DIMENSION THEORY OF GENERAL SPACES

BY

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Summary. We shall solve the following three problems, which come from the attempt to develope harmonious dimension theory:

(1) Find a non-separable complete metric space which plays a role of the space of real line.

(2) Find universal spaces for generalized metric spaces within the class of stratifiable spaces.

(3) Find strongly zero dimensional spaces X and Y such that dimension $\dim (X \times Y)$ of the product is arbitrary high.

CONVENTIONS.

The Greak letters α, β will indicate ordinal numbers, and each ordinal will be taken to be the set of its predecessors. Thus, $\alpha \in \beta$ and $\alpha < \beta$ means the same thing. These same Greak letters will also stand for cardinal numbers. ω is the first infinite ordinal, and ω_1 is the first uncountable ordinal. We also use ω to denote the set of positive integers. No difficulty will result from these ambiguity. The cardinality of continuum is sometimes denoted by c.

A sequence $\{A_i\}$ of sets is called decreasing if $A_i \supset A_{i+1}$ for each i.

All spaces considered are assumed to be Tychonoff, and all maps are continuous. The dimension dim X of a normal space X means the covering dimension of it, which goes back to the idea of Čech [1] and Lebesgue [1]. (When X is non-normal, we adapt the dimension function due to Katětov [1].) In particular, we say that X is strongly 0-dimensional when dim X = 0.

We shall use several different topologies τ on the Cantor-set C, and we denote it by (C,τ). In particular, by (C,ε_0) we denote the usual Euclidean topology on C. For a topology τ on C and for a subset U of C, we say sometimes that U is τ-open if U is open in (C,τ).

We say that U is clopen if it is closed and open simultaniously. For a subset A of (C,τ) $\tau|A$ denotes its relative topology. The unit interval with the Euclidean topology is denoted by I (the letter ε stands for its topology). Exp (X) denotes the hyperspace consisting of all closed sets of a topological space X with the Vietoris topology.

For the undefined terminology, we refer to the reader to Engelking [1] and Kunen [1].

Introduction.

It is difficult to determine the origin of the concept of dimension (see Duda [1]). But, it is no doubt to believe that the discovery by Peano [1] (that is, a continuous map on the unit interval, the image of which is the full square) reminded the people that it is necessary to define the concept of dimension precisely.

Nowadays, we have some satisfactory dimension functions for a space X; ind X, Ind X, and dim X.

One of the most remarkable facts concerning their definitions is that we can assign dimensions to fairly general spaces. For example, we can define the above three dimension functions for every (not necessarily, metrizable) normal spaces.

On the other hand, we believe that our dimension functions are satisfactory, since for separable metric spaces we have a beautiful harmonious dimension theory (see Hurewicz and Wallman [1]).

Hence, one of the main prolbems in dimension theory is to construct a harmonious dimension theory for more general spaces.

From this point of view we have remarkable developments for metrizable spaces (see Morita [2] and Nagata [7]), and have important results for some stratifiable spaces (Nagami [6,7] and Oka [1]).

There are, however, many theorems, which are valid for separable metric spaces, but are open for more general (even, complete metric) spaces in dimension theory. In this thesis we shall discuss three such theorems, and shall solve them completely.

(1) There is no doubt to believe that the real line R is one of the most important and useful 1-dimensional spaces. Our first problem is to seak a non-separable complete metric space, which plays a role of the space R.

To specify our problem, we quickly review the strongly 0-dimensional case. For the strongly 0-dimensional spaces we believe that we have a satisfactory one. It is the Baire's 0-dimensional space. The reason, why it is, is that the Baire's 0-dimensional space $B(\alpha)$ of weight α satisfies not only that $B(\omega)$ (that is, the separable case) is homeomorphic to the familiar space, "the space of irrational numbers", but also that

(1) every strongly 0-dimensional complete metric space of weight α is homeomorphic to a closed subset of $B(\alpha)$ (Stone [1]).

Closed embeddings are frequently used in topology (for example, it is indispensable for ANR theory). In particular, the above result (1) is essentially used

in the recent work of Jayne and Rogers [1] to obtain their representation theorem.

On the other hand, it is known that every finite dimensional separable locally compact space can be closed-embedded in the finite iterated product of R (Isbell [1]).
(Note that we cannot embed an n-dimensional space in the same dimensional product R^n in general. We can, however, embed it into the 2n+1 dimensional product R^{2n+1}.)

Therefore, we can specify our problem as follows:

PROBLEM. For a given infinite cardinal number α find a 1-dimensional complete metric space Z_α of weight α such that
every 1-dimensional complete metric space of weight α can be closed-embedded in a finite iterated product of Z_α.

In chapter 1 we can obtain more general results together with the solution of this problem:

THEOREM 1.2. For a given infinite cardinal number α and every non-negative integer n there exists an n-dimensional complete metric space $Z_{n,\alpha}$ of weight α such that every n-dimensional complete metric space of weight α can be closed-embedded in the product $(Z_{n,\alpha})^2$.

(2) It is one of the most important theorems in dimension theory that there exist universal spaces for several subclasses C of separable metric spaces. (A space X is called a universal space for a class C if
(a) X is a member of C ;
(b) every member of C is homeomorphic to a subset of X.)

For example, the countable infinite product of the real lines R^{ω} is a universal space for all separable metric spaces. It is also known that there are universal spaces for several (non-separable) metric spaces (Nagata [7]).

One of the important properties of universal spaces is that it reduces a general problem to a special problem for a subset of a particular space. (That is, when we want to prove a problem for every separable metric space, it suffices to show it for a subset of R^{ω}. Hence, we can use some special properties of R^{ω} for this problem.)

We believe that one of the reasons, why we cannot complete a harmonious dimension theory for stratifiable spaces, is that it was not known that whether or not there exist universal spaces.

Hence, we specify our problem as follows:

PROBLEM. For every infinite cardinal number α find a universal space for strongly 0-dimensional stratifiable spaces of netweight α.

(Note that the netweight of a space X is equal to its weight when X is metrizable, and that it is known that when we classify stratifiable spaces, it is convenient to use its netweight instead of its weight.)

In contrast with the known results for metric spaces we shall give a complete negative solution to this problem in chapter 2 (Theorem 2.1). We shall also show that there are none for almost all of known important subclasses of stratifiable spaces. (When we use their weight instead of their netweight, we obtain similar results in Corollary 2.1.)

Therefore, our results show that there are unexpected deep gaps between the class of metric spaces and that of stratifiable spaces.

(3) The third problem is concerned with the inequality

$$(*) \quad \dim (X \times Y) \leq \dim X + \dim Y,$$

which is called the product theorem in dimension theory.

It is known that the equality in the above formula is not valid in general. Namely, it is Pontjagin [1] who constructed a 3-dimensional compact metric space $X \times Y$ with 2-dimensional factor spaces X and Y for the first time in the literature.

It is also known that there are higher dimensional examples (Kuz'minov [1]): For a given triple (m,n,k) of positive integers which satisfies a certain condition (that is, the Bockstein condition) there exist compact metric spaces X and Y such that $\dim X = m$, $\dim Y = n$, but $\dim (X \times Y) = m+n-k$.

There exists an algebraic characterization in terms of the homology groups of X that the equality is valid for every compact metric space Y (independently, Boltyanskii (see Kuz'minov [1]) and Kodama [1]).

On the other hand, it seems that almost all theorems, which say that the inequality is valid, have been established in a way parallel to the discovery of the theorems, which say that the product is normal (for example, Morita [1]). Hence, there had been long (for about 20 years) standing the following problem:

PROBLEM. When $X \times Y$ is normal does the inequality (*) hold ?

The first counter-example for the inequality (*), and hence which solves the above problem negatively, was constructed by Wage [1] , under the assumption of continuum hypothesis (CH). After that, Przymusinski [3] eliminates CH, and constructed a counter-example within ZFC.

Following their results, we specify our problem as follows.

HIGHER DIMENSIONAL PROBLEM. For a given pair of non-negative integers (m,n) and $k = 1,2,\ldots,\infty$, is there any normal product space $X \times Y$ such that $\dim X = m$, $\dim Y = n$, while $\dim (X \times Y) = m+n+k$?

We shall solve this prolbem affirmatively. In the case $k = \infty$ our example also solves a problem raised in Fedorčuk [2] under CH.

Therefore, we now have fairly complete information about the behaviour of the dimension function on the product spaces.

Chapter 1. Closed embeddings of complete metric spaces.

1 Closed embeddings of complete metric spaces.

DEFINITION 1.1. We say that a metric space X has the type (n,α) if X is of dimension n and of weight α. A (complete) metrizable space X of type (n,α) is called (complete) (n,α)-universal if it satisfies that

(i) every (completely) metrizable space of type (n,α) can be embedded in X as a (closed) subset.

When X is not necessarily of type (n,α), we say that a space X is (closed) (n,α)-embeddable if X satisfies (i).

There is no doubt to believe that the real line R is one of the most important useful 1-dimensional spaces among separable metric spaces. Our present attempt is to seak a non-separable 1-dimensional complete metric space T of weight α satisfies that

(*) for every n there exists an $m < \infty$ such that T^m is closed (n,α)-embeddable;

(**) T has some nice properties such as linearly orderable or ANR.

REMARK 1.1. (a) One of the frequently used non-separable 1-dimensional space is the hedgehog (or, star) space $J(\alpha)$ of weight α (see for example, Nagata [7, Definition IV. 8]). Since every $J(\alpha)$ is σ-locally compact, the space of irrationals P cannot be embedded as a closed subset in any finite iterated products of it (it is only known that the countable product $J(\alpha)^{\omega}$ is complete (∞,α)-universal). Therefore, $J(\alpha)$ does not serve our purpose.

(b) On the other hand, we have satisfactory spaces for the 0-dimensional case. We believe that the Baire's 0-dimensional space $B(\alpha)$ of weight α is the space we want. It is because $B(\omega)$ (that is, the separable case) is homeomorphic to P and every $B(\alpha)$ is complete $(0,\alpha)$-universal (Stone [1]).

(c) It is also known (Nagata [7, Theorem VI. 10]) that there are (n,α)-universal spaces for every (n,α). Finally, we note that Lipscomb [1] defined a complete metric space $L(\alpha)$ of weight α such that n+1 fold product $L(\alpha)^{n+1}$ contains an easy definable (n,α)-universal spaces. But, these embeddings are not necessarily closed.

We begin with the following lemma which is a slight variation of a lemma due to Nagata. Its proof is achieved in a parallel manner of Nagata [7, VI. 2 A)], so that we omit it.

LEMMA 1.1. Let M be an n-dimensional metric space, and let $\{U_i: i \in \omega\}$ and $\{F_i: i \in \omega\}$ be a collection of open, and of closed subsets of M, respectively, with $U_i \supset F_i$ for each i. Then, for every pair $u_i > v_i$ of real numbers there exists a collection

$$\mathcal{V}_i = \{V_i : u_i > t > v_i,\ t \text{ is rational}\}$$

of open sets in M for each i satisfying

(i) $F_i \subset V_{i,r} \subset \overline{V}_{i,r} \subset V_{i,s} \subset \overline{V}_{i,s} \subset U_i$ if $r < s$;

(ii) $\overline{V}_{i,r} = \cap \{V_{i,s}: s > r\}$ and $V_{i,r} = \cup \{\overline{V}_{i,s}: s < r\}$;

(iii) $\operatorname{ord} \{\partial V : V \in \mathcal{V}\} \leq n$, where $\mathcal{V} = \cup_{i=1}^{\infty} \{\mathcal{V}_i\}$.

Next, we shall prove the following theorem. We introduce the following spacial subset Z_n of R^∞. Let Z_n be the set of points in R^∞ at most n of whose coordinates are rational. Then, Z_n is completely metrizable, since it is a G_δ-subset of R^∞. It is also known that Z_n is n-dimensional (cf. Nagata [7, p. 155]).

THEOREM 1.1. Let M be any completely metrizable space which is (n,α)-universal. Then, the product space $M \times Z_n$ is closed (n,α)-universal.

Proof. Let X be any completely metrizable space of type (n,α). Since M is (n,α)-universal, we may think of X as a subspace of M. Then, by Lavrentiev's theorem (Kuratowski [1, § 35, II]) X is a G_δ-subset of M. If $X = M$, our proof is complete, since $M \times \{q\}$ is a closed subspace homeomorphic to X, where q is a point of Z_n. So we may assume that $M \setminus X \neq \emptyset$. Put

$$M \setminus X = \cup_{i=1}^{\infty} F_i \text{ and } F_i \subset F_{i+1},$$

where each F_i is a closed subset of M. First, we shall define a continuous map $f_i: X \to R$ for each i such that

(1) for each number $r > 1$ and every point $z \in F_i$ there exists a $\delta > 0$ such that $f_i(x) > r$ for any $x \in X \cap B_\delta(z)$;

(2) $f = \Pi_{i=1}^{\infty} f_i: X \to R^\omega$ is a Z_n-valued map (that is, f is a continuous map into Z_n).

Let $\{r_m\}$ be a strictly decreasing sequence of irrational numbers which converges to 0. Put

$$F_{i,m} = Cl_M(B_{r_{m+1}}(F_i)), \text{ and } U_{i,m} = B_{r_m}(F_i).$$

Then, for the collections $\{F_{i,m}: i,m \in \omega\}$, $\{U_{i,m}: i,m \in \omega\}$, and each pair of irrationals $u_{i,m} = r_m$ and $v_{i,m} = r_{m+1}$ we can apply Lemma 1.1 to get a collection

$$\mathcal{V}_{i,m} = \{V_{i,m,t}: r_m > t > r_{m+1},\ t \text{ is rational}\}$$

of open sets in M which satisfies (i) - (iii) of the lemma for each pair of integer i and m. For each i we shall define a map $f_i: X \to R$ as follows.

(3) $f_i(x) = 1/r_1$ if $x \notin U_{i,1}$,

$f_i(x) = 1/r_m$ if $x \in U_{i,m} \setminus \cup \mathcal{V}_{i,m}$,

and

$f_i(x) = 1/\inf\{r: x \in V_{i,m,r}\}$ if $x \in \cup \mathcal{V}_{i,m} \setminus F_{i,m}$.

Then, it is easy to see (Nagata [1]) that f_i is continuous and that

(4) $f_i(x)$ = rational if and only if $x \in \partial V$ for some $V \in \mathcal{V}_i$, where $\mathcal{V}_i = \cup_{m=1}^{\infty} \mathcal{V}_{i,m}$.

From the definition (3) of f_i, we have $f_i(x) \geq 1/r_m$ for every $x \in U_{i,m}$, so that (1) holds. We can also show that (2) holds, since $\{r_m\}$ is strictly decreasing, (4) holds, and $\operatorname{ord} \partial \mathcal{V} \leq n$ by (iii). Let

$$g : X \to M \times Z_n \text{ be } g(x) = (x, f(x)).$$

Then, it is easy to see that g is an embedding (cf. Kuratowski [1, §21 XIII, Theorem]). Let us show that g is a closed embedding. Take any point

$$(y,z) \notin g(X).$$

At first, we consider the case $y \in X$. Then, we have $z \neq f(y)$, since $(y,z) \neq g(y) = (y,f(y))$. Therefore, there exists an i such that $z_i \neq f_i(y)$, where $z = (z_i) \in Z_n$. Take two open sets U and V in R such

that $z_i \in U$, $f_i(y) \in V$ and $U \cap V = \emptyset$. Since f_i is continuous, there exists a $\delta > 0$ such that $f_i(X \cap B_\delta(y)) \subset V$. Put

$$W = B_\delta(y) \times \pi_i^{-1}(U),$$

where $\pi_i : Z_n \to R$ is a restriction of the natural projection from R^ω to the n-th factor space. Then, W is an open neighborhood of (y,z) and it is easy to see that $W \cap g(X) = \emptyset$.

Next, we consider the case $y \notin X$. In this case we shall use the property (1). Since $y \notin X$, there exists an i with $y \in F_i$. Then, for $r = |z_i| + 1$ there exists a $\delta > 0$ such that $f_i(x) > r$ for any $x \in X \cap B_\delta(y)$. Put

$$W = B_\delta(y) \times \pi_i^{-1}(-\infty, r).$$

Then, W is an open neighborhood of (y,z) and $W \cap g(X) = \emptyset$. Thus, the proof is complete.

THEOREM 1.2. For every non negative integer n and every infinite cardinal α, there exists a (n,α)-universal space $Z_{n,\alpha}$ such that the 2n-dimensional space $(Z_{n,\alpha})^2$ is closed (n,α)-embeddable.

Proof. For the case $\alpha = \omega$ (that is, for the separable case), put $M = Z_n$. Then, since Z_n is (n,ω)-universal (e.g. Nagata [7, Theorem IV. 8.]), Z_n satisfies Theorem 1.1, so that $Z_{n,\omega} = Z_n$ satisfies the theorem for the case $\alpha = \omega$.

For the case $\alpha > \omega$, let S(A) be the hedgehog space with its weight $\alpha = |A|$ (see Nagata [7, Definition VI. 6] for the definition of the hedgehog space). Let $E_1 = [0,1]_1$ and $E_2 = [0,1]_2$ be some distinct segments in S(A). Put

$$T(A) = S(A) \setminus \{1_i : i = 1,2\},$$

where each 1_i is the end point of E_i, $i = 1,2$. Put

$$Z(A) = \prod_{m=1}^{\infty} T(A)_m,$$

where each $T(A)_m$ is a copy of T(A). Since $(E_1 \cup E_2) \cap T(A)$ is homeomorphic to the real line R, we may think of R^ω as a closed subset of the completely metrizable space Z(A). Let $Z_n(A)$ be the set of points in Z(A) at most n of whose non-zero coordinates are rational. Then, $Z_n(A)$ is (n,α)-universal, since it contains a topological copy of the universal space $K_n(A)$ defined in [7, Theorem VI. 10], and vice versa.

On the other hand, $Z_n(A)$ is completely metrizable, since it is a G_δ-subset of $Z(A)$. Since $R^\omega \cap Z_n(A)$ is homeomorphic to Z_n,

(5) $Z_n(A)$ contains Z_n as a closed subspace.

Put

$$M = Z_n(A).$$

Then, M satisfies Theorem 1.1, so that $Z_{n,\alpha} = Z_n(A)$ also satisfies our theorem from (5). This completes the proof.

REMARK 1.2. (a) Since each Baire's 0-dimensional space $B(\alpha)$ of weight $\alpha \geq \omega$ contains the space of irrationals Z_n as a closed subset and $B(\alpha)^2$ is homeomorphic to itself, Stone's result in Remark 1 (b) follows from Theorem 1.2.

(b) Recently, above Stone's theorem is used by Jayne and Rogers [1] to obtain their representation theorem.

(c) Let $S(\omega)$ be the hedgehog space of weight ω. Put $F_i = \{1_k : k \leq i\}$, where 1_k is the end point of each segment of $S(\omega)$. For the space $X = S(\omega) \setminus \cup_{i=1}^{\infty} F_i$ put

$$f_i(x) = 1/\rho(x,F_i) \text{ for each } x \in X.$$

Then, $f(0) = (1,1,\ldots)$, where 0 is the origin of $S(\omega)$ and $f = \Pi_{i=1}^{\infty} f_i : X \to R^\omega$. Therefore, the closed embeddings used in Kuratowski [1, §21 XIII, Theorem] does not necessarily satisfy our condition (2).

(d) For spaces which are not necessarily completely metrizable, the corresponding results are not valid, even for separable spaces. In fact, assume that there exists a finite dimensional separable metric space Z such that every 0-dimensional separable metric space can be embedded as a closed subset in Z. Since Z is separable, the power of the set of all its closed subsets is at most the cardinality of continuum c. But, it is known that there exists a family of power 2^c which is homeomorphic to any other (Kuratowski [1, §35, I, Therorem 5]). This is a contradiction.

(e) The following problem is communicated from R. Brown:

If $Z_{n,\alpha}$ is (n,α)-universal, then $(Z_{n,\alpha})^2$ is closed (n,α)-universal ?

Since there exist compact (n,ω)-universal spaces, it is not the case when $\alpha = \omega$.

(f) Professor R. Engelking [3] has kindly communicated the problem whether or not there exists a complete (n,α)-universal space. He also announced in [3] that R. Pol showed that it is the case for $\alpha = \omega$. Quite recently, A. Waśko [1] showed that for every infinite cardinal number α our space $Z_{n,\alpha}$ in Theorem 1.2 is complete (n,α)-universal.

Using her resul, we have the following theorem (Tsuda [5, Theorem 2]) which improves a result of J. Nagata.

THEOREM 1.3. The n+1 fold product of 1-dimensional space $Z_{1,\alpha}$ is closed (n,α)-embeddable.

Therefore, our space $Z_{1\alpha}$ has the property (*). On the other hand, though our space cannot satisfy the property (**), we have the following theorem, using a result of Kodama [5]:

THEOREM 1.4 (Tsuda [5, Corollary 3]). There exists 2-dimensional complete ANR metric space whose n+1 fold product is closed (n,α)-embeddable.

We conclude this chapter with the following simple improvement of a theorem of Fréchet (see, Engelking [2, Problem 1.3. G (b)]), which will be used in the next chapter.

THEOREM 1.5. Every metric space consisting of countable points can be embedded in the space of rational numbers Q as a closed subset.

Proof. Let X be arbitrary metric space consisting of countable points. Then, by the theorem of Fréchet we may think X is a subspace of Q. Put

$$Z = Q^2 \setminus \{0\}\times(Q \setminus X).$$

Then, X is a closed subset of Z. On the other hand, by a theorem of Sierpinski (Engelking [2, Problem 1.3. G (b)]) Z is homeomorphic to Q. This completes the proof.

Chapter 2. Non-existence of universal spaces for some stratifiable spaces.

2 Non-existence of universal spaces for some stratifiable spaces.

One of remarkable properties of metric spaces is that there are universal spaces with a given weight (for example, the Baire's 0-dimensional space and generalized Hilbert spaces). The purpose of this chapter is to show that there are none for some stratifiable spaces. We also discuss some positive result at the end of this chapter.

THEOREM 2.1. There are no universal spaces for the following subclasses of stratifiable spaces with a given network weight $\alpha \geq \omega$;

(1) σ -discrete stratifiable spaces,

(2) M_0-spaces in the sense of Heath and Junnila [1],

(3) stratifiable μ-spaces (Mizokami [1]),

(4) stratifiable spaces with encircling nets (Oka [1]),

(5) (strongly 0-dimensional) stratifiable spaces,

(6) (strongly 0-dimensional) L-spaces in the sense of Nagami [6],

(7) (strongly 0-dimensional) free L-spaces in the sense of Nagami [7].

When we use a given weight instead of a given network weight, we have a similar result:

COROLLARY 2.1. For any given infinite cardinal number α there exists a cardinal number $\beta > \alpha$ such that there exist no universal spaces for the above classes (1) - (7) with a given weight β.

When we consider more restricted subclasses, contained in Lašnev spaces or Nagata spaces, we also have a similar result:

THEOREM 2.2. There are no universal spaces for either countable Lašnev spaces or first-countable, separable, strongly 0-dimensional stratifiable spaces.

COROLLARY 2.2. There are no universal spaces for separable Lašnev spaces.

We start with the following proposition.

PROPOSITION 2.1. For every infinite cardinal number α there exists a family S of cardinality $2^{2^{\alpha}}$, none of whose members are homeomorphic to each other, and which consists of σ-discrete stratifiable spaces S with $|S| = nw(S) = \alpha$ and $w(S) \leq 2^{\alpha}$.

Proof. Let $\{\alpha_i\}$ be countable disjoint copies of α. Then, for each ultrafilter p on α, let S_p be the space $\alpha_0 \cup \{p\}$, where $\alpha_0 = \cup \{\alpha_i\}$, with the following topology τ_p.

(i) $U \in \tau_p$ if and only if $p \notin U$, or $p \in U$ and there exist an $n \in \omega$ and an $F \in p$ such that $U \cap \alpha_i \supset F$ for every $i \geq n$.

Then, since each α_i is a clopen discrete subspace of S_p, it is readily seen that S_p is a σ-discrete stratifiable space and that $|S_p| = nw(S_p) = \alpha$ and $w(S_p) \leq 2^{\alpha}$. Next, we shall show that

(ii) the family of all S_q's which are homeomorphic to a fixed S_p is of the cardinality at most 2^{α}.

