ON HEWITT REALCOMPACTIFICATIONS OF PRODUCTS

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0. Conventions

All spaces considered are assumed to be completely regular Hausdorff, and all maps are continuous. Notation and terminology will be used as in [E] *and* [GJ].

For a space X, C(X) (C*(X)) denotes the ring of all real-valued (bounded) continuous functions on X. A subspace Y of X is said to be C-embedded (C*-embedded) in X if every element of C(Y) (C*(Y)) admits a continuous extension over X, and Y is said to be z-embedded in X if every zero-set of Y is the restriction to Y of a zero-set of X.

For an ordinal α, $W(\alpha)$ denotes the set of all ordinals less than α, topologized with the order topology, and ω_0 (ω_1) denotes the initial ordinal of $\aleph_0$ ($\aleph_1$).

For a set A, $|A|$ denotes the cardinality of A, and m_1 stands for the first measurable cardinal. Since m_1 (if it exists) is greater than any non-measurable cardinal, that $|A|$ is non-measurable is denoted by $|A| < m_1$. If m is a cardinal number, then m^+ is the smallest cardinal greater than m, and $D(m)$ is the discrete space of cardinality m. The set of positive integers is denoted by N.

There is an index of terms and symbols at the end of the paper.

CHAPTER 0

INTRODUCTION AND PRELIMINARIES

1. Introduction

In 1948, Hewitt [H_5] introduced the notion of realcompact spaces, under the name of Q-spaces, as an aid in studying the ring C(X)[1]. He showed that for each space X there exists a unique realcompact space υX in which X is dense and C-embedded. The space υX is called the Hewitt realcompactification of X, and it plays the same role in the theory of C(X) as the Stone-Čech compactification βX does in the theory of $C^*(X)$. Among several characterizations of the class of realcompact spaces, it is most intuitional and convenient to view it as the class $\bar{C}$ of closed subspaces of products of real lines R, that is, as the epireflective hull of $C = \{R\}$ (this characterization was first proved by Shirota [S_2] in 1952). When viewed as above, the Hewitt realcompactification υX of X is the epireflection of X into $\bar{C}$, that is, it is the unique realcompact space containing X densely such that each map f from X into a realcompact space Y admits a continuous extension υf: $\upsilon X \to Y$. For more detailed information on υX, the reader is referred to chapter 8 of [GJ].

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1 Nachbin also defined independently (but not published) the same class of spaces in terms of uniformities, and so realcompact spaces are sometimes called Hewitt-Nachbin spaces.

An important problem in the theory of Hewitt realcompactifications concerns when the relation $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ is valid. This is motivated by Glicksberg's theorem that settled the analogous question in the theory of Stone-Čech compactifications:

1.1 THEOREM (Glicksberg [G_2]). If X and Y are infinite spaces, then $X \times Y$ is pseudocompact if and only if $\beta(X \times Y) = \beta X \times \beta Y$ holds.

The problem of finding a corresponding condition on $X \times Y$ in order that $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ was first considered by Comfort and Negrepontis [CN_1] in 1966. Although from that time the problem was attacked by many researchers (e.g., Blair, Comfort, Hager, Hušek, McArthur), it does not come to a satisfactory solution, and there appears to be no simple answer, as we shall see in the next section. In 1968 Comfort, however, established the following theorem that is one of the most important results about this problem:

1.2 THEOREM (Comfort [C_3]). Let X be a locally compact, realcompact space of non-measurable cardinal. Then $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each space Y.

In 1970 McArthur [M_1] attempted to characterize the class $\mathcal{R}$ of all spaces X such that $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each space Y, and he conjectured, in view of 1.2, that $\mathcal{R}$ is precisely the class of all locally compact, realcompact spaces of non-measurable cardinals. Theorem 1.2 asserts that a locally compact, realcompact space of non-measurable cardinal is a member of R. Conversely, McArthur [M_1] and Hušek [H_6], [H_8] independently proved that every member of R is realcompact, and Hušek also noted in [H_6], [H_7] that every member of R must be of non-measurable cardinal. In chapter 1 below, we complete his conjecture by showing that if X is not locally compact, then $\upsilon(X \times Y) \neq \upsilon X \times \upsilon Y$ for some space Y. This solution leads us to the following new problem:

1.3 PROBLEM. Characterize the class $R(\ P\ )$, where P is a given topological property, of all spaces X such that $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each P-space Y.

The author believes that full knowledge of these classes $R(\ P\ )$ would supply a deficiency of the theory due to the absence of an analogue of Glicksberg's theorem. This paper aims at making a systematic study of characterizations of the classes $R(\ P\ )$ for various familiar topological properties P. In fact, Problem 1.3 is settled for the following values of P: compact, metacompact, subparacompact, $P(m)$, k, locally compact, locally pseudocompact, first countable, Moore, metrizable, locally compact metrizable, discrete, 0-dimensional; definitions of these properties are stated when they first occur in the course of the paper.

The primary organization of the paper is into six chapters. Chapters 1, 2 and 4 present characterizations of the classes $R(\ P\ )$ stated above; the major theorems are formulated without proof at the first sections. The theory developed there has many applications. For instance, in case X satisfies the countable chain condition and Y is metrizable, we can settle the problem of when $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ is valid (16.6). These chapters end with quinary open questions which ask about possible extensions of our results. Chapter 3 studies the following two problems:

(I) For maps $f_i \colon X_i \to Y_i$ ($i = 1, 2$), when does $\upsilon(X_1 \times X_2) = \upsilon X_1 \times \upsilon X_2$ imply $\upsilon(Y_1 \times Y_2) = \upsilon Y_1 \times \upsilon Y_2$?

(II) For spaces X and Y, when is $X \times Y$ z-embedded in $X \times \beta Y$?

As will be stated in section 3, barring the existence of measurable cardinals, $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds whenever $X \times Y$ is z-embedded in $X \times \beta Y$. Therefore consideration of (II) yeilds several sufficient conditions in order that $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$, and that of (I) allows us to deduce new sufficient conditions from old. Some miscellaneous remarks are collected in chapter 5. The remaining sections in the present chapter is devoted to prerequisite materials. In the last one, we quote Hušek's theorems that are useful for dealing with the delicate questions caused by the existense of measurable cardinals, from which characterizations of $\mathcal{R}$(compact) and $\mathcal{R}$(discrete) are deduced.

2. The situation surrounding $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$

The relationship between $\upsilon(X \times Y)$ and $\upsilon X \times \upsilon Y$ is complicated, and many features appear pathological compared with the case of Stone-Čech compactifications. This section is devoted to a brief explanation of the awkward situation. In the interest of simplicity each cardinal is assumed to be non-measurable only in this section.

We begin by making mention of a witty work due to Hušek. He showed in [H_9] that there exist spaces X_i and Y_i ($i = 1, 2$) such that $X_1 \times Y_1$ and $X_2 \times Y_2$ are mutually homeomorphic yet $\upsilon(X_1 \times Y_1) = \upsilon X_1 \times \upsilon Y_1$ and $\upsilon(X_2 \times Y_2) \neq \upsilon X_2 \times \upsilon Y_2$. This fact disappoints one's hope of finding topological properties of $X \times Y$ equivalent to $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$. His construction is very simple; let R be a locally compact, realcompact space (e.g., the real line), and P, Q a pair of spaces for which $\upsilon(P \times Q) \neq \upsilon P \times \upsilon Q$. Then it follows from 1.2 that $\upsilon(R \times (P \times Q)) = \upsilon R \times \upsilon(P \times Q)$, while $\upsilon((R \times P) \times Q) \neq \upsilon(R \times P) \times \upsilon Q$. If one use a finite space instead of R, then the same holds good of the Stone-Čech compactifications, and this fact can be viewed as the reason why Glicksberg's theorem (1.1) applies only infinite spaces. From this point of view, one might reasonably ask whether or not, for spaces X, Y neither of which is locally compact and realcompact, the relation $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ can be determined by topological properties of $X \times Y$. The answer is still negative; in fact, we shall give a counter example in section 23. Consequently it is necessary for us to concern ourselves more deeply with the properties of each factors X, Y than merely with that of $X \times Y$ in order to determine whether or not the relation $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds.

The problem of when $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ is valid has been approached chiefly from two points of view. We next review these two customary methods, and show that they are both inadequate. In [H_3] Hager approached the problem from the point of view of uniform spaces. If we denote by υ_X (β_X) the weak uniformity generated by $C(X)$ ($C^*(X)$) on X, then υX (βX) is known to be the completion of X with respect to υ_X (β_X). These use of uniformities will be found in chapter 15 of [GJ]. In the case of β the situation is simple; in fact, it was proved in [I_2] that $\beta(X \times Y) = \beta X \times \beta Y$ if and only if $\beta_{X \times Y} = \beta_X \times \beta_Y$, where $\beta_X \times \beta_Y$ is the product uniformity generated by β_X and β_Y. Since $\upsilon X \times \upsilon Y$ can be viewed as the completion of $X \times Y$ with respect to $\upsilon_X \times \upsilon_Y$, it is evident that $\upsilon_{X \times Y} = \upsilon_X \times \upsilon_Y$ implies $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$. Unfortunately the converse is not true in general: Onuchic [O_5] proved that X is either pseudocompact or a P-space[2] whenever $\upsilon_{X \times X} = \upsilon_X \times \upsilon_X$, and the real line R has neither of these properties, whereas $\upsilon(R \times R) = \upsilon R \times \upsilon R$ ($= R \times R$). Therefore the relation $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ need not be described in terms of uniform spaces at least in the most concrete case. Among other

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2 A P-space is a space in which every G_δ-set is open.

things, in [H_3], Hager characterized those pairs X, Y of spaces such that $\upsilon_{X \times Y} = \upsilon_X \times \upsilon_Y$.

The second approach to the problem is through function spaces, and was employed effectively in [CN_1], [H_7], [H_8], [H_9]. In particular, Hušek [H_7] offered another proof of Comfort's theorem (1.2) from this point of view. An outline of his proof might be sketched here to show the standard use of function spaces: Let X be a locally compact, realcompact space, and Y a space. Let $f \in C(X \times Y)$. For our end it suffices to show that f admits a continuous extension over $X \times \upsilon Y$. Let C(X) be endowed with the compact-open topology; then C(X) is realcompact. The map $\hat{f}: Y \to C(X)$ defined by $\hat{f}(y) = f|(X \times \{y\})$ is continuous, and so it can be extended to a continuous map $\hat{g}: \upsilon Y \to C(X)$ because C(X) is realcompact. Then, since X is locally compact, the desired extension g of f is obtained by defining $g((x, y)) = [\hat{g}(y)](x)$. Hence the proof is complete. We now call an ordered pair (X, Y) of spaces a υ-pair if $X \times Y$ is C-embedded in $X \times \upsilon Y$. Obviously $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds if and only if both (X, Y) and (υY, X) are υ-pairs. Let C(Y, Z) denote the set of all continuous maps from Y into Z, and $C_t(X)$ the space C(X) with a topology t. The argument of the proof suggests that (X, Y) is a υ-pair whenever there exists a completely regular Hausdorff topology t for C(X) satisfying the following three conditions:

(1) $C_t(X)$ is realcompact,

(2) $C(X \times Y) \subset C(Y, C_t(X))$,

(3) $C(\upsilon Y, C_t(X)) \subset C(X \times \upsilon Y)$,

where the inclusions stand for the canonical injections (i.e., (2) and (3) mean that $\hat{f} \in C(Y, C_t(X))$ for each $f \in C(X \times Y)$, and $f \in C(X \times \upsilon Y)$ for each $\hat{f} \in C(\upsilon Y, C_t(X))$, respectively). In [$H_9$] Hušek asked whether or not the existence of such a topology t for C(X) is necessary for a pair (X, Y) to be a υ-pair. In section 23, we answer this question negatively by showing that if X is not locally compact, then there exists a space Y such that (X, Y) is a υ-pair but C(X) has no such a topology. Consequently, in this sense, the relation $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ can not be always described in terms of function spaces.

We nevertheless nourish a hope of finding a all-around theory to attack the problem. Actually various new methods for studying the relation $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ will be developed in this paper, although our goal is still far to see.

3. Definitions and results from the literatures

For convenience, we list certain basic definitions and facts that will be used in the sequal.

3.1 We begin with a brief account of measurable cardinals. A cardinal number m is said to be measurable if a discrete space $D(m)$ of cardinality m admits a countably additive $\{0, 1\}$-valued measure μ such that $\mu(D(m)) = 1$ and $\mu(\{d\}) = 0$ for each $d \in D(m)$. It is known (cf. [GJ]) that a discrete space is realcompact if and only if its cardinal is non-measurable. The class of non--measurable cardinals contains $\aleph_0$, and is closed under standard operations of cardinal arithmetic. For detailed treatment of non-measurable cardinals, see chapter 12 of [GJ]. The assuption that each cardinal is non-measurable is known to be consistent with the usual axioms of set theory; however, the consistency of the existence of measurable cardinals with the usual axioms of set theory remains an open question (cf. [S_1]).

3.2 Let X be a space. The weight of X is the minimal cardinality of a base for X and is denoted by $w(X)$. The density of X, denoted by $d(X)$, is the minimal cardinality of a dense subset of X. If $x \in X$, the character at x in X, written $\chi(x, X)$, is the minimal cardinality of a neighborhood base at x, and $\chi(X) = \sup\{ \chi(x, X) \mid x \in X \}$ is called the character of X. The cellularity of X, denoted by $c(X)$, is defined by $c(X) = \sup\{ |\mathcal{U}| \mid \mathcal{U}$ is a disjoint family of non-empty open sets in X }. The following inequalities are well known (cf. [E, 1.5.1, 1.5.6, 3.5.3]):

3.2.1 $c(X) \leqq d(X) \leqq w(X)$, $\chi(X) \leqq w(X)$.

3.2.2 $|X| \leqq \exp w(X) \leqq \exp \exp d(X)$.

3.2.3 $w(\upsilon X) \leqq w(\beta X) \leqq \exp d(X)$.

3.3 The following results are used throughout the paper.

3.3.1 If Y is C-embedded in X, then $\upsilon Y = cl_{\upsilon X} Y$ ([GJ, 8.10]).

3.3.2 If G is a cozero-set of υX, then $\upsilon(G \cap X) = G$ ([B_2, 5.2]).

3.3.3 A space X is pseudocompact if and only if $\upsilon X = \beta X$, and hence a pseudocompact realcompact space is compact ([GJ, 8A4]).

3.3.4 For a space X, $\upsilon X = \{ x \in \beta X \mid$ every zero-set of βX containing x meets X } ([GJ, 8.8]).

3.3.5 If Y_i is G_δ-dense in X_i[3] for i = 1, 2, then $Y_1 \times Y_2$ is G_δ-dense in $X_1 \times X_2$ ([CN_1, 5.1]).

3.3.6 A z-embedded subspace is C-embedded if and only if it is completely separated[4] from every zero-set disjoint from it ([BH_1, 3.6]).

3.3.7 If a space X admits a complete uniformity and $|X| < m_1$, then X is realcompact. In particular, a paracompact space X with $|X| < m_1$ is realcompact (cf. [S_2], [GJ, 15.20]).

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3 Every non-empty G_δ-set of X_i meets Y_i.

4 Recall from [GJ] that two subsets A, B of a space X are said to be completely separated in X if there exists $f \in C(X)$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.

3.3.8 If D is a discrete space with $|D| \geqq m_1$, then υD is not a k-space ([C_3, p.115]).

3.3.9 A pseudocompact paracompact space is compact (cf. [E, 3.10.21 and 5.1.20]).

3.3.10 Every metrizable space has a σ-discrete base. Conversely a space having a σ-locally finite base is metrizable (cf. [E, 4.4.3 and 4.4.7]).

3.4 By 3.3.3 above, we can interpret Glicksberg's theorem (1.1) as one sufficient condition in order that $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$; actually some weaker conditions are known to be sufficient. The following chart describes the situation. Recall from [H_7] that a space X is pseudo-m_1-compact if each discrete family of non-empty open sets in X is of non-measurable cardinal. The projection π_X: $X \times Y \to X$ is z-closed if it carries zero-sets to closed sets ([T_2]). (*) indicates that the implication requires the assumption that either $|Y| < m_1$ or X is pseudo-m_1-compact.

$$
\begin{tikzcd}
\text{(a) } \beta(X \times Y) = \beta X \times \beta Y \arrow[d, leftrightarrow] & \text{(f) } \pi_X \text{ is z-closed} \arrow[d, leftrightarrow] \\
\text{(b) } X \times Y \text{ is } C^*\text{-embedded in } \beta X \times \beta Y \arrow[r] \arrow[d] & \text{(g) } X \times Y \text{ is } C^*\text{-embedded in } X \times \beta Y \arrow[d] \\
\text{(c) } X \times Y \text{ is z-embedded in } \beta X \times \beta Y \arrow[r] \arrow[d] & \text{(h) } X \times Y \text{ is z-embedded in } X \times \beta Y \arrow[d] \arrow[dl, "(*)"] \\
\text{(d) } X \times Y \text{ is C-embedded in } \upsilon X \times \upsilon Y \arrow[r] \arrow[d, leftrightarrow] & \text{(i) } X \times Y \text{ is C-embedded in } X \times \upsilon Y \\
\text{(e) } \upsilon(X \times Y) = \upsilon X \times \upsilon Y &
\end{tikzcd}
$$

Here, (a) $\leftrightarrow$ (b) and (d) $\leftrightarrow$ (e) are virtually the definitions of β and υ, respectively. The equivalence (f) $\leftrightarrow$ (g) was first proved by Hager and Mrówka [HM], and the proof was described in [H_2]; (g) $\to$ (f) was also shown in [I_4]. In [CN_1], Comfort and Negrepontis proved a weak form of (f) $\to$ (g), and that (g) implies (d) if $|Y| < m_1$. It was proved by Hušek [H_9] that (g) also implies (d) if X is pseudo-m_1-compact. Blair has proved in [B_2] that (h) implies (d) under the assumption (*). Section 15 below concerns the question of what topological properties of X and Y make (h) true. A study of (c) was made in [H_1] and [BH_3], where (c) $\to$ (d) was noted. (h) $\to$ (i) as well as (c) $\to$ (d) follows from 3.3.4, 3.3.5 and 3.3.6. Other implications are obvious.

3.5 The following concepts and results will be used, without reference, in chapters 2, 3 and 4. A map $f\colon X \to Y$ is called (countably) bi-quotient if, whenever $y \in Y$ and $\mathcal{U}$ is a (countable) cover of $f^{-1}(y)$ by open sets of X, then finitely many $f(U)$, with $U \in \mathcal{U}$, cover a neighborhood of y in Y ([M_2], [SM]). Recall from [A_2] that pseudo-open maps are defined by restricting the family $\mathcal{U}$ in the above definition to have only one element. The following chart summarizes the relationship of these maps to other more familiar ones:

$$\begin{tikzcd}
& \text{perfect} \arrow[r] \arrow[d] & \text{quasi-perfect} \arrow[r] \arrow[d] & \text{closed} \arrow[d] & \\
\text{open} \arrow[r] & \text{bi-quotient} \arrow[r] & \text{countably bi-quotient} \arrow[r] & \text{pseudo-open} \arrow[r] & \text{quotient}
\end{tikzcd}$$

A space is called (countably) bi-sequential if it is the (countably) bi-quotient image of a metrizable space ([M_3], [S_3]). Recall from [M_4] that a paracompact M-space (= a paracompact p-space in the sense of Arhangel'skiĭ [A_3]) is a space which has a perfect map onto a metrizable space. A locally compact, paracompact space is a paracompact M-space ([A_3], [M_4]). A space is called a (countably) bi-k-space if it is the (countably) bi-quotient image of a paracompact M-space ([M_3]). The pseudo-open image of a metrizable (resp. paracompact M-) space is called a Fréchet (resp. singly bi-k-) space ([A_3], [M_3]). Internal characterizations of these spaces can be found in [M_3].

$$
\begin{tikzcd}
\text{first countable} \arrow[r] & \text{bi-sequential} \arrow[r] \arrow[d] & \text{countably bi-sequential} \arrow[r] \arrow[d] & \text{Fréchet} \arrow[d] & \\
\text{paracompact } M \arrow[r] & \text{bi-k} \arrow[r] & \text{countably bi-k} \arrow[r] & \text{singly bi-k} \arrow[r] & k \\
 & \text{locally compact} \arrow[u] & & &
\end{tikzcd}
$$

3.5.1 If X and Y are bi-k-spaces, then so is X × Y ([M_3, 3E4]).

3.5.2 If X is a bi-k-space and Y is a countably bi-k-space, then X × Y is a k-space ([M_3, p.114]).

3.5.3 If X is a locally compact space, then X × Y is a k-space for each k-space Y ([C_1]).

If f_i: $X_i \to Y_i$ is a map for i = 1, 2, then the product map f = $f_1 \times f_2$ from $X_1 \times X_2$ to $Y_1 \times Y_2$ is defined by $f((x_1, x_2))$ =

$(f_1(x_1), f_2(x_2))$ for $(x_1, x_2) \in X_1 \times X_2$. While the product of two quotient maps need not be quotient, the class of bi-quotient maps is well behaved with respect to products:

3.5.4 If $f_i: X_i \to Y_i$ ($i = 1, 2$) are bi-quotient maps, then so is $f_1 \times f_2$ ([M_2, 1.2]).

3.5.5 If $f_1: X_1 \to Y_1$ is a countably bi-quotient map and if f_2: $X_2 \to Y_2$ is a bi-quotient map onto a bi-sequential space Y_2, then $f_1 \times f_2$ is countably bi-quotient ([M_3, p.110]).

3.5.6 If X is locally compact, then $id_X \times g$ is a quotient map for each quotient map $g: Y \to Z$, where id_X is the identity map of X ([W_2]).

In 3.5.3 and 3.5.6, the converse is also valid. See Michael, Local compactness and cartesian products of quotient maps and k-spaces, Ann. Inst. Fourier, Grenoble 18.2 (1968), 281-286.

4. Hušek's theorems and characterizations of $\mathcal{R}$(compact) and $\mathcal{R}$(discrete)

For later use, we quote two theorems due to Hušek [H_7], from which characterizations of $\mathcal{R}$(compact) and $\mathcal{R}$(discrete) are deduced. They are useful to exclude difficulty originated in the possible existence of measurable cardinals from our theory.

4.1 THEOREM (Hušek). Let Y be locally compact, realcompact. Then $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds if and only if either $|Y| < m_1$ or X is pseudo-m_1-compact.

4.2 THEOREM (Hušek). Let Y be discrete. Then $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds if and only if either $|X| < m_1$ or $|Y| < m_1$.

As immediate consequences of the above theorems and 3.3.2 we have the following results that characterize $\mathcal{R}$(compact) and $\mathcal{R}$(discrete). Let $D^*(m)$ denote the one-point compactification of the discrete space $D(m)$ of cardinality m.

4.3 THEOREM. The following conditions on a space X are equivalent:

(a) X is pseudo-m_1-compact.

(b) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each locally compact, realcompact space Y.

(c) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each compact space Y with $w(Y) \leqq d(X)$.

(d) $\upsilon(X \times D^*(d(X))) = \upsilon X \times \upsilon D^*(d(X))$.

4.4 THEOREM. The following conditions on a space X are equivalent:

(a) $|X| < m_1$.

(b) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each discrete space Y.

(c) $\upsilon(X \times D(d(X))) = \upsilon X \times \upsilon D(d(X))$.

4.5 REMARK. 4.4 was noted in [H_7] implicitly, and was stated by the author in [O_2]. The special case $\upsilon(D(m_1) \times \beta D(m_1)) \neq \upsilon D(m_1) \times \beta D(m_1)$ was earlier observed in [CN_1]. In chapter 4, it will be shown that in 4.4 (b) "discrete" can be weakened to "locally compact, metrizable".

CHAPTER 1

CHARACTERIZATIONS OF THE CLASSES $\mathcal{R}$, $\mathcal{R}$(metacompact), $\mathcal{R}$(subparacompact), $\mathcal{R}$($P(m)$) AND $\mathcal{R}$(metacompact $P(m)$)

For an infinite cardinal m, a $P(m)$-space is a space each of whose point lies in the interior of every intersection of less than m its neighborhoods. Every space is a $P(\aleph_0)$-space, and a $P(\aleph_1)$-space usually is called a P-space. Recall that a space is metacompact (resp. subparacompact) if each open cover has a point-finite open (resp. σ-locally finite closed) refinement (cf. [AS], [E]).

5. Main theorems

In this section, we state main theorems of this chapter and discuss some remarks. The proofs will be given later in section 7. The first one proves McArthur's conjecture mentioned in the introduction.

5.1 THEOREM. The following conditions on a space X are equivalent:

(a) X is locally compact, realcompact and $|X| < m_1$.

(b) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each space Y.

(c) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each metacompact space Y with $w(Y) \leqq w(\upsilon X)\cdot\aleph_1$.

(d) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each subparacompact space Y with $w(Y) \leqq w(\upsilon X)\cdot \exp \aleph_0$.

5.2 REMARK. In 5.1, (a) $\rightarrow$ (b) is Comfort's theorem (1.2). Conversely it was proved in [H_6], [H_8], [M_1] that X is realcompact whenever X satisfies (b), and Hušek noted in [H_6], [H_7] that such a space X must be of non-measurable cardinal. The equivalence of (a) and (b) has been completed by the author in [O_2].

5.3 REMARKS. (1) A space is called 0-dimensional if it has a base consisting of open-and-closed sets ([E]). As the reader will observe in the proof, 5.1 remains true if 0-dimensionality is added to the conditions on Y in (b), (c) and (d).

(2) The author does not know if exp $\aleph_0$ can be replaced by $\aleph_1$ in (d).

For an infinite cardinal m, a space is weakly-m-compact if each open cover has a subfamily of cardinality less than m with dense union. This notion was introduced by Frolík [F_1] under a different name (cf. also [H_2]). A weakly-$\aleph_0$-compact space is precisely a compact space, and Lindelöf spaces and separable spaces are weakly-$\aleph_1$-compact. For an infinite cardinal m, let m^* denote the smallest regular cardinal not less than m (i.e., $m^* = m$ if m is regular and $m^* = m^+$ if m is singular). The following theorem gives characterizations of R(P(m)) and R(metacompact P(m)).

5.4 THEOREM. The following conditions on a space X are equivalent:

(a) Each point of υX has a neighborhood G in υX such that $G \cap X$ is weakly-m^*-compact, and $|X| < m_1$.

(b) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each $P(m)$-space Y.

(c) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each metacompact $P(m)$-space Y with $w(Y) \leqq \exp w(\upsilon X)$.

5.5 REMARKS. (1) A space X is called locally weakly-m-compact if each point has a weakly-m-compact neighborhood. As far as I know, this notion first appears in [H_9]. Obviously 5.4 (a) implies local weak-m^*-compactness of X and υX, but the converse need not be true if $m^* > \aleph_0$. In fact, let $m^* > \aleph_0$, and let us set $X = W(\omega_\alpha)$, where ω_α is the initial ordinal of m^*; then both X and υX ($= W(\omega_\alpha + 1)$) are locally compact, but X does not satisfy 5.4 (a).

