{1}{p^2}\right),$$

$\nu_\infty^{(\infty)}=\nu_\infty^{(\infty)}(n)=$ the number of cusps on $R(\Gamma(n, 2n))$ lying over $\pi(\infty)$,

$\nu_\infty^{(1)}=\nu_\infty^{(1)}(n)=$ the number of cusps on $R(\Gamma(n, 2n))$ lying over $\pi(1)$,

$$\nu=\nu(n)=\frac{1}{2}n^2\prod_{p|n}\left(1-\frac{1}{p^2}\right).$$

By Proposition 3, we have $\nu_\infty^{(\infty)}(n)=\nu_\infty^{(1)}(n)=\nu(n)$. Since $\bar{\Gamma}(n, 2n)$ is a normal subgroup of $\bar{\Gamma}(1, 2)$, we have

$$R(\Gamma(1,2)=(\bar{\Gamma}(1,2)/\bar{\Gamma}(n, 2n))\backslash R(\Gamma(n, 2n)).$$

Futhermore, we put

$$\varepsilon=\varepsilon_{R(\Gamma(n,2n))},$$

$$\bar{\varepsilon}=\bar{\varepsilon}_{R(\Gamma(n,2n)),\ \bar{\Gamma}(1,2)/\bar{\Gamma}(n,2n)}.$$

Then it is easy to show that the number of the Weierstrass points on $R(\Gamma(n, 2n))$ is given by

$$\bar{\varepsilon}(\pi(\sqrt{-1}))\frac{\mu}{2}+\bar{\varepsilon}(\pi(\infty))\nu+\bar{\varepsilon}(\pi(1))\nu+\sum_P{}'\ \bar{\varepsilon}(P)\mu \tag{1}$$

where $\sum_P{}'$ means that P runs over all points on $R(\Gamma(1,2))$ except for $\pi(1)$, $\pi(\infty)$ and and $\pi(\sqrt{-1})$. Suppose that $R(\Gamma(n, 2n))$ is hyperelliptic. Then (1) is equal to $2g(\Gamma(n, 2n))+2$. Therefore

$$8=\nu(n)\cdot(nx+2y+2z+2nw+4-n) \tag{2}$$

must have an integral solution on x, y, z and w. But $1/\prod_{p|n}\left(1-\frac{1}{p^2}\right)\leqq\zeta(2)$ where $\zeta(s)$ is Riemann's zeta function. Hence $\nu(n)\geqq n^2/2\zeta(2)=3n^2/\pi^2$. Hence, in the case of $n\geqq 7$, we have $\nu(n)>8$. In the case of $n=5$, we see that $\nu(5)=12$. Therefore $\nu(n)>8$ for all our cases. This is a contradiction.

Next, let n be an even integer, Our proof is similar to the first case. We put,

$$\pi=\pi_{\Gamma(1)},$$

$$\mu=\mu(n)=[\bar{\Gamma}(1):\ \bar{\Gamma}(n, 2n)]=2n^3\prod_{p|n}\left(1-\frac{1}{p^2}\right),$$

$$\nu=\nu(n)=n^2\prod_{p|n}\left(1-\frac{1}{p^2}\right),$$

$$\varepsilon=\varepsilon_{R(\Gamma(n,2n))},$$

$$\bar{\varepsilon}=\bar{\varepsilon}_{R(\Gamma(n,2n)),\bar{\Gamma}(1)/\bar{\Gamma}(n,2n)}.$$

Then the number of the Weierstrass points on $R(\Gamma(n, 2n))$ is given by

$$\bar{\varepsilon}(\pi(\sqrt{-1}))\cdot\frac{\mu}{2}+\bar{\varepsilon}(\pi(e^{2\pi\sqrt{-1}/3}))\cdot\frac{\mu}{3}+\bar{\varepsilon}(\pi(\infty))\cdot\nu+\sum_{P}{}'\,\bar{\varepsilon}(P)\cdot\mu \tag{3}$$

where $\sum_{P}'$ means that P runs over all points on $R(\Gamma(1))$ except for $\pi(\infty)$, $\pi(\sqrt{-1})$ and $\pi(e^{2\pi\sqrt{-1}/3})$. Suppose that $R(\Gamma(n, 2n))$ is hyperelliptic. Then

$$12=\nu(n)\cdot(3nx+2ny+3z+6nw+3-n) \tag{4}$$

must have an integral solution on x, y, z and w. By the simular discussion to the first proof, we have $\nu(n)>12$ if $n\geqq 6$. In the case of $n=4$, since $\nu(4)=12$, $2=12x+8y+3z+24w$ must have an integral solution on x, y z and w. But it is impossible. In any case, (4) has no integral solution. This is a contradiction. *Q.E.D.*

COROLLARY 5. *On $R(\Gamma(4,8))$, its cusps coincide with its Weierstrass points, and the gap sequence at any Weierstrass points is* $\{1, 2, 5\}$.

PROOF. By the formula (3) in the proof of Theorem 4, the number of the Weierstrass points on $R(\Gamma(4,8))$ is given by

$$4(12\bar{\varepsilon}(\pi(\sqrt{-1}))+8\bar{\varepsilon}(\pi(e^{2\pi\sqrt{-1}/3}))+3\bar{\varepsilon}(\pi(\infty))+24\sum_{P}{}'\bar{\varepsilon}(P))\,.$$

Since the genus of $R(\Gamma(4,8))$ is 3, the number of the Weierstrass points is at most 24. Therefore $\bar{\varepsilon}(\pi(\infty))$ is 1 and $\bar{\varepsilon}(P)$ is 0 for any other point P. Since the number of the Weierstrass points is 12, all gap sequences are $\{1, 2, 5\}$. *Q.E.D.*

§ 3. The answer to the second problem

The main purpose in this section is to prove the following theorem.

THEOREM 6. *Let n be an even integer grater than or equal to* 4. *Then any cusp of $R(\Gamma(n, 2n))$ is a Weierstrass point. Furthermore the Weierstrass points coincide with the cusps provided that $n=4$.*

Before starting our proof, we state two lemmas without proofs.

LEMMA 7. (Accola [1], Komiya [4]) *Let $R\longrightarrow R'$ be an abelian covering of Riemann surfaces of type (p, p) where p is a prime integer, and let R and R' have the genera g and g' respectively, and the genera of all intermediate Riemann surfaces between R and R' be $g_1, \cdots, g_{p+1}$. Then we have the following formula:*

$$\sum_{j=1}^{p+1} g_j=g+pg'\,.$$

LEMMA 8. (Schoeneberg [9]) *Let R be a Riemann surface of genus $g\geqq 2$, and σ be an automorphism of order n. Suppose that P is a fixed point of σ on R, and* *that the gap sequence at P is* $\{n_1, \cdots, n_g\}$. *Then the genus of* $\langle\sigma\rangle\backslash R$ *coincides with the cardinality of the set*

$$\{j \mid n_j \equiv 0 \mod. n \quad 1 \leqslant j \leqslant g\}.$$

Here $\langle\sigma\rangle$ *means the automorphism group of* R *generated by* σ.

PROOF OF THEOREM 6. In the case of $n=4$, the statement of the theorem was already proven by Corollary 5. So, from now on, we assume that n is an even integer grater than 4. Since each element of $\bar{\Gamma}(1)/\bar{\Gamma}(n, 2n)$ induces a permutation on the set of all cusps on $R(\Gamma(n, 2n))$ and the group of these permutations is transitive, it suffices to prove that some cusp is a Weierstrass point. By Lemma 2, we see that $R(\Gamma(n, 2n)) \longrightarrow R(\Gamma(n))$ is an abelin covering of type (2, 2). We put,

$$\sigma_1 = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix},$$

$$\sigma_3 = \begin{pmatrix} 1-n & n \\ -n & 1+n \end{pmatrix},$$

$$H_i = \Gamma(n, 2n) + \Gamma(n, 2n)\sigma_i \quad (i=1, 2, 3).$$

Then we see the following diagram and relations:

Galois group of $R(\Gamma(n, 2n))$ over $R(\Gamma(n))$

$\|$

$\bar{\Gamma}(n)/\bar{\Gamma}(n, 2n)$

$\bar{H}_1/\bar{\Gamma}(n, 2n)$ $\quad$ $\bar{H}_2/\bar{\Gamma}(n, 2n)$ $\quad$ $\bar{H}_3/\bar{\Gamma}(n, 2n)$

$\{1\}$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} H_1 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} = H_3, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} H_2 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} = H_3.$$

Hence $R(H_i)$ $(i=1, 2, 3)$ have the same genus as one another, say g'. By Lemma 7, we get the equation $3g' = g(\Gamma(n, 2n)) + 2g(\Gamma(n))$, so

$$g' = 1 + \frac{1}{12} n^2 (n-4) \prod_{p \mid n} \left(1 - \frac{1}{p^2}\right).$$

Furthermore, we see that the automorphism $\bar{\sigma}_i$ mod. $\bar{\Gamma}(n, 2n)$ is of order 2 and that it has a fixed point which is a cusp. Therefore, by Lemma 8, it suffices to show that the inequality

$$g(\Gamma(n, 2n))-2g'=\frac{1}{6}n^2\prod_{p|n}\left(1-\frac{1}{p^2}\right)-1\geqq 2$$

holds. It is clear that the above inequality holds if $n\geqq 6$. Finally,

$$2g(\Gamma(n, 2n))+2-(\text{the number of the cusps on } R(\Gamma(n, 2n)))$$

$$=4+\frac{1}{3}n^2(n-6)\prod_{p|n}\left(1-\frac{1}{p^2}\right)>0$$

under the condition $n\geqq 6$. This implies that there exists a Weierstrass point which is not a cusp under the condition $n\geqq 6$. *Q.E.D.*

Finally we shall state two remarks. The first remark is a counter-example against the conclusion of Theorem 6 for an odd integer case, and the second remark concerns itself with Komiya and Kuribayashi's result.

REMARK 1. If n is an odd integer, then the conclusion of Theorem 6 is not always true. In fact, we saw that $\sigma=\begin{pmatrix}1-n & n\\ -n & 1+n\end{pmatrix}$ acts on $R(\Gamma(2n))$ as an automorphism of order 2 having a fixed point $\pi_{\Gamma(2n)}(1)$ and that $R(\Gamma(n, 2n))=\langle\sigma\rangle\backslash R(\Gamma(2n))$. On the other hand, the gap sequence at $\pi_{\Gamma(10)}(1)$ on $R(\Gamma(10))$ is

$$\{1, 2, \cdots, 8, 9, 11, 13, 17, 19\} \quad \text{(Lewittes [7])}.$$

Hence the gap sequence at $\pi_{\Gamma(5,10)}(1)$ on $R(\Gamma(5, 10))$ is

$$\{1, 2, 3, 4\}.$$

Therefore $\pi_{\Gamma(5,10)}(1)$ is not a Weierstrass point, but it is a cusp.