Let S_q and S_r be homeomorphic to a fixed S_p. Then, take any homeomorphisms $h_q: S_p \to S_q$ and $h_r: S_p \to S_r$. To show (ii) it suffices to show that

(iii) $h_q|\alpha_0 \neq h_r|\alpha_0$ if $q \neq r$.

On the contrary, assume that there exists a permutation $h: \alpha_0 \to \alpha_0$ such that $h = h_q|\alpha_0 = h_r|\alpha_0$. Since $q \neq r$, we can assume without loss of generality that there exists an $F \in q \setminus r$. Put

$F_0 = \cup_{i=1}^{\infty} F$, where each F_i is a copy of F in α_i, and put

$$U_0 = \{q\} \cup F_0.$$

Then, U_0 is a neighborhood of q by the definition (i) of τ_q. Hence, $h_r h_q^{-1}(U_0)$ is a neighborhood of r, and by the definition(i) there exist an $n \in \omega$ and a $G \in r$ such that $G \subset \alpha_i \cap h_r h_q^{-1}(U_0)$ for $i \geq n$. Hence, $F_n \supset G$, since $\alpha_n \cap h_r h_q^{-1}(U_0) = \alpha_n \cap h\, h^{-1}(U_0) = F_n$.

Since F_n is a copy of F and r is an ultrafilter, we have $F \in r$. This contradiction shows that (iii) holds. Since (ii) holds and there exist 2^{2^α} many ultrafilters on α (Engelking [1, Theroem 3.6.11]), there exists a family $S = \{S_p : p \in \Lambda\}$, with $|\Lambda| = 2^{2^\alpha}$, none of whose members are homeomorphic to each other. Thus, our family S is the required one.

REMARK 2.1. For the case $\alpha = \omega$ we can show the above proposition much easier, using subspaces $\omega \cup \{p\}$ of the Stone-Čech compactification $\beta\omega$ (cf. Nagami [6] and Shelah and Rudin [1]).

Proofs of Theorem 2.1 and Corollary 2.1. The proofs for all the cases of Theorem 2.1 and Corollary 2.1 are derived simultaneously from the following observation, since every class (i+1) contains the preceding class (i) except i = 5, the class (3) contains the class (7), and the collection S in Proposition 2.1 consists of σ-discrete L-spaces (cf. references cited in Theorem 2.1).

Let us denote by $[A]^{\kappa}$ the family consisting of all subsets of a set A having cardinality κ. If κ and λ are infinte and $\kappa \leq \lambda$, then $|[\lambda]^{\kappa}| = \lambda^{\kappa}$. It follows that if X is a stratifiable space (more generally, a paracompact σ-space) with $nw(X) \leq 2^{\kappa}$, where κ is infinite, then $|X| \leq |[2^{\kappa}]^{\omega}| = |(2^{\kappa})^{\omega}| = 2^{\kappa}$; consequently, $|[X]^{\kappa}| \leq 2^{\kappa}$, and we see that X cannot contain copies of all the spaces in Proposition 2.1, with $\alpha = \kappa$.

Proofs of Theorem 2.2 and Corollary 2.2. It suffices to show that there exists a family $\mathcal{T}$ (respectively, $\mathcal{U}$) of cardinality 2^c, where $c = 2^\omega$, none of whose members are homeomorphic to each other, and which consists of countable Lašnev spaces (respectively, σ-compact, first countable, separable, strongly 0-dimensional, stratifiable spaces).

Let C be the Cantor set in the real line with usual Euclidean topology, and let $0 \in C$. Take a countable dense subset D of C, and let $\{x_n\} \subset C$ be a sequence converging to 0. At first, we show the existence of $\mathcal{T}$. Put

$F = \{0\} \times C$, and $A = \{(x_n, d_k): k \leq n, \text{ and } n \in \omega\}$, where $D = \{d_k: k \in \omega\}$. Then, A is a countable discrete subset in C^2 with $Cl(A) \supset F$. For every subset $T \subset F$ let

$$Y_T = T \cup A.$$

Take a point $p_T \notin A$, and let

$$X_T = \{p_T\} \cup A.$$

Define a function $\phi_T: Y_T \to X_T$ as

$$\phi_T(T) = p_T \text{ and } \phi_T(a) = a \text{ for every } a \in A.$$

We topologize the set X_T as

U is open in X_T if $\phi_T^{-1}(U)$ is open in Y_T.

By the definition of ϕ_T, ϕ_T is a closed map between Y_T and X_T. Thus, each X_T is a countable Lašnev space.

By the proof of Proposition 2.1, one can show that there exists a family $\mathcal{T} = \{X_T : T \in \mathcal{F}\}$, with $|\mathcal{F}| = 2^c$, none of whose members are homeomorphic to each other.

Next, we show the existence of u. We shall modify the examples constructed in van Douwen and Przymusinski [1]. Let $\mathcal{F}$ be a free ultrafilter on ω. Then, enumerate the family of all elements of $\mathcal{F}$ as $\{F_s : s \in C\}$, and for each $s \in C$ and $m \in \omega$ choose a $q_s(m) \in D$ such that $0 < |s - q_s(m)| < 1/m$, and put

$$D_s = \{q_s(m) : m \in \omega\}, \text{ and } E_s = \{x_n : n \in F_s\}.$$

We topologize the set

$$\Delta_F = (C \times \{0\}) \cup (D \times \{x_n : n \in \omega\})$$

as follows. Points of $D \times \{x_n : n \in \omega\}$ are isolated, and basic neighborhoods of point $(s,0) \in C \times \{0\}$ have the form

$$B_m(s) = \{(x,y) \in \Delta_F : |s-x| < 1/m\} \setminus (D_s \times E_s \cup \{s\} \times \{x_n : n \in \omega\}),$$

for $m \in \omega$. Then, it is known (van Douwen and Przymusiński [1]) that each Δ_F is first countable, σ-compact, strongly 0-dimensional, cosmic. We can show without difficulty that each Δ_F is stratifiable. Here, we show moreover that it admits a free L-structure. Let $\mathcal{V} = \{V_i : i \in \omega\}$ be a countable clopen base of C. Put

$$L_0 = C \times \{0\}, \; L_{2i} = V_i \times C, \text{ and } L_{2i+1} = \{u_i\},$$

where $\Delta_F \setminus L_0 = \{u_i : i \in \omega\}$.

Put

$$\mathcal{U}_{L_0} = \{\{u\}: u \notin L_0\}, \text{ and } \mathcal{U}_{L_i} = \{\Delta_F \setminus L_i\}$$

for each $L_i \neq L_0$. Then, one can easily checks that

$$(\mathcal{L} = \{L_i : i \in \omega\}, \mathcal{U}_L : L \in \mathcal{L})$$

is a free L-structure of Δ_F. Again, by the proof of Proposition 2.1, we can show that there exists a family $\mathcal{U} = \{\Delta_F : F \in \Lambda\}$, with $|\Lambda| = 2^c$, none of whose members are homeomorphic to each other. This completes the proof of Theroem 2.2.

The proof for Corollary goes on a way parallel in Corollary 2.1.

Professor Junnila [2] has kindly communicated to the author the following problem.

PROBLEM 2.1. Does there exist a universal space for closed images of countable metric spaces ?

Though the above problem is still open, the following proposition shows that it is <u>enough</u> to consider the case of "closed images of the space of <u>natural numbers</u> Q".

PROPOSITION 2.2. Every closed map $f:X \to Y$, where X is a countable metric space, is a restriction of a closed map from Q to a countable space containing Y.

Proof. By Theorem 1.5 we can assume that X is a closed subset of Q. Then, consider the following decomposition of Q;

$$\{\{x\} : x \notin X\} \cup \{f^{-1}(y) : y \in Y\}.$$

Then, the decomposition space S and the natural quotient map $q: Q \to S$ satisfies that

$$q|X = f.$$

It is easy to see that q is a closed map. This completes the proof.

The following problem is communicated by T. Nogura:

PROBLEM 2.2. Can every closed images of countable metric space be embedded in the Ξ-product Ξ_0 of countably many Q's ?

It is known that (San-ou [1]) Ξ_0 is a stratifiable space consisting of countable points.

Finally, we discuss some positive result. We believe that the class of simplicial complexes with weak topology $|K|_W$ is one of the most important classes among those of generalized metric spaces. In particular, it is known that (Ceder [1]) every $|K|_W$ is stratifiable, and that (Cauty [1]) every $|K|_W$ is ANR(stratifiable).

For a given cardinal α let $|K(\alpha)|_W$ be the full complex with its vertices A, where the cardinality A is equal to α. Let Z_α be the countable Cartesian product of $|K(\alpha)|_W$. Then, we consider the class C defined as follows:

$X \in C$ if and only if X is a subset of Z_α for some cardinal number α.

It is easy to see that

(1) if $A \subset X \in C$, then $A \in C$ (that is, subset hereditary);

(2) if $X_i \in C$, then $\Pi X_i \in C$ (that is, countably productive);

(3) every (not necessary full) complex belongs to C.

Here, we show that every metric space belongs to C:

THEOREM 2.3. Every metric space of weight α can be embedded in Z_α.

Proof. It suffies to see that the hedgehog space S(A) of weight $\alpha = |A|$ can be embedded in Z_α. Put $S(A) = \cup_{\lambda \in A} I_\lambda$, where each I_λ is a copy of unit interval I, and $I_\lambda \cap I_\mu = \{0\}$ for distinct λ and μ.

On the other hand, for each vertex $\lambda \in A$ and an integer $i \in \omega$, where λ is distinct from a fixed vertex 0, put $\lambda_i = (\underbrace{0,\ldots,0}_{i},\lambda,0,\ldots) \in Z_\alpha$.

Since each factor $|K|_W$ of Z_α has the ordinary convex structure, we can define

$t\lambda_i + (1-t)\lambda_{i+1}$ for each λ_i, λ_{i+1} and $0 \leq t \leq 1$. Then, put also

$$J_{\lambda,i} = \{t\lambda_i + (1-t)\lambda_{i+1} : 0 \leq t \leq 1\}.$$

Note that $J_\lambda = \cup_{i=1}^{\infty} J_{\lambda,i} \cup \{(0,0,\ldots)\}$ is homeomorphic to the unit interval. It is not dificult to see that $\cup_{\lambda \in A} J_\lambda$ is homeomorphic to $S(A)$. This completes the proof.

REMARK 2.3. (a) By the definition of C we see that the space Z_α is a universal space for the members in C of weight α.

(b) The class C is not contained in either the class of L-spaces or free L-spaces, since the full complex is not a free L-space by Nagami and Tsuda [1]. On the other hand, every (strongly 0-dimensional) member of it is an EM_3(M_0-space) respectively, since every member of C is a μ-space.

PROBLEM 2.3. Find an inner characterization of the members of the class C.

Chapter 3. Normal spaces $X \times Y$ satisfying the inequality $\dim (X \times Y) > \dim X + \dim Y$.

In Chapter 3 we shall discuss the counter-examples for the inequality

(*) $\dim (X \times Y) \leq \dim X + \dim Y$.

Here, we mainly discuss the problem under the condition that $X \times Y$ is normal, since it seems that there is still room to discuss what the suitable dimension function for non-normal spaces is (see Nagata [7], and compare the two treatments in Engelking [1 and 2] of the dimension function due to Katětov [1].)

We believe, however, that it is an interesting area of dimension theory to treat the cases when the product spaces are non-normal. Hence, we shall mention it in the final section.

The problem whether or not the inequality (*) is valid if $X \times Y$ is normal has been standing (at least 20 years) (for example, see Gillman and Jerison [1]).

It is Wage [1], who constructed for the first time the following counter-example for the inequality (*), assuming Continuum Hypothesis(CH):

EXAMPLE 1 (CH). There exist two spaces X and Y such that $\dim (X \times Y) > \dim X + \dim Y = 0$ and that $X \times Y$ is perfectly normal and locally compact.

Eliminating the assumption CH, Przymusiński [1,3] constructed the following counter-example, without any set-theoretic assumptions beyond ZFC:

EXAMPLE 2. There exists a separable, first-countable, Lindelöf space X such that dim X = 0 and X^2 is normal, while dim $X^2 > 0$.

Since Wage [1] discoverd a counter-example for the inequality (*) for the first time, we say that a triple $(X,Y,X \times Y)$ is a Wage-type example when it does not satisfy the inequality (*).

The construction of the Wage-type examples in Examples 1 and 2 can be outlined as follows.

At first, we show that there exists a "special" separable complete metric space $\tilde{T}$, which we want to call the 1-dimensional Wage's metric space. Then, we extend it to a metric space T. Next, we approximate T by a product topology of $X \times Y$. Finally, using some properties of our approximation, we calculate the dimensions of factors X and Y, and of the product $X \times Y$.

To solve our problem, we follow the above procedure. That is:

§§3,4. We shall find at first an n-dimensional Wage's metric space and next find its extension.

§§§ 5,6,7. We shall perform a delicate and complicated approximation of it.

§§§ 9,10,11. We can show that not only dim $(X \times Y)$ is positive, but also it is exactly equal to n. Hence, we must work very hard, using the properties derived from our approximations.

One of the main points of our construction is to find the metric spaces in §4. Such spaces will be produced in Theorems 4.1 and 4.2. Their completeness is heavily relates the elimination of CH. We shall explain this point in some detail.

A space which satisfies all the conditions in Theorem 4.1 except completeness would be easily obtained by a simple modification of Rubin, Scori, and Walsh [1] construction. But, if we use such a space, we must assume CH throughout our constructions.

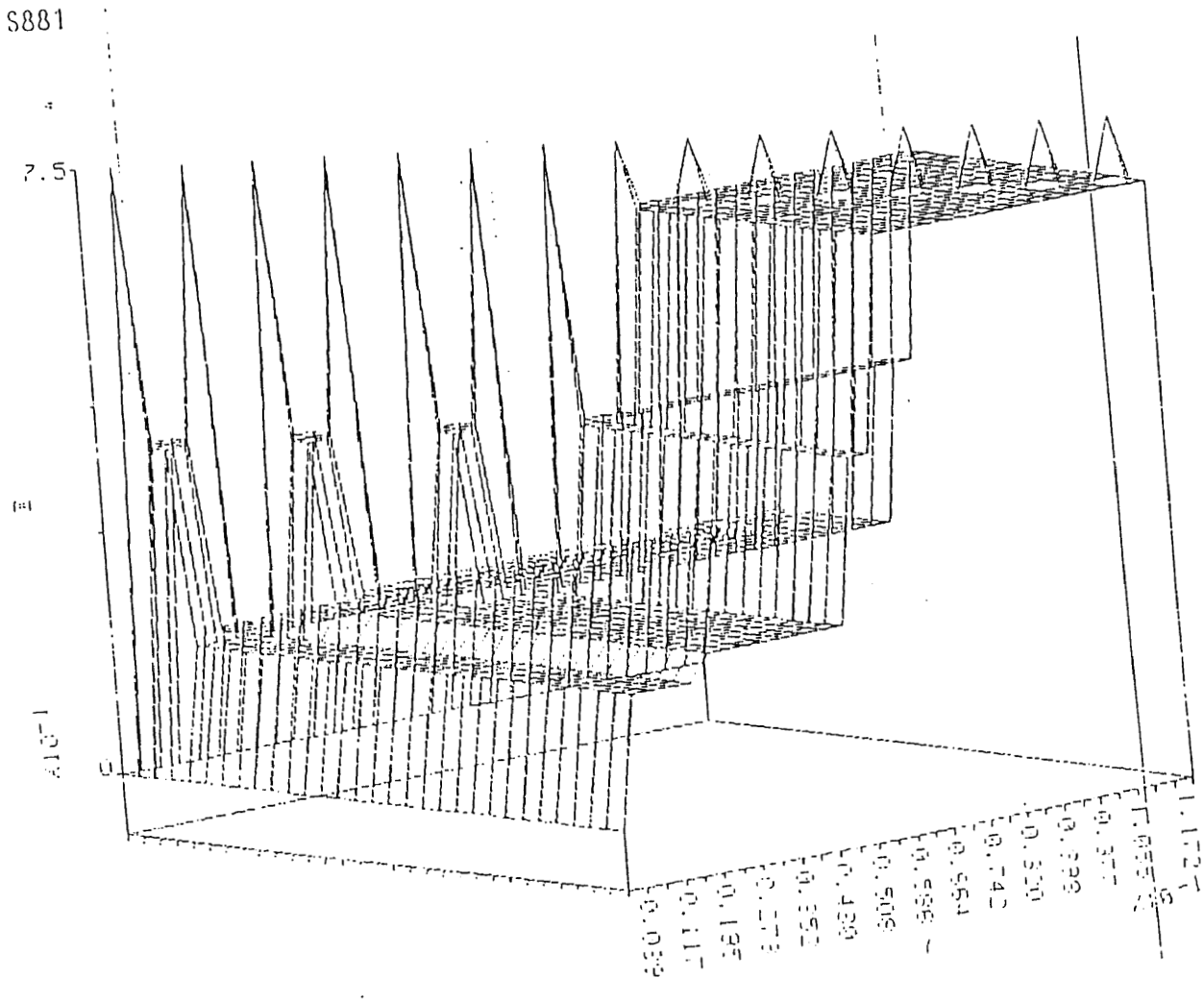
Moreover, the fact that our space satisfies the condition (1) in Theorem 4.1 is deeply related to make dim $(X \times Y) \geq n$. So, we must try to find a space which has the property (1) and completeness, simultaneously.

Since the space we found is satisfactory for our purpose, we can use both of approximation techniques

due to Wage [1,2] (which is also called the factorization technique in there), and due to Przymusiński [3]. (There is, however, some room to improve our space for producing other Wage-type examples (see the appendix and Remark 4.2(b)).)

Though we can proceed our approximation in a way parallel to their ones, rough constructions sometimes cause some errors to make our product space being normal (see the footenote 7 in Przymusiński [6]). Therefore, we must do the delicate and complicated inductive constructions very carefully. The techniques used here go back to the idea of Kunen (as cited in Przymusiński [5]) and that of van Douwen [1], respectively. In Chapter 3 we will construct both of heigher dimensional version of Examples 1 and 2.

3. A 1-dimensional Wage's metric space.



3. A 1-dimensional Wage's metric space.

The first Wage's metric space was constructed in Wage [1] as a 1-dimensional subspace of 2-dimensional cube I^2. In [2] Wage gives a more elementary one as follows:

THEOREM 3.1. There exists a 1-dimensional separable metric topology $\tilde{\rho}$ on C such that

(1) $\tilde{\rho}$ is the topology generated by the subbase $\varepsilon_0 \cup \{h^{-1}(U) : U \in \varepsilon\}$, where $h:C \to I$ is a lower semi-continuous function;

(2) there exists a pair of disjoint closed sets A_1 and B_1 in $(C, \tilde{\rho})$ such that every separation S between A_1 and B_1 has the cardinality of continuum.

Since we shall show the above result as the special case of Theorem 4.1, there is no need to prove it here. But we believe that Wage's function h in [2] is defined more explicitely than ours, and is interesting in itself (see also explanation of computer graphics I & II). Therefore, we give here the defintion of the function h in [2] together with the proof of (2). We begin with the definition of the function h. Let Q be a countable dense subset in the open interval (0,1). Let

$$f : C \to \Pi_{q \in Q} \{0,1\}_q$$

be a natural homeomorphism between the Cantor set and the countable product of 2-point set $\{0,1\}$. Then, define $h:C \to I$ as follows.

$$h(x) = \text{Sup } \{q: \pi_q f(x) = 1\} ,$$

where $\pi_q: \Pi_{q \in Q} \{0,1\}_q \to \{0,1\}_q$ is the natural projection. Then, it is easy to see that h is lower semi-continuous.

Proof of (2) (due to Wage [1]). Put

$$A_1 = h^{-1}(0) \text{ and } B_1 = h^{-1}(1).$$

Let S be any separation between A_1 and B_1. Then, there exist two open sets U and V in X such that $U \cup V = X \setminus S$, $U \cap V = \emptyset$, $U \supset A$, and $V \supset B$.

Fix $x \in U$ and a positive integer n. Consider the following procedure for generating x', x", B', and B":

Let B be an ε_0-clopen neighborhood of x with its diameter $< 1/n$. Define

$$\lambda = \text{Sup } \{ h(y) : y \in B \cap U \}.$$

It follows from the definition of h that $\lambda > h(x)$.

Fix two distinct points x' and x" of $B \cap U$ such that $|h(x') - \lambda|$ and $|h(x") - \lambda|$ are less than $|h(x) - \lambda|/2$. (Note that $h(x')$, $h(x") > h(x)$.) Since h is lower semi-continuous, choose disjoint ε_0-clopen sets B' and B" such that diameters of B' and B" are less than $1/n+1$, $x' \in B'$, $x" \in B"$, $B' \cup B" \subset B$, and $h(t) > h(x)$ for every $t \in B' \cup B"$.

Fix $x_0 \in U$ and $n_0 = 1$. We can inductively generate sequences $\{x_n\}$ and $\{B_n', B_n''\}$ from x_0 and n_0 using the above procedure: Having chosed x_m for each $m < k$, fix $n_k = k$ and x_k as either x_{k-1}' or x_{k-1}''. Since we have two choises at each step, there are c possible such sequences. The construction guarantees that each sequence has a different (unique) limit point in the Euclidean topology ε_0. Let $x = \lim x_n$. By construction, $\{h(x_n)\}$ is monotone increasing. Hence, $h(x) \leq \lim h(x_n)$, since h is lower semi-continuous. On the other hand, by the definition of h, $h(x) \geq \lim h(x_n)$.
Therefore, $h(x) = \lim h(x_n)$, and hence $x \in Cl_{\tilde{\rho}}(U)$, since $x_n \in U$. We complete the proof of (2) by showing that $x \notin U$. Suppose that $x \in U$. Then, there are real numbers a, b, and an ε_0-clopen set B_n such that

$$x \in B_n \cap h^{-1}(a,b) \subset U.$$

This implies that $\lambda \geq b$ for each λ defined in the procedure above in the construction of $\{x_m\}_{m \geq n}$. But, this contradicts at some step, the condition that $|h(x') - \lambda|$ and $|h(x'') - \lambda|$ are less than $|h(x) - \lambda|/2$.

REMARK 3.1 (a) (due to Wage [1]). It is probably easist to think of $\tilde{\rho}$ as the restriction of the Euclidian topology on I^2 to the subset

$$\tilde{X} = \{(x,h(x)) : x \in C\}$$

(h(x) stands for the "height" of x in I^2).

(b) Comparing two Wage's metric spaces in [1, 2], we would say that the latter is better, since topologies on Cantor set C are more convenient to determine the dimension of itself, the factors, and products (see §§ 9, 10, 11).

Explanation of computer graphics I.

We shall explain here the following computer graphics given in the next page, which is performed by Mrs. K. Nogura. We believe that the graphs show some aspects of Wage's metric space given in the preceding section.

