# SOME RESULTS ON RINGS OF AUTOMORPHIC FORMS

Shigeaki Tsuyumine

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It has been a quite hard problem to know the structures of rings of automorphic forms. Indeed the cases when the structures have been explicitly determined are not many, even for the Siegel or Hilbert modular forms which are considered to be significant in connection with the moduli spaces of abelian varieties. In 1971, Eichler stated that a ring of automorphic forms was Cohen-Macaulay in a number of individual case when it had been explicitly constructed, (which is still true up to now days,) and posed the problem which of them are Cohen-Macaulay; Projective varieties and modular forms, Lecture notes in Math., 210; On the graded ring of modular forms, Acta arith., 18. He derived even some consequence of the 'hypothesis' of the rings of automorphic forms being Cohen-Macaulay. Since a ring of automorphic forms is a graded ring, it is Cohen-Macaulay if and only if it is free over some (equivalently any) polynomial subring generated by homogeneous elements over which it is finite. It looks a good point of view in order to analyze the structure of a graded ring of automorphic forms.

Freitag first answered to Eichler's problem and showed that the ring of Hilbert modular forms of dimension $\geq 3$ is not Cohen-Macaulay; Lokale und globale Invarianten der Hilbertschen Modulgruppen, Invent. math., 17 (1972). We consider Eichler's problem in more general case. Cohomological methods play an important role in every case, since there is a cohomological criterion for a normal graded ring to be Cohen-Macaulay and since the Serre duality theorem for cohomology groups is closely related with Cohen-Macaulayness.

We give the summary of our result. Let $R = \bigoplus_{k\geq 0} R_k$ denote a graded ring of automorphic forms.

(1) Hilbert modular forms. We get exactly when the graded ring $R^{(s)} = \bigoplus_{k\equiv 0\,(s)} R_k$ is Cohen-Macaulay for $s \geq 2$, including the case of symmetric Hilbert modular forms. In particular, if the dimension is two, then all $R^{(2)}$ are Cohen-Macaulay, which is the answer also to the problem posed by Thomas and Vasquez; Rings of Hilbert modular forms, Compositio Math., 48 (1983). Together with their result, it follows that the graded rings $R^{(2)}$ of Hilbert modular forms of dimension 2 are also Gorenstein.

(2) Siegel modular forms. The graded ring $R$ of Siegel modular forms is not Cohen-Macaulay when the degree is greater than three. Samuel stated "All the examples of U.F.D.'s I know are Cohen-Macaulay. Is is true in general?" in his paper; On unique factorization domains, Illinoi J. Math., 5 (1961). Several mathematicains constructed counter-examples (e.g., Freitag and Kiehl, Algebraiche Eigenschaften der lokal Ringe in der Hilbertschen Modulgruppen, Invent. math., 24 (1974)). By Freitag's result; Stabile Modulformen, Math. Ann., 230 (1977), Die Irreducibilität der Schottky relation (Bemerkungen zu einem Satz von Igusa), Archiv der Math., 40 (1983), $R$ is a U.F.D. if the degree is greater than

two, and thus R's of degree $> 3$ give new negative examples to Samuel's question. R is not Cohen-Macaulay also for the principal congruence subgroup of degree 2,3, if its level is large enough. Our argument implies that the Satake compactification of the moduli space of abelian varieties of dimension $> 3$ is not a Cohen-Macaulay scheme. It gives some generalization of the result of Igusa; On the theory of compactifications, Summer Institute on Algebraic Geometry, Woods Hole, 1964, where he showed that it does not admit a finite nonsingular covering.

(3) Automorphic forms on a bounded symmetric domain. Let $\mathfrak{D}$ be a bounded symmetric domain, and $\Gamma$ a neat arithmetic group acting on $\mathfrak{D}$. Let $\mathfrak{D}'$ (resp. $\mathfrak{D}''$) be the highest (resp. the second highest) dimensional rational boundary component. Suppose one of the following;

(i) $m := \dim \mathfrak{D} - \dim \mathfrak{D}' \geq 3$ is odd,

(ii) $m \geq 4$ is even, and $\dim \mathfrak{D}'' \leq \dim \mathfrak{D}' - 2$.

Then the ring of automorphic forms on $\mathfrak{D}$ for $\Gamma$ is not Cohen-Macaulay.

These work were done while the author was staying at the Johns Hopkins University, and after that at Harvard University. He wishes to express his hearty thanks to the members of Department of Mathematics of each University for their hospitalities, and to the Educational Project for Japanese Mathematical Scientists for financial support.

# PART I

# Rings of Modular Forms (On Eichler's Problem)

Shigeaki Tsuyumine

In his paper [ 4 ] or lecture note [ 3 ], Eichler asked the problem when the ring of modular forms is Cohen-Macaulay. We shall try to investigate it for Hilbert or Siegel modular case.

When the dimension $n$ is one, any ring of modular forms for an arithmetic group is Cohen-Macaulay, indeed a normal (graded) ring of Krull dimension two is always Cohen-Macaulay. So we consider the case $n > 1$. Unfortunately rings of modular forms do not always have this nice property. In the case of (symmetric or not) Hilbert modular forms it is essentially Freitag's result (see 7.1 Satz [ 6 ] and Proposition A in § 1.1). Let $A(\Gamma) = \bigoplus_k A(\Gamma)_k$ be the ring of Hilbert modular forms for a group $\Gamma$. Then the same question for $A(\Gamma)^{(2)} = \bigoplus_{k \equiv 2\ (0)} A(\Gamma)_k$ with $n = 2$ was raised by Thomas and Vasquez [ 20 ], in which it is shown by using the criterion due to Stanley [ 18 ],[ 19 ] that $A(\Gamma)^{(2)}$ is also Gorenstein if it is Cohen-Macaulay under some condition on $\Gamma$. Also Eichler derived some consequence of the 'hypothesis' of $A(\Gamma)^{(2)}$ being Cohen-Macaulay with $n = 2$ in [ 3 ].

In this paper we shall show this affirmatively, and moreover get when $A(\Gamma)^{(r)}$ is Cohen-Macaulay for general $n$ and $r \geq 2$, as well as the case of symmetric Hilbert modular forms (Theorem 1). Furthermore if $n = 2$ and if $\Gamma$ acts on freely on $H^2$, the necessary and sufficient condition for $A(\Gamma)$ to be Cohen-Macaulay is given as

$$(1) \qquad \dim A(\Gamma)_1 = \tfrac{1}{2}\left(-\tfrac{1}{2}\,\zeta_K(-1)\cdot a + \chi + h\right)$$

where $K$ is a corresponding real quadratic field, $\zeta_K$ is its zeta function, $a = [SL_2(O_K): \Gamma]$, $O_K$ being the ring of integers of $K$, $h$ is the number of the cusps and $\chi$ is the arithmetic genus of the non-singular model of the Hilbert modular surface.

Let us refer to the case of Siegel modular forms, and let $A(\Gamma)$ denote the ring of Siegel modular forms for an arithmetic group $\Gamma$. When the degree $n$ of the Siegel space is two, if $\Gamma$ possesses, as its ~~normal subgroup,~~ the principal congruence subgroup $\Gamma_2(2)$ of level two, then $A(\Gamma)$ is Cohen-Macaulay. But the Cohen-Macaulayness does not always hold, indeed $A(\Gamma_2(\ell))$ is such an example if $\ell \geq 6$. When $n = 3$, $A(\Gamma_3(\ell))$, $\ell \geq 3$, is no longer Cohen-Macaulay. We shall show these by disproving the Serre duality theorem for $\mathrm{Proj}(A(\Gamma_n(\ell)))$ which should hold if $A(\Gamma_n(\ell))$ would be Cohen-Macaulay.

This work was done while the author was staying at Harvard University. He whishes to express his hearty thanks to the members of the department of mathematics of Harvard University for their hospitality, and to Educational Project for Japan Mathematical Sientists for the financial support.

## § 1. Preliminaries

1. Let $k$ be a field, and $R$ be a noetherian $k$-algebra. We call $R$ Cohen-Macaulay if any ideal generated by a regular sequence

has no embedded prime. A k-scheme is called Cohen-Macaulay if all of its local rings are Cohen-Macaulay.

When $R$ is a graded algebra, we have the following (for the detail see Serre [ 16 ] Theorem 2 IV-20, Hochster and Robert [ 13 ] § 1 (d));

Proposition A. *Let* $R = \bigoplus_{m \geq 0} R_m$ $(R_0 = k)$ *be a normal noetherian graded* k-*algebra of dimension* $N + 1$. *Then the following conditions are equivalent*

(i) $R$ *is Cohen-Macaulay.*

(ii) *For some (equivalently any) system of homogeneous elements* $x_0, \ldots, x_N$ *such that* $R$ *is integral over* $k[x_0, \ldots, x_N]$, $R$ *is free over it.*

(iii) *Let* $X = \mathrm{Proj}(R)$, *and* $\mathcal{O}_X$ *be its structure sheaf. Then the cohomology group* $H^\nu(X, \mathcal{O}_X(m))$ *vanishes for* $1 \leq \nu \leq N-1$ *and for every* $m \in \mathbb{Z}$, *where* $\mathcal{O}_X(m)$ *is Serre's twisting sheaf.*

As an easy consequence of this, we get the following;

Corollary. *Let* $R$ *be as in the proposotion, and let* $r$ *be an integer. Then* $R^{(r)} = \bigoplus_{m \equiv 0 \ (r)} R_m$ *is Cohen-Macaulay if and only if* $H^\nu(X, \mathcal{O}_X(m)) = 0$ for $1 \leq \nu \leq N-1$, $m \equiv 0 \mod r$.

Next one is a part of the famous results in [ 12 ].

Proposition B. Let $G$ be a finite group acting $k$-linearly on $R$. Suppose either $\mathrm{char}(k) = 0$, or the order of $G$ is coprime to $\mathrm{char}(k)$. If $R$ is Cohen-Macaulay, then so is the invariant subring $R^G$.

If we use the notation of Proposition A, then $X$ is a Cohen-Macaulay scheme if and only if $H^{\nu}(X,\mathcal{O}_X(m))$ vanishes for $\nu < N$, $m \ll 0$ (see for example, the proof of Theorem 7.6 Chap III Hartshorne [ 9 ]). So if $R$ is a Cohen-Macaulay algebra, then $X = \mathrm{Proj}(R)$ is a Cohen-Macaulay scheme. The converse is not necessarily true, indeed an $N$-dimensional projective manifold over $\mathbb{C}$ carrying non-trivial holomorphic $p$-forms $(0 < p < N)$ is a such example.

2. We shall prepare two lemmas for the later use.

Lemma 1. Let $D$ be a domain in $\mathbb{C}^n$ and $S$ be a finite group acting on $D$ as holomorphic automorphisms. Let $\pi : D \to Y = D/S$ be the quotient. Take an automorphy factor $\rho(g,z)$ $g \in S$, $z \in D$ and consider an action on $\mathcal{O}_D$ as

(2) $$f(z) \longmapsto \rho(g,z)^{-1} f(gz).$$

If $\mathcal{F}$ denotes the invariant subsheaf of $\pi_*(\mathcal{O}_D)$ under this action, then we have

$$i_*(\mathcal{F}|_{Y_0}) = \mathcal{F}$$

where $Y_0$ is the regular open subset of $Y$ with the inclusion map $i$.

*Proof.* Since all non-zero sections $f$ of $\mathfrak{F}$ over an open subset $V$ have $V$ as their support i.e., $\{P \in V \mid f_P = 0 \text{ in } \mathfrak{F} \otimes \mathcal{O}_{Y,P}\} = \phi$, $\mathfrak{F}$ is a subsheaf of $i_*(\mathfrak{F}|_{Y_0})$. $Y' \doteq Y - Y_0$ is of codimension $\geq 2$, since $Y$ is a normal complex space. Hence any holomorphic function $g$ on $\pi^{-1}(V) \cap (D - \pi^{-1}(Y'))$ is extendable to whole $\pi^{-1}(V)$, and moreover if $g$ satisfies (2) on $\pi^{-1}(V) \cap (D - \pi^{-1}(Y'))$, then the extension of $g$ also satisfies (2) on $\pi^{-1}(V)$. This shows that the injection of $\mathfrak{F}$ to $i_*(\mathfrak{F}|_{Y_0})$ is surjective. q.e.d.

The following is an easy consequence of Corollary to Proposition 5.2.3 Grothendieck [ 8 ].

Lemma 2.* Let $Y$ be a separated scheme over $\mathbb{C}$, and let $\mathfrak{F}$ be a coherent sheaf over $Y$. Let $G$ be a finite group acting on $Y, \mathfrak{F}$ compatibly, and $\pi: Y \to Y/G$ be the quotient morphism. Then we have

$$H^\nu(Y, \mathfrak{F})^G = H^\nu(Y/G, (\pi_*\mathfrak{F})^G).$$

## § 2. Hilbert modular forms

3. Let $K$ be a totally real algebraic number field of degree $n$ $(>1)$, and $O_K$ be the ring of integers. $SL_2(O_K)$ acts on the product $H^n$ of $n$ copies of the upper half plane $H = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ by the modular substitution;

$$z = (z_1, \ldots, z_n) \longrightarrow Mz \doteq \left(\frac{\alpha^{(1)} z_1 + \beta^{(1)}}{\gamma^{(1)} z_1 + \delta^{(1)}}, \ldots, \frac{\alpha^{(n)} z_n + \beta^{(n)}}{\gamma^{(n)} z_n + \delta^{(n)}}\right) \quad \text{for } M = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \in SL_2(O_K$$

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*The author was imformed this by Prof. T. Oda.

where $\alpha^{(1)},..,\alpha^{(n)}$ denote the conjugates of $\alpha \in K$ in some fixed order. Let $\Gamma$ be a subgroup of $SL_2(O_K)$ of finite index. A holomorphic function $f$ on $H^n$ is called a Hilbert modular form for $\Gamma$ of weight $k$ if it satisfies

(3) $f(Mz) = \prod_{i=1}^{n} (\gamma^{(i)} z_i + \delta^{(i)})^k f(z)$ for any $M = \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \in \Gamma$.

