On Plurigenera of Two-Dimensional Normal Singularities

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CONTENTS

INTRODUCTION

Let (X, x) be a normal n-dimensional isolated singularity over the complex number field $\mathbb{C}$ and $f\colon (M, A) \to (X, x)$ a resolution of the singularity (X, x) with the exceptional locus $A = f^{-1}(x)$. We say a resolution f is good if A is a divisor of normal crossings. The geometric genus of the singularity (X, x) is defined by $p_g(X, x) = \dim_{\mathbb{C}}(R^{n-1}f_*\mathcal{O}_M)_x$. Watanabe [Wt1] introduced pluri-genera $\{\delta_m(X, x)\}_{m\in\mathbb{N}}$ which carry more precise information of the singularity, where $\mathbb{N}$ is the set of positive integers. The pluri-genera $\{\delta_m(X, x)\}_{m\in\mathbb{N}}$ can be computed on a good resolution, and $\delta_1(X, x) = p_g(X, x)$. Our motive problems of this paper are the following: (1) What is the information of a singularity which determine the plurigenera? (2) Can the plurigenera $\{\delta_m\}_{m\in\mathbb{N}}$ be determined by $\{\delta_m\}_{m\in N}$ for some finite subset N (determined by the singularity) of $\mathbb{N}$?

In this paper, we work only on surface singularities, so "a singularity" always means a Stein germ of a normal surface singularity over $\mathbb{C}$. A singularity (X, x) is said to be rational (resp. elliptic) if $p_g(X, x) = 0$ (resp. 1). A singularity (X, x) is said to be Gorenstein if there exists a non-vanishing 2-form which is holomorphic on $X - \{x\}$; equivalently, the canonical divisor on X is a Cartier divisor. A complete intersection singularity is a Gorenstein singularity.

Section 1 gives a brief summary of a definition and results about the plurigenera.

In Section 2, we discuss the Riemann-Roch theorem, the Zariski decomposition of $K + A$ and vanishing theorems. We get that $H^1(\mathcal{O}_M(2K + A)) = 0$, which induce most results.

In Section 3, using the result of Section 2, we prove that the plurigenera of a $\mathbb{Q}$-Gorenstein singularity are expressed by finitely many polynomials, and that the plurigenera of rational and Gorenstein singularities are determined by p_g and the weighted dual graphs.

Section 4 and 5 give a criterion, in terms of the plurigenera, for a singularity to be a log-canonical singularity. In fact, we have the following: (1) If $\delta_4(X, x) = \delta_6(X, x) = 0$, then (X, x) is a quotient singularity; (2) If $\delta_{14}(X, x) = 0$, or $0 = \delta_1(X, x) < \delta_2(X, x)$ and $\delta_{14}(X, x) = 1$, then (X, x) is a log-canonical singularity; (3) If $\delta_1(X, x) = \delta_4(X, x) = \delta_6(X, x) = 1$, then (X, x) is a simple elliptic or cusp singularity. In Section 4, we see that the second plurigenus controls the weighted dual graphs. The second plurigenus of a hypersurface singularity is studied in Section 7.

We note that the results of Section 3, 4 and 5 give partial answers to the problems above.

In Section 6, we summarize the definitions and basic facts of equisingular deformations which will be used in Section 7. We show that a minimally elliptic singularity with a star-shaped weighted dual graph is a fibre of an equisingular deformation of a quasi-homogeneous minimally elliptic singularity.

In Section 7, we consider relations among the invariants δ_2, p_g, μ (Milnor number), τ (Tjurina number) and the modality. For complete intersections with $p_g > 0$, we have the equality

$$\delta_2 = h^1(S) + \mu - \tau - p_g + 1,$$

where $h^1(S)$ is the dimension of the equisingular deformation space of the singularity in case $p_g = 1$. As a corollary, we have that for an elliptic hypersurface singularity, the second plurigenus is less than or equal to the modality, and if the weighted dual graph of the singularity is a star-shaped, then the second plurigenus is equal to the modality if $\delta_2 \leq 2$. However, there exists an elliptic hypersurface singularity with $\delta_2 = 2$ whose weighted dual graph is not a star-shaped graph. A minimally elliptic singularity with $\delta_2 = 1$ is a simple elliptic or cusp singularity, or a singularity with a star-shaped graph. We list the dual graphs of minimally elliptic singularities with $\delta_2 \leq 2$ and elliptic hypersurface singularities with star-shaped graphs.