Let $\{q_i : 1 \leq i \leq N\}$ be a collection of N rational numbers. For each N-tuple $x = (x_1,\ldots,x_N)$ consisting of 0, 1 sequence we assign $\tilde{\phi}(x)$ by the following formula:

$$\tilde{\phi}(x) = \text{Max}\ \{q_i : 1 \leq i \leq N \text{ and } x_i = 1\}.$$

We also put

$$x^* = \Sigma_{i=1}^{N}\ x_i/2^i.$$

Let

$$C^* = \{x^* : x \in \{0,1\}^N\}.$$

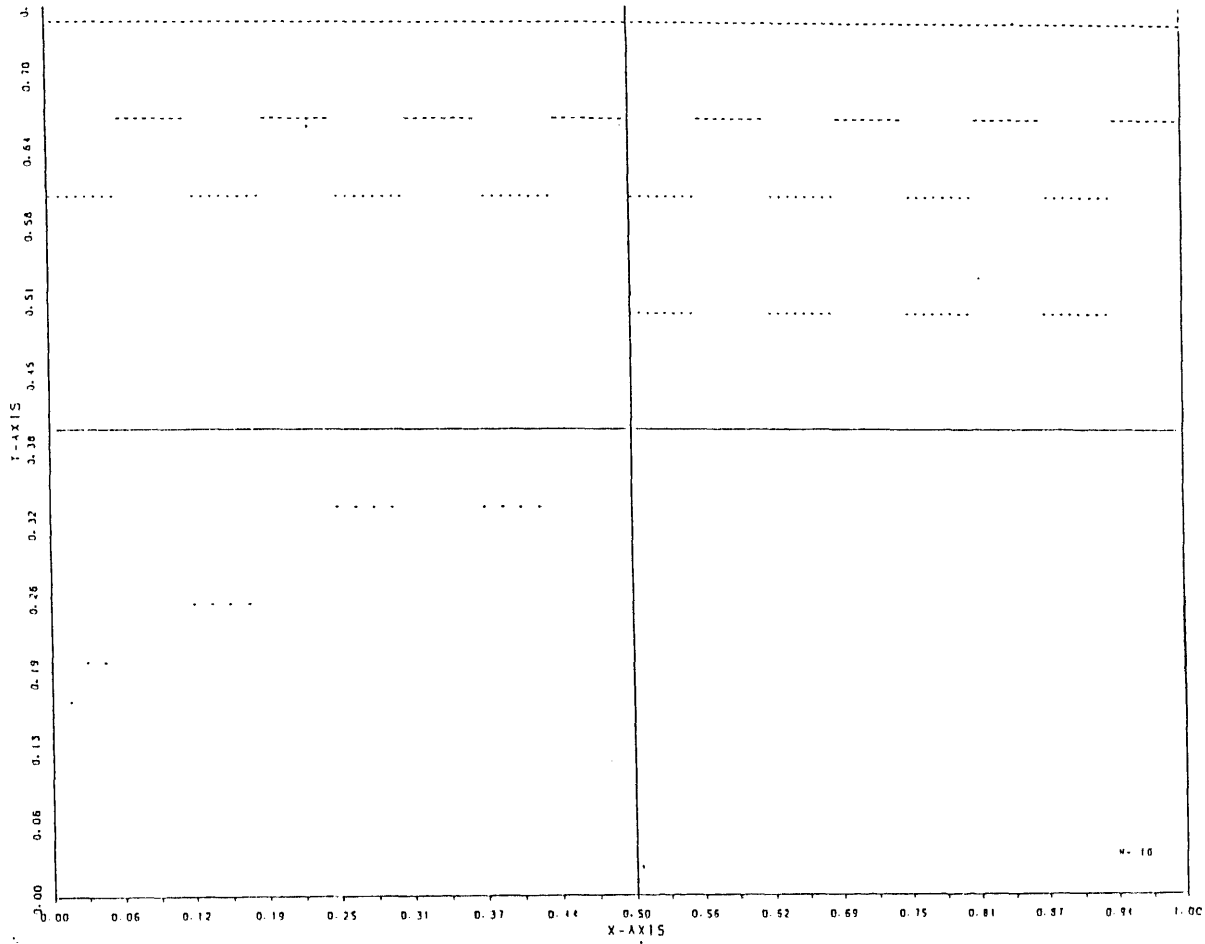
Then, the graphs consist of

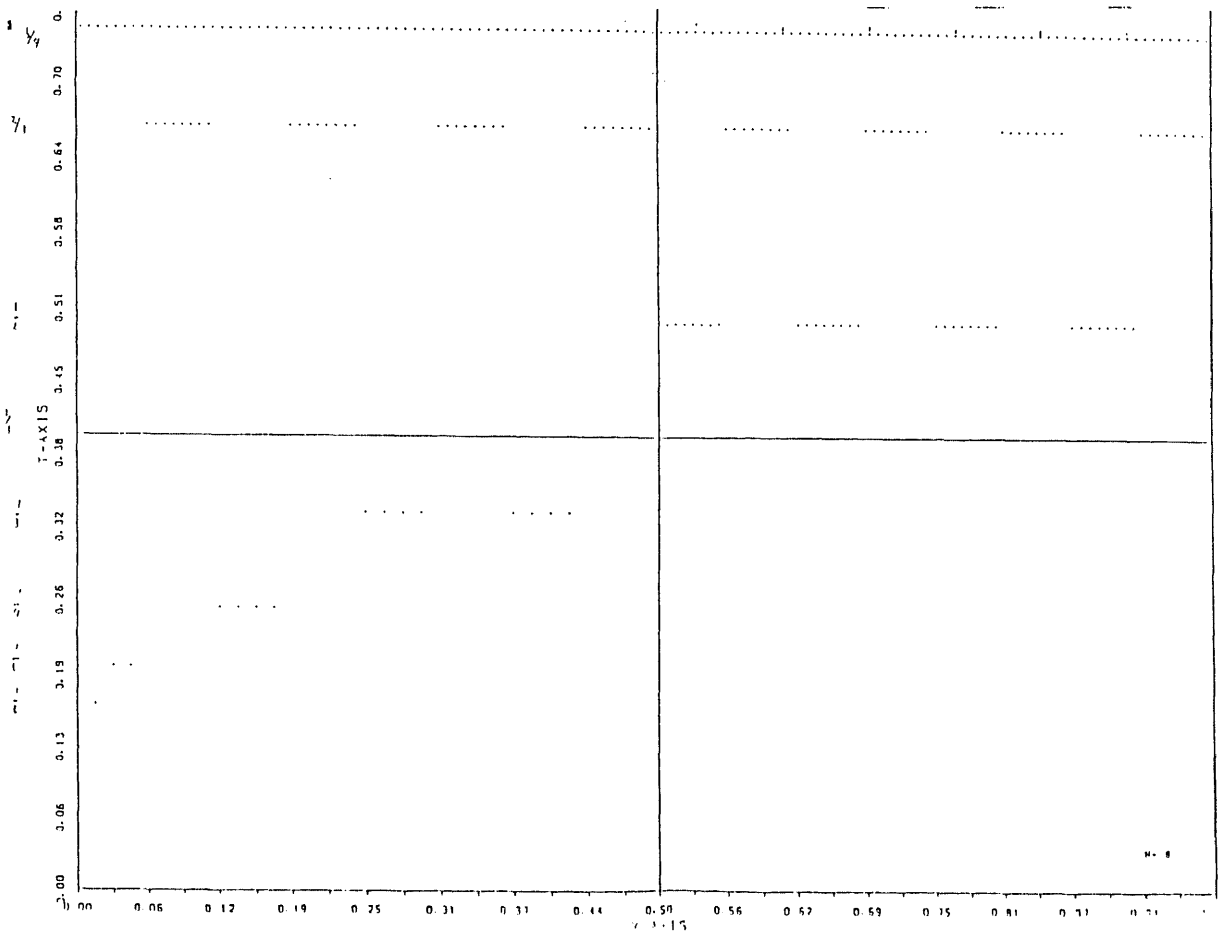
$$\tilde{G}^* = \{(x^*, \tilde{\phi}(x)) : x^* \in C^*\},$$

where N = 8 in the bottom half, and N = 10 in the upper half, respectively. (We put $(q_1,\ldots,q_{10}) = (1/2,1/3,1/4,2/3,1/5,1/6,2/5,3/4,3/5,1/7)$.) We may think that the graph $\tilde{G}^*$ is the restriction of the Wage's metric space in Theorem 3.1 to the following subset

$$\{(x,h(x)) : x \in \{0,1\}^N \times \{0\} \times \{0\} \times \ldots\},$$

where the first N-terms of $\{q_i\}$ in Theorem 3.1 are equal to the above ones.


Y-AXIS
X-AXIS
M= 10


Y-AXIS
M= 8

For each i let $\mathcal{B}_i$ be a finite disjoint clopen cover of (C,ε_0) with its mesh 1/i, and let each $\mathcal{B}_{i+1}$ refines $\mathcal{B}_i$. Then, since h is lower semi-continuous, there exists a monotone increasing continuous functions $\{h_i : C \to I\}$ such that $h_i(x) \uparrow h(x)$ and $h_i \mid B$ is constant for each $B \in \mathcal{B}_i$. Now, we define a function $H:C^2 \to I$ as follows.

$$H(x,y) = \begin{cases} h(x) \text{ if } x = y, \\ 0 \quad \text{if } (x,y) \notin \cup\{B^2 : B \in \mathcal{B}_1\}, \\ h_i(x) = h_i(y) \\ \text{if } (x,y) \in \cup\{B^2 : B \in \mathcal{B}_i\} \setminus \cup\{B^2 : B \in \mathcal{B}_{i+1}\}. \end{cases}$$

Let ρ be the topology on C^2 generated by the subbase $\varepsilon_0 \cup \{H^{-1}(U) : U \in \varepsilon\}$. Then, (C^2, ρ) is homeomorphic to the set $\{((x,y),H(x,y)) : (x,y) \in C^2\}$ with the restriction of the Euclidean topology on $C^2 \times I$. Hence,

THEOREM 3.2. (C^2, ρ) is a complete separable metric space satisfying

(1) $\tilde{\rho} = \rho \mid \Delta$, where $\Delta = \{(x,x) : x \in C\}$;

(2) $\rho \mid C^2 \setminus \Delta$ is homeomorphic to usual Euclidean topology on it;

(3) both topologies $\rho \mid (\{t\} \times C)$ and $\rho \mid (C \times \{t\})$ are homeomorphic to ε_0;

(4) ρ is coaser than both topologies $\tilde{\rho} \times \varepsilon_0$ and $\varepsilon_0 \times \tilde{\rho}$.

We say that the above extension (C^2, ρ) is a natural extension of Wage's metric space $(C, \tilde{\rho})$. In Wage [1,2] diffrent kinds of extensions are used, and the property (4) is not mentioned there. We give here its proof, since it is deeply concerned in constructing our examples with a pseudocompact (or, metric) factor.

Proof of (4). We shall show that ρ is coaser than $\tilde{\rho} \times \varepsilon_0$, (the proof for $\varepsilon_0 \times \tilde{\rho}$ goes on in a parallel way). It suffices to show that for any point (x,x) and a given $\delta > 0$ there exists a ρ-open neighborhood U of x and an ε_0-open neighborhood B of x such that

(i) $|H(y,z) - H(x,x)| < \delta$ for every $(y,z) \in U \times B$.

Since $h_i(x) \uparrow h(x)$, there exists an n such that $0 \leq h(x) - h_n(x) < \delta/2$. Let $B \in \mathcal{B}_j$, $j \geq n$, be an ε_0-open neighborhood of x such that $|h_n(y) - h_n(x)| < \delta/2$ for every $y \in B$. Let $U \subset B$ be a $\tilde{\rho}$-neighbornood of x such that $|h(x) - h(z)| < \delta/2$ for every $z \in U$. Then, note that $H(y,z) \leq h(y)$ and $|h_n(x) - h(y)| < \delta$ for $(y,z) \in U \times B$. Thus, for $(y,z) \in (Y \times B) \cap \Delta$, (i) holds, since $H(y,z) = h(y)$. Let $(y,z) \in U \times B \setminus \Delta$, and put $H(y,z) = h_k(y) = h_k(z)$ for $k \geq j$. Then, if $h(y) > h(x)$, (i) holds, since $h_n(y) \leq H(y,z) \leq h(y) \leq h(x) + \delta/2$. If $h(y) \leq h(x)$, then (i) holds, since

$$h_n(y) \leq H(y,z) \leq h(y) \leq h(x).$$

REMARK 3.2. The above natural extension, using an increasing sequence of continuous functions, is a slight modification of the construction given by Przymusiński [3]. The extension suggested in Wage [1,2] are too complicated to verfify the property (4) in Theorem 3.2.

Explanation of computer grapics II.

We shall explain here the following computer graphics in the next page, together with the ones given in the front pages of every section. We believe that they show some outlines of the natural extension given in the preceding section as well as the extension given in Wage [2].

We shall use same notations given in Explanation I. For each $x, y \in \{0,1\}^N$, we define

$\phi(x,y) = \text{Min}\{\phi(z)$: for all z such that $x^* \leq z^* \leq y^*\}$.

Then, the graph consists of

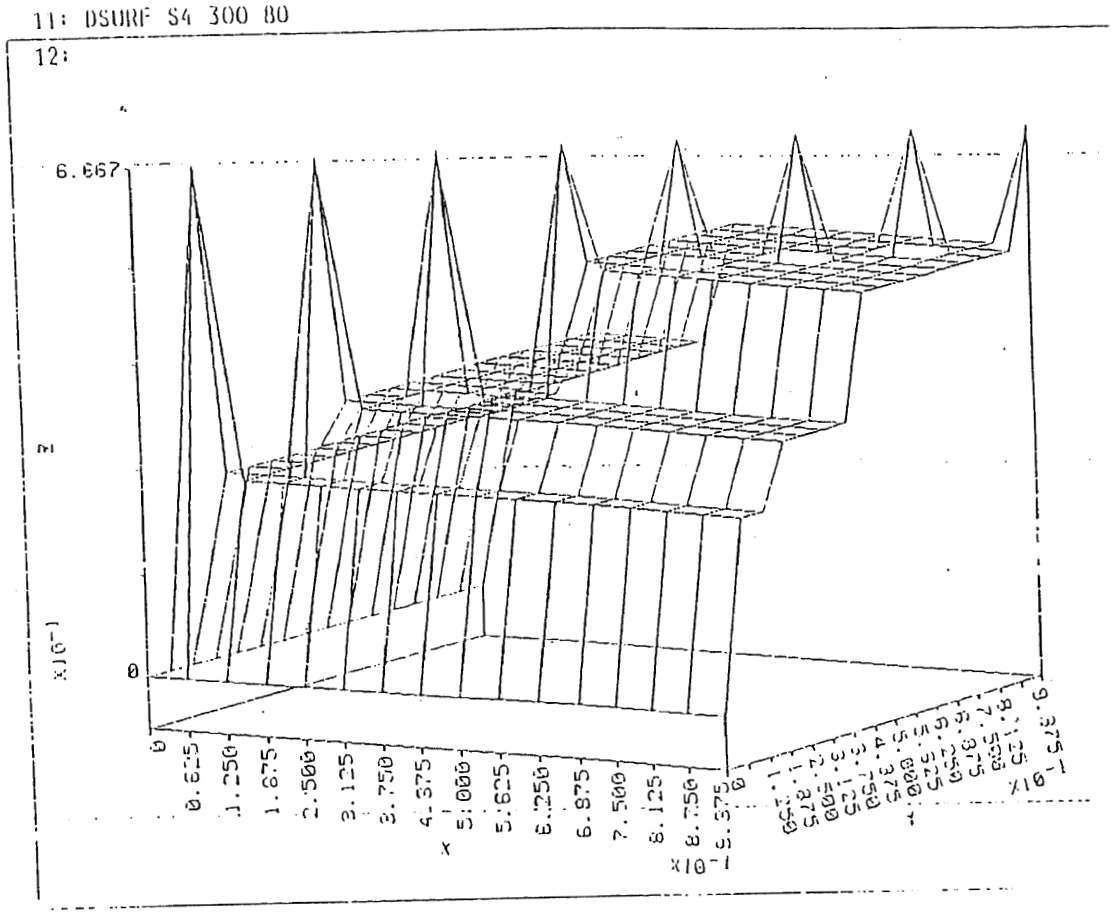
$$G^* = \{(x^*, y^*, \phi(x,y)) : x,y \in \{0,1\}^N\}.$$

The one given in the upper half of the next page is performed for the case $N = 4$. The bottom half is the corresponding Wage's metric space given in explanation I.

The graphics in every section are the ones performed for the case $N = 10$. In this case, however, they only show the graphs around the diagonal (they consist of 9 pieces), since the whole graphs need the capacity beyond our computer.

We may regard that the graph G^* is the restriction of the natural extention of Wage's metric space to the subset

$$\{(x,y,H(x,y)) : x,y \in \{0,1\}^N \times \{0\} \times \{0\} \times \ldots\}.$$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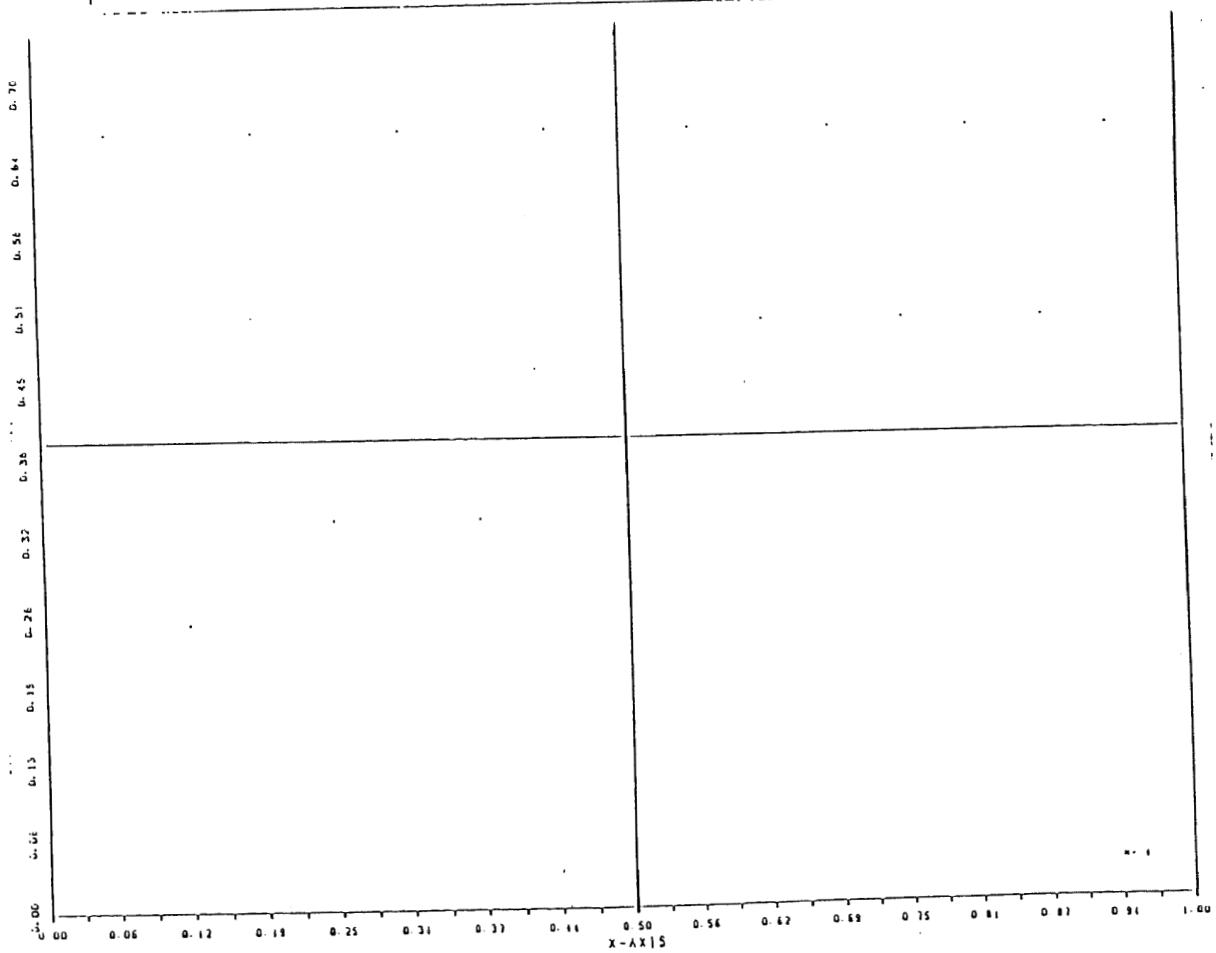
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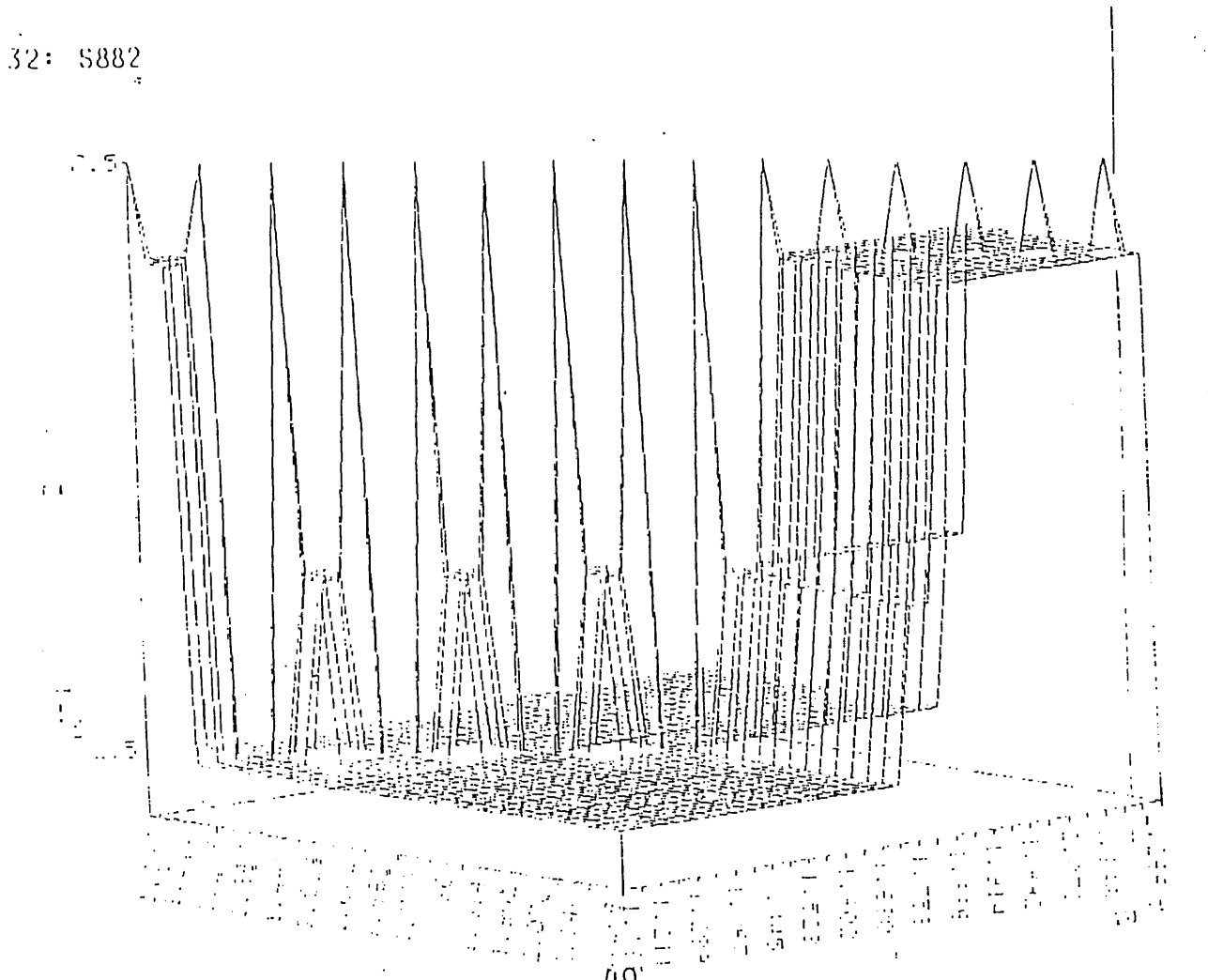
X-AXIS

4. n-dimensional Wage's metric spaces.



4. n-dimensional Wage's metric spaces.

We begin with the following two lemmas due to Mazurkiewicz [1].

LEMMA 4.1. Let $h:C \to I^n$ be a Baire function of class 1. Then, the graph $G = \{(x,h(x)):x \in C\}$ is a G_δ-set of I^{n+1} and hence, is completely metrizable.

LEMMA 4.2. There exists a function $\eta : \mathrm{Exp}(I^n) \to I^n$ such that

(a) $\eta(M) \in M$ for each $M \in \mathrm{Exp}(I^n)$,

(b) if $M_k \supset M_{k+1}$ and $M_k \in \mathrm{Exp}(I^n)$ for each $k \in \omega$, then $\lim_{k\to\infty} \eta(M_k) = \eta(\bigcap_{k=1}^{\infty} M)$ in I^n.