The symmetric group $\mathfrak{S}_n$ of $n$ letters acts on $H^n$ as permutations of the coordinates

$$z = (z_1,..,z_n) \longrightarrow \sigma z = (z_{\sigma(1)},..,z_{\sigma(n)}) \quad \sigma \in \mathfrak{S}_n.$$

The automorphism group $Aut(K/\mathbb{Q})$ can be regarded as a subgroup of $\mathfrak{S}_n$ because it acts on $n$-tuples $(\alpha^{(1)},..,\alpha^{(n)})$ as permutations, i.e., for $\sigma \in Aut(K/\mathbb{Q})$ $((\sigma\alpha)^{(1)},..,(\sigma\alpha)^{(n)})$ is nothing else but the permutation of $(\alpha^{(1)},..,\alpha^{(n)})$. Let us fix some subgroup $S$ of $Aut(K/\mathbb{Q}) \subset \mathfrak{S}_n$, and let $\hat{\Gamma}$ be the composite of $S$ and $\Gamma$ as groups acting on $H^n$. In what follows, we shall always suppose $\Gamma = \hat{\Gamma} \cap SL_2(O_K)$, in other words

(4) $\sigma\Gamma\sigma^{-1} = \Gamma$ for any $\sigma \in S$.

A holomorphic function $f$ on $H^n$ is called a (symmetric) Hilbert modular form for $\hat{\Gamma}$ if it satisfies both (3) and the identity $f(\sigma z) = f(z)$ for $\sigma \in S$. We shall denote by $A(\hat{\Gamma}) = \oplus A(\hat{\Gamma})_k$, the graded $\mathbb{C}$-algebra of Hilbert modular forms for $\hat{\Gamma}$, $A(\hat{\Gamma})_k$ being the vector space of Hilbert modular forms of weight $k$, and denote by $A(\hat{\Gamma})^{(r)}$, the subring $\bigoplus_{k \equiv 0\ (r)} A(\hat{\Gamma})_k$.

4. Let $h$ denote the number of the cusps for $\hat{\Gamma}$. $X = H^n/\hat{\Gamma}$ is compactified by adding $h$ points, and we get a normal projective

variety $X^*$, which is isomorphic to $\mathrm{Proj}(A(\hat{\Gamma}))$. We shall denote by $X_0$, the regular open subset of $X$, hence of $X^*$.

Let $\mathcal{L}(i)$ denote the coherent sheaf on $X^*$ corresponding to modular forms of weight $i \in \underset{\sim}{Z}$, and let $\mathcal{L} = \mathcal{L}(1)$. Obviously we have $\mathcal{L}(i) \otimes \mathcal{L}(j) \subset \mathcal{L}(i+j)$ for $i, j \geq 0$. Let $\pi : \tilde{X} \longrightarrow X^*$ be a desingularization. The canonical coherent sheaf $K_{X^*}$ on $X^*$ is given by $K_{X^*} = \pi_* K_{\tilde{X}}$, $K_{\tilde{X}}$ being the canonical invertible sheaf on $\tilde{X}$. $K_{X^*}$ is determined up to desingularizations (Grauert-Riemenschneider [ 7 ]). We shall need also the dualizing sheaf $\omega_{X^*}$ which gives rise to the functorial isomorphism $\mathrm{Hom}(\mathcal{F}, \omega_{X^*}) = H^n(X^*, \mathcal{F})^\vee$ for coherent sheaves $\mathcal{F}$. Again by [ 7 ], $\omega_{X^*}$ equals $i_* K_{X_0}$ where $i$ denotes the inclusion of $X_0$ to $X^*$. If $X^*$ is Cohen-Macaulay, then there are natural isomorphisms $H^\nu(X^*, \mathcal{F}) = H^{n-\nu}(X^*, \mathcal{F}^\vee \otimes \omega_{X^*})^\vee$ for any locally free sheaf $\mathcal{F}$ and for its dual $\mathcal{F}^\vee$. We have the canonical inclusion $K_{X^*} \subset \omega_{X^*}$ (loc. cit). Moreover by Freitag [ 5 ] Satz 1 we have an equality $K_{X^*}|_X = \omega_{X^*}|_X$.

If $S$ is a subgroup of the alternating group, then $\omega_{X^*}$, $\mathcal{L}(2)$ are isomorphic. Let us show this. Let $X^0$ be the open subset of $X$ which is the complement of the fixed points set. Obviously $X^0 \subset X_0$. For an open subset $U$ of $X^0$ if $f$ is a section of $\Gamma(U, \mathcal{L}(2))$, then $f\,dz_1 \wedge \cdots \wedge dz_n$ gives a section of $\Gamma(U, K_{X_0})$ and vise verse. So $K_{X_0}$ and $\mathcal{L}(2)$ are isomorphic on $X^0$. Since the codimension of $X^0$ in $X$ is larger than or equal to two as one can easily see, $K_{X_0}$, $\mathcal{L}(2)$ are isomorphic on $X_0$ by the extendability of holomorphic functions. So by Lemma 1 we get $\omega_{X^*}|_X = \mathcal{L}(2)|_X$. Let $\infty$ be any cusp, and let

$U$ be an open neighborhood at $\infty$. Then a section $f$ of $\Gamma(U-\{\infty\},\mathcal{L}(j))$, $j \in Z$, admits Fourier expansion

$$f(z) = c_0 + \sum_\lambda c_\lambda \exp(2\pi\sqrt{-1}(\lambda^{(1)}z_1 + \cdots + \lambda^{(n)}z_n))$$

where $\lambda$ varies over some lattice of totally positive numbers in $K$. So $f$ is holomorphic at $\infty$, and hence we get $i_*(\mathcal{L}(j)|_{U-\{\infty\}}) = \mathcal{L}(j)$, $i$ being the inclusion $U-\{\infty\} \longrightarrow U$. Thus

$$\mathcal{L}(2) = \omega_{X^*}.$$

Suppose that $S$ is a group not contined in the alternating group. Let $S'$ be the normal subgroup of $S$ of index two which is a subgroup of the alternating group. Let $\psi: (H^n/\hat{\Gamma}')^* \longrightarrow X^* = (H^n/\hat{\Gamma})^*$ be the canonical projection where $\hat{\Gamma}' = S'\cdot\Gamma$. If $\mathcal{L}'(2)$ is the coherent sheaf on $(H^n/\hat{\Gamma}')^*$ corresponding to modular forms of weight two, then we define a coherent sheaf $\mathcal{L}(2)_-$ on $X^*$ by

(5) $\quad \Gamma(U,\mathcal{L}(2)_-) = \{f \in \Gamma(\psi^{-1}(U),\mathcal{L}'(2)) \mid f(\sigma z) = \mathrm{sgn}(\sigma) f(z),\ \sigma \in S\}$,

$U$ being an open subset of $X^*$. Any section of $\mathcal{L}(2)_-$ vanishes along the fixed points set under the action of the group $S/S'$. Then $\mathcal{L}(2)_-|X_0$ is isomorphic to $K_{X_0}$ by the similar argument as above and by [ 5 ] the proof of Hilfsssatz 4, and moreover

$$\mathcal{L}(2)_- = \omega_{X^*}.$$

5. Let $X^* = (H^n/\hat{\Gamma})^*$, $\mathcal{L}(i)$, $\mathcal{L} = \mathcal{L}(1)$ be as above.

Lemma 3. Assume that $\Gamma = \hat{\Gamma} \cap SL_2(O_K)$ acts freely on $H^n$. Then

i) $\mathcal{L}(i)$ is invertible, and $\mathcal{L}^i = \mathcal{L}(i)$,

ii) $H^\nu(X^*,\mathcal{L}(i)) = 0$ for $i \geq 2$, $\nu > 0$, $(i,\nu) \neq (2,n)$.

Proof. Let $\pi:\tilde{X} \longrightarrow X^*$ be the desingularization. Then the cohomology group $H^{\nu}(\tilde{X}, K_{\tilde{X}})$ $(= H^{n-\nu}(\tilde{X}, \mathcal{O}_{\tilde{X}})^{\vee})$ vanishes for $0 < \nu < n$ (cf. [ 6 ]). Since all the higher direct image sheaves $R^{\nu}\pi_* K_{\tilde{X}}$ $(\nu \geq 1)$ vanish by [ 7 ], also $H^{\nu}(X^*, K_{X^*})$ $0 < \nu < n$ vanish by using the Leray spectral sequence.

At first let us suppose $\hat{\Gamma} = \Gamma$. Then i) is obvious, and this implies that $\mathcal{L}$ is ample. We have an exact sequence

$$0 \longrightarrow K_{X^*} \longrightarrow \mathcal{L}^2 \longrightarrow \mathcal{L}^2/K_{X^*} \longrightarrow 0,$$

where $\mathcal{L}^2/K_{X^*}$ is supported only at cusps by the observation of § 2.4. Tensoring $\mathcal{L}^i$ and taking the long exact sequence

$$\longrightarrow H^{\nu}(X^*, \mathcal{L}^i \otimes K_{X^*}) \longrightarrow H^{\nu}(X^*, \mathcal{L}^{2+i}) \longrightarrow H^{\nu}(X^*, \mathcal{L}^i \otimes (\mathcal{L}^2/K_{X^*})) \longrightarrow,$$

we get the desired result since $H^{\nu}(X^*, \mathcal{L}^i \otimes (\mathcal{L}^2/K_{X^*}))$, $\nu > 0$, vanishes, and since also $H^{\nu}(X^*, \mathcal{L}^i \otimes K_{X^*})$, $i \geq 0$, $\nu > 0$, $(i,\nu) \neq (0,n)$, vanishes by the generalized Kodaira vanishing theorem (cf. [ 7 ] Satz 2.1) and by the above observation.

Let us consider the general case. Let us put $Y = H^n/\Gamma$, and let $\mathfrak{M}$ be the invertivle sheaf on $Y^*$ corresponding to modular forms of weight one. We have shown above that i),ii) hold for $Y^*, \mathfrak{M}$. To prove i) for $X^*, \mathcal{L}$, it is enough to show that for any point $x$ of $X^*$, there is an neighborhood $V$ at $x$ such that $\Gamma(V, \mathcal{L})$ has a section not vanishing at $x$ as a function. If $x$ is not a ramification point of the canonical projection $p: Y^* \longrightarrow X^*$, then nothing is a problem. If $x$ is such a point, then we can take a point $y \in Y$ with $x = p(y)$, and its neighborhood $W$ such that $\Gamma(W, \mathfrak{M})$ has a section $f$ not vanishing at $y$. Then $g = \Sigma\ \sigma f$, $\sigma$

running over the stabilizer subgroup at $y$ of $S = \hat{\Gamma}/\Gamma$, is a desired element, indeed if we take a sufficiently small neighborhood $V$ at $x$, then $g|_V$ is a section of $\Gamma(V,\mathcal{L})$ whose value $g(x)$ at $x$ is not zero. This shows i). ii) is a direct consequence of Lemma 2, noticing $\mathcal{L} = (p_*\mathfrak{M})^S$. q.e.d.

For any $\hat{\Gamma}$, there is a normal subgroup $\hat{\Gamma}'$ of finite index such that $\hat{\Gamma}' \cap SL_2(O_K)$ acts freely on $H^n$. Then $\hat{\Gamma}/\hat{\Gamma}'$ acts on $(H^n/\hat{\Gamma}')^*$ as a finite automorphism group, and the quotient morphism $p: (H^n/\hat{\Gamma}')^* \longrightarrow (H^n/\hat{\Gamma})^* = X^*$ is induced. If $\mathcal{L}'(i)$ is the invertible sheaf on $(H^n/\hat{\Gamma}')^*$ corresponding to modular forms of weight $i$, then $\mathcal{L}(i)$ equals to $(p_*\mathcal{L}'(i))^{\hat{\Gamma}/\hat{\Gamma}'}$. So by Lemma 3 and Lemma 2, we have the following;

Proposition 1.

$$H^{\nu}(X^*,\mathcal{L}(i)) = 0 \quad \underline{\text{for}} \quad i \geq 2,\ \nu > 0,\ (i,\nu) \neq (2,n).$$

6. Our main theorem is as follows;

Theorem 1. <u>Let</u> $n$, $\Gamma$, $\hat{\Gamma}$, $A(\hat{\Gamma})$ <u>be as in</u> § 2.3, <u>and let</u> $r$ <u>be under the condition</u> (4). <u>Then</u> $A(\hat{\Gamma})^{(r)}$ <u>is Cohen-Macaulay for any</u> (<u>equivalently some</u>) $r \geq 2$ <u>if and only if</u>

$$(6)\qquad \begin{array}{ll} n \leq 2, & \underline{\text{or}} \\ n = 3 \quad \hat{\Gamma}_\infty/\Gamma_\infty \cong \mathbb{Z}/3\mathbb{Z} & \underline{\text{at each cusp}}\ \infty,\ \underline{\text{or}} \\ n = 4 \quad \hat{\Gamma}_\infty/\Gamma_\infty \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \underline{\text{at each cusp}}\ \infty, \end{array}$$

where $\hat{\Gamma}_\infty$ (resp. $\Gamma_\infty$) is denoting the stabilizer subgroup of $\hat{\Gamma}$ (resp. $\Gamma$) at $\infty$.

