Some Approximations to Non-Central Distributions

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A dissertation submitted to the Doctoral Program
in Mathematics, the University of Tsukuba
in partial fulfillment of the requirements for the
degree of Doctor of Philosophy (Science)

JANUARY, 1997

PREFACE

In statistical estimation and testing hypothesis, non-central distributions play an important part. In this thesis, we consider new higher order approximations to percentage points of the non-central t, χ^2 and F distributions and the distribution of the sample correlation coefficient.

It is well known that the t-statistic with a non-central t-distribution is useful in the testing problem on difference between normal means μ_1 and μ_2, which also derives a confidence interval for it. In order to consider such an inference we need a percentage point of the non-central t-distribution. It is quite difficult to obtain the value analytically, however, since the density is complex and the cumulative distribution function is expressed using incomplete beta functions. This brings a difficulty in obtaining analytically percentage points of the non-central t-distribution. In order to get numerically percentage points, we also need a large-scale computation. Hence it is very useful to find better approximation formulae of percentage points of the non-central t-distributions. From the viewpoint of the numerical precision of the well-known approximation formulae for a percentage point of non-central t-distribution $t(\nu, \delta)$ with ν degrees of freedom and a non-centrality parameter δ, they are not poor when the absolute value of $\eta := \delta/\sqrt{2\nu + \delta^2}$ is close to 0, but they are poor when $|\eta|$ is close to 1, that is, the non-centrality δ is large.

We present a new higher order approximation formula developed by Akahira [A95], which is derived from the Cornish-Fisher expansion for the statistic based on a linear combination of a normal random variable and a chi-random variable in Chapter 1. This new approximation formula is numerically better than the others.

We derive new higher order approximations to a percentage point of the non-central χ^2 distribution in Chapter 2 and that of the non-central F distribution in Chapter 3. In general, when the power of F test is calculated in the analysis of variance, the non-central F distribution is needed. The non-central χ^2 distribution is regarded as a limit of non-central F distribution and also appears when one obtains asymptotic distributions of χ^2 test statistics of goodness of fit and test statistics in testing the hypothesis on discrete distributions under the alternative hypothesis. We obtain new higher order approximation formulae of percentage points for them using the Cornish-Fisher expansion for the statistic based on a linear combination of chi-random variables in similar way to Chapter 1. Further, the numerical comparison of these formulae with the former shows that the new approximation formulae behave better than the others.

We present a new higher order approximation to a percentage point of the distribution of the sample correlation coefficient in Chapter 4. The new higher order approximation formula is derived from the Cornish-Fisher expansion for the statistic based on a linear combination of a normal random variable and a chi-random variable in a similar way to Chapter 1. In numerical calculations, the higher order approximation formula is seen to be that it dominates the normal approximation, the approximation by Fisher's **Z**-transformation, etc. and gives almost precise values in various cases.

CONTENTS

CHAPTER 1

On the New Approximation to the Non-Central t-Distributions

1.1. Introduction

Suppose that $X_1, \ldots, X_{n_1}$ and $Y_1, \ldots, Y_{n_2}$ are random samples from normal distributions with means μ_1 and μ_2, respectively, and a common variance σ^2. Letting

$$\bar{X} := \frac{1}{n_1}\sum_{i=1}^{n_1} X_i, \quad \bar{Y} := \frac{1}{n_2}\sum_{i=1}^{n_2} Y_i,$$

$$S_1^2 := \frac{1}{n_1}\sum_{i=1}^{n_1}(X_i - \bar{X})^2, \quad S_2^2 := \frac{1}{n_2}\sum_{i=1}^{n_2}(Y_i - \bar{Y})^2,$$

we define the statistic T by

$$T := \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{n_1+n_2}{n_1 n_2}}\sqrt{\frac{n_1 S_1^2 + n_2 S_2^2}{n_1+n_2-2}}}.$$

Then it is well known that T has a non-central t-distribution $t(\nu, \delta)$ with ν degrees of freedom and a non-centrality parameter δ, where $\nu := n_1 + n_2 - 2$, $\quad \delta := \sqrt{n_1 n_2/(n_1+n_2)}D$ and $D := (\mu_1 - \mu_2)/\sigma$ and that T plays an important part in the testing problem on difference between μ_1 and μ_2, which also derives a confidence interval for it. In order to consider such an inference we need a percentage point of the non-central t-distribution. It is quite difficult to obtain the value analytically, however since the density has the following form:

$$f_T(t; \nu, \delta) := \sum_{k=0}^{\infty} e^{-\frac{\delta^2}{2}} \frac{(\sqrt{2}\delta)^k}{k!} \frac{\Gamma[(\nu+k+1)/2]}{\sqrt{\pi\nu}\Gamma(\frac{\nu}{2})} \left(\frac{t}{\sqrt{\nu}}\right)^k \left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+k+1}{2}}$$

for $-\infty < t < \infty$. (For tables on percentage points of the non-central t-distributions, see e.g. Bagui [B93], IMS [I74] and Yamauti *et al.* [Y72].)

Hence, we will consider approximation formulae for a percentage point of the non-central t-distributions(Owen [Ow68] and Johnson *et al.* [JKB95]). In this chapter we discuss the approximation formulae of Jennett and Welch [JeW39], Johnson and Welch [JoW39], and van Eeden [V61], which are well known among others. We also present a new higher order approximation formula developed by Akahira [A95], which is derived from the Cornish-Fisher expansion for the statistic based on a linear combination of a normal random variable and a chi-random variable ([AST95]). Akahira [A95] illustrated that this new approximation formula is numerically better than the others. In addition we discuss the existence and uniqueness of a solution of the equation on the new approximation formula since it has an implicit form ([AST95]). More detailed numerical comparison of the formulae above are made and tables on values of percentage points computed from the new approximation formula are given.

1.2. Approximations to a percentage point of non-central t-distribution

Suppose that X is a normal random variable with mean δ and variance 1, νS^2 is a random variable according to a chi-square distribution with ν degrees of freedom, and X and νS^2 are independent. Denoting

$$T_{\nu,\delta} := \frac{X}{\sqrt{S^2}},$$

we see that $T_{\nu,\delta}$ has a non-central t-distribution $t(\nu, \delta)$. Letting $S := \sqrt{S^2}$, we have $T_{\nu,\delta} = X/S$. Among approximation formulae for a percentage point of the non-central t-distribution, we present some well-known formulae. Since

$$P\{T_{\nu,\delta} \leq t\} = P\{X - tS \leq 0\},$$

we see that S is asymptotically normally distributed with mean

(1.1) $$b_\nu := E[S] = \sqrt{\frac{2}{\nu}}\frac{\Gamma[(\nu+1)/2]}{\Gamma(\nu/2)},$$

and variance

(1.2) $$V[S] = 1 - b_\nu^2.$$

When ν is large, $X - tS$ is asymptotically normally distributed with mean $\delta - tb_\nu$ and variance $1 + t^2(1 - b_\nu^2)$. For any α with $0 < \alpha < 1$, we have

$$\alpha = P\{T_{\nu,\delta} > t_\alpha\} \fallingdotseq 1 - \Phi\Big(\frac{t_\alpha b_\nu - \delta}{\sqrt{1 + t_\alpha^2(1 - b_\nu^2)}}\Big),$$

hence

$$(1.3) \qquad t_\alpha \fallingdotseq \frac{\delta b_\nu + u_\alpha\sqrt{b_\nu^2 + (1 - b_\nu^2)(\delta^2 - u_\alpha^2)}}{b_\nu^2 - u_\alpha^2(1 - b_\nu^2)} \qquad \text{(Jennett and Welch [JeW39])},$$

where $\Phi(x)$ is the standard normal distribution function and u_α is the upper 100α percentile of the standard normal distribution. Letting $b_\nu \fallingdotseq 1$ and $1 - b_\nu^2 \fallingdotseq 1/2\nu$ in (1.3), we get

$$(1.4) \qquad t_\alpha \fallingdotseq \frac{\delta + u_\alpha\sqrt{1 + (\delta^2 - u_\alpha^2)/(2\nu)}}{1 - u_\alpha^2/(2\nu)} \qquad \text{(Johnson and Welch [JoW39])}.$$

(Masuyama [Ma51] obtained values of this approximation using an improved binomial probability paper.) Now, we will note the approximation formula

$$(1.5) \qquad t_\alpha \fallingdotseq \delta + u_\alpha + \frac{1}{\nu}B_1(u_\alpha) + \frac{1}{\nu^2}B_2(u_\alpha) \qquad \text{(van Eeden [V61])},$$

where

$$B_1(u_\alpha) = \frac{1}{4}\{u_\alpha^3 + u_\alpha + \delta(2u_\alpha^2 + 1) + \delta^2 u_\alpha\},$$

$$B_2(u_\alpha) = \frac{1}{96}\{5u_\alpha^5 + 16u_\alpha^3 + 3u_\alpha + 3\delta(4u_\alpha^4 + 12u_\alpha^2 + 1) + 6\delta^2(u_\alpha^3 + 4u_\alpha) - 4\delta^3(u_\alpha^2 - 1) - 3\delta^4 u_\alpha\}.$$

It is stated in Shibata [Sh81] that the approximation formula (1.5) coincides with that derived from the Taylor expansion of the characteristic function of $T_{\nu,\delta} - \delta$ with a chi-square statistic.

From the viewpoint of the numerical precision of the approximation formulae (1.3), (1.4) and (1.5), they are not poor when the absolute value of $\eta := \delta/\sqrt{2\nu + \delta^2}$ is close to 0, but they are poor when $|\eta|$ is close to 1, that is, the non-centrality δ is large (see Table 1.1).

Recently, Akahira [A95] proposed a new higher order approximation formula for a percentage point of the non-central t-distribution and showed that the formula were numerically better than (1.3), (1.4) and (1.5) and also worked well when $|\eta|$ is close to 1. Now, by following Akahira [A95], we present a derivation of the new formula. Letting t_α be the upper 100α percentage point of the non-central t-distribution and $Z := X - \delta$, we have

$$\begin{aligned} 1 - \alpha &= P\{T_{\nu,\delta} < t_\alpha\} = P\{X/S < t_\alpha\} \\ &= P\{(Z+\delta)/S < t_\alpha\} = P\{Z + \delta < t_\alpha S\} \\ &= P\{Z - t_\alpha S < -\delta\}. \end{aligned}$$

Since $E[Z - t_\alpha S] = t_\alpha b_\nu$ and $V[Z - t_\alpha S] = 1 + t_\alpha^2(1 - b_\nu^2)$, we obtain

$$(1.6) \qquad 1 - \alpha = P\left\{ \frac{Z - t_\alpha(S - b_\nu)}{\sqrt{1 + t_\alpha^2(1 - b_\nu^2)}} < \frac{t_\alpha b_\nu - \delta}{\sqrt{1 + t_\alpha^2(1 - b_\nu^2)}} \right\},$$

and letting

$$(1.7) \qquad W := \frac{Z - t_\alpha(S - b_\nu)}{\sqrt{1 + t_\alpha^2(1 - b_\nu^2)}},$$

we easily see that

$$E[W] = 0, \quad V[W] = 1.$$

Note that the statistic W is based on the chi-statistic S. In order to use the Cornish-Fisher expansion for the statistic W up to the order $o(\nu^{-3})$, we need the third and fourth cumulants of W up to the same order. (Johnson and Welch [JoW39] suggested the cumulants of $Z - t_\alpha S$ for the asymptotic expansion of its distribution.)

Lemma 1.1 ([A95]). *The third and fourth cumulants of $Z - t_\alpha(S_\nu - b_\nu)$ are given by*

$$\kappa_{3,\nu}(Z - t_\alpha(S_\nu - b_\nu)) = t_\alpha^3 b_\nu \left\{ 2(1 - b_\nu^2) - \frac{1}{\nu} \right\},$$

and

$$\kappa_{4,\nu}(Z - t_\alpha(S_\nu - b_\nu)) = 2t_\alpha^4 \left[(1 - b_\nu^2)\{2 - 3(1 - b_\nu^2)\} + \frac{2}{\nu}(1 - b_\nu^2) - \frac{1}{\nu} \right],$$

respectively.

Proof. Let μ'_r be the rth moment about zero of $Z - t_\alpha(S_\nu - b_\nu)$, we get $\kappa_{3,\nu}(Z - t_\alpha(S_\nu - b_\nu)) = \mu'_3 - 3\mu'_2\mu'_1 + 2{\mu'_1}^3$. However, since $\mu'_1 = E(Z - t_\alpha(S_\nu - b_\nu)) = 0$, we see $\kappa_{3,\nu}(Z - t_\alpha(S_\nu - b_\nu)) = \mu'_3$. Since $E(S) = b_\nu, E(S^2) = 1 - b_\nu^2, E(S^3) = (1 + 1/\nu)b_\nu$ and that Z and S are independent, we have

$$\begin{aligned}\kappa_{3,\nu}(Z - t_\alpha(S_\nu - b_\nu)) &= E[\{Z - t_\alpha(S_\nu - b_\nu)\}^3] \\ &= E[-\{t_\alpha(S_\nu - b_\nu)\}^3] \\ &= -t_\alpha^3\{E[S_\nu^3] - 3b_\nu E[S_\nu^2] + 3b_\nu^2 E[S_\nu] - b_\nu^3\} \\ &= t_\alpha^3 b_\nu \Big\{2(1 - b_\nu^2) - \frac{1}{\nu}\Big\}.\end{aligned}$$

Since $E[S_n^4] = 1 + (2/n)$ and $E(Z^4) = 3$, we have

$$\begin{aligned}&\kappa_{4,\nu}(Z - t_\alpha(S_\nu - b_\nu)) \\ &= \mu'_4 - 3{\mu'_2}^2 \\ &= E[(Z - t_\alpha(S_\nu - b_\nu))^4] - 3\{E[(Z - t_\alpha(S_\nu - b_\nu))^2]\}^2 \\ &= E(Z^4) + 6t_\alpha^2 E[(S_\nu - b_\nu)^2] + t_\alpha^4 E[(S_\nu - b_\nu)^4] - 3\{E(Z^2) + t_\alpha^2 E[(S_\nu - b_\nu)^2]\}^2 \\ &= 3 + 6t_\alpha^2(1 - b_\nu^2) + t_\alpha^4\{E(S^4) - 4b_\nu E(S^3) + 6b_\nu^2 E(S^2) - 4b_\nu^3 E(S) + b_\nu^4\} \\ &\quad - 3\{1 + t_\alpha^2(1 - b_\nu^2)\}^2 \\ &= 2t_\alpha^4\Big[(1 - b_\nu^2)\{2 - 3(1 - b_\nu^2)\} + \frac{2}{\nu}(1 - b_\nu^2) - \frac{1}{\nu}\Big]. \qquad \square\end{aligned}$$

Lemma 1.2 ([A95]). *The third and fourth cumulants of W are given by*

$$\kappa_{3,\nu}(W) = -\frac{1}{4\{1 + t_\alpha^2(1 - b_\nu^2)\}^{3/2}}\Big\{\frac{1}{\nu^2} + \frac{1}{4\nu^3} + O\Big(\frac{1}{\nu^4}\Big)\Big\},$$

and

$$\kappa_{4,\nu}(W) = O\Big(\frac{1}{\nu^4}\Big),$$

respectively.

Proof. Since, by the Stirling formula,

$$\Gamma(\nu) = \sqrt{2\pi}\nu^{\nu-1/2}e^{-\nu}\Big\{1 + \frac{1}{12\nu} + \frac{1}{288\nu^2} - \frac{139}{51840\nu^3} + O\Big(\frac{1}{\nu^4}\Big)\Big\}$$

for large ν, we obtain

$$
\begin{aligned}
(1.8)\quad b_\nu &= \sqrt{\frac{2}{\nu}}\Gamma\Big(\frac{\nu+1}{2}\Big)/\Gamma\Big(\frac{\nu}{2}\Big) \\
&= \Big(1+\frac{1}{\nu}\Big)^{\nu/2} e^{-1/2}\Big\{1+\frac{1}{6(\nu+1)}+\frac{1}{72(\nu+1)^2}-\frac{139}{6480(\nu+1)^3}-\cdots\Big\} \\
&\quad\cdot\Big\{1+\sum_{k=1}^{\infty}\Big(-\frac{1}{6\nu}-\frac{1}{72\nu^2}+\frac{139}{6480\nu^3}+\cdots\Big)^k\Big\} \\
&= \exp\Big(-\frac{1}{4\nu}+\frac{1}{6\nu^2}-\frac{1}{8\nu^3}+\cdots\Big) \\
&\quad\cdot\Big\{1+\frac{1}{6}\Big(\frac{1}{\nu}-\frac{1}{\nu^2}+\frac{1}{\nu^3}\Big)+\frac{1}{72}\Big(\frac{1}{\nu^2}-\frac{2}{\nu^3}\Big)-\frac{139}{6480}\frac{1}{\nu^3}+O\Big(\frac{1}{\nu^4}\Big)\Big\} \\
&\quad\cdot\Big\{1+\sum_{k=1}^{\infty}\Big(-\frac{1}{6\nu}-\frac{1}{72\nu^2}+\frac{139}{6480\nu^3}+\cdots\Big)^k\Big\} \\
&= \Big(1-\frac{1}{4\nu}+\frac{19}{96\nu^2}-\frac{65}{384\nu^3}+O\Big(\frac{1}{\nu^4}\Big)\Big)\Big(1+\frac{1}{6\nu}-\frac{11}{72\nu^2}+\frac{761}{6480\nu^3}+O\Big(\frac{1}{\nu^4}\Big)\Big) \\
&\quad\cdot\Big(1-\frac{1}{6\nu}+\frac{1}{72\nu^2}+\frac{139}{6480\nu^3}+O\Big(\frac{1}{\nu^4}\Big)\Big) \\
&= 1-\frac{1}{4\nu}+\frac{1}{32\nu^2}+\frac{5}{128\nu^3}+O\Big(\frac{1}{\nu^4}\Big),
\end{aligned}
$$

hence

$$
\begin{aligned}
(1.9)\qquad 1-b_\nu^2 &= (1-b_\nu^2)(1+b_\nu) = (1-b_\nu^2)\{2-(1-b_\nu^2)\} \\
&= \frac{1}{2\nu}-\frac{1}{8\nu^2}-\frac{1}{16\nu^3}+O\Big(\frac{1}{\nu^4}\Big).
\end{aligned}
$$