Proof of Lemma 4.2. Let $\psi : C \to I$ be an order preserving continuous surjective map (for example, see Engelking [2, Problem 1.3.D]). Put

$$\phi = \psi^n : C^n \to I^n.$$

Let us regard C^n as a subset of the real line. Put for each $M \in \mathrm{Exp}(I^n)$

(*) $\eta(M) = \phi(\min(\phi^{-1}(M)))$,

where minimum is taken with respect to the usual order of the real line. Note that our definition (*) is well-defined, since $\min(\phi^{-1}(M)) \in C^n$ for every above M. Then, one can show that η satisfies (a), (b).

THEOREM 4.1. (An n-dimensional Wage's metric space). For every $n = 1,2,\ldots,\infty$, there exists a separable completely metrizable, n-dimensional space $(C, \tilde{\rho}(n))$ such that

(0) $\tilde{\rho}(n)$ is the topology generated by the subbase $\varepsilon_0 \cup \{h^{-1}(U) : U \in \varepsilon_0\}$, where $h:C \to I^n$ is a Baire function of class 1, and h is lower semi-continuous when $n = 1$;

(1) there exists a collection $\{(A_i,B_i) : i = 1,\ldots,n\}$ of n-pairs of disjoint closed sets which satisfies that, if for each i, a closed set S_i separates A_i and B_i, then the set $\bigcap_{i=1}^{n} S_i$ has the cardinality of continuum.

Proof. At first, we consider the case $n < \infty$. Put

$$\tilde{A}_{n+1} = \pi_1^{-1}(0) \text{ and } \tilde{B}_{n+1} = \pi_1^{-1}(1),$$

where $\pi_1:I^{n+1} \to I_1$ is the natural projection to the first factor. Let $\mathcal{T}$ be the collection of all continua meeting both $\tilde{A}_{n+1}$ and $\tilde{B}_{n+1}$ in I^{n+1}. Then, $\mathcal{T}$ is closed in $\mathrm{Exp}(I^{n+1})$, and is compact metrizable, and hence there exists a continuous map g from (C, ε_0) onto $\mathcal{T}$ such that

(2) $|g^{-1}(T)| = c$ for any $T \in \mathcal{T}$.

By the definition of $\mathcal{T}$ we note that

(3) for each $p \in C$ we have $T \cap \pi_1^{-1}(p) \neq \emptyset$ for any $T \in \mathcal{T}$.

Let $\Pi^n : I^{n+1} \to I^n$ be the projection to the last n factors. Then, using Lemma 4.2 and property (3) we can obtain a function $h:C \to I^n$ such that

$$h(p) = \eta(\Pi^n(g(p) \cap \pi_1^{-1}(p)))$$

for each $p \in C$. Put

$$\tilde{X} = \{(p,h(p)) : p \in C\}.$$

Then, from (2) and (3) we have

(4) $|T \cap \tilde{X}| = c$ for each $T \in \mathcal{T}$.

To see that h satisfies (0), let $\{\mathcal{B}_i\}$ be a sequence of clopen disjoint covers of (C, ε_0) with its mesh 1/i and we assume that $\mathcal{B}_{i+1}$ refines $\mathcal{B}_i$. Define a function $h_k : C \to I^n$ by the following formula:

$$h_k(t) = \eta(\Pi^n(\cup\{g(p) \cap \pi_1^{-1}(p) : p \in B\}))$$

for each $t \in B \in \mathcal{B}_k$. Since g is continuous and B is clopen, the set $\cup\{g(p) \cap \pi_1^{-1}(p) : p \in B\}$ is closed in I^{n+1}. Hence $h_k : (C,\varepsilon_0) \to I^n$ is well-defined and is continuous. By Lemma 4.2(b) we can see that h is the limit function of $\{h_k\}$, so that h is a Baire function of class 1, and hence X is a G_δ-set in I^{n+1} by Lemma 4.1.

When n = 1, one can show without difficulty that h is lower semi-continuous, since $\{h_k\}$ is monotone increasing in this case by the definition (*) in Lemma 4.2.

To verify the property (1), let

$A_k = \tilde{X} \cap \pi_{k+1}^{-1}[0,1/7]$ and $B_k = \tilde{X} \cap \pi_{k+1}^{-1}[6/7,1]$

for each $k = 1,\ldots,n$. Then, each A_k and B_k are non-empty, since $\pi_{k+1}^{-1}(0)$ and $\pi_{k+1}^{-1}(1)$ are continua meeting both $\tilde{A}_{n+1}$ and $\tilde{B}_{n+1}$. Let S_k be a separation of A_k and B_k for each k and let $\tilde{S}_k$ be a separation of

$$\tilde{A}_k = \pi_{k+1}^{-1}(0) \text{ and } \tilde{B}_k = \pi_{k+1}^{-1}(1)$$

which extends S_k (that is, $\tilde{S}_k$ separates $\tilde{A}_k$ and $\tilde{B}_k$ in I^{n+1} and $\tilde{S}_k \cap \tilde{X} \subset S_k$). Then, since $\cap \tilde{S}_k$ contains a continuum meeting both $\tilde{A}_{n+1}$ and $\tilde{B}_{n+1}$ by Rubin-Schori-Walsh [1], there exists a $T \in \mathcal{T}$ such that

$$T \subset \cap \tilde{S}_k \text{ and } T \cap \tilde{X} \subset \cap \tilde{S}_k \cap \tilde{X} \subset \cap S_k.$$

From (4) $|\tilde{X} \cap T| = c$ so that $|\cap S_k| = c$, and hence for the case $n < \infty$ the theorem has been established.

For the case $n = \infty$ we replace I^{n+1} by the Hilbert cube I^{∞}. Then, the construction goes on analogously.

REMARK 4.1. (a) The above theorem is a slight improvement of Tsuda [1, Theorem 2.1]. Since $(C,\tilde{\rho}(1))$ in our Theorem 4.1 satisfies Theorem 3.1, we believe that our Theorem 4.1 is an n-dimensional version of Theorem 3.1.

(b) Using Lemma 4.1, Mazurkiewicz [1] has constructed n-dimensional totally disconnected, separable, complete metric space. The lemma is also used elsewhere efficiently (see Krasinkiewicz [1]).

(c) Comparing our spaces with other spaces used in Engelking-Pol [1] or charalambous [2], ours have several advantages. In the former paper they used a separable complete metric space due to R. Pol [1] which is not A-weak infinite dimensional. In their setting their space satisfies our theorem (see Section 5), and we believe that the property (1) is the key point in their case, too. We also note that we cannot apply their method for the finite dimensional case (see Section 5 why), because of lacking the property (1).

On the other hand, though we can obtain his result much easier (see Section 8), we cannot use the space in Charalambous [2] to produce Wage-type examples, because his space is not completely metrizable, and hence we cannot obtain the following natural extension for his space.

We shall embed $(C,\tilde{\rho}(n))$ in the natural extension $(C^2,\rho(n))$ as follows: Using the functions $\{h_k\}$ in the proof of Theorem 4.1, we get a function $H : C^2 \to I^n$ and the topology $\rho(n)$ in the same manner given in § 3. Therefore, we have:

THEOREM 4.2. For every $n = 1,2,\ldots,\infty$, $(C^2,\rho(n))$ is a complete separable metric space satisfying

(1) $\rho(n)|\Delta = \tilde{\rho}(n)$, where $\Delta = \{(x,x) : x \in C\}$;

(2) $\rho(n)$ $C^2 \setminus \Delta$ is homeomorphic to the usual Euclidean topology on it;

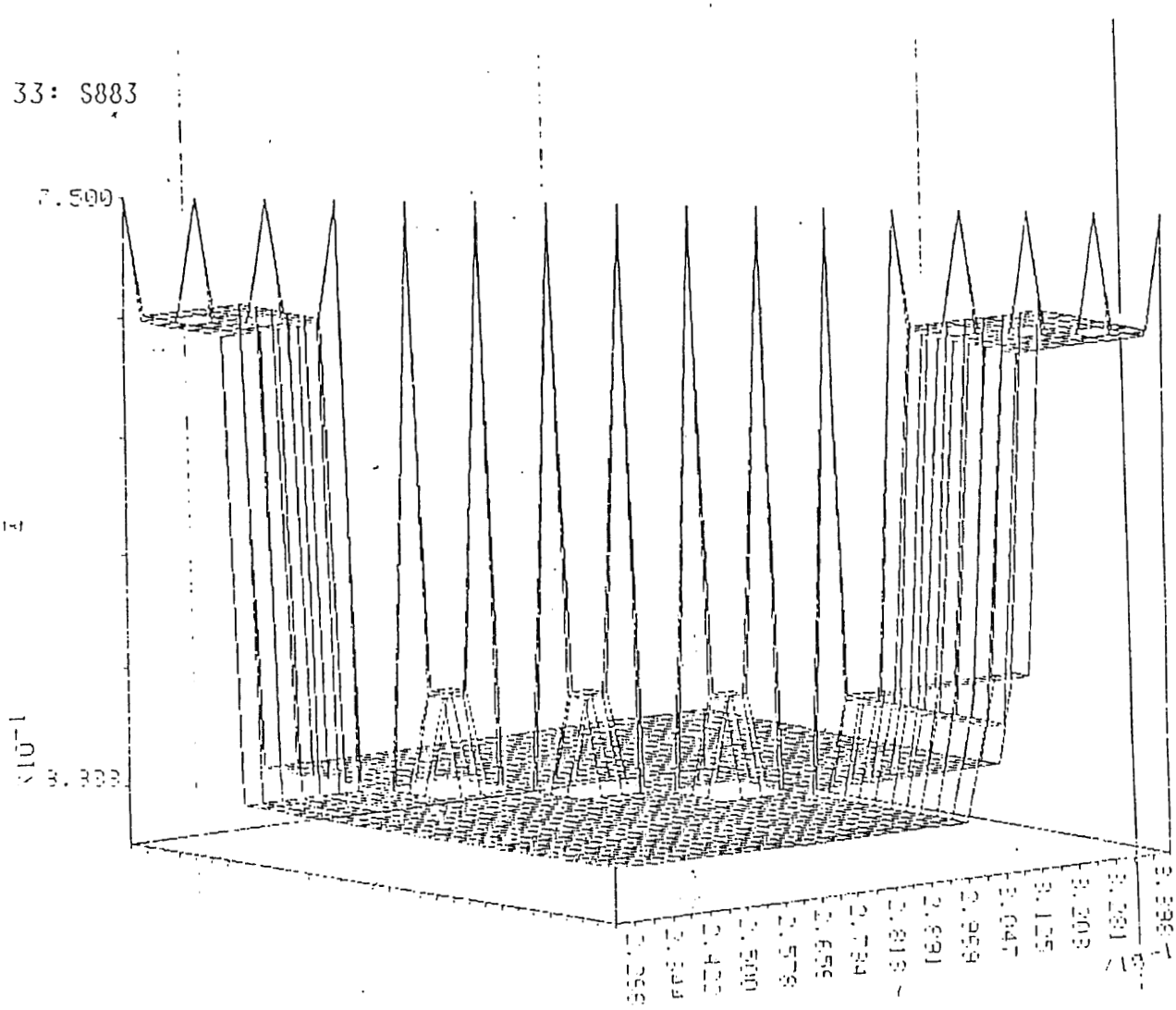
(3) both topologies $\rho(n)|(\{t\} \times C)$ and $\rho(n)|(C \times \{t\})$ are homeomorphic to ε_0;

(4) when $n = 1$, $\rho(n)$ is coaser than both $\rho(n) \times \varepsilon_0$ and $\varepsilon_0 \times \rho(n)$.

REMARK 4.2. (a) The extension space given in Engelking & Pol[1] is some modification of the one used in Wage [1]. Therefore, when $n = 1$, our method has an advantage (see §3).

(b) It is an open problem whether or not the n-fold product $(C,\rho)^n$ of Wage's metric space (C,ρ) in Theorem 3.1 satisfies the condition (2) in Theorem 4.1. If it were the case, there could be a space satifying Theorem 4.2 together with the condition (4) for every n.

5. Some consequences of Kunen construction and those of van Douwen construction.



5. Some consequences of Kunen construction and those of van Douwen construction.

In the following 1) we assume the continuum hypothesis, CH. While, in 2) we do not assume any set-theoretic assumptions beyond ZFC.

1) Some consequences of Kunen construction.

In this place we review the outline of the machine given in Juhász, Kunen & Rudin[1] for refining given topologies. While it is the technique for some general topologies, we specify it for our Wage's metric topologies $\tilde{\rho}(n)$ on C. Let $\{x_\alpha : \alpha < \omega_1\}$ be a well-ordering of C, and put $X_\alpha = \{x_\beta : \beta < \alpha\}$. Then, we have:

LEMMA 5.1 (CH). There exists an enumeration $\{S_\alpha : \alpha < \omega_1\}$ of all countable subsets of C (respectively, C^2) such that $S_\alpha \subset X_\alpha$ (respectively, $S_\alpha \subset (X_\alpha)^2$).

Proof. Let $\mathcal{T} = \{T_\alpha : \alpha < \omega_1\}$ be a well-ordering of the collection of all countable subsets of C. Then, for each α we define an ordinal $f(\alpha) < \omega_1$ by the following formula:

$$f(\alpha) = \min \{\beta : \beta \notin \{f(\gamma) : \gamma < \alpha\} \text{ and } T_\alpha \subset X_\beta\}.$$

Then, since each T_α is countable, $f(\alpha)$ is well-defined. It is easy to see that $f:\omega_1 \to \omega_1$ is bijective. Put

$$S_\alpha = T_{f^{-1}(\alpha)}.$$

Then, it is easy to see that $\{S_\alpha : \alpha < \omega_1\}$ is a required well-ordering. The assertion for C^2 follows in a way parallel.

It is shown in Juhász, Kunen & Rudin[1] that (see also § 6) there exists a topology $\tilde{\nu}$ on C such that

(i) $\tilde{\nu}$ is finer than $\tilde{\rho}(n)$,

(ii) for each $\alpha < \beta$, if $x_\beta \in Cl_{\tilde{\rho}(n)}(S_\alpha)$, then $x_\beta \in Cl_{\tilde{\nu}}(S_\alpha)$.

From (ii) we see (Juhász, Kunen & Rudin[1]) that

(#) if A is $\tilde{\nu}$-closed, then the set $Cl_{\tilde{\rho}(n)}(A) \setminus A$ is countable.

We believe that the above property (#) is one of the most important properties of Kunen construction, so that we conclude this section to see the following lemma, which uses (#) extensively:

LEMMA 5.2. $\dim (C,\tilde{\nu}) = \mathrm{Ind}\,(C,\tilde{\nu}) = n$. Moreover, for the case $n = \infty$ $(C,\tilde{\nu})$ is strongly infinite dimensional.

Proof. Let $\{(A_i,B_i) : 1 \leq i \leq n\}$ be the n-pairs of disjoint $\tilde{\rho}(n)$-closed sets in Theorem 4.1 (1). Then, take n-pairs of $\tilde{\rho}(n)$-open sets $\{(U_i,V_i)\}$ such that

(2) $A_i \subset U_i$, $B_i \subset V_i$, and $M_i \cap N_i = \emptyset$, where

$M_i = Cl_{\tilde{\rho}(n)}(U_i)$ and $N_i = Cl_{\tilde{\rho}(n)}(V_i)$.

Then, M_i and N_i are $\tilde{\nu}$-closed, since (i) holds. Let T_i be a closed separation between M_i and N_i in $(C,\tilde{\nu})$. Then, put $M_i \subset G_i$, $N_i \subset H_i$, $C \setminus T_i = G_i \cup H_i$, for some open disjoint sets G_i and H_i. Let

$S_i = Cl_{\tilde{\rho}(n)}(T_i) \cup (Cl_{\tilde{\rho}(n)}(G_i) \cap Cl_{\tilde{\rho}(n)}(H_i))$.

Then, S_i is a $\rho(n)$-closed separation between A_i and B_i from (2). It follows from (#) that

(3) the set $S_i \setminus T_i$ is countable.

Hence, the set $\cap S_i$ has the cardinality of continuum from Theorem 4.1 (1). Therefore, the set $\cap T_i$ also has the cardinality of continuum from (3). Hence, $\dim (C,\tilde{\nu}) \geq n$. On the other hand, it follows from Fedorčuk [1, Lemmas 7,9] that $\dim (C,\tilde{\nu}) \leq \mathrm{Ind}\ (C,\tilde{\nu}) \leq n$. This completes the proof.

REMARK 5.1. (a) We follow Engelking & Pol [1, p. 16] to say that the construction given here "Kunen construction". It is also called "Juhasz-Kunen-Rudin construction" in E. Pol [1].

(b) As mentioned in Remark 4.1 (c) our condition (1) in Theorem 4.1 is crucial at least for finite dimensional cases. Indeed, let (C,ρ) be the n-dimensional space in Charalambous [2] or that of Engelking & Pol [1], which does not necessarily satisfy (1). Then, we cannot

determine exactly the dimension of Kunen construction (C,ν) for it. As is shown in Fedorčuk [1] (see also Gruenhage & Pol[1]), we can only say that

$n-1 \leq \dim (C,\nu) \leq n$ and $n-1 \leq \mathrm{Ind}\,(C,\nu) \leq n$.

(c) Let (Δ,ρ) be the strongly infinite dimensional complete separable metric space given in Engelking & Pol [1]. Then, it is known that its Kunen construction is strongly infinite dimensional (see Gruenhage & Pol[1]). Moreover, we have:

PROPOSITION 5.1. Every strongly infinite dimensional complete metric space satisfies the property (1) in Thoerem 4.1.

Proof. Let $\{(A_i,B_i) : i \in \omega\}$ be a countable collection of pairs of disjoint closed sets satisfying

(4) if a closed set S_i saparates A_i and B_i for each i, then $\bigcap S_i \neq \emptyset$.

We show that the collection $\{(A_{i+1},B_{i+1}) : i \in \omega\}$ satisfies (1) in Theorem 4.1. Indeed, let S_i be a closed separation between A_i and B_i. We must show that the set $\bigcap_{i\geq 1} S_i$ has the cardinality of continuum.

On the contrary, assume that $|\bigcap_{i\geq 1} S_i| < c$. Then, $\bigcap_{i\geq 1} S_i$ is countable, since every uncountable complete separable metric space has the cardinality of continuum. Hence,

dim $(\bigcap_{i\geq 1} S_i) \leq 0$, and therefore, there exists a separation S_0 between A_0 and B_0 which does not meet the set $\bigcap_{i\geq 1} S_i$. Then, the collection $\{S_i : i \in \omega\}$ does not satisfy (4). This contradiction completes the proof.

2) Some consequences from van Douwen construction.

In this place we review the techniques of van Douwen [1], which does not require the continuum hypothesis. As in the preceding section, we specify the construction for our Wage's metric topologies $\tilde{\rho}(n)$, though it can be applied for some general topologies (see also § 8).

One of the key points of van Douwen construction is

(i) it is a topology $\tilde{\tau}$ on C finer than $\tilde{\rho}(n)$,

(ii) if A and B are disjoint τ-closed sets, then the set $Cl_{\tilde{\rho}(n)}(A) \cap Cl_{\tilde{\rho}(n)}(B)$ is countable.

Though the property (ii) is weaker than (ii) in Kunen's, we can deduce from it that $(C,\tilde{\tau})$ is countably paracompact, collectionwise normal (see §10). Here, we shall show the following lemma, using (ii):

LEMMA 5.3 (Tsuda [1, Lemma 5.6]). When $n = \infty$ $(C,\tilde{\tau})$ is not strongly countable dimensional.

Proof. If we assume that $(C,\tilde{\tau})$ is strongly countable-dimensional, there exists a countable collection $\{H_i\}$ of closed subspaces of $(C,\tilde{\tau})$ such that $C = \cup H_i$, and $\dim H_i \leq m_i$ for some integer m_i for each i. By Theorem 4.1 there exists a countable collection $\{(A_i,B_i)\}$ of pairs of disjoint $\tilde{\rho}(n)$-closed sets such that

(1) if for each i, $\tilde{\rho}(n)$-closed set S_i separates A_i and B_i, then $|\cap S_i| = c$.

Take disjoint $\tilde{\rho}(n)$-open sets U_i and V_i satisfying Lemma 5.2 (2). Then, since $\dim H_i \leq m_i$, for each m_i+1 pairs

$$\{(M_k,N_k) : i + \Sigma_{j=1}^{i-1} m_j \leq k \leq i + \Sigma_{j=1}^{i} m_j\},$$

there exists a zero set separation T_k of $H_i \cap M_k$ and $H_i \cap N_k$ in H_i such that

(2) $\cap \{T_k : i + \Sigma_{j=1}^{k-1} m_j \leq k \leq i + \Sigma_{j=1}^{i} m_j\} = \emptyset$.

By (i) we have a $\tilde{\rho}(n)$-closed separation $\tilde{S}_k$ of $A_k \cap H_i$ and $B_k \cap H_i$ such that

(3) $|\tilde{S}_i \setminus T_i| \leq \omega$, and $\tilde{S}_i \subset (C \setminus (M_i \cup N_i))$.

By the hereditarily normality of $\tilde{\rho}(n)$, there exists a $\tilde{\rho}(n)$-closed separation S_k of A_k and B_k such that $\tilde{S}_k = S_k \cap H_i$. Then, from (1) $|\cap S_k| = c$, and from (3) $|\cap \tilde{S}_k| = c$. Since $C = \cup H_i$, there exists an H_i such that $|H_i \cap (\cap S_k)| = c$. On the other hand, for each k such that $i + \Sigma_{j=1}^{i-1} m_j \leq k \leq i + \Sigma_{j=1}^{i} m_j$ we have $|\tilde{S}_k \cap H_i \setminus T_k| \leq \omega$ from (2), and hence from (3) we have

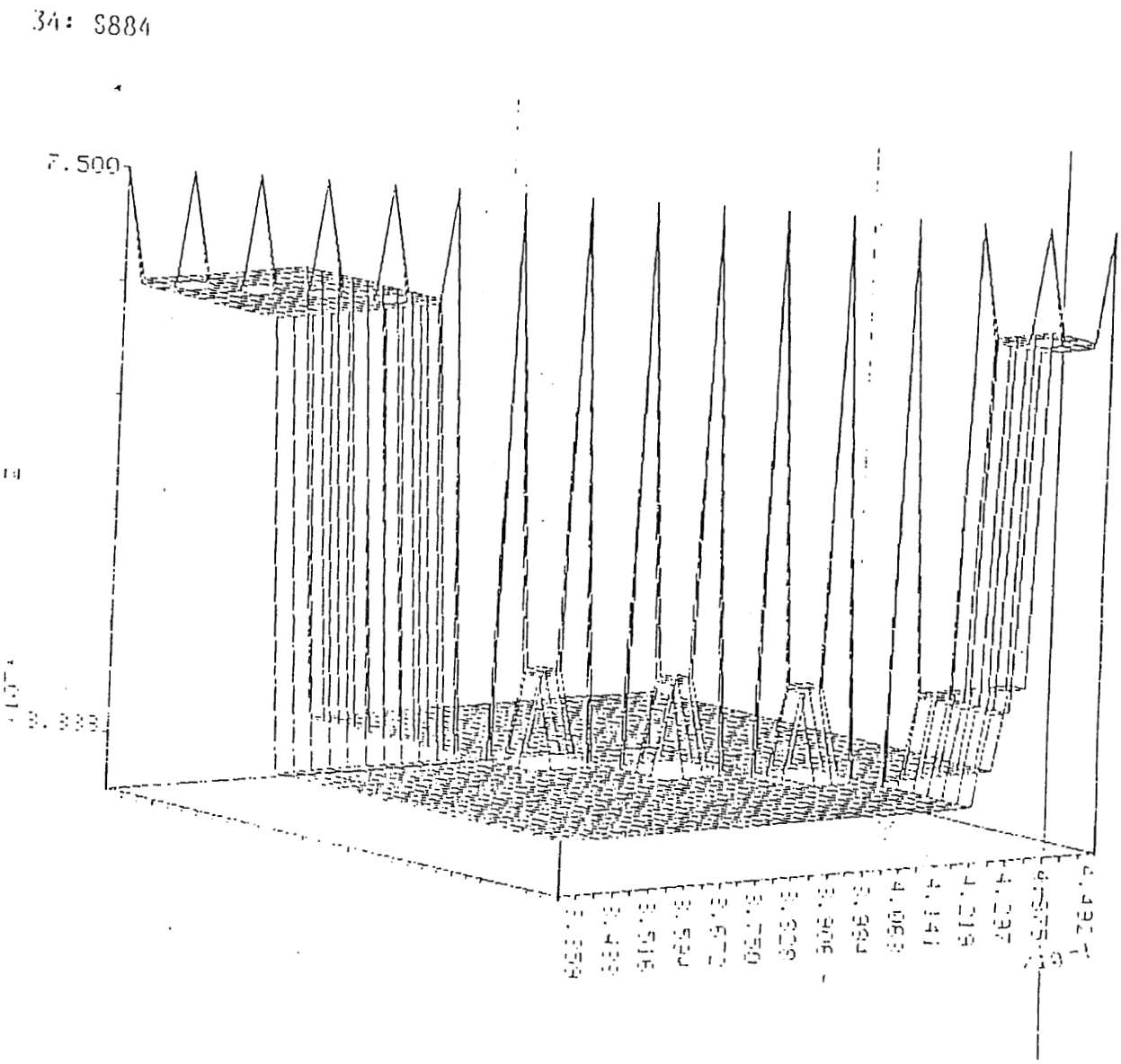
$|H_i \cap (\cap S_k)| \leq \omega$. This contradiction shows that $(C,\tilde{\nu})$ is not strongly countable-dimensional.