REMARK 2. Recently, A. Kuribayashi and K. Komiya discovered that there are exactly two Riemann surfaces up to isomorphism which are of genus 3 having exactly 12 Weierstrass points [5]. Our $R(\Gamma(4, 8))$ is of genus 3 and has exactly 12 Weierstrass points. It coincides with

$$X^4+Y^4+Z^4=0 \quad \text{in } \boldsymbol{P}^2 \qquad (\sharp)$$

in their paper (cf. Igusa [3]).

On the other hand, we see that $\Gamma(4, 8)$ acts on H fixed point free. Furthermore, we saw that the cusps of $R(\Gamma(4, 8))$ coincides with the Weierstrass points and that $\Gamma(4, 8)$ is a normal subgroup of $\Gamma(1)$. Therefore the automorphism group of the curve $(\sharp)$ is isomorphic to $\bar{\Gamma}(1)/\bar{\Gamma}(4, 8)$.

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P A R T 2

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ON THE EQUATIONS DEFINING A PROJECTIVE CURVE EMBEDDED BY A NON-SPECIAL DIVISOR

By
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Introduction. Let C be a complete reduced irreducible curve of arithmetic genus g over an algebraically closed field K. Let L be a very ample invertible sheaf of degree d on C, and let ϕ_L: $C\subsetneq P^{h^0(L)-1}$ be the projective embedding by means of a basis of $\Gamma(L)$. Then the following results are known:

(A) Assume that C is smooth over K.

(0) (D. Mumford [5]) L is normally generated, if $d\geq 2g+1$.

(1) (B. Saint-Donat [7]) The largest homogeneous ideal I defining $\phi_L(C)$, i.e., $I=\mathrm{Ker}[S\Gamma(L)\rightarrow\bigoplus_{m\geq 0}\Gamma(L^m)]$, is generated by its elements of deree 2, if $d\geq 2g+2$.

(2) (B. Saint-Donat [7]) I is generated by its elements of degree 2 and 3, if $d\geq 2g+1$.

(B) (T. Fujita [1]) The statements (0) and (1) in (A) are true without the assumption that C is smooth over K.

The purposes of the present paper are that we improve the second result (2) of Saint-Donat and that we construct some related examples (corollary 1.4, Example 2.4 and Proposition 3.1).

Notation and Terminology. We fix an algebraically closed field K of characteristic $p\geqq 0$ throughout the paper. We use the word "variety" to mean a reduced irreducible scheme of finite type and proper over K, and "curve" to mean a variety of dimention 1.

For a finite dimensional vector space V over K, S^mV means the m-th symmetric power of V and SV means the symmetric algebra of V, i.e., $SV=\bigoplus_{m\geqq 0}S^mV$.

Let L be an invertible sheaf on a projective variety X. We denote by L^m the m-th tensor product $L^{\otimes m}$. For the vector space of global sections $\Gamma(L)$, we define I and I_m $(m\geqq 1)$, by

$$I=I(L)=\mathrm{Ker}[S\Gamma(L)\rightarrow\bigoplus_{m\geqq 0}\Gamma(L^m)],$$

and

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$$I_m = I_m(L) = \mathrm{Ker}[S^m\Gamma(L) \to \Gamma(L^m)].$$

Let $L_1, \cdots, L_m$ be invertible sheaves on X. Then $\mathfrak{R}(L_1, \cdots, L_m)$ means the kernel of the natural map:

$$\Gamma(L_1)\otimes\cdots\otimes\Gamma(L_m) \to \Gamma(L_1\otimes\cdots\otimes L_m).$$

§ 1. Generality.

Let X be a projective variety, and let L be an ample invertible sheaf on X. If the canonical map $\Gamma(L)^{\otimes m}\to\Gamma(L^m)$ is surjective for all positive integers m, then L is called a normally generated ample invertible sheaf.

We will establish a criterion for surjectivity of the natural map $I_m(L)\otimes\Gamma(L) \to I_{m+1}(L)$ for a normally generated ample invertible sheaf L.

LEMMA 1.1. *Let V be a finite dimensional vectar space, and let r be a positive integer greater than* 1. *Then we have*

$$\mathrm{Ker}[V^{\otimes(r+1)}\to S^{r+1}V] = \mathrm{Ker}[V^{\otimes r}\to S^r V]\otimes V + V\otimes\mathrm{Ker}[V^{\otimes r}\to S^r V].$$

A proof of the lemma is easy, so we omit its proof.

PROPOSITION 1.2. *Let L be a normally generated ample invertible sheaf on a variety X. If m is a positive integer greater than* 1, *then the following conditions are equivalent*:

(1) $\Gamma(L)\otimes\mathfrak{R}(L^{m-1}, L) \xrightarrow{\xi} \mathfrak{R}(L^m, L)$ *is surjective,*

(2) $\mathfrak{R}(\overbrace{L, \cdots, L}^{m+1}) = \mathfrak{R}(\overbrace{L, \cdots, L}^{m})\otimes\Gamma(L) + \Gamma(L)\otimes\mathfrak{R}(\overbrace{L, \cdots, L}^{m})$,

(3) $I_m(L)\otimes\Gamma(L) \xrightarrow{q} I_{m+1}(L)$ *is surjective.*

PROOF[(*)]. We consider the following exact diagram

$$\begin{array}{ccccccc}
0\to\mathfrak{R}(\overbrace{L,\cdots,L}^{m})\otimes\Gamma(L) & \xrightarrow{\alpha} & \mathfrak{R}(\overbrace{L,\cdots,L}^{m+1}) & \longrightarrow & \mathfrak{R}(L^m, L) & \to 0 \\
 & & \uparrow{\scriptstyle\xi'} & & \uparrow{\scriptstyle\xi} & \\
 & & \Gamma(L)\otimes\mathfrak{R}(\overbrace{L,\cdots,L}^{m}) & \to & \Gamma(L)\otimes\mathfrak{R}(L^{m-1}, L) & \to 0 \\
 & & \uparrow & & & \\
 & & 0 & & &
\end{array}$$

It is easy to check that $\mathfrak{R}(\overbrace{L, \cdots, L}^{m+1}) = Im(\alpha) + Im(\xi')$ if and only if ξ is surjective.

Next, we will prove the equivalence (2)$\Longleftrightarrow$(3). Note that the canonical map π_r: $\mathfrak{R}(\overbrace{L, \cdots, L}^{r}) \to I_r(L)$ is surjective for any integers $r\geqq 2$. For a given $f\in I_{m+1}(L)$,

(*) The proof of the first part (1)$\Leftrightarrow$(2), has been fairly simplified by an idea of Dr. Sekiguchi.

We can find $s\in\mathfrak{R}(\overbrace{L,\cdots,L}^{m+1})$ such that $\pi_{m+1}(s)=f$. By (2), we have $s=\sum_i\beta_i\otimes s_i+\sum_j t_j\otimes\gamma_j$ for suitable elements β_i, $\gamma_j\in\mathfrak{R}(\overbrace{L,\cdots,L}^{m})$ and s_i, $t_j\in\Gamma(L)$, so we have $f=q(\sum_i\pi_m(\beta_i)\otimes s_i+\sum_j\pi_m(\gamma_j)\otimes t_j)$. Hence (2) implies (3). To prove the implication (3)$\Rightarrow$(2), it suffices to show the inclusion relation

$$\mathfrak{R}(\overbrace{L,\cdots,L}^{m+1})\subset\mathfrak{R}(\overbrace{L,\cdots,L}^{m})\otimes\Gamma(L)+\Gamma(L)\otimes\mathfrak{R}(\overbrace{L,\cdots,L}^{m}).$$

Let s be an element of $\mathfrak{R}(\overbrace{L,\cdots,L}^{m+1})$. Then by (3), there exist $t_j\in\mathfrak{R}(\overbrace{L,\cdots,L}^{m})$ and $s_j\in\Gamma(L)$ such that $\pi_{m+1}(s)=q(\sum_j\pi_m(t_j)\otimes s_j)$.
Hence

$$s-\sum_j t_j\otimes s_j\in\mathrm{Ker}(\Gamma(L)^{\otimes(m+1)}\to S^{m+1}\Gamma(L)).$$

Since by Lemma 1.1,

$$\begin{aligned}&\mathrm{Ker}(\Gamma(L)^{\otimes(m+1)}\to S^{m+1}\Gamma(L))\\&=\mathrm{Ker}(\Gamma(L)^{\otimes m}\to S^m\Gamma(L))\otimes\Gamma(L)+\Gamma(L)\otimes\mathrm{Ker}(\Gamma(L)^{\otimes m}\to S^m\Gamma(L))\\&\subset\mathfrak{R}(\overbrace{L,\cdots,L}^{m})\otimes\Gamma(L)+\Gamma(L)\otimes\mathfrak{R}(\overbrace{L,\cdots,L}^{m}),\end{aligned}$$

so we have

$$s\in\mathfrak{R}(\overbrace{L,\cdots,L}^{m})\otimes\Gamma(L)+\Gamma(L)\otimes\mathfrak{R}(\overbrace{L,\cdots,L}^{m}).\qquad\text{Q.E.D.}$$

COROLLARY 1.3. *Let L be a normally generated ample invertible sheaf on an n-dimensional variety X. Assume that $H^i(X, L^j)=(0)$ for any integers $i, j\geq 1$. Then the homogeneous ideal* I(L) *is generated by* $I_2,\cdots,I_{n+3}$.

PROOF. By Propostion 1.2, it suffices to prove that the natural map $\Gamma(L)\otimes\mathfrak{R}(L^{m-1}, L)\to\mathfrak{R}(L^m, L)$ is surjective for any integer $m\geqq n+3$. It is just the theorem of Mumford [5, Theorem 5].

COROLLARY 1.4. *Let L be a normally generated ample invertible sheaf on a curve C. Assume that $H^1(C, L)=(0)$. Then $I(L)$ is generated by I_2 and I_3.*

PROOF. By Proposition 1.2 and Corollary 1.3, it suffices to show that the natural map $\Gamma(L)\otimes\mathfrak{R}(L^2, L)\to\mathfrak{R}(L^3, L)$ is surjective. It is a direct consequence of the following lemma.