From (1.8), (1.9) and Lemma 1.1 we have

$$
\begin{aligned}
\kappa_{3,\nu}(W) &= \frac{1}{\{1+t_\alpha^2(1-b_\nu^2)\}^{3/2}}\kappa_{3,\nu}(Z-t_\alpha(S_\nu-b_\nu)) \\
&= \frac{t_\alpha^3 b_\nu}{\{1+t_\alpha^2(1-b_\nu^2)\}^{3/2}}\Big\{2(1-b_\nu^2)-\frac{1}{\nu}\Big\} \\
&= \frac{t_\alpha^3 b_\nu}{\{1+t_\alpha^2(1-b_\nu^2)\}^{3/2}}\Big\{1-\frac{1}{4\nu}+\frac{1}{32\nu^2}+\frac{5}{128\nu^3}+O\Big(\frac{1}{\nu^4}\Big)\Big\} \\
&\quad\cdot\Big\{2\Big(\frac{1}{2\nu}-\frac{1}{8\nu^2}-\frac{1}{16\nu^3}\Big)-\frac{1}{\nu}+O\Big(\frac{1}{\nu^4}\Big)\Big\} \\
&= -\frac{t_\alpha^3 b_\nu}{\{1+t_\alpha^2(1-b_\nu^2)\}^{3/2}}\Big\{\frac{1}{\nu^2}+\frac{1}{4\nu^3}+O\Big(\frac{1}{\nu^4}\Big)\Big\}.
\end{aligned}
$$

It also follows from Lemma 1.1 and (1.9) that

$$\begin{aligned}
\kappa_{4,\nu}(W) &= \frac{1}{\{1+t_\alpha^2(1-b_\nu^2)\}^2}\kappa_{4,\nu}(Z - t_\alpha(S_\nu - b_\nu)) \\
&= \frac{2t_\alpha^4}{\{1+t_\alpha^2(1-b_\nu^2)\}^2}\Big[(1-b_\nu^2)\{2-3(1-b_\nu^2)\} + \frac{2}{\nu}(1-b_\nu^2) - \frac{1}{\nu}\Big] \\
&= \frac{2t_\alpha^4}{\{1+t_\alpha^2(1-b_\nu^2)\}^2}\Big[\Big(\frac{1}{2\nu} - \frac{1}{8\nu^2} - \frac{1}{16\nu^3}\Big)\{2 - 3\Big(\frac{1}{2\nu} - \frac{1}{8\nu^2}\Big)\} \\
&\quad + \frac{2}{\nu}\Big(\frac{1}{2\nu} - \frac{1}{8\nu^2}\Big) - \frac{1}{\nu} + O\Big(\frac{1}{\nu^4}\Big)\Big] \\
&= O\Big(\frac{1}{\nu^4}\Big). \qquad \square
\end{aligned}$$

Hence, we have the following theorem.

Theorem 1.1([A95]). *The upper 100α percentile t_α of the non-central t-distribution with ν degrees of freedom and a non-centrality δ can be derived from the formula*

$$\frac{t_\alpha b_\nu - \delta}{\sqrt{1+t_\alpha^2(1-b_\nu^2)}} = u_\alpha - \frac{t_\alpha^3(u_\alpha^2-1)}{24\{1+t_\alpha^2(1-b_\nu^2)\}^{\frac{3}{2}}}\Big\{\frac{1}{\nu^2} + \frac{1}{4\nu^3} + O(\frac{1}{\nu^4})\Big\}. \tag{1.10}$$

Proof. From (1.6), (1.7) and Lemma 1.2, we have by the Cornish-Fisher expansion

$$\begin{aligned}
\frac{t_\alpha b_\nu - \delta}{\sqrt{1+t_\alpha^2(1-b_\nu^2)}} &= u_\alpha + \frac{1}{6}\kappa_{3,\nu}[W](u_\alpha^2-1) + \frac{1}{24}\kappa_{4,\nu}[W](u_\alpha^3 - 3u_\alpha) + O\Big(\frac{1}{\nu^4}\Big) \\
&= u_\alpha - \frac{t_\alpha^3(u_\alpha^2-1)}{24\{1+t_\alpha^2(1-b_\nu^2)\}^{\frac{3}{2}}}\Big\{\frac{1}{\nu^2} + \frac{1}{4\nu^3} + O\Big(\frac{1}{\nu^4}\Big)\Big\}. \qquad \square
\end{aligned}$$

If we ignore the second term of the right-hand side of (1.10), i.e.

$$\frac{t_\alpha b_\nu - \delta}{\sqrt{1+t_\alpha^2(1-b_\nu^2)}} \fallingdotseq u_\alpha$$

then we have (1.3). Letting $b_\nu \fallingdotseq 1$ and $1 - b_\nu^2 \fallingdotseq 1/(2\nu)$, we get (1.4). As previously stated, the van Eeden formula (1.5) is considered to be based on the chi-square statistic. The approximation formula (1.10) is obtained up to the order $O(\nu^{-4})$ and derived from the statistic W based on a linear combination of a normal random variable and a chi statistic S. On the other hand, from Kendall *et al.* [KSO94] (p. 545) it is well known that a chi statistic tends to normality with considerably greater rapidity than a chi-square statistic. Hence the approximation formula (1.10) is better on theoretical grounds than that obtained on the basis of the chi-square random variable like (1.5). From Theorem 1.1 we can obtain the lower confidence limit and the confidence interval for the non-centrality δ.

Corollary 1.1([A95]). *Let T be a statistic according to the non-central t- distribution with ν degrees of freedom and a non-centrality parameter δ. Then the lower confidence limit $\hat{\delta}$ of level $1-\alpha$ and the confidence interval $[\underline{\delta}, \bar{\delta}]$ of the non-centrality parameter δ of level $1-\alpha$ are given by*

$$\hat{\delta} = b_\nu T - u_\alpha \sqrt{1+(1-b_\nu^2)T^2} + \frac{(u_\alpha^2-1)T^3}{24\{1+(1-b_\nu^2)T^2\}}\left(\frac{1}{\nu^2}+\frac{1}{4\nu^3}\right) + O_p\left(\frac{1}{\nu^4}\right),$$

$$\underline{\delta} = b_\nu T - u_{\alpha/2} \sqrt{1+(1-b_\nu^2)T^2} + \frac{(u_{\alpha/2}^2-1)T^3}{24\{1+(1-b_\nu^2)T^2\}}\left(\frac{1}{\nu^2}+\frac{1}{4\nu^3}\right) + O_p\left(\frac{1}{\nu^4}\right),$$

$$\bar{\delta} = b_\nu T + u_{\alpha/2} \sqrt{1+(1-b_\nu^2)T^2} - \frac{(u_{\alpha/2}^2-1)T^3}{24\{1+(1-b_\nu^2)T^2\}}\left(\frac{1}{\nu^2}+\frac{1}{4\nu^3}\right) + O_p\left(\frac{1}{\nu^4}\right).$$

The proof is straightforward from Theorem 1.1.

1.3. Evaluation of new approximation formulae in comparison with others for the non-central t-distribution

In order to compare (1.10) with (1.3), (1.4) and (1.5), we have their numerical calculation when α is 0.10, 0.05 and 0.01, ν is 4, 9, 16, 36. (A similar table on (1.3), (1.4) and (1.5) for $\alpha = 0.05$ is given by Shibata [Sh81].) The errors of the approximation formulae are given as Table 1.1, where the true values are referred from Yamauti *et al.* [Y72] and the values of (1.10) are calculated by Newton's method in *Mathematica* for Macintosh. As is seen in Table 1.1, the approximation formula (1.10) dominates the others where the absolute value of η is close to 1 and gives precise values on the whole. For example, if α=0.05, ν is more than 9, and $|\eta|$ is less than 0.70, then the absolute values of errors are less than 0.005. The values of the upper percentiles of a non-central t-distribution for the t-statistic are given using the approximation formula (1.10) in Table 1.2. In cases where α=0.025, D is 2.0, 1.0 and 0.5, and n_1, n_2 are from 7 to 30 at unit interval. If D is 2.0, then $|\eta|$ is less than 0.621, hence the values in Table 1.2 are reliable. Table 1.2 may be useful for the argument between statistical significant difference and clinical significant difference (see Kubo *et al.* [KSIK91]).

Table 1.1. The errors of the approximation formulae of t_α for $\eta = \delta/\sqrt{2\nu+\delta^2}$. $(\alpha = 0.10)$

ν	η	true value	(1.10)	(1.3)	(1.4)	(1.5)
4	0.9	11.636	-0.033	0.290	-0.661	-1.894
	0.7	5.966	-0.004	0.130	-0.341	-0.102
	0.5	4.002	0.001	0.069	-0.234	-0.039
	0.3	2.814	0.001	0.033	-0.170	-0.020
	0.1	1.924	0.000	0.013	-0.120	-0.009
	-0.1	1.162	0.000	0.003	-0.073	-0.002
	-0.3	0.432	0.000	-0.000	-0.027	-0.000
	-0.5	-0.363	0.000	-0.000	0.021	-0.000
	-0.7	-1.428	0.000	-0.004	0.076	-0.008
	-0.9	-3.900	-0.002	-0.032	0.178	-1.217
9	0.9	13.178	-0.009	0.099	-0.331	-1.860
	0.7	6.677	-0.001	0.040	-0.173	-0.050
	0.5	4.384	0.000	0.019	-0.117	-0.006
	0.3	2.965	0.000	0.008	-0.081	-0.002
	0.1	1.873	0.000	0.003	-0.052	-0.001
	-0.1	0.908	0.000	0.000	-0.025	-0.000
	-0.3	-0.054	0.000	-0.000	0.001	-0.000
	-0.5	-1.154	0.000	-0.000	0.030	-0.000
	-0.7	-2.713	0.000	-0.004	0.065	-0.020
	-0.9	-6.570	-0.002	-0.026	0.138	-1.486
16	0.9	15.613	-0.004	0.057	-0.217	-1.875
	0.7	7.825	-0.001	0.021	-0.114	-0.047
	0.5	5.048	0.000	0.009	-0.076	-0.003
	0.3	3.309	0.000	0.003	-0.051	-0.001
	0.1	1.953	0.000	0.000	-0.031	-0.001
	-0.1	0.735	0.000	0.000	-0.012	-0.000
	-0.3	-0.500	0.000	-0.000	0.007	-0.000
	-0.5	-1.942	0.000	-0.000	0.029	-0.000
	-0.7	-4.033	0.000	-0.004	0.055	-0.028
	-0.9	-9.340	-0.002	-0.022	0.112	-1.602
36	0.9	21.058	-0.001	0.031	-0.127	-1.882
	0.7	10.403	0.000	0.011	-0.066	-0.046
	0.5	6.558	0.000	0.004	-0.043	-0.002
	0.3	4.119	0.000	0.002	-0.028	0.000
	0.1	2.191	0.000	0.000	-0.015	0.000
	-0.1	0.433	0.000	0.000	-0.003	0.000
	-0.3	-1.380	0.000	0.000	0.010	0.000
	-0.5	-3.535	0.000	-0.001	0.023	-0.001
	-0.7	-6.727	0.000	-0.003	0.041	-0.035
	-0.9	-15.009	-0.001	-0.016	0.082	-1.702

Table 1.1 (Continued). ($\alpha = 0.05$)

ν	η	true value	(1.10)	(1.3)	(1.4)	(1.5)
4	0.9	14.301	0.079	1.517	0.039	-2.937
	0.7	7.417	0.018	0.666	-0.062	-0.243
	0.5	5.053	0.017	0.359	-0.106	-0.104
	0.3	3.636	0.006	0.186	-0.124	-0.056
	0.1	2.587	0.001	0.082	-0.120	-0.028
	-0.1	1.703	0.000	0.026	-0.095	-0.010
	-0.3	0.877	0.000	0.004	-0.053	-0.001
	-0.5	0.013	0.000	-0.000	-0.001	-0.000
	-0.7	-1.079	0.000	-0.005	0.056	-0.003
	-0.9	-3.433	0.001	-0.058	0.124	-1.225
9	0.9	14.829	0.010	0.388	-0.142	-2.633
	0.7	7.606	0.005	0.155	-0.107	-0.090
	0.5	5.082	0.002	0.075	-0.094	-0.016
	0.3	3.535	0.000	0.033	-0.079	-0.007
	0.1	2.357	0.000	0.011	-0.060	-0.003
	-0.1	1.329	0.000	0.003	-0.036	-0.000
	-0.3	0.324	0.000	0.000	-0.009	0.000
	-0.5	-0.798	0.000	-0.000	0.021	-0.000
	-0.7	-2.341	0.000	-0.008	0.051	-0.017
	-0.9	-6.028	0.000	-0.052	0.096	-1.660
16	0.9	16.962	0.003	0.200	-0.115	-2.583
	0.7	8.598	0.001	0.075	-0.080	-0.074
	0.5	5.641	0.000	0.033	-0.066	-0.007
	0.3	3.804	0.000	0.014	-0.051	-0.001
	0.1	2.385	0.000	0.004	-0.035	-0.000
	-0.1	1.125	0.000	0.000	-0.017	-0.000
	-0.3	-0.136	0.000	0.000	0.002	0.000
	-0.5	-1.582	0.000	-0.001	0.022	-0.001
	-0.7	-3.640	0.000	-0.009	0.044	-0.027
	-0.9	-8.748	0.001	-0.045	0.078	-1.867
36	0.9	22.186	0.000	0.096	-0.075	-2.534
	0.7	11.062	0.001	0.034	-0.050	-0.068
	0.5	7.075	0.000	0.013	-0.039	-0.005
	0.3	4.562	0.000	0.005	-0.029	-0.000
	0.1	2.589	0.000	0.001	-0.017	-0.000
	-0.1	0.805	0.000	-0.000	-0.006	-0.000
	-0.3	-1.019	0.000	-0.001	0.020	-0.001
	-0.5	-3.165	0.000	-0.001	0.020	-0.001
	-0.7	-6.305	0.000	-0.008	0.033	-0.038
	-0.9	-14.353	0.000	-0.037	0.056	-2.058

Table 1.1 (Continued). ($\alpha = 0.01$)

ν	η	true value	(1.10)	(1.3)	(1.4)	(1.5)
4	0.9	22.240	1.734	18.278	11.212	-6.077
	0.7	11.683	0.715	8.432	5.036	-0.993
	0.5	8.095	0.351	4.852	2.781	-0.522
	0.3	5.967	0.156	2.727	1.456	-0.314
	0.1	4.412	0.052	1.348	0.620	-0.182
	-0.1	3.128	0.007	0.523	0.147	-0.089
	-0.3	1.968	-0.002	0.129	-0.044	-0.029
	-0.5	0.828	-0.000	0.008	-0.048	-0.002
	-0.7	-0.445	0.000	-0.001	0.025	-0.000
	-0.9	-2.711	0.005	-0.083	0.060	-1.038
9	0.9	18.849	0.198	1.961	1.087	-4.337
	0.7	9.837	0.067	0.826	0.396	-0.229
	0.5	6.728	0.026	0.430	0.155	-0.066
	0.3	4.848	0.008	0.216	0.032	-0.033
	0.1	3.439	0.001	0.093	-0.027	-0.017
	-0.1	2.235	0.000	0.029	-0.043	-0.005
	-0.3	1.093	0.000	0.003	-0.029	-0.001
	-0.5	-0.126	0.000	0.000	0.004	0.000
	-0.7	-1.695	0.000	-0.008	0.035	-0.007
	-0.9	-5.150	0.005	-0.089	0.034	-1.799
16	0.9	19.993	0.063	0.807	0.384	-4.054
	0.7	10.313	0.018	0.318	0.108	-0.147
	0.5	6.934	0.005	0.152	0.018	-0.022
	0.3	4.862	0.001	0.069	-0.021	-0.008
	0.1	3.284	0.001	0.026	-0.032	-0.003
	-0.1	1.910	-0.000	0.005	-0.027	-0.001
	-0.3	0.568	0.000	-0.000	-0.009	-0.001
	-0.5	-0.923	-0.000	-0.001	0.013	-0.000
	-0.7	-2.956	0.000	-0.012	0.030	-0.019
	-0.9	-7.758	0.004	-0.087	0.020	-2.212
36	0.9	24.549	0.017	0.326	0.124	-3.844
	0.7	12.429	0.003	0.118	0.018	-0.117
	0.5	8.131	0.001	0.051	-0.012	-0.010
	0.3	5.452	0.000	0.020	-0.022	-0.001
	0.1	3.374	-0.000	0.005	-0.020	-0.001
	-0.1	1.520	0.000	0.001	-0.010	0.000
	-0.3	-0.343	0.000	0.000	0.003	0.000
	-0.5	-2.490	-0.000	-0.002	0.014	-0.001
	-0.7	-5.560	0.000	-0.015	0.021	-0.039
	-0.9	-13.222	0.003	-0.076	0.008	-2.619

Table 1.2. The values of $t_{0.025}$. for $\nu = n_1 + n_2 - 2$ and $\delta = \sqrt{n_1 n_2 / n_1 + n_2} D$. (D=2.0)