REMARK 5.1. Recently, we learned in Engelking & Pol [1] that every normal strongly countable-dimensional space is A-weakly infinite-dimensional. Moreover, it is known that (van Douwen [2, Theorem 3.4 (a)]) a normal space which is the union of countably many closed A-weak infinite dimensional subspaces is A-weak infinite dimensional. Therefore, our lemma also follows from their results.

By a way parallel in the above lemma we have:

LEMMA 5.4. $\dim (C,\tilde{\tau}) = \mathrm{Ind}\ (C,\tilde{\tau}) = n$ when $n < \infty$.

6. Wage's factorization technique.



6. Wage's factorization technique.

In this section we approximate $\rho(n)$ in §4 by the method of Wage sugested in Wage [1,2]. (Note that we follow the improved method due to Przymusiński [5]. See also Remark 6.1 (a).) Our aim is to show that there exist two topologies $\nu(i)$ on C such that

(i) $\nu = \nu(1) \times \nu(2)$ is finer than $\rho(n)$, and

(ii) if A and B are disjoint ν-closed sets, then $|Cl_{\rho(n)}(A) \cap Cl_{\rho(n)}(B)|_2 \leq \omega$, where $|S|_2$ denotes the 2-cardinality of the set $S \subset C^2$ defined in Przymusiński [2].

Since we want to make our factors same (that is, $X = Y$), we also make our topologies satisfy

(iii) if A and B are disjoint $\nu(i)^2$-closed sets, then $|Cl_{\varepsilon_0^2}(A) \cap Cl_{\varepsilon_0^2}(B)|_2 \leq \omega$.

To achieve ν satisfying (i)-(iii), we need the following lemma, which is also used for other examples in §§10,11.

LEMMA 6.1. Let G_m be the (1/m)-neighborhood of $z_0 = (x_0, x_0)$ with respect to the metric $\rho(n)$, and let D be a countable $\tilde{\rho}(n)$-dense set in C. Let (r_m) and $(z_m = (s_m, t_m))$ be two sequences convergent to the point z_0 in $\rho(n)$ and in ε_0^2, respectively. Then, there exist two sequences (x_m), (y_m) in C and three sequences of descreasing ε_0-clopen sets

$\{H_m\}$, $\{U_m\}$, $\{V_m\}$ such that

(1) $(x_0,y_m) \in H_m \times V_m \subset G_m, (x_m,x_0) \in U_m \times H_m \subset G_m$;

(2) $D^2 \cap (X_m \times Y_m)$ and $R_m \cap (X_m \times Y_m)$ are non-empty;

(3) $Z_m \cap X_m^2$, $Z_m \cap Y_m^2$ are also non-empty,

where P_m denotes the set $\{p_k : k \geq m\}$, for a sequence (p_k).

Proof. We show the assertion by induction on k, and suppose that we have chosen x_i, y_i, H_i, U_i, V_i, for $i < 6k$. We shall choose them for all m satisfying $6k \leq m < 6(k+1)$. By Thoerm 4.2(3) we can always choose ε_0-clopen set $H_m' \subset H_{m-1}$ such that

$$H_m' \times \{x_0\} \cup \{x_0\} \times H_m' \subset G_m.$$

(a) For $m = 6k$ (respectively, $m = 6k+1$) we define x_m and y_m(respectively, x_{m+1} and y_{m+1}) by taking any point $r_k = (x_m,y_m) \in R_m \cap G_m \cap (H_m')^2$ (respectively, any point $(x_m,y_m) = G_m \cap D^2 \cap (H_m')^2$), since (r_k) is $\rho(n)$-converging (respetively, D^2 is $\rho(n)$-dense). Then, take any H_{m+i}, U_{m+i}, V_{m+i} satisfying (1), $i = 0,1$.

(b) For $m = 6k+2$, we define x_m, x_{m+1} by taking any point $z_k = (x_m,x_{m+1}) \in Z_m \cap (H_m')^2$. Take ε_0-clopen sets H_{m+i}, U_{m+i} such that $H_m' \supset H_m \supset H_{m+1}$, $(x_{m+i},x_0) \in U_{m+i} \times H_{m+i} \subset G_{m+i}$. Take any point $y_{m+i} = t_k \in H_{m+i}$, since (t_k) is ε_0-converging to x_0. Let V_{m+i} be any ε_0-clopen set satisfying (1), $i = 0,1$.

(c) For $m = 6k+4$, define (y_m, y_{m+1}) by taking any point $z_k = (y_m, y_{m+1}) \in Z_m \cap (H_m')^2$. Take ε_0-clopen sets H_{m+i}, V_{m+i} such that $H_m' \supset H_m \supset H_{m+1}$, $(x_0, y_{m+i}) \in H_{m+i} \times V_{m+i} \subset G_{m+i}$. Then, take any point point $x_{m+i} = s_k \in H_{m+i}$, since (s_k) is ε_0-converging to x_0. Let U_{m+i} be any ε_0-clopen set satisfying (1), $i = 0,1$.

This completes the proof of our lemma.

In the sequel, we assume CH and show that there <u>is</u> a topology satisfying (i)-(iii).

Since we assume CH, let $\{x_\alpha : \alpha < \omega_1\}$ be a well-ordering of C and let

$$X_\alpha = \{x_\beta : \beta < \alpha\}.$$

It follows from Lemma 5.1 that the family of all countable subsets $\{A_\alpha : \alpha < \omega_1\}$ of C^2 can be well-ordered so that

$$A_\alpha \subset (X_\alpha)^2.$$

For each x_α we define two neighborhood bases $\{N_m(x_\alpha)_i\}$ $i = 1,2$, so that the following inductive assumptions (0)-(3) are satisfied:

(0) $N_m(x_\alpha)_i \cap D \neq \emptyset$ for each m,i, where D is a fixed countable $\tilde{\rho}(n)$-dense set.

(1) The $\rho(n)$-diameter of $B_m = N_m(x_\alpha)_1 \times N_m(x_\alpha)_2$ is less than 1/m.

(2) The ε_0-diameter of $N_m(x_\alpha)_i$ is less than 1/m, $N_m(x_\alpha)_i$ is a compact set consisting of countable points, and for any $y \in N_m(x_\alpha)_i$ there exists an integer k such that $N_k(y)_i \subset N_m(x_\alpha)_i$.

(3) Let $\nu(i,\alpha)$ be the topology on X_α whose local base is $\{N_m(x_\beta)_i : \beta < \alpha$ and $m \in \omega\}$. Then, for every $i = 1,2$, and $\beta,\gamma < \alpha$

(a) if $(x_\alpha,x_\alpha) \in Cl_{\rho(n)}(A_\gamma)$, then $A_\gamma \cap B_m \neq \emptyset$ for any $m \in \omega$;

(b) if $(x_\alpha,x_\alpha) \in Cl_{\varepsilon_0{}^2}(A_\gamma)$, then $A_\gamma \cap (N_m(x_\alpha)_i)^2 \neq \emptyset$ for any $m \in \omega$;

(c) if $(x_\alpha,x_\beta) \in Cl_{\varepsilon_0 \times \nu(i,\alpha)}(A_\gamma)$, then $A_\gamma \cap B_m \neq \emptyset$ for any $m \in \omega$;

(d) if $(x_\beta,x_\alpha) \in Cl_{\nu(i,\alpha) \times \varepsilon_0}(A_\gamma)$, then $A_\gamma \cap B_m \neq \emptyset$ for any $m \in \omega$.

Suppose that two such neighborhood bases have been constructed for each x_β such that $\beta < \alpha$.

Let R^1_α be the collection of D and all A_γ such that $\gamma < \alpha$ and $(x_\alpha,x_\alpha) \in Cl_{\rho(n)}(A_\gamma)$.

Let R^2_α be the collection of all A_γ such that $\gamma < \alpha$, $A_\gamma \notin R^1_\alpha$, and $(x_\alpha,x_\alpha) \in Cl_{\varepsilon_0{}^2}(A_\gamma)$.

Let S^i_α $(i = 1,2)$ be the collection of all A_γ such that $\gamma < \alpha$, and $(x_\alpha,x_\beta) \in Cl_{\varepsilon_0 \times \nu(i,\alpha)}(A_\gamma)$ for some $\beta < \alpha$.

Let T^i_α $(i = 1,2)$ be the collection of all A_γ such that $\gamma < \alpha$, and $(x_\beta,x_\alpha) \in Cl_{\nu(i,\alpha) \times \varepsilon_0}(A_\gamma)$ for some $\beta < \alpha$.

Put

$$K_\alpha = \bigcup_{i=1}^{2} R_\alpha^i \cup S_\alpha^i \cup T_\alpha^i.$$

Then, take a convergent set $\lambda(A)$ for each $A \in K_\alpha$, which converges to one of the points (x_α, x_α), (x_α, x_β), and (x_β, x_α) according to the collection it belongs to. Put

$$\zeta(A) = \pi_1(\lambda(A)) \text{ and } \xi(A) = \pi_2(\lambda(A)),$$

where $\pi_i : C^2 \to C$ is the projection to the i-th factor. Then, by Lemma 6.1 we can choose a sequence $\langle (x_m, x_m) \rangle$ converging to (x_α, x_α), which satisfies the following (4):

(4) Let G_m be the (1/m)-neighborhood of (x_α, x_α) with respect to the metric $\rho(n)$. Then, there exist ε_U-clopen sets H_m, U_m, and V_m such that

(a) $(x_m, y_m) \in G_m$, $(x_\alpha, y_m) \in H_m \times V_m \subset G_m$, and $(x_m, x_\alpha) \in U_m \times H_m \subset G_m$;

(b) $H_m \supset H_{m+1}$ and $H_m \supset U_{m+1}, V_{m+1}$;

(c) $x_m \neq x_k$ and $y_m \neq y_k$ for $m \neq k$;

(d) let $\lambda = \{(x_m, y_m)\}$, $\zeta = \{x_m\}$, $\xi = \{y_m\}$, then

(1) $\lambda \cap \lambda(A) = \emptyset$ for each $A \in R_\alpha^1$,

(2) $\zeta^2 \cap \lambda(A) \neq \emptyset$ and $\xi^2 \cap \lambda(A) \neq \emptyset$ for each $A \in R_\alpha^2$,

(3) $\zeta \cap \zeta(A) \neq \emptyset$ for each $A \in S_\alpha^i$,

(4) $\xi \cap \xi(A) \neq \emptyset$ for each $A \in T_\alpha^i$;

(e) when $S_\alpha^i = \emptyset$ or $T_\alpha^i = \emptyset$, we put $U_m = V_m = \emptyset$, and we do not define x_m, y_m for the corresponding m.

Since x_m, $y_m < x_\alpha$, there exist two integers i(m) and j(m) by our inductive assumptions such that

$$N_{i(m)}(x_m)_1 \subset U_m,\ N_{j(m)}(y_m)_2 \subset V_m, \text{ and}$$
$$N_{i(m)}(x_m)_1 \times N_{j(m)}(y_m)_2 \subset G_m.$$

Put

$$N_m(x_\alpha)_1 = \{x_\alpha\} \cup (\cup \{N_{i(k)}(x_k)_1 : k \geq m\}),$$

and

$$N_m(x_\alpha)_2 = \{x_\alpha\} \cup (\cup \{N_{j(k)}(y_k) : k \geq m\}).$$

Then, we shall show that (1) holds. By our construction

$$L_1 = \bigcup_{k \geq m} N_{i(k)}(x_k)_1 \times N_{j(k)}(y_k)_2 \subset \bigcup_{k \geq m} G_k,$$
$$L_2 = N_{i(m)}(x_m)_1 \times (\bigcup_{k \geq m} N_{j(k)}(y_k)_2) \subset N_{i(m)}(x_m)_1 \times H_m,$$
$$L_3 = \bigcup_{k \geq m} (N_{i(k)}(x_k)_1 \times (\bigcup_{m \leq s \leq k} N_{i(s)}(y_s)_2)) \subset \bigcup_{m \leq s < k} H_s \times V_s,$$

and

$$L_4 = \bigcup_{k \geq m} (N_{i(k)}(x_k)_1 \times (\bigcup_{s \geq k} N_{j(k)}(y_k)_2)) \subset \bigcup_{k \geq m} U_k \times H_k.$$

Therefore, (1) holds, since $B_m = \{(x_\alpha, x_\alpha)\} \cup (\bigcup_{i=1}^{4} L_i)$. It is easy to see that $\{N_m(x_\beta)_i : \beta \leq \alpha,\ m \in \omega\}$, $i = 1,2$, satisfy all the remaining conditions. Thus, our inductive construction is completed. Let $\nu(i)$ be the topology whose local base is $\{N_m(x)_i : x \in C\}$. Then, they satisfy that

(5) $\nu = \nu(1) \times \nu(2)$ is finer than $\rho(n)$;

and

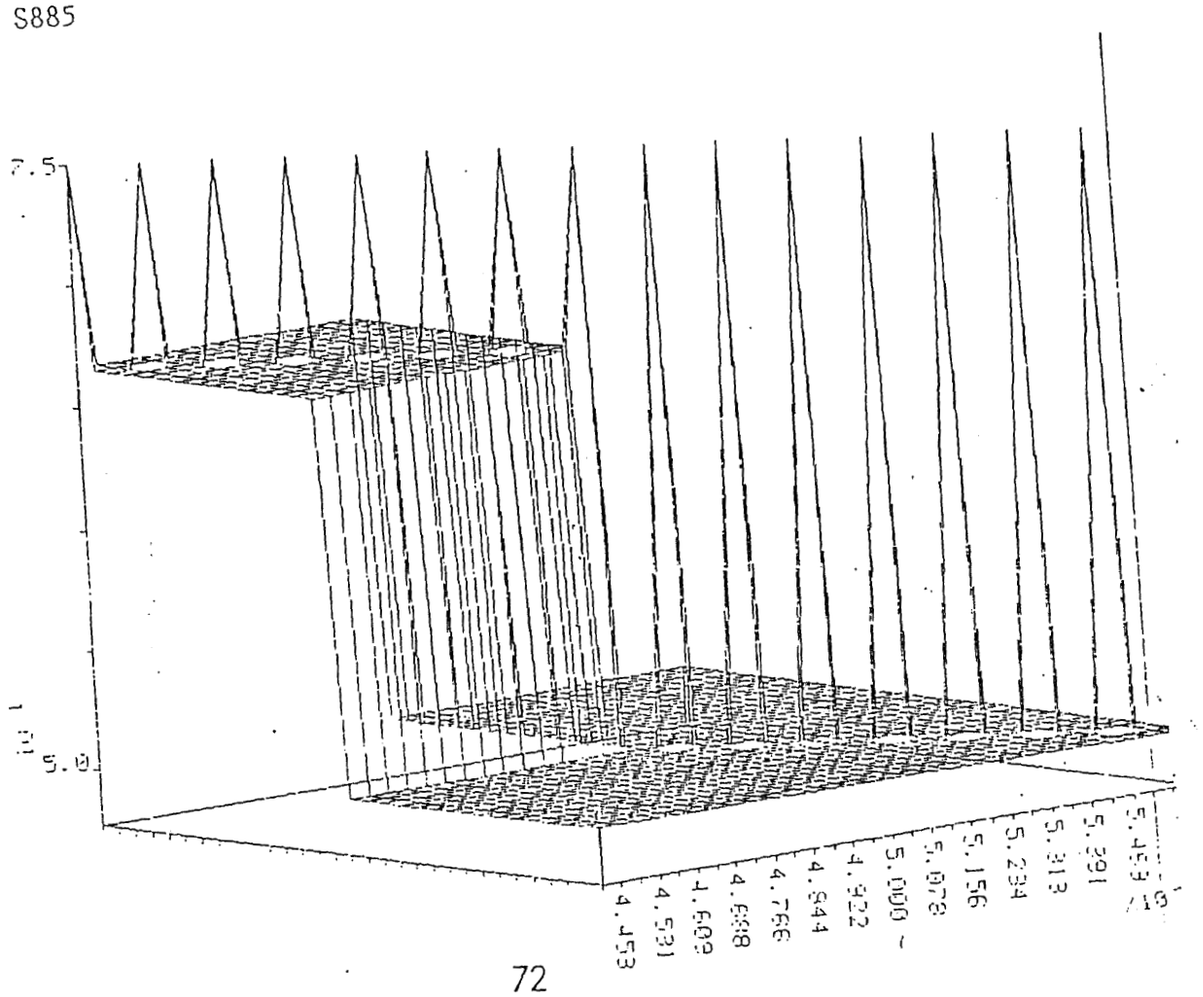
(6) $|Cl_{\rho(n)}(A) \setminus A|_2 \leq \omega$ for any ν-closed set A.

From (3)(b)

(7) $|Cl_{\varepsilon_0^2}(A) \setminus A|_2 \leq \omega$ for any δ_i^2-closed set A, where $\delta_i = \nu(i)$, $i = 1,2$.

REMARK 6.1. (a) In the above construction we use the method due to Przymusiński [5], since the procedure suggested in Wage [1] is not sufficient (see, Remark 9.1).
(b) There are some applications of Wage's factorization technique outside of dimension theory (Rudin [2], Wage [1,2]).
(c) Be careful that there are <u>other</u> famous "factorization " theorems in dimension theory (for example, that of Mardešić [1] and that of Pasynkov [3]). The meaning "facotrization" in there is different from ours.

7. Przymusiński's method.



7. Przymusiński's method.

In this section we shall define topologies τ_i on C so that $\tau = \tau_1 \times \tau_2$ approximates $\rho(n)$. Our method is due to Przymusiński [3], which eliminated the continuum hypothesis from the argument in the preceding section, and is based on a technique of van Douwen instead of Kunen.

For our inductive construction we need the following subset P of C^2. Let $F = \{(A,B)_\lambda : \lambda \in \Lambda\}$ be the collection of all pairs of countable subsets of C^2 such that

(0) $|Cl_{\rho(n)}(A) \cap Cl_{\rho(n)}(B)|_2 > \omega$.

Note that for any $(A,B)_\lambda \in F$ we have by Przymusiński [2]

$$|Cl_{\rho(n)}(A) \cap Cl_{\rho(n)}(B)|_2 = c.$$

In the sequel let $<$ be a well-ordering of C and let fix a $\rho(n)$-dense countable set D. Then, define for $x \in C$

$$I(x) = \{y \in C : y < x\} \cup D.$$

Then, for each $\lambda \in \Lambda$, since $|A \cup B| = \omega$, there exists an $x_\lambda \in C$ such that $A \cup B \subset I(x_\lambda)^2$, where $(A,B) = (A,B)_\lambda$. Hence, we can construct a transfinite sequence $\langle p_\lambda \rangle$ of points in C^2 satisfying

$p_\lambda \in Cl_{\rho(n)}(A) \cap Cl_{\rho(n)}(B) \setminus I(x_\lambda)^2$, where $(A,B) = (A,B)_\lambda$, and

if $\lambda \neq \mu$, then $\hat{p}_\lambda \cap \hat{p}_\mu = \emptyset$, where $\hat{p}$ denotes the set $\{p_1, p_2\}$ of C for $p = (p_1, p_2)$. Put

$$P = \{p_\lambda : \lambda \in \Lambda\}.$$

Since D is a countable $\rho(n)$-dense set of C, in a way parallel in the preceding section we can choose a sequence $<(x_m,y_m)>$ for each $p \in P$, satisfying the following condition (1).

(1) Let G_m be the (1/m)-neighborhood of $p = p_\lambda = (x_0,y_0)$ with respect to $\rho(n)$. Then,

(a) $(x_m,y_m) \in G_m$, $x_m \neq x_k$ and $y_m \neq y_k$ for $m \neq k$, and $\{(x_{3m},y_{3m})\} \subset A$, $\{(x_{3m+1},y_{3m+1})\} \subset B$, where $(A,B) = (A,B)_\lambda$ and $\{(x_{3m+2},y_{3m+2})\} \subset D$;

(b) there exist ε_0-clopen sets H_m, K_m, U_m, and V_m such that $p \in H_m \times K_m$, $(x_0,y_m) \in H_m \times V_m \subset G_m$, and $(x_m,y_0) \in U_m \times K_m \subset G_m$;

(c) $H_m \supset H_{m+1}$, $K_m \supset K_{m+1}$, $H_m \supset U_{m+1}$, and $K_m \supset V_{m+1}$.

Now, by transfinite induction on $<$ we assign to each point of C two neighborhood bases $\{B_m(x)_i\}$, $i = 1,2$ so that the following inductive assumptions (2) and (3) are satisfies.

(2) Each $B_m(x)_i$ is a compact set consisting of countable points, and for any $y \in B_m(x)_i$ there exists a k such that $B_m(x)_i \supset B_k(y)_i$.

(3) The diameter of $B_m = B_m(x)_1 \times B_m(x)_2$ is less than 1/m with respect to the metric $\rho(n)$.

Let each point of D be isolated, and suppose that $x \in C$ and such bases of neighborhoods have been constructed for each $y < x$. Then, we consider the following cases.

(a) $x = p_1 = p_2$ for some (unique) $p \in P$.

(b) $x = p_i$ for some (unique) $p \in P$ and $p_1 \neq p_2$.

(c) $x \neq p_i$ for any $p \in P$ and i.

In the cases (a) and (b) we have $x_m, y_m \prec p_i$ for $i = 1,2$ and $m \in \omega$, because $p_\lambda \notin I(x_\lambda)^2$ and $x_m, y_m \in I(x_\lambda)$. Thus inductive assumptions can be applied so that there exist integers $k(m)$ for each m such that

$B_{k(m)}(x_m)_1 \subset U_m$, $B_{k(m)}(y_m)_2 \subset V_m$, and

$B_{k(m)}(x_m)_1 \times B_{k(m)}(y_m)_2 \subset G_m$.

The last inclusion follows from the following observation. If $x_m = y_m$, then the inclusion follows from (3). If $x_m \neq y_m$, then $(x_m, y_m) \in \mathrm{Int}_{\varepsilon_0{}^2}(G_m)$. Therefore, the inclusion follows from the fact that both x_m and y_m have neighborhood bases finer than ε_0.

Case (a). Put for each $i = 1,2$ and $m \in \omega$

$$B_m(x)_i = \{x\} \cup (\bigcup_{t \geq m} B_{k(t)}(x_t)_i).$$

Case (b). If $x = p_1$, put for each $m \in \omega$

$B_m(x)_1 = \{x\} \cup (\bigcup_{t \geq m} B_{k(t)}(x_t))$, and $B_m(x)_2 = \{x\}$.