(2) In 5.4 (a), m^* cannot be replaced by m in general. To show this, let m be a singular cardinal with $m < m_1$, and let X be the quotient space obtained from $W(\omega_0 + 1) \times D(m)$ by collapsing the set $\{\omega_0\} \times D(m)$ to a single point. Then X is weakly -m^*-compact realcompact but not locally weakly-m-compact. Thus it follows from 5.4 that $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each $P(m)$--space Y, but X does not satisfy (a) with m^* changed to m.

In case X is itself a $P(m)$-space, we have the following theorem, from which the equivalence of (a) and (b) in 5.1 can be deduced as the special case $m = \aleph_0$.

5.6 THEOREM. For an infinite cardinal m, the following conditions on a $P(m)$-space X are equivalent:

(a) X is locally weakly-m^*-compact, realcompact and $|X| < m_1$.

(b) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each $P(m)$-space Y.

(c) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each metacompact $P(m)$-space Y with $w(Y) \leqq \exp w(X)$.

5.7 REMARK. In 5.6, (a) $\rightarrow$ (b) was proved by Hušek [H_9].

5.8 REMARKS. (1) In chapter 4 below, it will be proved that if X is a first countable realcompact space with $|X| < m_1$, then $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each normal countably paracompact space Y. This fact asserts that a member of $\mathcal{R}$(normal countably paracompact) is not necessary locally compact, and hence we can not adapt the above theorems by making Y a normal countably paracompact space instead of metacompact. Since a σ-locally finite family of a $P(\aleph_1)$-space is closure preserving[5], it follows from Michael's theorem [E, 5.1.G] that a subparacompact $P(\aleph_1)$-space is paracompact. Therefore, in case $m > \aleph_0$, 5.4 and 5.6 are not

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5 A family $\mathcal{F}$ of subsets of a space X is closure preserving if $cl_X(\bigcup\{ F \mid F \in \mathcal{F}' \}) = \bigcup\{ cl_X F \mid F \in \mathcal{F}' \}$ for each $\mathcal{F}' \subset \mathcal{F}$ (cf. [E, 5.1.G]).

true for subparacompact spaces. Furthermore, since normal metacompact spaces and normal subparacompact spaces are known to be countably paracompact ($[G_1]$), in 5.1 (c), (d), 5.4 (c) and 5.6 (c), one cannot add normality to the conditions on Y. However, it is fully expected that $\mathcal{R} = \mathcal{R}$(normal) = $\mathcal{R}$(countably paracompact); the possibility will be discussed in section 8 and 9.2 below.

(2) The author does not know whether, in the case $m > \aleph_0$ of 5.4 and 5.6, exp $w(\upsilon X)$ and exp $w(X)$ can be replaced by $w(\upsilon X)\cdot\aleph_1$, respectively.

6. Tools for proving theorems

When proving theorems stated in the preceding section, since a mild variant of Comfort's procedure ($[C_3]$) proves (a) $\to$ (b), the remaining issue is how to find a space Y such that $\upsilon(X \times Y) \neq \upsilon X \times \upsilon Y$ when (a) fails. The following two tasks are imposed on us:

Task 1. If X has a point having no weakly-m^*-compact neighborhood, then find a suitable space Y such that $X \times Y$ is not C-embedded in $X \times \upsilon Y$.

Task 2. If $\upsilon X - X$ has a point having no neighborhood in υX whose restriction to X is weakly-m^*-compact, then find a suitable space Y such that $X \times Y$ is not C-embedded in $\upsilon X \times Y$.

In this section, we present useful tools for performing these tasks.

6.1 NOTATION. Let k, m be infinite cardinals with $k \geqq m$, and D a set of cardinality k. Let Σ be the family of all subsets of D whose cardinality is less than m. Then $\Sigma^+(k, m)$ denotes the space $\Sigma \cup \{s\}$ topologized as follows: Each point of Σ is isolated and $\{ J(\sigma) \mid \sigma \in \Sigma \}$, where $J(\sigma) = \{s\} \cup \{ \sigma' \in \Sigma \mid \sigma' \supset \sigma \}$, is a neighborhood base at s. In case m is a regular cardinal, $\Sigma^+(k, m)$ is a $P(m)$-space.

6.2 NOTATION. Let Z be a space, and let $\Sigma^+ = \Sigma^+(k, m)$ be the space defined in 6.1. Then $F(Z, \Sigma^+(k, m))$ (or simply $F(Z, \Sigma^+)$) denotes the space obtained from the product space $Z \times \Sigma^+$ by letting each point of $Z \times \Sigma$ be isolated.

6.3 FACTS. The following simple facts about these constructions are listed without proofs:

6.3.1 $|\Sigma^+(k, m)| = w(\Sigma^+(k, m)) \leqq \exp k$, in particular, $|\Sigma^+(k, \aleph_0)| = w(\Sigma^+(k, \aleph_0)) = k$.

6.3.2 $|F(Z, \Sigma^+)| = |Z| \cdot |\Sigma^+|$, $w(F(Z, \Sigma^+)) = |Z| \cdot w(Z) \cdot w(\Sigma^+)$.

6.3.3 If Z has one of the following properties, then so does $F(Z, \Sigma^+)$: metacompactness, subparacompactness, normality, countable paracompactness, 0-dimensionality.

6.3.4 If both Z and Σ^+ are $P(m)$-spaces, then so is $F(Z, \Sigma^+)$.

6.4 THEOREM. Let X be a space having a point x_0, with $\chi(x_0, X) \leqq n$, that has no weakly-m-compact neighborhood. Let Z be a space having a locally finite family $\mathcal{F}$ of closed subsets in Z, with $|\mathcal{F}| = n$, such that $\bigcap\{ cl_{\upsilon Z}F \mid F \in \mathcal{F} \} \neq \emptyset$. Then $X \times Y$ is not C-embedded in $X \times \upsilon Y$, where $Y = F(Z, \Sigma^+(w(X), m))$.

Proof. With the notation in 6.1, $\Sigma^+(w(X), m) = \Sigma \cup \{s\}$, and the sets of the form $J(\sigma) = \{ \sigma' \in \Sigma \mid \sigma' \supset \sigma \}$, $\sigma \in \Sigma$, are basic neighborhoods of s. Let $\{ G_\lambda \mid \lambda \in \Lambda \}$ be a neighborhood base at x_0 in X with $|\Lambda| = n$. For each $\lambda \in \Lambda$, $cl_X G_\lambda$ is not weakly--m-compact, and thus there exists an open cover $\mathcal{U}_\lambda$ of X such that no subfamily of cardinality less than m has dense union in G_λ.

Since it can be assumed without loss of generality that $|\mathcal{U}_\lambda| = w(X)$, we denote the collection of all subfamilies of $\mathcal{U}_\lambda$ whose cardinality is less than m by $\{ \mathcal{U}_{\lambda\sigma} \mid \sigma \in \Sigma \}$, where $\sigma \subset \sigma'$ if and only if $\mathcal{U}_{\lambda\sigma} \subset \mathcal{U}_{\lambda\sigma'}$. If we set $H_{\lambda\sigma} = G_\lambda - \mathrm{cl}_X(\bigcup\{ U \mid U \in \mathcal{U}_{\lambda\sigma} \})$ for each $\sigma \in \Sigma$, then $H_{\lambda\sigma}$ is a non-empty open set. Pick $x_{\lambda\sigma} \in H_{\lambda\sigma}$, and choose $f_{\lambda\sigma} \in C(X)$ such that $f_{\lambda\sigma}(x_{\lambda\sigma}) = 0$ and $f_{\lambda\sigma}(X - H_{\lambda\sigma}) = \{1\}$. On the other hand, since $|\mathcal{F}| = n$, it can be written $\mathcal{F} = \{ F_\lambda \mid \lambda \in \Lambda \}$. For each $\lambda \in \Lambda$ and each $\sigma \in \Sigma$, $F_\lambda \times \{\sigma\}$ is an open-and--closed subset of Y ($= F(Z, \Sigma^+(w(X), m))$), so there is $g_{\lambda\sigma} \in C(Y)$ such that $g_{\lambda\sigma}(F_\lambda \times \{\sigma\}) = \{0\}$ and $g_{\lambda\sigma}(Y - (F_\lambda \times \{\sigma\})) = \{1\}$. For each $\lambda \in \Lambda$ and each $\sigma \in \Sigma$, let us set

$$J_{\lambda\sigma} = \{x_{\lambda\sigma}\} \times (F_\lambda \times \{\sigma\}) \subset X \times Y,$$

$$K_{\lambda\sigma} = H_{\lambda\sigma} \times (F_\lambda \times \{\sigma\}) \subset X \times Y,$$

$$h_{\lambda\sigma}((x, y)) = \min \{1, f_{\lambda\sigma}(x) + g_{\lambda\sigma}(y)\}; \quad (x, y) \in X \times Y.$$

Then $h_{\lambda\sigma} \in C(X \times Y)$, $h_{\lambda\sigma}(J_{\lambda\sigma}) = \{0\}$ and $h_{\lambda\sigma}((X \times Y) - K_{\lambda\sigma}) = \{1\}$.

Claim 1. $\mathcal{K} = \{ K_{\lambda\sigma} \mid \lambda \in \Lambda, \sigma \in \Sigma \}$ is locally finite in $X \times Y$: To prove this claim, let $p = (x, y) \in X \times Y$; then $y = (z, \tau)$ for some $z \in Z$ and $\tau \in \Sigma \cup \{s\}$. Since $\mathcal{F}$ is locally finite, z has a neighborhood $G(z)$ in Z which meets only finitely many members, say $F_{\lambda_1}, \ldots, F_{\lambda_n}$, of $\mathcal{F}$. In case $\tau \in \Sigma$, $X \times (G(z) \times \{\tau\})$ is a neighborhood of p which meets only $K_{\lambda_1\tau}, \ldots, K_{\lambda_n\tau}$, and so we only need consider the case $\tau = s$. For each $i = 1, \ldots, n$, choose $U_i \in \mathcal{U}_{\lambda_i}$ with $x \in U_i$; then $\{U_i\} = \mathcal{U}_{\lambda_i\sigma_i}$ for some $\sigma_i \in \Sigma$. If we set $G(x) = U_1 \cap \ldots \cap U_n$, then $G(x)$ is a neighborhood of x such that $G(x) \cap H_{\lambda_i \sigma_i} = \emptyset$ for each $i = 1, \ldots, n$. Let us set $G(p) = G(x) \times (G(z) \times J(\sigma_0))$, where $\sigma_0 = \sigma_1 \cup \ldots \cup \sigma_n$. Then $G(p)$ is a neighborhood of p in $X \times Y$ which meets no member of $\mathcal{K}$. For, if $G(p) \cap K_{\lambda\sigma} \neq \emptyset$, then $G(x) \cap H_{\lambda\sigma} \neq \emptyset$, $\sigma \supset \sigma_0$, and $\lambda = \lambda_i$ for some i. Since $\mathcal{U}_{\lambda_i \sigma} \supset \mathcal{U}_{\lambda_i \sigma_0} \supset \mathcal{U}_{\lambda_i \sigma_i}$, it follows that $H_{\lambda\sigma} = H_{\lambda_i \sigma} \subset H_{\lambda_i \sigma_i}$, and hence $G(x) \cap H_{\lambda_i \sigma_i} \neq \emptyset$. This is a contradiction, that proves claim 1.

Define a function h on $X \times Y$ by

$$h(q) = \inf \{ h_{\lambda\sigma}(q) \mid \lambda \in \Lambda,\ \sigma \in \Sigma \}; \quad q \in X \times Y.$$

Then h is continuous, since $\mathcal{K}$ is locally finite.

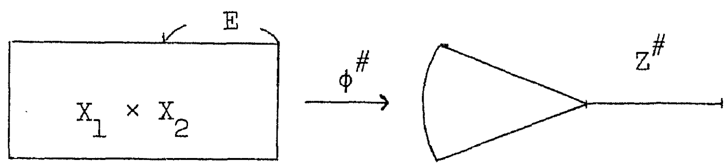
Claim 2. The function h admits no continuous extension over $X \times \upsilon Y$: To prove this claim, choose $y_0 \in \bigcap \{ \mathrm{cl}_{\upsilon Z} F_\lambda \mid \lambda \in \Lambda \}$; then local finiteness of $\mathcal{F}$ implies that $y_0 \in \upsilon Z - Z$. Since $Z \times \{s\}$ is C-embedded in Y, it follows from 3.3.1 that $\upsilon Z = \upsilon(Z \times \{s\}) \subset \upsilon Y$, and hence $y_0 \in \upsilon Y - Y$. Let $V \times W$ be a given neighborhood of (x_0, y_0) in $X \times \upsilon Y$. There is $\lambda \in \Lambda$ with $G_\lambda \subset V$, and then $W \cap (F_\lambda \times \{s\}) \neq \emptyset$. Find $z \in F_\lambda$ and $\sigma \in \Sigma$ such that $(z, s) \in W$ and $(z, \sigma) \in W$. Then both $p_1 = (x_{\lambda\sigma}, (z, \sigma))$ and $p_2 = (x_0, (z, s))$ belong to $V \times W$ and $h(p_1) = 0$, while $h(p_2) = 1$. This shows that h does not extend continuously to (x_0, y_0). Hence the proof is complete.

6.5 REMARKS. (1) Let X and Z be spaces satisfying the conditions of 6.4. Then one might ask whether $X \times Z$ is not C-embedded in $X \times \upsilon Z$ or not. Example 16.7 provides a negative answer to this question.

(2) A similar space to the space $Y = F(Z, \Sigma^+)$ was used by Przymusinski [P_1] to show the existence of normal non-weak cb-spaces.

6.6 The preceding theorem reduces Task 1 to the problem of finding a space Z which has a locally finite family $\mathcal{F}$ of closed subsets such that $\bigcap\{ \mathrm{cl}_{\upsilon Z} F \mid F \in \mathcal{F} \} \neq \emptyset$. Our next work is to construct such spaces Z. The following definition is needed for this end.

6.6.1 DEFINITION. Let X_i be a space with a base $\mathcal{B}_i$ for i = 1, 2. Let E be a closed subset of X_1, and let $Z^{\#}$ be the quotient space obtained from $X_1 \times X_2$ by collapsing the set $\{e\} \times X_2$ to a point for each $e \in E$. Let $\phi^{\#}: X_1 \times X_2 \to Z^{\#}$ be the quotient map.



Define τ to be the new topology for $Z^{\#}$ having the collection $\bigcup\{ \mathcal{B}(B) \mid B \in \mathcal{B}_1 \}$ as a base, where

$$\mathcal{B}(B) = \begin{cases} \{ \phi^{\#}(B \times X_2) \} & \text{if } B \cap E \neq \emptyset, \\ \{ \phi^{\#}(B \times B') \mid B' \in \mathcal{B}_2 \} & \text{if } B \cap E = \emptyset. \end{cases}$$

We call τ the strong topology with respect to this identification, and denote the space $Z^{\#}$ equipped with τ by Z. Then the natural map $\phi: X_1 \times X_2 \to Z$ is continuous, and $w(Z) = w(X_1) \cdot w(X_2)$.

6.6.2 FACT. For every two infinite cardinals m and n, there exists a 0-dimensional metacompact $P(m)$-space $Z = Z_1(n, m)$, with $|Z| = w(Z) = n\cdot(m^*)^+$, that has a discrete family F of closed subsets in Z such that $|F| = n$ and $\bigcap\{ cl_{\upsilon Z}F \mid F\in F \} \neq \emptyset$.

Proof. Let ω_α (resp. ω_β) be the initial ordinal of $(m^*)^+$ (resp. m^*). Define T_1 (resp. T_2) to be the subspace of $W(\omega_\alpha + 1)$ (resp. $W(\omega_\beta + 1)$) obtained by deleting all non-isolated points except ω_α (resp. ω_β). Let us set $T = (T_1 \times T_2) - \{t_0\}$, where $t_0 = (\omega_\alpha, \omega_\beta)$. Then T is a 0-dimensional metacompact $P(m)$-space with $|T| = w(T) = (m^*)^+$.

Claim 1. The space T is C-embedded in $T_1 \times T_2$, and hence $\upsilon T \supset T_1 \times T_2$: To prove this claim, let $f \in C(T)$. Since ω_α is a $P((m^*)^+)$-point[6], we can find a neighborhood G of ω_α in T_1 such that f is constant on $G \times \{\gamma\}$ for each $\gamma \in T_2 - \{\omega_\beta\}$. Then f takes on the constant value r on $(G - \{\omega_\alpha\}) \times \{\omega_\beta\}$. Extend f over $T_1 \times T_2$ by setting $f(t_0) = r$. Then the extension is continuous, and thus T is C-embedded in $T_1 \times T_2$, from which $\upsilon T \supset T_1 \times T_2$ follows.

Let us set $E = \{\omega_\alpha\} \times (T_2 - \{\omega_\beta\})$ and $F = (T_1 - \{\omega_\alpha\}) \times \{\omega_\beta\}$; then E and F are disjoint closed subsets of T such that

(1) $\quad t_0 \in cl_{\upsilon T}E \cap cl_{\upsilon T}F$.

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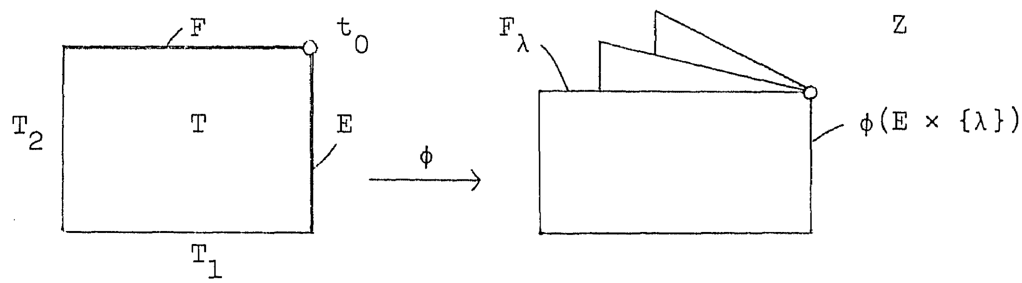
6 A $P(m)$-point is a point which lies in the interior of every intersection of less than m its neighborhoods.

Let Λ be the discrete space of cardinality n, and $Z^{\#}$ be the quotient space obtained from $T \times \Lambda$ by collapsing the set $\{e\} \times \Lambda$ to a point for each $e \in E$. Define Z to be the space $Z^{\#}$ with the strong topology with respect to this identification. Then it is easily checked that Z is a 0-dimensional metacompact $P(m)$-space with $|Z| = w(Z) = n \cdot (m^*)^+$. Let ϕ: $T \times \Lambda \to Z$ be the natural map. Setting $F_\lambda = \phi(F \times \{\lambda\})$ for each $\lambda \in \Lambda$, we have a discrete family $\{ F_\lambda \mid \lambda \in \Lambda \}$ of closed subsets in Z.



Claim 2. $\bigcap \{ cl_{\upsilon Z} F_\lambda \mid \lambda \in \Lambda \} \neq \emptyset$: Although this claim is apparent in the light of the above picture, we make a rigorous proof for later use. There exists a continuous extension Φ: $\upsilon(T \times \Lambda) \to \upsilon Z$ of ϕ. Claim 1 tells that $\upsilon(T \times \Lambda) \supset (T_1 \times T_2) \times \Lambda$. If we set $z(\lambda) = \Phi((t_0, \lambda))$ for each $\lambda \in \Lambda$, then

$$(2) \quad z(\lambda) \in \Phi(cl_{\upsilon(T \times \Lambda)}(F \times \{\lambda\})) \subset$$
$$\subset cl_{\upsilon Z}\phi(F \times \{\lambda\}) = cl_{\upsilon Z}F_\lambda .$$

We now show that $z(\lambda) = z(\mu)$ for each $\lambda, \mu \in \Lambda$. Suppose that $z(\lambda_1) \neq z(\lambda_2)$ for some $\lambda_1, \lambda_2 \in \Lambda$; then they have disjoint neighborhoods U_i, respectively. Since $V_i = \Phi^{-1}(U_i)$, $i = 1, 2$, are neighborhoods of (t_0, λ_i), there exist neighborhoods G_i of t_0 in $T_1 \times T_2$ such that $G_i \times \{\lambda_i\} \subset V_i$ for $i = 1, 2$. Then since

$\Phi(E \times \{\lambda_1\}) = \Phi(E \times \{\lambda_2\})$ and $U_1 \cap U_2 = \emptyset$, $G_1 \cap G_2 \cap E$ is empty, which contradicts the fact that $t_0 \in \mathrm{cl}_{\upsilon T} E$. Hence $z(\lambda) = z(\mu)$ for each $\lambda, \mu \in \Lambda$. This fact combined with (2) proves claim 2. Therefore Z is the desired space $Z_1(n, m)$.

A space X is called a Moore space (or a developable space) if there exists a countable collection $\{ \mathcal{U}_n \mid n \in N \}$ of open covers such that $\{ \mathrm{St}(x, \mathcal{U}_n) \mid n \in N \}$ is a neighborhood base for each $x \in X$. Here $\mathrm{St}(x, \mathcal{U}_n) = \bigcup \{ U \in \mathcal{U}_n \mid U \ni x \}$. The collection $\{ \mathcal{U}_n \mid n \in N \}$ is called a development for X (cf. [AS], [E]). It is known ($[B_4]$) that a Moore space is subparacompact.

6.6.3 FACT. For each infinite cardinal n, there exists a 0-dimensional Moore space $Z = Z_2(n)$, with $|Z| = w(Z) = n \cdot \exp \aleph_0$, that has a discrete family $\mathcal{F}$ of closed subsets in Z such that $|\mathcal{F}| = n$ and $\bigcap \{ \mathrm{cl}_{\upsilon Z} F \mid F \in \mathcal{F} \} \neq \emptyset$.

Proof. Let Ψ be the space of Isbell [GJ, 5I, p.79], which is made as follows: Choose a maximal family $\mathcal{S}$ of infinite subsets of N such that the intersection of any two is finite (apply Zorn's lemma to [E, 3.6.18]). Then $|\mathcal{S}| = \exp \aleph_0$. Let $D = \{ \omega_S \mid S \in \mathcal{S} \}$ be a new set of disjoint points, and define $\Psi = N \cup D$ with the following topology: Each point of N is isolated, while a neighborhood of ω_S is any set containing ω_S and all but finite number of points of S. Then Ψ is 0-dimensional, and it was proved in [GJ] that Ψ is pseudocompact (i.e., $\upsilon\Psi = \beta\Psi$ by 3.3.3) but not countably compact. Further Ψ is known to be a Moore space;

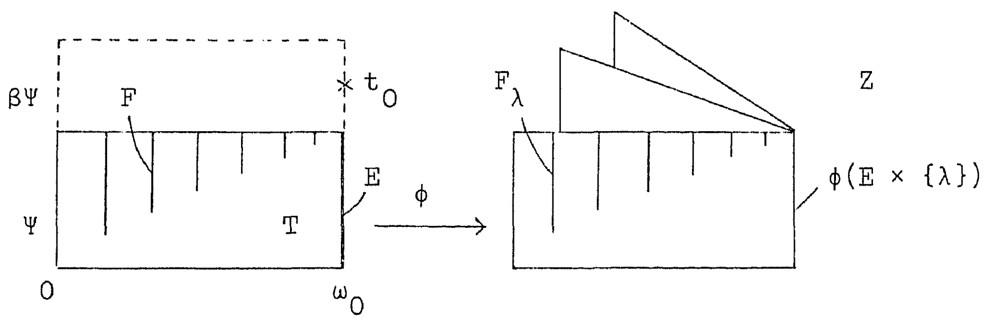
in fact, we have a development $\{ \mathcal{G}_n \mid n \in N \}$ for Ψ by defining $\mathcal{G}_n = \{ (\{\omega_S\} \cup S) - K_n \mid S \in \mathcal{S} \} \cup \{ \{n\} \mid n \in N \}$, where $K_n = \{ i \in N \mid i \leqq n \}$, for each $n \in N$. Let us set $T = W(\omega_0 + 1) \times \Psi$. Then T is a 0-dimensional Moore space with $|T| = w(T) = \exp \aleph_0$. Since $W(\omega_0 + 1)$ is compact, it follows from 1.2 that

$$(1) \qquad \upsilon T = W(\omega_0 + 1) \times \upsilon\Psi \ (= W(\omega_0 + 1) \times \beta\Psi).$$

Since Ψ is not countably compact, there exists a decreasing sequence $\{ H_n \mid n < \omega_0 \}$ of closed subsets in Ψ with empty intersection. Since $\upsilon\Psi = \beta\Psi$, we can find a point $p_0 \in \bigcap \{ \mathrm{cl}_{\upsilon\Psi} H_n \mid n < \omega_0 \}$. Let us set $E = \{\omega_0\} \times \Psi$, $F = \bigcup \{ \{n\} \times H_n \mid n < \omega_0 \}$ and $t_0 = (\omega_0, p_0)$; then E and F are disjoint closed subsets of T such that

$$(2) \qquad t_0 \in \mathrm{cl}_{\upsilon T} E \cap \mathrm{cl}_{\upsilon T} F.$$

Let Λ be the discrete space of cardinality κ. We make a space Z from $T \times \Lambda$ by the same way as in the proof of 6.6.2. Let $\phi: T \times \Lambda \to Z$ be the natural map.



Then the resulting space Z is a 0-dimensional Moore space with $|Z| = w(Z) = \kappa \cdot \exp \aleph_0$. Indeed, denoting the development for T by $\{ \mathcal{V}_n \mid n \in N \}$, we have a development $\{ \mathcal{U}_n \mid n \in N \}$ for Z by defining $\mathcal{U}_n = \bigcup\{ \mathcal{U}(V) \mid V \in \mathcal{V}_n \}$ for each $n \in N$, where

$$\mathcal{U}(V) = \begin{cases} \{ \phi(V \times \Lambda) \} & \text{if } V \cap E \neq \emptyset, \\ \{ \phi(V \times \{\lambda\}) \mid \lambda \in \Lambda \} & \text{if } V \cap E = \emptyset. \end{cases}$$

Let us set $F_\lambda = \phi(F \times \{\lambda\})$ for each $\lambda \in \Lambda$. Then $\{ F_\lambda \mid \lambda \in \Lambda \}$ is a discrete family of closed subsets in Z, and a similar argument to that of 6.6.2 shows that $\bigcap\{ \mathrm{cl}_{\upsilon Z} F_\lambda \mid \lambda \in \Lambda \} \neq \emptyset$. Hence Z is proved to be the desired space $Z_2(n)$.

We now proceed to Task 2.

6.7 NOTATION. Let k, m be infinite cardinals with $k \geqq m$, and $\Sigma^+(k, m) = \Sigma \cup \{s\}$ the space defined in 6.1. Then $\Sigma^*_n(k, m)$ denotes the quotient space obtained from $\Sigma^+(k, m) \times D(n)$ by collapsing the set $\{s\} \times D(n)$ to a point y_0. The space $\Sigma^*_n(k, m)$ is a 0-dimensional paracompact space, and it is a $P(m)$-space in case m is a regular cardinal. Note that $|\Sigma^*_n(k, m)| = k \cdot n$ and $w(\Sigma^*_n(k, m)) = (w(\Sigma^+(k, m)))^n$.