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Notation and terminology

Let Y be a normal variety over $\mathbb{C}$, $\mathcal{M}$ a sheaf of $\mathcal{O}_Y$-modules, D a divisor on Y and F a closed subset of Y. The $\mathcal{O}_Y(D)$ denotes the reflexive sheaf of rank one (invertible sheaf if D is a Cartier divisor) corresponding to the divisor D. Then we use the following notation:

$$\mathcal{M}(D) = \mathcal{M} \bigotimes_{\mathcal{O}_Y} \mathcal{O}_Y(D),$$

$$H^i(\mathcal{M}) = H^i(Y, \mathcal{M}), \quad H^i_F(\mathcal{M}) = H^i_F(Y, \mathcal{M}),$$

$$h^i(\mathcal{M}) = \dim_{\mathbb{C}} H^i(\mathcal{M}), \quad h^i_F(\mathcal{M}) = \dim_{\mathbb{C}} H^i_F(\mathcal{M}),$$

where H^i_F denotes the local cohomology groups with supports in Y. If D is written as $D = \sum n_i D_i$, where D_i are prime divisors and $n_i \neq 0$, then we define the support of D denoted by Supp(D) as Supp$(D) = \bigcup D_i$.

The minimal resolution of a singularity is one which has no non-singular rational curve with the self-intersection number -1. There exists a unique minimal resolution. A resolution is minimal if and only if the canonical divisor on the resolution space is nef (see (2.2)).

A singularity is called a simple elliptic (resp. cusp) singularity if the exceptional set of the minimal resolution is an elliptic curve (resp. a rational curve with a node or a cycle of non-singular rational curves).

Finally, a deformation of a variety Y is a flat morphism $\pi\colon \overline{Y} \to T$ such that there exists a point of T, usually denoted o, such that $\pi^{-1}(o)$ is isomorphic to Y.

1. Preliminaries

(1.1) Let (X, x) be a singularity and $f\colon (M, A) \to (X, x)$ a resolution. We denote by K the canonical divisor on M, and set $U = X - \{x\} \cong M - A$.

We will describe the definition of the plurigenera and basic results.

Definition 1.2 (Watanabe [Wt1]). We define the pluri-genera $\{\delta_m(X,x)\}_{m \in \mathbb{N}}$ by

$$\delta_m(X, x) = \dim_{\mathbb{C}} H^0(\mathcal{O}_U(mK_X))/L^{2/m}(U),$$

where $L^{2/m}(U)$ denote the set of all $L^{2/m}$-integrable m-ple holomorphic 2-forms on U. Note that $\delta_m(X, x) < \infty$ for all $m \in \mathbb{N}$

Proposition 1.3 (Watanabe [Wt1, p. 67]). *If $f\colon (M, A) \to (X, x)$ is a good resolution, then $\delta_m(X, x)$ is expressed as*

$$\delta_m(X, x) = \dim_{\mathbb{C}} H^0(\mathcal{O}_U(mK))/H^0(\mathcal{O}_M(mK + (m-1)A)).$$

Theorem 1.4 (Watanabe [Wt1, Theorem 2.8]). *Let A' be a connected proper subvariety of A, and (X', x') be the singularity obtained by contracting A' in M. Then for all $m \in \mathbb{N}$, we have*

$$\delta_m(X, x) \geq \delta_m(X', x').$$

Theorem 1.5 (Ishii [I4]). *Let $\pi\colon \overline{X} \to (\mathbb{C}, 0)$ be a small deformation of a singularity $(X, x) = \pi^{-1}(0)$. Let $Y = \pi^{-1}(c)$, $c \in \mathbb{C}$ near 0, and $\{y_i\}$ the set of singular points of Y. Then for all $m \in \mathbb{N}$, we have*

$$\delta_m(X, x) \geq \sum \delta_m(Y, y_i).$$

(1.6) A singularity (X, x) is said to be quasi-homogeneous, q-h for short, if it has a good $\mathbb{C}^*$-action; equivalently, $X = \operatorname{Spec}(R)$, for some (positively) graded ring R.

Let (X, x) be a q-h singularity and $f\colon (M, A) \to (X, x)$ the minimal good resolution (cf. (2.1)). It is well known that the weighted dual graph of (X, x) is a star-shaped graph (cf. (2.1)). The weighted dual graph of a cyclic quotient singularity is regarded as a star-shaped graph without central curves (note that it is a chain of rational curves).