LEMMA 1.5. (T. Fujita [1, Lemma 1.8]) *Let L, M and N be invertible sheaves on a curve C. Assume that $H^1(C, M\otimes L^{-1})=(0)$ and that $\Gamma(L)$ is base point free and that the natural map $\Gamma(M\otimes L^{-1})\otimes\Gamma(N)\to\Gamma(M\otimes N\otimes L^{-1})$ is surjective. Then the natural map $\Gamma(L)\otimes\mathfrak{R}(M, N)\to\mathfrak{R}(L\otimes M, N)$ is surjective.*

REMARK 1.6. Let L be an invertible sheaf of degree d on a curve C. If $d\geqq 2g+1$, then L is a normally generated ample invertible sheaf with $H^1(C, L)=(0)$. Therefore by Corollary 1.4, $I(L)$ is generated by I_2 and I_3,

This is another proof of the second result of Saint-Donat (c.f Introduction (2)).

§ 2. Example I.

In this section we use the word "curve" to mean a smooth curve over K. We assume that the characteristic of the ground field K is not 2. The purpose of this section is to show that the first result of Saint-Donat (see Introduction (1)) is the best possible for each genus $g\geqq 1$, namely, there exists a curve C of genus g with invertible sheaf L on C of degree $2g+1$ such that the homogeneous ideal $I(L)$ is not generated by $I_2(L)$.

REMARK 2.1. Let C be a curve of genus 1 or 2, and let L be an invertible sheaf of degree $2g+1$ on C. Then the homogeneous ideal $I(L)$ is not generated by $I_2(L)$.

Indeed, C is embedded by $\Gamma(L)$ to $\boldsymbol{P}^2$ if the genus is 1 (resp. to $\boldsymbol{P}^3$ if the genus is 2), but the dimension of $I_2(L)$ is 0 if the genus is 1 (resp. is 1 if the genus is 2).

From now on, we fix a hyper-elliptic curve C of genus $g\geqq 3$. Let $K(C)$ be the function field of C. Since the characteristic of the ground field K is not 2, there exist functions x, $y\in K(C)$ such that $K(C)=K(x, y)$ with a relation

$$y^2=(x-a_1)\cdot(x-a_2)\cdots(x-a_{2g+1}).$$

Let P_∞ be the closed point on C such that $x(P_\infty)=\infty$, and let $L=\mathcal{O}_C((2g+1)P_\infty)$ For any divisor D on C, we regard $\mathcal{O}_C(D)$ as a subsheaf of $K(C)$ in the canonical way. Then we have that the $g+2$ functions

$$\{1, x, \cdots, x^g, y\}$$

forms a basis of $\Gamma(L)$ and that the $\frac{1}{2}(g+2)(g+3)$ elements

$$\left\{\begin{array}{llllll} 1\odot 1 & & & & & \\ 1\odot x & x\odot x & & & & \\ 1\odot x^2 & x\odot x^2 & x^2\odot x^2 & & & \\ \vdots & \vdots & \vdots & & & \\ 1\odot x^g & x\odot x^g & x^2\odot x^g & \cdots\cdots & x^g\odot x^g & \\ 1\odot y & x\odot y & x^2\odot y & \cdots\cdots & x^g\odot y & y\odot y \end{array}\right\}$$

forms a basis of $S^2\Gamma(L)$, where the symbol $\odot$ means a symmetric product.

PROPOSITION 2.2. *The vector space*

$$I_2(L)=\mathrm{Ker}[S^2\Gamma(L)\to\Gamma(L^2)]$$

is generated by $\{x^i\odot x^j-x^{i-1}\odot x^{j+1}\mid 1\leqq i\leqq j\leqq g-1\}$ *over* K.

PROOF. It is easy to show that the above set is included in I_2. Let V be a subspace of I_2 generated by the above set, and let W be asubspace of $S^2\Gamma(L)$ generated by the following elements:

$$\left\{\begin{matrix} 1\odot 1 & 1\odot x, & \cdots\cdots, & 1\odot x^g \\ x\odot x^g & x^2\odot x^g, & \cdots\cdots, & x^g\odot x^g \\ 1\odot y & x\odot y, & \cdots\cdots, & x^g\odot y \\ y\odot y & & & \end{matrix}\right\}.$$

Then the natural map $W\to S^2\Gamma(L)/V$ is surjective. Indeed, if $i\leqq g-j$, then

$$x^i\odot x^j\equiv x^{i-1}\odot x^{j+1}\equiv\cdots\cdots\equiv 1\odot x^{j+i}\quad \mathrm{mod.}\ V,$$

and if $i>g-j$, then

$$x^i\odot x^j\equiv x^{i+1}\odot x^{j-1}\equiv\cdots\cdots\equiv x^{i+j-g}\odot x^g\quad \mathrm{mod.}\ V.$$

Hence we have $\dim[S^2\Gamma(L)]-\dim(V)\leqq\dim(W)$, so we have

$$\dim(V)\geqq\frac{1}{2}g(g-1)=\dim(I_2).$$

Since $I_2\supset V$, we have $I_2=V$. Q.E.D.

COROLLARY 2.3. *Let* $\{X_0, X_1, \cdots, X_g, Y\}$ *be a homogeneous coordinate of the projective space* $\boldsymbol{P}^{g+1}$ *corresponding to a basis* $\{1, x, \cdots, x^g, y\}$ *of* $\Gamma(L)$. *Then the vector space of quadrics vanishing on* $\phi_L(C)$ *is generated by the quadrics*

$$\{X_iX_j-X_{i-1}X_{j+1}\mid 1\leqq i\leqq j\leqq g-1\}$$

over K.

EXAMPLE 2.4. Let (C, L) be the above curve with invertible sheaf. Then the degree L is $2g+1$, but the homogeneous ideal $I(L)$ is not generated by $I_2(L)$.

In fact, if the homogeneous ideal $I(L)$ is generated by I_2, then

$$\phi_L(C)=\bigcap_{1\leqq i\leqq j\leqq g-1} V(X_iX_j-X_{i-1}X_{j+1})$$

by Corollary 2.3, where $V(X_iX_j-X_{i-1}X_{j+1})$ is the set of zeros of $X_iX_j-X_{i-1}X_{j+1}$ in $\boldsymbol{P}^{g+1}$. Let H be the linear subvariety of $\boldsymbol{P}^{g+1}$ defined by the equations:

$$X_0-X_1,\ X_1-X_2,\ \cdots,\ X_{g-1}-X_g.$$

Then $H\cong \boldsymbol{P}^1$, and

$$H\subset \bigcap_{1\leqq i\leqq j\leqq q-1} V(X_iX_j-X_{i-1}X_{j+1}).$$

Hence we have $H=\phi_L(C)$, because $H\subset\phi_L(C)$ and $\phi_L(C)$ is irreducible. This contradicts $g\geqq 1$.

§ 3. Example II.

We continue assuming that the characteristic of the ground field K is not 2 and that a "curve" means a smooth curve over K.

In this section we will show that there are many examples of curves of genus g with invertible sheaf of degree $2g$ on which Corollary 1.4 works effectively. Note that since the degree of L is $2g$, the condition $H^1(C, L)=(0)$ in Corollary 1.4 is automatically satisfied. Therefore our problem is reduced to constructing many curves of genus g which have a normally generated ample invertible sheaf of degree $2g$.

PROPOSITION 3.1. *Let C be a curve of genus $g\geqq 5$. Suppose that there exists an invertible sheaf M of degree $g-1$ on C such that $\Gamma(M)$ is a base point free pencil. Then almost all invertible sheaves of degree $2g$ on C are ample with normal generation.*

The following lemma, B. Saint-Donat [8] called it "base point free pencil trick", plays an important role in the proof of our proposition.

LEMMA 3.2. (Mumford [5, p. 57], Saint-Donat [8, Lemma 2.6]) *Let M and N be invertible sheaves on a curve. Suppose that $\Gamma(M)$ is a base point free pencil. Then we have an isomorphism*

$$\mathrm{Ker}[\Gamma(M)\otimes\Gamma(N)\rightarrow\Gamma(M\otimes N)]\cong\Gamma(N\otimes M^{-1}).$$

We will use the following notation.

$\mathrm{Pic}^d(C)$: the connected component of the Picard scheme of C whose member represents an invertible sheaf of degree d,

G_d^r: the closed subvariety of $\mathrm{Pic}^d(C)$ representing the set of invertible sheaves of degree d and of projective dimension $\geqq r$,

F_d^r: the closed subvariety of $\mathrm{Pic}^d(C)$ defined by the image of the morphism

$$G_{d-1}^r\times C\ni(L, P)\longrightarrow L(P)\in\mathrm{Pic}^d(C)$$

(if $G_{d-1}^r=\phi$, then F_d^r means the void subset).

Note that $F_d^r\subset G_d^r$ and that if $r\geqq 1$, the set $G_d^r-F_d^r$ represents the set of inrertible sheaves free from base points, of degree d and of proejctive dimension r.

PROOF OF PROPOSITION 3.1. There exists an invertible sheaf M_0 of degree -1 such that $\Gamma(M_0)$ is a base point free pencil and $M_0^2 \neq \omega$, where ω is the ca- ıonical sheaf on C. Indeed, since $G^1_{g-1}-F^1_{g-1}$ is non-empty open in G^1_{g-1} by our ıssumption and since

$$\dim G^1_{g-1} \geqq g-4 \geqq 1 \quad [4, \text{ Theorem } 1],$$

$G^1_{g-1}-F^1_{g-1}$ has infinitely many elements. So there exists such an invertible sheaf. Ve put

$$V=G^1_{g+1}-F^1_{g+1}=\mathrm{Pic}^{g+1}(C)-F^1_{g+1}, \text{ and}$$
$$U=\{N\otimes M_0 \mid N\in V\}\subset \mathrm{Pic}^{2g}(C).$$

)bviously, V is non-empty open in $\mathrm{Pic}^{g+1}(C)$. Hence U is non-empty open in $\mathrm{Pic}^{2g}(C)$. We will show that any invertible sheaf in U is ample with normal generation. Let L be an invertible sheaf in U. By the generalized lemma of Castelnuvo [5, Theorem 2], we have natural map $\Gamma(L^m)\otimes\Gamma(L)\to\Gamma(L^{m+1})$ is surjective for $m\geqq 2$. Therefore it suffices to show that the natural map $\Gamma(L)\otimes\Gamma(L)\to\Gamma(L^2)$ is surjective. Consider the commutative diagram

$$\begin{array}{ccc}
\Gamma(L\otimes M_0^{-1})\otimes\Gamma(M_0)\otimes\Gamma(L) & \xrightarrow{1\otimes\psi_1} & \Gamma(L\otimes M_0^{-1})\otimes\Gamma(M_0\otimes L) \\
\downarrow & & \downarrow{\scriptstyle \psi_2} \\
\Gamma(L)\otimes\Gamma(L) & \longrightarrow & \Gamma(L^2),
\end{array}$$

vhere ψ_1 is the natural map $\Gamma(M_0)\otimes\Gamma(L)\to\Gamma(M_0\otimes L)$. By Lemma 3.2, we have Ker $\psi_1\cong\Gamma(L\otimes M_0^{-1})$ and $\mathrm{Ker}\,\psi_2\cong\Gamma(M_0^2)$. Therefore we have

$$\dim(\mathrm{Ker}\,\psi_1)=\dim[\Gamma(L\otimes M_0^{-1})]=2,$$
$$\dim[\Gamma(M_0)\otimes\Gamma(L)]=2(g+1),$$
$$\dim[\Gamma(M_0\otimes L)]=2g,$$
$$\dim(\mathrm{Ker}\,\psi_2)=\dim[\Gamma(M_0^2)]=g-1 \quad (\text{Note that } M_0^2\neq\omega),$$
$$\dim[\Gamma(L\otimes M_0^{-1})\otimes\Gamma(M_0\otimes L)]=4g \quad \text{and}$$
$$\dim[\Gamma(L^2)]=3g+1.$$

Ience ψ_1 and ψ_2 are surjective, and hence the natural map $\Gamma(L)\otimes\Gamma(L)\to\Gamma(L^2)$ is urjective. Q.E.D.