$n_1 \backslash n_2$	5	6	7	8	9	10	11	12	13
5	7.1309								
6	7.1013	7.1445							
7	7.0693	7.1682	7.2366						
8	7.0379	7.1803	7.2853	7.3648					
9	7.0081	7.1854	7.3205	7.4262	7.5335				
10	6.9804	7.1859	7.3464	7.4745	7.5999	7.6645			
11	6.9548	7.1837	7.3655	7.5130	7.6544	7.7362	7.8222		
12	6.9311	7.1796	7.3798	7.5441	7.6995	7.7968	7.8957	7.9810	
13	6.9093	7.1744	7.3903	7.5694	7.7373	7.8485	7.9591	8.0551	8.1392
14	6.8893	7.1685	7.3981	7.5902	7.7691	7.8930	8.0141	8.1201	8.2133
15	6.8707	7.1622	7.4038	7.6074	7.7963	7.9314	8.0623	8.1773	8.2791
16	6.8536	7.1557	7.4079	7.6218	7.8196	7.9650	8.1048	8.2281	8.3378
17	6.8377	7.1492	7.4107	7.6338	7.8396	7.9945	8.1423	8.2734	8.3904
18	6.8229	7.1426	7.4125	7.6439	7.8570	8.0204	8.1758	8.3140	8.4377
19	6.8092	7.1361	7.4135	7.6524	7.8722	8.0434	8.2057	8.3505	8.4806
20	6.7963	7.1298	7.4140	7.6596	7.8854	8.0639	8.2325	8.3835	8.5194
21	6.7843	7.1236	7.4139	7.6657	7.8971	8.0822	8.2567	8.4134	8.5549
22	6.7731	7.1176	7.4134	7.6709	7.9074	8.0987	8.2786	8.4406	8.5873
23	6.7625	7.1118	7.4126	7.6753	7.9165	8.1135	8.2985	8.4655	8.6170
24	6.7526	7.1062	7.4116	7.6790	7.9246	8.1268	8.3166	8.4882	8.6443
25	6.7433	7.1008	7.4104	7.6822	7.9318	8.1389	8.3332	8.5091	8.6694
26	6.7345	7.0955	7.4091	7.6849	7.9383	8.1499	8.3483	8.5284	8.6927
27	6.7261	7.0905	7.4076	7.6872	7.9440	8.1600	8.3622	8.5462	8.7142
28	6.7182	7.0856	7.4060	7.6891	7.9492	8.1691	8.3750	8.5626	8.7343
29	6.7108	7.0809	7.4044	7.6907	7.9538	8.1775	8.3868	8.5778	8.7529
30	6.7037	7.0764	7.4027	7.6921	7.9580	8.1852	8.3978	8.5920	8.7702

Table 1.2. (Continued). (D=2.0)

$n_1 \backslash n_2$	14	15	16	17	18	19	20	21	22
14	8.2960								
15	8.3697	8.4509							
16	8.4358	8.5239	8.6035						
17	8.4953	8.5900	8.6758	8.7643					
18	8.5491	8.6499	8.7415	8.8352	8.9114				
19	8.5980	8.7046	8.8016	8.9002	8.9814	9.0562			
20	8.6425	8.7545	8.8568	8.9600	9.0459	9.1253	9.1986		
21	8.6833	8.8003	8.9075	9.0151	9.1056	9.1892	9.2667	9.3387	
22	8.7207	8.8425	8.9543	9.0660	9.1608	9.2485	9.3300	9.4058	9.4765
23	8.7550	8.8814	8.9975	9.1132	9.2121	9.3037	9.3890	9.4684	9.5426
24	8.7867	8.9174	9.0377	9.1571	9.2598	9.3552	9.4441	9.5270	9.6046
25	8.8161	8.9508	9.0750	9.1980	9.3044	9.4034	9.4957	9.5819	9.6627
26	8.8433	8.9818	9.1097	9.2361	9.3460	9.4484	9.5440	9.6335	9.7174
27	8.8685	9.0107	9.1422	9.2717	9.3850	9.4907	9.5895	9.6820	9.7689
28	8.8921	9.0377	9.1725	9.3051	9.4217	9.5304	9.6322	9.7277	9.8174
29	8.9140	9.0629	9.2009	9.3365	9.4561	9.5678	9.6726	9.7708	9.8633
30	8.9345	9.0865	9.2276	9.3659	9.4885	9.6031	9.7106	9.8116	9.9067

$n_1 \backslash n_2$	23	24	25	26	27	28	29	30
23	9.6121							
24	9.6773	9.7456						
25	9.7385	9.8098	9.8769					
26	9.7962	9.8703	9.9403	10.006				
27	9.8505	9.9275	10.000	10.069	10.134			
28	9.9019	9.9815	10.057	10.128	10.195	10.259		
29	9.9504	10.033	10.110	10.184	10.254	10.320	10.383	
30	9.9964	10.081	10.161	10.237	10.309	10.378	10.443	10.505

Table 1.2. (Continued). (D=1.0)

$n_1 \backslash n_2$	5	6	7	8	9	10	11	12	13
5	4.5993								
6	4.5656	4.5711							
7	4.5362	4.5715	4.5953						
8	4.5108	4.5691	4.6119	4.6444					
9	4.4887	4.5652	4.6237	4.6696	4.7063				
10	4.4694	4.5607	4.6322	4.6894	4.7362	4.7750			
11	4.4524	4.5559	4.6383	4.7054	4.7610	4.8076	4.8474		
12	4.4374	4.5510	4.6428	4.7184	4.7817	4.8354	4.8815	4.9214	
13	4.4240	4.5462	4.6460	4.7290	4.7992	4.8592	4.9111	4.9564	4.9962
14	4.4120	4.5416	4.6483	4.7379	4.8142	4.8799	4.9370	4.9871	5.0315
15	4.4012	4.5371	4.6500	4.7454	4.8271	4.8979	4.9598	5.0144	5.0629
16	4.3914	4.5328	4.6511	4.7517	4.8383	4.9137	4.9800	5.0388	5.0911
17	4.3825	4.5288	4.6518	4.7570	4.8481	4.9278	4.9981	5.0606	5.1165
18	4.3744	4.5249	4.6522	4.7616	4.8567	4.9403	5.0142	5.0803	5.1395
19	4.3669	4.5213	4.6524	4.7656	4.8644	4.9514	5.0288	5.0981	5.1604
20	4.3600	4.5178	4.6524	4.7690	4.8711	4.9615	5.0420	5.1143	5.1795
21	4.3537	4.5145	4.6523	4.7720	4.8772	4.9705	5.0540	5.1290	5.1970
22	4.3479	4.5114	4.6520	4.7746	4.8826	4.9787	5.0649	5.1426	5.2131
23	4.3424	4.5085	4.6517	4.7769	4.8875	4.9862	5.0749	5.1550	5.2279
24	4.3373	4.5057	4.6513	4.7789	4.8920	4.9930	5.0840	5.1665	5.2415
25	4.3326	4.5031	4.6508	4.7807	4.8960	4.9993	5.0925	5.1771	5.2542
26	4.3282	4.5006	4.6504	4.7823	4.8997	5.0050	5.1003	5.1869	5.2660
27	4.3241	4.4982	4.6498	4.7837	4.9030	5.0103	5.1075	5.1960	5.2770
28	4.3202	4.4959	4.6493	4.7849	4.9061	5.0152	5.1142	5.2045	5.2872
29	4.3165	4.4938	4.6487	4.7860	4.9089	5.0197	5.1204	5.2124	5.2968
30	4.3131	4.4917	4.6482	4.7870	4.9115	5.0239	5.1262	5.2198	5.3058

Table 1.2. (Continued). (D=1.0)

$n_1 \backslash n_2$	14	15	16	17	18	19	20	21	22
14	5.0710								
15	5.1063	5.1453							
16	5.1381	5.1805	5.2189						
17	5.1669	5.2125	5.2539	5.2917					
18	5.1931	5.2416	5.2859	5.3264	5.3636				
19	5.2169	5.2683	5.3152	5.3582	5.3978	5.4344			
20	5.2387	5.2927	5.3422	5.3876	5.4295	5.4683	5.5042		
21	5.2588	5.3153	5.3671	5.4148	5.4589	5.4997	5.5377	5.5730	
22	5.2773	5.3361	5.3902	5.4401	5.4862	5.5290	5.5689	5.6061	5.6408
23	5.2944	5.3554	5.4116	5.4635	5.5116	5.5564	5.5980	5.6370	5.6734
24	5.3102	5.3733	5.4316	5.4854	5.5354	5.5819	5.6253	5.6659	5.7040
25	5.3250	5.3900	5.4502	5.5059	5.5576	5.6059	5.6510	5.6932	5.7328
26	5.3387	5.4056	5.4675	5.5250	5.5785	5.6284	5.6750	5.7188	5.7599
27	5.3515	5.4202	5.4838	5.5430	5.5981	5.6495	5.6977	5.7430	5.7855
28	5.3634	5.4338	5.4991	5.5599	5.6165	5.6695	5.7192	5.7658	5.8097
29	5.3746	5.4467	5.5135	5.5758	5.6339	5.6883	5.7394	5.7874	5.8326
30	5.3852	5.4587	5.5271	5.5908	5.6503	5.7061	5.7585	5.8079	5.8544

$n_1 \backslash n_2$	23	24	25	26	27	28	29	30
23	5.7076							
24	5.7397	5.7734						
25	5.7700	5.8051	5.8382					
26	5.7986	5.8351	5.8695	5.9021				
27	5.8256	5.8634	5.8991	5.9330	5.9651			
28	5.8511	5.8902	5.9272	5.9623	5.9956	6.0272		
29	5.8753	5.9157	5.9539	5.9902	6.0246	6.0573	6.0885	
30	5.8983	5.9399	5.9793	6.0166	6.0522	6.0860	6.1182	6.1489

Table 1.2. (Continued). (D=0.5)

$n_1 \backslash n_2$	5	6	7	8	9	10	11	12	13
5	3.4076								
6	3.3715	3.3587							
7	3.3423	3.3469	3.3486						
8	3.3183	3.3362	3.3488	3.3578					
9	3.2983	3.3267	3.3480	3.3645	3.3775				
10	3.2813	3.3181	3.3467	3.3695	3.3880	3.4032			
11	3.2667	3.3104	3.3451	3.3733	3.3965	3.4160	3.4326		
12	3.2540	3.3036	3.3434	3.3762	3.4036	3.4269	3.4469	3.4642	
13	3.2430	3.2974	3.3417	3.3785	3.4096	3.4362	3.4592	3.4793	3.4970
14	3.2332	3.2917	3.3399	3.3803	3.4146	3.4442	3.4700	3.4927	3.5127
15	3.2245	3.2866	3.3382	3.3817	3.4189	3.4513	3.4795	3.5045	3.5267
16	3.2168	3.2820	3.3365	3.3828	3.4227	3.4574	3.4880	3.5151	3.5393
17	3.2098	3.2777	3.3348	3.3836	3.4259	3.4629	3.4955	3.5246	3.5506
18	3.2035	3.2738	3.3333	3.3843	3.4287	3.4677	3.5023	3.5332	3.5609
19	3.1978	3.2702	3.3318	3.3848	3.4312	3.4721	3.5084	3.5410	3.5703
20	3.1925	3.2669	3.3303	3.3853	3.4334	3.4760	3.5139	3.5480	3.5789
21	3.1877	3.2638	3.3290	3.3856	3.4353	3.4795	3.5190	3.5545	3.5867
22	3.1833	3.2610	3.3277	3.3858	3.4371	3.4827	3.5236	3.5605	3.5940
23	3.1792	3.2583	3.3264	3.3860	3.4386	3.4856	3.5278	3.5659	3.6007
24	3.1755	3.2558	3.3253	3.3861	3.4400	3.4882	3.5316	3.5710	3.6069
25	3.1720	3.2535	3.3241	3.3862	3.4413	3.4907	3.5352	3.5757	3.6126
26	3.1687	3.2513	3.3231	3.3862	3.4424	3.4929	3.5385	3.5800	3.6180
27	3.1657	3.2493	3.3221	3.3862	3.4435	3.4950	3.5416	3.5841	3.6230
28	3.1628	3.2474	3.3211	3.3862	3.4444	3.4969	3.5444	3.5878	3.6276
29	3.1602	3.2456	3.3201	3.3862	3.4453	3.4986	3.5471	3.5914	3.6320
30	3.1577	3.2438	3.3193	3.3862	3.4461	3.5003	3.5496	3.5947	3.6361

Table 1.2. (Continued). (D=0.5)

$n_1 \backslash n_2$	14	15	16	17	18	19	20	21	22
14	3.5306								
15	3.5466	3.5645							
16	3.5610	3.5807	3.5985						
17	3.5741	3.5954	3.6147	3.6324					
18	3.5860	3.6088	3.6296	3.6487	3.6662				
19	3.5969	3.6211	3.6433	3.6637	3.6824	3.6998			
20	3.6069	3.6325	3.6559	3.6775	3.6974	3.7158	3.7330		
21	3.6161	3.6429	3.6676	3.6903	3.7113	3.7308	3.7489	3.7659	
22	3.6246	3.6526	3.6784	3.7022	3.7242	3.7447	3.7638	3.7817	3.7984
23	3.6324	3.6616	3.6884	3.7133	3.7363	3.7578	3.7778	3.7965	3.8140
24	3.6397	3.6699	3.6978	3.7236	3.7476	3.7700	3.7909	3.8104	3.8288
25	3.6465	3.6777	3.7066	3.7333	3.7582	3.7814	3.8032	3.8235	3.8426
26	3.6528	3.6850	3.7148	3.7424	3.7682	3.7922	3.8147	3.8359	3.8557
27	3.6588	3.6918	3.7225	3.7510	3.7775	3.8024	3.8257	3.8475	3.8681
28	3.6643	3.6982	3.7297	3.7590	3.7864	3.8120	3.8360	3.8585	3.8798
29	3.6695	3.7043	3.7365	3.7666	3.7947	3.8210	3.8457	3.8690	3.8909
30	3.6744	3.7099	3.7430	3.7738	3.8026	3.8296	3.8550	3.8789	3.9015
$n_1 \backslash n_2$	23	24	25	26	27	28	29	30	
23	3.8305								
24	3.8460	3.8623							
25	3.8606	3.8776	3.8937						
26	3.8745	3.8921	3.9088	3.9247					
27	3.8875	3.9059	3.9232	3.9397	3.9553				
28	3.8999	3.9189	3.9369	3.9539	3.9701	3.9855			
29	3.9116	3.9312	3.9498	3.9675	3.9842	4.0002	4.0154		
30	3.9228	3.9430	3.9622	3.9804	3.9977	4.0142	4.0299	4.0449	

CHAPTER 2

Approximations to Non-Central χ^2 Distribution

2.1. Introduction

Approximations to a percentage point of the non-central distribution have been discussed by many authors (see, *e.g.* Ashour and Abdel-Samad [AA90], Bagui [B93], Cox and Reid [CR87], Guirguis [G90], Johnson *et al.* [JKB95], Moon [Mo86], Shibata [Sh81], Takeuchi [Ta75], Wang and Gray [WG93] and Yamauti *et al.* [Y72]), but they do not necessarily suffice for use. We are interested in the approximations to the non-central distributions since they seems to be comparatively uncultivated.

The non-central χ^2 distribution is regarded as a limit of non-central F distribution and also appears when one obtains asymptotic distributions of χ^2 test statistics of goodness of fit and test statistics in testing the hypothesis on discrete distributions under the alternative hypothesis. Therefore, percentage points of the non-central χ^2 distributions play an important part in testing the hypothesis and obtaining the confidence interval.

We consider approximations to the non-central χ^2 distribution $\chi^2(\nu, \xi)$ with ν degrees of freedom whose density is given by

$$p_{\chi^2}(x; \nu, \xi) := e^{-\frac{\xi}{2}} \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{\xi}{2}\right)^k \frac{2^{-(\nu+2k)/2}}{\Gamma(k+\nu/2)} x^{(\nu+2k)/2-1} e^{-x/2}$$

for $0 < x < \infty$, $\nu = 1, 2, \ldots$, $-\infty < \xi < \infty$, where $\Gamma(\cdot)$ is gamma function. Then the cumulative distribution function is expressed using incomplete beta functions. This brings a difficulty in obtaining analytically percentage points of the non-central distribution. In order to get numerically percentage points, we need a large-scale computation. Hence it is very useful to find better approximation formulae of percentage points of the non-central χ^2 distribution.

In this chapter, in a similar way to Chapter 1, we obtain new higher order approximation formulae of percentage points of the non-central χ^2 distribution using the Cornish-Fisher expansion for the statistic based on a chi-random variable ([To96]). We also have an approximation formula of a percentage point of the non-central χ^2 distribution by a direct application of the Cornish-Fisher expansion for the chi-square statistic. Further, the numerical comparison of these formulae with the former shows that the new approximation formulae behave better than the others.

2.2. Approximations to a percentage point of non-central χ^2 distribution

We denote by $\chi^2(\alpha;\nu,\xi)$ an upper 100α percentile of non-central χ^2 distribution $\chi^2(\nu,\xi)$. Among approximation formulae for the percentage point, we present some well-known formulae following Shibata [Sh81] (see also, *e.g.* Johnson *et al.* [JKB95]). First we have

$$\chi^2(\alpha;\nu,\xi) = (\nu+\xi)(\mu+\sigma u_\alpha)^{1/h} \quad \text{(Sankaran [Sa63])}, \tag{2.1}$$

where u_α is the upper 100α percentile of the standard normal distribution,

$$h = 1 - \frac{2(\nu+\xi)(\nu+3\xi)}{3(\nu+2\xi)^2},$$

$$\mu = 1 + h(h-1)\frac{(\nu+2\xi)}{(\nu+\xi)^2} + h(h-1)(h-2)(1-3h)\frac{(\nu+2\xi)^2}{2(\nu+\xi)^4},$$

and

$$\sigma = \sqrt{h^2\frac{2(\nu+2\xi)}{(\nu+\xi)^2} + h^2(h-1)(1-3h)\frac{2(\nu+2\xi)^2}{(\nu+\xi)^4}}.$$

This is a generalization of Wilson-Hilferty's approximation formula.