We set $A = A_0 + \sum_{i=1}^{\beta} S_i$, where A_0 is the central curve, and S_i the branches. The curves of S_i are denoted by $A_{i,j}$, $1 \leq j \leq r_i$, where $A_0 \cdot A_{i,1} = A_{i,j} \cdot A_{i,j+1} = 1$ $(j = 1, \ldots, r_i - 1)$. Let $b_{i,j} = -A_{i,j} \cdot A_{i,j}$. For each branch S_i, positive integers e_i and d_i are defined by

$$d_i/e_i = b_{i,1} - \cfrac{1}{b_{i,2} - \cfrac{1}{\cdots - \cfrac{1}{b_{i,r_i}}}}$$

where $e_i < d_i$, and e_i and d_i are relatively prime.

For any integers $m \geq 1$ and $k \geq 0$, we define the divisors on A_0 by

$$D_m^{(k)} = kD - \sum_{i=1}^{\beta} [(ke_i + m(d_i - 1))/d_i] P_i,$$

where D is any divisor such that $\mathcal{O}_{A_0}(D)$ is the conormal sheaf of A_0, $P_i = A_0 \bigcap A_{i,1}$, and for any $a \in \mathbb{R}$, $[a]$ is the greatest integer not more than a.

The following is the extended version of Pinkham's formula (cf. [P1, Theorem 5.7]).

Theorem 1.7 (Watanabe [Wt2, Corollary 2.22]). *In the situation above,*

$$\delta_m(X,x) = \sum_{k\geq 0} h^0(\mathcal{O}_{A_0}(mK_{A_0} - D_m^{(k)})).$$

Theorem 1.8 (Tomaru [TSH]). *In the situation above, let g be the genus of the central curve A_0. Then q-h singularities are classified as follows by the plurigenera:*

δ_m	*structure*
When $m \longrightarrow \infty$, δ_m diverges with second order	(1) $g \geq 2$ (2) $g = 1$ and $\beta \geq 1$ (3) $g = 0$ and $\sum_{i=1}^{\beta}(d_i - 1)/d_i > 2$
$\delta_m = 1$ *for any* $m \in \mathbb{N}$	$g = 1$ and $\beta = 0$ *(i.e., simple elliptic singularities)*
$\delta_m = 0$ *if* $m \not\equiv 0 \pmod d$, $\delta_m = 1$ *if* $m \equiv 0 \pmod d$	$g = 0$ and $\sum_{i=1}^{\beta}(d_i - 1)/d_i = 2$
$\delta_m = 0$ *for any* $m \in \mathbb{N}$	$g = 0$ and $\sum_{i=1}^{\beta}(d_i - 1)/d_i < 2$ *or cyclic quotient singularities*

where d is the least common multiple of $d_1, \ldots, d_\beta$.

Lemma 1.9. *Let (X,x) be a Gorenstein singularity. Then*

$$\delta_m(X,x) \leq \delta_{m+1}(X,x)$$

for all $m \in \mathbb{N}$.

Proof. If (X,x) is a rational double point, then $\delta_m(X,x) = 0$ for all $m \in \mathbb{N}$ (cf. Theorem 4.3). Assume that (X,x) is not a rational double point and f is minimal good. Then $\mathrm{Supp}(K) = A$ and $K + A \leq 0$. There exists an inclusion

$$\mathcal{O}_M(mK + (m-1)A) \supset \mathcal{O}_M((m+1)K + mA).$$

Since $H^0(\mathcal{O}_U(mK)) = H^0(\mathcal{O}_M)$, we have

$$\delta_m(X,x) = \dim_{\mathbb{C}} H^0(\mathcal{O}_M)/H^0(\mathcal{O}_M(mK + (m-1)A)).$$

Hence

$$\begin{aligned}&\delta_{m+1}(X,x) - \delta_m(X,x)\\ &= \dim_{\mathbb{C}} H^0(\mathcal{O}_M(mK + (m-1)A))/H^0(\mathcal{O}_M((m+1)K + mA)) \geq 0.\end{aligned}$$

□

Lemma 1.10. *Let (X, x) be a singularity, m and n positive integers. Then*

$$\delta_m(X, x) \leq \delta_{mn}(X, x).$$

Proof. If $\delta_m(X, x) = 0$, then we are done. Assume $\delta_m(X, x) \neq 0$. Let $f\colon (M, A) \to (X, x)$ be a good resolution and $A = \bigcup_{i=1}^{t} A_i$ the decomposition of A into irreducible components. Let ν_i be the minimal value of $v_{A_i}(\omega)$ for all $\omega \in H^0(\mathcal{O}_U(mK))$, where v_{A_i} is the valuation associated to the divisor A_i. Since $\delta_m(X, x) < \infty$, we see that $\nu_i \neq -\infty$. Then there exists an element ω_1 such that $v_{A_i}(\omega_1) = \nu_i$ for all i. Assume that $\omega_1, \dots, \omega_k \in H^0(\mathcal{O}_U(mK))$ are linearly independent in