6.8 THEOREM. Let X be a locally weakly-m-compact space. If $\upsilon X - X$ has a point x_0, with $\chi(x_0, \upsilon X) \leqq n$, that has no neighborhood in υX whose restriction to X is weakly-m-compact, then $X \times Y$ is not C-embedded in $\upsilon X \times Y$, where $Y = \Sigma^*_n(w(X), m)$.

Proof. Let x_0 be a point of $\upsilon X - X$ satisfying the stated conditions. Let $\{ G_\lambda \mid \lambda \in \Lambda \}$ be a neighborhood base at x_0 in υX with $|\Lambda| = n$. For each $\lambda \in \Lambda$, $X \cap \mathrm{cl}_{\upsilon X} G_\lambda$ is not weakly-m-compact, and thus there exists an open cover $\mathcal{U}_\lambda$ of X such that no subfamily of cardinality less than m has dense union in $X \cap G_\lambda$. Let us set $\Sigma^+(w(X), m) = \Sigma \cup \{s\}$. Since it can be assumed without loss of generality that $|\mathcal{U}_\lambda| = w(X)$, we denote the collection of all subfamilies of $\mathcal{U}_\lambda$ whose cardinality is less than m by $\{ \mathcal{U}_{\lambda\sigma} \mid \sigma \in \Sigma \}$, where $\sigma \subset \sigma'$ if and only if $\mathcal{U}_{\lambda\sigma} \subset \mathcal{U}_{\lambda\sigma'}$. If we set $H_{\lambda\sigma} = (X \cap G_\lambda) - cl_X(\bigcup\{ U \mid U \in \mathcal{U}_{\lambda\sigma} \})$ for each $\sigma \in \Sigma$, then $H_{\lambda\sigma}$ is a non-empty open set. Pick $x_{\lambda\sigma} \in H_{\lambda\sigma}$, and choose $f_{\lambda\sigma} \in C(X)$ such that $f_{\lambda\sigma}(x_{\lambda\sigma}) = 0$ and $f_{\lambda\sigma}(X - H_{\lambda\sigma}) = \{1\}$. Topologize Λ with the discrete topology. Since $|\Lambda| = n$, $Y = \Sigma^*_n(w(X), m)$ is made from $(\Sigma \cup \{s\}) \times \Lambda$ by collapsing the set $\{s\} \times \Lambda$ to a point $y_0 \in Y$. Let ψ: $(\Sigma \cup \{s\}) \times \Lambda \to Y$ be the quotient map. If we set $\psi((\sigma, \lambda)) = y_{\lambda\sigma}$ for each $\lambda \in \Lambda$ and each $\sigma \in \Sigma$, since $y_{\lambda\sigma}$ is an isolated point of Y, there exists $g_{\lambda\sigma} \in C(Y)$ such that $g_{\lambda\sigma}(y_{\lambda\sigma}) = 0$ and $g_{\lambda\sigma}(Y - \{y_{\lambda\sigma}\}) = \{1\}$. For each $\lambda \in \Lambda$ and each $\sigma \in \Sigma$, let us set

$$p_{\lambda\sigma} = (x_{\lambda\sigma}, y_{\lambda\sigma}) \in X \times Y,$$

$$K_{\lambda\sigma} = H_{\lambda\sigma} \times \{y_{\lambda\sigma}\} \subset X \times Y,$$

$$h_{\lambda\sigma}((x, y)) = \min\{1, f_{\lambda\sigma}(x)+g_{\lambda\sigma}(y)\}; \quad (x, y) \in X \times Y.$$

Then $h_{\lambda\sigma} \in C(X \times Y)$, $h_{\lambda\sigma}(p_{\lambda\sigma}) = 0$ and $h_{\lambda\sigma}((X \times Y) - K_{\lambda\sigma}) = \{1\}$.

Claim 1. $\mathcal{K} = \{ K_{\lambda\sigma} \mid \lambda \in \Lambda, \sigma \in \Sigma \}$ is locally finite in $X \times Y$: To prove this claim, let $p = (x, y) \in X \times Y$. In case $y = y_{\lambda\sigma}$ for some $\lambda \in \Lambda$ and $\sigma \in \Sigma$, $X \times \{y\}$ is a neighborhood of p which meets only $K_{\lambda\sigma} \in \mathcal{K}$. In case $y = y_0$, choose a weakly-m-compact neighborhood $G(x)$ of x; then for each $\lambda \in \Lambda$ there is $\sigma_\lambda \in \Sigma$ such that $G(x) \subset cl_X(\bigcup\{ U \mid U \in \mathcal{U}_{\lambda\sigma_\lambda} \})$. Let us set $G(y) = \bigcup\{ \psi(J(\sigma_\lambda) \times \{\lambda\}) \mid \lambda \in \Lambda \}$, where $J(\sigma_\lambda) = \{s\} \cup \{ \sigma \in \Sigma \mid \sigma \supset \sigma_\lambda \}$. Then $G(y)$ is a neighborhood of y in Y, and $G(x) \times G(y)$ meets no member of $\mathcal{K}$. Hence $\mathcal{K}$ is proved to be locally finite in $X \times Y$.

Define a function h on $X \times Y$ by

$$h(q) = \inf \{ h_{\lambda\sigma}(q) \mid \lambda \in \Lambda, \sigma \in \Sigma \};\ q \in X \times Y.$$

Then h is continuous, since $\mathcal{K}$ is locally finite.

Claim 2. The function h admits no continuous extension over $\upsilon X \times Y$: To prove this claim, let $V \times W$ be a given basic neighborhood of (x_0, y_0) in $\upsilon X \times Y$. There is $\lambda \in \Lambda$ with $G_\lambda \subset V$. Choose $\sigma \in \Sigma$ with $y_{\lambda\sigma} \in W$. Then both $p_1 = (x_{\lambda\sigma}, y_{\lambda\sigma})$ and $p_2 = (x_{\lambda\sigma}, y_0)$ belong to $V \times W$ and $h(p_1) = 0$, while $h(p_2) = 1$. This shows that h does not extend continuously to (x_0, y_0). Hence the proof is complete.

6.9 For practical use, we summarize essential information about spaces constructed above. Let k, m and n be infinite cardinals.

6.9.1 If $Z_1 = Z_1(n, m^*)$, $\Sigma^+ = \Sigma^+(k, m^*)$ and $n \cdot m^* \leqq k$, then $F(Z_1, \Sigma^+)$ is a 0-dimensional metacompact $P(m)$-space such that

$$\begin{aligned} w(F(Z_1, \Sigma^+)) &= |Z_1| \cdot w(Z_1) \cdot w(\Sigma^+) \\ &= |Z_1| \cdot |\Sigma^+| \ (= |F(Z_1, \Sigma^+)| \ ) \\ &\leqq n \cdot (m^*)^+ \cdot \exp k = \exp k. \end{aligned}$$

In case $m = \aleph_0$, since $w(\Sigma^+) = k$,

$$w(F(Z_1, \Sigma^+)) = |F(Z_1, \Sigma^+)| = n \cdot (\aleph_0)^+ \cdot k = k \cdot \aleph_1.$$

6.9.2 If $Z_2 = Z_2(n)$, $\Sigma^+ = \Sigma^+(k, \aleph_0)$ and $n \leqq k$, then, since a Moore space is subparacompact, $F(Z_2, \Sigma^+)$ is a 0-dimensional subparacompact space such that

$$\begin{aligned} w(F(Z_2, \Sigma^+)) &= |Z_2| \cdot w(Z_2) \cdot w(\Sigma^+) \\ &= |Z_2| \cdot |\Sigma^+| \ (= |F(Z_2, \Sigma^+)| \ ) \\ &\leqq n \cdot (\exp \aleph_0) \cdot k = k \cdot \exp \aleph_0. \end{aligned}$$

6.9.3 If $\Sigma^+ = \Sigma^+(k, m^*)$, $\Sigma^* = \Sigma^*_n(k, m^*)$ and $m^* \leqq k$, then Σ^* is a a 0-dimensional paracompact $P(m)$-space such that

$$w(\Sigma^*) = (w(\Sigma^+))^n \leqq (\exp k)^n \text{ and}$$

$$|\Sigma^*| = n \cdot |\Sigma^+| \leqq n \cdot \exp k.$$

6.10 We conclude this section by applying our theory to answer the question of Hušek [H_7, p.326]: Do there exist minimal cardinals m, n for which $|X| = m$, $|Y| = n$ and $\upsilon(X \times Y) \neq \upsilon X \times \upsilon Y$? The following example provides a positive answer to this question.

6.10.1 EXAMPLE. Let Q be the space of all rational numbers with the usual topology. There exists a 0-dimensional metacompact space Y with $|Y| = w(Y) = \aleph_1$ such that $\upsilon(Q \times Y) \neq Q \times \upsilon Y$.

Proof. We take for Y the space $F(Z_1(\aleph_0, \aleph_0), \Sigma^+(\aleph_0, \aleph_0))$ defined above. Then, by 6.9.1, Y satisfies the stated conditions. Since $w(Q) = \chi(Q) = \aleph_0$, it follows from 6.4 that $\upsilon(Q \times Y) \neq Q \times \upsilon Y$.

6.10.2 REMARK. This question was first answered by the author in $[O_3]$; however, the space Y exhibited there is not metacompact and $w(Y) = \exp \aleph_0$. In 11.5.1, we shall also make a 0-dimensional locally compact space Y with $|Y| = w(Y) = \aleph_1$ such that $\upsilon(Q \times Y) \neq Q \times \upsilon Y$ (see also 15.8.1).

7. Proofs of theorems (5.1, 5.4 and 5.6)

7.1 PREREQUISITES. We make use of the following results:

7.1.1 ([HM], [H_2]) If X is weakly-m*-compact and Y is a P(m)-space, then the projection π_Y: X × Y → Y is z-closed. (It should be noted that a space is a P(m)-space if and only if it is a P(m*)-space.)

7.1.2 ([H_9]) Let $m < m_1$. If X is a P(m)-space, so is υX.

We begin with the detailed theorems which constitute parts of 5.1, 5.4 and 5.6.

7.2 THEOREM. For each infinite cardinal m, the following conditions on a space X, with $|X| < m_1$, are equivalent:

(a) X is locally weakly-m*-compact.

(b) X × Y is C-embedded in X × υY for each P(m)-space Y.

(c) X × Y is C-embedded in X × υY for each 0-dimensional metacompact P(m)-space Y with $w(Y) \leqq \exp w(X)$.

In case $m = \aleph_0$, the following conditions are also equivalent to the above:

(c') X × Y is C-embedded in X × υY for each 0-dimensional metacompact space Y with $w(Y) \leqq w(X)\cdot\aleph_1$.

(d) X × Y is C-embedded in X × υY for each 0-dimensional subparacompact space Y with $w(Y) \leqq w(X)\cdot\exp\aleph_0$.

Proof. (a) → (b). Let Y be a P(m)-space. Since X is locally weakly-m*-compact, it suffices to show that S × Y is

C-embedded in $S \times \upsilon Y$ for each weakly-m^*-compact subset S of X. Let S be a given weakly-m^*-compact subset of X; then $\pi_Y: S \times Y \to Y$ is z-closed by 7.1.1. Since $|S| < m_1$, it follows from 3.4 that $S \times Y$ is C-embedded in $S \times \upsilon Y$.

(b) $\to$ (c). Obvious.

(c) $\to$ (a). Suppose on the contrary that X is not locally weakly-m^*-compact at $x_0 \in X$. Let $n = \chi(x_0, X)$, and let us set $Y = F(Z_1(n, m^*), \Sigma^+(w(X), m^*))$, as defined in the preceding section. Then, by 6.9.1, Y is a 0-dimensional metacompact $P(m)$-space with $w(Y) \leqq \exp w(X)$. It follows from 6.4 that $X \times Y$ is not C-embedded in $X \times \upsilon Y$. Thus we have (a).

In case $m = \aleph_0$, (b) $\to$ (c') and (b) $\to$ (d) are obvious.

(c') $\to$ (a). Apply the proof of (c) $\to$ (a) to the case $m = \aleph_0$. Then $w(Y) \leqq w(X) \cdot \aleph_1$, as noted in 6.9.1.

(d) $\to$ (a). The proof is the same as that of (c) $\to$ (a) if one consider Y to be $F(Z_2(n), \Sigma^+(w(X), \aleph_0))$ (then, by 6.9.2, Y is a 0-dimensional subparacompact space with $w(Y) \leqq w(X) \cdot \exp \aleph_0$). Hence the proof is complete.

7.3 REMARK. In 7.2, (a) $\to$ (b) is essentially due to Hušek, who showed in [H_9] that a υ-pair (X, Y) of a locally weakly-m-compact space X with $|X| < m_1$ and a $P(m)$-space Y can be described in terms of function spaces (see section 2). For $m = \aleph_0$, (a) $\to$ (b) was earlier proved by Comfort in [C_3].

7.4 REMARK. In the preceding proof, the assumption that $|X| < m_1$ is useful only for the implication (a) $\to$ (b).

7.5 THEOREM. For each infinite cardinal m, the following conditions on a locally weakly-m^*-compact space X are equivalent:

(a) Each point of $\upsilon X - X$ has a neighborhood G in υX such that $G \cap X$ is weakly-m^*-compact.

(b) $X \times Y$ is C-embedded in $\upsilon X \times Y$ for each $P(m)$-space Y.

(c) $X \times Y$ is C-embedded in $\upsilon X \times Y$ for each 0-dimensional paracompact $P(m)$-space Y with $w(Y) \leqq \exp w(\upsilon X)$.

Proof. (a) $\to$ (b). Let Y be a $P(m)$-space, and let $f \in C(X \times Y)$. For our end, it suffices to find, for each $x \in \upsilon X - X$, a neighborhood G of x such that f can be continuously extended over $(X \times Y) \cup (G \times Y)$. Let $x \in \upsilon X - X$. Choose a cozero-set neighborhood G of x in υX such that $X \cap \mathrm{cl}_{\upsilon X} G$ is weakly-m^*-compact. If we set $X_1 = X \cap \mathrm{cl}_{\upsilon X} G$, then $\pi_Y \colon X_1 \times Y \to Y$ is z-closed by 7.1.1, and so it follows from 3.4 that $X_1 \times Y$ is C-embedded in $\upsilon X_1 \times Y$. There is a cozero-set G_1 of υX_1 with $G_1 \cap X_1 = G \cap X$. Then f can be continuously extended over $(X \times Y) \cup (G_1 \times Y)$. By 3.3.2,

$$G_1 = \upsilon(G_1 \cap X_1) = \upsilon(G \cap X) = G.$$

Hence f admits a continuous extension over $(X \times Y) \cup (G \times Y)$, as required.

(b) $\to$ (c). Obvious.

(c) $\to$ (a). Assume that (a) is false at some $x_0 \in \upsilon X - X$. We take for Y the space $\Sigma^*_n(w(X), m^*)$ defined in 6.7, where $n =$

$\chi(x_0, \upsilon X)$. Then, by 6.9.3, Y is a 0-dimensional paracompact $P(m)$-space with $w(Y) \leqq \exp w(\upsilon X)$. Since X is locally weakly-m^*-compact, it follows from 6.8 that $X \times Y$ is not C-embedded in $\upsilon X \times Y$. Hence the proof is complete.

It might be interesting to know whether in 7.5 the assumption that X is locally weakly-m^*-compact can be omitted or not. The following theorem shows that this omission is possible for $m < m_1$ if we allow X to be a $P(m)$-space.

7.6 THEOREM. Let X be a $P(m)$-space, where $m < m_1$. If $X \times Y$ is C-embedded in $\upsilon X \times Y$ for each 0-dimensional paracompact $P(m)$--space Y with $w(Y) \leqq \chi(\upsilon X)$, then X is realcompact.

Proof. For $m = \aleph_0$, this theorem was proved by McArthur in [M_1]. Our proof is a slight modification of his proof. Suppose on the contrary that X is not realcompact. Choose $x_0 \in \upsilon X - X$, and let $\{ G_\lambda \mid \lambda \in \Lambda \}$ be a neighborhood base at x_0 in υX with $|\Lambda| \leqq \chi(\upsilon X)$. Let us set $Y = \Lambda \cup \{y_0\}$, topologized as follows: Each point of Λ is isolated and $\{ J(\lambda) \mid \lambda \in \Lambda \}$, where $J(\lambda) = \{y_0\} \cup \{ \lambda' \in \Lambda \mid G_{\lambda'} \subset G_\lambda \}$, is a neighborhood base at y_0. Then Y is a 0-dimensional paracompact space with $w(Y) \leqq \chi(\upsilon X)$. By 7.1.2, υX is a $P(m)$-space, which implies that Y is a $P(m)$-space. For each $\lambda \in \Lambda$, choose $f_\lambda \in C(\upsilon X)$ such that $f_\lambda(x_0) = 0$ and $f_\lambda(\upsilon X - G_\lambda) = \{1\}$. Define a function h on $X \times Y$ by

$$h((x,\ y)) = \begin{cases} 1 & \text{if } y = y_0, \\ f_\lambda(x) & \text{if } y = \lambda \in \Lambda. \end{cases}$$

Then $h \in C(X \times Y)$, and it is easily checked that h cannot be extended continuously to $(x_0,\ y_0) \in \upsilon X \times Y$. This contradiction completes the proof.

We are now in a position to prove theorems stated in section 5. Before proving 5.1, we give proofs of 5.4 and 5.6.

7.7 Proof of Theorem 5.4. (a) $\rightarrow$ (b). Let X be a space satisfying 5.4 (a), and let Y be a $P(m)$-space. By 7.5, $X \times Y$ is C-embedded in $\upsilon X \times Y$. On the other hand, since υX is locally weakly-m^*-compact, it follows from 7.2 that $\upsilon X \times Y$ is C-embedded in $\upsilon X \times \upsilon Y$, and so $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds. (b) $\rightarrow$ (c) is clear. (c) $\rightarrow$ (a). Since $|X| < m_1$ by 4.4, it follows from 7.2 and 7.5 that X satisfies (a). Hence the proof is complete.

7.8 Proof of Theorem 5.6. (a) $\rightarrow$ (b) is the special case of (a) $\rightarrow$ (b) in 5.4. (b) $\rightarrow$ (c) is obvious. (c) $\rightarrow$ (a). Since $|X| < m_1$ by 4.4, it follows from 7.2 that X is locally weakly-m^*--compact. It remains to prove that X is realcompact. In case $m \geqq m_1$, X is discrete, because X is a $P(m)$-space with $|X| < m_1$. Since a discrete space of non-measurable cardinal is realcompact, X is realcompact. In case $m < m_1$, since $\chi(\upsilon X) \leqq \exp w(X)$ by 3.2.3, it follows from 7.6 that X is realcompact. Hence the proof is complete.

7.9 Proof of Theorem 5.1. (a) $\to$ (b) follows from the case $m = \aleph_0$ of 5.6. (b) $\to$ (c) and (b) $\to$ (d) are obvious. Both (c) $\to$ (a) and (d) $\to$ (a) follow from 4.4, 7.2 and 7.6. Hence the proof is complete.

8. On R(normal) and R(countably paracompact)

In 5.8, we have conjectured that $R = R$(normal) = R(countably paracompact). In this section, we prove these equalities in the class of first countable spaces.

8.1 THEOREM. Let X be a first countable space. Then the following conditions on X are equivalent:

(a) X is locally compact, realcompact and $|X| < m_1$.

(b) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each normal space Y.

(c) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each countably paracompact space Y.

Proof. Both (a) → (b) and (a) → (c) are the results of 1.2 (or 5.1). When proving (b) → (a) and (c) → (a), that X is realcompact and $|X| < m_1$ follows from 7.6 and 4.4. To show the local compactness of X, in view of 6.3.3 and 6.4, it suffices to observe the following two facts.

8.1.1 FACT. There exists a normal space $Z = Z_3(\aleph_0)$ that has a countable locally finite family $\{ F_n \mid n \in N \}$ of closed subsets in Z such that $\bigcap \{ cl_{\upsilon Z} F_n \mid n \in N \} \neq \emptyset$.

Proof. Let Z be Rudin's example of a Dowker (i.e., normal but not countably paracompact) space (cf. [R_1]). It was shown in [R_1] that Z has a decreasing sequence $\{ D_n \mid n < \omega_0 \}$ of closed subsets with empty intersection such that $\bigcap \{ cl_{\upsilon Z} F_n \mid$

$n < \omega_0 \} \neq \emptyset$. Since $\{ D_n \mid n < \omega_0 \}$ is clearly locally finite, Z is the desired space $Z_3(\aleph_0)$.

8.1.2 FACT. There exists a countably paracompact space $Z = Z_4(\aleph_0)$ that has a countable locally finite family $\{ F_n \mid n \in N \}$ of closed subsets in Z such that $\bigcap \{ \mathrm{cl}_{\upsilon Z} F_n \mid n \in N \} \neq \emptyset$.

Proof. We utilize the space Y due to Comfort $[C_2$, p.99] (independently due to Mack and Johnson [MJ]). The space Y is constructed as the quotient space obtained from the product space $T_0 = N \times W(\omega_1 + 1) \times W(\omega_1 + 1)$ by identifying, for each $k \in N$ and each $\gamma \leqq \omega_1$, the two points (k, ω_1, γ) and $(k+1, \gamma, \omega_1)$. Let f: $T_0 \to Y$ be the quotient map, and let us set $Z = Y - \{y_0\}$, where y_0 is the center point $f((1, \omega_1, \omega_1))$ ($= f((k, \omega_1, \omega_1))$). Then he proved that $\upsilon Z = Y$. Let us set $T = T_0 - f^{-1}(y_0)$ and $g = f|T$. Since T is countably paracompact, and since g is a perfect map from T onto Z, Z is countably paracompact. Setting

$$F_n = f(\{ i \mid i \geqq n \} \times W(\omega_1 + 1) \times W(\omega_1 + 1)) \cap Z$$

for each $n \in N$, we obtain a locally finite family $\{ F_n \mid n \in N \}$ of closed subsets in Z such that $y_0 \in \bigcap \{ \mathrm{cl}_{\upsilon Z} F_n \mid n \in N \}$. Hence Z is the desired space $Z_4(\aleph_0)$. Facts 8.1.1 and 8.1.2 complete the proof of 8.1.

8.2 REMARK. In 8.1, "normal" or "countably paracompact" can not be replaced by "normal countably paracompact" (see 5.8).

9. Problems and remarks

9.1 PROBLEM. Is it true that $\mathcal{R} = \mathcal{R}($ normal $) = \mathcal{R}($ countably paracompact $)$? More generally, for each infinite cardinal m, is it true that $\mathcal{R}(\ \mathrm{P}(m)\ ) = \mathcal{R}($ normal $\mathrm{P}(m)\ ) = \mathcal{R}($ countably paracompact $\mathrm{P}(m)\ )$?

9.2 PROBLEM. A space $Z = Z(n)$ which has a locally finite family $\mathcal{F}$ of closed subsets such that $|\mathcal{F}| = n$ and $\bigcap\{\ \mathrm{cl}_{\upsilon Z}F \mid F \in \mathcal{F}\ \} \neq \emptyset$ has played an important role throughout this chapter. It would be nice to have examples of $Z(n)$ which have various other properties. Among other things, we want the following ones:

(1) a normal $\mathrm{P}(m)$-space $Z(n)$,

(2) a countably paracompact $\mathrm{P}(m)$-space $Z(n)$,

(3) a metacompact subparacompact space $Z(n)$ with $|Z(n)| = w(Z(n)) = n\cdot\aleph_1$.

The existence of (1) and (2) guarantees 9.1 to be true, and that of (3) enable us to replace "metacompact" by "metacompact subparacompact" in 5.1 (c). The constructions developed in 6.6.2 and 6.6.3 cannot be duplicated to make (1), because every pairwise disjoint closed subsets of a normal space T have disjoint closures in υT. It should be emphasized that (1) must be a Dowker space (see 8.2). Recently, Rudin $[R_2]$ has proved that for each infinite cardinal κ there exists a normal space Z, called a κ-Dowker space, which has a monotone decreasing family $\{\ F_\alpha \mid \alpha < \omega(\kappa)\ \}$ of closed subsets with empty intersection such that

$\bigcap\{ \mathrm{cl}_{\upsilon Z}F_\alpha \mid \alpha < \omega(\kappa) \} \neq \emptyset$, where $\omega(\kappa)$ denotes the initial ordinal of κ. The author does not know whether the κ-Dowker space has the desired locally finite families or not.

9.3 PROBLEM. Do there exist any other conditions on Y for which 5.1 remains valid ?

9.4 We temporarily say that a space X is an $\alpha(m)$-space if each point of υX has a neighborhood G in υX such that $G \cap X$ is weakly-m-compact.

9.4.1 PROBLEM. Characterize $R(\ \alpha(m)\ )$ for each $\alpha > \aleph_0$.

Since an $\alpha(\aleph_0)$-space is precisely a locally compact, realcompact space, 9.4.1 can be viewed as an extension of the problem of characterizing $R($ locally compact, realcompact $)$ (see 4.3). The following proposition shows that $R(\ \alpha(m)\ ) = R($ weakly-m-compact $)$.

9.4.2 PROPOSITION. For each infinite cardinal m, the following conditions on a space X are equivalent:

(a) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each $\alpha(m)$-space Y.

(b) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each weakly-m-compact space Y.

Proof. Since a weakly-m-compact space is an $\alpha(m)$-space, (a) $\rightarrow$ (b) is obvious. To prove (b) $\rightarrow$ (a), let Y be an $\alpha(m)$-space, and let $f \in C(X \times Y)$. For each $y \in \upsilon Y$, choose a cozero-set neighborhood G of y such that $Y \cap \mathrm{cl}_{\upsilon Y}G$ is weakly-m-compact. If we set

$Y_1 = Y \cap cl_{\upsilon Y} G$, then $\upsilon(X \times Y_1) = \upsilon X \times \upsilon Y_1$ holds by (b). There is a cozero-set G_1 of υY_1 with $G_1 \cap Y_1 = G \cap Y$. By 3.2.2, $G_1 = \upsilon(G_1 \cap Y_1) = \upsilon(G \cap Y) = G$, and so f can be continuously extended over $(X \times Y) \cup (\upsilon X \times G)$. Since $y \in \upsilon Y$ was arbitrary, this shows that f admits a continuous extension over $\upsilon X \times \upsilon Y$. Hence the proof is complete.

9.5 PROBLEM. Characterize $\mathcal{R}$(realcompact)[7]. Hušek [H_6], [H_8] and McArthur [M_1] proved that each member X of this class, with $|X| < m_1$, is realcompact (cf. also 7.6); however, the characterization is not yet known in complete form.

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7 This problem has been posed by the author in [O_3].

CHAPTER 2

CHARACTERIZATIONS OF THE CLASSES $\mathcal{R}$(k), $\mathcal{R}$(locally compact), $\mathcal{R}$(Moore) AND $\mathcal{R}$(locally pseudocompact)

A space X is called a k-space if it has the weak topology determined by the family of its compact subsets, i.e., $S \subset X$ is open if and only if $S \cap K$ is open in K for each compact subset K of X (cf. [E]). Locally compact spaces and first countable spaces are k-spaces. Note that a Moore space is first countable, and hence it is a k-space.