Letting $\chi^2_{\nu,\xi}$ be a random variable according to a non-central χ^2 distribution $\chi^2(\nu,\xi)$ and χ^2_m a random variable according to a central χ^2 distribution $\chi^2(m)$ with m degrees of freedom. If the distribution of $\chi^2_{\nu,\xi}/c_1$ is approximated to the central χ^2 distribution $\chi^2(m)$ so that their first and second cumulants are equal, then we have

$$\chi^2(\alpha;\nu,\xi) = c_1\chi^2(\alpha;m) \quad \text{(Patnaik [Pa49])}, \tag{2.2}$$

where

$$c_1 = \frac{\nu+2\xi}{\nu+\xi}, \qquad m = \frac{(\nu+\xi)^2}{\nu+2\xi}.$$

Furthermore, if the distribution of $(\chi^2_{\nu,\xi} - b)/c_2$ is approximated to the central χ^2 distribution $\chi^2(n)$ so that their first, second and third cumulants are equal, then we have

$$\chi^2(\alpha;\nu,\xi) = c_2\chi^2(\alpha;n) + b \quad \text{(Pearson [Pe59])}, \tag{2.3}$$

where

$$b = -\frac{\xi^2}{\nu+3\xi}, \qquad c_2 = \frac{\nu+3\xi}{\nu+2\xi}, \qquad n = \frac{(\nu+2\xi)^3}{(\nu+3\xi)^2}.$$

Now, denote by $\chi^2(\alpha;\nu)$ the upper 100α percentile of the central χ^2 distribution $\chi^2(\nu)$. Then it is known that by the Cornish-Fisher expansion

$$\begin{aligned}\chi^2(\alpha;\nu) =&\nu + \sqrt{2\nu}u_\alpha + \frac{2}{3}(u_\alpha^2 - 1) + \frac{1}{9\sqrt{2\nu}}(u_\alpha^3 - 7u_\alpha)\\ &- \frac{2}{405\nu}(3u_\alpha^4 + 7u_\alpha^2 - 16) + o\Big(\frac{1}{\nu}\Big).\end{aligned}$$

Next, we derive a new approximation from (2.3) using the Cornish-Fisher expansion for the statistic based on the chi-random variable. Let $\chi^2(\alpha;\nu,\xi)$ be an upper 100α percentile of $\chi^2(\nu,\xi)$. Then it follows from (2.3) that for sufficiently large ν

$$\begin{aligned}1-\alpha &\approx P\{\chi^2_{\nu,\xi} < \chi^2(\alpha;\nu,\xi)\}\\ &= P\Big\{\frac{\chi^2_{\nu,\xi} - b}{c_2} < \frac{\chi^2(\alpha;\nu,\xi) - b}{c_2}\Big\}.\end{aligned} \tag{2.4}$$

Put $x_\alpha := (\chi^2(\alpha;\nu,\xi) - b)/c_2$ and $X := (\chi^2_{\nu,\xi} - b)/c_2$. Then X is asymptotically equal to χ^2_n. Letting $S_n := \sqrt{\chi^2_n/n}$, we have from (1.1) and (1.2), $E[S_n] = b_n = \sqrt{\frac{2}{n}}\Gamma[(n+1/2)]/\Gamma(n/2)$ and $V(S_n) = 1 - b_n^2$. From (2.4) we obtain for sufficiently large ν

$$\begin{aligned}1-\alpha &\approx P\Big\{S_n < \sqrt{\frac{x_\alpha}{n}}\Big\}\\ &= P\Big\{\frac{S_n - b_n}{\sqrt{1-b_n^2}} < \frac{\sqrt{x_\alpha/n} - b_n}{\sqrt{1-b_n^2}}\Big\}.\end{aligned} \tag{2.5}$$

Putting

$$Y := \frac{S_n - b_n}{\sqrt{1-b_n^2}},$$

we easily see that $E[Y] = 0$ and $V(Y) = 1$. In order to use the Cornish-Fisher expansion for the statistic Y up to the order $o(n^{-3/2})$, we need the third and fourth cumulants of Y

up to the same order.

Lemma 2.1([A95]). *The third and fourth cumulants of $S_n - b_n$ are given by*

$$\kappa_{3,n}(S_n - b_n) = -b_n\Big\{2(1-b_n^2) - \frac{1}{n}\Big\},$$

and

$$\kappa_{4,n}(S_n - b_n) = (1-b_n^2)\{4 - 6(1-b_n^2)\} + \frac{4}{n}(1-b_n^2) - \frac{2}{n},$$

respectively.

Proof. We have from (1.1), (1.2) and Lemma 1.1

$$\begin{aligned}\kappa_{3,n}(S_n - b_n) &= E[(S_n - b_n)^3]\\ &= E[S_n^3] - 3b_n E[S_n^2] + 3b_n^2 E[S_n] - b_n^3\\ &= -b_n\Big\{2(1-b_n^2) - \frac{1}{n}\Big\}.\end{aligned}$$

It also follows from (1.1), (1.2) and Lemma 1.1 that

$$\begin{aligned}\kappa_{4,n}(S_n - b_n) &= E[(S_n - b_n)^4] - 3\{E[(S_n - b_n)^2]\}^2\\ &= \frac{2}{n}(1-2b_n^2) + (1-b_n^2)(1+3b_n^2) - 3(1-b_n^2)^2\\ &= (1-b_n^2)\{4 - 6(1-b_n^2)\} + \frac{4}{n}(1-b_n^2) - \frac{2}{n}. \qquad \square\end{aligned}$$

Lemma 2.2([A95]). *The third and fourth cumulants of Y are given by*

$$\kappa_{3,n}(Y) = \frac{1}{4(1-b_n^2)^{3/2}}\Big(\frac{1}{n^2} + \frac{1}{4n^3}\Big) + O\Big(\frac{1}{n^{5/2}}\Big),$$

and

$$\kappa_{4,n}(Y) = O\Big(\frac{1}{n^2}\Big),$$

respectively.

Proof. From Lemmas 1.2 and 2.1 we have

$$\begin{aligned}\kappa_{3,n}(Y) &= \frac{1}{(1-b_n^2)^{3/2}}\kappa_{3,n}(S_n - b_n)\\ &= -\frac{b_n}{(1-b_n^2)^{3/2}}\Big\{2(1-b_n^2) - \frac{1}{n}\Big\}\\ &= \frac{1}{4(1-b_n^2)^{3/2}}\Big\{\frac{1}{n^2} + \frac{1}{4n^3} + O\Big(\frac{1}{n^4}\Big)\Big\}\\ &= \frac{1}{4(1-b_n^2)^{3/2}}\Big(\frac{1}{n^2} + \frac{1}{4n^3}\Big) + O\Big(\frac{1}{n^{5/2}}\Big).\end{aligned}$$

It also follows from Lemmas 1.2 and 2.1 that

$$\begin{aligned}
\kappa_{4,n}(Y) &= \frac{1}{(1-b_n^2)^2}\kappa_{4,n}(S_n - b_n) \\
&= \frac{1}{(1-b_n^2)^2}\Big[(1-b_n^2)\{4-6(1-b_n^2)\} + \frac{4}{n}(1-b_n^2) - \frac{2}{n}\Big] \\
&= O\Big(\frac{1}{n^2}\Big). \qquad \square
\end{aligned}$$

Theorem 2.1. *The upper* 100α *percentile* $\chi^2(\alpha;\nu,\xi)$ *of the non-central* χ^2 *distribution with* ν *degrees of freedom and a non-centrality parameter* ξ *can be derived from the formula*

$$\begin{aligned}
(2.6) \qquad \chi^2(\alpha;\nu,\xi) =& b + c_2 n\Big\{b_n + u_\alpha\sqrt{1-b_n^2} + \frac{u_\alpha^2 - 1}{24(1-b_n^2)}\Big(\frac{1}{n^2} + \frac{1}{4n^3}\Big) \\
& - \frac{2u_\alpha^3 - 5u_\alpha}{576(1-b_n^2)^{5/2}n^4} + O\Big(\frac{1}{n^{5/2}}\Big)\Big\}^2.
\end{aligned}$$

Proof. From (2.5) and Lemma 2.2 we have by the Cornish-Fisher expansion

$$\begin{aligned}
\frac{\sqrt{x_\alpha/n} - b_n}{\sqrt{1-b_n^2}} &= u_\alpha + \frac{1}{6}\kappa_{3,n}(Y)(u_\alpha^2 - 1) + \frac{1}{24}\kappa_{4,n}(Y)(u_\alpha^3 - 3u_\alpha) \\
&\quad - \frac{1}{36}\kappa_{3,n}^2(Y)(2u_\alpha^3 - 5u_\alpha) + O\Big(\frac{1}{n^2}\Big) \\
&= u_\alpha + \frac{u_\alpha^2 - 1}{24(1-b_n^2)^{3/2}}\Big(\frac{1}{n^2} + \frac{1}{4n^3}\Big) - \frac{2u_\alpha^3 - 5u_\alpha}{576(1-b_n^2)^3 n^4} + O\Big(\frac{1}{n^2}\Big).
\end{aligned}$$

Since $x_\alpha = (\chi^2(\alpha;\nu,\xi) - b)/c_2$, the result follows. $\square$

Remark 2.1. From Theorem 2.1 we consider a formula

$$(2.7) \qquad \chi^2(\alpha;\nu,\xi) \fallingdotseq b + c_2 n\Big\{b_n + u_\alpha\sqrt{1-b_n^2} + \frac{u_\alpha^2 - 1}{24(1-b_n^2)}\Big(\frac{1}{n^2} + \frac{1}{4n^3}\Big)\Big\}^2,$$

which seems to behave numerically better than the formula (2.6).

Next we consider another approximation formula of the percentage point of the non-central χ^2 distribution up to the order $o(\nu^{-1})$, using directly the Cornish-Fisher expansion for $\chi^2_{\nu,\xi}$. Since $E[\chi^2_{\nu,\xi}] = \nu + \xi$ and $V(\chi^2_{\nu,\xi}) = 2(\nu + 2\xi)$, letting

$$U := \frac{\chi^2_{\nu,\xi} - (\nu + \xi)}{\sqrt{2(\nu + 2\xi)}},$$

we easily see that $E[U] = 0$ and $V(U) = 1$. The third, fourth and fifth cumulants of U are also given by

$$\kappa_{3,U} = \frac{2\sqrt{2}(\nu+3\xi)}{(\nu+2\xi)^{3/2}}, \qquad \kappa_{4,U} = \frac{12(\nu+4\xi)}{(\nu+2\xi)^2}, \qquad \kappa_{5,U} = \frac{48\sqrt{2}(\nu+5\xi)}{(\nu+2\xi)^{5/2}},$$

respectively. By the Cornish-Fisher expansion for U, we have the upper 100α percentile $x(\alpha; U)$ of the distribution of U as follows.

$$x(\alpha; U) = u_\alpha + B_1(u_\alpha) + B_2(u_\alpha) + B_3(u_\alpha) + O(\nu^{-2}),$$

where

$$\begin{aligned}
B_1(u_\alpha) &= \frac{\sqrt{2}(\nu+3\xi)}{3(\nu+2\xi)^{3/2}}(u_\alpha^2 - 1),\\
B_2(u_\alpha) &= \frac{\nu+4\xi}{2(\nu+2\xi)^2}(u_\alpha^3 - 3u_\alpha) - \frac{2(\nu+3\xi)^2}{9(\nu+2\xi)^3}(2u_\alpha^3 - 5u_\alpha),\\
B_3(u_\alpha) &= \frac{2\sqrt{2}(\nu+5\xi)}{5(\nu+2\xi)^{5/2}}(u_\alpha^4 - 6u_\alpha^2 + 3) - \frac{\sqrt{2}(\nu+3\xi)(\nu+4\xi)}{(\nu+2\xi)^{7/2}}(u_\alpha^4 - 5u_\alpha^2 + 2)\\
&\quad + \frac{4\sqrt{2}(\nu+3\xi)^3}{81(\nu+2\xi)^{9/2}}(12u_\alpha^4 - 53u_\alpha^2 + 17).
\end{aligned}$$

Since $\chi^2(\alpha; \nu, \xi) = \nu + \xi + \sqrt{2(\nu+2\xi)}\, x(\alpha; U)$, then we have

$$\begin{aligned}
(2.8)\quad \chi^2(\alpha; \nu, \xi) &= \nu + \xi + u_\alpha\sqrt{2(\nu+2\xi)} + \frac{2(\nu+3\xi)}{3(\nu+2\xi)}(u_\alpha^2 - 1)\\
&\quad + \frac{\sqrt{2}(\nu+4\xi)}{2(\nu+2\xi)^{3/2}}(u_\alpha^3 - 3u_\alpha) - \frac{2\sqrt{2}(\nu+3\xi)^2}{9(\nu+2\xi)^{5/2}}(2u_\alpha^3 - 5u_\alpha)\\
&\quad + \frac{4(\nu+5\xi)}{5(\nu+2\xi)^2}(u_\alpha^4 - 6u_\alpha^2 + 3) - \frac{2(\nu+3\xi)(\nu+4\xi)}{(\nu+2\xi)^3}(u_\alpha^4 - 5u_\alpha^2 + 2)\\
&\quad + \frac{8(\nu+3\xi)^3}{81(\nu+2\xi)^4}(12u_\alpha^4 - 53u_\alpha^2 + 17) + O(\nu^{-3/2}).
\end{aligned}$$

2.3. Evaluation of new approximation formulae in comparison with others for the non-central χ^2 distribution

In order to compare (2.7) and (2.8) with (2.1), (2.2) and (2.3) for the upper 100α percentile $\chi^2(\alpha, \nu, \xi)$ of the non-central χ^2 distribution, we have the numerical calculation when α is 0.05, ν is 10, 15, 20, 25, 30, 40, 50 and ξ is 1.0, 3.0, 5.0, 10.0, 15.0, 20.0, 25.0. Then the errors of the approximation formulae are given as Table 2.1, where some true values are searchingly obtained using *Mathematica*. (A similar table on (2.1), (2.2) and (2.3) is also given by Shibata [Sh81].) As is seen in Table 2.1, the approximation formula (2.8) based on a direct application of the Cornish-Fisher (C-F) expansion for $\chi^2_{\nu,\xi}$ dominates the others, and the new approximation one (2.7) also behaves comparatively better than the former (2.1), (2.2) and (2.3).

Table 2.1. Errors of the approximation formulae of the upper 5 percentile of $\chi^2(\nu, \xi)$

ν	ξ	true value	(2.7)	C - F (2.8)	Sankaran (2.1)	Patnaik (2.2)	Pearson (2.3)
10	1.0	20.0936	0.004	-0.001	-0.018	0.001	-0.005
	3.0	23.4593	-0.005	0.001	-0.020	0.020	-0.013
	5.0	26.6362	-0.010	0.002	-0.018	0.048	-0.018
	10.0	34.0886	-0.015	0.002	-0.012	0.116	-0.021
	15.0	41.1226	-0.015	0.002	-0.007	0.171	-0.021
	20.0	47.8997	-0.015	0.002	-0.004	0.214	-0.020
	25.0	54.4983	-0.014	0.001	-0.002	0.248	-0.019
15	1.0	26.6377	0.004	-0.001	-0.012	0.001	-0.002
	3.0	29.7989	-0.001	0.000	-0.014	0.013	-0.006
	5.0	32.8370	-0.004	0.001	-0.014	0.032	-0.010
	10.0	40.0756	-0.009	0.001	-0.012	0.084	-0.014
	15.0	46.9825	-0.011	0.001	-0.009	0.132	-0.016
	20.0	53.6727	-0.011	0.001	-0.006	0.172	-0.016
	25.0	60.2068	-0.011	0.001	-0.004	0.205	-0.015

Table 2.1 (Continued).

ν	ξ	true value	(2.7)	C - F (2.8)	Sankaran (2.1)	Patnaik (2.2)	Pearson (2.3)
20	1.0	32.9650	0.003	-0.001	-0.009	0.001	-0.001
	3.0	35.9912	0.001	-0.000	-0.011	0.009	-0.004
	5.0	38.9294	-0.002	0.000	-0.011	0.023	-0.006
	10.0	46.0004	-0.006	0.001	-0.011	0.064	-0.010
	15.0	52.8005	-0.008	0.001	-0.009	0.105	-0.012
	20.0	59.4148	-0.009	0.001	-0.007	0.141	-0.013
	25.0	65.8913	-0.009	0.001	-0.005	0.172	-0.013
25	1.0	39.1472	0.003	-0.000	-0.008	0.000	-0.001
	3.0	42.0759	0.001	-0.000	-0.009	0.007	-0.002
	5.0	44.9377	-0.000	0.000	-0.009	0.017	-0.004
	10.0	51.8731	-0.004	0.000	-0.009	0.051	-0.008
	15.0	58.5820	-0.006	0.001	-0.008	0.085	-0.009
	20.0	65.1293	-0.007	0.001	-0.007	0.118	-0.010
	25.0	71.5540	-0.007	0.001	-0.006	0.147	-0.011
30	1.0	45.2234	0.003	-0.000	-0.007	0.000	-0.001
	3.0	48.0776	0.002	-0.000	-0.007	0.005	-0.002
	5.0	50.8787	0.000	-0.000	-0.008	0.014	-0.003
	10.0	57.7014	-0.002	0.000	-0.008	0.041	-0.006
	15.0	64.3312	-0.004	0.000	-0.008	0.071	-0.007
	20.0	70.8191	-0.005	0.000	-0.007	0.100	-0.009
	25.0	77.1968	-0.006	0.001	-0.006	0.127	-0.009
40	1.0	57.1469	0.002	-0.000	-0.005	0.000	-0.000
	3.0	59.8929	0.002	-0.000	-0.006	0.003	-0.001
	5.0	62.6027	0.001	-0.000	-0.006	0.009	-0.002
	10.0	69.2477	-0.001	0.000	-0.007	0.028	-0.003
	15.0	75.7469	-0.002	0.000	-0.007	0.051	-0.005
	20.0	82.1332	-0.003	0.000	-0.006	0.075	-0.006
	25.0	88.4291	-0.004	0.000	-0.006	0.097	-0.007
50	1.0	68.8510	0.002	-0.000	-0.004	0.000	-0.000
	3.0	71.5211	0.002	-0.000	-0.005	0.002	-0.001
	5.0	74.1646	0.001	-0.000	-0.005	0.006	-0.001
	10.0	80.6747	0.000	0.000	-0.005	0.021	-0.002
	15.0	87.0696	-0.001	0.000	-0.006	0.039	-0.004
	20.0	93.3722	-0.002	0.000	-0.006	0.058	-0.004
	25.0	99.5989	-0.003	0.000	-0.005	0.077	-0.005

CHAPTER 3

Approximations to Non-Central F Distribution

3.1. Introduction

In general, when the power of F test is calculated in the analysis of variance, the non-central F distribution is needed. Indeed, suppose that, for each $i = 1, \dots, p$, X_{ij} $(j = 1, \dots, q)$ are independently, identically and normally distributed random variables with mean μ_i and variance σ^2. In case of testing the null hypothesis $\mu_i = \mu$ $(i = 1, \dots, p)$, it is known to use $T := q\sum_{i=1}^{p}(\bar{X}_{i\cdot} - \bar{X}_{\cdot\cdot})^2/S^2$ as a test statistic, where

$$\bar{X}_{i\cdot} := \frac{1}{q}\sum_{j=1}^{q} X_{ij} \quad (i = 1, \dots, p), \qquad \bar{X}_{\cdot\cdot} := \frac{1}{pq}\sum_{i=1}^{p}\sum_{j=1}^{q} X_{ij},$$

$$S^2 := \frac{1}{p(q-1)}\sum_{i=1}^{p}\sum_{j=1}^{q}(X_{ij} - \bar{X}_{i\cdot})^2.$$

Then, under the alternative hypothesis, T is distributed according to a non-central F distribution with a non-centrality parameter $\lambda := q\sum_{i=1}^{p}(\mu_i - \bar{\mu})^2/\sigma^2$, where $\bar{\mu} := \sum_{i=1}^{p}\mu_i/p$. Therefore, percentage points of the non-central F distributions play an important part in testing the hypothesis and obtaining the confidence interval.