Next, we will give a sufficient condition for a curve to have an invertible heaf M of degree $g-1$ such that $\Gamma(M)$ is a base point free pencil. Our result ın it is a direct consequence of the following theorem of Martens and Mumford.

THEOREM OF MARTENS AND MUMFORD [6, Appendix]. *Let C be a curve of* *renus* $g\geqq 5$. *Then there exists integer* d, $3\leqq d\leqq g-2$, *such that* $\dim G^1_d \geqq d-3$ *if* *and only if C is hyperelliptic, or trigonal, or double covering of an elliptic curve* ($g\geqq 6$), *or non-singular plane quintic.*

PROPOSITION 3.3. *Let C be a curve of genus* $g\geqq 5$ *neither hyperelliptic, nor trigonal, nor double covering of an elliptic curve* ($g\geqq 6$), *nor non-singular plane quintic. Then there exists an invertible sheaf M of degree* $g-1$ *on C such that* $\Gamma(M)$ *is a base point free pencil.*

PROOF. We must prove that $G_{g-1}^1-F_{g-1}^1\neq\phi$ in our case. For this, it suffices to show that $\dim G_{g-1}^1>\dim F_{g-1}^1$. By the results of Martens, Kleiman and Laksov [4, Theorem 1 and 3, Theorem 5], we have

$$g-3\geqq\dim G_{g-1}^1\geqq g-4, \text{ and}$$
$$g-4\geqq\dim G_{g-2}^1\geqq g-6.$$

Note that if $G_{g-2}^1\neq\phi$, then

$$\dim F_{g-1}^1=\dim G_{g-2}^1+1 \quad [4, \text{p. } 115]$$

and that if $G_{g-2}^1=\phi$, then $F_{g-1}^1=\phi$. Suppose that $\dim G_{g-1}^1=\dim F_{g-1}^1$. Then $\dim G_{g-2}^1\geqq g-5$. This contradicts the theorem of Martens and Mumford. Q.E.D.

Finally, we state an elementary remark relative to our topic.

REMARK 3.4. If C is a curve of genus $g\geqq 4$, then there exists a non-special very ample invertible sheaf on C which is not normally generated.

Indeed, for a non-special normally generated ample invertible sheaf L, we have

$$\deg L\geqq g+\frac{1}{2}+\sqrt{2g+\frac{1}{4}}$$

because $\dim S^2\Gamma(L)\geqq\dim\Gamma(L^2)$. On the other hand, by the theorem of Halphen [2, Theorem 1.2], there exists a non-special very ample invertible sheaf of degree d, if $d\geqq g+3$.

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P A R T 3

ON PROJECTIVE NORMALITY AND DEFINING EQUATIONS OF A PROJECTIVE CURVE OF GENUS THREE EMBEDDED BY A COMPLETE LINEAR SYSTEM

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Introduction. Let $\Phi_L : C \hookrightarrow \mathbb{P}^{h^0(L)-1}$ be the projective embedding of a complete non-singular curve C of genus g by means of $\Gamma(L)$, where L is a very ample invertible sheaf on C. We will study the homogeneous coordinate ring and the ideal $I(L)$ of definition of $\Phi_L(C)$ in the case $g=3$. Our results are summarized in the following table. (If the genus of C is less than three, answers to the same kind of problems are easy.) In the table we will say that the homogeneous ideal $I(L)$ is generated strictly by its elements of degrees $\nu_1,\dots,\nu_m$ if $I(L)$ is generated by its elements of degrees $\nu_1,\dots,\nu_m$ and $I(L)$ is not generated by its elements of degrees $\nu_1,\dots,\hat{\nu}_j,\dots,\nu_m$ for any ν_j $(1 \leqq j \leqq m)$, where $\hat{\nu}_j$ means that ν_j is omitted.

$d \leqq 3$	There is no very ample invertible sheaf of degree $d \leqq 3$ on C.
$d = 4$	If C is hyperelliptic, then C has no very ample invertible sheaf of degree 4. If C is non-hyperelliptic, then there is only one very ample invertible sheaf of degree 4 on C, which is the canonical sheaf ω_C. $\Phi_{\omega_C}(C)$ is projectively normal. The homogeneous ideal $I(\omega_C)$ is generated strictly by its elements of degree 4.
$d = 5$	There is no very ample invertible sheaf of degree 5 on C.
$d = 6$	The set of very ample invertible sheaves of degree 6 on C coincides with $$\mathrm{Pic}^6(C) - \{\omega_C(P+Q) \mid P, Q \in C\}.$$ If C is hyperelliptic, then for a very ample invertible sheaf L of degree 6 on C, $\Phi_L(C)$ is not projectively normal and the homogeneous ideal I(L) is generated strictly by its elements of degrees 2 and 4. If C is non-hyperelliptic, then for a very ample invertible sheaf L of degree 6 on C, $\Phi_L(C)$ is projectively normal and the homogeneous ideal I(L) is generated strictly by its elements of degree 3.
$d = 7$	Any invertible sheaf of degree 7 on C is very ample. For an invertible sheaf L of degree 7 on C, $\Phi_L(C)$ is projectively normal and the homogeneous ideal I(L) is generated strictly by its elements of degrees 2 and 3.

$d \geqq 8$	Any invertible sheaf of degree $d \geqq 8$ on C is very ample. For an invertible sheaf L of degree $d \geqq 8$, $\Phi_L(C)$ is projectively normal and the homogeneous ideal I(L) is generated strictly by its elements of degree 2.

Notation and Terminology. We fix an algebraically closed field K. We use the word "curve" to mean a complete non-singular curve over K. For a finite dimensional vector space V, $S^m V$ means the m-th symmetric power of V. Let L be an invertible sheaf on a curve C. We denote by L^m the m-th tensor product $L^{\otimes m}$. For the vector space of global sections $\Gamma(L)$, we define $I_m(L)$ (or simply I_m) and I(L), by

$$I_m(L) = \mathrm{Ker}[S^m\Gamma(L) \longrightarrow \Gamma(L^m)]$$

and

$$I(L) = \bigoplus_{m\geq 0} I_m(L).$$

We denote by ω_C the canonical invertible sheaf on C, and by $\mathrm{Pic}^d(C)$ the set of invertible sheaves of degree d on C. For a coherent sheaf $\mathcal{F}$ on C, $h^i(\mathcal{F})$ is the dimension of the vector space $H^i(C,\mathcal{F})$ over K.

§1. Known facts.

This section consists of two parts. In the first part we will state some general facts concerning our problems. In the second part we will determine the set of very ample invertible sheaves on a curve of genus three.

Let L be an invertible sheaf on a projective variety X. According to Mumford [4], we say that L is normally generated if L is ample and the natural map $\Gamma(L)^{\otimes m} \longrightarrow \Gamma(L^m)$ is surjective for any positive integer m. Obviously, $\Gamma(L)^{\otimes m} \longrightarrow \Gamma(L^m)$ is surjective for all $m \geq 1$ if and only if $\Gamma(L^m)\otimes\Gamma(L) \longrightarrow \Gamma(L^{m+1})$ is surjective for all $m \geq 1$. If X is a normal variety and L is normally generated, then L is very ample and $\Phi_L(C)$ is projectively normal, and the converse is true too.

The following theorem was proved by Mumford [4, Corollary to Theorem 6].

Theorem 1.1. *Let* L *be an invertible sheaf of degree* d *on a curve of genus* g. *If* $d \geq 2g+1$, *then* L *is normally generated*.

A proof of the following "Noether's Theorem" is found in [6].

Theorem 1.2. *Let* C *be a curve. Then the following conditions are equivalent*:

(1) C *is non-hyperelliptic, and*

(2) *the canonical sheaf* ω_C *is normally generated*.

Concerning the ideal of definition $I(L)$ of $\Phi_L(C)$, Saint-Donat [5] proved,

Theorem 1.3. *Let* L *be an invertible sheaf of degree* d *on a curve of genus* g.

(a) *If* $d \geq 2g+1$, *then* $I(L)$ *is generated by* I_2 *and* I_3.

(b) *If* $d \geq 2g+2$, *then* $I(L)$ *is generated by* I_2.

In the previous paper [3], we learned a slight generalization of Theorem 1.3 (a):

Theorem 1.4. *If* L *is a normally generated invertible sheaf on a curve* C *with* $H^1(C,L)=(0)$, *then* $I(L)$ *is generated by* I_2 *and* I_3.

An invertible sheaf L on C is very ample if and only if $\Gamma(L)$ separates two distinct points and infinitely near points, so we have:

Proposition 1.5. An invertible sheaf on C is very ample if and only if

$$h^0(C,L(-P-Q)) = h^0(C,L)-2$$

for any $P, Q \in C$ (including the case $P=Q$).

A precise proof of Proposition 1.5 can be found in [2, IV Proposition 3.1].

Corollary 1.5.1. If L is an invertible sheaf on a curve of genus g, whose degree is not less than 2g+1, then L is very ample.

Corollary 1.5.2. An invertible sheaf L of degree 2g on a curve C of genus g is not very ample if and only if L is isomorphic to $\omega_C(P+Q)$ for some points $P, Q \in C$ (may be $P=Q$).

The following two propositions are useful to determine the set of very ample invertible sheaves on a curve of genus three. The first one is "Halphen's Theorem" [2, IV Proposition 6.1], and the second one is famous as "Clifford's Theorem".