10. Main theorems

In this section, we state main theorems of this chapter and discuss some remarks. The proofs will be given later in section 12.

10.1 THEOREM. The following conditions on a space X are equivalent:

(a) υX is locally compact and $|X| < m_1$.

(b) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each k-space Y.

(c) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each locally compact space Y with $w(Y) \leqq \chi(\upsilon X) \cdot \aleph_1$.

(d) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each Moore space Y with $w(Y) \leqq \chi(\upsilon X) \cdot \exp \aleph_0$.

10.2 REMARK. In 10.1, (a) → (b) was proved by Hušek [H_8], and it was also proved in [C_3] under the assumption that $|X \times Y| < m_1$. Conversely, in [O_3], the author has proved (c) → (a) without the cardinality condition on Y. For (b) → (a), see also [O_2].

10.3 REMARKS. (1) As the reader will observe in the proof, 10.1 remains valid if 0-dimensionality is added to the conditions on Y in (b), (c) and (d).

(2) It will be shown in 13.1 that in (d) "Moore" can be replaced by "locally compact, Moore" if $\chi(\upsilon X) \leqq \exp \aleph_0$; however, the author does not know whether this replacement is possible in general (cf. 13.1.1).

(3) Recall from [C_3] that a space is locally pseudocompact if each point has a pseudocompact neighborhood. If υX is locally compact, then X is locally pseudocompact, but the converse is false (see [C_3]). As for the problem of when it will occur that υX is locally compact, see [B_3], [C_2], [C_3], [H_4], [I_6], [I_7] and [N_1].

10.4 EXAMPLE. In 10.1 (c), local compactness of Y cannot be weakened to local pseudocompactness. Let X_i, i = 1, 2, be pseudocompact spaces with $|X_i| < m_1$ such that $X_1 \times X_2$ is not pseudocompact (e.g., see [E, p.265]). Then $\upsilon X_i = \beta X_i$ by 3.3.3, and so υX_i are compact, but it follows from 1.1 that $\upsilon(X_1 \times X_2) \neq \upsilon X_1 \times \upsilon X_2$. Consequently, R(locally compact) $\neq$ R(locally pseudocompact).

The next theorem asserts that $\mathcal{R}$(locally pseudocompact) = $\mathcal{R}$(locally compact)$\cap\mathcal{R}$(pseudocompact). We now say that a space is a pseudo-k-space if it has the weak topology determined by the family of its pseudocompact subsets. Locally pseudocompact spaces and k-spaces are pseudo-k-spaces. The class of all spaces X such that X × Y is pseudocompact for each pseudocompact space Y is denoted by $\mathfrak{P}$, and was intensively studied by Frolik [F_2] and Noble [N_2]. We call a member of $\mathfrak{P}$ a strongly pseudocompact space.

10.5 THEOREM. The following conditions on a space X are equivalent:

(a) Each point of υX has a neighborhood G in υX such that $G\cap X$ is strongly pseudocompact, and $|X| < m_1$.

(b) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each pseudo-k-space Y.

(c) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each locally pseudocompact space Y.

(d) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each locally compact space Y and for each pseudocompact space Y.

10.6 REMARKS. (1) Let us say that a space is locally strongly pseudocompact if each point has a strongly pseudocompact neighborhood. Since a pseudocompact space which is locally strongly pseudocompact is strongly pseudocompact ([F_2]), 10.5 (a) is formally rephrased as follows: (a') X is a locally strongly pseudocompact space such that υX is locally compact and $|X| < m_1$.

In particular, this is the case if X is a k-space such that υX is locally compact and $|X| < m_1$ (cf. [T_2] and 10.3 (3)).

(2) The author does not know whether 10.5 remains true with the cardinality condition "$w(Y) \leqq \chi(\upsilon X)\cdot\aleph_1$" added.

10.7 REMARK. A space is called a quasi-k-space if it has the weak topology determined by the family of its countably compact subsets. We leave it to the reader to find a version of 10.5 for quasi-k-spaces. It will be proved, with essentially the same proof, that $\mathcal{R}$(quasi-k) = $\mathcal{R}$(locally countably compact) = $\mathcal{R}$(locally compact) $\cap$ $\mathcal{R}$(countably compact).

11. Tools for proving theorems

In this chapter, pseudo-$\mathfrak{m}$-compactness plays key roles in place of weak-$\mathfrak{m}$-compactness. Recall from [I_2] that, for each infinite cardinal $\mathfrak{m}$, a space X is pseudo-$\mathfrak{m}$-compact if each locally finite (or discrete, equivalently) family of non-empty open sets in X has cardinality less than $\mathfrak{m}$. A space is called locally pseudo-$\mathfrak{m}$-compact if each point has a pseudo-$\mathfrak{m}$-compact neighborhood. A (locally) pseudo-$\aleph_0$-compact space is precisely a (locally) pseudocompact space. The relationship between pseudo-$\mathfrak{m}$-compact spaces and weakly-$\mathfrak{m}$-compact spaces is summarized as follows:

$$\begin{tikzcd}
\text{compact} \arrow[r] \arrow[d, leftrightarrow] & \text{Lindelöf} \arrow[dr] & & \text{separable} \arrow[dl] \\
\text{weakly-}\aleph_0\text{-compact} \arrow[rr] \arrow[d] & & \text{weakly-}\aleph_1\text{-compact} \arrow[r] \arrow[d] & \cdots \\
\text{pseudo-}\aleph_0\text{-compact} \arrow[rr] \arrow[d, leftrightarrow] & & \text{pseudo-}\aleph_1\text{-compact} \arrow[r] & \cdots \\
\text{pseudocompact} & & &
\end{tikzcd}$$

The purpose of this section is to present useful tools for proving theorems stated in the preceding section. The central issue of the proofs is how to construct a suitable space Y such that $\upsilon(X \times Y) \neq \upsilon X \times \upsilon Y$ when υX is not locally compact. In section 6, we have built up such a space Y by attaching isolated points to the outside of a space Z which has a locally finite family $\mathcal{F}$ such that $\bigcap\{ \mathrm{cl}_{\upsilon Z} F \mid F \in \mathcal{F} \} \neq \emptyset$. In this section, we turn our attention to the inside construction of the space Y, and observe how to project the problem of extending a continuous function on $X \times Y$ over $\upsilon X \times \upsilon Y$ to the problem of finding local pseudo-m-compactness of X (or υX). This observation enables us to make the desired space Y, and yields a number of necessary conditions in order that $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ be valid, which will be used again in chapter 4 to characterize R(metrizable). The pattern of attack is much the same as that in section 6 except that pseudo-m-compactness replaces weak-m-compactness. The concepts and results 11.1 ∿ 11.3, 11.4 (2) have been presented by the author in [O_3].

A family $\{ F_\alpha \}$ of subsets of a space X is called expandable if there is a locally finite family $\{ H_\alpha \}$ of open sets in X with $F_\alpha \subset H_\alpha$ for each α. We introduce a new class of expandable families.

11.1 DEFINITION. Let m be a cardinal. A family $\{ F_\alpha \mid \alpha \in A \}$ of subsets of a space X is $D(m)$-expandable if there exists a locally finite family $\{ H_\alpha \mid \alpha \in A \}$ of open sets with $F_\alpha \subset H_\alpha$ for each $\alpha \in A$ and each F_α is the union of at most m subsets each of which is completely separated from $X - H_\alpha$.

In this definition we may replace "at most m" by "just m". As a space considered in this paper is completely regular Hausdorff, every expandable family in X is $D(|X|)$-expandable, and a CZ-expandable family defined in [O_1] is $D(1)$-expandable. Of course, $D(m)$-expandable families are $D(n)$-expandable whenever $n \geqq m$.

11.2 THEOREM. Let $X \times Y$ be C-embedded in $X \times \upsilon Y$. If there exists a $D(m)$-expandable family $\mathcal{F}$ in Y such that $|\mathcal{F}| = n$ and $\bigcap\{ \mathrm{cl}_{\upsilon Y} F \mid F \in \mathcal{F} \} \neq \emptyset$, then each point $x \in X$, with $\chi(x, X) \leqq n$, has a pseudo-m-compact neighborhood.

Proof. Suppose on the contrary that there exists a point $x_0 \in X$, with $\chi(x_0, X) \leqq n$, which has no pseudo-m-compact neighborhood. Let $\{ G_\lambda \mid \lambda \in \Lambda \}$ be a neighborhood base at x_0 in X with $|\Lambda| = n$. Then, for each $\lambda \in \Lambda$, $\mathrm{cl}_X G_\lambda$ is not pseudo-m-compact, and thus there is a locally finite family $\{ G'_{\lambda\mu} \mid \mu \in M_\lambda \}$ of non-empty open sets in $\mathrm{cl}_X G_\lambda$ with $|M_\lambda| = m$. Setting $G_{\lambda\mu} = G'_{\lambda\mu} \cap G_\lambda$ for each $\mu \in M_\lambda$, we have a locally finite family $\{ G_{\lambda\mu} \mid \mu \in M_\lambda \}$ of non-empty open sets in X. It can be assumed without loss of generality that $x_0 \notin \bigcap\{ G_{\lambda\mu} \mid \mu \in M_\lambda \}$. For each $\mu \in M_\lambda$, pick $x_{\lambda\mu} \in G_{\lambda\mu}$, and choose $f_{\lambda\mu} \in C(X)$ such that $f_{\lambda\mu}(x_{\lambda\mu}) = 0$ and $f_{\lambda\mu}(X - G_{\lambda\mu}) = \{1\}$. On the other hand, since $|\mathcal{F}| = n$, we may write $\mathcal{F} = \{ F_\lambda \mid \lambda \in \Lambda \}$. Then there exists a locally finite family $\{ H_\lambda \mid \lambda \in \Lambda \}$ of open sets in Y with $F_\lambda \subset H_\lambda$ for each $\lambda \in \Lambda$, and each F_λ is a union of m subsets each of which is completely separated from $Y - H_\lambda$. We express it by $F_\lambda = \bigcup\{ F_{\lambda\mu} \mid \mu \in M_\lambda \}$, that is, there is $g_{\lambda\mu} \in C(Y)$ such that $g_{\lambda\mu}(F_{\lambda\mu}) = \{0\}$ and $g_{\lambda\mu}(Y - H_\lambda) = \{1\}$. For each $\lambda \in \Lambda$ and each $\mu \in M_\lambda$, let us set

$$J_{\lambda\mu} = \{x_{\lambda\mu}\} \times F_{\lambda\mu},$$

$$K_{\lambda\mu} = G_{\lambda\mu} \times H_{\lambda},$$

$$h_{\lambda\mu}((x, y)) = \min \{1, f_{\lambda\mu}(x)+g_{\lambda\mu}(y)\}; \quad (x, y) \in X \times Y.$$

Then $h_{\lambda\mu} \in C(X \times Y)$, $h_{\lambda\mu}(J_{\lambda\mu}) = \{0\}$ and $h_{\lambda\mu}((X \times Y) - K_{\lambda\mu}) = \{1\}$. It is easily checked that $\{ K_{\lambda\mu} \mid \mu \in M_{\lambda}, \lambda \in \Lambda \}$ is locally finite in $X \times Y$. Therefore if we define a function h on $X \times Y$ by

$$h(q) = \inf \{ h_{\lambda\mu}(q) \mid \mu \in M_{\lambda}, \lambda \in \Lambda \}; \quad (x, y) \in X \times Y,$$

then h is continuous. Let us choose $y_0 \in \bigcap\{ \mathrm{cl}_{\upsilon Y} F_{\lambda} \mid \lambda \in \Lambda \}$; then $y_0 \in \upsilon Y - Y$, since $\mathcal{F}$ is locally finite in Y. We now show that h admits no continuous extension to the point $p_0 = (x_0, y_0) \in X \times \upsilon Y$. Let $V \times W$ be a given basic neighborhood of p_0 in $X \times \upsilon Y$. There is $\lambda \in \Lambda$ with $G_{\lambda} \subset V$, and $W \cap F_{\lambda\mu} \neq \emptyset$ for some $\mu \in M_{\lambda}$. Choose $y \in W \cap F_{\lambda\mu}$. Then both $p_1 = (x_{\lambda\mu}, y)$ and $p_2 = (x_0, y)$ belong to $V \times W$ and $h(p_1) = 0$, while $h(p_2) = 1$. This shows that h does not extend continuously to p_0, which contradicts the assumption that $X \times Y$ is C-embedded in $X \times \upsilon Y$. Hence the proof is complete.

11.3 COROLLARY. Let $X \times Y$ be C-embedded in $X \times \upsilon Y$. If there exists a locally finite family $\mathcal{H}$ of non-empty open sets in Y such that $|\mathcal{H}| = n$ and $\bigcap\{ \mathrm{cl}_{\upsilon Y} H \mid H \in \mathcal{H} \} \neq \emptyset$, then each point $x \in X$, with $\chi(x, X) \leqq n$, has a pseudo-c(Y)-compact neighborhood.

Proof. Let $\mathcal{H} = \{ H_{\lambda} \mid \lambda \in \Lambda \}$, and choose $y_0 \in \bigcap\{ \mathrm{cl}_{\upsilon Y} H_{\lambda} \mid \lambda \in \Lambda \}$. For each $\lambda \in \Lambda$, by Zorn's lemma, there exists a maximal disjoint family $\mathcal{F}_{\lambda}$ of non-empty open sets of H_{λ} such that each $F \in \mathcal{F}_{\lambda}$ is completely separated from $Y - H_{\lambda}$. Let us set $F_{\lambda} = \bigcup\{ F \mid F \in \mathcal{F}_{\lambda} \}$. For each $\lambda \in \Lambda$, the maximality of $\mathcal{F}_{\lambda}$ implies that $y_0 \in \mathrm{cl}_{\upsilon Y} F_\lambda$. Since $|F_\lambda| \leqq c(Y)$, $\{ F_\lambda \mid \lambda \in \Lambda \}$ is a $D(c(Y))$--expandable family in Y such that $|\Lambda| = n$ and $\bigcap \{ \mathrm{cl}_{\upsilon Y} F_\lambda \mid \lambda \in \Lambda \} \neq \emptyset$. Hence this corollary follows from 11.2.

Roughly speaking, 11.3 suggests that if $X \times Y$ is C-embedded in $X \times \upsilon Y$, then either X is locally compact or every locally finite family of non-empty open sets in Y is locally finite in υY. A detailed study of the latter property will be made in chapter 4. The surprising fact is that the condition is not only necessary but also sufficient provided suitable conditions are imposed (see 16.6).

11.4 REMARKS. (1) In 11.2 (resp. 11.3), the condition "$\bigcap \{ \mathrm{cl}_{\upsilon Y} F \mid F \in \mathcal{F} \} \neq \emptyset$" (resp. "$\bigcap \{ \mathrm{cl}_{\upsilon Y} H \mid H \in \mathcal{H} \} \neq \emptyset$") can be weakened to the following statement: There exists a point of υY each of whose neighborhood meets all but finitely many members of $\mathcal{F}$ (resp. $\mathcal{H}$).

(2) Let us say that a family $\mathcal{G}$ of subsets of a space X converges to $x \in X$ if each neighborhood of x contains some member of $\mathcal{G}$, and that a subspace S of a space X is relatively pseudo--m-compact in X if each locally finite family $\mathcal{U}$ of non-empty open sets in X, with $S \cap U \neq \emptyset$ for each $U \in \mathcal{U}$, has cardinality less than m. 11.2 (resp. 11.3) remains true if the conclusion is strengthened as follows: Each convergent family $\mathcal{G}$ of subsets in X, with $|\mathcal{G}| \leqq n$, has a member which is relatively pseudo-m--compact (resp. relatively pseudo-$c(Y)$-compact) in X.

11.5. Our next work is to construct spaces Y which have a $D(\aleph_0)$-
-expandable family $\mathcal{F}$ such that $\bigcap\{ \mathrm{cl}_{\upsilon Y}F \mid F \in \mathcal{F} \} \neq \emptyset$.

11.5.1 FACT. For each infinite cardinal κ, there exists a 0-
-dimensional locally compact space $Y = Y_1(\kappa)$, with $|Y| = w(Y) = \kappa\cdot\aleph_1$, that has a $D(\aleph_0)$-expandable family $\mathcal{F}$ such that $|\mathcal{F}| = \kappa$ and $\bigcap\{ \mathrm{cl}_{\upsilon Y}F \mid F \in \mathcal{F} \} \neq \emptyset$.

Proof. Let Λ be a discrete space of cardinality κ, and $\Lambda^* = \{\infty\}\cup\Lambda$ the one-point compactification of Λ. Let $Z^{\#}$ be the quotient space obtained from $T = (W(\omega_1 + 1) \times W(\omega_0 + 1)) \times \Lambda^*$ by collapsing the set $\{(\omega_1, \beta)\} \times \Lambda^*$ to a point $z(\beta) \in Z^{\#}$ for each $\beta \leqq \omega_0$. Define Z_0 to be the space $Z^{\#}$ with the strong topology with respect to this identification (in the sense of 6.6.1). Let $\phi\colon T \to Z_0$ be the natural map, and let us set

$$Z = Z_0 - \phi(\{ ((\gamma, \omega_0), \infty) \mid \gamma \leqq \omega_1 \}).$$

Then Z is a 0-dimensional locally compact space with $|Z| = w(Z) = \kappa\cdot\aleph_1$. Since $z(\beta)$ is a $P(\aleph_1)$-point[8] for each $\beta < \omega_0$, it is easily checked that Z is C-embedded in $Z\cup\{y_0\}$, where $y_0 = z(\omega_0)$, and so $y_0 \in \upsilon Z - Z$. Setting $E_\lambda = \phi(\{ ((\gamma, \omega_0), \lambda) \mid \gamma < \omega_1 \})$ for each $\lambda \in \Lambda$, we have a discrete family $\{ E_\lambda \mid \lambda \in \Lambda \}$ of closed subsets in Z such that $y_0 \in \bigcap\{ \mathrm{cl}_{\upsilon Z}E_\lambda \mid \lambda \in \Lambda \}$. Define the subspace Y of the product space $Z \times W(\omega_0 + 1)$ as follows:

$$Y = (Z \times \{\omega_0\})\cup(\bigcup\{ E_\lambda \times W(\omega_0 + 1) \mid \lambda \in \Lambda \}).$$

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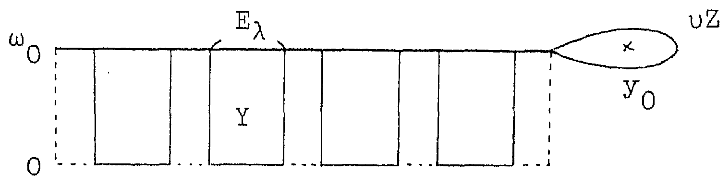
8 See footnote of p.28.

Then Y is a 0-dimensional locally compact space with $|Y| = w(Y) = n\cdot\aleph_1$, because Y is closed in $Z \times W(\omega_0 + 1)$. It remains to show the existence of a $D(\aleph_0)$-expandable family satisfying the stated conditions. Since $Z \times \{\omega_0\}$ is C-embedded in Y, it follows from 3.3.1 that $\upsilon Z = \upsilon(Z \times \{\omega_0\}) \subset \upsilon Y$, and hence $y_0 \in \upsilon Y - Y$.



Setting $F_\lambda = E_\lambda \times W(\omega_0)$ for each $\lambda \in \Lambda$, we have a locally finite family $\mathcal{F} = \{ F_\lambda \mid \lambda \in \Lambda \}$ of open sets in Y such that $y_0 \in \bigcap\{ \mathrm{cl}_{\upsilon Y} F_\lambda \mid \lambda \in \Lambda \}$. Then, since each F_λ is a union of countably many open-and-closed subsets in Y, $\mathcal{F}$ is a $D(\aleph_0)$-expandable family in Y. Hence Y is proved to be the desired space $Y_1(n)$.

11.5.2 REMARKS. (1) Since $Y_1(n)$ is the union of n many pseudo-compact subspaces, it is pseudo-n^+-compact.

(2) A 0-dimensional locally compact space having the same $D(\aleph_0)$-expandable family as above has been constructed by the author in $[O_3]$ to prove a weak form of 10.1 (c) $\to$ (a); however, the cardinality and the weight of that space are larger than those of the above.

11.5.3 FACT. For each infinite cardinal n, there exists a 0-dimensional Moore space $Y = Y_2(n)$, with $|Y| = w(Y) = n\cdot\exp\aleph_0$, that has a $D(\aleph_0)$-expandable family $\mathcal{F}$ such that $|\mathcal{F}| = n$ and

$\bigcap\{ \mathrm{cl}_{\upsilon Y} F \mid F \in \mathcal{F} \} \neq \emptyset$.

Proof. Let Z be the 0-dimensional Moore space $Z_2(n)$ constructed in 6.6.3. Then $|Z| = w(Z) = n \cdot \exp \aleph_0$, and Z has a discrete family $\{ E_\lambda \mid \lambda \in \Lambda \}$ of closed subsets in Z such that $|\Lambda| = n$ and $\bigcap\{ \mathrm{cl}_{\upsilon Z} E_\lambda \mid \lambda \in \Lambda \} \neq \emptyset$. Define the subspace Y of the product space $Z \times W(\omega_0 + 1)$ as follows:

$$Y = (Z \times \{\omega_0\}) \cup (\bigcup\{ E_\lambda \times W(\omega_0 + 1) \mid \lambda \in \Lambda \}).$$

Then Y is a 0-dimensional Moore space with $|Y| = w(Y) = n \cdot \exp \aleph_0$. Setting $F_\lambda = E_\lambda \times W(\omega_0)$ for each $\lambda \in \Lambda$, we have a $D(\aleph_0)$-expandable family $\{ F_\lambda \mid \lambda \in \Lambda \}$ in Y such that $\bigcap\{ \mathrm{cl}_{\upsilon Y} F_\lambda \mid \lambda \in \Lambda \} \neq \emptyset$. Hence Y is the desired space $Y_2(n)$.

We now proceed to the discussion analogous to the latter part of section 6. A family $\{ F_\alpha \}$ of subsets of a space X is called hereditarily closure-preserving if any family $\{ E_\alpha \}$ with $E_\alpha \subset F_\alpha$ is closure-preserving.

11.6 DEFINITION. Let m be a cardinal. A family $\{ F_\alpha \mid \alpha \in A \}$ of subsets of a space X is weakly $D(m)$-expandable if there exists a point-finite hereditarily closure-preserving family $\{ H_\alpha \mid \alpha \in A \}$ of open sets with $F_\alpha \subset H_\alpha$ for each $\alpha \in A$ and each F_α is the union of at most m subsets each of which is completely separated from $X - H_\alpha$.

11.7 THEOREM. Let X be a locally pseudocompact space, and let $X \times Y$ be C-embedded in $\upsilon X \times Y$. If there exists a weakly $D(m)$-expandable family F in Y such that $|F| = n$ and $\bigcap\{ \mathrm{cl}_Y F \mid F \in F \} \neq \emptyset$, then each point $x \in \upsilon X - X$, with $\chi(x, \upsilon X) \leqq n$, has a pseudo-$m$-compact neighborhood in υX.

Proof. Suppose on the contrary that there exists a point $x_0 \in \upsilon X - X$, with $\chi(x_0, \upsilon X) \leqq n$, which has no pseudo-$m$-compact neighborhood in υX. Let $\{ G_\lambda \mid \lambda \in \Lambda \}$ be a neighborhood base at x_0 in υX with $|\Lambda| = n$. For each $\lambda \in \Lambda$, similarly to the proof of 11.2, we can find a locally finite (in υX) family $\{ G_{\lambda\mu} \mid \mu \in M_\lambda \}$ of non-empty open sets in G_λ with $|M_\lambda| = m$. For each $\mu \in M_\lambda$, pick $x_{\lambda\mu} \in G_{\lambda\mu} \cap X$, and choose $f_{\lambda\mu} \in C(X)$ such that $f_{\lambda\mu}(x_{\lambda\mu}) = 0$ and $f_{\lambda\mu}(X - G_{\lambda\mu}) = \{1\}$. On the other hand, since $|F| = n$, we may write $F = \{ F_\lambda \mid \lambda \in \Lambda \}$. Then there is a point-finite hereditarily closure-preserving family $\{ H_\lambda \mid \lambda \in \Lambda \}$ of open sets in Y with $F_\lambda \subset H_\lambda$ for each $\lambda \in \Lambda$, and each F_λ is expressed as $F_\lambda = \bigcup\{ F_{\lambda\mu} \mid \mu \in M_\lambda \}$, where there is $g_{\lambda\mu} \in C(Y)$ such that $g_{\lambda\mu}(F_{\lambda\mu}) = \{0\}$ and $g_{\lambda\mu}(Y - H_\lambda) = \{2\}$. For each $\lambda \in \Lambda$ and each $\mu \in M_\lambda$, let us set $H_{\lambda\mu} = \{ y \in Y \mid |g_{\lambda\mu}(y)| < 1 \}$, and set

$$J_{\lambda\mu} = \{x_{\lambda\mu}\} \times F_{\lambda\mu} \subset X \times Y,$$

$$K_{\lambda\mu} = G_{\lambda\mu} \times H_{\lambda\mu} \subset X \times Y,$$

$$h_{\lambda\mu}((x, y)) = \min \{1, f_{\lambda\mu}(x)+g_{\lambda\mu}(y)\}; \quad (x, y) \in X \times Y.$$

Then $h_{\lambda\mu} \in C(X \times Y)$, $h_{\lambda\mu}(J_{\lambda\mu}) = \{0\}$ and $h_{\lambda\mu}((X \times Y) - K_{\lambda\mu}) = \{1\}$. We now show that $K = \{ K_{\lambda\mu} \mid \mu \in M_\lambda, \lambda \in \Lambda \}$ is locally finite in

$X \times Y$. Let $p = (x, y) \in X \times Y$. Since $\{ H_\lambda \}$ is point-finite, y is contained in only finitely many H_λ, say $H_{\lambda_1}, \dots, H_{\lambda_n}$. Choose a pseudocompact neighborhood $U(x)$ of x in X; then for each $\lambda \in \Lambda$ there is a finite subsets M'_λ of M_λ such that $U(x) \cap G_{\lambda\mu} = \emptyset$ for each $\mu \in M_\lambda - M'_\lambda$. Setting $E_\lambda = \bigcup \{ cl_Y H_{\lambda\mu} \mid \mu \in M'_\lambda \}$ for each $\lambda \in \Lambda$, we have closed subsets E_λ of X with $E_\lambda \subset H_\lambda$. Let us set $U(y) = Y - \bigcup \{ E_\lambda \mid \lambda \neq \lambda_1, \dots, \lambda_n \}$. Then, since $\{ H_\lambda \}$ is hereditarily closure-preserving, $U(y)$ is a neighborhood of y such that $U(x) \times U(y)$ meets only finitely many members of $\mathcal{K}$. Hence $\mathcal{K}$ is proved to be locally finite in $X \times Y$. Therefore if we define a function h on $X \times Y$ by

$$h(q) = \inf \{ h_{\lambda\mu}(q) \mid \mu \in M_\lambda, \ \lambda \in \Lambda \}; \quad q \in X \times Y,$$

then h is continuous. Let us choose $y_0 \in \bigcap \{ cl_Y F_\lambda \mid \lambda \in \Lambda \}$. Then a similar argument to that of 11.2 shows that h admits no continuous extension to the point $(x_0, y_0) \in \upsilon X \times Y$, which contradicts the assumption that $X \times Y$ is C-embedded in $\upsilon X \times Y$. Hence the proof is complete.