We consider approximations to the non-central F distribution $F(\nu_1, \nu_2, \lambda)$ with (ν_1, ν_2) degrees of freedom and a non-centrality parameter λ whose density is given by

$$\begin{aligned} p_F(x; \nu_1, \nu_2, \lambda) :=& \frac{e^{-\lambda/2}\nu_1^{\nu_1/2}\nu_2^{\nu_2/2}}{B(\nu_1/2, \nu_2/2)} x^{\nu_1/2-1}(\nu_2 + \nu_1 x)^{-(\nu_1+\nu_2)/2} \\ &\cdot \sum_{k=0}^{\infty}\left\{\frac{\lambda\nu_1 x}{2(\nu_2 + \nu_1 x)}\right\}^k \left(\frac{1}{k!}\right)\frac{B(\nu_1/2, \nu_2/2)}{B(\nu_1/2 + k, \nu_2/2)} \end{aligned}$$

for $0 < x < \infty$, $\nu_1, \nu_2 = 1, 2, \dots$, $\lambda > 0$, where $B(\cdot, \cdot)$ is beta function. Then the cumulative distribution function is expressed using incomplete beta functions. This brings a difficulty in obtaining analytically percentage points of the non-central distributions. In order to get numerically percentage points, we also need a large-scale computation. Hence it is very useful to find better approximation formulae of percentage points of the non-central F distributions.

In this chapter, in a similar way to Chapter 1, we obtain new higher order approximation formula of percentage points of the non-central F distributions using the Cornish-Fisher expansion for the statistic based on a linear combination of chi-random variables ([To96]). Further, the numerical comparison of these formulae with the former shows that the new approximation formulae behave better than the others.

3.2. Approximation to a percentage point of the non-central F distribution

First, we present some well-known formulae for a percentage point of the non-central F distribution following Shibata[Sh81] (see also [JKB95]). Suppose that X_1 and X_2 are independent random variables according to a non-central χ^2 distribution $\chi^2(\nu_1, \lambda)$ and a central χ^2 distribution $\chi^2(\nu_2)$, respectively. Denote

$$F_{\nu_1,\nu_2,\lambda} := \frac{X_1/\nu_1}{X_2/\nu_2}.$$

Then we see that the random variable $F_{\nu_1,\nu_2,\lambda}$ has a non-central F distribution $F(\nu_1, \nu_2, \lambda)$. Let $X_1 = c_1 X'$ with some constant c_1. If the distribution of X' is approximated to the $\chi^2(m)$, then we have, from (2.2), $c_1 = (\nu_1 + 2\lambda)/(\nu_1 + \lambda)$ and $m = (\nu_1 + \lambda)^2/(\nu_1 + 2\lambda)$. Since for sufficiently large ν_1 and ν_2

$$F_{\nu_1,\nu_2,\lambda} \approx \left(1 + \frac{\lambda}{\nu_1}\right) \frac{X'/m}{X_2/\nu_2},$$

we have

$$P\{F_{\nu_1,\nu_2,\lambda} > f\} \approx P\Big\{F_{m,\nu_2} > \frac{\nu_1 f}{\nu_1 + \lambda}\Big\} \quad \text{(Patnaik [Pa49])}, \tag{3.1}$$

where F_{m,ν_2} is a random variable according to a central F distribution $F(m, \nu_2)$ with (m, ν_2) degrees of freedom. By Wilson-Hilferty's approximation we have

$$\begin{aligned} P\{F_{m,\nu_2} \le t\} &= P\Big\{\frac{X_0/m}{X_2/\nu_2} \le t\Big\} \\ &= P\Big\{\Big(\frac{X_0}{m}\Big)^{1/3} - t^{1/3}\Big(\frac{X_2}{\nu_2}\Big)^{1/3} \le 0\Big\}, \end{aligned}$$

where X_0 is a random variable with a central χ^2 distribution $\chi^2(m)$ which is independent of X_2. Since, for sufficiently large ν_1 and ν_2, the distribution of $(X_0/m)^{1/3} - t^{1/3}(X_2/\nu_2)^{1/3}$

is approximated to the normal distribution with mean $1-\{2/(9\nu_1)\}-t^{1/3}[1-\{2/(9\nu_2)\}]$ and variance $\{2/(9\nu_1)\}+t^{2/3}\{2/(9\nu_2)\}$, it follows that

$$P\{F_{m,\nu_2}\le t\} \fallingdotseq \Phi\left(\frac{(1-\frac{2}{9\nu_2})t^{1/3}-(1-\frac{2}{9\nu_1})}{\sqrt{\frac{2}{9\nu_1}+t^{2/3}\cdot\frac{2}{9\nu_2}}}\right)$$

which is said to be Paulson's approximation formula (see [Sh81]). Using the Paulson approximation formula we have from (3.1) for sufficiently large ν_1 and ν_2

$$\text{(3.2)}\qquad P\{F_{\nu_1,\nu_2,\lambda}>f\}\approx 1-\Phi\left[\frac{(1-d)z^{1/3}-(1-a)}{\sqrt{a+dz^{2/3}}}\right]\quad\text{(Severo-Zelen [SZ60])},$$

where

$$z=\frac{\nu_1 f}{\nu_1+\lambda},\qquad a=\frac{2}{9m}=\frac{2(\nu_1+2\lambda)}{9(\nu_1+\lambda)^2},\qquad d=\frac{2}{9\nu_2}.$$

If the distribution of $(F_{\nu_1,\nu_2,\lambda}-\rho)/\gamma$ is approximated to the central F distribution $F(\nu^*,\nu_2)$ so that their first, second and third cumulants are equal, then we have for sufficiently large ν_1 and ν_2

$$\text{(3.3)}\qquad P\{F_{\nu_1,\nu_2,\lambda}>f\}\approx P\left\{F_{\nu^*,\nu_2}>\frac{(f-\rho)}{\gamma}\right\}\quad\text{(Tiku [Ti65])},$$

where

$$\nu^*=\frac{1}{2}(\nu_2-2)\left(\sqrt{\frac{H^2}{H^2-4K^3}}-1\right),$$

$$\gamma=\frac{\nu^* H}{\nu_1(2\nu^*+\nu_2-2)K},\qquad \rho=\frac{\nu_2(1+(\lambda/\nu_1)-\gamma)}{\nu_2-2}$$

with $H=2(\nu_1+\lambda)^3+3(\nu_1+\lambda)(\nu_1+2\lambda)(\nu_2-2)+(\nu_1+3\lambda)(\nu_2-2)^2$ and $K=(\nu_1+\lambda)^2+(\nu_1+2\lambda)(\nu_2-2)$.

Next, in a similar way to Chapter 1, we derive a new approximation formula of a percentage point of the non-central F distribution from (3.3) by the Cornish-Fisher expansion for a statistic based on a linear combination of two chi-random variables. Let S be a random variable according to the χ^2 distribution $\chi^2(\nu^*)$, which is independent of the random variable X_2, and put $f'_\alpha := (f_\alpha-\rho)/\gamma$. Then it follows from (3.3) that for sufficiently large ν_1 and ν_2

$$
\begin{aligned}
(3.4)\quad 1-\alpha &\approx P\Big\{\frac{S/\nu^*}{X_2/\nu_2} < f'_\alpha\Big\} \\
&= P\Big\{\sqrt{\frac{S}{\nu^*}} - \sqrt{f'_\alpha}\sqrt{\frac{X_2}{\nu_2}} < 0\Big\} \\
&= P\Big\{\frac{\sqrt{S/\nu^*} - b_{\nu^*} - \sqrt{f'_\alpha}(\sqrt{X_2/\nu_2} - b_{\nu_2})}{\sqrt{1 - b_{\nu^*}^2 + f'_\alpha(1 - b_{\nu_2}^2)}} < \frac{-b_{\nu^*} + \sqrt{f'_\alpha}b_{\nu_2}}{\sqrt{1 - b_{\nu^*}^2 + f'_\alpha(1 - b_{\nu_2}^2)}}\Big\}.
\end{aligned}
$$

Letting $S_{\nu^*} := \sqrt{S/\nu^*}$, $S'_{\nu_2} := \sqrt{X_2/\nu_2}$ and

$$
W = \frac{S_{\nu^*} - b_{\nu^*} - \sqrt{f'_\alpha}(S'_{\nu_2} - b_{\nu_2})}{\sqrt{1 - b_{\nu^*}^2 + f'_\alpha(1 - b_{\nu_2}^2)}}.
$$

we easily see that $E[W] = 0$ and $V(W) = 1$. We assume that $O(\nu_1/\nu_2) = 1$ for sufficiently large ν_1 and ν_2. Now, we calculate the third and fourth cumulants of W and derive a new approximation using the Cornish-Fisher expansion in a similar way to Theorem 1.1.

Lemma 3.1. *The third and fourth cumulants of W are given by*

$$
\begin{aligned}
\kappa_3(W) =& \frac{1}{4\{1 - b_{\nu^*}^2 + f'_\alpha(1 - b_{\nu_2}^2)\}^{3/2}} \\
&\cdot \Big\{\frac{1}{\nu^{*2}} + \frac{1}{4\nu^{*3}} - f_\alpha'^{3/2}\Big(\frac{1}{\nu_2{}^2} + \frac{1}{4\nu_2{}^3}\Big)\Big\} + O\Big(\frac{1}{\nu_2{}^{5/2}}\Big)
\end{aligned}
$$

and

$$
\kappa_4(W) = O\Big(\frac{1}{\nu_2{}^2}\Big),
$$

respectively.

Proof. Since $S_{\nu^*} - b_{\nu^*}$ and $S'_{\nu_2} - b_{\nu_2}$ are independent, we have from Lemmas 1.1 and 1.2

$$
\begin{aligned}
\kappa_3(W) &= \frac{1}{\{1 - b_{\nu^*}^2 + f'_\alpha(1 - b_{\nu_2}^2)\}^{3/2}}\kappa_3((S_{\nu^*} - b_{\nu^*}) - \sqrt{f'_\alpha}(S'_{\nu_2} - b_{\nu_2})) \\
&= \frac{1}{\{1 - b_{\nu^*}^2 + f'_\alpha(1 - b_{\nu_2}^2)\}^{3/2}}\Big\{\kappa_3(S_{\nu^*} - b_{\nu^*}) - f_\alpha'^{3/2}\kappa_3(S'_{\nu_2} - b_{\nu_2})\Big\} \\
&= \frac{1}{4\{1 - b_{\nu^*}^2 + f'_\alpha(1 - b_{\nu_2}^2)\}^{3/2}} \\
&\quad \cdot \Big\{\Big(\frac{1}{\nu^{*2}} + \frac{1}{4\nu^{*3}}\Big) - f_\alpha'^{3/2}\Big(\frac{1}{\nu_2{}^2} + \frac{1}{4\nu_2{}^3}\Big)\Big\} + O\Big(\frac{1}{\nu_2{}^{5/2}}\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
\kappa_4(W) &= \frac{1}{\{1 - b_{\nu^*}^2 + f'_\alpha(1 - b_{\nu_2}^2)\}^2}\kappa_4((S_{\nu^*} - b_{\nu^*}) - \sqrt{f'_\alpha}(S'_{\nu_2} - b_{\nu_2})) \\
&= O\Big(\frac{1}{\nu_2^2}\Big). \quad \square
\end{aligned}
$$

Theorem 3.1. *The upper* 100α *percentile* f_α *of the non-central* F *distribution with* (ν_1, ν_2) *degrees of freedom and a non-centrality parameter* λ *can be derived from the formula*

$$
\begin{aligned}
(3.5)\qquad & -\frac{b_{\nu^*} - \sqrt{f'_\alpha}b_{\nu_2}}{\sqrt{1 - b_{\nu^*}^2 + f'_\alpha(1 - b_{\nu_2}^2)}} \\
&= u_\alpha + \frac{u_\alpha^2 - 1}{24\{1 - b_{\nu^*}^2 + f'_\alpha(1 - b_{\nu_2}^2)\}^{3/2}} \cdot \Big\{\frac{1}{\nu^{*2}} + \frac{1}{4\nu^{*3}} - f_\alpha'^{3/2}\Big(\frac{1}{\nu_2{}^2} + \frac{1}{4\nu_2{}^3}\Big)\Big\} \\
&\quad - \frac{2u_\alpha^3 - 5u_\alpha}{576\{1 - b_{\nu^*}^2 + f'_\alpha(1 - b_{\nu_2}^2)\}^3}\Big(\frac{1}{\nu^{*2}} - \frac{f_\alpha'^{3/2}}{\nu_2{}^2}\Big)^2 + O\Big(\frac{1}{\nu_2{}^2}\Big),
\end{aligned}
$$

where $f'_\alpha = (f_\alpha - \rho)/\gamma$.

Proof. From (3.4) and Lemma 3.1, we have by the Cornish-Fisher expansion

$$
\begin{aligned}
& -\frac{b_{\nu^*} - \sqrt{f'_\alpha}b_{\nu_2}}{\sqrt{1 - b_{\nu^*}^2 + f'_\alpha(1 - b_{\nu_2}^2)}} \\
&= u_\alpha + \frac{1}{6}\kappa_3(W)(u_\alpha^2 - 1) + \frac{1}{24}\kappa_4(W)(u_\alpha^3 - 3u_\alpha) - \frac{1}{36}\kappa_3^2(W)(2u_\alpha^3 - 5u_\alpha) + O\Big(\frac{1}{\nu_2^2}\Big) \\
&= u_\alpha + \frac{u_\alpha^2 - 1}{24\{1 - b_{\nu^*}^2 + f'_\alpha(1 - b_{\nu_2}^2)\}^{3/2}}\Big\{\frac{1}{\nu^{*2}} + \frac{1}{4\nu^{*3}} - f_\alpha'^{3/2}\Big(\frac{1}{\nu_2{}^2} + \frac{1}{4\nu_2{}^3}\Big)\Big\} \\
&\quad - \frac{2u_\alpha^3 - 5u_\alpha}{576\{1 - b_{\nu^*}^2 + f'_\alpha(1 - b_{\nu_2}^2)\}^3}\Big(\frac{1}{\nu^{*2}} - \frac{f_\alpha'^{3/2}}{\nu_2{}^2}\Big)^2 + O\Big(\frac{1}{\nu_2{}^2}\Big). \quad \square
\end{aligned}
$$

3.3. Evaluation of new approximation formulae in comparison with others for the non-central F distribution

First, we compare (3.5) with others for the upper 100α percentile f_α of the non-central F distribution $F(\nu_1, \nu_2, \lambda)$. From (3.2), we have

$$
(3.6)\qquad f_\alpha \fallingdotseq \Big(1 + \frac{\lambda}{\nu_1}\Big)\Big\{\frac{(1-a)(1-d) + u_\alpha\sqrt{(1-a)^2 d + (1-d)^2 a - adu_\alpha^2}}{(1-d)^2 - du_\alpha^2}\Big\}^3.
$$

From (3.3) and the Paulson approximation formula, we obtain

$$f_\alpha \fallingdotseq \gamma \left\{ \frac{(1-a')(1-d)+u_\alpha\sqrt{(1-a')^2 d+(1-d)^2 a' - a' d u_\alpha^2}}{(1-d)^2 - d u_\alpha^2} \right\}^3 + \rho, \tag{3.7}$$

where $a' = 2/(9\nu^*)$.

In order to compare a new approximation formula (3.5) with (3.6) and (3.7), we give the errors of the formulae as Table 3.1 when $\alpha = 0.05, \nu_1$ and ν_2 are 3, 5, 10, 20, 30, 60 and $\sqrt{\lambda/\nu_1}$ is $1, \sqrt{2}, 2$, where the true values are searchingly obtained using *Mathematica*. As is seen in Table 3.1, the approximation formula (3.5) is better than the others. It gives much better values on the whole especially when ν_2 is small for each value of ν_1. Note that the formula (3.5) is given as an implicit form of f_α, but the approximate values of f_α can be easily computed using the Newton method.