By Thomas and Vasquez [ 20 ] Theorem 2 we also get the following corollary;

Corollary. Let $n = 2$, and let $\Gamma$ be $SL_2(O_K)$ or its torsion free subgroup. Then $A(\Gamma)^{(2)}$ is Gorenstein.

To prove Theorem 1 the following is a key proposition.

Proposition C (Freitag [ 6 ]). The condition (6) and the following two conditons (a),(b) are all equivalent to each other.

(a) $X^*$ is Cohen-Macaulay,

(b) $H^\nu(X^*, \mathcal{O}_{X^*}) = 0$ for $0 < \nu < n$.

By the fact we saw in § 1.1, $A(\hat{\Gamma})^{(r)}$ cannot be Cohen-Macaulay for any $r$ unless (6) is the case. In [ 20 ] it is shown that $A(\Gamma)^{(2)}$ is never Cohen-Macaulay under some condition on $\Gamma$ with $n = 3$. But it is not even Cohen-Macaulay.

7. Proof of Theorem 1. By Proposition A and C it is enough to show merely 'if' part. We shall give the proofs of two kinds, however one is available only for $n = 2$. At first we assume $n = 2$.

Then $X^*$ is a normal surface and hence it is Cohen-Macaulay. We may assume $\Gamma$ $(= \hat{\Gamma} \cap SL_2(O_K))$ acts freely on $H^2$ by replacing $\hat{\Gamma}$ by a normal subgroup $\hat{\Gamma}'$ of finite index if necessary. Indeed if $A(\hat{\Gamma}')^{(r)}$ is Cohen-Macaulay, then so is $A(\hat{\Gamma})^{(r)}$ by Proposition B because $A(\hat{\Gamma})^{(r)}$ is the invariant subring of $A(\hat{\Gamma}')^{(r)}$ under the action of $\hat{\Gamma}/\hat{\Gamma}'$. So we may assume that $\mathcal{L}$ is an ample invertible sheaf by Lemma 3. Since $X^*$ is Cohen-Macaulay, we have an isomorphism between the cohomology groups

$$H^{\nu}(X^*, \mathcal{L}^{-i}) \simeq H^{n-\nu}(X^*, \mathcal{L}^{i} \otimes \omega_{X^*})^{\vee}$$

by Serre's duality theorem. As we have seen in § 2.4, $K_{X^*}$ is a subsheaf of $\omega_{X^*}$ and $\omega_{X^*}/K_{X^*}$ is supported only at cusps. Now the similar argument as in the proof of Lemma 3 will derive the vanising of the cohomology groups $H^{n-\nu}(X^*, \mathcal{L}^{i} \otimes \omega_{X^*})$ for $n - \nu > 0$, $i > 0$, $(i, n-\nu) \neq (0, n)$ (use $\omega_{X^*}$ instead of $\mathcal{L}^2$). So $H^{\nu}(X^*, \mathcal{L}^{-i})$ vanishes for $0 < \nu < n$, $i > 0$. Together with Proposition 1 and Proposition C (b) we get

$$H^{\nu}(X^*, \mathcal{L}^{i}) = 0 \quad \text{for} \quad 0 < \nu < n, \quad i \equiv 0 \bmod r,$$

where $r$ is any integer greater than one. By Corollary to Proposition A this implies that $A(\hat{\Gamma})^{(r)}$ is Cohen-Macaulay, and our assertion is proved when $n = 2$.

In the case $n = 3,4$ the above argument does not work since $(H^n/\hat{\Gamma}')^*$ may not be Cohen-Macaulay even if so is $(H^n/\hat{\Gamma})^*$, where $\hat{\Gamma}'$ is a subgroup of $\hat{\Gamma}$. Let us take a normal subgroup $\Gamma'$ of $\hat{\Gamma}$ which acts freely on $H^n$. Then by the virture of [ 1 ] we have a smooth toroidal compactification $\overline{X'}$ of $X' = H^n/\Gamma'$, on which, we may assume, the finite quotient group $\hat{\Gamma}/\Gamma'$ acts in the natural way (cf. [ 1 ], [ 22 ]). Let us put $\overline{X} = \overline{X'}/(\hat{\Gamma}/\Gamma')$ that has only quotient singularities. $\overline{X}$ (resp. $\overline{X'}$) has $X = H^n/\hat{\Gamma}$ (resp. $X'$) as its Zariski open subset. Let $\pi$ (resp. $\pi'$) be the morphism of the blowing up $\overline{X} \longrightarrow X^*$ (resp. $\overline{X'} \longrightarrow X'^*$), and let $\psi$ (resp. $\overline{\psi}$) be the quotient map of $X'$ to $X^*$ (resp. $\overline{X'}$ to $\overline{X}$). We have a commutative diagram

$$\begin{array}{ccc} \overline{X}' & \xrightarrow{\overline{\psi}} & \overline{X} \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ X'^* & \xrightarrow[\psi]{} & X^* \end{array}$$

We shall show that the morphism $\pi$ enjoys

$$R^{\nu}\pi_*\mathcal{O}_{\overline{X}} = 0 \quad 0 < \nu < n-1,$$

by using our assumption of $X^*$ being Cohen-Macaulay. Let $\tilde{\pi}:\tilde{X} \longrightarrow \overline{X}$ be the desingularization. Since $\overline{X}$ has only ratinal singularities, the higher direct image sheaves $R^{\nu}\pi_*\mathcal{O}_{\tilde{X}}$ $\nu > 0$ vanish. $\pi\circ\tilde{\pi}:\tilde{X} \longrightarrow X^*$ is the desingularization of $X^*$, and by the same reason as above $R^{\nu}(\pi\circ\tilde{\pi})_*\mathcal{O}_{\tilde{X}}|_X$ vanish for $\nu > 0$. Since $(R^{\nu}(\pi\circ\tilde{\pi})_*\mathcal{O}_X)_{\infty} = 0$ for $0 < \nu < n-1$ if the local ring at a cusp $\infty$ is Cohen-Macaulay as Freitag showed [ 6 ] 4.5, $R^{\nu}(\pi\circ\tilde{\pi})_*\mathcal{O}_{\tilde{X}}$ vanish for $0 < \nu < n-1$. Considering the Leray spectral sequence $E_2^{p,q} = R^p\pi_*(R^q\tilde{\pi}_*\mathcal{O}_{\tilde{X}}) \Longrightarrow R^{p+q}(\pi\circ\tilde{\pi})_*\mathcal{O}_{\tilde{X}}$, we get the vanishing of $R^{\nu}\pi_*\mathcal{O}_{\overline{X}}$ for $0 < \nu < n-1$.

If $\mathcal{L}'(i)$ is the invertible sheaf on $X'^*$ corresponding to modular forms of weight $i$, then $(\overline{\psi}_*\pi'^*\mathcal{L}'(i))^{\hat{\Gamma}/\Gamma'}$ is equal to $\pi^*\mathcal{L}(i)$ by using the facts that (i) $\pi,\pi'$ are birational, (ii) $X^*, X'^*$ are normal, and (iii) $\psi_*\mathcal{L}'(i)^{\hat{\Gamma}/\Gamma'} = \mathcal{L}(i)$. $H^{\nu}(\overline{X'}, \pi'^*\mathcal{L}'(i))$ vanishes for $\nu < n,\ i < 0$ by the generalized Kodaira vanishing theorem [ 7 ]. So applying Lemma 2, we have $H^{\nu}(\overline{X}, \pi^{*}\mathcal{L}(i)) = 0$ for $\nu < n,\ i < 0$. Since $\mathcal{L}(i)$ is locally free near at each cusp, the projection formula $R^{\nu}\pi_{*}\mathcal{L}(i) = \mathcal{L}(i) \otimes R^{\nu}\pi_{*}\,{}_{\overline{X}}$ holds and hence there exists the Leray spectral sequence $E_2^{p,q} = H^p(X^{*}, \mathcal{L}(i) \otimes R^q\pi_{*}\mathcal{O}_{\overline{X}}) \Longrightarrow H^{p+q}(X, \pi^{*}\mathcal{L}(i))$. It follows from this that $H^{\nu}(X^{*}, \mathcal{L}(i)) = 0$ for $\nu < n,\ i < 0$. Now the same argument as above shows our assertion. q.e.d.

In the above proof, we have shown under the condition (6) that $H^{\nu}(X^{*}, \mathcal{L}(i))$ vanishes for $0 < \nu < n$ if $i \neq 1$. As a consequence of this we get the following;

Proposition 2. $A(\hat{\Gamma})$ <u>is Cohen-Macaulay if and only if</u>

$$H^{\nu}(X^{*}, \mathcal{L}) = 0 \quad \underline{\text{for}} \quad 0 < \nu < n$$

<u>together with the condition</u> (6).

8. In what follows we always assume $n = 2$, and that $\Gamma$ acts freely on $H^2$. Let $a$ be the index $a = [SL_2(O_K):\Gamma]$, and let $\chi$ be the arithmetic genus $\sum_{\nu=0}^{2} (-1)^{\nu} \dim H^{\nu}(\tilde{X}, \mathcal{O}_{\tilde{X}})$ where $\tilde{X}$ is the non-singular model of $X^{*} = (H^2/\Gamma)^{*}$. $\chi$ is equal to $1 +$ dim. of the space of cusp froms of weight two. By Shimizu [ 17 ] (see also Hirzebruch [ 10 ] § 2 Theorem, Freitag [ 6 ] 7.2 Satz) we have Hilbert polynomial $P(k)$ of $A(\Gamma)$:

(7) $$P(k) = \tfrac{1}{2}\cdot\zeta_K(-1)\cdot a\, k(k-2) + \chi + h,$$

where $\zeta_K$ is the zeta function of K, and h is the number of cusps. $P(k)$ gives the dimension of $A(\Gamma)_k$ for $k \geq 3$, and $P(2)$ equals $\dim A(\Gamma)_2 + 1$. $P(k)$ must be equal to the Euler-Poincaé characteristics $\chi(\mathcal{L}^k) = \sum_{\nu=0}^{2} (-1)^\nu \dim H^\nu(X^*, \mathcal{L}^k)$, which is known to be a polynomial of k (cf. [15]). Hence we have

$$-\tfrac{1}{2}\zeta_K(-1)\cdot a + \chi + h = \dim H^0(X^*,\mathcal{L}) - \dim H^1(X^*,\mathcal{L}) + \dim H^2(X^*,\mathcal{L}).$$

Since $\mathcal{L}^2$ is now Serre's dualizing sheaf (§ 2.4), $H^2(X^*,\mathcal{L})$ is just dual to $H^0(X^*,\mathcal{L})$. So we obtain

$$\dim A(\Gamma)_1 = \tfrac{1}{2}(-\tfrac{1}{2}\zeta_K(-1)\, a + \chi + h) + \tfrac{1}{2}\dim H^1(X^*,\mathcal{L}).$$

Especially the inequality

$$\dim A(\Gamma)_1 \geq \tfrac{1}{2}(-\tfrac{1}{2}\zeta_K(-1)\, a + \chi + h)$$

always holds, and $A(\Gamma)$ is Cohen-Macaulay if and only if the equality (1) holds by Proposition 2.

(1) is a nice equality in the following sence. If (1) is the case, then we can compute the generating function $Q(t) = \Sigma \dim A(\Gamma)_k t^k$ together with (7) and with $\dim A(\Gamma)_2 = P(2) - 1$ as

$$Q(t) = \frac{1}{(1-t)^3}\{1 + t^5 + (t + t^4)\{\tfrac{1}{2}(-\tfrac{1}{2}\zeta_K(-1)\, a + \chi - h) - 3\} + (t^2 + t^3)\{\tfrac{1}{2}(3/2\,\zeta_K(-1)\, a - \chi - h) - 2\}\}.$$

It is easy to see $Q(t)$ satisfies $-t^2 Q(t^{-1}) = Q(t)$. By Stanley [18] this implies that $A(\Gamma)$ is Gorenstein.

9. Let $X^*$, $\Gamma$ be as above. Let $\hat{X}^*$ denote $(H^2/\hat{\Gamma})^*$ where $\hat{\Gamma} = \mathfrak{S}_2\cdot\Gamma$, and let $\hat{\Gamma}$ be the invertible sheaf on $\hat{X}$ corresponding to symmetric Hilbert modular forms of weight one. Let us suppose (4).

Then if $p: X^* \longrightarrow \hat{X}^*$ is the canonical projection, we have the direct decomposition

$$p_* \mathcal{L} = \hat{\mathcal{L}} \oplus \mathcal{L}_-$$

where $\mathcal{L}_-$ is the coherent sheaf given in the similar way as (5). Since $\hat{\mathcal{L}} \otimes \mathcal{L}_- = \mathcal{L}(2)_-$ , it gives Serre's dualizing sheaf on $\hat{X}^*$ (§ 2.4). Thus we have

$$H^1(\hat{X}^*, \mathcal{L}_-) \cong \operatorname{Ext}^1(\mathcal{O}_{\hat{X}^*}, \mathcal{L}_-) \cong \operatorname{Ext}^1(\hat{\mathcal{L}}, \hat{\mathcal{L}} \otimes \mathcal{L}_-) \cong H^1(\hat{X}^*, \hat{\mathcal{L}})^\vee$$

and hence

$$\begin{aligned} H^1(X^*, \mathcal{L}) &\cong H^1(\hat{X}^*, p_*\mathcal{L}) \cong H^1(\hat{X}^*, \hat{\mathcal{L}}) \oplus H^1(\hat{X}^*, \mathcal{L}_-) \\ &\cong H^1(\hat{X}^*, \hat{\mathcal{L}}) \oplus H^1(\hat{X}^*, \hat{\mathcal{L}})^\vee, \end{aligned}$$

and hence $\dim H^1(X^*, \mathcal{L}) = 2 \dim H^1(\hat{X}^*, \hat{\mathcal{L}})$. Thus $A(\Gamma)$ and $A(\hat{\Gamma})$ are Cohen-Macaulay or not alike by Proposition 2. Summing up the above we shall state it as the proposition.