11.8 COROLLARY. Let X be a locally pseudocompact space, and let $X \times Y$ be C-embedded in $\upsilon X \times Y$. If there exists a point--finite hereditarily closure-preserving family $\mathcal{H}$ of open sets in Y such that $|\mathcal{H}| = \mathfrak{n}$ and $\bigcap \{ cl_Y H \mid H \in \mathcal{H} \} \neq \emptyset$, then each point $x \in \upsilon X - X$, with $\chi(x, \upsilon X) \leqq \mathfrak{n}$, has a pseudo-$c(Y)$-compact neighborhood in υX.

Proof. The proof goes just as that of 11.3.

11.9 REMARK. For 11.7 and 11.8, the analogous generalizations to those of 11.2 and 11.3, stated in 11.4, are all possible.

A space is called a Lašnev space if it is the image of a metrizable space under a closed map (cf. [L]).

11.10 FACT. For each infinite cardinal n, there exists a 0-dimensional Lašnev space $Y = Y_3(n)$, with $w(Y) \leqq \exp n$, that has a weakly $D(\aleph_0)$-expandable family $\mathcal{F}$ such that $|\mathcal{F}| = n$ and $\bigcap\{ \mathrm{cl}_Y F \mid F \in \mathcal{F} \} \neq \emptyset$.

Proof. Let Λ be a discrete space of cardinality n, and let Y be the quotient space obtained from $W(\omega_0 + 1) \times \Lambda$ by collapsing the set $\{\omega_0\} \times \Lambda$ to a point y_0. Then, the quotient map $\phi \colon W(\omega_0 + 1) \times \Lambda \to Y$ being a closed map, Y is a 0-dimensional Lašnev space with $w(Y) \leqq \exp n$. Setting $F_\lambda = \phi(W(\omega_0) \times \{\lambda\})$ for each $\lambda \in \Lambda$, we have a weakly $D(\aleph_0)$-expandable family $\{ F_\lambda \mid \lambda \in \Lambda \}$ in Y such that $y_0 \in \bigcap\{ \mathrm{cl}_Y F_\lambda \mid \lambda \in \Lambda \} \neq \emptyset$. Hence Y is the desired space $Y_3(n)$.

12. Proofs of theorems (10.1 and 10.5)

A subspace of a space X is said to be relatively pseudocompact in X if it is relatively pseudo-$\aleph_0$-compact in the sense of 11.4 (2). A regular closed set is the closure of an open set.

12.1 PREREQUISITES. We make use of the following results:

12.1.1 (cf. [C_3, 4.1]) A relatively pseudocompact regular closed subspace is pseudocompact.

12.1.2 ([M_6]) A subspace S of a space X is relatively pseudocompact in X if and only if $cl_{\beta X} S \subset \upsilon X$.

12.1.3 If X is locally compact, then $X \times Y$ is a pseudo-k-space for each pseudo-k-space Y.

12.1.3 can be proved similarly to 3.5.3 due to Cohen [C_1] (cf. also [E, 3.3.27]). The following result is essentially due to Comfort [C_3], who proved it under the assumption that each compact subset of Y is of non-measurable cardinal; for completeness we include a proof.

12.1.4 If $\upsilon X \times Y$ is a k-space, then $X \times Y$ is C-embedded in $\upsilon X \times Y$.

Proof. We first show that X is pseudo-m_1-compact. If this were not the case, then there is a discrete family $\{ G_\lambda \mid \lambda \in \Lambda \}$ of non-empty open sets in X with $|\Lambda| = m_1$. Choose $x_\lambda \in G_\lambda$ for each $\lambda \in \Lambda$, and set $D = \{ x_\lambda \mid \lambda \in \Lambda \}$. Since D is C-embedded in X, it follows from 3.3.1 that $\upsilon D = cl_{\upsilon X} D$, and so υD is a k-

-space. This contradicts 3.3.8. Hence X is pseudo-m_1-compact. For our end, let $f \in C(X \times Y)$. For each $y \in Y$, $f|(X \times \{y\})$ has a continuous extension g_y over $\upsilon X \times \{y\}$. We define a function g on $\upsilon X \times Y$ by

$$g((x, y)) = g_y(x); \quad (x, y) \in \upsilon X \times Y.$$

Then $g|(X \times Y) = f$. In order to prove $g \in C(\upsilon X \times Y)$, $\upsilon X \times Y$ being a k-space, it suffices to show that the restriction of g to each compact subset K of $\upsilon X \times Y$ is continuous. Since $\pi_Y(K)$ is compact and X is pseudo-m_1-compact, it follows from 4.1 that $X \times \pi_Y(K)$ is C-embedded in $\upsilon X \times \pi_Y(K)$. This fact shows that $g|(\upsilon X \times \pi_Y(K))$ is continuous, and thus so is $g|K$. That completes the proof.

We devide 10.1 into two detailed theorems:

12.2 THEOREM. The following conditions on a space X, with $|X| < m_1$, are equivalent:

(a) X is locally pseudocompact.

(b) $X \times Y$ is C-embedded in $X \times \upsilon Y$ for each k-space Y.

(c) $X \times Y$ is C-embedded in $X \times \upsilon Y$ for each 0-dimensional locally compact space Y with $w(Y) \leqq \chi(X)\cdot\aleph_1$.

(d) $X \times Y$ is C-embedded in $X \times \upsilon Y$ for each 0-dimensional Moore space Y with $w(Y) \leqq \chi(X)\cdot\exp\aleph_0$.

Proof. (a) $\rightarrow$ (b). Let Y be a k-space. Since X is locally pseudocompact, it suffices to show that $S \times Y$ is C-embedded in $S \times \upsilon Y$ for each pseudocompact subset S of X. Let S be a given pseudocompact subset of X. Since υS is compact by 3.3.3, it follows from 3.5.3 and 12.1.4 that $S \times Y$ is C-embedded in $\upsilon S \times Y$. Since $|\upsilon S| < m_1$, this fact combined with 5.1 shows that $\upsilon(S \times Y) = \upsilon S \times \upsilon Y$, and hence $S \times Y$ is C-embedded in $S \times \upsilon Y$.

(b) $\to$ (c), (b) $\to$ (d). Obvious.

(c) $\to$ (a). Suppose on the contrary that X is not locally pseudocompact at $x_0 \in X$. Let Y be the 0-dimensional locally compact space $Y_1(n)$ constructed in 11.5.1, where $n = \chi(x_0, X)$. Then $w(Y) \leqq n \cdot \aleph_1 \leqq \chi(X) \cdot \aleph_1$, and it follows from 11.2 that $X \times Y$ is not C-embedded in $X \times \upsilon Y$. Thus we have (a).

(d) $\to$ (a). The proof is the same as above if one use $Y_2(n)$ constructed in 11.5.3 instead of $Y_1(n)$. Hence the proof is complete.

12.3 REMARK. The equivalence of (a), (b) and (c) has been proved by the author in [O_2], [O_3] without the cardinality condition on Y.

12.4 REMARK. In the preceding proof, the assumption that $|X| < m_1$ is useful only for the implication (a) $\to$ (b).

12.5 THEOREM. The following conditions on a locally pseudocompact space X are equivalent:

(a) Each point of $\upsilon X - X$ has a compact neighborhood in υX.

(b) $X \times Y$ is C-embedded in $\upsilon X \times Y$ for each k-space Y.

(c) $X \times Y$ is C-embedded in $\upsilon X \times Y$ for each 0-dimensional Lašnev space Y with $w(Y) \leqq \exp \chi(\upsilon X)$.

Proof. (a) $\rightarrow$ (b). Let Y be a k-space, and let $x \in \upsilon X - X$. Choose a cozero-set neighborhood G of x in υX such that $cl_{\upsilon X} G$ is compact. Then $\upsilon(G \cap X) = G$ by 3.3.2. Since G is locally compact, it follows from 3.5.3 and 12.1.4 that $(G \cap X) \times Y$ is C-embedded in $G \times Y$. Since $x \in \upsilon X - X$ was arbitrary, this implies that $X \times Y$ is C-embedded in $\upsilon X \times Y$.

(b) $\rightarrow$ (c). Every Lašnev space is a k-space (cf. 3.5).

(c) $\rightarrow$ (a). Assume that (a) is false at some $x_0 \in \upsilon X - X$. Then by 3.3.3 x_0 has no pseudo-$\aleph_0$-compact neighborhood in υX. Let Y be the 0-dimensional Lašnev space $Y_3(n)$ constructed in 11.10, where $n = \chi(x_0, \upsilon X)$. Then $w(Y) \leqq \exp n \leqq \exp \chi(\upsilon X)$, and it follows from 11.7 that $X \times Y$ is not C-embedded in $\upsilon X \times Y$. Hence the proof is complete.

12.6 REMARKS. (1) Since a Lašnev space is a singly bi-k-space, 12.5 remains true if "k-space" is replaced by "singly bi-k-space", but it cannot be strengthened to "countably bi-k-space". In fact, Kato proved in $[K_1]$ that the space $X = \beta R - \beta N$, where R is the real line, has the following properties.

(i) X is a locally compact, M-space (in the sense of Morita $[M_4]$) with $|X| < m_1$, and hence υX is a paracompact M--space (cf. $[M_5, 4.4]$).

(ii) υX is not locally compact, and hence $\upsilon X - X$ contains a point having no compact neighborhood in υX.

Then it follows from 3.5.2 and 12.1.4 that $X \times Y$ is C-embedded in $\upsilon X \times Y$ for each countably bi-k-space Y.

(2) It is open whether 12.5 can be proved without assuming local pseudocompactness of X. In the preceding proof, this assumption plays no role everywhere except for the place in which 11.7 is used. Although (c) $\rightarrow$ (a) is not needed for proving 10.1, the question is of independent interest, because it suggests the possibility that one can characterize the class $\mathcal{R}$(Lašnev).

12.7 Proof of Theorem 10.1. (a) $\rightarrow$ (b) follows from 12.2 and 12.5. (b) $\rightarrow$ (c) and (b) $\rightarrow$ (d) are obvious. (c) $\rightarrow$ (a). By 4.4, $|X| < m_1$. It follows from 12.2 that υX is locally pseudocompact. Then υX is locally compact, because of 3.3.3. The proof of (d) $\rightarrow$ (a) is quite similar to that of (c) $\rightarrow$ (a). Hence the proof is complete.

Theorem 10.5 as well as 10.1 is devided into two parts; we require one more lemma that gives an analogue of [P_2, 4.1] for Hewitt realcompactifications.

12.8 LEMMA. If $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds, then the product $A \times B$ of two relatively pseudocompact subsets $A \subset X$ and $B \subset Y$ is relatively pseudocompact in $X \times Y$.

Proof. Let A and B be relatively pseudocompact subsets of X and Y, respectively. Then, by 12.1.2, $cl_{\beta X} A \subset \upsilon X$ and $cl_{\beta Y} B \subset \upsilon Y$. Since $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$, it follows that

$$\mathrm{cl}_{\beta(X \times Y)}(A \times B) \subset \mathrm{cl}_{\beta X}A \times \mathrm{cl}_{\beta Y}B \subset \upsilon(X \times Y).$$

Hence $A \times B$ is relatively pseudocompact in $X \times Y$ by 12.1.2 again.

12.9 THEOREM. The following conditions on a space X, with $|X| < m_1$, are equivalent:

(a) X is locally strongly pseudocompact.

(b) $X \times Y$ is C-embedded in $X \times \upsilon Y$ for each pseudo-k-space Y.

(c) $X \times Y$ is C-embedded in $X \times \upsilon Y$ for each locally pseudocompact space Y.

(d) $X \times Y$ is C-embedded in $X \times \upsilon Y$ for each locally compact space Y and for each pseudocompact space Y.

Proof. (a) $\to$ (b). Let Y be a pseudo-k-space, and let S be a strongly pseudocompact subset of X. Since X is locally strongly pseudocompact, it suffices to show that $S \times Y$ is C-embedded in $S \times \upsilon Y$. For each pseudocompact subset K of Y, $S \times K$ is pseudocompact, and so $S \times K$ is C-embedded in $\upsilon S \times K$ by 1.1. Then, $\upsilon S \times Y$ ($= \beta S \times Y$) being a pseudo-k-space by 12.1.3, the same argument as in the proof of 12.1.4 assures us that $S \times Y$ is C-embedded in $\upsilon S \times Y$. Since υS is compact and $|\upsilon S| < m_1$, it follows from 5.1 that $\upsilon S \times Y$ is C-embedded in $\upsilon S \times \upsilon Y$. Thus $S \times Y$ is proved to be C-embedded in $S \times \upsilon Y$.

(b) $\to$ (c) $\to$ (d). Obvious.

(d) $\to$ (a). Suppose on the contrary that X is not locally strongly pseudocompact at $x_0 \in X$. By 12.2, x_0 has an open neighborhood G such that $\mathrm{cl}_X G$ is pseudocompact. Then $\mathrm{cl}_X G \times Y$ is not pseudocompact for some pseudocompact space Y. Since $cl_X G \times Y$ is a regular closed set of $X \times Y$, it is not relatively pseudocompact in $X \times Y$ by 12.1.1, and so it follows from 12.8 that $\upsilon(X \times Y) \neq \upsilon X \times \upsilon Y$. Since υY is compact by 3.3.3 and $|X| < m_1$, it follows from 4.1 that $X \times \upsilon Y$ is C-embedded in $\upsilon X \times \upsilon Y$, which shows that $X \times Y$ is not C-embedded in $X \times \upsilon Y$. This contradiction completes the proof.

12.10 THEOREM. The following conditions on a locally pseudocompact space X are equivalent:

(a) Each point of $\upsilon X - X$ has a neighborhood G in υX such that $G \cap X$ is strongly pseudocompact.

(b) $X \times Y$ is C-embedded in $\upsilon X \times Y$ for each pseudo-k-space Y.

(c) $X \times Y$ is C-embedded in $\upsilon X \times Y$ for each k-space Y and for each pseudocompact space Y.

Proof. (a) $\rightarrow$ (b). Let Y be a pseudo-k-space, and let $x \in \upsilon X - X$. Choose a cozero-set neighborhood G of x in υX such that $cl_{\upsilon X} G \cap X$ is strongly pseudocompact. If we set $X_1 = G \cap X$, then X_1 is locally strongly pseudocompact and $\upsilon X_1 = G$ by 3.3.2. Since $cl_{\beta X} G \subset \upsilon X$ by 12.1.2, υX_1 is locally compact. We now show that $X_1 \times Y$ is C-embedded is $\upsilon X_1 \times Y$. To do this, since $\upsilon X_1 \times Y$ is a pseudo-k-space by 12.1.3, it suffices to prove that $X_1 \times K$ is C-embedded in $\upsilon X_1 \times K$ for each pseudocompact subset K of Y. Let K be a given pseudocompact subset of Y. Then, for each strongly pseudocompact subset S of X_1, $S \times K$ is C-embedded in

$S \times \upsilon K$ by 1.1. Since X_1 is locally strongly pseudocompact, this shows that $X_1 \times K$ is C-embedded in $X_1 \times \upsilon K$. Note that υK is compact by 3.3.3. Then, $\upsilon X_1 \times \upsilon K$ being locally compact, it follows from 12.1.4 that $X_1 \times \upsilon K$ is C-embedded in $\upsilon X_1 \times \upsilon K$. Thus $X_1 \times K$ is proved to be C-embedded in $\upsilon X_1 \times K$, and hence $X_1 \times Y$ ($= (G \cap X) \times Y$) is C-embedded in $\upsilon X_1 \times Y$ ($= G \times Y$). Since $x \in \upsilon X - X$ was arbitrary, this implies that $X \times Y$ is C-embedded in $\upsilon X \times Y$.

(b) → (c). Obvious.

(c) → (a). Assume that (a) is false at some $x_0 \in \upsilon X - X$. Since X is locally pseudocompact, each point of X has a compact neighborhood in υX by 12.1.2 (cf. also [C_3]). This fact combined with 12.5 implies that υX is locally compact. Choose an open neighborhood G of x_0 in υX such that $\mathrm{cl}_{\upsilon X} G$ is compact. If we set $X_1 = \mathrm{cl}_{\upsilon X} G \cap X$, X_1 is pseudocompact by 12.1.1 and 12.1.2, and so $X_1 \times Y$ is not pseudocompact for some pseudocompact space Y. Then, $X_1 \times Y$ being regular closed in $X \times Y$, it follows from 12.1.1 and 12.8 that $\upsilon(X \times Y) \neq \upsilon X \times \upsilon Y$. Since υY is compact by 3.3.3, $\upsilon X \times \upsilon Y$ is locally compact. Hence it follows from 12.1.4 that $\upsilon X \times Y$ is C-embedded in $\upsilon X \times \upsilon Y$, which implies that $X \times Y$ is not C-embedded in $\upsilon X \times Y$. This contradiction completes the proof.

12.11 Proof of Theorem 10.5. (a) → (b) follows from 12.9 and 12.10. (b) → (c) and (c) → (d) are obvious. (d) → (a) is the result of 4.4, 12.9 and 12.10. Hence the proof is complete.

13. Problems and remarks

13.1 Recall the Isbell's space $\Psi = N \cup D$, where $D = \{ \omega_S \mid S \in \mathcal{S} \}$, explained in the proof of 6.6.3. In [M_8], Mrówka showed that a maximal family $\mathcal{S}$ can be chosen so that $\beta\Psi$ is the one-point compactification. Let Ψ be such the space. Then, as noted in 6.6.3, Ψ is a pseudocompact Moore space, and so $\upsilon\Psi = \beta\Psi$ by 3.3.3. Further it is easily checked that Ψ is locally compact. Dividing D into a pairwise disjoint family of countable subsets, we have a discrete family $\{ D_\lambda \mid \lambda \in \Lambda \}$ of closed subsets in Ψ such that $|\Lambda| = \exp \aleph_0$ and $\bigcap \{ \mathrm{cl}_{\upsilon\Psi} D_\lambda \mid \lambda \in \Lambda \} \neq \emptyset$. Then, following the same procedure as in 6.6.3, we can make a locally compact Moore space Y that has a $D(\aleph_0)$-expandable family $\mathcal{F}$ such that $|\mathcal{F}| = \exp \aleph_0$ and $\bigcap \{ \mathrm{cl}_{\upsilon Y} F \mid F \in \mathcal{F} \} \neq \emptyset$. This fact can be combined with 10.1 and 11.2 to yeild the following result:

13.1.1 PROPOSITION. The following conditions on a space X, with $\chi(\upsilon X) \leqq \exp \aleph_0$, are equivalent:

(a) υX is locally compact and $|X| < m_1$.

(b) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each locally compact Moore space Y.

13.1.2 PROBLEM. Does 13.1.1 remain true if the condition $\chi(\upsilon X) \leqq \exp \aleph_0$ is omitted ?

13.2 PROBLEM. Do there exist any other conditions on Y for which 10.1 remains valid ?

13.3 Following $[I_8]$, we say that a space X is υ-locally compact if υX is locally compact.

13.3.1 PROBLEM. Characterize $\mathcal{R}$(υ-locally compact)[9].

In $[H_2]$, Hager raised the problem: What property of X is necessary and sufficient that for each pseudo-m-compact space Y, the projection π_X: X × Y → X is z-closed ? By the following proposition, we remark that 13.1.1 essentially is the case $m = \aleph_0$ of his problem, and that $\mathcal{R}$(υ-locally compact) = $\mathcal{R}$(pseudocompact).

13.3.2 PROPOSITION. The following conditions on a space X are equivalent:

(a) X is pseudo-m_1-compact and π_X: X × Y → X is z-closed for each pseudocompact space Y.

(b) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each pseudocompact space Y.

(c) $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds for each υ-locally compact space Y.

Proof. (a) → (b) follows from 3.4. If υY is locally compact, then by 12.1.1 and 12.1.2 each point of υY has a neighborhood G in υY such that $G \cap Y$ is pseudocompact. Thus the proof of (b) → (c) follows the same argument as in the proof of 9.4.2. (c) → (a). By 4.3, X is pseudo-m_1-compact. Let Y be a pseudo-

9 This problem has been posed by the author in $[O_3]$.

compact space; then $X \times Y$ is C-embedded in $X \times \upsilon Y$. Since $\upsilon Y = \beta Y$ by 3.3.3, it follows from 3.4 that $\pi_X : X \times Y \to X$ is z-closed. Hence the proof is complete.

13.4 PROBLEM. Does 11.2 remain true if "$\bigcap\{ \mathrm{cl}_{\upsilon Y} F \mid F \in \mathcal{F} \} \neq \emptyset$" is weakened to the condition "$\mathcal{F}$ is not locally finite at some $y \in \upsilon Y - Y$" ? 11.2 and 11.3 would be more useful if this replacement is possible (cf. 11.4 and 17.2).

13.5 PROBLEM. In [C_3], Comfort showed that a locally pseudo-compact space is characterized as a C-embedded subspace of a locally compact space. Our question is whether a pseudo-k-space is characterized as a C-embedded subspace of a k-space or not.

CHAPTER 3

MAPPING THEOREMS AND z-EMBEDDING IN $X \times \beta Y$

As preliminaries to the next chapter, we discuss two topics which are interesting in themselves. All the results and problems in this chapter, except 14.8, 15.7 and 15.8 (B), have been stated in [O_2], [O_3] and [O_4].

14. Mapping theorems I

One line of attack on the problem about υ is to seek for the analogue of a known result about β. We now fix our eyes upon the following fact which is an immediate consequence of Glicksberg's theorem (1.1).

14.1 FACT. Let $f_i: X_i \to Y_i$ ($i = 1, 2$) be onto maps. Then $\beta(X_1 \times X_2) = \beta X_1 \times \beta X_2$ implies $\beta(Y_1 \times Y_2) = \beta Y_1 \times \beta Y_2$.

The analogous result for υ is in general false. In fact, let X be the discrete space of cardinality $\aleph_1$, and f the one-to-one map from X onto $Y = W(\omega_1)$; then $\upsilon(Y \times Z) \neq \upsilon Y \times \upsilon Z$ for some realcompact space Z by 7.6, while $\upsilon(X \times Z) = \upsilon X \times \upsilon Z$.

As is well known, for a map $f: X \to Y$, there exists a continuous extension $\upsilon f: \upsilon X \to \upsilon Y$ ($\beta f: \beta X \to \beta Y$) of f ([GJ]). The following theorem is the fundamental result of this section.

14.2 THEOREM. Let $f_i: X_i \to Y_i$ ($i = 1, 2$) be onto maps. If $\upsilon f_1 \times \upsilon f_2$ is a quotient map from $\upsilon X_1 \times \upsilon X_2$ onto $\upsilon Y_1 \times \upsilon Y_2$, then $\upsilon(X_1 \times X_2) = \upsilon X_1 \times \upsilon X_2$ implies $\upsilon(Y_1 \times Y_2) = \upsilon Y_1 \times \upsilon Y_2$.

More generally, we have the following theorem:

14.3 THEOREM. Let $F_i: \widetilde{X}_i \to \widetilde{Y}_i$ ($i = 1, 2$) be onto maps such that $F = F_1 \times F_2$ is a quotient map, and let X_i (resp. $Y_i = F_i(X_i)$) be dense C-embedded subspaces of $\widetilde{X}_i$ (resp. $\widetilde{Y}_i$). If $X_1 \times X_2$ is C-embedded in $\widetilde{X}_1 \times \widetilde{X}_2$, then $Y_1 \times Y_2$ is C-embedded in $\widetilde{Y}_1 \times \widetilde{Y}_2$.

Proof. Let us set $f_i = F_i | X_i$ ($i = 1, 2$) and $f = f_1 \times f_2$. To show that $Y_1 \times Y_2$ is C-embedded in $\widetilde{Y}_1 \times \widetilde{Y}_2$, let $g \in C(Y_1 \times Y_2)$. Since $h = g \circ f \in C(X_1 \times X_2)$, by our assumption, there exists $H \in C(\widetilde{X}_1 \times \widetilde{X}_2)$ such that $H | (X_1 \times X_2) = h$. We shall show that

$$(*) \begin{cases} H \text{ takes on the constant value } t_p \text{ on} \\ F^{-1}(p) \text{ for each } p \in \widetilde{Y}_1 \times \widetilde{Y}_2. \end{cases}$$

Let $x \in X_1$; then $h(x, \cdot) = g(f_1(x), \cdot) \circ f_2$, where $h(x, \cdot) = h | (\{x\} \times X_2)$. Since $g(f_1(x), \cdot) \in C(Y_2)$, it has a continuous extension G_x over $\widetilde{Y}_2$. Then, X_2 being dense in $\widetilde{X}_2$, $H(x, \cdot) = G_x \circ F_2$. Hence it follows that $H(x, \cdot)$ is constant on $\{x\} \times F_2^{-1}(y)$ for each $y \in \widetilde{Y}_2$. This implies that H is constant on $f_1^{-1}(y_1) \times F_2^{-1}(y_2)$ for each $(y_1, y_2) \in Y_1 \times \widetilde{Y}_2$. Similarly, H is constant on $F_1^{-1}(y_1) \times f_2^{-1}(y_2)$ for each $(y_1, y_2) \in \widetilde{Y}_1 \times Y_2$. To see (*), let $p = (y_1, y_2) \in \widetilde{Y}_1 \times \widetilde{Y}_2$. Then it follows from these facts that

$$H(x, \cdot) = H(x', \cdot) \text{ for each } x, x' \in F_1^{-1}(y_1),$$
$$H(\cdot, x) = H(\cdot, x') \text{ for each } x, x' \in F_2^{-1}(y_2),$$

and from which (*) is proved. Define a function G on $\widetilde{Y}_1 \times \widetilde{Y}_2$ by $G(p) = t_p$ for $p \in Y_1 \times Y_2$. Then $H = G \circ F$ and $G|(Y_1 \times Y_2) = g$. Since F is a quotient map and H is continuous, it follows that G is continuous (cf. [E, 2.4.2]). This completes the proof.

14.4 REMARK. 14.3 remains true if "C-embedded" is replaced by "C*-embedded". Then 14.1 can be viewed as a corollary of 14.3, because $\beta f_1 \times \beta f_2$ is always a perfect map from $\beta X_1 \times \beta X_2$ onto $\beta Y_1 \times \beta Y_2$.

In [I_3], Ishii proved that if f is an open perfect onto map, so is υf. This leads to the following corollary of 14.2.

14.5 COROLLARY. If $f_i: X_i \to Y_i$ is an open perfect onto map for i = 1, 2, then $\upsilon(X_1 \times X_2) = \upsilon X_1 \times \upsilon X_2$ implies $\upsilon(Y_1 \times Y_2) = \upsilon Y_1 \times \upsilon Y_2$.