Proposition 1.6. Let C be a curve of genus $g \geq 2$, and let d be an integer. Then C has a very ample invertible sheaf L of degree d with $h^1(L)=0$ if and only if $d \geq g+3$.

Proposition 1.7. Let L be an invertible sheaf on C with $h^0(L)>0$ and $h^1(L)>0$. Then

$$2(h^0(L)-1)\leq \deg L.$$

Furthermore, equality occurs if and only if either $L\cong\mathcal{O}_C$ or $L\cong\omega_C$ or C is hyperelliptic and $L\cong(f^*\mathcal{O}_{P^1}(1))^{\otimes r}$ $(0\leq r\leq g-1)$, where $f:C\longrightarrow P^1$ is a double covering.

Corollary 1.7.1. Let C be a curve of genus $g\geq 1$, and let L be an invertible sheaf on C with $h^0(L)>0$ and $h^1(L)>0$. Then

$$h^0(L)\leq g.$$

Furthermore, equality occurs if and only if $L\cong\omega_C$.

Remark 1.8. Let L be an invertible sheaf on a curve of genus $g\leq 2$. Then L is very ample if and only if $\deg L\geq 2g+1$.

Proof. In the case of $g=0$ or 1, our remark is proved easily. If $g=2$ and L is very ample, then we have $h^1(L)=0$ by Corollary 1.7.1. Therefore our remark follows from Corollary 1.5.1 and Proposition 1.6.

Proposition 1.9. Let C be a curve of genus three. Then we have,

d	The set of very ample invertible sheaves of degree d on C.
$d\leq 3$	None.
$d=4$	None, if C is hyperelliptic. $\{\omega_C\}$, if C is non-hyperelliptic.
$d=5$	None.
$d=6$	$\mathrm{Pic}^6(C)-\{\omega_C(P+Q)\mid P,Q\in C\}$.
$d\geq 7$	$\mathrm{Pic}^d(C)$.

Proof. In the case of $d \geq 6$, our results follows from Corollaries 1.5.1 and 1.5.2. By Halphen's Theorem there is no very ample invertible sheaf L of degree $d \leq 5$ with $h^1(L)=0$. By virtue of Corollary 1.7.1, a possibility of a very ample invertible sheaf L of degree $d \leq 5$ with $h^1(L) > 0$ is only the canonical invertible sheaf ω_C. On the other hand, ω_C is very ample if and only if C is non-hyperelliptic. This completes the proof.

§2. Projective normality.

In this section we will determine the set of normally generated invertible sheaves on a curve C of genus three. The answer to the same kind of problem for a curve of genus $g \leq 2$ is easy. Indeed, by Remark 1.8 and by Theorem 1.1 an invertible sheaf L is normally generated if and only if L is very ample.

In the case of genus three, by Theorem 1.1 an invertible sheaf L is normally generated if $\deg L \geq 7$, and by Theorem 1.2 the canonical invertible sheaf ω_C is normally generated if C is non-hyperelliptic. Therefore, to show our table it suffices to prove the following theorem.

Theorem 2.1. *Let C be a curve of genus three, and let L be a very ample invertible sheaf of degree 6 on C. Then L is normally generated if and only if C is non-hyperelliptic.*

<u>Proof</u>. (Step 1) First we will show that L is normally generated if and only if $\Phi_L(C)$ is not contained any quadric surface in $\mathbb{P}^3$. Indeed L is normally generated if and only if $\hat{\Gamma}(L^m)\otimes\hat{\Gamma}(L) \longrightarrow \hat{\Gamma}(L^{m+1})$ is surjective for all $m \geq 1$. By the lemma of Castelnuovo [4], these maps are surjective when $m \geq 2$. Hence,

$$
\begin{aligned}
& L \text{ is normally generated,} \\
\Longleftrightarrow\ & \hat{\Gamma}(L)\otimes\hat{\Gamma}(L) \longrightarrow \hat{\Gamma}(L^2) \text{ is surjective,} \\
\Longleftrightarrow\ & S^2\hat{\Gamma}(L) \longrightarrow \hat{\Gamma}(L^2) \text{ is surjective.}
\end{aligned}
$$

Since $\dim S^2\hat{\Gamma}(L) = \dim \hat{\Gamma}(L^2)$, these conditions are equivalent to the condition that $S^2\hat{\Gamma}(L) \longrightarrow \hat{\Gamma}(L^2)$ is injective. The last condition means that $\Phi_L(C)$ is not contained any quadric surface in $\mathbb{P}^3$.

It is well known that a quadric surface in $\mathbb{P}^3$ is a union of planes (may be non-reduced) or an irreducible quadric cone, which is a projective cone of a 2-uple embedding of $\mathbb{P}^1$, or a non-singular quadric surface, which is a Segre embedding of $\mathbb{P}^1\times\mathbb{P}^1$ into $\mathbb{P}^3$. Obviousely, a union of planes dose not contain $\Phi_L(C)$. In the next step, we will show that an irreducible quadric cone does not contain $\Phi_L(C)$ too.

(Step 2) Let F be an irreducible quadric cone with vertex O in $\mathbb{P}^3$. Let

$$\mathbb{P}^3\times\mathbb{P}^2 \supset \tilde{F} \xrightarrow{\pi} F \subset \mathbb{P}^3$$

be the monoidal transformation of F with center O. Then $p_2|\tilde{F}$ factors through a 2-uple embedding of $\mathbb{P}^1$:

$$
\begin{array}{ccc}
\tilde{F} & \xrightarrow{\quad p_2|F \quad} & \mathbb{P}^2 \\
 & {}_{q}\searrow \quad \nearrow & \\
 & \mathbb{P}^1 &
\end{array}
$$

and then $F \xrightarrow{q} \mathbb{P}^1$ coincides with the geometrically ruled surface $\mathrm{Proj}(S(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2))) \longrightarrow \mathbb{P}^1$ [2, Example 2.11.4].

If C_0 is the inverse image $\widetilde{\pi}^{-1}(0)$ of 0 and if f is a fibre of q of a point, then Pic(F) is isomorphic to $\mathbb{Z}C_0 \oplus \mathbb{Z}f$. A canonical divisor K_F of F is linearly equivalent to $-2C_0-4f$, and the intersection pairing on F is given by

$$C_0^2 = -2, \quad C_0.f = 1 \quad \text{and} \quad f^2 = 0 \quad [1, p.33].$$

Let D be a curve of genus g on F, and let $\widetilde{D}$ be the strict transform of D on $\widetilde{F}$. Assume that $\widetilde{D}$ is linearly equivalent to aC_0+bf. Then by the adjunction formula, we have

$$2g-2 = -2(a^2-ab+b).$$

If the vertex 0 lies on D, then $1 = \widetilde{D}.C_0 = -2a+b$. Therefore we have $g = a(a-1)$, so g is even. If the vertex 0 does not lie on D, then $0 = \widetilde{D}.C_0 = -2a+b$. Therefore we have $g = (a-1)^2$, so g is a square number. We conclude that any curve of genus 3 does not lie on F.

(Step 3) In this step, we will show that if a curve C of genus 3 and degree 6 in $\mathbb{P}^3$ lies on a non-singular quadric surface F, then C is hyperelliptic.

First, note that

$$\mathrm{Pic}(F) = \mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = p_1{}^*\mathrm{Pic}(\mathbb{P}^1) \oplus p_2{}^*\mathrm{Pic}(\mathbb{P}^1) = \mathbb{Z} \oplus \mathbb{Z},$$

where $p_1{}^*\mathcal{O}_{\mathbb{P}^1}(1)$ corresponds to (1,0) and $p_2{}^*\mathcal{O}_{\mathbb{P}^1}(1)$ corresponds to (0,1). Obviously, a canonical divisor K_F corresponds to (-2,-2), and a hyperplane section on F corresponds to (1,1). The intersection pairing on F is given by $D.D' = ab' + ba'$ for two divisors D and D' corresponding to (a,b) and (a',b') respectively.

Assume that C corresponds to (a,b). Then we have

$$6 = \deg_{\mathbb{P}^3} C = (C.H)_{\mathbb{P}^3} = (C.H|_F)_F = a+b,$$

where H is a hyperplane of $\mathbb{P}^3$, and

$$2\cdot 3-2 = C.(C+K_F) = 2ab-2a-2b.$$

Hence, we have "a=4, b=2" or "a=2, b=4". Since $F = \mathbb{P}^1 \times \mathbb{P}^1$, we may assume that C corresponds to $(4,2)$. Consider the diagram:

$$C \subset F = \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{p_1} \mathbb{P}^1 .$$

If $f: C \to \mathbb{P}^1$ is defined by the restriction of p_1 to C, then f is surjective and then

$$\deg f = \deg f^*\mathcal{O}_{\mathbb{P}^1}(1) = \deg p_1^*\mathcal{O}_{\mathbb{P}^1}(1)|C = 2.$$

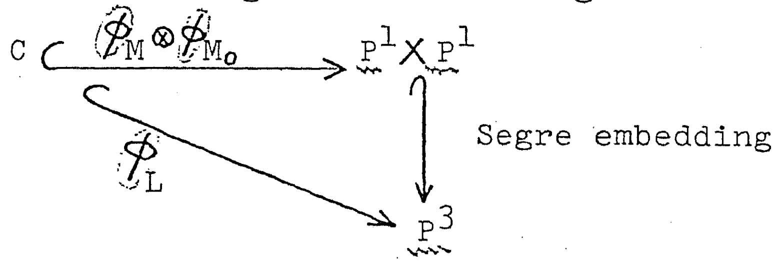
Therefore C is hyperelliptic.

(Step 4) The final step, for a given hyperelliptic curve C of genus 3 and a given very ample invertible sheaf L of degree 6 on C, we construct a non-singular quadric surface in $\mathbb{P}^3$ containing $\Phi_L(C)$.

Since C is hyperelliptic, there is a morphism $f: C \to \mathbb{P}^1$ of degree 2. We put $M_0 = f^*\mathcal{O}_{\mathbb{P}^1}(1)$, and $M = L \otimes M_0^{-1}$. Then the canonical map

$$(\#) \qquad \Gamma(M) \otimes \Gamma(M_0) \longrightarrow \Gamma(L)$$

is an isomorphism. To prove this, note that $\Gamma(M_0)$ is a base point free pencil. By the "base point free pencil trick" [6], we have an isomorphism $\mathrm{Ker}[\Gamma(M) \otimes \Gamma(M_0) \to \Gamma(L)] \cong \Gamma(M \otimes M_0^{-1})$. Assume that $\Gamma(M \otimes M_0^{-1}) \neq (0)$. Then there are two points P and Q on C such that $M \otimes M_0^{-1} \cong \mathcal{O}_C(P+Q)$. Hence $L \cong M_0^{\,2}(P+Q) \cong \omega_C(P+Q)$. This contradicts the very ampleness of L. Therefore the map $(\#)$ is injective. On the other hand $\dim(\Gamma(M) \otimes \Gamma(M_0)) = \dim \Gamma(L)$, so the map $(\#)$ is an isomorphism. By the isomorphism $(\#)$ we obtain the following commutative diagram:



This completes our proof.