Proposition 3. *Let $K$ be a real quadratic field, and $\Gamma$ be a subgroup of $SL_2(O_K)$ of finite index acting freely on $H^2$. Then the following are equivalent:*

(a) $A(\Gamma)$ *is Gorenstein.*

(b) $A(\Gamma)$ *is Cohen-Macaulay.*

(c) *The equality* (1) $\dim A(\Gamma)_1 = \frac{1}{2}(-\frac{1}{2}\zeta_K(-1)a + \chi + h)$ *holds.*

*Assuming* (4) *for* $S = \mathfrak{S}_2$,

(d) $A(\hat{\Gamma})$ *is Cohen-Macaulay.*

The known examples of a full ring $A(\Gamma)$ for above $\Gamma$ are quite a few yet. At any rate such examples in Hirzebruch [ 11 ], which are

$K = Q(\sqrt{5})$, $\Gamma = \Gamma(\sqrt{5}) = \{M \in SL_2(O_K) \mid M \equiv 1_2 \bmod \sqrt{5}\}$, and

$K = Q(\sqrt{2})$, $\Gamma = \Gamma(2)\cdot\left\langle\begin{pmatrix}1+\sqrt{2} & \\ & 1-\sqrt{2}\end{pmatrix}\right\rangle$, $\Gamma(2) = \{M \in SL_2(O_K) \mid M \equiv 1_2 \bmod 2\}$,

are satisfying the conditions in Proposition 3. It may not be unreasonable to expect it in more general case.

10. Let $\Gamma$ be as above. By the mehtod of [ 20 ], we can show that $A(\Gamma)$ may possibly be a complete intersection ring only in a finite number of cases as following. The index $a = [SL_2(O_K):\Gamma]$ is divisible by 6 because $SL_2(O_K)$ has torsion points of order 2,3, on the other hand $\Gamma$ not (cf. Hirzebruch [ 10 ] § 1.7). Let us put $a = 6f$. Then $A(\Gamma)$ may possibly be a complete intersection ring only if $(2\zeta_K(-1)f, h+\chi)$ is one of

$(32/3,32),(8,26),(16/3,20),(8/3,14),(4,16),(2,11),$

$(4/3,8),(10/3,15),(4/3,10),(2/3,7),(2/3,9)$.

Considering the values of the zeta function at $-1$, this cannot happen if the discriminant of $K$ is larger than 105. We skip the proof, which will be almost the same as in [ 20 ].

## § 3 Siegel modular forms

11. Let $H_n$ be the Siegel space of degree $n$, i.e., $\{Z \in M_n(\mathbb{C}) \mid {}^tZ = Z, \operatorname{Im} Z > 0\}$. The symplectic group $Sp_n(\mathbb{R})$ acts on $H_n$ by the usual modular substitution

$$Z \longmapsto MZ = (AZ+B)(CZ+D)^{-1} \qquad M = \begin{pmatrix}A & B\\ C & D\end{pmatrix} \in Sp_n(\mathbb{R}).$$

We shall denote by $\Gamma_n(\ell)$, the principal congruence subgroup of level $\ell$; $\{M \in Sp_n(\mathbb{Z}) \mid M \equiv 1_{2n} \bmod \ell\}$.

Let $f$ be a holomorphic function on $H_n$. $f$ is called Siegel modular form of weight $k$ for a congruence subgroup , if it satisfies

$$f(MZ) = |CZ + D|^k f(Z) \quad \text{for} \quad M = \binom{AB}{CD} \in \Gamma .$$

When $n = 1$, we need an additional condition that $f$ is holomorphic also at cusps, which is automatic if $n > 1$. We denote by $A(\Gamma) = \bigoplus_{k \geq 0} A(\Gamma)_k$ (resp. $S(\Gamma) = \bigoplus_{k \geq 0} S(\Gamma)_k$), the graded ring of modular forms (resp. the graded ideal of cusp forms).

Let $X = H_n/\Gamma$, and let $X^*$ be its Satake compactification, which is normal projective variety isomorphic to $\mathrm{Proj}(A(\Gamma))$.

12. Let $\Gamma$ be neat, and let $\mathcal{L}$ be an invertible sheaf on $X^*$ corresponding to modular forms of weight one. The regular open subset of $X^*$ coincides with $X$, and $\mathcal{L}^{n+1}_{|X}$ is isomorphic to the canonical invertible sheaf $K_X$ on $X$. Then by [ 7 ], the dualizing sheaf $\omega_{X^*}$ on $X^*$ is given by $i_* \mathcal{L}^{n+1}_{|X}$, $i$ being the inclusion map of $X$ to $X^*$, where $\omega_{X^*}$ gives rise to the functorial isomorphism $\mathrm{Hom}(\mathcal{F}, \omega_{X^*}) \simeq H^{n(n+1)/2}(X^*, \mathcal{F})^\vee$ for coherent sheaves $\mathcal{F}$ on $X^*$. Here we note

$$\omega_{X^*} \simeq \mathcal{L}^{n+1}$$

by Koecher's principle. So if $X^*$ is Cohen-Macaulay, then $H^\nu(X^*, \mathcal{L}^k)$ is isomorphic to the dual of $H^{n(n+1)/2 - \nu}(X^*, \mathcal{L}^{n+1-k})$ and hence $P(k) = (-1)^{n(n+1)/2} P(n+1-k)$, $P(k)$ denoting the Hilbert polynomial of the graded ring $A(\Gamma)$ or equivalently $\chi(\mathcal{L}^k)$.

On the other hand it is shown in [ 2 ] Vol 2 - 16 that

$$P(k) = \dim A(\Gamma)_k = \dim S(\Gamma)_k + \sum_{\Gamma' \subset Sp_{n-1}(\underset{\sim}{R})} \dim S(\Gamma')_k$$

$$+ \cdots + \sum_{\Gamma' \subset Sp_1(\underset{\sim}{R})} \dim S(\Gamma')_k + \#(0 \text{ dimensional cusps}$$

for $k \gg 0$ where $\Gamma'$ varies over the set of all the subgroups attached to cusps of $X^*$. (The above is shown in [ 2 ] for $k \gg 0$, $k \equiv 0$ mod 2. However both sides must be numerical polynomials of $k$ for $k \gg 0$, so we get the above formula.)

13. Let us consider the case $n = 2$. $A(\Gamma_2(2))$ was shown to be Cohen-Macaulay in Igusa [ 14 ]. So for any arithmetic group $\Gamma$ containing $\Gamma_2(2)$ as a normal subgroup, $A(\Gamma)$ is Cohen-Macaulay by Proposition B. However the Cohen-Macaulayness fails for $\Gamma_2(\ell)$ $\ell \geq 6$. We shall show it.

Let $X^*$ be the Satake compactification of $H^2/\Gamma_2(\ell)$ for some $\ell \geq 3$, and let $P(k)$ be the Hilbert polynomial for $A(\Gamma_2(\ell))$. Then if $X^*$ is Cohen-Macaulay, we would have $P(3/2) = 0$ since $P(k) = -P(3-k)$ by the observation in § 3.12. By Yamazaki [ 23 ] we can actually calculate $P(k)$ and hence $P(3/2)$;

$$P(3/2) = 2^{-4}3^{-1}\ell^4 \prod_{p|\ell}(1-p^{4})\{(\ell^3 - 6\ell^2)\prod_{p|\ell}(1-p^{-2}) + 2^4 3\}.$$

This is not zero if $\ell \geq 6$, so in this case $X^*$ cannot be Cohen-Macaulay and hence $A(\Gamma_2(\ell))$ $\ell \geq 6$ are not Cohen-Macaulay algebras.

The similar argument works also for $\Gamma = \Gamma_3(\ell)$ $\ell \geq 3$ by using the formula by Tsushima [ 21 ]. Indeed if $(H_3/\Gamma_3(\ell))^*$ were Cohen-

Macaulay, then the Hilbert polynomial $P(k)$ of the graded ring $A(\Gamma_3(\ell))$ would satisfy $P(k-2) - P(2-k) = 0$ by the observation in § 3.12. However actually we have :

$$P(k-2) - P(2-k) = 2^{-7}3^{-3}5^{-1}\,\ell^{16} \prod_{p\mid\ell}(1-p^{-2})(1-p^{-4})(1-p^{-6})\,k^3 + O(k^2).$$

So $(H_3/\Gamma_3(\ell))^*$ is not Cohen-Macaulay. We obtain the following;

Proposition 4. Let $\Gamma = \Gamma_n(\ell)$ with $n = 2$, $\ell \geq 6$ or $n = 3$, $\ell \geq 3$. Then the Satake compactification of $H_n/\Gamma_n(\ell)$ is not a Cohen-Macaulay variety. Especialy if $A(\Gamma)$ denotes the ring of Siegel modular forms for $\Gamma$, then $A(\Gamma)^{(r)}$ is not Cohen-Macaulay for any integer $r$.

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# PART II

# Rings of automorphic forms which are not Cohen-Macaulay, I

Shigeaki Tsuyumine

By Noether's normalization theorem, a noetherian graded algebra $R$ has a polynomial subring $S$ generated by homogeneous elements such that $R$ is finite over $S$. It is known (see, for instance Stanley [ 24 ], § 3) that $R$ is Cohen-Macaulay (C.-M., for short) if and only if $R$ is free over any (equivalently some) such $S$. Thus it is meaningful to ask which of the graded rings of automorphic forms are C.-M. This is a problem posed by Eichler [ 4 ],[ 5 ]. Igusa [ 16 ] determined the structure of the graded rings of Siegel modular forms of degree two for subgroups of $\Gamma_2(2)$, and Resnikoff and Tai [ 20 ],[ 26 ] determined the structure of the graded rings of automorphic forms on the complex 2-ball for some arithmetic group. These rings turn out to be C.-M. However Freitag [ 6 ] showed that the ring of Hilbert modular forms of degree $\geq 3$ is not C.-M., while in our previous paper [ 27 ], we proved that the ring of Hilbert modular forms of even weight and of degree two is C.-M. In this paper we show that the ring of automorphic forms fails to be C.-M. for a large class of neat arithmetic groups as well as for the Siegel modular group $\Gamma_g = Sp_{2g}(\mathbb{Z})$, $g \geq 4$.

Samuel [ 22 ] stated "All the examples of U.F.D.'s I know are C.-M. Is it true in general?" (see Lipman [ 19 ] for the history of this question). In the case of characteristic zero, Freitag and Kiehl [ 9 ] gave a negative answer to this question of Samuel by constructing analytic local rings which are U.F.D.'s of dimension 60 and depth 3, hence not C.-M. As far as we know these are the only previously known examples. As Freitag [ 7 ],[ 8 ] has shown, the ring of Siegel modular forms for $\Gamma_g$ ($g \geq 3$) is U.F.D. Hence our result shows that they furnish negative examples to Samuel's question in arbitrary high dimension.

To prove our assertion it is enough to prove that the Baily-Borel compactification of the corresponding quotient space is not a C.-M. scheme, where a C.-M. scheme is a scheme whose local rings are all C.-M. This gives some generalization of the result of Igusa [ 17 ], where he shows that the Baily-Borel compactification does not admit a finite nonsingular covering under some condition on the bounded symmetric domain and the arithmetic group.

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## § 1. Main result

1.1 Let $H_g$ be the Siegel space of degree $g$, i.e., $\{ Z \in M_g(\mathbb{C}) |$

${}^tZ = Z$, $\mathrm{Im}\, Z > 0\}$. The symplectic group $Sp_{2g}(\mathbb{R}) = \{ M = \binom{AB}{CD} \in M_{2g}(\mathbb{R}) \mid A{}^tD - B{}^tC = 1_g,\ A{}^tB = B{}^tA,\ C{}^tD = D{}^tC\}$ acts on $H_g$ by the <u>symplectic transformation</u>

$$Z \longrightarrow MZ = (AZ+B)(CZ+D)^{-1} \qquad M = \binom{AB}{CD} \in Sp_{2g}(\mathbb{R}).$$

Let $\Gamma_g$ denote the Siegel modular group $Sp_{2g}(\mathbb{Z})$. A holomorphic function $f$ on $H_g$ is called a <u>Siegel modular form</u> of weight $k$ if it satisfies

$$f(MZ) = |CZ+D|^k f(Z) \qquad M = \binom{AB}{CD} \in \Gamma_g$$

and if it is holomorphic also at cusps (the last condition is automatic if $g > 1$). Let $A(\Gamma_g) = \bigoplus_{k\geq 0} A(\Gamma_g)_k$ denote the graded ring of Siegel modular forms. The quotient space $H_g/\Gamma_g$ is the coarse moduli space of the principally polarized abelein varieties over $\mathbb{C}$ of dimension $g$. It has the natural compactification $(H_g/\Gamma_g)^*$ called the <u>Satake compactification</u> which is isomorphic to $\mathrm{Proj}(A(\Gamma_g))$, and set-theoretically equals

$$H_g/\Gamma_g \cup H_{g-1}/\Gamma_{g-1} \cup \cdots \cup H_1/\Gamma_1 \cup \{\text{a point}\}.$$

Theorem 1. <u>Let</u> $g \geq 4$. <u>Then the Satake compactification</u> $(H_g/\Gamma_g)^*$ <u>is not a Cohen-Macaulay scheme</u>. <u>For the graded ring</u> $A(\Gamma_g)$ <u>of Siegel modular forms, the ring</u> $A(\Gamma_g)^{(r)} = \bigoplus_{k\equiv 0\ (r)} A(\Gamma_g)_k$ <u>is not Cohen-Macaulay for any integer</u> $r$.