14.6 THEOREM. Among the following conditions on a space X, (a) → (b) → (c) are valid. Conversely, (c) → (a) holds if $|X| < m_1$.

(a) υX is locally compact.

(b) For each space Y satisfying $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ and each quotient image Z of Y, $\upsilon(X \times Z) = \upsilon X \times \upsilon Z$ holds.

(c) As in (b), with "perfect" instead of "quotient".

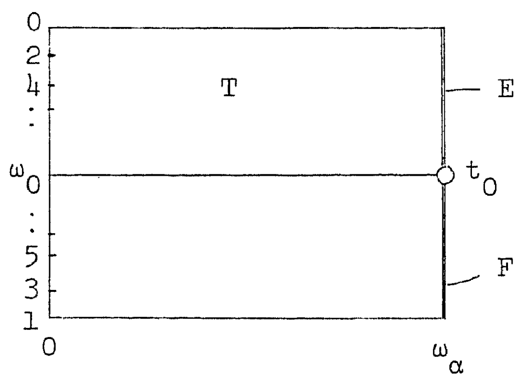
Proof. (a) $\rightarrow$ (b). Let Y be a space satisfying $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$, and let Z be the image of Y under a quotient map f. Since υX is locally compact, by 3.5.6, $id_{\upsilon X} \times f$ is a quotient map. It follows from 14.3 that $X \times Z$ is C-embedded in $\upsilon X \times Z$. If we apply 4.1 to our case, then either $|\upsilon X| < m_1$ or Y is pseudo-m_1-compact. If Y is pseudo-m_1-compact, then so is Z. Hence it follows from 4.1 that $\upsilon(\upsilon X \times Z) = \upsilon X \times \upsilon Z$. Thus we have $\upsilon(X \times Z) = \upsilon X \times \upsilon Z$.

(b) $\rightarrow$ (c). Obvious.

(c) $\rightarrow$ (a). Suppose that $|X| < m_1$ and υX is not locally compact at $x_0 \in \upsilon X$. By 3.3.3, x_0 has no pseudo-$\aleph_0$-compact neighborhood in υX. Let $n = \max \{ |\upsilon X|, \chi(x_0, \upsilon X) \}$; then $n < m_1$. Let ω_α be the initial ordinal of n^+, and let us set

$$T = (W(\omega_\alpha + 1) \times W(\omega_0 + 1)) - \{t_0\},$$

where $t_0 = (\omega_\alpha, \omega_0)$. Then $\upsilon T = W(\omega_\alpha + 1) \times W(\omega_0 + 1)$. Let $E' = \{ 2n \mid n < \omega_0 \}$ and $F' = \{ 2n+1 \mid n < \omega_0 \}$. Setting $E = \{\omega_\alpha\} \times E'$ and $F = \{\omega_\alpha\} \times F'$, we have disjoint closed subsets E and F of T such that $t_0 \in cl_{\upsilon T} E \cap cl_{\upsilon T} F$.



Let Λ be a discrete space of cardinality $\mathfrak{n}$, and let S be the quotient space obtained from $R = T \times \Lambda$ by collapsing the set $\{e\} \times \Lambda$ to a point $s(e) \in S$ for each $e \in E$. We note that

$$(1) \qquad \upsilon R = (W(\omega_\alpha + 1) \times W(\omega_0 + 1)) \times \Lambda.$$

Let $g: R \to S$ be the quotient map, and let us set $E_0 = \{ s(e) \mid e \in E \}$ and $F_\lambda = g(F \times \{\lambda\})$ for each $\lambda \in \Lambda$. Then a similar argument to that of 6.6.2 shows that

$$(2) \qquad \bigcap\{ \mathrm{cl}_{\upsilon S} F_\lambda \mid \lambda \in \Lambda \} \neq \emptyset.$$

Now g is a closed map and $g^{-1}(s)$ is compact for each $s \in S - E_0$. If we set $G = g((W(\omega_\alpha + 1) \times E') \times \Lambda)$, then G is a cozero-set of S, and so $G = S \cap G^*$ for some cozero-set G^* of βS. Let us set $Z = S \cup G^*$. We now need the following lemma.

LEMMA. Let $X \supset X_1 \supset X_2$. Suppose that X_2 is dense in X and is C-embedded in X_1. Then, for each open set H of X, $X_2 \cup H$ is C-embedded in $X_1 \cup H$.

The proof is left to the reader, since it requires only routine verification. We continue the proof of 14.6. Since $G \cap F_\lambda = \emptyset$, the above lemma shows that $\mathrm{cl}_{\upsilon S} F_\lambda = \mathrm{cl}_{\upsilon Z} F_\lambda$ for each $\lambda \in \Lambda$. Hence $\bigcap\{ \mathrm{cl}_{\upsilon Z} F_\lambda \mid \lambda \in \Lambda \} \neq \emptyset$ by (2). Setting $G_\lambda = g((W(\omega_\alpha + 1) \times F') \times \{\lambda\})$ for each $\lambda \in \Lambda$, we obtain a locally finite family $\{ G_\lambda \mid \lambda \in \Lambda \}$ of open sets in Z with $G_\lambda \supset F_\lambda$, and thus $\{ F_\lambda \mid \lambda \in \Lambda \}$ is a $D(\aleph_0)$-expandable family in Z. Since $\chi(x_0, \upsilon X) \leqq |\Lambda|$, it follows from 11.2 that $\upsilon(X \times Z) \neq \upsilon X \times \upsilon Z$. For our end, it suffices to show that Z is the perfect image of a space Y satisfying $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$. There exists the extension $\beta g\colon \beta R \to \beta S$ of g. Let us set $Y = R \cup H^*$, where $H^* = (\beta g)^{-1}(G^*)$, and set $f = (\beta g)|Y$. Since $G^* \supset E_0$, it is easily checked that f is a perfect map from Y onto Z. On the other hand, since H^* is a cozero-set of βR, $\upsilon Y = \upsilon R \cup H^*$ by 3.3.4. Since Y is a locally compact space with $|Y| < m_1$, it follows from 7.2 that $X \times Y$ is C-embedded in $\upsilon X \times Y$. It remains to prove that $\upsilon X \times Y$ is C-embedded in $\upsilon X \times \upsilon Y$. Since $|\upsilon X| < n^+$, $\upsilon X \times W(\omega_\alpha) \times W(\omega_0 + 1)$ is C-embedded in $\upsilon X \times W(\omega_\alpha + 1) \times W(\omega_0 + 1)$. Thus by (1) $\upsilon X \times R$ is C-embedded in $\upsilon X \times \upsilon R$. Since $\upsilon X \times H^*$ is an open set of $\upsilon X \times \beta R$, it follows from the above lemma that

$$(\upsilon X \times R) \cup (\upsilon X \times H^*) \ (= \upsilon X \times Y \ )$$

is C-embedded in

$$(\upsilon X \times \upsilon R) \cup (\upsilon X \times H^*) \ (= \upsilon X \times \upsilon Y \ ).$$

Hence the proof is complete.

14.7 REMARK. In case $|X| \geqq m_1$, (c) $\to$ (a) in 14.6 need not be true. In fact, it follows from 4.2 that $D(m_1)$ satisfies (c), but $\upsilon D(m_1)$ is not a k-space by 3.3.8.

14.8 COROLLARY. Let $f_i\colon X_i \to Y_i$ ($i = 1, 2$) be quotient onto maps. If both υX_1 and υY_2 are locally compact, then $\upsilon(X_1 \times X_2) = \upsilon X_1 \times \upsilon X_2$ implies $\upsilon(Y_1 \times Y_2) = \upsilon Y_1 \times \upsilon Y_2$.

Proof. This is proved by using 14.6 repeatedly.

Since the product of bi-quotient onto maps is again a bi-quotient map (cf. 3.5.4), 14.2 bring to mind the question of when υf is a bi-quotient onto map. In section 20, we answer this question by considering the special case where f is a perfect map, and give another virsion of 14.6 which is apparently stronger. On the other hand, very little can be said about the inverse invariance of the equality $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ under maps. We conclude this section with the following problem and counter-example about it.

14.9 PROBLEM. Let $f_i: X_i \to Y_i$ ($i = 1, 2$) be onto maps. When does $\upsilon(Y_1 \times Y_2) = \upsilon Y_1 \times \upsilon Y_2$ imply $\upsilon(X_1 \times X_2) = \upsilon X_1 \times \upsilon X_2$?

14.10 EXAMPLE. Let $f: Y \to Z$ be perfect onto map. Then $\upsilon(X \times Z) = \upsilon X \times \upsilon Z$ does not necessary imply $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ even when $id_{\upsilon X} \times \upsilon f$ is a quotient onto map and υX is compact. To see this, let us set $X = W(\omega_1)$; then, by 7.6 and 3.3.7, there exists a realcompact space Y_1 such that $\upsilon(X \times Y_1) \neq \upsilon X \times \upsilon Y_1$. By [$N_2$, 2.3], Y_1 can be embedded as a closed subspace of a pseudocompact space Y_2. Let $i: Y_1 \to Y_2$ be the embedding. Let us set $Y = Y_1 \oplus Y_2$ and $Z = Y_2$, where the symbol $\oplus$ means the topological sum. Define $f: Y \to Z$ by $f(y) = i(y)$ if $y \in Y_1$ and $f(y) = y$ if $y \in Y_2$. Then f is a perfect map and υf is a quotient map from υY ($= Y_1 \oplus \upsilon Y_2$) onto υZ ($= \upsilon Y_2$). Since X is locally compact, it follows from [E, 3.10.26] that $X \times Z$ is pseudocompact, and so

$\upsilon(X \times Z) = \upsilon X \times \upsilon Z$ by 1.1. On the other hand, $\upsilon(X \times Y) \neq \upsilon X \times \upsilon Y$ obviously. Further, υX ($= W(\omega_1 + 1)$) being compact, it follows from 3.5.6 that $id_{\upsilon X} \times f$ is a quotient map.

15. z-embedding in $X \times \beta Y$

As noted in 3.4, barring the existence of measurable cardinals, $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds whenever $X \times Y$ is z-embedded in $X \times \beta Y$. In this section, we consider the problem of when $X \times Y$ is z-embedded in $X \times \beta Y$. The analogous problem for $\beta X \times \beta Y$ has been discussed by Blair and Hager in [BH_3]. Our results refine their works, and give new characterizations of metrizable spaces.

15.1 PREREQUISITES. We make use of the following results:
15.1.1 (Terasawa) For each finite cozero-set cover $\mathcal{G}$ of the product space $X \times Y$ of a space X with a compact space Y, there exist a locally finite cozero-set cover $\{ U_\alpha \mid \alpha \in A \}$ of X and finite open covers $\mathcal{V}_\alpha$, $\alpha \in A$, of Y such that $\{ U_\alpha \times V \mid V \in \mathcal{V}_\alpha, \alpha \in A \}$ is a refinement of $\mathcal{G}$ (the proof was described in [CC]).
15.1.2 ([F_3]) An open perfect map carries a zero-set to a zero-set.

Following [H_1], by a cozero-rectangle in a product $X \times Y$, we mean a set of the form $U \times V$, where U and V are cozero-sets in X and Y, respectively.

15.2 PROPOSITION. The following conditions on a product space $X \times Y$ are equivalent:

(a) $X \times Y$ is z-embedded in $X \times \beta Y$.

(b) Each cozero-set of $X \times Y$ is the union of a family $\{ U_\alpha \times V_\alpha \mid$

$\alpha \in A$ } of cozero-rectangles in $X \times Y$ such that $\{ U_\alpha \mid \alpha \in A \}$ is σ-locally finite in X.

(c) Each finite cozero-set cover of $X \times Y$ has a refinement $\{ U_\alpha \times V_\alpha \mid \alpha \in A \}$ by cozero-rectangles in $X \times Y$ such that $\{ U_\alpha \mid \alpha \in A \}$ is σ-locally finite in X.

Proof. (a) $\to$ (b). Let G be a cozero-set of $X \times Y$. Then by (a) there is a cozero-set $G^* = \{ p \in X \times \beta Y \mid g(p) \neq 0 \}$, where $g \in C(X \times \beta Y)$, of $X \times \beta Y$ with $G^* \cap (X \times Y) = G$. For each $n \in N$, let us set

$$K_n = \{ p \in X \times \beta Y \mid |g(p)| > 1/n \},$$
$$L_n = \{ p \in X \times \beta Y \mid |g(p)| < 1/n \},$$

and $\mathcal{G}_n = \{ K_{n+1}, L_n \}$. Then $\mathcal{G}_n$ is a cozero-set cover of $X \times \beta Y$, and hence by 15.1.1 there exist a locally finite cozero-set cover $\{ U_{n\alpha} \mid \alpha \in A_n \}$ of X and finite open covers $\mathcal{W}_{n\alpha}$, $\alpha \in A_n$, of βY such that $\{ U_{n\alpha} \times W \mid W \in \mathcal{W}_{n\alpha}, \alpha \in A_n \}$ is a refinement of $\mathcal{G}_n$. Since βY is compact, $\mathcal{W}_{n\alpha}$ has a refinement $\mathcal{V}_{n\alpha}$ consisting of finitely many cozero-sets of βY. For each $\alpha \in A$, if we set

$$V_{n\alpha} = \bigcup \{ V \in \mathcal{V}_{n\alpha} \mid U_{n\alpha} \times V \subset K_{n+1} \} \cap Y,$$

then $V_{n\alpha}$ is a cozero-set of Y such that

$$(X \times Y) \cap K_n \subset \bigcup \{ U_{n\alpha} \times V_{n\alpha} \mid \alpha \in A_n \} \subset K_{n+1}.$$

Since $G^* = \bigcup \{ K_n \mid n \in N \}$, it follows that $G = \bigcup \{ U_{n\alpha} \times V_{n\alpha} \mid \alpha \in A_n, n \in N \}$, and then $\{ U_{n\alpha} \mid \alpha \in A_n, n \in N \}$ is σ-locally finite in X. Thus we have (b).

(b) $\to$ (c). Let $\mathcal{G} = \{ G_i \mid i \in I \}$ be a finite cozero-set cover of $X \times Y$. By (b), each G_i is the union of a family $\{ U_{i\alpha} \times V_{i\alpha} \mid \alpha \in A_i \}$ of cozero-rectangles in $X \times Y$ such that $\{ U_{i\alpha} \mid \alpha \in A_i \}$ is σ-locally finite in X. Then $\{ U_{i\alpha} \times V_{i\alpha} \mid \alpha \in A_i, i \in I \}$ is the desired refinement of G.

(c) $\to$ (b). The proof goes just as that of (a) $\to$ (b) if one use (c) instead of 15.1.1.

(b) $\to$ (a). Let G be a cozero-set of $X \times Y$. It suffices to show that $G = G^* \cap (X \times Y)$ for some cozero-set G^* of $X \times \beta Y$. By (b), G is the union of a family $\{ U_\alpha \times V_\alpha \mid \alpha \in A \}$ of cozero--rectangles in $X \times Y$ such that $\{ U_\alpha \mid \alpha \in A \}$ is σ-locally finite in X. For each $\alpha \in A$, there is a cozero-set V^*_α of βY with $V^*_\alpha \cap Y = V_\alpha$. Let us set $G^* = \bigcup \{ U_\alpha \times V^*_\alpha \mid \alpha \in A \}$. Since cozero-sets are closed under σ-locally finite union, G^* is a cozero-set of $X \times \beta Y$ with $G^* \cap (X \times Y) = G$. Hence the proof is complete.

The following theorems 15.3 and 15.5 are main results of this section. A space X is said to be extremally disconnected if the closure of every open set is open (cf. [GJ, 1H]). For the notions of P-spaces in the sense of Morita and of M'-spaces in the sense of Isiwata, the reader is referred to [M_4] and [I_5], respectively. In particular, an M'-space is known to be precisely a C-embedded subspace of the product of a metric space with a compact space (see [I0]).

15.3 THEOREM. The following conditions on a space X are equivalent:

(a) X is metrizable.

(b) $X \times Y$ is z-embedded in $X \times \beta Y$ for each extremally disconnected space Y.

(c) $X \times Y$ is z-embedded in $X \times \beta Y$ for each normal P-space (in the sense of Morita [M_4]) Y.

(d) $X \times Y$ is z-embedded in $X \times \beta Y$ for each M'-space (in the sense of Isiwata [M_5]) Y.

(e) $X \times Y$ is z-embedded in $X \times \beta Y$ for each discrete space Y with $|Y| \leqq |X| \cdot w(X)$.

Proof. (a) $\rightarrow$ (b). Let Y be an extremally disconnected space, and let $G = \{ p \in X \times Y \mid g(p) \neq 0 \}$ ($g \in C(X \times Y)$) be a cozero-set of $X \times Y$. By 3.3.10, X has a σ-locally finite base $\{ U_\alpha \mid \alpha \in A \}$. Note that each U_α is a cozero-set. For each $n \in N$ and each $\alpha \in A$, let us set

$$V_{n\alpha} = \bigcup \{ H \mid H \text{ is an open set in } Y \text{ such that } U_\alpha \times H \subset G_n \},$$

where $G_n = \{ p \in X \times Y \mid |g(p)| > 1/n \}$. Then $V_{n\alpha}$ is open in Y and $U_\alpha \times \mathrm{cl}_Y V_{n\alpha} \subset G_{n+1}$. Since Y is extremally disconnected, $\mathrm{cl}_Y V_{n\alpha}$ is open-and-closed in Y. Hence, setting $V_\alpha = \bigcup \{ \mathrm{cl}_Y V_{n\alpha} \mid n \in N \}$, we have a family $\{ V_\alpha \mid \alpha \in A \}$ of cozero-sets in Y such that $G = \bigcup \{ U_\alpha \times V_\alpha \mid \alpha \in A \}$. Thus it follows from 15.2 that $X \times Y$ is z-embedded in $X \times \beta Y$.

(b) $\to$ (e). Obvious.

(e) $\to$ (a). The proof is a slight modification of that of [BH_3, 3.1]. Let $\mathcal{B}$ be a base for X, and topologize the set $Y = \{ (x, B) \mid x \in B \in \mathcal{B} \}$ with its discrete topology. Then $|Y| \leqq |X| \cdot w(X)$. For each $(x, B) \in Y$, there is $f_{(x, B)} \in C(X)$ such that $f_{(x, B)}(x) = 1$ and $f_{(x, B)}(X - B) = \{0\}$. Define $g \in C(X \times Y)$ by

$$g(x', (x, B)) = f_{(x, B)}(x'),$$

and set $G = \{ p \in X \times Y \mid g(p) \neq 0 \}$. Since $X \times Y$ is z-embedded in $X \times \beta Y$, by 15.2, G is the union of a family $\{ U_\alpha \times V_\alpha \mid \alpha \in A \}$ of cozero-rectangles in $X \times Y$ such that $\{ U_\alpha \mid \alpha \in A \}$ is σ-locally finite in X. It is easily checked that $\{ U_\alpha \mid \alpha \in A \}$ forms a base for X. Thus X is metrizable by 3.3.10.

Since the equivalence of (a), (c) and (d) is incidental in our subsequent discussions, we state only out-line of the proof. The proof of (a) $\to$ (c) is the same as that of [M_4, 4.4] if one use 15.2 (c). To prove (a) $\to$ (d), let Y be an M'-space. Then by [M_5, 4.4] (cf. also [I_5]) there exists a paracompact M-space $\widetilde{Y}$ with $Y \subset \widetilde{Y} \subset \upsilon Y$. Since $X \times \widetilde{Y}$ is a k-space by 3.5.2, a similar argument to that of 12.1.4 shows that $X \times Y$ is C-embedded in $X \times \widetilde{Y}$. Since a paracompact M-space is a normal P-space ([M_4, 6.3]), $X \times \widetilde{Y}$ is z-embedded in $X \times \beta\widetilde{Y}$ ($= X \times \beta Y$). Consequently $X \times Y$ is z-embedded in $X \times \beta Y$. (c) $\to$ (e) and (d) $\to$ (e) are obvious. Hence the proof is complete.

15.4 REMARK. T. Hoshina informed me that $\dim S \leqq \dim X$ holds whenever S is z-embedded in X[10]. This fact can be combined with [M_7, Theorem 1] to yeild the following result: If $X \times Y$ is z-embedded in $X \times \beta Y$, then $\dim (X \times Y) \leqq \dim X + \dim Y$. On the other hand, Wage has proved in [W_1] that there exist a metric space X and a Lindelöf space Y such that $\dim (X \times Y) > \dim X + \dim Y$. Therefore the product $X \times Y$ of a metric space X with a Lindelöf space Y need not be z-embedded in $X \times \beta Y$.

15.5 THEOREM. The following conditions on a space X are equivalent:

(a) X is locally compact, metrizable.

(b) $X \times Y$ is z-embedded in $X \times \beta Y$ for each space Y.

(c) $X \times Y$ is z-embedded in $X \times \beta Y$ for each locally compact, Moore space Y with $w(Y) \leqq |X| \cdot w(X) \cdot \exp \aleph_0$.

Proof. (a) → (b). Let Y be a space. By 3.3.10, X has a σ-locally finite base $\{ U_\alpha \mid \alpha \in A \}$ such that $\mathrm{cl}_X U_\alpha$ is compact for each $\alpha \in A$. To prove 15.2 (b), let G be a cozero-set of $X \times Y$. For each $\alpha \in A$, let us set

$$V_\alpha = \bigcup \{ H \mid H \text{ is an open set in } Y \text{ such that } \mathrm{cl}_X U_\alpha \times H \subset G \}.$$

Then $G = \bigcup \{ U_\alpha \times V_\alpha \mid \alpha \in A \}$. It suffices to show that each V_α

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10 For a space X, $\dim X$ denotes the covering dimension of X (cf. chapter 7 of [E]).

is a cozero-set of Y. Since $cl_X U_\alpha$ is compact, the projection π_α from $cl_X U_\alpha \times Y$ to Y is an open perfect map and $V_\alpha = Y - \pi_\alpha((cl_X U_\alpha \times Y) - G)$. Thus it follows from 15.1.2 that V_α is a cozero-set of Y.

(b) $\to$ (c). Obvious.

(c) $\to$ (a). By 15.3, X is metrizable. Suppose that X is not locally compact at $x_0 \in X$. Then $\chi(x_0, X) \leqq \aleph_0$ and x_0 has no pseudo-$\aleph_0$-compact neighborhood in X by 3.3.9, As noted in 13.1, there exists a locally compact Moore space Y, with $w(Y) \leqq \exp \aleph_0$, that has a $D(\aleph_0)$-expandable family $\mathcal{F}$ such that $|\mathcal{F}| = \exp \aleph_0$ and $\bigcap \{ cl_{\upsilon Y} F \mid F \in \mathcal{F} \} \neq \emptyset$. Then it follows from 11.2 that $X \times Y$ is not C-embedded in $X \times \upsilon Y$, and so $X \times Y$ is not z-embedded in $X \times \beta Y$ by 3.4. This contradiction completes the proof.

15.6 REMARK. In [BH_3, 2.2], Blair and Hager essentially proved the equivalence of the following three conditions:

(a) X is pseudo-$\aleph_1$-compact.

(b) $X \times \beta Y$ is z-embedded in $\beta X \times \beta Y$ for each space Y.

(c) $X \times \beta D(\aleph_1)$ is z-embedded in $\beta X \times \beta D(\aleph_1)$.

Since pseudo-$\aleph_1$-compactness coincides with separability for metrizable spaces, one can obtain characterizations of separable (resp. separable locally compact) metrizable spaces by replacing "$X \times \beta Y$" by "$\beta X \times \beta Y$" in 15.3 (resp. 15.5). For separable case, some of those were proved in [BH_3].

15.7 PROBLEM. Do there exist any other conditions on Y for which 15.3 and 15.5 remain valid ?

15.8 We can apply our theory to answer the following questions posed by Blair and Hager in [BH_3].

(A) Does the following condition (d') imply that $X \times Y$ is z-embedded in $\beta X \times \beta Y$? (d') For each $f \in C(X \times Y)$ and $\varepsilon > 0$, there is a countable open rectangular cover $\{ G_n \}$ of $X \times Y$ such that $\sup \{ |f(p) - f(q)| \mid p, q \in G_n \} < \varepsilon$ for each n.

(B) If X is σ-compact and Y is pseudo-$\aleph_1$-compact, then is $X \times Y$ z-embedded in $\beta X \times \beta Y$?

In [BH_3], they proved that if X is a separable metrizable space, then $X \times Y$ satisfies (d') for each space Y. Hence the following example answers (A) and (B), simultaneously, in the negative.

15.8.1 EXAMPLE. Let Q be the space of all rational numbers with the usual topology. Then there exists a 0-dimensional pseudo-$\aleph_1$--compact locally compact space Y with $|Y| = w(Y) = \aleph_1$ such that $\upsilon(Q \times Y) \neq Q \times \upsilon Y$ (and hence $Q \times Y$ is not z-embedded in $\beta Q \times \beta Y$ by 3.4).

Proof. We take for Y the space $Y_1(\aleph_0)$ constructed in 11. 5.1. Then Y satisfies the stated conditions by 11.5.1 and 11.5.2 (1). Since $\chi(Q) = \aleph_0$ and Q is not locally pseudo-$\aleph_0$-compact, it follows from 11.2 that $\upsilon(Q \times Y) \neq Q \times \upsilon Y$.

15.8.2 REMARK. (A) was first answered by the author in [O_2].

CHAPTER 4

CHARACTERIZATIONS OF THE CLASSES

$\mathcal{R}$(metrizable) AND $\mathcal{R}$(locally compact metrizable)

Two distinct lines of investigation in chapter 3 will be combined. In dealing with $\mathcal{R}(P(m))$ and $\mathcal{R}(k)$, some kinds of local compactness were the central ideas. By contrast, in this chapter weak cb*-property plays a crucial role. A space X is called a weak cb*-space if for each decreasing sequence $\{ F_n \mid n \in N \}$ of regular closed sets in X with empty intersection, $\bigcap\{ cl_{\upsilon X} F_n \mid n \in N \} = \emptyset$ holds. This notion first appeared in [HW] without a name, and was named by Isiwata in $[I_8]$. Recall from [MJ] that a space X is a weak cb-space if for each decreasing sequence $\{ F_n \mid n \in N \}$ of regular closed sets in X with empty intersection, there is a decreasing sequence $\{ Z_n \mid n \in N \}$ of zero-sets in X with empty intersection such that $F_n \subset Z_n$ for each $n \in N$. Weak cb*-property is a simultaneous generalization of weak cb-property and realcompactness. Since normal countably paracompact spaces, extremally disconnected spaces and pseudocompact spaces (or more generally, M'-spaces in the sense of Isiwata $[I_5]$) are weak cb-spaces, they are weak cb*-spaces. The most important fact about weak cb*-spaces X is that $\upsilon E(X) = E(\upsilon X)$ is valid, where E(X) denotes the absolute of X. A brief account of absolutes and various characterizations of weak cb*--spaces are given in section 17 below.