If $x = p_2$, put for each $m \in \omega$

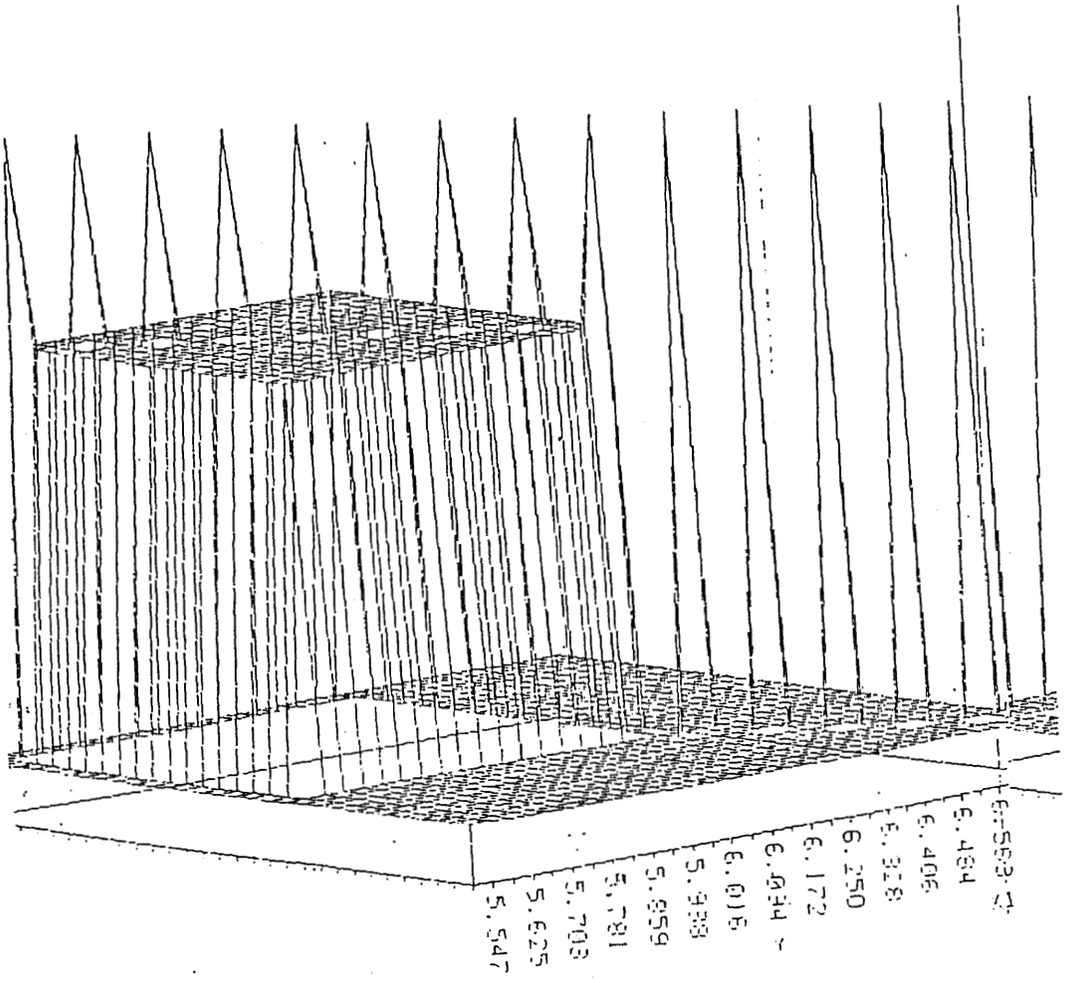
$B_m(x)_1 = \{x\}$, and $B_m(x)_2 = \{x\} \cup (\bigcup_{t \geq m} B_{k(t)}(y_t)_2)$.

Case (c). Put $B_m(x)_i = \{x\}$ for each $i = 1,2$ and $m \in \omega$.

Because there does not exist any decreasing chains in a well-ordered set and (1) holds, we can show that $\{B_m(x)_i\}$ satisfies (2) and (3) for each i in a way parallel in the preceding section. Thus, inductive construction has completed.

From (2) and (3) $\{B_m(x)_i : m \in \omega, x \in C\}$ constitutes a local base on C and it produces a Hausdorff topology τ_i for each i = 1,2. This completes the constructions of τ_1 and τ_2.

8. An application to inverse sequences.



8. An application to inverse sequences.

The purpose of this section is to give some application of the Wage's metric spaces in §4 and of the technique of van Douwen in §5. We begin with the following theorem due to Charalambous [1]:

THEOREM 8.1. There exists an inverse sequence $\{X_i, f_{ij}\}$ with the limit space X such that each X_i is 0-dimensional, first countable, separable, Lindelöf, while X is normal with dim X = 1.

REMARK 8.1. (a) We do not give here the proof of the theorem, since we shall show an n-dimensional version of it below which contains the above result as its corollary.

(b) Before Charalambous it is already known in Engelking [1, Problem 6.3.25] that

(#) the limit of an inverse sequence of strongly 0-dimensional spaces need not be strongly 0-dimensional.

Comparing these two constructions, Charalambous construction has the advantage of nice factor spaces X_i. On the other hand, by Engelking's construction we have a sequence with a nice limit space: there exists an inverse sequence, which satisfies (#), and whose limit is metrizable.

Indeed, let X be the Roy's 1-dimensional space. Since X has a clopen base of cardinality of continuum c, we can assume that X is a subspace of the generalized Cantor-set C^C. Take a suitable sequence $\{x_i\}$ of points in C^C such that the sequence $\{\Sigma(x_i)\}$ of Σ-products with base point x_i is mutually disjoint. Put

$$X_i = X \cup \{\Sigma(x_k) : k \geq i\}.$$

Then, each X_i is strongly 0-dimensional, since X_i is C*-embedded in C^C (Engelking [1, Problem 3.11.23(c)]).

Let $f_{ij} : X_j \to X_i$ be the inclusion map. Then, it is readily seen that X is the limit of the inverse sequence $\{X_i, f_{ij}\}$.

THEOREM 8.2. For every $k = 1,2,\ldots,\infty$ there exists an inverse sequence $\{X_i, f_{ij}\}$ with the limit space X such that each X_i is 0-dimensional, first countable, Lindelöf, while X is (separable) countably paracompact, collection-wise normal with dim X = Ind X = k.

Proof. Let $(C,\tilde{\rho}(k))$ be the k-dimensional Wage's metric space in Theorem 4.1. Then, take a sequence $\{E_i : i \in \omega\}$ of pairwise disjoint Bernstein sets satisfying that

(1) the set $A \cap E_i$ has the cardinality of continuum for any $\tilde{\rho}(k)$-closed uncountable set A.

Let τ_i be the approximation of the topology $\mu_i = \tilde{\rho}(k)|E_i$ on E_i due to van Douwen(see §5, 2)). Thus,

(2) τ_i is first countable, separable, having a clopen base, collectionwise normal, countably paracompact, and the set $Cl_{\mu_i}(A) \cap Cl_{\mu_i}(B)$ is countable whenever A and B are disjoint τ_i-closed sets.

Now, we shall define topologies υ_i on C, which is generated by the subbase $\varepsilon_0 \cup \{\tau_j : j \leq i\}$. Let υ_∞ be the topology on C which is generated by $\{\tau_j : j \in \omega\}$. Put for each i

$$X_i = (C, \upsilon_i), \text{ and let}$$

$$f_{ij}: X_j \to X_i \text{ be the identiy function.}$$

Then, by (1) and (2) each X_i is first countable, 0-dimensional, separable, and Lindelöf. By the definition, f_{ij} is continuous. Let X_∞ be the limit of the inverse sequence $\{X_i, f_{ij}\}$. Then, it can be seen from the construction that

$$X_\infty \simeq (C, \upsilon_\infty) \simeq \overset{\infty}{\underset{i=1}{\oplus}} (E_i, \upsilon_i).$$

Hence, by (2) X_∞ is (separable,) countably paracompact, collectionwise normal, and dim X_∞ = Ind X_∞ = k by the proof of Lemma 5.4.

REMARK 8.2. (a) The above construction is a slight improvement of the one given in Tsuda [1] (separability is not achieved there). In Charalambous [2] similar examples are given, independently. Comparing ours and his construction, we believe that our construction is much simpler. For example, we can show that the limit space is collectionwise normal and countably paracompact much easier than he does in Charalambous [2].
(b) When we consider more special case when the limit space is X^{ω} and its factors are X^{i}, we have no Wage-type example from the following theorem due to Nagami.

THEOREM 8.1 (Nagami [5]). Let each X_i be a normal space with $\dim X_i \leq n$. Let its limit X_∞ be countably paracompact and each bonding map f_{ij} be open. Then, X_∞ is a normal space with $\dim X_\infty \leq n$.

9. Perfect normality of products.

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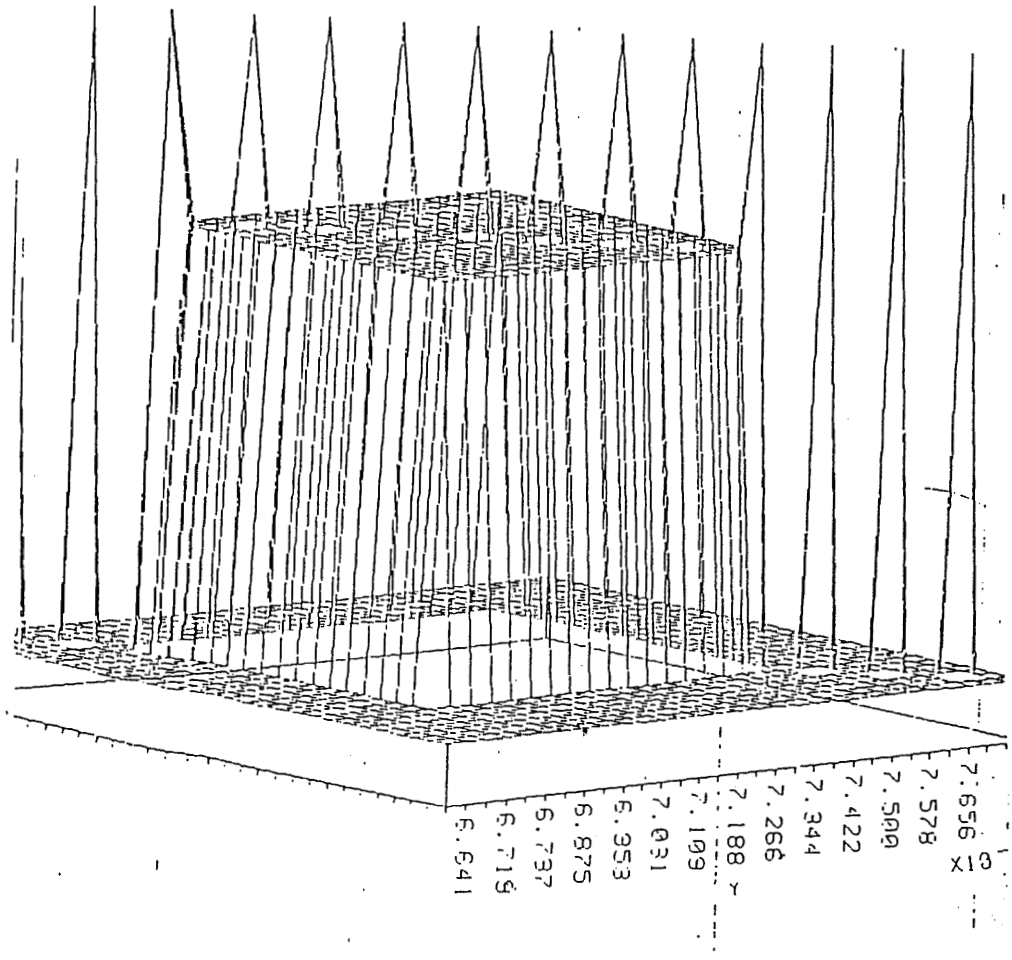


9. Perfect normality of products.

We shall show the following theorem under continuum hypothesis, CH:

THEOREM 9.1 (CH). For every $n = 1,2,\ldots,\infty$, there exists a space $X(n)$ such that

$$\dim X(n)^2 = \operatorname{Ind} X(n)^2 = n > \dim X(n) = 0,$$

and $X(n)^2$ is perfectly normal, separable, first-countable, and locally compact. Moreover, $X(\infty)^2$ is not countable dimensional.

Proof. Put

$$X(n) = (C,\nu(1)) \oplus (C,\nu(2)),$$

where $\nu(i)$ is defined in §6. Hence, $X(n)^2$ is separable, first-countable, locally compact, regular and 0-dimensional by the definition. Let us show that at first $X(n)^2$ is perfectly normal. We begin with the following lemma essentially due to Kunen.

LEMMA 9.1. For every i and every subset A of C there exists a $\gamma < \omega_1$ such that

if $\alpha \geq \gamma$ and $x_\alpha \in \mathrm{Cl}_{\varepsilon_0}(A)$, then $x_\alpha \in \mathrm{Cl}_{\nu(i)}(A)$.

LEMMA 9.2. For every $A \subset C^2$ there exists a $\lambda < \omega_1$ such that if $\alpha \neq \beta$, $\alpha,\beta \geq \lambda$, and $(x_\alpha,x_\beta) \in Cl_{\varepsilon_0^2}(A)$, then $(x_\alpha,x_\beta) \in (Cl_{\nu(k)\times\varepsilon}(A)) \cap (Cl_{\varepsilon\times\nu(k)}(A))$, $k = 1,2$.

LEMMA 9.3. For every $A \subset C^2$ there exists a γ such that if $(x_\alpha,x_\beta) \in Cl_{\rho(n)}(A)$ and $\alpha,\beta \geq \gamma$, then $(x_\alpha,x_\beta) \in Cl_\nu(A)$.

LEMMA 9.4. For every i and $A \subset C^2$ there exists a γ such that if $(x_\alpha,x_\beta) \in Cl_{\varepsilon_0^2}(A)$ and $\alpha,\beta \geq \gamma$, then $(x_\alpha,x_\beta) \in Cl_{\nu(i)^2}(A)$.

Proof of Lemma 9.1. Let L be a countable Euclidean dense subset of A. Then, by (3) (a) in §6, if $x_\alpha \in Cl_{\varepsilon_0}(A)$ with $\alpha > \gamma$, then $(x_\alpha,x_\alpha) \in Cl_{\rho(n)}(A^2) = Cl_{\rho(n)}(L^2)$. Since $L^2 = A_\gamma$ for some γ, each neighborhood of x_α in $\nu(i)$ contians a subset of L by the definition of $\nu(i)$ and (3) (a) in §6.

The proofs for the remaining lemmas follow from (5), (6) in §6 and by ways parallel to Przymusiński [5, Lemmas 1,2,3].

Now, we shall show that $X(n)^2$ is perfectly normal. At first, we show that $X(n)$ is perfectly normal by a method due to Juhász, Kunen and Rudin [1].

It suffices to see that each $\nu(i)$ is perfectly normal. Let A be a $\nu(i)$-closed set. Then, the set $Cl_{\varepsilon_0}(A) \setminus A$ is countable by Lemma 9.1. Since $Cl_{\varepsilon_0}(A)$ is ε_0-G_δ and hence $\nu(i)$-G_δ, so A is a $\nu(i)$-G_δ. Let H, K be $\nu(i)$-closed disjoint. To see that they can be separated, it is sufficient to produce a countable cover of $(C,\nu(i))$ by $\nu(i)$-open sets U such that $Cl_{\nu(i)}(U)$ intersects at most one of H and K; call such U "nice". By Lemma 9.1 the set $S = Cl_{\varepsilon_0}(H) \cap Cl_{\varepsilon_0}(K)$ is countable, and around each its points we may put a nice U. Since $C \setminus S$ is ε_0-Lindelöf, we may cover it with a countable collection of nice U (which are in fact ε_0-open and whose ε_0-closure intersect at most one of H and K). Thus, these two collections together produce the desired cover. Therefore, X(n) is perfectly normal.

We see that $X(n)^2$ is perfectly normal by a way parallel to the above observation, since ν is finer than $\rho(n)$, $|Cl_{\rho(n)}(A) \setminus A|_2 \leq \omega$ for any ν-closed set A, and $|Cl_{\varepsilon_0{}^2}(A) \setminus A|_2 \leq \omega$ for any $\nu(i)^2$-closed set A (see §6 (i) - (iii)).

By Lemma 5.2 we have $\dim X(n)^2 \geq n$. (It is easy to see that $\dim X(n) = 0$.) By Fedorčuk [1, Lemmas 7,9] $\dim X(n)^2 \leq \operatorname{Ind} X(n)^2 \leq n$. By Lemma 5.2 $X(\infty)^2$ is not countable dimensional. This completesthe proof.

REMARK 9.1. (1) Theorem 9.1 solves a problem raised in Fedorčuk [2, Problem 6] affirmatively. Similar example is outlined in Engelking and Pol [1, Example 4.1].

(2) Our present construction of X(n) is a slight improvement of the construction given in Tsuda [2]. The main differences are

(i) our space X(n) is separable,

(ii) while two different factor spaces X and Y are constructed in Tsuda [2], we can construct here one satisfying that X = Y.

(3) Since the product of $X(n)^2$ and k-dimensional cube I^k is perfectly normal (Engelking [1, Problem 4.5.16(b)]), we have: For every n = 1,2,3,..., ∞, and

for every pair of non-negative integers (ℓ,m)

there exist spaces X and Y such that

(a) dim X = ℓ and dim Y = m, while dim (X × Y) = m+n+ℓ;

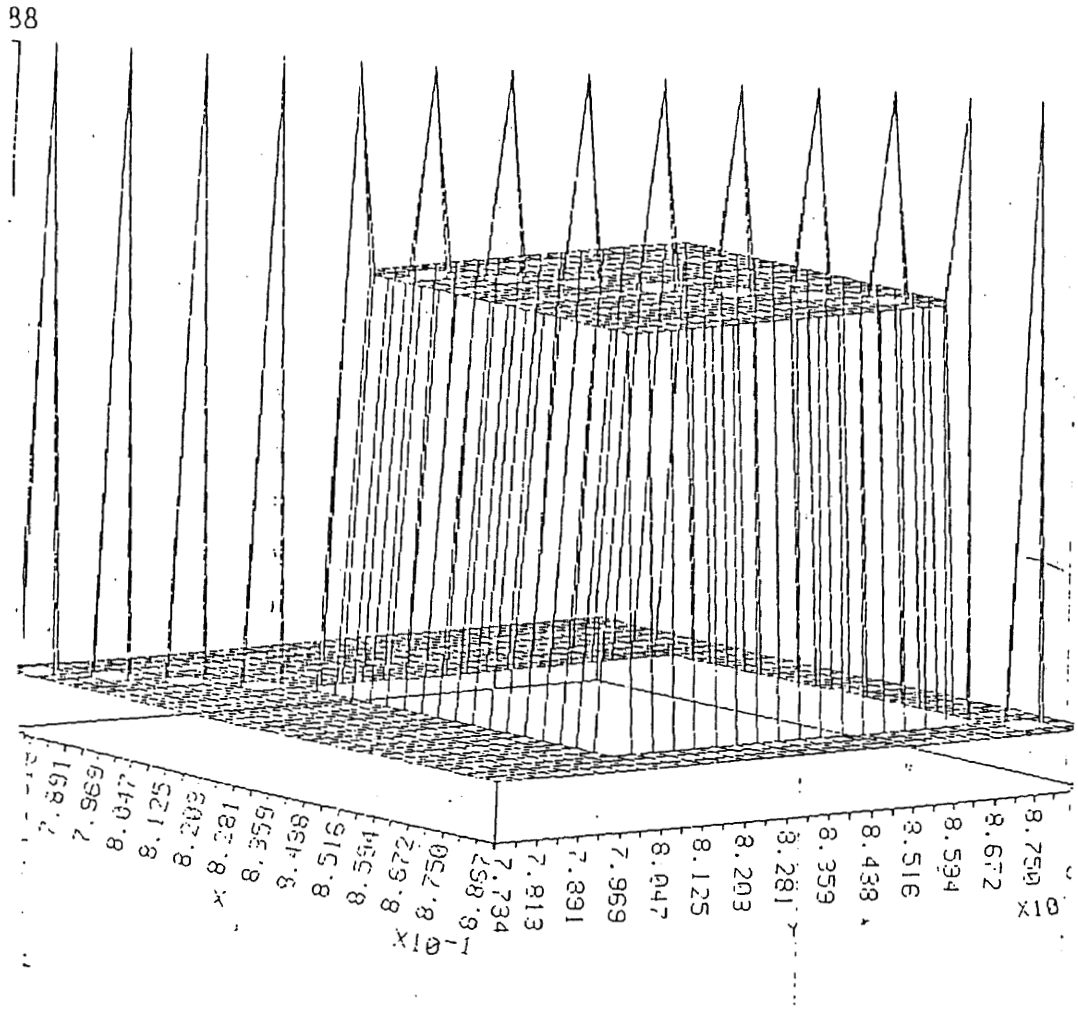
(b) X × Y is locally compact, separable, and perfectly normal.

(4) As mentioned in Wage[2] the first Wage-type example in the literature is the one satisfying Theorem 9.1 for the case n = 1. The construction suggested in Wage [1] remains <u>correct</u> if we do not force X = Y (in other word, in our terminology his proof that ν(1) × ν(2) is normal <u>is</u> correct, though the method to make X^2 being normal was incorrect (cited in Przymusiński [6, Footenote 7]).

Since he said nothing about Lemma 9.4, we cannot conclude that $\nu(i)^2$ is normal. By the lemma we can show that it is normal.

(5) It is annouced in Wage [1] that Theorem 9.1 holds for general dimension.

10. Local compactness and collectionwise normality of products.



10. Local compactness and collectionwise normality of products.

In this section we shall prove the following theorem without any set-theoretic assumptions beyond ZFC:

THEOREM 10.1. For every $n = 1,2,\ldots,\infty$, there exist two first-countable, separable, locally compact, locally countable spaces X and Y such that

(a) $X \times Y$ is countably paracompact, collectionwise normal;

(b) $\dim X = \dim Y = 0$, while $\dim (X \times Y)$ and $\mathrm{Ind}\,(X \times Y)$ are exactly equal to n. Moreover, our space $X \times Y$ in the case $n = \infty$ is not strongly countable dimensional.

Proof. We put

$$X = (C,\tau_1) \text{ and } Y = (C,\tau_2),$$

where τ_i is defined in §7. By the construction it follows that $X \times Y$ is first-countable, separable, locally compact, and locally countable. We shall show at first that it is countably paracompact and collectionwise normal. We need the following lemma.

LEMMA 10.1. (a) If A and B are closed disjoint sets in X or Y, then the set $Cl_{\varepsilon_0}(A) \cap Cl_{\varepsilon_0}(B)$ is countable.

(b) Let A and B be closed disjoint sets in $X \times Y$. Then,

$$|Cl_{\rho(n)}(A) \cap Cl_{\rho(n)}(B)|_2 \leq \omega.$$

Proof. (a) The assertion follows from the fact that if $|Cl_{\varepsilon_0}(A) \cap Cl_{\varepsilon_0}(B)| = c$, then $|Cl_{\rho(n)}(A^2) \cap Cl_{\rho(n)}(B^2)|_2 > \omega$.
(b) It follows from the definition of τ(see §7).

By the above lemma we can prove that each of X and Y is countably paracompact, collectionwise normal, and ω_1-compact (that is, every closed discrete sets in it is countable). Hence, we can show that $X \times Y$ is also countably paracompact and collectionwise normal by the technique of Przymusiński [1,3].

Next, we determine the dimension of our products and factors. We need the following lemma.

LEMMA 10.2. (a) Let U be a cozero-set of X or Y. Then, the set $U \cap Cl_{\varepsilon_0}(C \setminus U)$ is countable.
(b) Let Z be a zero-set of $(\Delta, \tau|\Delta)$. Then, the set $Cl_{\tilde{\rho}(n)}(Z) \setminus Z$ is countable.
(c) Let A be closed in $X \times Y$ and $U \subset A$ be a cozero-set in A. Then, $|U \cap Cl_{\rho(n)}(A \setminus U)|_2 \leq \omega$.

The proof goes on similarly as Lemma 10.1, so we omit it.

Now, we show that dim X = dim Y = 0. Let U be a cozero-set in X (or, Y). Then, by Lemma 10.2(a) the set $C \setminus Cl_{\varepsilon_0}(C \setminus U)$ is countable. Then, there exists an ε_0-clopen collection $\{G_i: i \in \omega\}$ such that $\cup_{i\in\omega} G_i = C \setminus Cl_{\varepsilon_0}(C \setminus U)$ and $|U \setminus \cup G_i| \leq \omega$. Therefore,

$$U = \cup \{G_i: i \in \omega\} \cup \{B(y): y \in U \setminus \cup G_i\},$$

where B(y) is a clopen neighborhood of y in X (or Y) satisfying $B(y) \subset U$. Hence, dim X = dim Y = 0 by Terasawa [1, Theorem 1].