§3. Defining equations.

In this section we will study the homogeneous ideal $I(L)$ for a curve of genus $g \leqq 3$ with a very ample invertible sheaf L.

Remark 3.1. Let C be a curve of genus $g \leqq 2$, and let L be a very ample invertible sheaf of degree d on C.

(a) If $d \geqq 2g+2$, then $I(L)$ is generated strictly by I_2.

(b) If $g=2$ and $d=5$ $(=2g+1)$, then $I(L)$ is generated strictly by I_2 and I_3.

(c) If $g=1$ and $d=3$ $(=2g+1)$, then $I(L)$ is generated strictly by I_3.

(d) If $g=0$ and $d=1$ $(=2g+1)$, then $I(L) = (0)$.

A proof of this remark is easy, so we omit it.

Theorem 3.2. Let L be a very ample invertible sheaf of degree d on a curve C of genus three.

(a) If $d \geqq 8$, then $I(L)$ is generated strictly by I_2.

(b) If $d = 7$, then $I(L)$ is generated strictly by I_2 and I_3.

(c) If C is non-hyperelliptic and $d=6$, then $I(L)$ is generated strictly by I_3.

(d) If C is non-hyperelliptic and $L=\omega_C$, then $I(\omega_C)$ is generated strictly by I_4.

Proof. (a) It is a special case of Theorem 1.3 (b).

(b) By Theorem 1.3 (a), $I(L)$ is generated by I_2 and I_3. Assume that $I(L)$ is generated by I_2. Since $\dim I_2(L) = 3$, $\Phi_L(C)$ is a complete intersections of three quadric hypersurfaces in P^4. Therefore we have $\deg_{P^4} \Phi_L(C) = 8$. This contradicts the fact $\deg L = 7$.

(c) By Theorem 1.4, $I(L)$ is generated by I_2 and I_3. On the other hand, by the proof of Theorem 2.1 we have $I_2(L) = (0)$.

(d) It is well known that $\Phi_{\omega_C}(C)$ is plane quartic.

Q.E.D.

By this theorem, to show our table it suffices to prove the following theorem.

Theorem 3.3. Let L be a very ample invertible sheaf of degree 6 on a hyperelliptic curve C of genus 3. Then $I(L)$ is generated strictly by I_2 and I_4.

A proof of the theorem will be given at the last part of this section.

Let $M_1,\dots,M_r$ be invertible sheaves on a projective variety. $R(M_1,\dots,M_r)$ denotes the kernel of the canonical map:

$$\Gamma(M_1)\otimes\dots\otimes\Gamma(M_r) \longrightarrow \Gamma(M_1\otimes\dots\otimes M_r).$$

Lemma 3.4. *Let* L *be an ample invertible sheaf on a projective variety, and let* m *be a positive integer greater than* 1. *Assume that*

$$\Gamma(L)^{\otimes(m-1)} \longrightarrow \Gamma(L^{m-1})$$

and

$$\Gamma(L)^{\otimes m} \longrightarrow \Gamma(L^{m})$$

are surjective. Then

$$\Gamma(L)\otimes R(L^{m-1},L) \longrightarrow R(L^{m},L)$$

is surjective if and only if

$$I_m(L)\otimes\Gamma(L) \longrightarrow I_{m+1}(L)$$

is surjective.

Definition 3.5. (1) Let X be a normal closed subvariety of $\mathbb{P}^N$, and let $R(X) = \bigoplus_{i=0}^{\infty} R(X)_i$ be the homogeneous coordinate ring of X. $\widetilde{R(X)}$ denotes the normalization of $R(X)$. It is well known that $\widetilde{R(X)}$ is a graded ring too. We can define the non-negative integer $n(X\subset\mathbb{P}^N)$ by

$$n(X\subset\mathbb{P}^N) = \mathrm{Min}\left\{ n\in\mathbb{N} \;\middle|\; \widetilde{R(X)}_i = R(X)_i \text{ for any } i\geq n \right\}.$$

(2) Let L be a very ample invertible sheaf on the normal projective variety X. We define the non-negative integer $n(L)$ by

$$n(L) = n(X \overset{\Phi_L}{\hookrightarrow} \mathbb{P}^{h^0(L)-1}).$$

It is easy to show that

$$n(L) = \mathrm{Min}\left\{ n\in\mathbb{N} \;\middle|\; \Gamma(L)^{\otimes i} \longrightarrow \Gamma(L^i) \text{ is surjective for any } i\geq n \right\}$$

Corollary 3.6. *Let* L *be a very ample invertible sheaf on an n-dimensional projective variety* X. *Assume that* $H^1(X, L^j) = (0)$ *for any integers* $i, j > 0$. *If* $\alpha = \mathrm{Max}(n+3, n(L)+1)$ *then* $I(L)$ *is generated by* $I_2, \ldots, I_\alpha$.

Proofs of Lemma 3.4 and Corollary 3.6 are similar to those of [3, Proposition 1.2 and Corollary 1.3].

Next, we will calculate $n(L)$ for a very ample invertible sheaf L of degree 6 on a hyperelliptic curve of genus 3.

Proposition 3.7. *Let* L *be a very ample invertible sheaf of degree 6 on a hyperelliptic curve* C *of genus* 3. *Then* $\Gamma(L)^{\otimes m} \xrightarrow{\beta_m} \Gamma(L^m)$ *is surjective for any* $m \geq 3$, *i.e.,* $n(L) = 3$.

Proof.(*) We prove the surjectivity of $\beta_m : \Gamma(L)^{\otimes m} \longrightarrow \Gamma(L^m)$ ($m \geq 3$) by induction on m. For a given $m \geq 3$, we consider the following commutative diagram:

$$\begin{array}{ccc} \Gamma(L)^{\otimes(m+1)} & \xrightarrow{\beta_m \otimes 1} & \Gamma(L^m) \otimes \Gamma(L) \\ \downarrow \beta_{m+1} & \swarrow \gamma_m & \\ \Gamma(L^{m+1}) & & \end{array}$$

By the induction hypothesis β_m is surjective, and also $\beta_m \otimes 1$ is surjective. By the lemma of Castelnuovo γ_m is surjective, and also β_{m+1} is surjective. Therefore, to prove our assertion, it suffices to prove the surjectivity of β_3.

(*) The author expresses his heartfelt thanks to the referee for suggesting a simplification of the proof.

By Step 4 in the proof of Theorem 2.1, there is an irreducible quadric surface Q in P^3 containing $\Phi_L(C)$. The curve $\Phi_L(C)$ can not be contained in a quadric surface other than Q, because $\Phi_L(C)$ is not contained in any P^2 and $\deg \Phi_L(C) = 6$. Hence we have $I_2(L) = K.q$, where q is a quadratic form defining the quadric surface Q. If $\Phi_L(C)$ is contained in an irreducible cubic surface H, then $\Phi_L(C)$ coincides with the complete intersection $Q \cap H$, because $\Phi_L(C)$ and $Q \cap H$ have degree 6. But the genus of a curve which is a complete intersection of surfaces of degrees 2 and 3 is equal to 4. This is a contradiction. Therefore, we have $I_3(L) = K.q \odot s_1 \oplus \dots \oplus K.q \odot s_4$, where $\{s_1, \dots, s_4\}$ is a basis of $\Gamma(L)$ and the symbol $\odot$ means a symmetric product. Consider the exact sequence

$$0 \longrightarrow I_3(L) \longrightarrow S^3\Gamma(L) \xrightarrow{\beta_3} \Gamma(L^3) .$$

The left hand vector space has dimension 4 by the above result, the middle vector space has dimension 20, and right hand vector space has dimension 16 by the theorem of Riemann-Roch. So we conclude that β_3 is surjective. Q.E.D.

Proof of Theorem 3.3. By Corollary 3.6 and Proposition 3.7, $I(L)$ is generated by I_2, I_3 and I_4. By the proof of Proposition 3.7, $I_2 = K.q$ and $I_3 = K.q \odot s_1 \oplus \dots \oplus K.q \odot s_4$. Therefore, $I(L)$ is generated by I_2 and I_4. Obviously, I_2 does not generate $I(L)$. This completes the proof.

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P A R T 4

AUTOMORPHISMS OF PRIME ORDER OF CURVES

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If an order q of an automorphism of an algebraic curve of genus $g \geqslant 2$ is prime, then $q \leqslant 2g+1$. In this paper we determine all curves having an automorphism of order $2g+1$ when $2g+1$ is prime.

Introduction.

Let k be an algebraically closed field of characteristic $p \geqslant 0$, and let C be a complete non-singular curve of genus $g \geqslant 2$ over k. It is known that the group of automorphisms of C is finite. Furthermore, concerning a prime number which may be an order of an automorphism of C, we have the following result.

THEOREM 1. *If a prime number* q *is an order of an automorphism on* C, *then either* $q \leqslant g + 1$ *or* $q = 2g - 1$ $(g > 2)$ *or* $q = 2g + 1$.

The main purpose of the paper is to determine all curves of genus $g \geqslant 2$ which have an automorphism of prime order $q > g + 1$.

THEOREM 2. *Let* q *be a prime number.*

(a) *Assume that* $q = 2g + 1 \neq p$. *Then a curve* C *has an automorphism of order* q *if and only if* C *is birationally equivalent to one of the following plane curves:*

$$y^{m-r}(y-1)^r = x^q \qquad (1 \leqslant r < m \leqslant g + 1).$$

(b) *Assume that* $q = p = 2g + 1$. *Then a curve* C *has an automorphism of order* q *if and only if* C *is birationally equivalent to the plane curve*

$$y^2 = x^q - x.$$

(c) <u>Assume that</u> $q = 2g - 1$ <u>and</u> $g > 2$. <u>Then there is no automorphism of order</u> q <u>on</u> C.

Part (b) of Theorem 2 was given by Roquette [2], and part (c) was given by Accola [1] in the case k is the complex number field. In the case $g = 3$, Theorem 2 was established by Komiya's unpublished work.

The author expresses his sincere thanks to Professor K. Komiya who gave him useful suggestion.