Let $\mathfrak{D}$ be a bounded symmetric domain, and $\Gamma$ an arithmetic group acting on $\mathfrak{D}$. The quotient space $\mathfrak{D}/\Gamma$ has the natural compactification $(\mathfrak{D}/\Gamma)^*$, which is called the Baily-Borel compactification [ 2 ]. Let $j(\gamma,z)$ be the Jacobian of $\gamma \in \Gamma$, at a point $z \in \mathfrak{D}$, which is an automorphy factor. Let us fix some automorphy factor $\rho$ such that $\rho^{k_0} = j^{-1}$ for a positive integer $k_0$. A holomorphic function $f$ on $\mathfrak{D}$ is called an automorphic form for $\Gamma$ of weight $k$ if it satisfies

$$f(\gamma z) = \rho(\gamma,z)^k f(z) \qquad \text{for} \quad \gamma \in \Gamma$$

and if $f$ is holomorphic also at cusps (the last condition is automatic if $\operatorname{codim}((\mathfrak{D}/\Gamma)^* - (\mathfrak{D}/\Gamma)) \geq 2$). Then the compactification $(\mathfrak{D}/\Gamma)^*$ is isomorphic to the projective spectrum of the graded ring of automorphic forms for $\Gamma$. $\Gamma$ is said to be neat if, taking some (equivalently any) faithful representation $\psi$ of $\Gamma$ to $GL_m(\mathbb{C})$, the algebra generated over $\mathbb{Q}$, by the all the eigenvalues of $\psi(\gamma)$ is torsion free for every $\gamma \in \Gamma$. Any arithmetic group has a neat arithmetic subgroup of finite index.

Theorem 2. Let $\mathfrak{D}$ be a bounded symmetric domain, and $\Gamma$ a neat arithmetic group acting on $\mathfrak{D}$. Let $\mathfrak{D}'$ (resp. $\mathfrak{D}''$) be the highest (resp. the second highest) dimensional rational boundary component. Suppose $\operatorname{rank} \mathfrak{D}' = \operatorname{rank} \mathfrak{D} - 1$, and suppose one of the following holds;

(i) $m := \dim \mathfrak{D} - \dim \mathfrak{D}' \geq 3$ is odd,

(ii) $m \geq 4$ is even, and $\dim \mathfrak{D}'' \leq \dim \mathfrak{D}' - 2$.

Then the Baily-Borel compactification $(\mathfrak{D}/\Gamma)^*$ is not a Cohen-Macaulay scheme, and if $A(\Gamma)$ denotes the ring of automorphic forms, then $A(\Gamma)^{(r)}$ is not Cohen-Macaulay for any $r$.

Remark (i) Let $R = \bigoplus_k R_k$ be a graded algebra, and let $R^{(r)} = \bigoplus_{k \equiv 0 (r)} R_k$. Then it is standard that $\mathrm{Proj}(R) \simeq \mathrm{Proj}(R^{(r)})$ for any $r$, and that $\mathrm{Proj}(R)$ is a C.-M. scheme if $R$ is C.-M. (cf. [ 27 ], § 1). So the first assertion implies the second both in Theorem 1 and in Theorem 2.

(ii) In the case of characteristic zero, an invariant subring of a C.-M. ring under an action of a finite group is also C.-M. by Hochster and Eagon [ 15 ]. It follows from this and from Theorem 1 that the ring of Siegel modular forms for any normal subgroup of $\Gamma_g$ of finite index is not C.-M. if $g \geq 4$.

(iii) The proof of Theorem 1 is given in § 4, which is easily generalized to the following case. For a diagonal matrix

$$T = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_g \end{pmatrix}, \quad t_i \mid t_{i+1} \quad (i = 1, \ldots, g-1),$$

let

$$\Gamma_g(T) = \{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{2g}(\mathbb{Z}) \mid {}^tM \begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix} M = \begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix} \}.$$

$\Gamma_g(T)$ acts on $H_g$ by

$$Z \longrightarrow MZ = (TAT^{-1}Z + TB)(CT^{-1}Z + D)^{-1}, \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g(T).$$

Then the Satake compactification $(H_g/\Gamma_g(T))^*$ is not a C.-M. scheme for $g \geq 4$, and $A(\Gamma_g(T))^{(r)}$ is not C.-M. for any $r$, $A(\Gamma_g(T))$ denoting the graded ring of Siegel modular forms for $\Gamma_g(T)$.

(iv) The rings $A(\Gamma_1)$, $A(\Gamma_2)$ of Siegel modular forms of degree 1, 2 are known to be C.-M. (cf. Igusa [ 16 ]). On the other hand, the graded ring $A(\Gamma_3)$ of Siegel modular forms of degree three is believed to be not C.-M., however our method does not work in this case. It will be investigated in a later paper [ 28 ].

## § 2. The Proof of Theorem 2

2.1 Let $\mathfrak{D}$, $\mathfrak{D}'$, $\mathfrak{D}''$, $\Gamma$ be as in Theorem 2. Let $X = \mathfrak{D}/\Gamma$, and let $X^*$ be its Baily-Borel compactification. Set-theoretically, $D^* := X^* - X$ is the union of lower dimensional pieces

$$\mathfrak{D}'/\Gamma'_1, \ldots, \mathfrak{D}'/\Gamma'_s, \mathfrak{D}''/\Gamma''_1, \ldots, \mathfrak{D}''/\Gamma''_t, \mathfrak{D}'''/\Gamma'''_1, \ldots$$

similar to $\mathfrak{D}/\Gamma$. We denote by $X'$ the highest dimensional cusp $\mathfrak{D}'/\Gamma'_1 \cup \ldots \cup \mathfrak{D}'/\Gamma'_s$. Let $\bar{X}$, together with the morphism $\pi : \bar{X} \longrightarrow X^*$, be a toroidal compactification which was constructed by [ 1 ] and which is determined by a projective regular $\Gamma$-admissible decomposition of the associated polyhedral cone. $\pi$ coincides with the normalization of the blowing up of $X^*$ along some sheaf $\mathcal{J}^*$ of ideals with the support of $\mathcal{O}_{X^*}/\mathcal{J}^*$ contained in $D^*$. Hence $X$ is canonically contained in $\bar{X}$ on which $\pi$ induces the identity map. $D := \bar{X} - X$ is known to be a divisor with only normal crossings. The following is a direct consequence of the construction of $\bar{X}$, where it is essential that rank $\mathfrak{D}'$ = rank $\mathfrak{D} - 1$.

Lemma 1. i) <u>The fibre</u> $\pi^{-1}(x)$, $x \in X'$, <u>is an abelian variety of dimension</u> $m-1$, <u>where</u> $m = \dim \mathfrak{D} - \dim \mathfrak{D}'$.

ii) <u>Let</u> $\Gamma'$ <u>be any arithmetic group having</u> $\Gamma$ <u>as a normal subgroup, and let</u> $\mathcal{J}_{D^*}$ <u>be the sheaf of ideals determining the reduced subscheme</u> $D^*$. <u>Then by a suitable choice of a</u> $\Gamma$-

admissible decomposition, $\Gamma'/\Gamma$ acts naturally on $\bar{X}$, and $\mathcal{J}_{D*}{}^{r}$ equals $\mathcal{J}^{*}$ on $X \cup X'$ for a positive integer $r$. In this case, the quotient space $\bar{X}/(\Gamma'/\Gamma)$ gives a toroidal compactification of $\mathfrak{D}/\Gamma'$, and $\pi^{-1}(X \cup X')$ is the blowing up of $X \cup X'$ with respect to the sheaf of ideals defining the reduced subscheme $X'$.

2.2 Let $j$, $\rho$, $k_0$ be as in §.1. There is a coherent sheaf $\mathcal{L}(\rho^k)'$ on $X$ defined by

$$H^0(U, \mathcal{L}(\rho^k)') = \{f \in \mathcal{O}_{p^{-1}}(U) \mid f(\gamma z) = \rho(\gamma,z)^k f(z),\ \gamma \in \Gamma, z \in p^{-1}(U)\}$$

where $p$ is the projection of $\mathfrak{D}$ onto $X$, and $U$ is any open subset of $X$. Baily and Borel showed that $\mathcal{L}(j^{-1})'$ canonically extends to an ample invertible sheaf $\mathcal{L}(j^{-1})$ on $X^*$, since $\Gamma$ is neat (cf. Mumford [ 18 ], the proof of Proposition 3.4). At any rate there is an integer $k_1$ such that $k_1 \mid k_0$ and $\mathcal{L}(\rho^{k_1})'$ extends to an invertible sheaf $\mathcal{L}(\rho^{k_1})$ on $X^*$ satisfying $\mathcal{L}(\rho^{k_1})^{\otimes k_0/k_1} = \mathcal{L}(j^{-1})$ (for instance, take $k_0$ as $k_1$). Since $X^*$ is normal and projective, $X^*$ is isomorphic to $\mathrm{Proj}(\bigoplus_{k_1 \mid k} H^0(X^*, \mathcal{L}(\rho)^k))$. Our purpose is to show that this is not a C.-M. scheme, and so we may replace $\rho$ by $\rho^{k_1}$. In other words, we may assume

(i) $\mathcal{L} := \mathcal{L}(\rho)$ is an ample invertible sheaf,

(ii) $\mathcal{L}^{\otimes k} = \mathcal{L}(\rho^k)$, especially $\mathcal{L}^{\otimes k_0} = \mathcal{L}(j^{-1})$.

If $\mathrm{codim}(X^* - X) \geq 2$, then $\mathcal{L}^{\otimes k}$ equals the direct image $i_* \mathcal{L}(\rho^k)'$, $i$ being the inclusion of $X$ to $X^*$. $H^0(X^*, \mathcal{L}^{\otimes k})$ is just the space of automorphic forms of weight $k$. A global section of

$H^0(X^*, \mathcal{L}^{\otimes k} \otimes \mathcal{I}_{D^*})$ is called a cusp form of weight $k$. When $\mathfrak{D}$ is a point, the sapce of automorphic forms, or cusp forms of weight $k \geq 0$ is just $\mathbb{C}$. It is well-known that if $k \gg 0$, then $\dim_{\mathbb{C}} H^0(X^*, \mathcal{L}^{\otimes k})$ equals the sum of the dimensions of the spaces of cusp forms of weight $k$ on $X^*$, $(\mathfrak{D}'/\Gamma_1')^*$, $(\mathfrak{D}'/\Gamma_2')^*, \ldots$, where $\mathfrak{D}'/\Gamma_1'$, $\mathfrak{D}'/\Gamma_2', \ldots$ are all the members appering in the cusps of $X^*$.

Let us put $Q(k) := \chi(X^*, \mathcal{L}^{\otimes k})$, the Euler-Poincaré cheracteristic, which equals $\dim_{\mathbb{C}} H^0(X^*, \mathcal{L}^{\otimes k})$ for $k \gg 0$ since $\mathcal{L}$ is ample. Let $\mathcal{M} = \pi^* \mathcal{L}$. Then the canonical invertible sheaf $K_{\overline{X}}$ on $\overline{X}$ is isomorphic to $\mathcal{M}^{\otimes k_0} \otimes \mathcal{O}_{\overline{X}}(-D)$ (cf. [ 1 ], Chap. IV, § 1, Theorem 1). We put $P(k) := \chi(\overline{X}, \mathcal{M}^{\otimes k} \otimes \mathcal{O}_{\overline{X}}(-D))$. It equals $\dim_{\mathbb{C}} H^0(\overline{X}, \mathcal{M}^{\otimes k} \otimes \mathcal{O}_{\overline{X}}(-D))$ for $k \gg 0$ by the vanishing theorem of Kodaira type (Grauert and Riemenschneider [ 11 ]). Hence $P(k)$ for $k \gg 0$ is equal to the dimension of the space of cusp forms of weight $k$, since $H^0(\overline{X}, \mathcal{M}^{\otimes k} \otimes \mathcal{O}_{\overline{X}}(-D)) = H^0(X^*, \mathcal{L}^{\otimes k} \otimes \mathcal{I}_{D^*})$.