$$H^0(\mathcal{O}_U(mK))/H^0(\mathcal{O}_M(mK + (m-1)A)).$$

Let $a_1, \dots, a_k \in \mathbb{C}$ and $\omega = \sum_{i=1}^{k} a_i \omega_i$. By assumption, $v_{A_j}(\omega) \leq -m$ for some j. Since $\nu_j \leq m$, we have that $v_{A_j}(\omega_1^{n-1}\omega) \leq (n-1)\nu_j - m \leq -mn$. This means that $\omega_1^n, \omega_1^{n-1}\omega_2, \dots, \omega_1^{n-1}\omega_k$ are linearly independent in

$$H^0(\mathcal{O}_U(mnK))/H^0(\mathcal{O}_M(mnK + (mn-1)A)).$$

□

2. Cycles on the resolution space

(2.1) Let (X, x) be a singularity and $f\colon (M, A) \to (X, x)$ a resolution of the singularity (X, x). K denotes the canonical divisor on M. Let $A = \bigcup_{i=1}^{t} A_i$ be the decomposition of the exceptional set A into irreducible components.

A resolution $f\colon (M, A) \to (X, x)$ is called a minimal good resolution, if f is a smallest resolution for which A consists of non-singular curves intersecting among themselves transversally, with no three through one point. It is well known that there exists a unique minimal good resolution. Let us assume that $f\colon (M, A) \to (X, x)$ is the minimal good resolution of the singularity (X, x). The weighted dual graph of (X, x) is the graph such that each vertex of which represents a component of A weighted by the self-intersection number, while each edge connecting the vertices corresponding to A_i and A_j, $i \neq j$, corresponds to the point $A_i \bigcap A_j$. Giving the weighted dual graph is equivalent to giving the information of the genera of the A_i's and the intersection matrix $(A_i \cdot A_j)$. A string S in A is a chain of smooth rational curves $A_1, \dots, A_n$ so that $A_i \cdot A_{i+1} = 1$ for $i = 1, \cdots, n-1$, and these account for all intersections in A

among the A_i's, except that A_1 intersects exactly one other curve. The weighted dual graph of the singularity (X, x) is said to be star-shaped, if the divisor A is written as

$$A = A_0 + \sum S_j,$$

where A_0 is a curve and S_j are the maximal strings. Then A_0 is called a central curve, and S_j are called branches.

(2.2) We set

$$A_{\mathbb{Z}} = \bigoplus_{i=1}^{t} \mathbb{Z}A_i \quad \text{and} \quad A_{\mathbb{Q}} = \bigoplus_{i=1}^{t} \mathbb{Q}A_i,$$

where $\mathbb{Z}$ (resp. $\mathbb{Q}$) is the set of rational integers (resp. rational numbers). An element of $A_{\mathbb{Z}}$ (resp. $A_{\mathbb{Q}}$) is called a cycle (resp. $\mathbb{Q}$-cycle). There is a natural partial ordering between $\mathbb{Q}$-cycles defined by comparison of the coefficients. Let $V = \sum d_i A_i$ be a $\mathbb{Q}$-cycle. We set

$$\ulcorner V \urcorner = \sum \ulcorner d_i \urcorner A_i,$$

where the $\ulcorner d \urcorner$ denotes the least integer not less than d. V is said to be effective (resp. nef) if $d_i \geq 0$ (resp. $V \cdot A_i \geq 0$) for all i. A $\mathbb{Q}$-cycle V is said to be positive if $V \geq 0$ and $V \neq 0$. For any two positive cycles V and W, there exists an exact sequence

(2.2.1) $$0 \to \mathcal{O}_W \bigotimes_{\mathcal{O}_M} \mathcal{O}_M(-V) \to \mathcal{O}_{V+W} \to \mathcal{O}_V \to 0.$$

The Riemann-Roch theorem implies, for any positive cycle V and any invertible sheaf $\mathcal{L}$ on M,

$$\chi(\mathcal{O}_V) = h^0(\mathcal{O}_V) - h^1(\mathcal{O}_V) = -V \cdot (V + K)/2,$$

and

$$\chi(\mathcal{O}_V \bigotimes \mathcal{L}) = h^0(\mathcal{O}_V \bigotimes \mathcal{L}) - h^1(\mathcal{O}_V \bigotimes \mathcal{L}) = \mathcal{L} \cdot V + \chi(\mathcal{O}_V).$$

Let $\mathrm{Div}(M)$ be the group of divisors on M. Since the intersection matrix $(A_i \cdot A_j)$ is negative definite, there exists a homomorphism

$$\pi \colon \mathrm{Div}(M) \to A_{\mathbb{Q}},$$

defined by $\pi(F) \cdot A_i = F \cdot A_i$ for all i. We set $F_1 \cdot F_2 = \pi(F_1) \cdot \pi(F_2)$ for any $F_1, F_2 \in \mathrm{Div}(M)$.