16. Main theorems

We state main theorems of this chapter and discuss some remarks. The proofs will be given later in section 18. All the results in this section have been presented in [O_3]. Hereafter X^ω denotes the product of countably many copies of a space X.

16.1 THEOREM. The following conditions on a space X are equivalent:

(a) X is a weak cb*-space and $|X| < m_1$.

(b) $\upsilon(X \times T) = \upsilon X \times \upsilon T$ holds for each metrizable space T.

(c) $\upsilon(X \times D(d(X))^\omega) = \upsilon X \times \upsilon D(d(X))^\omega$.

16.2 REMARKS. (1) It is to be noted that $D(d(X))^\omega$ is a 0--dimensional metrizable space, and that $w(D(d(X))^\omega) = d(X)$ if $d(X) \geqq \aleph_0$ (cf. [E, 2.3.13]).

(2) In view of 10.1, "metrizable space" cannot be weakened to "Moore space" in (b). Further, it cannot be weakened to "Lašnev space". In fact, since an M-space is a weak cb*-space, Kato's example X (= $\beta R - \beta N$) quoted in 12.6 (1) is a weak cb*--space, but $\upsilon(X \times Y) \neq \upsilon X \times \upsilon Y$ for some Lašnev space Y by 12.5. In section 19, we shall, however, prove that $\mathcal{R}$(metrizable) = $\mathcal{R}$(paracompact M) under the assumption that each cardinal is non-measurable.

In the following theorem, the equivalence of (a) and (c) has been described in 4.4; however, we repeat it here to compare with 16.1 (c).

16.3 THEOREM. The following conditions on a space X are equivalent:

(a) $|X| < m_1$.

(b) $\upsilon(X \times T) = \upsilon X \times \upsilon T$ holds for each locally compact, metrizable space T.

(c) $\upsilon(X \times D(d(X))) = \upsilon X \times \upsilon D(d(X))$.

16.4 REMARK. It is open whether "locally compact, metrizable" can be weakened to "locally compact, paracompact" or not in (b) (cf. 21.1).

One more purpose of this chapter is to discuss the problem of finding necessary and sufficient conditions for X and Y in order that $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ holds in the restrictive situation when Y is a (locally compact) metrizable space. The following theorem generalizes Theorem 4.2 due to Hušek.

16.5 THEOREM. Suppose that T is a locally compact, metrizable space. Then $\upsilon(X \times T) = \upsilon X \times \upsilon T$ holds if and only if either $|X| < m_1$ or $|T| < m_1$.

It would be nice to have a theorem analogous to the above in the case where T is only assumed to be metrizable. Our next theorem gives a partial answer to this requirement.

16.6 THEOREM. Suppose that X satisfies the countable chain condition (i.e., $c(X) \leqq \aleph_0$) and T is a metrizable space. Then $\upsilon(X \times T) = \upsilon X \times \upsilon T$ holds if and only if (i) either $|X| < m_1$ or $|T| < m_1$ and (ii) either X is a weak cb*-space or T is locally compact.

16.7 EXAMPLE. 16.6 fails to be valid if we drop the assumption that $c(X) \leqq \aleph_0$ (in this sense 16.6 is best possible). To see this, we take for X the space $Z_4(\aleph_0)$ constructed in 8.1.2. The proof of 8.1.2 essentially shows that X is not a weak cb*-space. Let T be the space of all rational numbers with the usual topology. Obviously T is metrizable but not locally compact. To prove that $\upsilon(X \times T) = \upsilon X \times \upsilon T$ ($= \upsilon X \times T$), let $f \in C(X \times T)$. Recall that the only point x_0 in $\upsilon X - X$ is a P-point in υX. Since $|T| = \aleph_0$, x_0 has a neighborhood G in υX such that f takes on the constant value r_t on $(G \cap X) \times \{t\}$ for each $t \in T$. Extend f over $\upsilon X \times T$ by setting $f((x_0, t)) = r_t$ for $t \in T$. Then the extension is continuous, and so $X \times T$ is proved to be C-embedded in $\upsilon X \times T$. Hence $\upsilon(X \times T) = \upsilon X \times \upsilon T$ holds, but X is not a weak cb*-space and T is not locally compact.

17. Absolutes and weak cb*-spaces

17.1 Recall that a space X is extremally disconnected if the closure of every open set of X is open. A map $f: X \to Y$ is called irreducible if f carries a proper closed subset of X to a proper closed subset of Y. Associated with each space X, there exist an extremally disconnected space $E(X)$ and a perfect irreducible map e_X from $E(X)$ onto X. The space $E(X)$ is unique up to homeomorphism and is called the absolute (or projective cover) of X. More detailed information on absolutes may be found in [IF] and [S_5].

17.1.1 ([IF], [W_3]) For a space X, $\beta E(X) = E(\beta X)$ and $\beta e_X = e_{\beta X}$ hold, where βe_X is the extension of e_X over $\beta E(X)$.

17.1.2 ([IF], [S_5]) If $f: X \to Y$ is a perfect onto map, then there exists a perfect map h from $E(Y)$ onto a closed subset of X such that $e_Y = f \circ h$.

$$\begin{tikzcd} & E(Y) \arrow[dl, "h"'] \arrow[d, "e_Y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Although $E(\upsilon X)$ is usually different from $\upsilon E(X)$, the situation is simpler if X is a weak cb*-space. In the following theorem, the equivalence of (a) and (b) follows immediately from [HW, 2.4] and [HJ, 1.2] (cf. also [I_1]); for completeness we include a proof.

17.2 THEOREM. The following conditions on a space X are equivalent:

(a) X is a weak cb*-space.

(b) $\upsilon E(X) = E(\upsilon X)$ (and then $\upsilon e_X = e_{\upsilon X}$).

(c) $\upsilon e_X\colon \upsilon E(X) \to \upsilon X$ is a countably bi-quotient onto map.

(d) $X \times Y$ is C-embedded in $\upsilon X \times Y$ for each bi-sequential space Y.

(e) $X \times T$ is C-embedded in $\upsilon X \times T$ for each metrizable space T.

(f) $X \times D(c(X))^{\omega}$ is C-embedded in $\upsilon X \times D(c(X))^{\omega}$.

(g) Each countable locally finite family $\mathcal{G}$ of open sets in X is locally finite in υX.

(h) Each locally finite family $\mathcal{G}$ of open sets in X, with $|\mathcal{G}| < m_1$, is locally finite in υX.

Proof. (a) $\to$ (b). Since $\mathrm{cl}_{\upsilon E(X)}G = \mathrm{cl}_{\upsilon E(X)}(G \cap E(X))$ for each open set G in $\upsilon E(X)$, $\upsilon E(X)$ is extremally disconnected. By the uniqueness of $E(\upsilon X)$, it suffices to show that $\upsilon e_X\colon \upsilon E(X) \to \upsilon X$ is a perfect irreducible onto map. Clearly υe_X is irreducible. Suppose that υe_X is not perfect onto; then there exists $p_0 \in \beta E(X) - \upsilon E(X)$ such that $(\beta e_X)(p_0) \in \upsilon X$. By 3.3.4, there is a zero-set $Z = \{ p \in \beta E(X) \mid f(p) = 0 \}$ ($f \in C(\beta E(X))$) of $\beta E(X)$ with $p_0 \in Z \subset \beta E(X) - \upsilon E(X)$. For each $n \in \mathbb{N}$, let us set $H_n = X - e_X(E(X) - G_n)$, where $G_n = \{ p \in E(X) \mid |f(p)| < 1/n \}$. Then, e_X being a perfect irreducible map, H_n is a non-empty open set in X with $\mathrm{cl}_X H_n = e_X(\mathrm{cl}_{E(X)} G_n)$. Since $p_0 \in \mathrm{cl}_{\beta E(X)} G_n$, $(\beta e_X)(p_0) \in \mathrm{cl}_{\beta X} H_n$ for each

$n \in N$, and hence $(\beta e_X)(p_0) \in \bigcap\{ \, cl_{\upsilon X} H_n \mid n \in N \, \}$. This contradicts the fact that X is a weak cb*-space.

(b) → (c). As noted in 17.1.1, $\beta E(X) = E(\beta X)$ holds. If we set $\widetilde{X} = (\beta e_X)^{-1}(\upsilon X)$, then $\widetilde{X}$ is extremally disconnected and $e = (\beta e_X)|\widetilde{X}$ is a perfect irreducible map onto υX. Because of the uniqueness of $E(\upsilon X)$, $\widetilde{X} = E(\upsilon X)$ and $e = e_{\upsilon X}$. Since $\upsilon E(X) = E(\upsilon X)$ by (b), $\widetilde{X} = \upsilon E(X)$ and $e = \upsilon e_X$. Hence υe_X is a perfect onto map, and so it is a countably bi-quotient map.

(c) → (d). Let Y be a bi-sequential space; then there is a bi-quotient map f from a metrizable space T onto Y. By 15.3, $E(X) \times T$ is z-embedded in $\beta E(X) \times T$, and so it is C-embedded in $\upsilon E(X) \times T$ by 3.4. Since υe_X is a countably bi-quotient onto map, it follows from 3.5.5 that $\upsilon e_X \times f$ is a quotient map from $\upsilon E(X) \times T$ onto $\upsilon X \times T$. Hence it follows from 14.3 that $X \times Y$ is C-embedded in $\upsilon X \times Y$.

(d) → (e) → (f). Obvious.

(f) → (g). Suppose that there is a countable locally finite family $\{ \, G_n \mid n \in N \, \}$ of open sets in X such that it is not locally finite in υX. Then, setting $H_n = \bigcup\{ \, G_i \mid i \geqq n \, \}$ for each $n \in N$, we have a locally finite family $\{ \, H_n \mid n \in N \, \}$ of open sets in X such that $\bigcap\{ \, cl_{\upsilon X} H_n \mid n \in N \, \} \neq \emptyset$. Since $\chi(D(c(X))^{\omega}) = \aleph_0$ and each point of $D(c(X))^{\omega}$ has no pseudo-c(X)--compact neighborhood, it follows from 11.3 that $X \times D(c(X))^{\omega}$ is not C-embedded in $\upsilon X \times D(c(X))^{\omega}$. Thus we have (g).

Since (h) → (g) → (a) are obvious, it remains to prove that (b) implies (h). To do this, let $\{ G_\alpha \mid \alpha \in A \}$ be a locally finite family of open sets in X with $|A| < m_1$. We may assume without loss of generality that it covers X. Setting $H_\alpha = cl_{E(X)} e_X^{-1}(G_\alpha)$ for each $\alpha \in A$, we obtain a locally finite cover $\mathcal{H} = \{ H_\alpha \mid \alpha \in A \}$ of E(X) by open-and-closed subsets. For our end, since $e_{\upsilon X}$ is a perfect map from $\upsilon E(X)$ onto υX by (b), it suffices to show that $\mathcal{H}$ is locally finite in $\upsilon E(X)$. Define $\mathcal{A}$ to be the family of all finite subsets of A. If, for each $B \in \mathcal{A}$, we set

$$U(B) = \bigcap \{ H_\alpha \mid \alpha \in B \} - \bigcup \{ H_\alpha \mid \alpha \in A - B \},$$

then $\{ U(B) \mid B \in \mathcal{A} \}$ is a locally finite cover of E(X). Since each U(B) is open-and-closed in E(X), we can find a disjoint open cover $\{ V(B) \mid B \in \mathcal{A} \}$ of E(X) with $V(B) \subset U(B)$. For each $B \in \mathcal{A}$, let us set $W(B) = cl_{\upsilon E(X)} V(B)$. Then W(B) is an open-and--closed subset of $\upsilon E(X)$ with $W(B) \cap E(X) = V(B)$. We shall show that $\upsilon E(X) = \bigcup \{ W(B) \mid B \in \mathcal{A} \}$. Suppose that there exists $p_0 \in \upsilon E(X) - \bigcup \{ W(B) \mid B \in \mathcal{A} \}$. Let $P(\mathcal{A})$ denote the power set of $\mathcal{A}$, and define a function $\mu: P(\mathcal{A}) \to \{0, 1\}$ by

$$\mu(\mathcal{B}) = \begin{cases} 0 & \text{if } p_0 \notin cl_{\upsilon E(X)}(\bigcup \{ W(B) \mid B \in \mathcal{B} \}), \\ 1 & \text{if } p_0 \in cl_{\upsilon E(X)}(\bigcup \{ W(B) \mid B \in \mathcal{B} \}). \end{cases}$$

Then $\mu(\mathcal{A}) = 1$ and $\mu(\{B\}) = 0$ for each $B \in \mathcal{A}$. Since (b) implies (g), X satisfies (g), and so it follows that μ is a countably additive $\{0, 1\}$-valued measure on $\mathcal{A}$. This contradicts the fact that $|\mathcal{A}| < m_1$. Thus $\{ W(B) \mid B \in \mathcal{A} \}$ is proved to be an open cover of $\upsilon E(X)$. Since each U(B) meets only finitely many members of H, so is W(B). This implies that H is locally finite in $\upsilon E(X)$. Hence the proof is complete.

17.3 REMARK. The equivalence of (a) and (e) has been proved by the author in [O_3].

17.4 REMARKS. (1) In 17.2, (d) and (e) are to be compared with 12.5. In particular, 12.6 (1) shows that "bi-sequential" cannot be weakened to "Fréchet" in (d). Further, if the existence of measurable cardinals is assumed, then one can see that it cannot be weakened to even "countably bi-sequential". In fact, let Y be the one-point compactification of $X = D(m_1)$; then it is known [M_3, 10.15] that Y is countably bi-sequential, and X is clearly a weak cb*-space, while it follows from 4.2 that $X \times Y$ is not C-embedded in $\upsilon X \times Y$.

(2) The reader might ask whether the product $X \times T$ of a weak cb*-space X with a metrizable space T is z-embedded in $\beta X \times T$. Since Lindelöf spaces are weak cb*-spaces, 15.4 provides a negative answer to this question.

(3) A more direct proof of (a) $\rightarrow$ (e), not using E(X), can be given by slightly modifying the proof of 19.2 below.

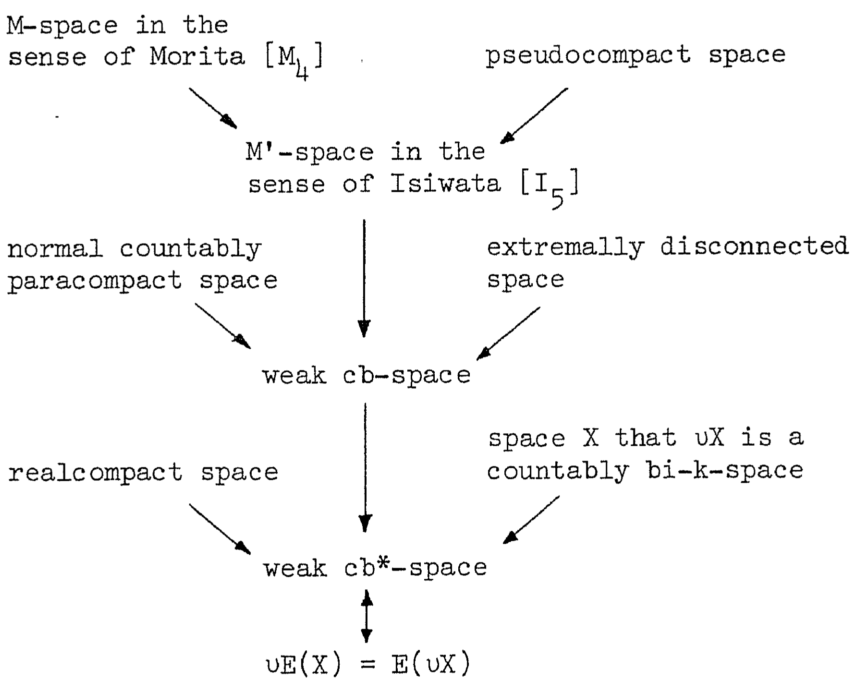
The following corollary improves a result of Woods [W_4, 2.10], who proved it with the assumption that υX is locally compact.

17.5 COROLLARY. If υX is a countably bi-k-space, then X is a weak cb*-space, and hence $\upsilon E(X) = E(\upsilon X)$.

Proof. Let T be a metrizable space; then $\upsilon X \times T$ is a k-space by 3.5.2, and so $X \times T$ is C-embedded in $\upsilon X \times T$ by 12.1.4. Hence it follows from 17.2 that X is a weak cb*-space.

17.6 REMARK. In 17.5, "countably bi-k" cannot be weakened to "singly bi-k". To see this, let X be the space $Z_4(\aleph_0)$ constructed in 8.1.2. Then, as noted in 16.7, X is not a weak cb*-space. On the other hand, υX is a singly bi-k-space, because it is the closed image of a locally compact, paracompact space.

17.7 The following chart provides a summary of the relationship of weak cb*-spaces to other more familiar ones:



18. Proofs of theorems (16.1, 16.3, 16.5 and 16.6)

18.1 Proof of Theorem 16.1. (a) $\to$ (b). Let X be a weak cb*-space with $|X| < m_1$ and T a metrizable space. Since $|E(X)| < m_1$, it follows from 3.4 and 15.3 that $\upsilon(E(X) \times T) = \upsilon E(X) \times \upsilon T$. By 17.2, υe_X: $\upsilon E(X) \to \upsilon X$ is perfect onto, and so $\upsilon e_X \times id_{\upsilon T}$ is a perfect map from $\upsilon E(X) \times \upsilon T$ onto $\upsilon X \times \upsilon T$. Hence it follows from 14.2 that $\upsilon(X \times T) = \upsilon X \times \upsilon T$.

(b) $\to$ (c). Obvious.

(c) $\to$ (a). Let $\{ F_n \mid n \in N \}$ be a decreasing sequence of regular closed sets in X with empty intersection. Clearly it is locally finite in X. Since $c(X) \leqq d(X)$, each point p of $D(d(X))^{\omega}$ has no pseudo-c(X)-compact neighborhood and $\chi(p, D(d(X))^{\omega}) = \aleph_0$. Since $X \times D(d(X))^{\omega}$ is C-embedded in $\upsilon X \times D(d(X))^{\omega}$, it follows from 11.3 that $\bigcap \{ cl_{\upsilon X} F_n \mid n \in N \} = \emptyset$, and thus X is a weak cb*-space. To prove that $|X| < m_1$, find a discrete family $\{ G_{\alpha} \mid \alpha \in A \}$ of non-empty open sets in $D(d(X))^{\omega}$ with $|A| = d(X)$. Pick $p_{\alpha} \in G_{\alpha}$ for each $\alpha \in A$, and set $D = \{ p_{\alpha} \mid \alpha \in A \}$. Then it is easily checked that $X \times D$ is C-embedded in $X \times D(d(X))^{\omega}$. Since $\upsilon D \subset \upsilon D(d(X))^{\omega}$ by 3.3.1, (c) implies that $\upsilon(X \times D) = \upsilon X \times \upsilon D$. Then it follows from 4.2 that $|X| < m_1$ or $|D| < m_1$. If $|D| < m_1$, then $|X| < m_1$ by 3.2.2. Hence the proof is complete.

18.2 Proof of Theorem 16.2. (a) $\to$ (b) follows from 3.4 and 15.5. (b) $\to$ (c) is obvious. (c) $\to$ (a) has been stated in 4.4. Hence the proof is complete.

18.3 REMARKS. (1) Michael proved that if X is a space and S is a closed subspace of a metrizable space T, then $X \times S$ is C-embedded in $X \times T$ (for the proof, see [S_4]). If one make use of this result to prove 16.1, then (c) $\rightarrow$ (a) turns a corollary of 4.4 and 17.2, because both $D(d(X))$ and $D(c(X))^{\omega}$ are closed subspaces of $D(d(X))^{\omega}$.

(2) In 16.1 and 16.3, "d(X)" cannot be weakened to "c(X)" in general. To see this, let X be the product of m_1 copies of D(2); then $c(X) \leqq \exp \aleph_0$ by [K_3, 5.4] (cf. also [E, 2.7.10]). Hence $\upsilon(X \times T) = \upsilon X \times \upsilon T$ ($= X \times T$) holds for each metrizable space T with $w(T) \leqq c(X)$, while $|X| = m_1$. Of course, if we assume that each cardinal is non-measurable, then the above replacement is possible.

18.4 Proof of Theorem 16.5. Since the sufficiency is straightforward by use of 3.4 and 15.5, we shall prove only the necessity. Suppose on the contrary that $|X| \geqq m_1$, $|T| \geqq m_1$ and $\upsilon(X \times T) = \upsilon X \times \upsilon T$. Then $w(T) \geqq m_1$ by 3.2.2, and hence by 3.3.10 we can find a discrete family $\{ G_\alpha \mid \alpha \in A \}$ of non-empty open sets in T with $|A| = m_1$. Pick $t_\alpha \in G_\alpha$ for each $\alpha \in A$. If we set $D = \{ t_\alpha \mid \alpha \in A \}$, then a similar argument to that of 18.1 shows that $\upsilon(X \times D) = \upsilon X \times \upsilon D$. Since $|D| = m_1$, this contradicts 4.2. Hence the proof is complete.

18.5 Proof of Theorem 16.6. Necessity: The proof that $|X| < m_1$ or $|T| < m_1$ is similar to that of 16.5. Suppose that T is not locally compact; then it is not locally pseudo-$\aleph_0$-compact by 3.3.9. Hence it follows from 11.3 that X must be a weak cb*-space. Sufficiency: In case X is a weak cb*-space and $|T| < m_1$, since T is realcompact by 3.3.7, it follows from 17.2 that $\upsilon(X \times T) = \upsilon X \times \upsilon T$ ($= \upsilon X \times T$). The remaining cases follow from 16.1 and 16.5, and hence the proof is complete.

19. On R(paracompact M)

In this section, we prove the equality R(paracompact M) = R(metrizable) under the assumption that each cardinal is non--measurable, from which several corollaries are deduced.

19.1 PREREQUISITES. We make use of the following results:

19.1.1 ($[T_1]$) Let S be a dense subspace of a space X. Then S is C*-embedded in X if and only if every two disjoint zero-sets of S have disjoint closures in X (cf. also [E, 3.2.1]).

19.1.2 ($[S_4]$) If X is a space and S is a compact subspace of a space Y, then $X \times S$ is C-embedded in $X \times Y$.

We denote the following condition on a space by (C).

(C) Each compact subset of the space is of non-measurable cardinal.

19.2 THEOREM. Let X be a weak cb*-space and Y a paracompact M-space. If either X is pseudo-m_1-compact or Y satisfies (C), then $X \times Y$ is C-embedded in $\upsilon X \times Y$.

Proof. By 3.3.4, 3.3.5 and 3.3.6, it suffices to prove that $X \times Y$ is C*-embedded in $\upsilon X \times Y$. To verify 19.1.1, let Z_i ($i = 1, 2$) be disjoint zero-sets in $X \times Y$; then there is $f \in C(X \times Y)$ such that $f(Z_1) = \{0\}$ and $f(Z_2) = \{1\}$. Let $p_0 = (x_0, y_0) \in \upsilon X \times Y$. We shall show that $p_0 \notin cl_{\upsilon X \times Y} Z_1 \cap cl_{\upsilon X \times Y} Z_2$. Since Y is a paracompact M-space, y_0 is contained in a compact subset

K that has a countable neighborhood base in Y. Then, since X is pseudo-m_1-compact or $|K| < m_1$, it follows from 4.1 that $\upsilon(X \times K) = \upsilon X \times K$, and so $f|(X \times K)$ is extended to $h \in C(\upsilon X \times K)$. Let us set

$$E_1 = \{ p \in X \times Y \mid f(p) \leqq 1/3 \},$$
$$E_2 = \{ p \in X \times Y \mid f(p) \geqq 2/3 \},$$
$$W_1 = \{ p \in \upsilon X \times K \mid h(p) < 2/3 \},$$
$$W_2 = \{ p \in \upsilon X \times K \mid h(p) > 1/3 \}.$$

Then $(X \times K) \cap Z_i \subset (X \times K) \cap E_i \subset (X \times K) \cap W_i$ and $\{ W_1, W_2 \}$ is a cozero-set cover of $\upsilon X \times K$. By 15.1.1, there exist a locally finite cozero-set cover $\mathcal{U}$ of υX and binary open covers $\mathcal{V}_U = \{ V_U(1), V_U(2) \}$, $U \in \mathcal{U}$, of K such that $U \times V_U(i) \subset W_i$ for i = 1, 2. Choose $U \in \mathcal{U}$ with $x_0 \in U$. Let us set $S = U \cap X$; then $\upsilon S = U$ by 3.3.2, and it is easily checked that S is a weak cb*-space. Since K is compact, there exist a cozero-set cover $\{ A(1), A(2) \}$ of K and zero-sets B(i) of K such that $A(i) \subset B(i) \subset V_U(i)$ for i = 1, 2.

(1) In case $y_0 \in B(1) \cap B(2)$, let us set $K_0 = B(1) \cap B(2)$; then $(S \times K_0) \cap (E_1 \cup E_2) = \emptyset$. Since K has a countable neighborhood base in Y, and since K_0 is a zero-set of K, K_0 also has a countable neighborhood base $\{ G_n \mid n \in N \}$ in Y with $G_n \supset G_{n+1}$. For each $n \in N$, let us set

$$H_n = \{ x \in S \mid x \text{ has a neighborhood } H \text{ in } S \text{ such that } (H \times G_n) \cap (E_1 \cup E_2) = \emptyset \},$$
$$F_n = \mathrm{cl}_S(S - \mathrm{cl}_S H_n).$$

Then $(\mathrm{cl}_S H_n \times G_n) \cap (Z_1 \cup Z_2) = \emptyset$, and $\{ H_n \mid n \in N \}$ is a increasing open cover of S, because K_0 is compact. Thus $\{ F_n \mid n \in N \}$ is a decreasing sequence of regular closed sets in S with empty intersection. Then, S being a weak cb*-space, $\bigcap \{ cl_{\upsilon S} F_n \mid n \in N \} = \emptyset$, and so $x_0 \notin cl_{\upsilon S} F_n$ for some $n \in N$. If we set $J = \upsilon S - cl_{\upsilon S} F_n$ ($= U - cl_U F_n$), then $J \times G_n$ is a neighborhood of p_0 in $\upsilon X \times Y$ such that $(J \times G_n) \cap (Z_1 \cup Z_2) = \emptyset$ since $J \cap S \subset cl_S H_n$. Hence $p_0 \notin cl_{\upsilon X \times Y} Z_1 \cup cl_{\upsilon X \times Y} Z_2$.

(2) In case $y_0 \notin B(1) \cap B(2)$, then $y_0 \notin A(1)$ or $y_0 \notin A(2)$; we assume that $y_0 \notin A(2)$. If we set $K_0 = K - A(2)$, then K_0 is a zero-set of K containing y_0 such that $(S \times K_0) \cap E_2 = \emptyset$. Hence, by the same argument as above, we can prove that $p_0 \notin cl_{\upsilon X \times Y} Z_2$.

Therefore $cl_{\upsilon X \times Y} Z_1 \cap cl_{\upsilon X \times Y} Z_2 = \emptyset$, and so it follows from 19.1.1 that $X \times Y$ is C*-embedded in $\upsilon X \times Y$. Hence the proof is complete.