Next, we compare (3.9) with others for the upper 100α percentile f_α of the doubly non-central F distribution $F(\nu_1, \nu_2, \lambda_, \lambda_2)$. In order to compare a new approximation formula (3.9) with (3.8), we have their numerical calculation when ν_1 is 4, 9, 16, 36, ν_2 is 15, 30 ,60, λ_1 and λ_2 are 0.5, 1.5, 3 (ν_1=2), 0.5, 2, 5 (ν_1=4), 0.5, 4, 9 (ν_1=8), respectively . In fact, f_α in Table 3.2 is a 100α percentile of the central F distribution $F(\nu_1, \nu_2)$. (A similar table is given by Bulgren [Bu71].) The errors of the approximation formulae are given as Table 3.2, where the true values are referred from Bulgren [Bu71] and the values of (3.8) and (3.9) are calculated by Newton's method in *Mathematica* for Macintosh. As is seen in Table 3.2, the new approximation one (3.9) also behaves comparatively better than the former (3.8)

Table 3.1. Errors of the approximation formulae of the upper 5 percentile of $F(\nu_1, \nu_2; \lambda)$

$\sqrt{\lambda/\nu_1}$	ν_1	ν_2	true value	(3.5)	(3.6)	(3.7)
1	3	3	18.2747	0.351	1.136	1.233
		5	10.4166	0.046	0.091	0.134
		10	6.9705	-0.007	-0.015	-0.006
		20	5.7339	-0.015	-0.014	-0.022
		30	5.3761	-0.016	-0.011	-0.025
		60	5.0420	-0.016	-0.007	-0.027
1	5	3	17.8235	0.313	1.187	1.234
		5	9.8105	0.042	0.118	0.138
		10	6.3248	0.000	0.003	0.006
		20	5.0761	-0.006	0.000	-0.004
		30	4.7142	-0.007	0.002	-0.009
		60	4.3754	-0.007	0.006	-0.010
1	10	3	17.4527	0.283	1.215	1.234
		5	9.3008	0.036	0.130	0.138
		10	5.7593	0.003	0.010	0.011
		20	4.4788	-0.001	0.003	-0.000
		30	4.1030	-0.002	0.004	-0.002
		60	3.7476	-0.002	0.006	-0.003
1	20	3	17.2565	0.266	1.224	1.233
		5	9.0240	0.032	0.134	0.136
		10	5.4371	0.003	0.011	0.012
		20	4.1206	0.000	0.003	0.001
		30	3.7264	-0.000	0.003	-0.000
		60	3.3461	-0.000	0.004	-0.001
1	30	3	17.1894	0.261	1.227	1.232
		5	8.9281	0.030	0.134	0.136
		10	5.3217	0.003	0.011	0.012
		20	3.9869	0.000	0.002	0.001
		30	3.5822	0.000	0.002	0.000
		60	3.1861	-0.000	0.002	-0.000
1	60	3	17.1216	0.255	1.229	1.231
		5	8.8301	0.029	0.134	0.135
		10	5.2015	0.003	0.011	0.012
		20	3.8431	0.000	0.002	0.001
		30	3.4237	0.000	0.001	0.000
		60	3.0028	-0.000	0.001	-0.000

Table 3.1 (Continued).

$\sqrt{\lambda/\nu_1}$	ν_1	ν_2	true value	(3.5)	(3.6)	(3.7)
$\sqrt{2}$	3	3	27.0002	0.480	1.737	1.880
		5	15.0610	0.065	0.152	0.212
		10	9.8467	-0.003	-0.004	0.006
		20	7.9754	-0.014	0.001	-0.017
		30	7.4329	-0.015	0.008	-0.020
		60	6.9253	-0.016	0.018	-0.023
$\sqrt{2}$	5	3	26.4595	0.439	1.797	1.869
		5	14.3345	0.058	0.183	0.211
		10	9.0639	0.004	0.012	0.015
		20	7.1679	-0.004	0.009	-0.004
		30	6.6152	-0.006	0.013	-0.007
		60	6.0957	-0.006	0.019	-0.009
$\sqrt{2}$	10	3	26.0297	0.407	1.828	1.859
		5	13.7404	0.050	0.197	0.208
		10	8.3943	0.005	0.017	0.018
		20	6.4479	-0.000	0.007	0.001
		30	5.8716	-0.001	0.008	-0.001
		60	5.3217	-0.002	0.012	-0.002
$\sqrt{2}$	20	3	25.8071	0.390	1.839	1.853
		5	13.4247	0.046	0.200	0.205
		10	8.0216	0.005	0.017	0.018
		20	6.0253	0.001	0.004	0.002
		30	5.4219	0.000	0.004	0.000
		60	4.8330	-0.000	0.006	-0.000
$\sqrt{2}$	30	3	25.7317	0.384	1.842	1.850
		5	13.3165	0.044	0.201	0.203
		10	7.8900	0.004	0.017	0.018
		20	5.8703	0.001	0.003	0.002
		30	5.2525	0.000	0.002	0.001
		60	4.6409	-0.000	0.004	-0.000
$\sqrt{2}$	60	3	25.6558	0.379	1.844	1.848
		5	13.2066	0.042	0.201	0.202
		10	7.7542	0.004	0.017	0.017
		20	5.7057	0.001	0.002	0.002
		30	5.0691	0.000	0.001	0.001
		60	4.4242	0.000	0.001	0.000

Table 3.1 (Continued).

$\sqrt{\lambda}/\nu_1$	ν_1	ν_2	true value	(3.5)	(3.6)	(3.7)
2	3	3	44.2219	0.725	2.972	3.146
		5	24.0496	0.096	0.293	0.359
		10	15.2720	0.006	0.017	0.025
		20	12.1132	-0.008	0.021	-0.008
		30	11.1926	-0.011	0.032	-0.013
		60	10.3273	-0.012	0.047	-0.016
2	5	3	43.6026	0.684	3.030	3.121
		5	23.2049	0.086	0.320	0.352
		10	14.3360	0.008	0.026	0.030
		20	11.1215	-0.001	0.017	0.000
		30	10.1754	-0.003	0.023	-0.003
		60	9.2778	-0.004	0.034	-0.006
2	10	3	43.1236	0.653	3.059	3.100
		5	22.5334	0.077	0.332	0.345
		10	13.5584	0.008	0.029	0.030
		20	10.2593	0.001	0.009	0.003
		30	9.2693	-0.000	0.011	0.000
		60	8.3113	-0.001	0.017	-0.001
2	20	3	42.8794	0.637	3.069	3.088
		5	22.1841	0.072	0.335	0.340
		10	13.1375	0.007	0.029	0.030
		20	9.7684	0.001	0.005	0.003
		30	8.7370	0.000	0.005	0.001
		60	7.7144	-0.000	0.007	-0.000
2	30	3	42.7974	0.632	3.072	3.084
		5	22.0655	0.071	0.335	0.338
		10	12.9914	0.007	0.029	0.029
		20	9.5924	0.001	0.004	0.003
		30	8.5415	0.000	0.003	0.001
		60	7.4852	0.000	0.004	0.000
2	60	3	42.7150	0.626	3.074	3.080
		5	21.9458	0.069	0.335	0.337
		10	12.8421	0.006	0.028	0.029
		20	9.4086	0.001	0.004	0.003
		30	8.3339	0.000	0.002	0.001
		60	7.2326	0.000	0.001	0.000

CHAPTER 4

A New Higher Order Approximation to a Percentage Point of the Distribution of the Sample Correlation Coefficient

4.1. Introduction

In the inference on the correlation coefficient ρ of a bivariate normal distribution, percentage points of the distribution of the sample correlation coefficient play an important part. Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be independent and identically distributed random vectors according to the bivariate normal distribution. Then $R := \sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})/\sqrt{\sum_{i=1}^{n}(X_i - \bar{X})^2 \sum_{i=1}^{n}(Y_i - \bar{Y})^2}$ is called the sample correlation coefficient, where $\bar{X} = (1/n)\sum_{i=1}^{n} X_i$ and $\bar{Y} = (1/n)\sum_{i=1}^{n} Y_i$. The density of R can be obtained, but it is quite difficult to get a percentage point analytically since the density has a complicate form. Hence it is useful to consider approximation formulae for a percentage point of the distribution of R (see Johnson *et al.* [JKB95], Chapter 32). One of the well-known ways to obtain percentage points of the distribution of R is a normal approximation (see, *e.g.* Ruben [R66]). Indeed, for $0 < \alpha < 1$, we have $P\{R \leq r_\alpha\} \fallingdotseq 1 - \alpha$, where

$$u_\alpha \fallingdotseq \frac{p_\alpha\sqrt{2n-5} - q\sqrt{2n-3}}{\sqrt{p_\alpha^2 + q^2 + 2}}$$

with $p_\alpha := r_\alpha/\sqrt{1 - r_\alpha^2}$ and $q := \rho/\sqrt{1-\rho^2}$, which yields the upper 100α percentile r_α of the distribution of R with the upper 100α percentile u_α of the standard normal distribution. Another way is to use Fisher's transformation, *i.e.* $\mathbf{Z} = (1/2)\log((1+R)/(1-R))$. Indeed, since $\mathbf{Z}$ is asymptotically normally distributed with mean $\zeta = (1/2)\log((1+\rho)/(1-\rho))$ and variance $1/(n-3)$, the upper 100α percentile r_α is asymptotically given by $\zeta + (u_\alpha/\sqrt{n-3})$. The higher order asymptotic expansion for the distribution of the sample correlation coefficient is derived by Niki and Konishi [NK84] from the Fisher transformation, and it is extremely accurate and very complex as is stated in [JKB95].

In this chapter, in a similar way to Chapter 1, we derive a new approximation formula of the percentage point up to the order $o(n^{-3/2})$, using the Cornish-Fisher expansion for the statistic based on a linear combination of a normal random variable and chi-random variables ([AT96]). In numerical calculations, the higher order approximation formula is seen to be that it dominates the above normal approximation, the approximation by Fisher's **Z**-transformation, etc. and gives almost precise values in various cases of α and ρ even for $n = 10$ ([AT96]).

4.2. A new higher order approximation formula of a percentage point of the distribution of the sample correlation coefficient

In this section, first we derive a new higher order approximation formula of a percentage point of the distribution of the sample correlation coefficient using the Cornish-Fisher expansion.

Suppose that $(X_1, Y_1), \ldots, (X_n, Y_n)$ are independent and identically distributed random vectors according to a bivariate normal distribution with mean vector $(0, 0)$ and variances σ_1^2 and σ_2^2 and correlation coefficient ρ. Then it is known that the distribution of the sample correlation coefficient

$$R := \frac{\sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum_{i=1}^n (X_i - \bar{X})^2 \sum_{i=1}^n (Y_i - \bar{Y})^2}}$$

depends on only ρ, where $\bar{X} = (1/n)\sum_{i=1}^n X_i$ and $\bar{Y} = (1/n)\sum_{i=1}^n Y_i$. Letting $Y_i = \alpha X_i + U_i \quad (i = 1, \ldots, n)$ with $\alpha = \rho\sigma_2/\sigma_1$, we see that, for each $i = 1, \ldots, n$, X_i and U_i are independently and normally distributed with mean 0 and variances σ_1^2 and $\sigma_2^2(1-\rho^2)$, respectively. Putting

$$T := \frac{\sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^n (X_i - \bar{X})^2},$$

we see that the conditional distribution of T, given $X_1, \ldots, X_n$, is normal with mean α and variance $\sigma_2^2(1-\rho^2)/\sum_{i=1}^n (X_i - \bar{X})^2$. Let

$$Z := \frac{\sqrt{\sum_{i=1}^n (X_i - \bar{X})^2}}{\sigma_2\sqrt{1-\rho^2}}(T - \alpha) = \frac{\sqrt{\sum_{i=1}^n (X_i - \bar{X})^2}}{\sigma_1\sqrt{1-\rho^2}}\left(\frac{\sigma_1}{\sigma_2}T - \rho\right).$$

Then it is seen that Z is normally distributed with mean 0 and variance 1. Putting $S_1^2 := \sum_{i=1}^n (X_i - \bar{X})^2/\sigma_1^2$, we see that S_1^2 is distributed according to a chi-square distribution with $n-1$ degrees of freedom. Then we have

$$T = \frac{\sigma_2}{\sigma_1}\Big(\sqrt{1-\rho^2}\frac{Z}{S_1} + \rho\Big).$$

Putting

$$\begin{aligned} S_2^2 &:= \frac{1}{\sigma_2^2(1-\rho^2)}\Big\{\sum_{i=1}^n (Y_i - \bar{Y})^2 - T^2 \sum_{i=1}^n (X_i - \bar{X})^2\Big\} \\ &= \frac{1}{\sigma_2^2(1-\rho^2)}(1-R^2)\sum_{i=1}^n (Y_i - \bar{Y})^2, \end{aligned}$$

we see that S_2^2 is independent of $X_1, \ldots, X_n$ and Z and is distributed according to a chi-square distribution with $n-2$ degrees of freedom. We also have

$$R = T\sqrt{\frac{\sum_{i=1}^n (X_i - \bar{X})^2}{\sum_{i=1}^n (Y_i - \bar{Y})^2}} = \frac{\sigma_1 T S_1 \sqrt{1-R^2}}{\sigma_2\sqrt{1-\rho^2}S_2},$$

which yields

$$\frac{R}{\sqrt{1-R^2}} = \frac{\sigma_1 T S_1}{\sigma_2\sqrt{1-\rho^2}S_2} = \frac{Z}{S_2} + \frac{\rho}{\sqrt{1-\rho^2}}\frac{S_1}{S_2}.$$

Hence we obtain

$$\begin{aligned} P\{R \le r\} &= P\Big\{\frac{R}{\sqrt{1-R^2}} \le \frac{r}{\sqrt{1-r^2}}\Big\} \\ &= P\Big\{\frac{Z}{S_2} + \frac{\rho}{\sqrt{1-\rho^2}}\frac{S_1}{S_2} \le \frac{r}{\sqrt{1-r^2}}\Big\}. \end{aligned}$$

Putting

$$p := \frac{r}{\sqrt{1-r^2}}, \qquad q := \frac{\rho}{\sqrt{1-\rho^2}},$$

we have

$$P\{R \le r\} = P\{Z + qS_1 - pS_2 \le 0\}. \tag{4.1}$$

The above derivation is stated in Takeuchi [Ta75] to obtain the exact density function of R (see also Johnson *et al.* [JKB95]). It is also noted that Z, S_1^2 and S_2^2 are independent.

In order to get a higher order approximation formula of a percentage point of the distribution of R, we first have from (1.1) and (1.2)

$$(4.2) \qquad E(S_1) = \sqrt{n-1}b_{n-1}, \quad E(S_2) = \sqrt{n-2}b_{n-2},$$

$$V(S_1) = (n-1)(1-b_{n-1}^2), \quad V(S_2) = (n-2)(1-b_{n-2}^2),$$

and $V(S)$ denotes the variance of S. Since

$$(4.3) \qquad \begin{aligned} E(Z + qS_1 - pS_2) &= q\sqrt{n-1}b_{n-1} - p\sqrt{n-2}b_{n-2}, \\ V(Z + qS_1 - pS_2) &= V(Z) + q^2 V(S_1) + p^2 V(S_2) \\ &= 1 + q^2(n-1)(1-b_{n-1}^2) + p^2(n-2)(1-b_{n-2}^2), \end{aligned}$$

it follows that $Z + qS_1 - pS_2$ is normalized as

$$(4.4) \qquad W := \frac{Z + q(S_1 - \sqrt{n-1}b_{n-1}) - p(S_2 - \sqrt{n-2}b_{n-2})}{\{1 + q^2(n-1)(1-b_{n-1}^2) + p^2(n-2)(1-b_{n-2}^2)\}^{1/2}},$$

which implies

$$E(W) = 0, \qquad V(W) = 1.$$

Note that the statistic W is based on a linear combination of a normal random variable and chi-random variables.

For any α with $0 < \alpha < 1$, there exists a r_α such that $P\{R \le r_\alpha\} = 1 - \alpha$. The r_α is called the upper 100α percentile of the distribution of the sample correlation coefficient R. Then we have from (4.1) and (4.4)

$$(4.5) \qquad \begin{aligned} 1 - \alpha &= P\{R \le r_\alpha\} = P\{Z + qS_1 - p_\alpha S_2 \le 0\} \\ &= P\Big\{W_\alpha \le \frac{-q\sqrt{n-1}b_{n-1} + p_\alpha\sqrt{n-2}b_{n-2}}{\{1 + q^2(n-1)(1-b_{n-1}^2) + p_\alpha^2(n-2)(1-b_{n-2}^2)\}^{1/2}}\Big\}, \end{aligned}$$

where $p_\alpha = r_\alpha/\sqrt{1-r_\alpha^2}$ and W_α denotes W with p_α instead of p. In a similar way to Chapter 1, we obtain an approximation formula of the percentage point up to the order $o(n^{-3/2})$, using the Cornish-Fisher expansion for the statistic W . In order to do so we need the third and fourth cumulants of W.

Lemma 4.1. *The third and fourth cumulants of $Z + qS_1 - pS_2$ are given by*

$$\kappa_3(Z + qS_1 - pS_2)$$
$$= q^3(n-1)^{3/2}b_{n-1}\Big\{2(b_{n-1}^2 - 1) + \frac{1}{n-1}\Big\} - p^3(n-2)^{3/2}b_{n-2}\Big\{2(b_{n-2}^2 - 1) + \frac{1}{n-2}\Big\}$$

and

$$\kappa_4(Z + qS_1 - pS_2)$$
$$= 2q^4[(n-1)\{-1 + 2(1 - b_{n-1}^2)\} + (n-1)^2\{2(1 - b_{n-1}^2) - 3(1 - b_{n-1}^2)^2\}]$$
$$+ 2p^4[(n-2)\{-1 + 2(1 - b_{n-2}^2)\} + (n-2)^2\{2(1 - b_{n-2}^2) - 3(1 - b_{n-2}^2)^2\}],$$

respectively, for $n \geq 3$.