Proposition 1. <u>Let</u> $\Gamma$ <u>be a neat arithmetic group acting on</u> $\mathfrak{D}$. <u>If we denote</u> $n = \dim \mathfrak{D}$, $n' = \dim \mathfrak{D}'$, $n'' = \dim \mathfrak{D}''$, <u>then we have</u>

$$P(k + k_0) = (-1)^n P(-k) + O(k^{n'}).$$

<u>Under the additional assumption</u> $\operatorname{rank} \mathfrak{D}' = \operatorname{rank} \mathfrak{D} - 1$, <u>we have</u>

$$P(k + k_0) = (-1)^n P(-k) + O(k^{\max(n'', n'-m+1)}).$$

2.3 <u>Proof of Proposition 1</u>. Tensoring $\mathcal{M}^{\otimes (k+k_0)}$ with the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\overline{X}}(-D) \longrightarrow \mathcal{O}_{\overline{X}} \longrightarrow \mathcal{O}_D \longrightarrow 0,$$

we get

$$\chi(\overline{X}, \mathcal{M}^{\otimes(k+k_0)} \otimes \mathcal{O}_{\overline{X}}(-D)) = \chi(\overline{X}, \mathcal{M}^{\otimes(k+k_0)}) - \chi(D, \mathcal{M}^{\otimes(k+k_0)} \otimes \mathcal{O}_D).$$

Since $\chi(\overline{X}, \mathcal{M}^{\otimes(k+k_0)}) = (-1)^n P(-k)$ by the Serre duality theorem, we have

$$P(k + k_0) = (-1)^n P(-k) - \chi(D, \mathcal{M}^{\otimes(k+k_0)} \otimes \mathcal{O}_D).$$

We have the Leray spectral sequence

$$E_2^{p,q} = H^p(D^*, R^q\pi_*(\mathcal{M}^{\otimes(k+k_0)} \otimes \mathcal{O}_D)) \Longrightarrow H^{p+q}(D, \mathcal{M}^{\otimes(k+k_0)} \otimes \mathcal{O}_D).$$

By the projection formula, we have $H^p(D^*, R^q\pi_*(\mathcal{M}^{\otimes(k+k_0)} \otimes \mathcal{O}_D)) = H^p(D^*, \mathcal{L}^{\otimes(k+k_0)} \otimes R^q\pi_*\mathcal{O}_D)$, which vanishes for $p > 0$ and $k \gg 0$, because $\mathcal{L}$ is ample. Thus

$$H^0(D^*, \mathcal{L}^{\otimes(k+k_0)} \otimes R^i\pi_*\mathcal{O}_D) \simeq H^i(D, \mathcal{M}^{\otimes(k+k_0)} \otimes \mathcal{O}_D), \quad k \gg 0$$

and

$$\chi(D, \mathcal{M}^{\otimes(k+k_0)} \otimes \mathcal{O}_D) = \sum_{i=0}^{n-1} (-1)^i \dim_{\mathbb{C}} H^0(D^*, \mathcal{L}^{\otimes(k+k_0)} \otimes R^i\pi_*\mathcal{O}_D), \quad k \gg 0.$$

Since the dimension of $D^*$ equals $n'$, we immediately see the first assertion. Let us suppose $\operatorname{rank} \mathfrak{D}' = \operatorname{rank} \mathfrak{D} - 1$. As we recalled in Lemma 1, $\pi^{-1}(X')$ is flat over $X'$, and moreover its fibres are abelian varieties of dimension $m - 1$. By the base change theorem $R^i\pi_*\mathcal{O}_D$ is locally free on $X'$. By cup product we have a canonical homomorphism

$$\bigwedge^i R^1\pi_*\mathcal{O}_D \longrightarrow R^i\pi_*\mathcal{O}_D$$

on $X'$ (cf. [ 12 ], Chap. 0, 12.2). It is an isomorphism since so is the induced homomorphism on the fibre at each point. Since $\dim \pi^{-1}(x) = m - 1$ for $x \in X'$, the sheaf $R^i\pi_*\mathcal{O}_D$, $i > m - 1$, is supported on $D^* - X'$. So

$$\chi(D,\mathcal{M}^{\otimes(k+k_0)}\otimes\mathcal{O}_D) = \sum_{i=0}^{m-1}(-1)^i \dim_{\mathbb{C}} H^0(D^*,\mathcal{L}^{\otimes(k+k_0)}\otimes R^i\pi_*\mathcal{O}_D) + O(k^{n''}).$$

Now our assertion follows from the following lemma;

Lemma. Let $Y'$ be a normal irreducible projective variety of dimension $n'$ with an ample invertible sheaf $\mathcal{L}'$, and $Y''$ its subvariety of dimension $n''$. Let $Y^0 = Y' - Y''$.

(i) Let $\mathcal{F}, \mathcal{G}$ be coherent sheaves on $Y'$ such that $\mathcal{F}|_{Y^0} \simeq \mathcal{G}|_{Y^0}$. Then

$$\dim_{\mathbb{C}} H^0(Y',\mathcal{L}'^{\otimes k}\otimes\mathcal{F}) = \dim_{\mathbb{C}} H^0(Y',\mathcal{L}'^{\otimes k}\otimes\mathcal{G}) + O(k^{n''}).$$

(ii) Suppose $Y^0$ is nonsingular. Let $\mathcal{E}_1,\dots,\mathcal{E}_{m-1}$ be a coherent sheaves on $Y'$ such that $\mathcal{E}_1|_{Y^0}$ is locally free of rank $m-1$, and $\mathcal{E}_i|_{Y^0} \simeq \bigwedge^i \mathcal{E}_1|_{Y^0}$. Then

$$\sum_{i=0}^{m-1}(-1)^i \dim_{\mathbb{C}} H^0(Y',\mathcal{L}'^{\otimes k}\otimes\mathcal{E}_i) = O(k^{\max(n'',n'-m+1)}),$$

$\mathcal{E}_0$ being the structure sheaf $\mathcal{O}_{Y'}$.

Proof. To prove (i) we may assume that $\mathcal{F}, \mathcal{G}$ have no coherent subsheaves supported on $Y''$, and that $\mathcal{F}, \mathcal{G}$ are generated by their global sections. Let $\{s_i\}$ be global sections of $\mathcal{F}$. Then $s_i|_{Y^0}$ can be regarded as sections of $H^0(Y^0,\mathcal{G})$ via the isomorphism. Let $\{s_i'\}$ be the rational sections of $\mathcal{G}$ given as their extensions, and let $\tilde{\mathcal{G}}$ be the coherent sheaf generated by $\mathcal{G}$ and $\{s_i'\}$. Then we have two short exact sequences

$$0 \longrightarrow \mathcal{F} \longrightarrow \tilde{\mathcal{G}} \longrightarrow \tilde{\mathcal{G}}/\mathcal{F} \longrightarrow 0$$
$$0 \longrightarrow \mathcal{G} \longrightarrow \tilde{\mathcal{G}} \longrightarrow \tilde{\mathcal{G}}/\mathcal{G} \longrightarrow 0,$$

where $\tilde{\mathcal{G}}/\mathcal{F}, \tilde{\mathcal{G}}/\mathcal{G}$ are supported on $Y''$. Then

$$\dim_{\mathbb{C}} H^0(Y', \mathcal{L}'^{\otimes k} \otimes \mathcal{F}) = \dim_{\mathbb{C}} H^0(Y', \mathcal{L}'^{\otimes k} \otimes \tilde{\mathcal{G}}) + O(k^{n''}),$$

$$\dim_{\mathbb{C}} H^0(Y', \mathcal{L}'^{\otimes k} \otimes \mathcal{G}) = \dim_{\mathbb{C}} H^0(Y', \mathcal{L}'^{\otimes k} \otimes \tilde{\mathcal{G}}) + O(k^{n''}).$$

This shows (i). Now let us prove (ii). We may assume $\mathcal{E}_i$'s are torsion free. There is a proper modification $\phi: \tilde{Y} \longrightarrow Y$ such that $\tilde{Y}$ is a compact complex manifold with $\phi: \phi^{-1}(Y^0) \xrightarrow{\sim} Y^0$, and $\mathcal{E}_1' :=$ $\phi^* \mathcal{E}_1$ is locally free of rank $m-1$ (Riemenschneider [ 21 ]). By the Riemann-Roch Theorem we easily see that

$$\sum_{i=0}^{m-1} (-1)^i \chi(Y, \phi^* \mathcal{L}'^{\otimes k} \otimes \overset{i}{\Lambda} \mathcal{E}_1') = O(k^{n'-m+1}).$$

Then by the same argument as in the proof of Proposition 1 we have

$$\sum_{i=0}^{m-1} (-1)^i \dim_{\mathbb{C}} H^0(Y', \mathcal{L}'^{\otimes k} \otimes \phi_* \overset{i}{\Lambda} \mathcal{E}_1') = O(k^{\max(n'', n'-m+1)}).$$

We are done, since $\mathcal{E}_i$ and $\phi_* \overset{i}{\Lambda} \mathcal{E}_1'$ satisfy the condition in (i).
q.e.d.

2.4. Proof of Theorem 2. By Remark (i) of § 1, it is enough to show that $X^* = \mathrm{Proj}(\bigoplus_{k \geq 0} H^0(X^*, \mathcal{L}^{\otimes k}))$ cannot be a C.-M. scheme. The dualizing sheaf $\omega_{X^*}$ is the uniquely determined coherent sheaf on $X^*$ which gives rise to a functorial isomorphism $\mathrm{Hom}(\mathcal{F}, \omega_{X^*}) \cong H^n(X^*, \mathcal{F})^\vee$ for any coherent sheaf $\mathcal{F}$ (cf. Hartshorne [ 14 ]). By Grauert and Riemenschneider [ 11 ], $\omega_{X^*}$ coincides with $i_* K_X$, where $i$ is the canonical inclusion $X$ into $X^*$, and $K_X$ is the canonical invertible sheaf on $X$. Obviously $K_X = \mathcal{L}^{\otimes k_0}|_X$, and hence $\omega_{X^*} = i_*(\mathcal{L}^{\otimes k_0}|_X) = \mathcal{L}^{\otimes k_0}$ by Koecher's principle (cf. Serre [ 23 ]).

We suppose that $X^*$ is a C.-M. scheme. Then by [ 14 ], for instance, we have the Serre duality $H^i(X^*, \mathcal{L}^{\otimes(k+k_0)}) \simeq H^{n-i}(X^*, \mathcal{L}^{\otimes -k})^\vee$, and hence $Q(k+k_0) = (-1)^n Q(-k)$. If $P'(k)$ denotes the Hilbert polynomial for the space of cusp forms of weight $k$ on $X'$, then $Q(k) = P(k) + P'(k) + O(k^{n''})$. Now we can apply to $X'$ and $P'$ the first assertion of Proposition 1, and we get $P'(k+k_0) = (-1)^{n'} P'(-k) + O(k^{n''})$, where $k_0'$ is an integer such that $0 < k_0' < k_0$ (Baily and Borel [ 2 ], Proposition 1.11). Hence $P'(k)$ is of the form

$$P'(k) = c_0 (k - k_0'/2)^{n'} + c_2 (k - k_0'/2)^{n'-2} + \cdots + O(k^{n''}), \quad c_0 \neq 0.$$

Then, applying to $P(k)$ the second assertion of Proposition 1, we get $Q(k+k_0) - (-1)^n Q(-k) = \{P(k+k_0) - (-1)^n P(-k)\} + \{P'(k+k_0) - (-1)^n P'(-k)\} + O(k^{n''}) = P'(k+k_0) - (-1)^n P'(-k) + O(k^{\max(n'', n'-m+1)})$. Hence $Q(k+k_0) - (-1)^n Q(-k) = c_0 \{1 - (-1)^m\} k^{n'} + n'c_0 \{k_0 + (-1 + (-1)^{m-1}) k_0'/2\} k^{n'-1} + \cdots + O(k^{\max(n'', n'-m+1)})$, which cannot vanish by our assumption, hence we have a contradiction. So $X^*$ is not a C.-M. scheme. q.e.d.

## § 3. Siegel modular forms

3.1 Let $X := X_g := H_g/\Gamma_g$, and $X^*$ be its Satake compactification, which is set-theoretically equal to $X_g \cup X^*_{g-1} = X_g \cup X_{g-1} \cup \cdots \cup X_0$. The dimension $n$ of $X^*$ equals $g(g+1)/2$. For an integer $k$ such that $kg$ is even, let $\mathcal{L}(k)$ denote the coherent sheaf on $X^*$ corresponding to Siegel modular forms of weight $k$, i.e., the sheaf defined by

$$H^0(U, \mathcal{L}(k))$$
$$= \{f \in \mathcal{O}_{p^{-1}(U)} \mid \text{(i)} \quad f(MZ) = |CZ+D|^k f(Z) \quad \text{for} \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g,$$
$$Z \in p^{-1}(U \cap X), \quad \text{(ii)} \quad f \text{ extends holomorphically}$$
$$\text{to the intersection of } U \text{ and of the cusps}\},$$

for open sets $U$ of $X^*$, $p$ being the canonical projection of $H_g$ to $X$, where the second condition is automatic if $g > 1$. $H^0(X^*, \mathcal{L}(k))$ is the space of Siegel modular forms of weight $k$. It is easy to verify that $\mathcal{L}(k)$ is reflexive, and that if $k$ is even, then $\mathcal{L}(k)|_{X^*_{g-1}}$ is the coherent sheaf on $X^*_{g-1}$ corresponding to Siegel modular forms of weight $k$.

$H_r$, $0 \leq r \leq g-1$, can be regarded as a rational boundary component of $H_g$. Let $Z$ be a point of $H_r$ where $0 \leq r \leq g$. Then the group of matrices of the form

$$M = \left(\begin{array}{cc|cc} A' & 0 & B' & * \\ * & {}^tU^{-1} & * & * \\ \hline C' & 0 & D' & * \\ 0 & 0 & 0 & U \end{array}\right) \in \Gamma_g, \quad M' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \in \Gamma_r, \quad M'Z = Z,$$

is equal to the stabilizer group at $Z$ in $\Gamma_g$ up to conjugacy. Then the following is standard;

Lemma 2. Let $Z \in H_r$, $0 \leq r \leq g$. Then $\mathcal{L}(k)$ is invertible at the image of $Z$ in $X^*_g$ if $|C'Z + D'|^k |U|^k$ equals one for any $M \in \Gamma_g$ stabilizing $Z$ where $C', D', U$ are as above.

Corollary. There is a positive integer $N_0$ satisfying the following;

(i) $\mathcal{L}_0 := \mathcal{L}(N_0)$ is an ample invertible sheaf,

(ii) $\mathcal{L}(k + N_0) = \mathcal{L}(k) \otimes \mathcal{L}_0$ for all $k$,

(iii) the algebra $\bigoplus_{s \geq 0} H^0(X^*, \mathcal{L}_0^{\otimes s})$ is generated by $H^0(X^*, \mathcal{L}_0)$.

Since $\Gamma_g$ has fiexd points of even order, $N_0$ must be an even integer.

Let $X^0$ denote the Zariski open subset of $X$ consisting of the images of points in $H_g$ whose stabilizer in $\Gamma_g$ is trivial, i.e., $\{\pm 1_{2g}\}$. When $g \geq 3$, $X_g^0$ is just the smooth locus of $X_g$.