Next, we shall show that

LEMMA 10.3. $\dim (X \times Y) \geq n$.

Proof. Since Δ is closed in $X \times Y$, it suffices to show that $\dim (\Delta,\tau|\Delta) \geq n$. Let $Z = (\Delta,\tau|\Delta)$. Put $A_k' = \Delta \cap \pi_{k+2}^{-1}[0,1/3]$ and $B_k' = \Delta \cap \pi_{k+2}^{-1}[2/3,1]$ for each $1 \leq k \leq n$. (Remember that $\rho(n) = \tilde{\rho}(n)|\Delta$.) Then, it is obvious that each A_k' and B_k' are disjoint zero-set in Z. We shall show that if for each k, a zero-set S_k' separates A_k' and B_k' in Z, then $\cap S_k' \neq \emptyset$.
At first, for such a separation S_k' we shall show that there exists a closed separation S_k of A_k and B_k in $(\Delta,\tilde{\rho}(n))$ such that

(1) $S_k \setminus S_k'$ is a countable set for each k.

Because the zero-set S_k' separates A_k' and B_k', there exist two cozero-sets U_k' and V_k' for each k such that $U_k' \cap V_k' = \emptyset$, $U_k' \supset A_k'$, $V_k' \supset B_k'$, and $\Delta \setminus S_k' = U_k \cup V_k'$. Since $U_k' \cup S_k'$ is a zero-set, it follows from Lemma 10.2(b) that

(2) the set $Cl_{\tilde{\rho}(n)}(U_k') \setminus (U_k' \cup S_k')$ is countable, and

$B_k \cap (Cl_{\tilde{\rho}(n)}(U_k')) = \emptyset$ for each $k = 1,\ldots,n$.

(Remember that

$B_k = \Delta \cap \pi_{k+2}^{-1}[6/7,1]$ and $Cl_{\tilde{\rho}(n)}(U_k') \subset \Delta \cap \pi_{k+2}^{-1}[0,2/3]$.)

Put

$$V_k = \Delta \setminus Cl_{\tilde{\rho}(n)}(U_k').$$

Then, V_k is open in $(\Delta,\tilde{\rho}(n))$, $V_k \supset B_k$ and $A_k \cap Cl_{\tilde{\rho}(n)}(V_k) = \emptyset$. Finally, put

$$S_k = Cl_{\tilde{\rho}(n)}(V_k) \setminus V_k.$$

Then, the set $S_k \setminus S_k'$ is countable, since the inclusion $\Delta \cap \pi_{k+2}^{-1}[1/3,1] \supset V_k' \cup S_k' \supset V_k$ holds, (2) and Lemma 10.2 (b) holds. Thus, S_k is a closed separation of A_k and B_k satisfying (1). On the other hand, by Theorem 4.1 (1) we can see that $|\cap S_k| = c$. From (1) and the fact that $(\cap S_k) \setminus S_k' \subset S_k \setminus S_k'$, we have $|(\cap S_k) \cap (\cap S_k')| = c$, so that $\cap S_k' \neq \emptyset$. This completes the proof of Lemma 10.3.

Finally, we shall show that $\mathrm{Ind}\,(X \times Y) \leq n$. For this purpose we need the following lemma.

LEMMA 10.4. Let A be a closed set in $X \times Y$ such that $A = T \cup S$ for some $\rho(n)$-closed set T of $|T|_2 \leq \omega$, and a set S satisfying $\dim (S,\rho(n)|S) \leq m$ for some non-negative integer m. Then, $\dim (A,\tau|A) \leq \mathrm{Ind}\ (A,\tau|A) \leq m$.

Proof. Since A is normal, it suffices to show that $\mathrm{Ind}\ (A,\tau|A) \leq m$. We shall prove it by induction on m. Because $|T|_2 \leq \omega$, $\dim (S,\rho(n)|S) \leq m$, and the countable sum theorem holds, we first note that

(1) $\dim (A,\rho(n)|A) \leq m$.

(Remember that $\tilde{\rho}(n)|\{t\} \times C$ and $\tilde{\rho}(n)|C \times \{t\}$ are homeomorphic to (C,ε_0) for each $t \in C$ by Theorem 4.2.)

For the case $m = 0$ we take arbitrary cozero-set U in $(A,\tau|A)$. Then, the 2-cardinality of the set

$$F = U \cap \mathrm{Cl}_{\rho(n)}(A \setminus U)$$

is countable by Lemma 10.2 (c). For the $\rho(n)|A$-open set

$$U \setminus F = A \setminus \mathrm{Cl}_{\rho(n)}(A \setminus U)$$

we have a collection $\mathcal{U}$ of countably many $\rho(n)|A$-clopen sets such that

$$\cup\, \mathcal{U} = U \setminus F$$

from (1). For the set F we take a countable set $W \subset C$ such that

$$W \times C \cup C \times W \supset F.$$

Then, for each $y \in W$ we shall show that there exist two collections $\mathcal{U}(y)$ and $\mathcal{V}(y)$ consisting of countably

many $\tau|A$-open sets such that

(2) $U \supset \cup u(y) \supset U \cap (\{y\} \times C)$ and $U \supset \cup v(y) \supset U \cap (C \times \{y\})$.

Let G be a τ-open set such that $U = G \cap A$. Since U is a F_σ-set in $X \times Y$, put $U = \cup K_i$, where each K_i is τ-closed. Because dim Y = 0, we have a clopen set Z_i in Y for each K_i and G such that

(3) $K_i \cap (\{y\} \times C) \subset \{y\} \times Z_i \subset G \cap (\{y\} \times C)$.

Let $\{G_s : s \in \omega\}$ be a countable clopen base of y in Y. Put

$$U_s = \cup \{V : V \text{ is open in } Y \text{ and } G_s \times V \subset G\}.$$

Then, from (3) $\{U_s : s \in \omega\}$ is a countable open cover of Z_i. Because dim $Z_i = 0$ and Z_i is countably paracompact normal, there exists a disjoint clopen cover $W = \{W_s : s \in \omega\}$ of Z_i such that $W_s \subset U_s$. Put

$$u_i = \{A \cap (G_s \times W_s) : s \in \omega\}.$$

Then, u_i is a countable $\tau|A$-clopen cover of the set $K_i \cap (\{y\} \times C)$ satisfying $\cup u_i \subset U$. Similarly, for each i, we have a countable $\tau|A$-clopen collection v_i satisfying

$$K_i \cap (C \times \{y\}) \subset \cup v_i \subset U.$$

Finally, put

$$u(y) = \cup u_i \text{ and } v(y) = \cup v_i.$$

Thus, (2) holds. Then, $\dim (A, \tau|A) \leq 0$ by Terasawa [1, Theorem 1], since

$$U = \cup u \cup \{u(y), v(y) : y \in W\}.$$

This completes the proof for the case $m = 0$. Suppose that our lemma is true for $m-1$. For the case m let H and F be disjoint closed set in $(A,\tau|A)$, and let U and V be disjoint cozero-sets in it such that

(4) $U \supset H$, $V \supset F$, and $Cl_\tau(U) \cap Cl_\tau(V) = \emptyset$.

Put

$$T' = Cl_{\rho(n)}(U) \cap Cl_{\rho(n)}(V).$$

Then, $|T'|_2 \leq \omega$ by Lemma 10.1(b). Because $\dim (A,\rho(n)|A) \leq m$ from (1), we have an at most $(m-1)$-dimensional closed separation S' in the space $(A \setminus T', \rho(n)|A \setminus T')$ between the sets $Cl_{\rho(n)}(U) \setminus T'$ and $Cl_{\rho(n)}(V) \setminus T'$. (Note that $S' \cap (U \cup V) = \emptyset$.) Put

$$A' = S' \cup (T' \setminus U \cup V).$$

Then, A' is a closed separation of H and F in $(A,\tau|A)$ from (4) and the construction of S'. Hence, inductive assumption can be applied to the set A' so that $\operatorname{Ind} (A',\tau|A') \leq m-1$, and therefore, $\operatorname{Ind} (A,\tau|A) \leq m$. This completes the proof of our lemma.

To show that $\operatorname{Ind} (X \times Y) \leq n$ take two disjoint closed sets H and F in $X \times Y$. Then, by taking cozero-sets U and V satisfying (4), we have a closed separation A of H and F which satisfies all the conditions in Lemma 10.4 for the case $m = n-1$. Hence, H and F are separated by a closed set A of $\operatorname{Ind} A \leq n-1$, therefore $\operatorname{Ind} (X \times Y) \leq n$.

The assertion for the case $n = \infty$ follows form Lemma 5.3, since the topology $\tilde{\tau} = \tau|\Delta$ satisfies the conditions i) and ii) in §5, 2).

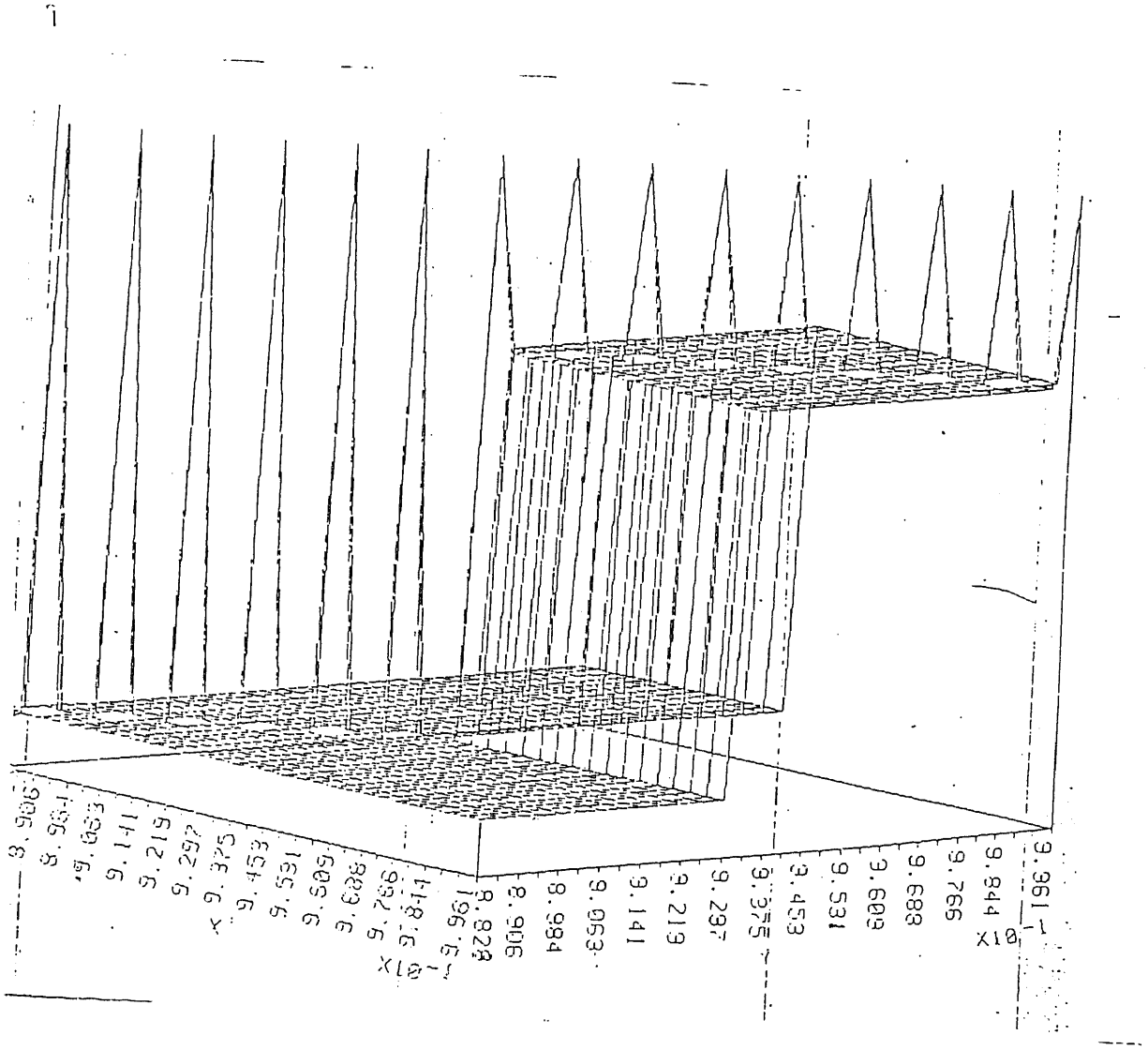
REMARK 10.1. (a) For the case $n = 1$ the existence of our example was suggested in Przymusiński [3].
(b) Our present examples are slight improvements of the ones given in Tsuda [1] (our examples are separable).
(c) Since $X \times Y$ is countably paracompact normal, we have: for every $n = 1,2,\ldots,\infty$, and for every pair of non-negative integers (ℓ,m) there exist two spaces X and Y such that
(i) $\dim X = \ell$ and $\dim Y = m$, while $\dim (X \times Y) = \ell+m+n$;
(ii) $X \times Y$ is locally compact, separable, countably paracompact and collectionwise normal.

One of the remaining problems in this section is:

PROBLEM 10.1. Whether or not we can construct one with $X = Y$.

In Przymusiński [3] or Engelking and Pol [1] it is announced that it is the case.

11. With Lindelöf factors and normality of products.



11. With Lindelöf factors and normality of products.

In this section we prove the following theorem without any set-theoretic assumptions beyond ZFC:

THEOREM 11.1. For every $n = 1, \ldots, \infty$, and integer $m \geq 1$ there exists a separable and first-countable space X such that

(i) X^m is Lindelöf and $\dim X^m = 0$;

(ii) X^{m+1} is normal but $\dim X^{m+1} = n$. Moreover, X^{m+1} is not strongly countable dimensional when $n = \infty$.

COROLLARY 11.1. For every $n = 1,2,\ldots,\infty$, there exists a separable and first-countable Lindelöf space X such that

(i) $\dim X = 0$;

(ii) X^2 is normal but $\dim X^2 = n$. Moreover, X^2 is not strongly countable dimensional when $n = \infty$.

At first, we shall show the following lemma, which is some version of the natural extension of $\tilde{p}(n)$ (cf. Theorem 4.2).

LEMMA 11.1. There exists a complete separable metric space $(C^{m+1},\bar{\rho}(n))$ satisfying

(1) $\bar{\rho}(n)|\Delta = \tilde{\rho}(n)$, where $\Delta = \{(x,x,\ldots,x) : x \in C\}$;

(2) $\bar{\rho}(n)|C^{m+1} \setminus \Delta$ is homeomorphic to the usual Euclidean topology on it;

(3) for each $t \in C$ (m+1) topologies $\bar{\rho}(n)|\pi_i^{-1}(t)$ are homeomorphic to ε_0, where $\pi_i : C^{m+1} \to C$ is the projection to the i-th factor.

Proof. Let $h:C \to I^n$ and $\mathcal{B}_i$ be the function and the development given in Theorem 4.1, respectively. We shall extend h to the function $\bar{H} : C^{m+1} \to I^n$ as follows.

$$\bar{H}((x_i)) = \begin{cases} h(x_i) \text{ if } (x_i) \in \Delta, \\ 0 \text{ if } (x_i) \in \cup\{B^{m+1} : B \in \mathcal{B}_1\}, \\ h_i(x_i) \\ \text{if } (x_i) \in \cup\{B^{m+1} : B \in \mathcal{B}_i\} \setminus \cup\{B^{m+1} : B \in \mathcal{B}_{i+1}\}. \end{cases}$$

Since $h|B$ is constant for each $B \in \mathcal{B}_i$, the above definition of $\bar{H}$ is well-defined. Let $\bar{\rho}(n)$ be the topology on C^{m+1} generated by the subbase

$$\varepsilon_0 \cup \{\bar{H}^{-1}(U) : U \in \varepsilon_0\}.$$

Then, $(C^{m+1},\bar{\rho}(n))$ is homeomorphic to the set

$$\{((x_i),\bar{H}((x_i))) : (x_i) \in C^{m+1}\}$$

with the restriction of the Euclidean topoligy on $C^{m+1} \times I^n$. By an argument parallel in Theorem 4.2 we see that (1)

and (2) hold and that $\bar{\rho}(n)$ is separable complete metric. Let us show that (3) holds. By the definition of H, we see that there exists a sequence of ε_0-open sets $\{U_i\}$ in $\pi_i^{-1}(t)$ such that $\cup U_i = \pi_i^{-1}(t) \setminus \{(t,\ldots,t)\}$, $\bar{H}|U_i$ is constant, and ε_0^m-distance between the point $(t,\ldots,t)$ and U_i is less than $1/i$.
Since $\lim_{\to\infty} \bar{H}(U_i) = \bar{H}(t,\ldots,t) = h(t)$, $\bar{H}|\pi_i^{-1}(t)$ is ε_0-continuous. Hence, (3) holds. This completes the proof.

Next, we need the following lemma.

LEMMA 11.2. There exist m+2 disjoint subsets $E_0,\ldots, E_{m+1}$ in C such that the m+1-cardinality of the set $A \cap (E_i)^{m+1}$ is the cardinality of continuum for any $\bar{\rho}(n)$-closed set A in C^{m+1} when A has uncountable m+1-cardinality.

Proof. Since $\bar{\rho}(n)$ is complete separable metric, the assertion follows from Przymusiński [2, Theorem 2].

On the other hand, we choose the following collection $\bar{P}$ of points from $(E_0)^{m+1}$, which plays the role of P given in section 7.

Let $F = \{(A,B)_\lambda : \lambda \in \Lambda\}$ be the collection of all pairs of countable subsets of C^{m+1} such that

$$|Cl_{\bar{\rho}(n)}(A) \cap Cl_{\bar{\rho}(n)}(B)|_{m+1} > \omega.$$

Let $<$ be a well-ordering of C and for each $x \in C$ we define $I(x) = \{y \in C : y < x\}$. Let D be some fixed countable $\rho(n)$-dense set in C. Then, by an argument parallel in Lemma 7.3 we can choose a transfinite sequence $<p_\lambda : \lambda \in \Lambda>$ of points in $(E_0)^{m+1}$ satisfying the following (4) and (5).

(4) $p_\lambda \in (E_0)^{m+1} \cap Cl_{\bar{\rho}(n)}(A) \cap Cl_{\bar{\rho}(n)}(B) \setminus ((I(x_\lambda) \cup D)^{m+1})$, where $(A,B) = (A,B)_\lambda$, and x_λ is some element of C such that $A \cup B \subset I(x_\lambda)^{m+1}$.

(5) If $\lambda \neq \mu$, then $\bar{p}_\lambda \cap \bar{p}_\mu = \emptyset$, where $\bar{p} = \{p_0,\ldots,p_m\}$ for $p = (p_0,\ldots,p_m)$.

Put $$\bar{P} = \{p_\lambda : \lambda \in \Lambda\}.$$

Finally, we can show by a way parallel to §§10, 11 that

(6) Let G_k be the (1/k)-neighborhood of $p_\lambda = (t_i)$ with respect to the metric $\bar{\rho}(n)$. Then, there exists a sequence $\{z_k = (z_{k,i})\}$ such that

(a) $z_k \in G_k$, $z_{k,i} \neq z_{s,i}$ for $s \neq k$, $\{(z_{3k,i})\} \subset A$, $\{(z_{3k+1,i})\} \subset B$, $z_{3k+2} \in D^{m+1}$, where $(A,B) = (A,B)_\lambda$;

(b) there exist $2(m+1)$ ε_0-clopen sets $H_{k,i}$ and $U_{k,i}$ such that

$p_\lambda \in \Pi_{i=1}^{m+1} H_{k,i}$, $(t_1,\ldots,z_k,\ldots,t_{m+1}) \in \Pi_{j\neq i} H_{k,j} \times U_{k,i} \subset G_k$ for every $1 \leq i \leq m+1$.

Proof of Theorem 11.1. 1) Construction of X.

At first, we construct m+1 local bases

$$\{H_k(x)_i : k \in \omega,\ x \in C\},\ 1 \leqq i \leqq m+1,$$

satisfying the following conditions as inductive assumptions.

(7) Each $H_k(x)_i$ is ε_0-closed, and for any $y \in H_k(x)_i$ there exists an $s \in \omega$ such that $H_s(y)_i \subset H_k(x)_i$.

(8) The diameter of $H_k(x)_1 \times \cdots \times H_k(x)_{m+1}$ is less than $1/k$ with respect to the metric $\bar{\rho}(n)$.

Suppose that $x \in C$ and such bases of neighborhoods have been constructed for each $y < x$.

We consider the following four cases:

(c) $(x,\ldots,x) = p_\lambda \in \Delta$ for some (unique) $\lambda \in \Lambda$. Put for each $k \in \omega$

$H_k(x)_i = \{x\} \cup \bigcup_{t \geq k} H_{s(t)}(z_{t,i})_i$, $1 \leqq i \leqq m+1$, where $k(t)$ is defined by a way parallel to the one in §11.

(d) $x = p_{\lambda,i}$, and $p_\lambda \in \Delta$. Put for each $k \in \omega$

$H_k(x)_i = \{x\}$ if $x \neq p_{\lambda,j}$, and

$H_k(x)_i = \{x\} \cup \bigcup_{t \geq k} H_{s(t)}(z_{t,i})_i$ if $x = p_{\lambda,i}$.

(e) $x \in E_i$, $1 \leqq i \leqq m+1$. Put $H_k(x)_j = \{x\}$ for $j \neq i$, and $H_k(x)_i$ is an ε_0-clopen neighborhood of x whose diameter with respect to ε_0 is less than $1/s$.

(f) x does not belong to the above three cases. Let $H_k(x)_i = \{x\}$ for $1 \leqq i \leqq m+1$, and $k \in \omega$.

It is easy to see that $\{H_k(x)_i : k \in \omega, x \in C\}$ satisfies (7). Thus, inductive construction has completed. From (7) and (8) they constitute local bases on C, and they produce Hausdoeff topologies σ_i. This completes the construction of σ_i.

2) Desired properties of X. We put

$$X = \oplus_{i=1}^{m+1}(C, \sigma_i).$$

Then, it follows from Przymusiński [1] that X^m is first-countable, separable, Lindelöf. We have $\dim X^m = 0$, and $\dim X^{m+1} = \operatorname{Ind} X^{m+1} = n$ by a way parallel to Theorem 10.1 (b). The assertion for $n = \infty$ follows from an argument parallel to Theorem 10.1 (b), also. This completes the proof.

REMARK 11.1. (1) For the case (m,1) our result was stated in Przymusiński [3] without proof. For the case (2,∞) our result was outlined in Engelking and Pol [1, Example 4.2]. Corollary 11.1 is a slight improvement of Tsuda [1, Theorem 1.2]. Here, our space is separable. It is also announced in Charalambous [2, Footnote at p. 648] that Engelking and Pol have independently showed this corollary.

(2) We have no Wage-type examples for the case $m = \infty$ (see also section 8):

THEOREM 11.2 (Nagami [5]). Let X^{ω} be normal. If each X^{i} is paracompact (respectively, Lindelöf), then X^{ω} is paracompact (respectively, Lindelöf).