Proof. Since, by Lemma 1.1,

$$E(S_1) = \sqrt{n-1}b_{n-1}, \quad E(S_2) = \sqrt{n-2}b_{n-2}, \quad E(S_1^2) = n-1, \quad E(S_2^2) = n-2,$$
$$E(S_1^3) = (n-1)^{3/2}\Big(1 + \frac{1}{n-1}\Big)b_{n-1}, \quad E(S_2^3) = (n-2)^{3/2}\Big(1 + \frac{1}{n-2}\Big)b_{n-2},$$

it follows that

$$\kappa_3(S_1) = E[(S_1 - \sqrt{n-1}b_{n-1})^3] = (n-1)^{3/2}b_{n-1}\Big\{2(b_{n-1}^2 - 1) + \frac{1}{n-1}\Big\},$$
$$\kappa_3(S_2) = E[(S_2 - \sqrt{n-2}b_{n-2})^3] = (n-2)^{3/2}b_{n-2}\Big\{2(b_{n-2}^2 - 1) + \frac{1}{n-2}\Big\}.$$

Then we have

$$\kappa_3(Z + qS_1 - pS_2)$$
$$= E[\{Z + q(S_1 - \sqrt{n-1}b_{n-1}) - p(S_2 - \sqrt{n-2}b_{n-2})\}^3]$$
$$= E(Z^3) + q^3E[(S_1 - \sqrt{n-1}b_{n-1})^3] - p^3E[(S_2 - \sqrt{n-2}b_{n-2})^3]$$
$$= q^3\kappa_3(S_1) - p^3\kappa_3(S_2)$$
$$= q^3(n-1)^{3/2}b_{n-1}\Big\{2(b_{n-1}^2 - 1) + \frac{1}{n-1}\Big\} - p^3(n-2)^{3/2}b_{n-2}\Big\{2(b_{n-2}^2 - 1) + \frac{1}{n-2}\Big\}.$$

Since, by Lemma 1.2,

$$E[(S_1-\sqrt{n-1}b_{n-1})^4]=2(n-1)(1-2b_{n-1}^2)+(n-1)^2(1-b_{n-1}^2)(1+3b_{n-1}^2),$$

$$E[(S_2-\sqrt{n-2}b_{n-2})^4]=2(n-2)(1-2b_{n-2}^2)+(n-2)^2(1-b_{n-2}^2)(1+3b_{n-2}^2),$$

it follows from (4.2) that

$$\begin{aligned}
(4.6)\qquad & E[\{Z+q(S_1-\sqrt{n-1}b_{n-1})-p(S_2-\sqrt{n-2}b_{n-2})\}^4] \\
&= E(Z^4)+q^4E[(S_1-\sqrt{n-1}b_{n-1})^4]+p^4E[(S_2-\sqrt{n-2}b_{n-2})^4] \\
&\quad +6\{q^2E(Z^2)V(S_1)+p^2q^2V(S_1)V(S_2)+p^2E(Z^2)V(S_2)\} \\
&= 3+q^4\{2(n-1)(1-2b_{n-1}^2)+(n-1)^2(1-b_{n-1}^2)(1+3b_{n-1}^2)\} \\
&\quad +p^4\{2(n-2)(1-2b_{n-2}^2)+(n-2)^2(1-b_{n-2}^2)(1+3b_{n-2}^2)\} \\
&\quad +6\{q^2(n-1)(1-b_{n-1}^2)+p^2q^2(n-1)(n-2)(1-b_{n-1}^2)(1-b_{n-2}^2) \\
&\quad +p^2(n-2)(1-b_{n-2}^2)\}.
\end{aligned}$$

From (4.3) and (4.6) we have

$$\begin{aligned}
&\kappa_4(Z+qS_1-pS_2) \\
&= E[\{Z+q(S_1-\sqrt{n-1}b_{n-1})-p(S_2-\sqrt{n-2}b_{n-2})\}^4] \\
&\quad -3\{V(Z+q(S_1-\sqrt{n-1}b_{n-1})-p(S_2-\sqrt{n-2}b_{n-2}))\}^2 \\
&= q^4[2(n-1)(1-2b_{n-1}^2)+(n-1)^2(1-b_{n-1}^2)\{4-3(1-b_{n-1}^2)\}-3(n-1)^2(1-b_{n-1}^2)^2] \\
&\quad +p^4[2(n-2)(1-2b_{n-2}^2)+(n-2)^2(1-b_{n-2}^2)\{4-3(1-b_{n-2}^2)\}-3(n-2)^2(1-b_{n-2}^2)^2].
\end{aligned}$$

□

Lemma 4.2. *For a sufficiently large n*

$$\begin{aligned}
V(Z+qS_1-pS_2) &= 1+(p^2+q^2)\Big\{\frac{1}{2}-\frac{1}{8n}-\frac{1}{16n^2}+O\Big(\frac{1}{n^3}\Big)\Big\}, \\
\kappa_3(Z+qS_1-pS_2) &= \frac{q^3-p^3}{4\sqrt{n}}\Big(1+\frac{1}{4n}\Big)+O\Big(\frac{1}{n^{5/2}}\Big), \\
\kappa_4(Z+qS_1-pS_2) &= O\Big(\frac{1}{n^2}\Big).
\end{aligned}$$

Proof. From (1.8), (1.9) and (4.3) we have for a sufficiently large n

$$\begin{aligned}
&V(Z+qS_1-pS_2)\\
&=1+q^2(n-1)(1-b_{n-1}^2)+p^2(n-2)(1-b_{n-2}^2)\\
&=1+q^2(n-1)\Big\{\frac{1}{2(n-1)}-\frac{1}{8(n-1)^2}-\frac{1}{16(n-1)^3}+O\Big(\frac{1}{(n-1)^4}\Big)\Big\}\\
&\qquad+p^2(n-2)\Big\{\frac{1}{2(n-2)}-\frac{1}{8(n-2)^2}-\frac{1}{16(n-2)^3}+O\Big(\frac{1}{(n-2)^4}\Big)\Big\}\\
&=1+(p^2+q^2)\Big\{\frac{1}{2}-\frac{1}{8n}-\frac{1}{16n^2}+O\Big(\frac{1}{n^3}\Big)\Big\}.
\end{aligned}$$

Since, by (1.8) and (1.9),

$$\nu^{3/2}b_\nu\Big\{2(b_\nu^2-1)+\frac{1}{\nu}\Big\}=\frac{1}{4\sqrt{\nu}}\Big(1+\frac{1}{4\nu}\Big)+O\Big(\frac{1}{\nu^{5/2}}\Big),$$

it follows from Lemma 4.1 that

$$\begin{aligned}
&\kappa_3(Z+qS_1-pS_2)\\
&=\frac{q^3}{4\sqrt{n-1}}\Big\{1+\frac{1}{4(n-1)}\Big\}-\frac{p^3}{4\sqrt{n-2}}\Big\{1+\frac{1}{4(n-2)}\Big\}+O\Big(\frac{1}{n^{5/2}}\Big)\\
&=\frac{q^3-p^3}{4\sqrt{n}}\Big(1+\frac{1}{4n}\Big)+O\Big(\frac{1}{n^{5/2}}\Big).
\end{aligned}$$

From Lemma 4.1 and (1.9) we have

$$\begin{aligned}
&\kappa_4(Z+qS_1-pS_2)\\
&=q^4\Big[2(n-1)\Big\{-1+\frac{1}{n-1}-\frac{1}{4(n-1)^2}-\frac{1}{8(n-1)^3}+O\Big(\frac{1}{n^4}\Big)\Big\}\\
&\qquad+(n-1)^2\Big\{4\Big(\frac{1}{2(n-1)}-\frac{1}{8(n-1)^2}-\frac{1}{16(n-1)^3}+O\Big(\frac{1}{n^4}\Big)\Big)\\
&\qquad-6\Big(\frac{1}{4(n-1)^2}-\frac{1}{8(n-1)^3}+O\Big(\frac{1}{n^4}\Big)\Big)\Big\}\Big]\\
&\quad+p^4\Big[2(n-2)\Big\{-1+\frac{1}{n-2}-\frac{1}{4(n-2)^2}-\frac{1}{8(n-2)^3}+O\Big(\frac{1}{n^4}\Big)\Big\}\\
&\qquad+(n-2)^2\Big\{4\Big(\frac{1}{2(n-2)}-\frac{1}{8(n-2)^2}-\frac{1}{16(n-2)^3}+O\Big(\frac{1}{n^4}\Big)\Big)\\
&\qquad-6\Big(\frac{1}{4(n-2)^2}-\frac{1}{8(n-2)^3}+O\Big(\frac{1}{n^4}\Big)\Big)\Big\}\Big]\\
&=O\Big(\frac{1}{n^2}\Big). \qquad \square
\end{aligned}$$

Lemma 4.3. *The third and fourth cumulants of W are given by*

$$\kappa_3(W) = \frac{q^3(n-1)^{3/2}b_{n-1}\Big\{2(b_{n-1}^2-1)+\frac{1}{n-1}\Big\} - p^3(n-2)^{3/2}b_{n-2}\Big\{2(b_{n-2}^2-1)+\frac{1}{n-2}\Big\}}{\{1+q^2(n-1)(1-b_{n-1}^2)+p^2(n-2)(1-b_{n-2}^2)\}^{3/2}}$$

for $n \geq 3$ and

$$\kappa_4(W) = O\Big(\frac{1}{n^2}\Big)$$

for a sufficiently large n.

The proof is straightforward from (4.4) and Lemmas 4.1 and 4.2. Using the Cornish-Fisher expansion, we can obtain a higher order approximation formula of a percentage point of the distribution of the sample correlation coefficient R. We denote W with p_α instead of p by W_α.

Theorem 4.1. *The upper 100α percentile r_α of the distribution of R can be derived from the formula*

(4.7)

$$\frac{-q\sqrt{n-1}b_{n-1}+p_\alpha\sqrt{n-2}b_{n-2}}{\{1+q^2(n-1)(1-b_{n-1}^2)+p_\alpha^2(n-2)(1-b_{n-2}^2)\}^{1/2}} = u_\alpha + \frac{1}{6}(u_\alpha^2-1)\kappa_3(W_\alpha) + O\Big(\frac{1}{n^2}\Big)$$

up to the order $o(n^{-3/2})$, where $p_\alpha = r_\alpha/\sqrt{1-r_\alpha^2}$ and u_α is the upper 100α percentile of the standard normal distribution and $\kappa_3(W_\alpha)$ is given in Lemma 4.3.

Proof. From (4.5) and Lemma 4.3 we have by the Cornish-Fisher expansion

$$\begin{aligned}
&\frac{-q\sqrt{n-1}b_{n-1}+p_\alpha\sqrt{n-2}b_{n-2}}{\{1+q^2(n-1)(1-b_{n-1}^2)+p_\alpha^2(n-2)(1-b_{n-2}^2)\}^{1/2}} \\
&= u_\alpha + \frac{1}{6}\kappa_3(W_\alpha)(u_\alpha^2-1) + \frac{1}{24}\kappa_4(W_\alpha)(u_\alpha^3-3u_\alpha) + O\Big(\frac{1}{n^2}\Big) \\
&= u_\alpha + \frac{1}{6}(u_\alpha^2-1)\kappa_3(W_\alpha) + O\Big(\frac{1}{n^2}\Big),
\end{aligned}$$

where W_α denotes W with p_α instead of p and $\kappa_3(W)$ is given in Lemma 4.3. □

If we ignore the second term of the right-hand side of (4.7), *i.e.*

$$\text{(4.8)} \qquad \frac{-q\sqrt{n-1}b_{n-1} + p_\alpha\sqrt{n-2}b_{n-2}}{\{1+q^2(n-1)(1-b_{n-1}^2) + p_\alpha^2(n-2)(1-b_{n-2}^2)\}^{1/2}} = u_\alpha + o(1),$$

which is called the first order approximation. Since, by (1.8), $\sqrt{\nu}b_\nu \fallingdotseq \sqrt{\nu-(1/2)}$, it follows that

$$\text{(4.9)} \qquad -q\sqrt{n-1}b_{n-1} + p_\alpha\sqrt{n-2}b_{n-2} \fallingdotseq p_\alpha\sqrt{n-\frac{5}{2}} - q\sqrt{n-\frac{3}{2}}.$$

It also follows from (4.3) and Lemma 4.2 that

$$\text{(4.10)} \qquad 1+q^2(n-1)(1-b_{n-1}^2) + p_\alpha^2(n-2)(1-b_{n-2}^2) = 1 + \frac{p_\alpha^2+q^2}{2} + O\Big(\frac{1}{n}\Big).$$

From (4.8), (4.9) and (4.10) we have

$$\frac{p_\alpha\sqrt{2n-5} - q\sqrt{2n-3}}{\sqrt{p_\alpha^2+q^2+2}} \fallingdotseq u_\alpha,$$

which yields the well-known normal approximation formula

$$\text{(4.11)} \qquad p_\alpha \fallingdotseq \frac{q\sqrt{(2n-3)(2n-5)} + u_\alpha\sqrt{(4n-8)q^2+2(2n-5)-u_\alpha^2q^2-2u_\alpha^2}}{2n-5-u_\alpha^2}$$

(see, e.g. Yamauti *et al.* [Y72]).

Next we consider an approximation to the percentage point by Fisher's **Z**-transformation. Let

$$\mathbf{Z} = \frac{1}{2}\log\Big(\frac{1+R}{1-R}\Big).$$

Then it is known that **Z** is asymptotically normally distributed with mean $(1/2)\log((1+\rho)/(1-\rho))$ and variance $1/(n-3)$ for a sufficiently large n. Hence we have for $0<\alpha<1$

$$1-\alpha = P\{R \le r_\alpha\} = P\{\mathbf{Z} \le z_\alpha\} \fallingdotseq \Phi(\sqrt{n-3}(z_\alpha-\zeta)),$$

where

$$z_\alpha = \frac{1}{2}\log\frac{1+r_\alpha}{1-r_\alpha}, \qquad \zeta = \frac{1}{2}\log\frac{1+\rho}{1-\rho}.$$

Then the upper 100α percentile r_α of the distribution of R is given by

$$\text{(4.12)} \qquad r_\alpha \fallingdotseq \frac{e^{2z_\alpha}-1}{e^{2z_\alpha}+1}.$$

with $z_\alpha = \zeta + (u_\alpha/\sqrt{n-3})$. See Johnson *et al.* [JKB95] for other approximations to the distribution of the sample correlation coefficient.

4.3. Evaluation of the new approximation formula in comparison with others for the sample correlation coefficient

In this section we numerically compare the higher order approximation formula (4.7) with the first order approximation (4.8), the normal approximation (4.11) and the approximation (4.12) by Fisher's **Z**-transformation when α = 0.990, 0.975, 0.950, 0.900, 0.750, 0.500, 0.250, 0.100, 0.050, 0.025, 0.010, ρ = 0.000, 0.100, 0.200, 0.300, 0.400, 0.500, 0.600, 0.700, 0.800, 0.900, 0.950, n =10, 20, 30. The errors of the approximation formulae are given as Tables 4.4 to 4.15, where the true values of percentage points of the distribution of the sample correlation coefficient are referred from Odeh [Od82] and made as Tables 4.1 to 4.3. The values of (4.7) and (4.8) are calculated by Newton's method in *Mathematica* for Macintosh. As is seen in Tables 4.4 to 4.15, the approximation formula (4.7) dominates the others and gives almost precise values in various cases of α and ρ even for n = 10. Further, for the above values of α, ρ in the case of n = 20 and 30, the errors of the formula (4.7) are ± 0.000 (see Tables 4.5 and 4.6). Hence the formula (4.7) can be recommended as a good one to derive percentage points of the distribution of the sample correlation coefficient.

Table 4.1. True values of percentage points of the distribution of sample correlation coefficient referred from Odeh (1982) when $n = 10$

$\rho\backslash\alpha$	.990	.975	.950	.900	.750	.250	.100	.050	.025	.010
.000	-.71546	-.63190	-.54936	-.44280	-.24230	.24230	.44280	.54936	.63190	.71546
.100	-.65913	-.56326	-.47035	-.35294	-.13946	.33984	.52436	.61942	.69167	.76362
.200	-.59262	-.48394	-.38091	-.25385	-.03131	.43214	.59848	.68177	.74403	.80514
.300	-.51325	-.39164	-.27930	-.14452	.08211	.51930	.66592	.73746	.79014	.84122
.400	-.41737	-.28345	-.16343	-.02386	.20064	.60146	.72735	.78735	.83096	.87278
.500	-.29999	-.15565	-.03083	.10927	.32408	.67880	.78340	.83221	.86726	.90057
.600	-.15407	-.00345	.12142	.25606	.45211	.75150	.83462	.87265	.89969	.92516
.700	.03046	.17930	.29667	.41763	.58430	.81976	.88147	.90922	.92877	.94705
.800	.26826	.40049	.49875	.59501	.72014	.88379	.92441	.94238	.95494	.96661
.900	.58091	.67004	.73180	.78900	.85895	.94380	.96380	.97253	.97858	.98417
.950	.77469	.82664	.86126	.89237	.92925	.97236	.98228	.98658	.98955	.99229

Table 4.2. True values of percentage points of the distribution of sample correlation coefficient referred from Odeh (1982) when $n = 20$

$\rho\backslash\alpha$	.990	.975	.950	.900	.750	.250	.100	.050	.025	.010
.000	-.51550	-.44376	-.37834	-.29921	-.16015	.16015	.29921	.37834	.44376	.51550
.100	-.43575	-.35722	-.28665	-.20262	-.05831	.25866	.38997	.46308	.52266	.58713
.200	-.34656	-.26203	-.18730	-.09978	.04687	.35383	.47527	.54151	.59478	.65173
.300	-.24634	-.15703	-.07948	.00971	.15535	.44569	.55547	.61420	.66087	.71019
.400	-.13317	-.04086	.03770	.12629	.26709	.53427	.63092	.68169	.72158	.76331
.500	-.00467	.08804	.16522	.25042	.38201	.61962	.70192	.74442	.77747	.81172
.600	.14210	.23152	.30417	.38256	.50001	.70179	.76878	.80282	.82904	.85597
.700	.31077	.39174	.45577	.52315	.62097	.78083	.83177	.85726	.87671	.89655
.800	.50591	.57121	.62131	.67264	.74475	.85684	.89114	.90807	.92088	.93385
.900	.73324	.77284	.80222	.83146	.87116	.92987	.94715	.95556	.96188	.96824
.950	.86117	.88300	.89890	.91450	.93529	.96529	.97395	.97815	.98128	.98443

Table 4.3. True values of percentage points of the distribution of sample correlation coefficient referred from Odeh (1982) when $n = 30$

$\rho\backslash\alpha$	.990	.975	.950	.900	.750	.250	.100	.050	.025	.010
.000	-.42257	-.36101	-.30606	-.24075	-.12808	.12808	.24075	.30606	.36101	.42257
.100	-.33517	-.26908	-.21081	-.14244	-.02663	.22689	.33432	.39550	.44636	.50270
.200	-.23958	-.16991	-.10930	-.03912	.07744	.32307	.42339	.47956	.52575	.57637
.300	-.13470	-.06273	-.00100	.06944	.18412	.41663	.50820	.55866	.59971	.64426
.400	-.01929	.05333	.11463	.18355	.29337	.50758	.58898	.63314	.66872	.70698
.500	.10816	.17922	.23821	.30346	.40517	.59593	.66592	.70334	.73321	.76506
.600	.24939	.31605	.37039	.42946	.51945	.68173	.73924	.76956	.79356	.81894
.700	.40647	.46506	.51187	.56183	.63616	.76500	.80914	.83210	.85012	.86905
.800	.58188	.62763	.66341	.70086	.75522	.84577	.87578	.89119	.90320	.91572
.900	.77855	.80536	.82584	.84683	.87653	.92409	.93935	.94709	.95307	.95928
.950	.88593	.90044	.91139	.92250	.93800	.96234	.97003	.97390	.97689	.97998