Lemma 3. The codimension of $X_g - X_g^0$ in $X_g$ is $g - 1$. Moreover, the image in $X_g$ of the fixed point set under the action of $M \in \Gamma_g$ with $M^2 \neq 1_{2g}$ is of codimension $\geq g$. Especially, $\mathcal{L}(k)$ for even $k$ is invertible except on a subvariety of codimension $\geq g$.

Proof. Let $M$ be a torsion element of $\Gamma_g$. $M$ can be diagonalized as

$$
{}^tU M U = \begin{pmatrix} \zeta^{t_1} & & & & & 0 \\ & \ddots & & & & \\ & & \zeta^{t_n} & & & \\ & & & \bar{\zeta}^{t_1} & & \\ & & & & \ddots & \\ 0 & & & & & \bar{\zeta}^{t_n} \end{pmatrix}
$$

where $\zeta$ is a root of unity and $U$ is a unitary matrix. Then the dimension of the fixed point set in $H_g$ under $M$ is given by

$$\# \{(t_i, t_j) \mid 1 \le i \le j \le g,\ \zeta^{t_i + t_j} = 1\}$$

(cf. Gottschling [ 10 ]). The first assertion follows immediately from this. If $M^2 \neq 1_{2g}$, then some $\zeta^{t_i} \neq \pm 1$, hence the second follows. $\mathcal{L}(k)$ is invertible at a point $Z_0$ fixed only by $M$'s such that $M^2 = 1_{2g}$. Indeed, letting $M = \binom{AB}{CD}$, we get $(CZ_0 + D)(C(MZ_0) + D) = 1_g$. Hence $(CZ_0 + D)^2 = 1_g$, thus $|CZ_0 + D|^k = 1$. q.e.d.

3.2 Let $\pi : \overline{X} \longrightarrow X^*$ be a toroidal compactification. Let $D := \overline{X} - X$, and $D^* := X^* - X = X_{g-1} \cup \cdots \cup X_0$. $\pi$ induces a map of $D$ to $D^*$, which we shall also denote by $\pi$.

Proposition 2. <u>If</u> $i > 0$, <u>then</u>

$$R^i \pi_* \mathcal{O}_{\overline{X}} \cong R^i \pi_* \mathcal{O}_D$$

<u>on</u> $X_{g-1}$.

<u>Proof</u>. Let $\Gamma_g(\ell) := \{M \in \Gamma_g \mid M \equiv 1_{2g} \bmod \ell\}$ be the principal congruence subgroup of level $\ell$. Let $X(\ell) = H_g/\Gamma_g(\ell)$. Denote by $X^*(\ell)$ and $\overline{X}(\ell)$ its Satake compactification and its toroidal compactification, respectively. Let $D^*(\ell) := X^*(\ell) - X(\ell)$, and $D(\ell) := \overline{X}(\ell) - X(\ell)$. We shall denote also by $\pi$, the morphism of $\overline{X}(\ell)$ to $X^*(\ell)$. Moreover, $X'(\ell)$ denotes the union of the highest dimensional cusps in $D^*(\ell)$, which is a disjoint union of copies of $H_{g-1}/\Gamma_{g-1}(\ell)$.

We first show $R^i \pi_* \mathcal{O}_{\overline{X}(\ell)} \cong R^i \pi_* \mathcal{O}_{D(\ell)}$ on $X'(\ell)$ for $i > 0$, provided that $\ell \geq 3$. Let $\mathcal{I}_{D^*(\ell)}$ be the sheaf of ideals of $D^*(\ell)$ with the reduced structure in $\mathcal{O}_{X^*(\ell)}$, and let $\mathcal{I} := \mathcal{I}_{D^*(\ell)}\mathcal{O}_{\overline{X}(\ell)}$. Here we note that we have the canonical injection of $\mathcal{O}_{X^*(\ell)}$ to $\mathcal{O}_{\overline{X}(\ell)}$. Since $\ell \geq 3$, we can apply Lemma 1 to our argument. So $\pi^{-1}(X(\ell) \cup X'(\ell))$ is the blowing up with respect to $\mathcal{I}_{D^*(\ell)}|_{X(\ell) \cup X'(\ell)}$, and hence $\mathcal{I}|_{\pi^{-1}(X(\ell) \cup X'(\ell))}$ is an invertible sheaf of ideals on $\pi^{-1}(X(\ell) \cup X'(\ell))$ defining $\pi^{-1}(X'(\ell))$. We have a short exact sequence

$$0 \longrightarrow \mathcal{I}^{j+1} \longrightarrow \mathcal{I}^{j} \longrightarrow \mathcal{I}^{j}/\mathcal{I}^{j+1} \longrightarrow 0,$$

where $\mathcal{I}^{j}/\mathcal{I}^{j+1}$ is an invertible sheaf on $\pi^{-1}(X'(\ell))$. We thus have a long exact sequence

$$\longrightarrow R^i\pi_*\mathcal{I}^{j+1} \xrightarrow{\alpha_{i,j}} R^i\pi_*\mathcal{I}^{j} \longrightarrow R^i\pi_*\mathcal{I}^{j}/\mathcal{I}^{j+1} \longrightarrow R^{i+1}\pi_*\mathcal{I}^{j+1} \longrightarrow .$$

For a point $x \in X'(\ell)$ and for $j > 0$, $(\mathcal{I}^{j}/\mathcal{I}^{j+1})_{\pi^{-1}(x)}$ is ample on $\pi^{-1}(x)$ by the definition of the blowing up, and hence the higher cohomology groups $H^i(\pi^{-1}(x), (\mathcal{I}^{j}/\mathcal{I}^{j+1})_{\pi^{-1}(x)})$, $i > 0$, vanish since $\pi^{-1}(x)$ is an abelian variety. By the base change thoerem $R^i\pi_*\mathcal{I}^{j}/\mathcal{I}^{j+1}$ vanishes at $x \in X'(\ell)$ if $i > 0$, $j > 0$, and hence $\alpha_{i,j}$ is surjective at $x$ for $i > 0$, $j > 0$. Since $R^i\pi_*\mathcal{I}^{j} = 0$ for $i > 0$, $j \gg 0$ ([ 12 ], Théorème (2.2.1), (ii)), it follows that $R^i\pi_*\mathcal{I}$, $i > 0$, vanishes on $X'(\ell)$. Considering the long exact sequence in the case $j = 0$, we get $R^i\pi_*\mathcal{O}_{\overline{X}(\ell)} \simeq R^i\pi_*\mathcal{O}_{D(\ell)}$ on $X'(\ell)$ for $i > 0$.

To prove the proposition we note that by general theory (cf. Grothendieck [ 13 ], Théorème 5.3.1, the proof of its corollary and Corollaire to Proposition 5.2.3), $H^i(Y/G, (\psi_*\mathcal{F})^G)$ and $H^i(Y, \mathcal{F})^G$

are canonically isomorphic, where $Y$ is a separated scheme over $\mathbb{C}$ with an action of a finite group $G$, and $\mathcal{F}$ is a coherent sheaf on $Y$ having an action of $G$ compatible with the action on $Y$, and $\psi: Y \to Y/G$ is the quotient morphism. Now let $U$ be an affine open subset of $X_{g-1} \subset X_g^*$ and let $\tilde{U}$ be the inverse image of $U$ in $X_{g-1}(\ell) = H_{g-1}/\Gamma_{g-1}(\ell) \subset X'(\ell)$. Then, letting $G$ to be the subgroup of $\Gamma_g/\Gamma_g(\ell)$ stabilizing $\tilde{U}$, we have

$$H^0(U, R^i\pi_*\mathcal{O}_{\overline{X}}) = H^i(\pi^{-1}(U), \mathcal{O}_{\overline{X}}) = H^i(\pi^{-1}(\tilde{U}), \mathcal{O}_{\overline{X}(\ell)})^G = H^0(\tilde{U}, R^i\pi_*\mathcal{O}_{\overline{X}(\ell)})^G,$$

$$H^0(U, R^i\pi_*\mathcal{O}_D) = H^i(\pi^{-1}(U), \mathcal{O}_D) = H^i(\pi^{-1}(\tilde{U}), \mathcal{O}_{D(\ell)})^G = H^0(\tilde{U}, R^i\pi_*\mathcal{O}_{D(\ell)})^G.$$

By what we saw above, the terms at extreme right hand side are canonically isomorphic if $i > 0$. Hence we are done. q.e.d.

Remark (i) As we easily see, Proposition 2 is true for a toroidal compactification of a quotient space $\mathfrak{D}/\Gamma$ of a bounded symmetric domain $\mathfrak{D}$ by an arithmetic group $\Gamma$ provided that rank $\mathfrak{D}'$ = rank $\mathfrak{D} - 1$, $\mathfrak{D}'$ being the highest dimensional rational boundary component of $\mathfrak{D}$.

(ii) Let $Z$ be a point of $H_{g-1}$ whose stabilizer in $\Gamma_{g-1}/\{\pm 1_{2g-2}\}$ is trivial, and let $y \in H_{g-1}/\Gamma_{g-1}(\ell) \subset X_g^*(\ell)$ be the corresponding point. Then the stabilizer subgroup $P$ at $y$ of $\Gamma_g$ is generated by $\Gamma_g(\ell)$ and matrices $M$ of the form

$$M = \pm\left(\begin{array}{cc|cc} 1_{g-1} & 0 & 0 & b \\ {}^t v & \pm 1 & {}^t b & e \\ \hline & & 1_{g-1} & -v \\ \multicolumn{2}{c|}{0} & 0 & \pm 1 \end{array}\right) \in \Gamma_g$$

Let $W$ (resp. $U$) be the group generated by $\Gamma_g(\ell)$ and matrices $M$ of the form

$$M = \left(\begin{array}{cc|cc} 1_{g-1} & 0 & & b \\ {}^t v & 1 & {}^t b & e \\ \hline & & 1_{g-1} & -v \\ \multicolumn{2}{c|}{0} & 0 & 1 \end{array}\right) \quad (\text{resp.} \left(\begin{array}{c|c} 1_g & \begin{matrix} 0 & 0 \\ 0 & e \end{matrix} \\ \hline 0 & 1_g \end{array}\right)).$$

Then we have inclusions of normal subgroups

$$\Gamma_g(\ell) \subset U \subset W \subset P.$$

$U$ acts trivially on the fibre $\pi^{-1}(y)$. Let us suppose $\ell \geq 3$. Then $\pi^{-1}(y)$ is isomorphic to an abelian variety $\mathbb{C}^{g-1}/(Z, 1_{g-1})(\ell Z)^{g-1}$. Regarding $z$ as an element of $\mathbb{C}^{g-1}$, an element $M$ of $W/U$ acts on $\pi^{-1}(y)$ as $z \longrightarrow z + Zv + b$. So the quotient of $\pi^{-1}(y)$ by $W$ is isomorphic to itself. Finally $P/W$ acts on the abelian variety as $z \longrightarrow \pm z$. It follows that the fibre $\pi^{-1}(x)$ for $x \in X^0_{g-1}$ is a $(g-1)$-dimensional Kummer variety, i.e., the quotient of a $(g-1)$-dimensional abelian variety by the group $\{\pm \mathrm{id}\}$, where $\pi$ is the morphism of $\overline{X}_g$ to $X^*_g$.

3.3 Let $N_0$ be as in Corollary to lemma 2. Then the Euler-Poincaré characetristic $\chi(X^*, \mathcal{L}(k + sN_0))$ is a numerical polynomial of $s$, since $\mathcal{L}(k + sN_0) = \mathcal{L}(k) \otimes \mathcal{L}_0^{\otimes s}$ and $\mathcal{L}_0$ is invertible. Let $Q(k) := \chi(X^*, \mathcal{L}(k))$. If $k$ is large enough, then $Q(k)$ gives the dimension of the space of Siegel modular forms of weight $k$, which equals $\sum_{r=0}^{g} \dim_{\mathbb{C}} \{\text{cusp forms of weight } k \text{ for } \Gamma_r\}$ if $k > 2g+1$ is even (cf. Cartan [ 3 ]).

We shall define $P(k)$ as follows. Fix an integer $k_1$ with

$0 \leq k_1 < N_0$. Then $\bigoplus_{s \geq 0}$ {cusp forms of weight $k_1 + sN_0$} is a graded module over the ring $\bigoplus_{s \geq 0} H^0(X^*, \mathcal{L}_0^{\otimes s})$. $P(k_1 + sN_0)$ is defined to be the Hilbert polynomial in $s$ for the graded module. Then $P(k)$ is well-defined for any $k$ and equals the dimension of the space of cusp forms of weight $k$ for $k \gg 0$ by definition.