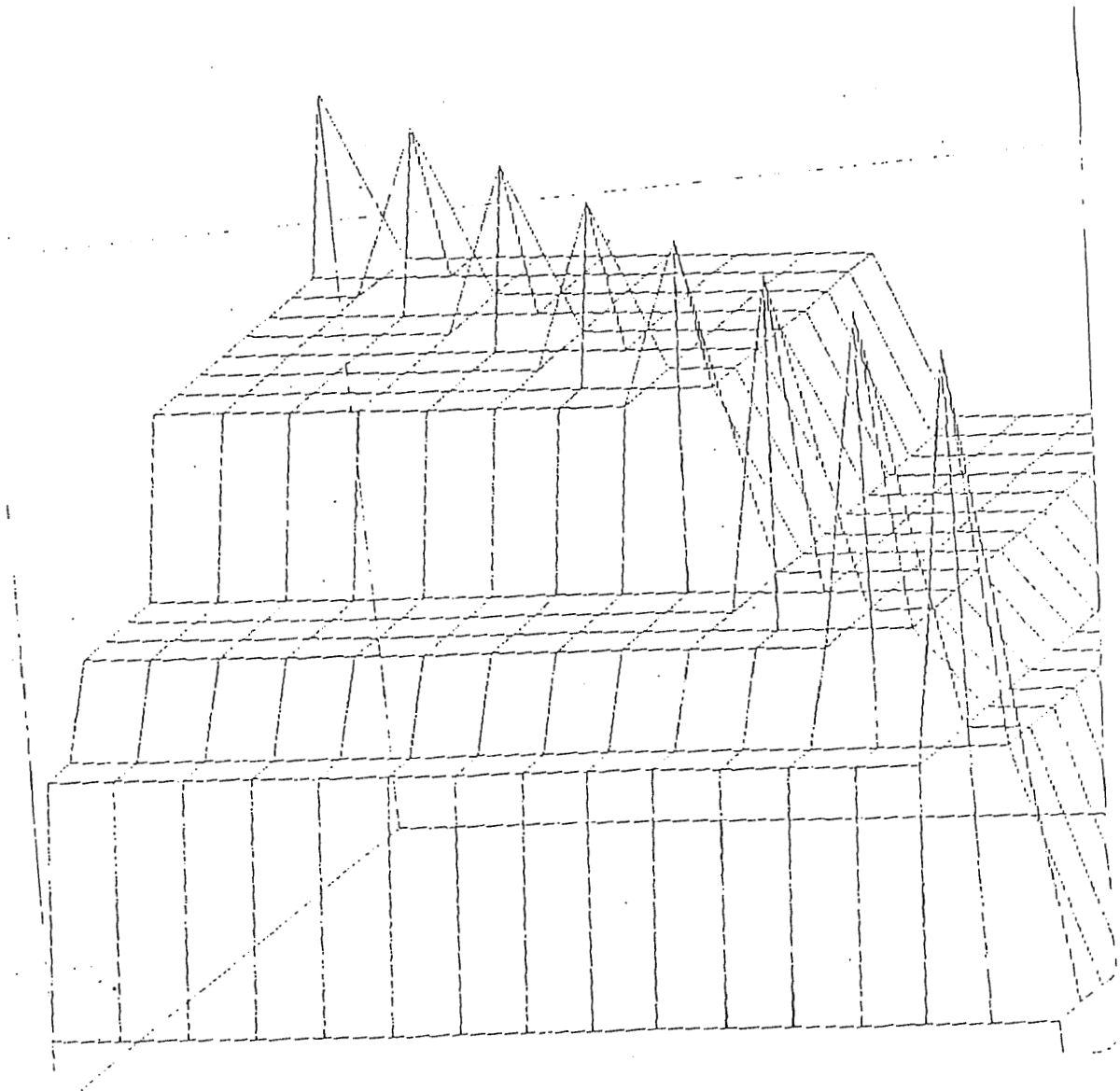
One of remaining problems in this section is:

PROBLEM 11.1. For every triple of non-negative integers (m,k,n) find a space X such that

(i) $\dim X^{i} = k$, and X^{i} is Lindelöf for each $i \leq m$;

(ii) X^{m+1} is normal but $\dim X^{m+1} = k+m$.

12. Non-normal Wage-type examples.



12. Non-normal Wage-type examples.

1) The purpose of this section is to show the following theorem due to Wage [2]. We also discuss some related problems.

THEOREM 12.1. There exists a separable metric space X and a first-countable separable Lindelöf space Y such that

$$\dim X = \dim Y = 0, \text{ and } \dim (X \times Y) = 1.$$

Proof. 1) Constructions of X and Y. Let $\tilde{\rho}$ and ρ be the Wage's metric topologies given in Theorem 3.1, and Theorem 3.2, respectively. Let E_1 and E_2 be two Bernstein sets given in Lemma 11.2 which also satisfy (0) $|A \cap (E_1 \times E_2)|_2 = c$ for any ρ-closed set $A \subset C \setminus \Delta$. Finally, let τ_0 be the approximation by the method of van Douwen (see § 5) of the topology $\tilde{\rho}_0$ of the restriction of $\tilde{\rho}$ to the usbset E_1. That is, τ_0 is first-countable, locally countable, locally compact, separable topology on E_1 finer than $\tilde{\rho}_0$ such that

(1) if A and B are disjoint τ_0-closed sets, then the set $Cl_{\tilde{\rho}_0}(A) \cap Cl_{\tilde{\rho}_0}(B)$ is countable.

Put

$$X = (E_1, \varepsilon'), \text{ and } Y = (C, \upsilon),$$

where $\varepsilon' = \varepsilon_0 | E_1$ and υ is the topology generated by the subbase $\varepsilon_0 \cup \tau_0$.

2) Dimension of factors and the product $X \times Y$. Since ε_0 is 0-dimensional separable metric topology, X is also 0-dimensional separable metrizable. On the other hand, since E_1 is a Bernstein set, Y is Lindelöf (see § 11). By the definition of Y, we see that Y is separable, first-countable, and has a clopen base. We shall show that $\dim (X \times Y) = 1$.

At first, we note that from Theorem 3.2 (4) the topology $\varepsilon' \times \upsilon$ is finer than the topology $\tilde{\rho} | E_1 \times C$. Hence, since $H | X \times Y$ is continuous, it can be seen in a parallel way as in § 3 that $\dim (X \times Y) > 0$. Therefore, it suffices to see that $\dim (X \times Y) \leq 1$. Let $\mathcal{U} = \{U_i\}$ be a finite cozero-sets cover of $X \times Y$, and let $\mathcal{F} = \{F_i\}$ be a finite zero-set shrinking of $\mathcal{U}$ such that $U_i \supset F_i$. We shall show at first that $\Delta' = \{(x,x): x \in E_1\}$ is covered by finite cozero-sets of order at most 2. Put

$$T = \bigcup (\mathrm{Cl}_{\tilde{\rho}_0}(\Delta' \setminus F_i) \cap \mathrm{Cl}_{\tilde{\rho}_0}(\Delta' \setminus U_i)).$$

Then, by (1) T is a countable set in Δ', and is τ_0-closed, so that when we consider the finite $\tilde{\rho}_0$-open cover $\{\Delta' \setminus \mathrm{Cl}_{\tilde{\rho}_0}(\Delta' \setminus U_i)\}$ of the space $\Delta' \setminus T$, we have a $\tilde{\rho}_0$-open cover $\mathcal{V}' = \{V_i'\}$ of order at most 2 such that

(1) $V_i' \subset \Delta' \setminus \mathrm{Cl}_{\tilde{\rho}_0}(\Delta' \setminus U_i)$ and $\bigcup \mathcal{V}' = \Delta' \setminus T$.

Let ρ_0 be the restriction of the topology ρ to the subset $E_1 \times C$. Then, by the following theorem we have a finite family of ρ_0-open sets G satisfying the fol'lowing (2).

THEOREM (K. Kuratowski [1, p. 226]). Given a family V' of $\tilde{\rho}_0$-open sets, there exists a family G of ρ_0-open sets such that

(2) $\Delta' \cap G_i = V_i'$ and the condition $V_{i_1}' \cap \ldots \cap V_{i_k}' = \emptyset$ implies $G_{i_1} \cap \ldots \cap G_{i_k} = \emptyset$, for every (finite) system of indices $i_1, \ldots, i_k$.

Then, G is a cozero-set collection of $X \times Y$ such that

(3) $(\cup G) \cap \Delta' = \cup V' = \Delta' \setminus T$.

For each $t \in T$ we can take a clopen set

$$N(t) = G(t) \times B(t),$$

where $G(t)$ is an ε'-clopen neighborhood of t and $B(t)$ is a τ_0-clopen neighborhood of t consisting countable points such that $N(t) \subset U_i$ for some $U_i \in U$. Note that $N(t)$ is Lindelöf. Put

$$S = \cup \{N(t) : t \in T\}.$$

Then, S is a 0-dimensional Lindelöf space, becuase $N(t)$ is Lindelöf and T is countalbe. Since S is a cozero-set containing the sero-set Δ' in $X \times Y$, there exists a finite clopen collection $S = \{S_i\}$ in $X \times Y$ such that

$$S_i \subset U_i \text{ and } \cup S \supset T.$$

Hence, the set $W = \cup G \cup \cup S$ is a cozero-set containing the set Δ'. Hence, there exists a zero-set Z and a cozero-set G in $X \times Y$ such that

$$\Delta' \subset G \subset Z \setminus W.$$

Because space $X \times Y \setminus \Delta'$ is 0-dimensional Lindelöf by (0), we can choose a clopen set K in $X \times Y \setminus \Delta'$ such that

$$K \cap Z = \emptyset \text{ and } K \subset X \times Y \setminus W.$$

Since $Cl_{X\times Y}(K) \subset X \times Y \setminus G$, K is also clopen in $X \times Y$. Put

$$V_i'' = (G_i \cup S_i \setminus K \cup \cup_{j<i} S_j).$$

Then, V_i'' is a cozero-set of $X \times Y$ such that

$$\cup V_i'' = X \times Y \setminus K \supset \Delta',$$

and its order is at most 2 by (2), (3). Because K is strongly 0-dimensional, there eixsts a finite disjoint collection $K = \{K_i\}$ of clopen sets such that

$$K_i \subset U_i \text{ and } \cup K = K.$$

At last, put

$$V_i = V_i'' \cup K_i$$

for each i. Then, it can be seen that $V = \{V_i\}$ is a cozero-sets refinement of U and its order is at most 2. This completes the proof of our theorem.

REMARK 12.1. (a) Our present construction of X and Y is a slight improvement of the costruction given in Wage [2]. In there some approximation on E_1 is used, which is defined more easier than ours. But, it seems that its definition is too simple to see that it satisfies our condition (1). Therefore, he can only announce that dim $(X \times Y)$ is positive. In our case, however, we can determine the dimension of our product space exacly, using the property (1) and Lemma 10.2 (b).

(b) One of the main characteristic properties of Wage-type examples with a metric factor is that its product cannot be normal (see Kodama [4]). In this point of view we can consider our space Y as some variation of the well-known Michael line C_{E_1}: Its topology is defined by declaring the set of the form $U \cup K$, where U is ε_0-open and $K \subset C \setminus E_1$, as its open set. (It is some times called the Hannerization in C with respect to E_1). Since it is known (Terasawa [1]) that dim $(X \times C_{E_1}) = 0$ for every strongly 0-dimensional metric space X, Michael line does not serve our purpose in the view of Wage-type examples. (From our constructions we see that the discrete topology on E_1 in C_{E_1} is too storng to construct Wage-type example with a metric factor.)

(c) There is another well-known Michael line C_Q, which is the Hannerization in C with respect to the countable

dense subset Q in C. Its advantage is that its product with the space of irrational numbers P is not normal. Hence, we can raise a following problem. (Note that our space X in Theorem 12.1 cannot be complete metric.)

PROBELM 12.1. Are there any Wage-type examples with a complete metric factor ?

We also note that we cannot replace complete metric by compact metric in the above problem, since $\dim (X \times I) = \dim X + 1$ for every X (see, Chiba and Chiba [1, Lemma 1] or Morita [4]).

We also note that there exists a strongly 0-dimensional Dwoker space (Dow and van Mill [1]).

We can specify the above problem as follows.

PROBELM 12.2. Are there any Wage-type examples with an irrational factor ?

As mentioned at the end of Remark 12.1 (b), when we attempt to prove this problem by modifying the Michael line, I believe that one of the key points is

(*) whether or not there exists a topology on P finer than $\tilde{\rho}$, which is 1-dimensional paracompact and has a clopen base.

We can raise one more problem for higher dimensional examples:

PROBLEM 12.3. Does there exist for every non-negative integer n a separable metric space X and a first-countable separable Lindelöf space Y such that

$\dim X = \dim Y = 0$, and $\dim (X \times Y) = n$?

2) With a pseudocompact factor.

One of the most remarkable facts among the previously known Wage-type examples is that all of them satisfy the first-countablity axiom (hence are k-spaces), and some of their products are also locally compact. The purpose of this section is to show that there exists another Wage-type example which cannot be a k-space.

By a theorem due to H. Tamano [1], we can specify our problem as follows.

PROBLEM 12.4. Are there any counter-examples for the inequality (*) with a pseudocompact factor ?

The following theorem solves this problem affirmatively:

THEOREM 14.2 (Tsuda [3]). Without any set-theoretic assumptions beyond ZFC there exist a hereditarily separable, hereditarily Lindelöf space X and a first-countable, locally compact, separable, pseudocompact space Y such that

$$\dim X = \dim Y = 0, \text{ while } \dim (X \times Y) > 0.$$

To prove our theorem we need the following another factorization thechnique:

THEOREM 12.2 (Tsuda [3, Key Lemma]). For each point $(x,x) \in C$ and each countable set λ which converges to x in the usual Euclidean topology on C, and for a given ρ-open neighborhood G of (x,x), there exists a $\tilde{\rho}$-clopen set B and a finite set F such that

$$(x,x) \in \underbrace{(\lambda \setminus F)}_{\cup\{x\}} \times B \subset G.$$

Now, we define our space X as the set C with the topology μ determined by assigning to each point $x \in C$ a base of neighborhoods consisting of those subset U of C which satisfy the condition :

(1) U contains some $\tilde{\rho}$-clopen set B with $x \in B$.

On the other hand, the space Y is a space which is an $N \cup R$ for some maximal almost disjoint collection (m.a.d. collection for short) R. By $N \cup R$ we mean a space which is defined in the following way on the set-theoretic union of a countable infinte set N and an almost disjoint collection R of infinite subsets of N:

each point of N is isolated and $\lambda \in R$ has a neighborhood base

$\{\{\lambda\} \cup (\lambda \setminus F) : F$ is a finite subset of $N\}$.

REMARK 12.2. (a) Spaces $N \cup R$ have been introduced by Mrówka [1], and have been found to have interesting properties Mrówka [2,3], Terasawa [2,3]. The use of spaces $N \cup R$ for dimension theory is not new. One of the remarkable results among them is the one of Terasawa [3].

(b) Our present example was given in Tsuda [3]. It gives a partial answer to the problem given in Przymusiński [6, Problem 18], that whether or not there exists a Wage-type example with a countably compact factor.

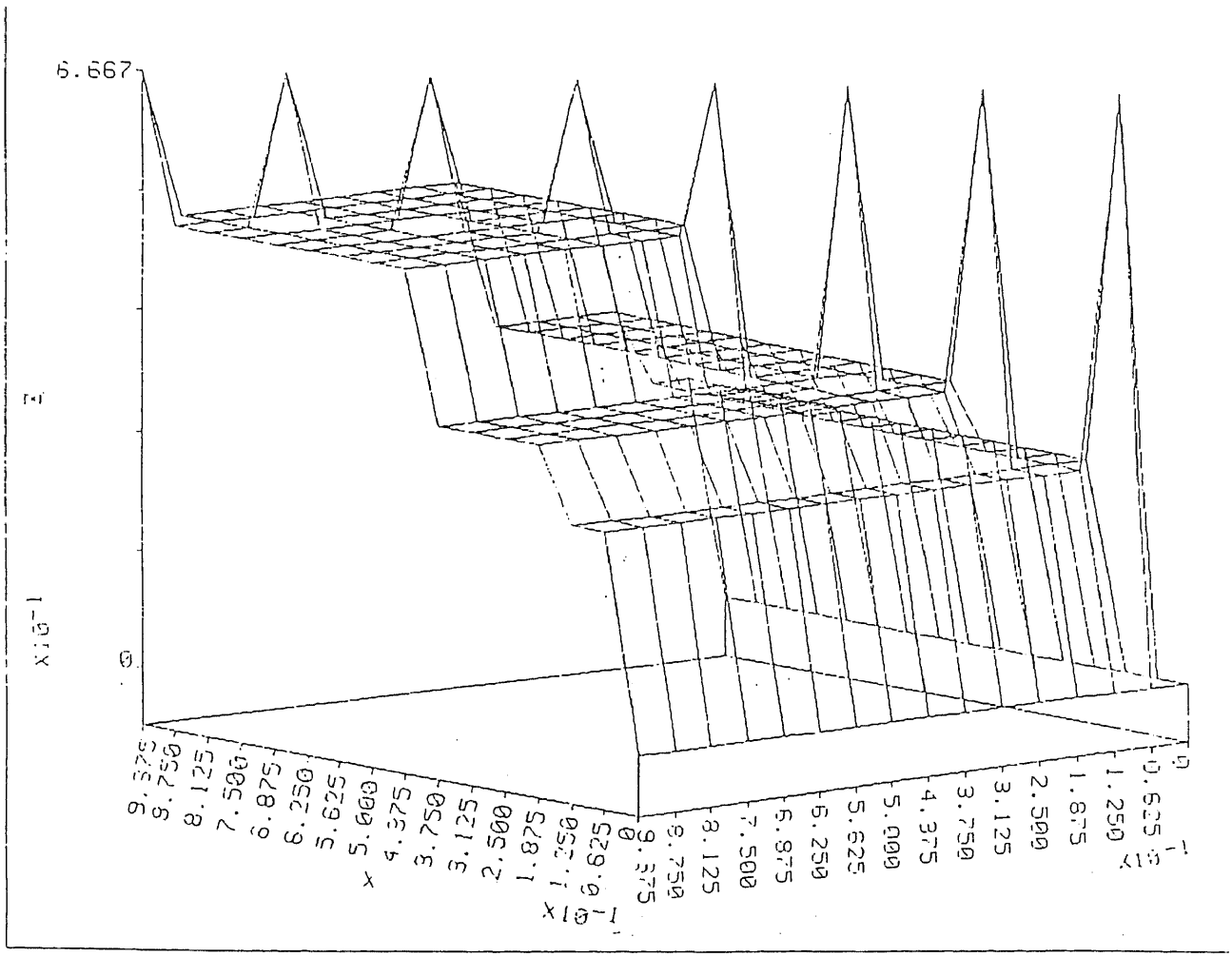
(c) We remark that our factor space X is perfectly normal, since it is a σ-space. We cannot, however, improve X to be also pseudocompact, since Y is frist-countable, locally compact. We also note that Y is neither normal nor countably compact, since Y has uncountable closed discrete subset. Hence, we can raise the following problem:

PROBLEM 12.5. Are there any Wage-type example with a pseudocompact factor whose product are also normal ?

We can raise one more problem (see also, Remark 8.1):

PROBLEM 12.6. For a given positive integer n are there any Wage-type examples with a pseudocompact factor which satisfy that $\dim X = \dim Y = 0$, while $\dim (X \times Y) = n$?

Appendix, Several product theorems.



Appendix. Several product theorems.

Here, we review some positive results for the inequality (*), because we can conclude from them some topological properties of our examples in Chapter 3 (for example, non-paracompactness of products, with non-metric factor, or with non-pseudocompact factor). It gives also some motivations of the attempt to look for other Wage-type examples in Chapter 3. In the sequel, we do not assume the normality of products unless otherwise specified. Hence, by the dimension of a space we mean the covering dimension due to Katětov [1]. Therefore, it coincides with the ordinaly covering dimension when the space is normal.

We begin with the relatively strong theorem due to Pasynkov. For this purpose we need the following interesting and importnat concept of the "(piecewise) rectangularity" of a product (Pasynkov [1 - 5]).

Definition A.1. A subset of the product space $X \times Y$ is said to be (piecewise) cozero rectangular if it is (a clopen subset of the set) of the form $U \times V$, where U and V are cozero sets of X and Y, respectively.

The product space is said to be (piecewise) rectangular if any finite cozero cover of it has a

σ-locally finite refinement consisting of (piecewise) cozero rectangular subsets.

THEOEM A.1 (Pasynkov [1 , 4]). Every (piecewise) rectangular product satisfies the inequality (*).

In each of the following cases (a) through (c), and (d) the product is rectangular and piecewise rectangular, respectively.

(a) X is a metric space and X × Y is countably paracompact and normal (Kodama [4]). Moreover, X is a paracompact p-space and X × Y is countably paracompact and normal (Filippov [1] and Pasynkov [1,3]).

(b) X is locally compact and paracompact (Morita [4]).

(c) The projection p_X: X × Y → X is a z-closed map (that is, $p_X(Z)$ is closed in X for every zero set Z of X × Y) (Filippov [1], Nagami [8], Pasynkov [2]).

(d) X × Y is completely paracompact.(That is, for any open cover $\mathcal{U}$ there exists a sequence $\mathcal{V}_i$ of star-finite open cover such that $\cup \mathcal{V}_i$ contains a refinement of $\mathcal{U}$. In particular, every Lindelöf space is completely paracompact.) Zolotarev [1].

REMARK A.1. (a) By a result of Rudin and Starbird [1] we can remove the assumption countably paracompactness in the case (a) when it is a metric factor. But, we cannot remove the normality of the product (see §12).
(b) When X is compact, the result is due to Terasawa (see Chiba and Chiba [1, Lemma 1]).
(c) The concept of the z-closed map was introduced by Ishiwata. The proof of Theorem 1.1 in Ishiwata [1] contains a proof that the inequality (*) is valid in the case (c), implicitly. In particular, p_X is z-closed, when X is a pseudocompact and Y is a k-space (implicitly, H. Tamano [1]).
(d) More general results can be found in Filippov [2], and Pasynkov [5].

The Pasynkov's theorem is relatively strong, but it is known that
(1) there exist non-rectangular storngly 0-dimensional products (Hoshina and Morita [1], Ohta [1], Przymusinski [7], K. Tamano [1], and Terasawa [1]).
Moreover, Ohta [1] showed a machine to produce normal non-rectangular products $X \times Y$ which satisfy the inequality (*) for every normal non-paracompact (not necessary strongly 0-dimensional) space X.

One of the remarkable consequences of the introduction of the piecewise rectangularity is that it is a necessary and sufficient condition for the validity of the inequality (*) when the factor spaces are strongly 0-dimensional. Hence, all the examples in (1) are piecewise rectangular. In Tsuda [7] we showed that for every positvie integer n there exists an n-dimensional, collectionwise normal, non-piecewise rectangular product which satisfies the inequality (*).

The idea we used there goes back to Chiba and Chiba [2], and Chiba [1].

On the other hand, there is a theorem which cannot be deduced from Theorem A.1.

THEOREM A.2 (Morita [6]). Let X be σ-locally compact paracompact. Then, the inequality (*) is valid.

The following problem is communicated by T. Goto.

PROBLEM A.1. Is the inequality (*) valid for the normal product with a Lašnev factor ?

It is known (Hoshina [1]) that the normality of product and countably paracompactness of it is equivalent in this case.

Finally, we mention infinite products. The following definition is suggested by Pasynkov [5] and Yajima [3].

DEFINITION A.2. A Cartesian product $X = \Pi_{\lambda \in \Lambda} X_\lambda$ is said to be (piecewise) cylindrical if each finite cozero cover of X has a σ-loclly finite refinement by sets of (a clopen subset of) the form $\pi_\xi^{-1}(U)$, where ξ is a finite subset of Λ and U is a cozero set in $X_\xi = \Pi_{\lambda \in \Lambda} X_\lambda$.

Then, the following theorem is obtained in a way parallel to Pasynkov [5] and Yajima [3].

THEOREM A.3. If X is piecewise cylindrical, then
(**) $\dim X = \text{Sup}\{\dim X_\xi : \xi \text{ is a finite subset of } \Lambda\}$.

Note that our definition is slight different from the piecewise rectangularity of infinte product due to Pasynkov [5]. The following result shows that it is sometimes easy to see piecewise cylindricality instead of picewise rectangularity.

Theorem A.4 (Filippov [2]). Every Cartesian product of paracompact p-spaces is piecewise cylindrical.

REMARK A.2. Filippov [2] announced that it is indeed picewise rectangular in the sense of Pasynkov. But, the proof given there is the one for our Theorem A.3.

COROLLARY A.1 (E. Pol [2]). Every Cartesian product of metric spaces satisfies (**).

REMARK A.3. It is known in Anderson and Kleisler [1] that for every n there exists an n-dimensional separable metric space X such that every finite product X^i is (and, hence countable product X^ω is) n-dimensional.

We conclude this section with the following problem:

PROBLEM A.2. Does every Cartesian product of paracompact Σ-spaces satisfy (**) ?

When we assume countable tightness, it is the case (Yajima [2]). It is also known that every finite product of paracompact Σ-spaces is rectangular (Pasynkov [2] , and see also, Nagami [2]).

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