<u>NOTATION</u>. We denote by $\mathrm{Aut}(C)$ the group of automorphisms of a complete non-singular curve C. Let σ be an automorphism of C. We denote by $\langle\sigma\rangle$ the subgroup of $\mathrm{Aut}(C)$ generated by σ, and by $\mathrm{Fix}(\sigma)$ the set of fixed points of σ. For a finite set X, we denote by $\#(X)$ the cardinality of X.

1. Proof of Theorem 1

Throughout the paper, we denote by p the characteristic of the ground field k, and by g the genus of a non-singular curve C over k. Furthermore, q means a prime number.

Let σ be an automorphism of order q of C, and g' the genus of $C/\langle\sigma\rangle$. By Hurwitz's formula, we have

$$(1) \qquad 2g - 2 = q(2g' - 2) + \deg R,$$

where R is the ramification divisor of a natural morphism $\pi: C \longrightarrow C/\langle\sigma\rangle$.

<u>Proof of Theorem 1</u>. First we assume that $q \neq p$. Since the ramification of π is tame, the formula (1) is rewritten as

$$(2) \qquad 2g - 2 = q(2g' - 2) + r(q - 1),$$

where $r = \#[\mathrm{Fix}(\sigma)]$. Using the formula (2), we estimate an upper bound for q. If $g' \geqslant 2$, then $2g - 2 \geqslant 2q + r(q - 1) \geqslant 2q$, so we have $q \leqslant g - 1$ $(\leqslant g + 1)$. By (2) it is impossible that $g' = 1$ and $r = 0$. If $g' = 1$ and

$r = 1$, then $q = 2g - 1$. If $g' = 1$ and $r \geqslant 2$, then $2g - 2 \geqslant 2(q - 1)$, so we have $q \leqslant g$ $(\leqslant g + 1)$. By (2), it is impossible that $g' = 0$ and $r \leqslant 2$. If $g' = 0$ and $r = 3$, then $q = 2g + 1$. If $g' = 0$ and $r \geqslant 4$, then $2g - 2 \geqslant -2q + 4(q - 1)$, so we have $q \leqslant g + 1$.

Next we assume that $q = p$. We use the following lemma.

LEMMA 1.1 (Serre [4]). *Let G be a subgroup of $\mathrm{Aut}(C)$, and let $\pi : C \longrightarrow C/G$ be a natural morphism. For a point P on C, let t be a local parameter at P, and s a local parameter at $\pi(P)$. Then,*

$$v_P\left(\frac{ds}{dt}\right) = \sum_{\sigma \in G_P \setminus \{1_C\}} v_P(t - \sigma^* t).$$

Here v_P is the valuation at P, and G_P is the isotropy subgroup at P of G.

Let P be a fixed point of σ, and t a local parameter at P. Since the order of σ coincides with the characteristic of k, $\sigma^* t$ may be expanded by t as

$$\sigma^* t = t + c_m t^m + c_{m+1} t^{m+1} + \dots \quad (c_m \neq 0).$$

Note that the above integer m does not depend on a choice of local parameter. Therefore we denote this integer by $m(P)$. By virtue of Lemma 1.1, we have

$$\begin{aligned} \deg R &= \sum_{P \in \mathrm{Fix}(\sigma)} \sum_{i=1}^{p-1} v_P(t_P - \sigma^{i*} t_P) \\ &= \sum_{P \in \mathrm{Fix}(\sigma)} m(P)(p - 1), \end{aligned}$$

where t_P is a local parameter at P. Hence the formula (1) is rewritten as

(3) $$2g - 2 = q(2g' - 2) + \sum_{P \in \mathrm{Fix}(\sigma)} m(P)(q - 1).$$

Using the formula (3), we can estimate an upper bound for q by similar arguments to the first case.

REMARK 1.2. By formulas (2) and (3) in the above proof, we have:

(A) *if* $q = 2g + 1$, *then*

(a) $q \neq p$, $g' = 0$ *and* $r = 3$

or

(b) $q = p$, $g' = 0$ _and_ $r = 1$;

(B) _if_ $q = 2g - 1$ _and_ $g > 2$, _then_

(c) $q \neq p$, $g' = 1$ _and_ $r = 1$.

Here r _is the cardinality of_ $\mathrm{Fix}(\sigma)$.

2. Lemmas

Let L be an invertible sheaf on a complete non-singular curve C. We denote by $\Gamma(L)$ the vector space of global sections of L. If L is generated by its global sections, we may define a morphism $\varphi_L : C \to \mathbb{P}(\Gamma(L))$ by means of $\Gamma(L)$, where $\mathbb{P}(\Gamma(L))$ is the projective space of hyperplanes in $\Gamma(L)$. Moreover, if L is stable under an automorphism σ of C, then there is a unique automorphism $\bar{\sigma}$ of $\mathbb{P}(\Gamma(L))$ such that $\varphi_L \circ \sigma = \bar{\sigma} \circ \varphi_L$.

For an invertible sheaf L, we denote by $\mathrm{Aut}^L(C)$ the subgroup

$$\{ \sigma \in \mathrm{Aut}(C) \mid \sigma^* L \cong L \}$$

of $\mathrm{Aut}(C)$.

By the above arguments, we may define a group homomorphism

$$r_L : \mathrm{Aut}^L(C) \longrightarrow \mathrm{Aut}(\mathbb{P}(\Gamma(L)))$$

by $\sigma \longmapsto \bar{\sigma}$, for an invertible sheaf L which is generated by its global sections.

LEMMA 2.1. _Let_ L _be an invertible sheaf of degree_ d _on_ C, _which is generated by its global sections_. _Suppose that_ $\dim \Gamma(L) = 2$. _If an automorphism_ σ _of order_ m _of_ C _satisfies conditions_

(i) $\sigma \in \mathrm{Aut}^L(C)$

and

(ii) $(m, d!) = 1$,

then we have:

(a) $\mathrm{Aut}^L(C) \supseteq \langle\sigma\rangle \xrightarrow{r_L|\langle\sigma\rangle} \langle\bar{\sigma}\rangle \subset \mathrm{Aut}(\mathbb{P}^1)$ _is an isomorphism, and_

(b) _for a point_ P _on_ C, $P \in \mathrm{Fix}(\sigma)$ _if and only if_ $\varphi_L(P) \in \mathrm{Fix}(\bar{\sigma})$.

Proof. Our assertions are trivial when $m = 1$, so we may assume that $m > 1$.

(a) Let G be the subgroup $\mathrm{Ker}\, r_L \cap \langle\sigma\rangle$ of $\mathrm{Aut}(C)$. Then $\varphi_L : C \longrightarrow \mathbb{P}^1$ factors through a natural morphism $f : C \longrightarrow C/G$. Since $\deg \varphi_L = d$ and $\deg f = \#(G)$, d is divided by $\#(G)$. By the assumption (ii), we conclude that $\#(G) = 1$.

(b) The "only if" part is trivial. If $\varphi_L(P) \in \mathrm{Fix}(\sigma)$, then $\varphi_L^{-1}(\varphi_L(P))$ is σ-stable as a divisor. Let $\{P_1, \ldots, P_{d'}\}$ be the support of the divisor $\varphi_L^{-1}(\varphi_L(P))$. Since $\deg \varphi_L = d$, we have $d' \leq d$. By σ-stability of the divisor $\varphi_L^{-1}(\varphi_L(P))$, we have a group homomorphism from $\langle\sigma\rangle$ to the permutation group on the set $\{P_1, \ldots, P_{d'}\}$, which is defined by

$$\sigma^j \longmapsto \begin{pmatrix} P_1 & , \ldots, & P_{d'} \\ \sigma^j(P_1) & , \ldots, & \sigma^j(P_{d'}) \end{pmatrix}.$$

By the assumption (ii), this homomorphism must be trivial, i.e., each of the P_i is σ-stable.

The following lemma is elementary, so we omit its proof.

LEMMA 2.2. Let τ be an automorphism of finite order, say m, on the projective line $\mathbb{P}^1_k$ over an algebraically closed field k of characteristic $p \geq 0$. When $p > 0$, we write $m = np^e$ with $(n, p) = 1$.

(A) If $p > 0$, then either $e = 0$ or $e = 1$. (In the latter case, we have $n = 1$.)

(B) If $m > 1$, then we have:

(a) $\#[\mathrm{Fix}(\tau)] = 2$ when "$p = 0$" or "$p > 0$ and $e = 0$";

(b) $\#[\mathrm{Fix}(\tau)] = 1$ when $p > 0$ and $e = 1$.

3. Proof of Theorem 2

PROPOSITION 3.1. *Let C be a complete non-singular curve of genus $g \geqslant 2$ over an algebraically closed field k of characteristic $p \geqslant 0$. Assume that $q = 2g + 1$ is a prime number different from p. Let L be an invertible sheaf of degree $d < q$ on C such that L is generated by its global sections and $\dim \Gamma(L) = 2$. If C has an automorphism $\sigma \in \mathrm{Aut}^L(C)$ of order q, then C is birationally equivalent to one of the following plane curves:*

$$y^{d-r}(y - 1)^r = x^q \qquad (1 \leqslant r < d).$$

Proof. By Remark 1.2, we may put

$$\mathrm{Fix}(\sigma) = \{P, Q, R\}.$$

Let x be a morphism $\varphi_L : C \longrightarrow \mathbb{P}^1$, and $\bar{\sigma}$ an automorphism of $\mathbb{P}^1$ such that $x \circ \sigma = \bar{\sigma} \circ x$. By Lemma 2.1 and Lemma 2.2, we may assume that

$$\mathrm{Fix}(\bar{\sigma}) = \{x(P), x(Q)\},$$

$$x^{-1}(x(P)) = dP$$

and

$$x^{-1}(x(Q)) = (d - r)Q + rR,$$

where r is the ramification index at R of x. We can select a coordinate of $\mathbb{P}^1$ such that $x(P) = \infty$ and $x(Q) = x(R) = 0$. Then we have

(4) $\quad \mathrm{div}\, x = (d - r)Q + rR - dP,$

where div x means the divisor of a function x. On the other hand, let y be a natural morphism $C \longrightarrow C/\langle\sigma\rangle = \mathbb{P}^1$. Since $y(P)$, $y(Q)$ and $y(R)$ differ from each other, we can select a coordinate of $\mathbb{P}^1$ such that $y(Q) = 0$, $y(R) = 1$ and $y(P) = \infty$. Then we have

(5) $\quad \mathrm{div}\, y = qQ - qP$

and

(6) $\quad \mathrm{div}\, (y - 1) = qR - qP.$

By (4), (5) and (6), we have a relation

$$y^{d-r}(y - 1)^r = cx^q$$

for a suitable non-zero constant $c \in k$. It is easy to show that the function field of C coincides with

$k(\sqrt[q]{c}.x,\ y)$. Therefore, C is birationally equivalent to the plane curve $y^{d-r}(y-1)^r = x^q$. This completes the proof.