Corollary to Proposition 2. Let $\mathcal{I}_D$ be the sheaf of ideals of $D$, and let $\mathcal{M}(k) = \pi^* \mathcal{L}(k)$. Under the condition $g \geq 3$ and $k$ even, we have

$$\chi(\overline{X}, \mathcal{M}(k) \otimes \mathcal{I}_D) = P(k) + O(k^{(g-1)(g-2)/2}).$$

Proof. Tensoring $\mathcal{M}(k)$ with the short exact sequence

$$0 \longrightarrow \mathcal{I}_D \longrightarrow \mathcal{O}_{\overline{X}} \longrightarrow \mathcal{O}_D \longrightarrow 0,$$

we have a short exact sequence

$$0 \longrightarrow \mathcal{M}(k) \otimes \mathcal{I}_D \longrightarrow \mathcal{M}(k) \longrightarrow \mathcal{M}(k) \otimes \mathcal{O}_D \longrightarrow 0.$$

Hence

$$\chi(\overline{X}, \mathcal{M}(k) \otimes \mathcal{I}_D) = \chi(\overline{X}, \mathcal{M}(k)) - \chi(D, \mathcal{M}(k) \otimes \mathcal{O}_D).$$

Now let us put $\mathcal{L}'(k) := \mathcal{L}(k) \otimes \mathcal{O}_{X^*_{g-1}}$, which equals the coherent sheaf corresponding to Siegel modular forms of weight $k$ on $X^*_{g-1}$, since $k$ is even. We have the Leray spectral sequence

$$E_2^{p,q} = H^p(X^*, R^q \pi_* \mathcal{M}(k)) \Longrightarrow H^{p+q}(\overline{X}, \mathcal{M}(k))$$

$$E_2^{p,q} = H^p(X^*_{g-1}, R^q \pi_* (\mathcal{M}(k) \otimes \mathcal{O}_D)) \Longrightarrow H^{p+q}(D, \mathcal{M}(k) \otimes \mathcal{O}_D).$$

By the same argument as in the proof of Proposition 1, we get

$$H^0(X^*, R^i \pi_* \mathcal{M}(k)) = H^i(\overline{X}, \mathcal{M}(k)), \quad k \gg 0,$$

$$H^0(X^*_{g-1}, R^i \pi_* (\mathcal{M}(k) \otimes \mathcal{O}_D)) = H^i(D, \mathcal{M}(k) \otimes \mathcal{O}_D), \quad k \gg 0.$$

Now by Lemma 3 and Proposition 2, both $R^i \pi_* \mathcal{M}(k)$ and

$R^i\pi_*(\mathcal{M}(k)\otimes \mathcal{O}_D)$ are isomorphic to $\mathcal{L}'(k)\otimes R^i\pi_*\mathcal{O}_D$ on $X^*_{g-1}$ minus a subvariety of codimension $\geq g-1$ if $i>0$. Thus

$$\dim_{\mathbb{C}} H^0(X^*,R^i\pi_*\mathcal{M}(k)) = \dim_{\mathbb{C}} H^0(X^*_{g-1},R^i\pi_*(\mathcal{M}(k)\otimes \mathcal{O}_D)) + O(k^{(g-1)(g-2)/2})$$

for $i>0$, hence

$$\begin{aligned}&\chi(\overline{X},\mathcal{M}(k)) - \chi(D,\mathcal{M}(k)\otimes\mathcal{O}_D)\\ &= \dim_{\mathbb{C}} H^0(X^*,\mathcal{L}(k)) - \dim_{\mathbb{C}} H^0(X^*_{g-1},\mathcal{L}'(k)) + O(k^{(g-1)(g-2)/2}),\ k\gg 0.\end{aligned}$$

We are done, since $P(k)$ equals $\dim_{\mathbb{C}} H^0(X^*,\mathcal{L}(k)) - \dim_{\mathbb{C}} H^0(X^*_{g-1},\mathcal{L}'(k))$ for $k\gg 0$. q.e.d.

Since $\overline{X}$ has only quotient singularities, the canonical coherent sheaf $K_{\overline{X}}$ (in the sence of Grauert-Riemenschneider [ 11 ]) and the dualizing sheaf coincide. Let $\overline{X}^0$ be the open subset of $\overline{X}$ whose points are not ramification points of the quotient morphism of $\overline{X}(\ell)$ to $\overline{X}$ for some $\ell\geq 3$. Then $\overline{X}-\overline{X}^0$ is just the singular locus, when $g\geq 3$ (Tai [ 25 ]).

Lemma 4. <u>Let</u> $g\geq 3$.

(i) <u>For the canonical injection</u> $i$ <u>of</u> $X^0$ <u>to</u> $X$, <u>we have</u> $i_*(\mathcal{L}(k)|_{X^0}) = \mathcal{L}(k)|_X$.

(ii) <u>For the canonical injection</u> $\overline{i}$ <u>of</u> $\overline{X}^0$ <u>to</u> $\overline{X}$, <u>we have</u>

$$K_{\overline{X}} = \overline{i}_*((\mathcal{M}(g+1)\otimes\mathcal{I}_D)|_{\overline{X}^0}) \quad \underline{\text{if}}\ g\ \underline{\text{is odd}},$$
$$K_{\overline{X}} = \overline{i}_*((\mathcal{M}(g+1)|_{\overline{X}^0}) \quad \underline{\text{if}}\ g\ \underline{\text{is even}}.$$

<u>Proof</u>. Since $g\geq 3$, $\mathrm{codim}(X-X^0)$ is greater than one. Then (i) is an easy consequence of the extendability of holomorphic

functions across a subvariety of codimension two. In the case (ii) with odd $g$, we have $K_{\overline{X}^0} = (\mathcal{M}(g+1)\otimes \mathcal{I}_D)|_{\overline{X}^0}$ by Tai [ 25 ], Theorem 1.1. If $g$ is even, then any section in $H^0(U,\mathcal{M}(g+1))$ for an open set $U$ with $U \cap D \neq \phi$, vanishes automatically at a point of $D$, so $K_{\overline{X}^0} = \mathcal{M}(g+1)|_{\overline{X}^0}$ (loc. cit). Then our assertion follows from Grauert and Riemenschneider [ 11 ]. q.e.d.

## § 4. Proof of Theorem 1

4.1 We shall prove Theorem 1 for even $g \geq 4$.

Proposition 3. <u>Let</u> $g \geq 4$ <u>be even, and</u> $N_0$ <u>be as in</u> Corollary <u>to</u> Lemma 2. <u>If</u> $k$ <u>is divisible by</u> $N_0$, <u>then</u>

$$Q(k+g+1) = (-1)^n Q(-k) + (2^{g-2}-1)\,\chi(X^*_{g-1}, \mathcal{L}'(k)) + O(k^{n-g-1}),$$

<u>where</u> $\mathcal{L}'(k)$ <u>is</u> $\mathcal{L}(k)\otimes \mathcal{O}_{X^*_{g-1}}$.

<u>Proof</u>. $N_0$ is an even integer, so $k+g+1$ is odd. Since any modular form for $\Gamma_g$ of odd weight is a cusp form, we have $Q(k+g+1) = P(k+g+1)$. $\mathcal{M}(g+1)$ and $K_{\overline{X}}$ are isomorphic on $\overline{X}^0$ by Lemma 4, (ii), and $H^0(\overline{X},\mathcal{M}(k+g+1)) = H^0(\overline{X}^0,\mathcal{M}(k+g+1)) = H^0(\overline{X},\mathcal{M}(k)\otimes K_{\overline{X}})$, since $\operatorname{codim}(\overline{X}-\overline{X}^0) \geq 2$. Thus

$$P(k+g+1) = \chi(X,\mathcal{M}(k)\otimes K_{\overline{X}})$$

by the vanishing theorem of Kodaira type [ 11 ]. By the Serre duality we have

$$\chi(\overline{X},\mathcal{M}(k)\otimes K_{\overline{X}}) = (-1)^n\,\chi(X,\mathcal{M}(-k)).$$

On the other hand, we have

$$\chi(\overline{X},\mathcal{M}(k)) = \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{C}} H^i(\overline{X},\mathcal{M}(k))$$
$$= \sum_{i=0}^{n} (-1)^i \dim_{\mathbb{C}} H^0(\overline{X}, \mathcal{L}(k)\otimes R^i\pi_*\mathcal{O}_{\overline{X}}), \quad k \gg 0,$$

by the same argument as in the proof of Proposition 1. Since the fibre $\pi^{-1}(x)$ for $x \in X^*_{g-1}$ is of dimension $g-1$, $R^i\pi_*\mathcal{O}_{\overline{X}}$ is supported on $X^*_{g-2}$ for $i \geq g$. Hence by Proposition 2,

$$\chi(\overline{X},\mathcal{M}(k)) = Q(k) + \sum_{i=1}^{g-1} (-1)^i \dim_{\mathbb{C}} H^0(\overline{X}, \mathcal{L}(k)\otimes R^i\pi_*\mathcal{O}_D)$$
$$+ O(k^{(g-1)(g-2)/2}).$$

$\pi : \pi^{-1}(X^0_{g-1}) \longrightarrow X^0_{g-1}$ is a fibre space of Kummer varieties, so $R^i\pi_*\mathcal{O}_D|_{X^0_{g-1}}$ is 0 if $i$ is odd or $i \geq g$, and it is a vector bundle of rank $\binom{g-1}{i}$ if $i < g$ is even. So by the Riemann-Roch theorem $\chi(\overline{X},\mathcal{M}(k)) = Q(k) + (2^{g-2}-1)\chi(X^*_{g-1},\mathcal{L}'(k)) + O(k^{n-g-1})$, since the sum of $\binom{g-1}{i}$ for $i = 2,4,..,g-2$ is equal to $2^{g-2}-1$.

Now

$$Q(k+g+1) = (-1)^n \chi(\overline{X},\mathcal{M}(-k))$$
$$= (-1)^n \{Q(-k) + (2^{g-2}-1)\chi(X^*_{g-1},\mathcal{L}'(-k)) + O(k^{n-g-1})\}$$
$$= (-1)^n \{Q(-k) + (-1)^{n-g}(2^{g-2}-1)\chi(X^*_{g-1},\mathcal{L}'(k))\}$$
$$+ O(k^{n-g-1}),$$

and we are done. q.e.d.

By Grauert and Riemenschneider [ 11 ], $i_*K_{X^0}$ gives the dualizing sheaf, $i$ being the inclusion of $X^0$ to $X^*$, as we saw in § 2.4. Since $K_{X^0} = \mathcal{L}(g+1)|_{X^0}$, and since $\operatorname{codim}(X^* - X^0) \geq n-g+1 > 2$, $i_*K_{X^0} = i_*(\mathcal{L}(g+1)|_{X^0}) = \mathcal{L}(g+1)$ by the extendability of holomorphic functions across a subvariety of codimension 2.

Now let us show that $X^*$ is not C.-M. If $X^*$ is C.-M., then we have for $k$ divisible by $N_0$

$$H^i(X^*, \mathcal{L}(k+g+1)) = H^i(X^*, \mathcal{L}(k) \otimes \mathcal{L}(g+1))$$
$$\simeq H^{n-i}(X^*, \mathcal{L}(-k))^\vee$$

by the Serre duality, and so $Q(k+g+1) = (-1)^n Q(-k)$. This contradicts Proposition 3. Hence $X^*$ is not C.-M.

4.2 Let us prove Theorem 1 for odd $g \geq 5$. The above argument works also for this case, so it is enough to show the following;

Proposition 4. Let $g \geq 3$ be odd. If $k$ is divisible by $N_0$, then

$$P(k) = (-1)^n P(-k+g+1) - 2^{g-2} \chi(X^*_{g-1}, \mathcal{L}'(k)) + O(k^{n-g-1}),$$
$$Q(k) = (-1)^n Q(-k+g+1) - (2^{g-2}-2) \chi(X^*_{g-1}, \mathcal{L}'(k)) + O(k^{n-g-1}).$$

Proof. By the short exact sequence

$$0 \longrightarrow \mathcal{M}(k) \otimes \mathcal{I}_D \longrightarrow \mathcal{M}(k) \longrightarrow \mathcal{M}(k) \otimes \mathcal{O}_D \longrightarrow 0,$$

we get

$$\chi(\bar{X}, \mathcal{M}(k) \otimes \mathcal{I}_D) = \chi(\bar{X}, \mathcal{M}(k)) - \chi(D, \mathcal{M}(k) \otimes \mathcal{O}_D).$$

Then

$$\chi(\bar{X}, \mathcal{M}(k) \otimes \mathcal{I}_D) = P(k) + O(k^{(g-1)(g-2)/2})$$

by Corollary to Proposition 2, and

$$\chi(\bar{X}, \mathcal{M}(k)) = (-1)^n \chi(\bar{X}, \mathcal{M}(-k) \otimes K_{\bar{X}})$$
$$= (-1)^n P(-k+g+1)$$
$$\chi(D, \mathcal{M}(k) \otimes \mathcal{O}_D) = \sum_{i=0}^{n-1} (-1)^i \dim_{\mathbb{C}} H^i(D, \mathcal{M}(k) \otimes \mathcal{O}_D)$$

$$= \sum_{i=0}^{g-1} (-1)^i \dim_{\mathbb{C}} H^0(X^*_{g-1}, \mathcal{L}(k) \otimes R^i \pi_* \mathcal{O}_D) + O(k^{(g-1)(g-2)/2})$$

$$= 2^{g-2} \chi(X^*_{g-1}, \mathcal{L}'(k)) + O(k^{n-g-1})$$

by an arrangement similar to that in Proposition 3. This gives the first assertion. Since $Q(k) = P(k) + \chi(X^*_{g-1}, \mathcal{L}'(k))$, we have

$$Q(k) - (-1)^n Q(-k+g+1)$$
$$= \{P(k) - (-1)^n P(-k+g+1)\}$$
$$+ \{\chi(X^*_{g-1}, \mathcal{L}'(k)) - (-1)^n \chi(X^*_{g-1}, \mathcal{L}'(-k+g+1))\}$$
$$= -(2^{g-2}-1)\chi(X^*_{g-1}, \mathcal{L}'(k)) - (-1)^n \chi(X^*_{g-1}, \mathcal{L}'(-k+g+1)) + O(k^{n-g-1}).$$

Here we note that $-(-1)^n \chi(X^*_{g-1}, \mathcal{L}'(-k+g+1)) = -(-1)^n(-1)^{n-g} \times \chi(X^*_{g-1}, \mathcal{L}'(k)) + O(k^{n-g-1}) = \chi(X^*_{g-1}, \mathcal{L}'(k)) + O(k^{n-g-1})$ because $g$ is odd. Then the second assertion follows immediately from this. q.e.d.

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Shigeaki Tsuyumine
2856-235 Sashiogi,
Omiya-shi,
Saitama 330,
Japan.