Table 4.4. The errors of the new higher order approximation formula (4.7) for n=10

$\rho\backslash\alpha$	.990	.975	.950	.900	.750	.250	.100	.050	.025	.010
.000	-.0002	-.0000	.0000	.0000	-.0000	.0000	.0000	-.0000	.0000	.0002
.100	-.0001	.0000	-.0000	-.0000	.0000	.0000	-.0000	.0000	.0001	.0002
.200	-.0000	-.0000	-.0000	.0000	-.0000	.0000	-.0000	.0000	.0001	.0004
.300	-.0000	-.0000	-.0000	-.0000	.0000	-.0000	.0000	.0001	.0002	.0005
.400	.0000	.0000	-.0000	.0000	.0000	-.0000	-.0000	.0001	.0002	.0005
.500	-.0000	-.0000	-.0000	-.0000	-.0000	.0000	.0000	.0001	.0002	.0006
.600	-.0001	-.0000	-.0000	-.0000	.0000	.0000	.0000	.0001	.0002	.0006
.700	-.0002	-.0001	-.0001	.0000	-.0000	-.0000	-.0000	.0001	.0002	.0005
.800	-.0007	-.0003	-.0001	-.0000	.0000	-.0000	.0000	.0001	.0002	.0004
.900	-.0015	-.0005	-.0002	-.0000	-.0000	-.0000	.0000	.0000	.0001	.0002
.950	-.0014	-.0004	-.0001	-.0000	-.0000	-.0000	-.0000	.0000	.0001	.0001

Table 4.5. The errors of the new higher order approximation formula (4.7) for n=20

$\rho\backslash\alpha$	.990	.975	.950	.900	.750	.250	.100	.050	.025	.010
.000	-.0000	.0000	-.0000	-.0000	.0000	-.0000	.0000	.0000	-.0000	.0000
.100	.0000	-.0000	.0000	-.0000	-.0000	-.0000	-.0000	-.0000	-.0000	.0000
.200	.0000	-.0000	-.0000	.0000	-.0000	.0000	-.0000	.0000	-.0000	.0000
.300	-.0000	-.0000	.0000	.0000	-.0000	-.0000	-.0000	.0000	-.0000	.0000
.400	.0000	.0000	-.0000	-.0000	-.0000	-.0000	.0000	-.0000	-.0000	.0001
.500	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	-.0000	.0001
.600	-.0000	.0000	-.0000	-.0000	.0000	-.0000	-.0000	.0000	.0000	.0001
.700	-.0001	.0000	-.0000	.0000	-.0000	.0000	-.0000	-.0000	.0000	.0001
.800	-.0001	-.0000	-.0000	.0000	.0000	.0000	.0000	-.0000	.0000	.0001
.900	-.0002	-.0001	.0000	-.0000	-.0000	-.0000	.0000	-.0000	.0000	.0000
.950	-.0001	-.0000	-.0000	-.0000	-.0000	-.0000	-.0000	.0000	-.0000	.0000

Table 4.6. The errors of the new higher order approximation formula (4.7) for n=30

$\rho\backslash\alpha$	.990	.975	.950	.900	.750	.250	.100	.050	.025	.010
.000	.0000	-.0000	.0000	-.0000	.0000	-.0000	.0000	-.0000	.0000	-.0000
.100	.0000	.0000	-.0000	-.0000	-.0000	-.0000	.0000	-.0000	-.0000	.0000
.200	.0000	-.0000	-.0000	-.0000	.0000	-.0000	-.0000	-.0000	-.0000	-.0000
.300	-.0000	-.0000	-.0000	.0000	.0000	.0000	.0000	-.0000	.0000	-.0000
.400	.0000	.0000	.0000	.0000	-.0000	-.0000	-.0000	.0000	.0000	.0000
.500	-.0000	.0000	.0000	-.0000	-.0000	.0000	.0000	.0000	.0000	-.0000
.600	-.0000	-.0000	-.0000	-.0000	-.0000	.0000	.0000	-.0000	-.0000	.0000
.700	-.0000	-.0000	-.0000	.0000	-.0000	-.0000	.0000	.0000	.0000	.0000
.800	-.0000	.0000	.0000	-.0000	.0000	-.0000	-.0000	-.0000	.0000	.0000
.900	-.0001	-.0000	.0000	.0000	.0000	-.0000	.0000	-.0000	-.0000	-.0000
.950	-.0000	.0000	-.0000	-.0000	.0000	.0000	.0000	.0000	-.0000	-.0000

Table 4.7. The errors of the first order approximation formula (4.8) for n=10

$\rho\backslash\alpha$	.990	.975	.950	.900	.750	.250	.100	.050	.025	.010
.000	-.0090	-.0040	-.0016	-.0003	.0000	-.0000	.0003	.0016	.0040	.0090
.100	-.0073	-.0029	-.0010	-.0002	.0000	-.0001	.0005	.0021	.0050	.0103
.200	-.0055	-.0019	-.0006	-.0001	.0000	-.0002	.0007	.0027	.0057	.0111
.300	-.0039	-.0013	-.0005	-.0001	.0001	-.0003	.0008	.0030	.0062	.0113
.400	-.0032	-.0014	-.0007	-.0003	.0002	-.0004	.0010	.0033	.0063	.0110
.500	-.0042	-.0025	-.0015	-.0005	.0003	-.0005	.0010	.0032	.0060	.0101
.600	-.0078	-.0049	-.0028	-.0009	.0004	-.0005	.0010	.0030	.0054	.0088
.700	-.0148	-.0086	-.0044	-.0013	.0004	-.0005	.0009	.0026	.0045	.0070
.800	-.0244	-.0124	-.0057	-.0014	.0004	-.0004	.0007	.0019	.0032	.0049
.900	-.0297	-.0127	-.0052	-.0012	.0003	-.0003	.0004	.0010	.0017	.0026
.950	-.0232	-.0090	-.0034	-.0007	.0002	-.0001	.0002	.0005	.0009	.0013

Table 4.8. The errors of the first order approximation formula (4.8) for n=20

$\rho\backslash\alpha$	.990	.975	.950	.900	.750	.250	.100	.050	.025	.010
.000	-.0014	-.0006	-.0002	-.0000	.0000	-.0000	.0000	.0002	.0006	.0014
.100	-.0009	-.0003	-.0001	-.0000	-.0000	-.0000	.0001	.0004	.0009	.0020
.200	-.0005	-.0002	-.0001	.0000	-.0000	-.0000	.0001	.0006	.0013	.0026
.300	-.0004	-.0002	-.0001	-.0000	-.0000	-.0001	.0002	.0008	.0016	.0030
.400	-.0008	-.0005	-.0003	-.0001	.0001	-.0001	.0003	.0009	.0018	.0033
.500	-.0017	-.0011	-.0006	-.0002	.0001	-.0001	.0003	.0010	.0019	.0034
.600	-.0031	-.0018	-.0009	-.0003	.0001	-.0002	.0003	.0010	.0018	.0032
.700	-.0049	-.0026	-.0013	-.0004	.0001	-.0002	.0003	.0009	.0017	.0028
.800	-.0064	-.0032	-.0015	-.0004	.0001	-.0001	.0002	.0007	.0013	.0021
.900	-.0059	-.0027	-.0012	-.0003	.0001	-.0001	.0001	.0004	.0007	.0012
.950	-.0040	-.0018	-.0007	-.0002	.0001	-.0000	.0001	.0002	.0004	.0006

Table 4.9. The errors of the first order approximation formula (4.8) for n=30

$\rho\backslash\alpha$	.990	.975	.950	.900	.750	.250	.100	.050	.025	.010
.000	-.0005	-.0002	-.0001	-.0000	.0000	-.0000	.0000	.0001	.0002	.0005
.100	-.0003	-.0001	-.0000	-.0000	-.0000	-.0000	.0000	.0001	.0004	.0008
.200	-.0001	-.0001	-.0000	-.0000	.0000	-.0000	.0001	.0003	.0006	.0011
.300	-.0002	-.0001	-.0001	.0000	.0000	-.0000	.0001	.0003	.0007	.0014
.400	-.0005	-.0003	-.0002	-.0001	.0000	-.0001	.0001	.0004	.0009	.0017
.500	-.0010	-.0006	-.0003	-.0001	.0000	-.0001	.0002	.0005	.0010	.0018
.600	-.0018	-.0010	-.0005	-.0002	.0001	-.0001	.0002	.0005	.0010	.0018
.700	-.0025	-.0014	-.0007	-.0002	.0001	-.0001	.0001	.0005	.0009	.0016
.800	-.0030	-.0015	-.0007	-.0002	.0001	-.0001	.0001	.0004	.0007	.0012
.900	-.0026	-.0012	-.0006	-.0001	.0000	-.0000	.0001	.0002	.0004	.0007
.950	-.0017	-.0008	-.0003	-.0001	.0000	.0000	.0000	.0001	.0002	.0004

Table 4.10. The errors of the normal approximation formula (4.11) for n=10

$\rho\backslash\alpha$	.990	.975	.950	.900	.750	.250	.100	.050	.025	.010
.000	-.0127	-.0067	-.0035	-.0015	-.0003	.0003	.0015	.0035	.0067	.0127
.100	-.0109	-.0053	-.0027	-.0011	-.0003	.0004	.0019	.0043	.0078	.0141
.200	-.0088	-.0041	-.0020	-.0009	-.0003	.0004	.0023	.0050	.0087	.0149
.300	-.0070	-.0034	-.0018	-.0010	-.0003	.0005	.0026	.0055	.0092	.0149
.400	-.0063	-.0035	-.0022	-.0013	-.0004	.0005	.0028	.0057	.0091	.0143
.500	-.0075	-.0050	-.0034	-.0020	-.0005	.0005	.0029	.0055	.0086	.0130
.600	-.0120	-.0082	-.0054	-.0029	-.0006	.0005	.0027	.0050	.0077	.0112
.700	-.0205	-.0131	-.0079	-.0038	-.0007	.0004	.0023	.0042	.0063	.0089
.800	-.0322	-.0181	-.0099	-.0043	-.0007	.0003	.0017	.0031	.0045	.0062
.900	-.0385	-.0184	-.0089	-.0034	-.0005	.0002	.0010	.0017	.0024	.0032
.950	-.0302	-.0130	-.0059	-.0022	-.0003	.0001	.0005	.0009	.0012	.0016

Table 4.11. The errors of the normal approximation formula (4.11) for n=20

$\rho\backslash\alpha$	.990	.975	.950	.900	.750	.250	.100	.050	.025	.010
.000	-.0020	-.0010	-.0005	-.0002	.0000	-.0000	.0002	.0005	.0010	.0020
.100	-.0014	-.0006	-.0003	-.0001	-.0000	.0001	.0003	.0007	.0014	.0027
.200	-.0009	-.0004	-.0002	-.0001	-.0000	.0001	.0004	.0010	.0019	.0034
.300	-.0008	-.0005	-.0003	-.0002	-.0001	.0001	.0006	.0013	.0023	.0039
.400	-.0013	-.0009	-.0006	-.0004	-.0001	.0001	.0007	.0015	.0025	.0042
.500	-.0024	-.0016	-.0011	-.0006	-.0001	.0001	.0008	.0016	.0026	.0042
.600	-.0041	-.0026	-.0016	-.0008	-.0002	.0001	.0008	.0016	.0025	.0040
.700	-.0063	-.0038	-.0022	-.0010	-.0002	.0001	.0007	.0014	.0023	.0034
.800	-.0081	-.0045	-.0025	-.0011	-.0002	.0001	.0006	.0011	.0017	.0026
.900	-.0075	-.0038	-.0020	-.0008	-.0001	.0000	.0003	.0007	.0010	.0014
.950	-.0050	-.0025	-.0012	-.0005	-.0001	.0000	.0002	.0003	.0005	.0008

Table 4.12. The errors of the normal approximation formula (4.11) for n=30

$\rho\backslash\alpha$	.990	.975	.950	.900	.750	.250	.100	.050	.025	.010
.000	-.0007	-.0003	-.0002	-.0001	-.0000	.0000	.0001	.0002	.0003	.0007
.100	-.0004	-.0002	-.0001	-.0000	-.0000	.0000	.0001	.0003	.0006	.0011
.200	-.0003	-.0001	-.0001	-.0001	-.0000	.0000	.0002	.0004	.0008	.0015
.300	-.0004	-.0002	-.0002	-.0001	-.0000	.0000	.0003	.0006	.0010	.0018
.400	-.0007	-.0005	-.0003	-.0002	-.0000	.0000	.0003	.0007	.0012	.0021
.500	-.0014	-.0009	-.0006	-.0003	-.0001	.0001	.0004	.0008	.0013	.0022
.600	-.0023	-.0014	-.0009	-.0004	-.0001	.0001	.0004	.0008	.0014	.0022
.700	-.0032	-.0019	-.0011	-.0005	-.0001	.0001	.0004	.0008	.0012	.0019
.800	-.0038	-.0021	-.0012	-.0005	-.0001	.0000	.0003	.0006	.0010	.0015
.900	-.0032	-.0017	-.0009	-.0004	-.0001	.0000	.0002	.0004	.0006	.0009
.950	-.0021	-.0011	-.0006	-.0002	-.0000	.0000	.0001	.0002	.0003	.0005

Table 4.13. The errors of the approximation formula (4.12)
by Fisher's **Z**-transformation for n=10

$\rho\backslash\alpha$	.990	.975	.950	.900	.750	.250	.100	.050	.025	.010
.000	.0094	.0023	-.0029	-.0069	-.0073	.0073	.0069	.0029	-.0023	-.0094
.100	.0070	-.0020	-.0084	-.0133	-.0139	.0012	.0017	-.0013	-.0053	-.0107
.200	.0034	-.0076	-.0151	-.0206	-.0208	-.0040	-.0024	-.0043	-.0072	-.0112
.300	-.0019	-.0147	-.0231	-.0286	-.0276	-.0080	-.0053	-.0063	-.0082	-.0110
.400	-.0091	-.0235	-.0321	-.0368	-.0335	-.0109	-.0072	-.0074	-.0085	-.0103
.500	-.0185	-.0335	-.0414	-.0444	-.0379	-.0124	-.0080	-.0076	-.0081	-.0092
.600	-.0299	-.0442	-.0501	-.0503	-.0399	-.0126	-.0079	-.0072	-.0072	-.0078
.700	-.0424	-.0535	-.0559	-.0524	-.0385	-.0114	-.0070	-.0061	-.0059	-.0061
.800	-.0524	-.0572	-.0550	-.0479	-.0324	-.0089	-.0053	-.0045	-.0042	-.0042
.900	-.0489	-.0461	-.0405	-.0326	-.0201	-.0051	-.0030	-.0024	-.0022	-.0021
.950	-.0338	-.0294	-.0246	-.0189	-.0111	-.0027	-.0016	-.0013	-.0011	-.0011

Table 4.14. The errors of the approximation formula (4.12) by Fisher's **Z**-transformation for n=20

$\rho \backslash \alpha$	.990	.975	.950	.900	.750	.250	.100	.050	.025	.010
.000	.0044	.0012	-.0007	-.0020	-.0020	.0020	.0020	.0007	-.0012	-.0044
.100	.0025	-.0012	-.0034	-.0048	-.0049	-.0007	-.0005	-.0015	-.0031	-.0057
.200	.0000	-.0040	-.0064	-.0079	-.0077	-.0031	-.0026	-.0033	-.0045	-.0066
.300	-.0030	-.0073	-.0097	-.0110	-.0104	-.0050	-.0041	-.0045	-.0054	-.0069
.400	-.0065	-.0108	-.0130	-.0139	-.0127	-.0064	-.0051	-.0052	-.0058	-.0069
.500	-.0102	-.0142	-.0160	-.0164	-.0143	-.0071	-.0056	-.0054	-.0057	-.0064
.600	-.0139	-.0171	-.0182	-.0178	-.0150	-.0072	-.0055	-.0052	-.0052	-.0056
.700	-.0166	-.0187	-.0189	-.0178	-.0143	-.0066	-.0049	-.0045	-.0044	-.0046
.800	-.0172	-.0178	-.0171	-.0155	-.0118	-.0052	-.0038	-.0034	-.0032	-.0033
.900	-.0131	-.0126	-.0115	-.0100	-.0072	-.0030	-.0021	-.0019	-.0017	-.0017
.950	-.0080	-.0074	-.0067	-.0056	-.0040	-.0016	-.0011	-.0010	-.0009	-.0009

Table 4.15. The errors of the approximation formula (4.12) by Fisher's **Z**-transformation for n=30

$\rho \backslash \alpha$	.990	.975	.950	.900	.750	.250	.100	.050	.025	.010
.000	.0026	.0007	-.0003	-.0010	-.0010	.0010	.0010	.0003	-.0007	-.0026
.100	.0011	-.0009	-.0021	-.0028	-.0028	-.0007	-.0006	-.0012	-.0021	-.0036
.200	-.0006	-.0028	-.0040	-.0048	-.0046	-.0023	-.0020	-.0024	-.0031	-.0044
.300	-.0026	-.0048	-.0060	-.0066	-.0063	-.0035	-.0031	-.0033	-.0038	-.0048
.400	-.0048	-.0069	-.0079	-.0084	-.0077	-.0045	-.0038	-.0039	-.0042	-.0049
.500	-.0069	-.0088	-.0096	-.0097	-.0087	-.0050	-.0042	-.0041	-.0043	-.0047
.600	-.0088	-.0102	-.0106	-.0104	-.0090	-.0051	-.0041	-.0039	-.0040	-.0042
.700	-.0099	-.0108	-.0108	-.0102	-.0085	-.0046	-.0037	-.0034	-.0034	-.0035
.800	-.0096	-.0098	-.0095	-.0088	-.0070	-.0037	-.0029	-.0026	-.0025	-.0025
.900	-.0069	-.0067	-.0062	-.0055	-.0043	-.0021	-.0016	-.0015	-.0014	-.0014
.950	-.0041	-.0038	-.0035	-.0031	-.0023	-.0011	-.0009	-.0008	-.0007	-.0007

ACKNOWLEDGMENTS

I would like to express my sincere thanks to Professor M. Akahira for giving me ideas and for helping me prepare this thesis. I would also like to thank Professor N. Sugiura for his kind encouragement.

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