LEMMA 3.2. <u>Let</u> C <u>be a plane curve over</u> k <u>defined by</u>

$$y^s(y-1)^r = x^t,$$

<u>where</u> $t > r + s$, $(t, r) = 1$, $(t, s) = 1$, $(t, r + s) = 1$, <u>and</u> $(t, p) = 1$ <u>whenever</u> $p > 0$. <u>Then the genus of the non-singular model of</u> C <u>is equal to</u> $(t-1)/2$.

A proof of the lemma is not difficult, so we omit it.

<u>Proof of Theorem</u> 2. First we prove "if" parts of (a) and (b). Assume that $q = 2g + 1 \neq p$. By Lemma 3.2, the genus of a non-singular curve C which is birationally equivalent to one of curves in (a) is equal to g. Define an automorphism σ of C by $\sigma^*(x) = \eta.x$ and $\sigma^*(y)=y$, where η is a primitive q-th root of 1. Then the order of σ is equal to q. Assume that $q = 2g + 1 = p$. It is obvious that the non-singular curve C which is birationally equivalent to the plane curve $y^2 = x^q - x$ is of genus g. Define an automorphism σ of C by $\sigma^*(x) = x + 1$ and $\sigma^*(y) = y$. Then the order of σ is equal to q.

Let q be a prime number such that one of assumptions in (a) or (b) or (c) occurs. Let σ be an automorphism of order q on C. In any case, by Remark 1.2 $\mathrm{Fix}(\sigma)$ is non-empty. We fix a point $P \in \mathrm{Fix}(\sigma)$. Let m be a positive integer defined by

$$m = \mathrm{Min}\{\, n \in \mathbb{Z} \mid \dim \Gamma(C, \underline{O}_C(mP)) = 2 \,\}.$$

If L is the invertible sheaf $\underline{O}_C(mP)$, then $\deg L \leqslant g + 1$, L is generated by its global sections, and $\sigma \in \mathrm{Aut}^L(C)$. Moreover, there is an automorphism $\bar{\sigma}$ of order q on $\mathbb{P}^1$ such that $\varphi_L \circ \sigma = \bar{\sigma} \circ \varphi_L$.

If $q = 2g + 1 \neq p$, then we can apply Proposition 3.1 for C, L and σ. Therefore, C is birationally equivalent to one of the following plane curves:

$$y^{m-r}(y-1)^r = x^q \quad (1 \leqslant r < m).$$

Assume that $q = 2g - 1$. By Remark 1.2 and Lemma 2.2, we have $\#[\mathrm{Fix}(\sigma)] = 1$ and $\#[\mathrm{Fix}(\overline{\sigma})] = 2$. By Lemma 2.1 (b), those contradict each other. Hence C does not have such an automorphism.

The remaining part of this proof, we shall devote to a proof of (b).

First we shall show that φ_L is a double covering. Since φ_L is separable and tame, by Hurwitz's formula we have

(7) $\quad 2g - 2 = -2m + (m - 1) + \sum_{Q \neq P} (e_Q - 1),$

where e_Q is the ramification index at Q of φ_L. For a point $\overline{Q} \in \mathbb{P}^1$, if $\varphi_L^{-1}(\overline{Q}) = e_1 Q_1 + \dots + e_r Q_r$, then $\varphi_L^{-1}(\overline{\sigma}(\overline{Q})) = e_1 \sigma(Q_1) + \dots + e_r \sigma(Q_r)$. Hence, there is a non-negative integer n such that $\sum_{Q \neq P} (e_Q - 1) = (2g + 1)n$. By the formula (7), we have

(8) $\quad 2g - 1 + m = (2g + 1)n.$

Under the condition $0 < m \leqslant g + 1$, the equation (8) has one and only one non-negative integral solution on m and n, which is "$m = 2$ and $n = 1$". Hence φ_L must be a double covering.

Since φ_L is a double covering over $\mathbb{P}^1$ and the characteristic of the ground field is not 2, there are $2g + 2$ ramification points on C of φ_L. Obviously, the fixed point P of σ is a ramification point of φ_L. Let $\{P_1, \dots, P_{2g+1}\}$ be the set of ramification points different from P. Since $\sigma^i(P_1)$ $(i = 0, \dots, 2g)$ must be a ramification point and P_1 is not a fixed point of σ, the set $\{P_1, \sigma(P_1), \dots, \sigma^{2g}(P_1)\}$ must coincide with the set $\{P_1, P_2, \dots, P_{2g+1}\}$. Therefore, we may assume that $\sigma(P_i) = P_{i+1}$ $(i = 1, \dots, 2g)$ and $\sigma(P_{2g+1}) = P_1$. Hence we can select a coordinate of $\mathbb{P}^1$ such that $x(P) = \infty$, and $x(P_i) = i$ $(i = 1, \dots, 2g + 1)$, where x is a morphism φ_L. Hence we have

(9) $\quad \mathrm{div}\,(x - i) = 2P_i - 2P \quad (i = 1, \dots, 2g + 1).$

Since $\{P_1, \dots, P_{2g+1}\} = \{P_1, \sigma(P_1), \dots \sigma^{2g}(P_1)\}$, we have

(10) $y^{-1}(y(P_1)) = P_1 + \ldots + P_{2g+1}$,
where y is a natural morphism $C \longrightarrow C/\langle\sigma\rangle = \mathbb{P}^1$. By (10) we can select a coordinate of $\mathbb{P}^1$ such that $y(P) = \infty$ and $y(P_1) = \ldots = y(P_{2g+1}) = 0$. Hence we have

(11) $\operatorname{div} y = P_1 + \ldots + P_{2g+1} - (2g + 1)P$.

By (9) and (11), we have a relation

$$cy^2 = x^{2g+1} - x$$

for a suitable non-zero constant $c \in k$. Obviously, the function field of C coincides with $k(\, x, \sqrt{c}.y\,)$. This completes the proof.

PROPOSITION 3.3. Under the same assumption to (a) in Theorem 2., we have the following results.

(a) Suppose that C is hyperelliptic. Then C has an automorphism of order q if and only if C is birationally equivalent to the plane curve

$$y(y - 1) = x^q.$$

(b) Suppose that C is trigonal, i.e., there is a morphism of degree three from C onto $\mathbb{P}^1$ and no morphism of degree less than three from C onto $\mathbb{P}^1$. Then C has an automorphism of order q if and only if C is birationally equivalent to the plane curve

$$y^2(y - 1) = x^q.$$

Proof. Both "if" parts in (a) and (b) are trivial, so we shall show "only if" parts in (a) and (b). Let σ be an automorphism of order q of C.

(a) It is well known that there is a unique invertible sheaf L of degree two with $\dim \Gamma(L) = 2$, which is generated by its global sections. Since $\sigma^*L \cong L$ by uniqueness of L, we can apply Proposition 3.1 for C, L and σ. Therefore, C is birationally equivalent to the plane curve $y(y - 1) = x^q$.

(b) First we shall show that there is a trigonal invertible sheaf L on C such that $\sigma^*L \cong L$. Here by a trigonal invertible sheaf, we understand an invrtible sheaf L of degree three with $\dim \Gamma(L) = 2$, which is generated by its global sections. When $g = 3$, $\omega_C(-P)$ has such properties, where ω_C is the canonical sheaf of C and $P \in Fix(\sigma)$. When $g = 4$, there are at most two trigonal invertible sheaves. (See Appendix below.) Since q is a prime number greater than two, for each trigonal invertible sheaf L we have $\sigma^*L \cong L$. When $g \geqslant 5$, there is a unique trigonal invertible sheaf L (see Appendix below.), so we have $\sigma^*L \cong L$.

In any case, we can apply Proposition 3.1 for C, σ and such a trigonal invertible sheaf. Therefore, we have that C is birationally equivalent to a plane curve $y(y-1)^2 = x^q$ or $y^2(y-1) = x^q$. But those curves are birationally equivalent to each other.

Appendix. Trigonal linear systems on trigonal curves

In this appendix, we shall show some known facts which were used in the proof of Proposition 3.3. Throughout the appendix, a curve will mean a complete non-singular curve over k. By a trigonal linear system (resp. hyperelliptic linear system) on a curve, we understand a linear system of projective dimension one and degree three (resp. two). By a trigonal curve we understand a curve which has a trigonal linear system and no hyperelliptic linear system. Note that any trigonal linear system on a trigonal curve is complete and free from base points. Therefore on a trigonal curve, there is a natural 1-1 correspondence between the set of trigonal invertible sheaves (in the sense of section 3) and the set of trigonal linear systems, modulo linearly equivalence.

PROPOSITION. Let C be a trigonal curve of genus g, and G^1_3 the set of trigonal invertible sheaves on C. Then:

(a) if $g = 3$, then $G^1_3 = \{\omega_C(-P) \mid P \in C\}$, where ω_C is the canonical sheaf of C;

(b) if $g = 4$, then $\#(G^1_3) = 1$ or 2;

(c) if $g \geqslant 5$, then $\#(G^1_3) = 1$.

Proof. Suppose that $g = 3$. It is obvious that an invertible sheaf $\omega_C(-P)$ is trigonal. Conversely, let L be a trigonal invertible sheaf on C. By the theorem of Riemann-Roch, we have $\dim \Gamma(\omega_C \otimes L^{-1}) = 1$. Since $\deg \omega_C \otimes L^{-1} = 1$, there is a point $P \in C$ such that $\omega_C \otimes L^{-1} \cong \underline{O}_C(P)$.

Next, suppose that $g \geqslant 4$. Let L and M be trigonal invertible sheaves on C. By the "base point free pencil trick" [3], we have

$$\mathrm{Ker}[\Gamma(L) \otimes \Gamma(M) \longrightarrow \Gamma(L \otimes M)] \cong \Gamma(L \otimes M^{-1}).$$

If $L \not\cong M$, then $\Gamma(L \otimes M^{-1}) = (0)$, so $\dim \Gamma(L \otimes M) \geqslant 4$. Hence we have

$$\dim \Gamma(\omega_C \otimes (L \otimes M)^{-1}) = \dim \Gamma(L \otimes M) - (6 + 1 - g) \geqslant g - 3 > 0,$$

i.e., $L \otimes M$ is special. Since C is non-hyperelliptic, $L \otimes M$ must be the canonical sheaf by Clifford's theorem. If $g \geqslant 5$, this is impossible. When $g = 4$, it shows $\#(G_3^1) \leqslant 2$.

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