19.3 REMARKS. (1) Since a paracompact M-space of non-measurable cardinal is realcompact by 3.3.7, the preceding theorem 19.2, together with 16.1, shows that $\mathcal{R}$(paracompact M) = $\mathcal{R}$(metrizable) under the assumption that all cardinals are non-measurable. The author does not know whether this equality holds or not in general.

(2) The converse of 19.2 is also valid; that is, if the product $X \times Y$ of a weak cb*-space X with a paracompact M-space Y is C-embedded in $\upsilon X \times Y$, then either X is pseudo-$\mathfrak{m}_1$-compact or Y satisfies (C). In fact, more generally, 4.1 combined with 19.1.2 shows that, for each spaces R, S, if $R \times S$ is C-embedded in $\upsilon R \times S$, then either R is pseudo-$\mathfrak{m}_1$-compact or S satisfies (C).

19.4 COROLLARY. Let X be a weak cb*-space and Y a bi-k-space. If either X is pseudo-m_1-compact or Y satisfies (C), then $X \times Y$ is C-embedded in $\upsilon X \times Y$.

Proof. There is a bi-quotient map f from a paracompact M-space T onto Y. If Y satisfies (C), then by the proof of [M_3, 3E3] T can be chosen in such a way that each point is contained in a compact subset K, with $|K| < m_1$, that has a countable neighborhood base in T. Hence, in any case, $X \times T$ is proved to be C-embedded in $\upsilon X \times T$ similarly to the proof of 19.2. Since $id_{\upsilon X} \times f$ is a quotient map by 3.5.4, it follows from 14.3 that $X \times Y$ is C-embedded in $\upsilon X \times Y$.

19.5 COROLLARY. Let X be a weak cb*-space and Y a realcompact bi-k-space. If either X is pseudo-m_1-compact or Y satisfies (C), then $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ ($= \upsilon X \times Y$) holds and $X \times Y$ is a weak cb*-space.

Proof. The first assertion follows from 19.4. If Y satisfies (C), then so is $Y \times D(c(X \times Y))^{\omega}$. Since $Y \times D(c(X \times Y))^{\omega}$ is a bi-k-space by 3.5.1, it follows from 19.4 that

$$X \times Y \times D(c(X \times Y))^{\omega}$$

is C-embedded in

$$\upsilon X \times Y \times D(c(X \times Y))^{\omega} \quad (= \upsilon(X \times Y) \times D(c(X \times Y))^{\omega}).$$

Hence $X \times Y$ is a weak cb*-space by 17.2.

In $[H_8]$, Hušek proved that if a locally compact realcompact space X satisfies (C), then $|X| < m_1$, and he asked whether this result remains true for realcompact k-spaces. Our next corollary provides a partial answer to this question.

19.6 COROLLARY. If X is a realcompact bi-k-space satisfying (C), then $|X| < m_1$.

Proof. This follows from 4.4 and 19.5.

20. Mapping theorems II

In this section the investigation of mapping theorems which was begun in section 14 is continued. We have an interest in the consequence of weak cb*-property, and summarize some of pleasant properties which result from the additional assumption. The following theorem gives other versions of 17.2.

20.1 THEOREM. The following conditions on a space Y are equivalent:

(a) Y is a weak cb*-space.

(b) For any perfect onto map $f: X \to Y$, there exists a closed subset X_0 of υX such that $(\upsilon f)|X_0$ is a perfect map from X_0 onto υY.

(c) For any perfect onto map $f: X \to Y$, $\upsilon f: \upsilon X \to \upsilon Y$ is a bi-quotient onto map.

(d) For any perfect onto map $f: X \to Y$, $\upsilon f: \upsilon X \to \upsilon Y$ is a countably bi-quotient onto map.

(e) For any perfect irreducible onto map $f: X \to Y$, $\upsilon f: \upsilon X \to \upsilon Y$ is a perfect onto map.

Proof. (a) $\to$ (b). Let $f: X \to Y$ be a perfect onto map. By 17.1.2 there exist a closed subset X_1 of X and a perfect map h from E(Y) onto X_1 such that $e_Y = f \circ h$. Let us set $X_2 = cl_{\beta X} X_1$ and $\tilde{f} = (\beta f)|X_2$. Since X_2 is compact, h has a continuous extension $\tilde{h}$ from $\beta E(Y)$ onto X_2. Then

$$(1) \qquad \beta e_Y = \tilde{f} \circ \tilde{h}.$$

Let us set $X_0 = X_2 \cap \upsilon X$. We shall prove that

$$(2) \qquad \tilde{h}(\upsilon E(Y)) = X_0.$$

Since $\tilde{h}$ is perfect, $\tilde{h}^{-1}(X_0)$ is realcompact and $\tilde{h}^{-1}(X_0) \supset E(Y)$. This implies that $\upsilon E(Y) \subset \tilde{h}^{-1}(X_0)$, and so $\tilde{h}(\upsilon E(Y)) \subset X_0$. To show the converse inclusion, suppose that there exists a point $p \in X_0 - \tilde{h}(\upsilon E(Y))$. Then $\tilde{h}(q) = p$ for some $q \in \beta E(Y) - \upsilon E(Y)$. Since Y is a weak cb*-space, it follows from 17.2 that υe_Y is a perfect map onto υY, and so $(\beta e_Y)(q) \in \beta Y - \upsilon Y$. Hence $\tilde{f}(p) \in \beta Y - \upsilon Y$ by (1), which contradicts the fact that $(\beta f)(\upsilon X) \subset \upsilon Y$. Thus (2) is proved. Since υe_Y: $\upsilon E(Y) \to \upsilon Y$ is perfect onto,

$$(3) \qquad \upsilon E(Y) = (\beta e_Y)^{-1}(\upsilon Y).$$

Hence it follows from (1), (2) and (3) that

$$X_0 = \tilde{h}(\upsilon E(Y)) = \tilde{h}((\beta e_Y)^{-1}(\upsilon Y)) = \tilde{f}^{-1}(\upsilon Y).$$

Since $\tilde{f}$ is perfect, this shows that $\tilde{f}|X_0$ ($= (\upsilon f)|X_0$) is a perfect map from X_0 onto υY.

(b) $\to$ (c). Let f: $X \to Y$ be a perfect onto map. By (b), there is a closed subset X_0 of υX such that $(\upsilon f)|X_0$ is a perfect map onto υY. Let $y \in \upsilon Y$, and let $\mathcal{U}$ be a cover of $(\upsilon f)^{-1}(y)$ by open sets in υX. Since $(\upsilon f)^{-1}(y) \cap X_0$ is compact, it is covered by a finite subfamily $\{ U_1, \ldots, U_n \}$ of $\mathcal{U}$. If we set

$$G = \upsilon Y - (\upsilon f)(X_0 - (U_1 \cup \ldots \cup U_n)),$$

then G is an open neighborhood of y in υY such that $G \subset (\upsilon f)(U_1) \cup \ldots \cup (\upsilon f)(U_n)$. Hence υf is a bi-quotient onto map.

(c) $\to$ (d) is obvious. (d) $\to$ (a) and (e) $\to$ (a) follow from 17.2. (a) $\to$ (e) can be proved similarly to that of 17.2 (a) $\to$ (b). Hence the proof is complete.

20.2 REMARK. The reader might ask whether, in 20.1 (e), "irreducible" can be omitted or not. The answer is negative. In fact, Isiwata [I_8] and the author independently proved that υf: $\upsilon X \to \upsilon Y$ is perfect onto for any perfect onto map f: $X \to Y$ if and only if Y satisfies the following condition (*):

(*) For every decreasing sequence $\{ F_n \mid n \in N \}$ of closed subsets in Y with empty intersection, $\bigcap \{ cl_{\upsilon Y} F_n \mid n \in N \}$ is empty.

A space Y satisfying (*) is called a cb*-space, and there exists a weak cb*-space that is not a cb*-space (e.g., $(W(\omega_1 + 1) \times W(\omega_0 + 1)) - \{(\omega_1, \omega_0)\}$). We note that normal countably paracompact spaces are cb*-spaces.

20.3 THEOREM. Among the following conditions on a space Y, (a) $\to$ (b) $\to$ (c) are valid. Conversely, (c) $\to$ (a) holds if $|Y| < m_1$.

(a) Y is a weak cb*-space.

(b) For each perfect onto map f: $X \to Y$ and each space Z satisfying $\upsilon(X \times Z) = \upsilon X \times \upsilon Z$, $\upsilon(Y \times Z) = \upsilon Y \times \upsilon Z$ holds.

(c) For each perfect onto map f: $X \to Y$ and each perfect map g: $Z_1 \to Z_2$ onto a weak cb*-space Z_2, $\upsilon(Y \times Z_2) = \upsilon Y \times \upsilon Z_2$ holds whenever $\upsilon(X \times Z_1) = \upsilon X \times \upsilon Z_1$.

Proof. (a) $\rightarrow$ (b). Since Y is a weak cb*-space, $\upsilon f\colon \upsilon X \rightarrow \upsilon Y$ is bi-quotient onto by 20.1. Then, by 3.5.4, $\upsilon f \times id_{\upsilon Z}$ is a bi-quotient map from $\upsilon X \times \upsilon Z$ onto $\upsilon Y \times \upsilon Z$. Hence it follows from 14.2 that $\upsilon(Y \times Z) = \upsilon Y \times \upsilon Z$.

(b) $\rightarrow$ (c). By (b), $\upsilon(Y \times Z_1) = \upsilon Y \times \upsilon Z_1$. Then, Z_2 being a weak cb*-space, $\upsilon(Y \times Z_2) = \upsilon Y \times \upsilon Z_2$ by (b) again.

(c) $\rightarrow$ (a). Suppose on the contrary that $|Y| < m_1$ and Y is not a weak cb*-space. By 16.1, there exists a metrizable space T such that $\upsilon(Y \times T) \neq \upsilon Y \times \upsilon T$. Let us set $X = E(Y)$ and $f = e_Y$. Then $f\colon X \rightarrow Y$ is a perfect onto map and $\upsilon(X \times T) = \upsilon X \times \upsilon T$, because X is a weak cb*-space with $|X| < m_1$. This contradicts (c), and hence the proof is complete.

20.4 REMARK. The author does not know whether (c) $\rightarrow$ (a) could be proved without the assumption that $|Y| < m_1$.

The preceding theorem raises the question of what happens if the second factor g is merely a quotient map. Our next result answers this question, and improves 14.6 above.

20.5 THEOREM. Among the following conditions on a space Y, (a) $\rightarrow$ (b) $\rightarrow$ (c) are valid. Conversely, (c) $\rightarrow$ (a) holds if $|Y| < m_1$.

(a) υY is locally compact.

(b) For each perfect onto map $f\colon X \rightarrow Y$ and each quotient onto map $g\colon Z_1 \rightarrow Z_2$, $\upsilon(Y \times Z_2) = \upsilon Y \times \upsilon Z_2$ holds whenever $\upsilon(X \times Z_1) = \upsilon X \times \upsilon Z_1$.

(c) For each perfect onto maps $f\colon X \to Y$ and $g\colon Z_1 \to Z_2$, $\upsilon(Y \times Z_2) = \upsilon Y \times \upsilon Z_2$ holds whenever $\upsilon(X \times Z_1) = \upsilon X \times \upsilon Z_1$.

Proof. (a) $\to$ (b). If υY is locally compact, then Y is a weak cb*-space by 17.5. Hence it follows from 20.3 that $\upsilon(Y \times Z_1) = \upsilon Y \times \upsilon Z_1$, and so $\upsilon(Y \times Z_2) = \upsilon Y \times \upsilon Z_2$ by 14.6. (b) $\to$ (c) is obvious. (c) $\to$ (a) follows from 14.6. Hence the proof is complete.

20.6 REMARK. As noted in 14.7, (c) $\to$ (a) cannot be proved without the assumption that $|Y| < m_1$.

21. Problems and remarks[11]

21.1 PROBLEMS. Do 16.1 and 16.6 remain true if "metrizable space" is weakened to "paracompact M-space" ? Do 16.3 and 16.5 remain true if "locally compact, metrizable space" is weakened to "locally compact, paracompact space" ?

To attack these problems, the following facts may be available (cf. [A_3], [M_4]): A locally compact, paracompact space (resp. a paracompact M-space) X has a perfect map f from X onto a locally compact, metrizable space (resp. a metrizable space) T. Moreover, since a metrizable space satisfies (*) in 20.2, $\upsilon f\colon \upsilon X \to \upsilon T$ is then a perfect onto map.

21.2 PROBLEM. Do there exist any other conditions on T for which 16.1 remains valid ?

21.3 PROBLEM. Find necessary and sufficient conditions for X and T in order that $\upsilon(X \times T) = \upsilon X \times \upsilon T$ holds in the case where T is a metrizable space.

21.4 PROBLEM. Characterize $\mathcal{R}$(weak cb*). We note that it follows from 14.2 and 17.2 that $\mathcal{R}$(weak cb*) = $\mathcal{R}$(extremally disconnected). Moreover, since every paracompact space is a

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[11] Problems 21.1, 21.3 and 21.4 have been posed by the author in [O_3]

weak cb*-space, it follows from 4.4 and 7.6 that every member of $\mathcal{R}$(weak cb*) is a realcompact space of non-measurable cardinal.

21.5 PROBLEM. Find characterizations of an onto map $f: X \to Y$ such that $\upsilon f: \upsilon X \to \upsilon Y$ is bi-quotient onto. We are interested in this problem in view of 22.1 (3) below.

CHAPTER 5

MISCELLANEOUS REMARKS

22. Common properties of $\mathcal{R}(\ \mathcal{P}\ )$

22.1 It seems that the classes $\mathcal{R}(\ \mathcal{P}\ )$ considered above have several common properties. We list some of these below: Let $\mathcal{P}$ be a topological property of spaces. Each assertion follows from results in the bracket.

(1) $\mathcal{R}(\ \mathcal{P}\ )$ includes all locally compact, realcompact spaces of non-measurable cardinals (1.2).

(2) $\mathcal{R}(\ \mathcal{P}\ )$ is closed under cozero-subspaces ([BH_2, 3.2]).

(3) $\mathcal{R}(\ \mathcal{P}\ )$ is closed under open perfect images (14.5); more generally, if $\upsilon f\colon \upsilon X \to \upsilon Y$ is bi-quotient onto, then $Y \in \mathcal{R}(\ \mathcal{P}\ )$ whenever $X \in \mathcal{R}(\ \mathcal{P}\ )$ (3.5.4 and 14.2).

(4) If $f\colon X \to Y$ is a perfect map onto a weak cb*-space Y, then $Y \in \mathcal{R}(\ \mathcal{P}\ )$ whenever $X \in \mathcal{R}(\ \mathcal{P}\ )$ (20.3).

(5) If each $\mathcal{P}$-space is υ-locally compact, then $\mathcal{R}(\ \mathcal{P}\ )$ is closed under quotient images (14.6).

(6) If $X \in \mathcal{R}(\ \mathcal{P}\ )$ and Y is a locally compact, realcompact space with $|Y| < m_1$, then $X \times Y \in \mathcal{R}(\ \mathcal{P}\ )$ (1.2).

22.2 REMARK. (1), (2), (3) and (5) have been stated by the author in [O_3].

23. A continuation of section 2

This section is devoted to the example and the result announced in section 2. In the interest of simplicity, we assume that all cardinals are non-measurable.

23.1 EXAMPLE. There exist spaces X_i and Y_i ($i = 1, 2$), neither of which is locally compact and realcompact, such that $X_1 \times Y_1$ and $X_2 \times Y_2$ are homeomorphic, but $\upsilon(X_1 \times Y_1) = \upsilon X_1 \times \upsilon Y_1$ and $\upsilon(X_2 \times Y_2) \neq \upsilon X_2 \times \upsilon Y_2$.

Proof. Let R be an arbitrary pseudocompact non-compact space, and let P be an arbitrary first countable non-weak cb*--space (e.g., $Y_2(\aleph_0)$ constructed in 11.5.3), and let $Q = D(d(R \times P))^{\omega}$. Then, since υR is compact by 3.3.3 and $P \times Q$ is a k-space, it follows from 10.1 that

$$\upsilon(R \times (P \times Q)) = \upsilon R \times \upsilon(P \times Q).$$

On the other hand, it follows from 16.1 that

$$\upsilon((R \times P) \times Q) \neq \upsilon(R \times P) \times \upsilon Q.$$

Therefore, if we set

$$X_1 = R, \qquad Y_1 = P \times Q,$$
$$X_2 = R \times P, \qquad Y_2 = Q,$$

then X_i and Y_i satisfy the stated conditions.

We next show the existence of a υ-pair which cannot be described in terms of function spaces. The following theorem gives a negative answer to Husek's problem, quoted in section 2. That (a) implies (b) was observed by Hušek in [H_9] (or [H_7]); however, we include a proof for completeness.

23.2 THEOREM. The following conditions on a space X are equivalent:

(a) X is locally compact.

(b) For each space Y, if (X, Y) is a υ-pair, then it can be described in terms of function spaces, that is, there exists a completely regular Hausdorff topology t for C(X) satisfying the following three conditions:

(1) $C_t(X)$ is realcompact,

(2) $C(X \times Y) \subset C(Y, C_t(X))$,

(3) $C(\upsilon Y, C_t(X)) \subset C(X \times \upsilon Y)$[12].

Before proving the theorem, let us agree on some terminology. Recall from [K_2] that a topology t for C(X) is jointly continuous if the real-valued function α on $X \times C_t(X)$ defined by $\alpha((x, f)) = f(x)$ is continuous. The function α is called the evaluation. Let $\mathcal{H}$ be a family of subsets of a space X. In C(X), the topology of uniform convergence on members of $\mathcal{H}$ is that having the family $\{ S(H, f, \varepsilon) \mid H \in \mathcal{H}, f \in C(X), \varepsilon > 0 \}$ as a subbase, where $S(H, f, \varepsilon) = \{ g \in C(X) \mid \sup_{x \in H} |f(x) - g(x)| < \varepsilon \}$.

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12 See section 2 (p.8).

Proof of Theorem 23.2. (a) $\to$ (b). Since X is locally compact, the compact-open topology t for C(X) satisfies (2) and (3) for each space Y (cf. [E, 3.4.8]). On the other hand, a locally compact space being a k-space, $C_t(X)$ has a complete uniformity by [K_2, Theorem 12, p.231]. Since all cardinals are assumed to be non-measurable, it follows from 3.3.7 that $C_t(X)$ is realcompact. Thus we have (b).

(b) $\to$ (a). Suppose on the contrary that X is not locally compact at $x_0 \in X$. Let $\{ G_\lambda \mid \lambda \in \Lambda \}$ be a neighborhood base at x_0 in X. Then, for each $\lambda \in \Lambda$, there exists an open cover $\mathcal{U}_\lambda$ of X such that no finite subfamily covers $\mathrm{cl}_X G_\lambda$. Let $\mathcal{H}_\lambda$ be the family of all open sets in X whose closures are contained in some members of $\mathcal{U}_\lambda$, and let $t(\lambda)$ be the topology for C(X) of uniform convergence on members of $\mathcal{H}_\lambda$. If we set $X_\lambda = C_{t(\lambda)}(X)$, then X_λ has a complete uniformity by [K_2, Theorem 10, p.228], and hence it is realcompact by 3.3.7. Since $t(\lambda)$ is jointly continuous by [K_2, Theorem 10, p.228], the evaluation $\alpha_\lambda : X \times X_\lambda \to R$ (= the real line) is continuous. Define Φ to be the family of all completely regular Hausdorff topologies t for C(X) such that $C_t(X)$ is realcompact. For each $t \in \Phi$, let $X_t = C_t(X)$, and let us set

$$Y = (\bigoplus\{ X_\lambda \mid \lambda \in \Lambda \}) \oplus (\bigoplus\{ X_t \mid t \in \Phi \}).$$

Then Y is realcompact, and so (X, Y) is a υ-pair. It remains to prove that (X, Y) cannot be described in terms of function spaces. To do this, let τ be an arbitrary completely regular Hausdorff topology for C(X) satisfying (1), and let $\alpha: X \times C_\tau(X) \to R$ be the evaluation. Then $\tau \in \Phi$. We shall show that either (2) or (3) fails for τ.

In case τ is jointly continuous, then pick an arbitrary element g of $C(X)$, and define a function $\hat{f}$ from υY (= Y) into $C_\tau(X)$ by

$$\hat{f}(h) = \begin{cases} g & \text{if } h \in X_\lambda,\ \lambda \in \Lambda, \\ g & \text{if } h \in X_t,\ t \in \Phi \text{ with } t \neq \tau, \\ h & \text{if } h \in X_\tau. \end{cases}$$

Then $\hat{f} \in C(\upsilon Y, C_\tau(X))$ obviously, but the real-valued function f on $X \times \upsilon Y$ defined by $f((x, h)) = [\hat{f}(h)](x)$ is not continuous, because $f|(X \times X_\tau) = \alpha$ and τ is not jointly continuous. Hence it follows that $C(\upsilon Y, C_\tau(X)) \not\subset C(X \times \upsilon Y)$, i.e., (3) fails.

In case τ is jointly continuous, then define a real-valued function f on $X \times Y$ by

$$f((x, h)) = \begin{cases} \alpha_\lambda((x, h)) & \text{if } h \in X_\lambda,\ \lambda \in \Lambda, \\ 0 & \text{if } h \in X_t,\ t \in \Phi. \end{cases}$$

Then $f \in C(X \times Y)$. To show that $\hat{f}\colon Y \to C_\tau(X)$ (= the function defined by $\hat{f}(y) = f|(X \times \{y\})$) is not continuous, assume that it is continuous. Choose a constant function $k \in C_\tau(X)$ with $k(X) = \{0\}$. Since $k(x_0) = \alpha((x_0, k)) = 0$ and α is continuous, there exist $\lambda \in \Lambda$ and τ-neighborhood V of k in $C_\tau(X)$ such that

$$|\alpha((x, h))| < 1 \quad \text{for each } (x, h) \in G_\lambda \times V.$$

Since $\hat{f}|X_\lambda\colon C_{t(\lambda)}(X) \to C_\tau(X)$ is the identity, and since it is continuous, there is a $t(\lambda)$-neighborhood W of k with $W \subset V$. Then some basic $t(\lambda)$-neighborhood

$$\bigcap\{\ S(H_i,\ k,\ \varepsilon_i)\ \mid\ i = 1,\ldots,\ n\ \}$$

of k must be contained in W. Since $cl_X G_\lambda$ cannot be covered by finitely many members of $\mathcal{U}_\lambda$, we can find a point $x_1 \in G_\lambda - \bigcup\{\ cl_X H_i\ \mid\ i = 1,\ldots,\ n\ \}$ and $j \in C(X)$ such that $j(x_1) = 1$ and $j(\bigcup\{\ cl_X H_i\ \mid\ i = 1,\ldots,\ n\ \}) = \{0\}$. Since $j \in W$, $(x_1,\ j) \in G_\lambda \times V$, but $\alpha((x_1,\ j)) = j(x_1) = 1$, that is a contradiction. Therefore $\hat{f}$ is not continuous, which shows that $C(X \times Y) \not\subset C(Y,\ C_\tau(X))$, i.e., (2) fails. Hence the proof is complete.

23.3 REMARKS. (1) The essential idea of the proof of (b) $\to$ (a) is due to Arens [A_1].

(2) In the proof of (b) $\to$ (a), we can deform Y into a non-realcompact space. Consider $Y \times W(\omega_\alpha)$ instead of Y, where ω_α is the initial ordinal of $(|X|\cdot|Y|)^+$; then the proof goes just as above.

24. Dieudonné topological completions of products

A space is called topologically complete if it is complete with respect to its finest uniformity. For each space X, there exists a unique topologically complete space γX containing X densely such that each map $f: X \to Y$ admits a continuous extension $\gamma f: \gamma X \to \gamma Y$. The space γX is called the Dieudonné topological completion of X, and γX as well as υX has been studied by several researchers. For a survey of γX, the reader is referred to [CN_2]. In general, a result about Hewitt realcompactifications of products has an obvious analogue for Dieudonné topological completions (cf. [B_1], [CH], [CN_2], [I_6], [M_5], [O_2], [P_2]), and most of our results about υ, except those of chapter 4, can also be transformed in a straightforward manner into results about γ. In this section, we only state the corresponding theorems to those of chapter 4 without proofs.

24.1 THEOREM. Consider the following conditions on a space X:

(1) Each locally finite family of open sets in X is locally finite in γX.

(2) $E(\gamma X) = \gamma E(X)$ (and then $e_{\gamma X} = \gamma e_X$).

(3) $\gamma e_X: \gamma E(X) \to \gamma X$ is a bi-quotient onto map.

(4) For any perfect onto map $f: Y \to X$, there exists a closed subset Y_0 of γY such that $(\gamma f)|Y_0$ is a perfect map from Y_0 onto γX.

(5) For any perfect onto map $f: Y \to X$, $\gamma f: \gamma Y \to \gamma X$ is a bi--quotient onto map.

(6) For any perfect irreducible onto map $f: Y \to X$, $\gamma f: \gamma Y \to \gamma X$ is a perfect onto map.

(7) For each perfect onto map $f: Y \to X$ and each space Z satisfying $\gamma(Y \times Z) = \gamma Y \times \gamma Z$, $\gamma(X \times Z) = \gamma X \times \gamma Z$ holds.

(8) For any perfect onto map $f: Y \to X$, $\gamma f: \gamma Y \to \gamma X$ is a countably bi-quotient onto map.

(9) $\gamma e_X: \gamma E(X) \to \gamma X$ is a countably bi-quotient onto map.

(10) $X \times Y$ is C-embedded in $\gamma X \times Y$ for each bi-k-space Y.

(11) $\gamma(X \times Y) = \gamma X \times Y$ holds for each paracompact M-space Y.

(12) $\gamma(X \times T) = \gamma X \times T$ holds for each metrizable space T.

(13) $\gamma(X \times D(c(X))^{\omega}) = \gamma X \times D(c(X))^{\omega}$.

(14) Each countable locally finite family of open sets in X is locally finite in γX.

(15) Each locally finite family $\mathcal{G}$ of open sets in X, with $|\mathcal{G}| < m_1$, is locally finite in γX.

Then these conditions are related as follows:

$$(1) \leftrightarrow (2) \leftrightarrow (3) \leftrightarrow (4) \leftrightarrow (5) \leftrightarrow (6) \leftrightarrow (7) \to (8) \to$$
$$\to (9) \to (10) \leftrightarrow (11) \leftrightarrow (12) \leftrightarrow (13) \leftrightarrow (14) \leftrightarrow (15).$$

24.2 REMARKS. (1) In case $|X| < m_1$, all of these conditions are equivalent. The author does not know whether $(10) \to (9) \to (8) \to (7)$ are true or not in general.

(2) In [O_1], the author called a space X satisfying (1) a weak b*-space, where a weak b*-space is defined by internal properties of X. Extremally disconnected spaces, M'-spaces in the sense of Isiwata [I_5], and collectionwise normal countably paracompact spaces are weak b*-spaces (cf. [O_1]).

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(*TERMS*)

(*SYMBOLS*)