Lemma-Definition 2.3 (cf. [M, 0.3]). For any $D \in \mathrm{Div}(M)$, there exists a unique cycle $\langle D \rangle \in A_{\mathbb{Z}}$ such that

(1) $\pi(D) - \langle D \rangle$ is nef,

(2) if $D' \in A_{\mathbb{Z}}$ and if $\pi(D) - D'$ is nef, then $\langle D \rangle \leq D'$.

(2.4) $\langle D \rangle$ can be obtained by means of a computation sequence as follows (cf. [TW, (6.12)]): $Z_0 = \ulcorner \pi(D) \urcorner$, $Z_i = Z_{i-1} + A_{j_i}$ when there exists an irreducible component A_{j_i} of A with $(\pi(D) - Z_{i-1}) \cdot A_{j_i} < 0$. Let Z_n be the last member in the above. Then $Z_n = \langle D \rangle$. Thus we have

(2.4.1) $$\langle D \rangle = \ulcorner \pi(D) \urcorner + \langle D - \ulcorner \pi(D) \urcorner \rangle.$$

Proposition-Definiton 2.5 (cf. [M, 1.3]). We set $\mathcal{E} = \{D \in \mathrm{Div}(M) \mid \langle D \rangle = 0\}$. We define maps $\alpha, \beta \colon \mathcal{E} \to \mathbb{Q}$ by

$$\alpha(D) = h^1(\mathcal{O}_M(D)) - p_g(X, x), \quad \beta(D) = D \cdot (D - K).$$

Then $\alpha(\mathcal{E})$ and $\beta(\mathcal{E})$ are finite subsets of $\mathbb{Q}$.

(2.6) Let F be a divisor on M, and $U = M - A$. We set

$$\chi_M(F) = \dim_{\mathbb{C}} H^0(\mathcal{O}_U(F))/H^0(\mathcal{O}_M(F)) + h^1(\mathcal{O}_M(F)).$$

By the Riemann-Roch theorem proved by Morales [M, 1.4] (which is a generalization of [Kt, Corollary 1]), we have

(2.6.1) $$\chi_M(F) = F \cdot (F - K)/2 + p_g(X, x) + \epsilon(F),$$

where $\epsilon(F) = \alpha(F - \langle F \rangle) + \beta(F - \langle F \rangle)/2$. We note that $\langle F - \langle F \rangle \rangle = 0$.

Theorem 2.7 (Sakai [Sa, Theorem A.1]). *Let F be a divisor on M. Then there exists a unique Zariski decompositon $\pi(F) = P + N$, where*

(1) *P is a nef $\mathbb{Q}$-cycle.*

(2) *N is an effective $\mathbb{Q}$-cycle.*

(3) *$P \cdot N = 0$, i.e., $P \cdot A_i = 0$ for all $A_i \subset \mathrm{Supp}(N)$.*

Theorem 2.8 (Sakai [Sa, Theorem A.2]). *In the situation above, we have*

$$H^1(\mathcal{O}_M(K + F - [N])) = 0.$$

(2.9) For the remainder of this section, $f\colon (M, A) \to (X, x)$ denote the minimal good resolution, and $P + N$ will only denote the Zariski decomposition of $K + A$.

It is well known that if (X, x) is a quotient singularity, then $P = 0$; if (X, x) is a simple elliptic or a cusp singularity, then $K + A = 0$. By definition (cf. Definition 3.2), $P = 0$, i.e., $N = \pi(K + A)$, for a log-canonical singularity. In all other cases, J. Wahl computed the effective part N by considering the strings.

Let $S = \sum_{i=1}^{n} A_i$ be a string as in (2.1). Let $S' = \sum_{i=1}^{n} a_i A_i$ be the $\mathbb{Q}$-cycle such that $S' \cdot A_n = -1$, $S' \cdot A_i = 0$ $(i < n)$. Then $a_i > 0$ for $i = 1, \dots, n$.

Theorem 2.10 (Wahl [Wh6, Proposition 2.3]). *Suppose (X, x) is not a quotient, simple elliptic, or cusp singularity. Let $\sum S'_j$ be the sum over each maximal strings S_j in A of the corresponding $\mathbb{Q}$-cycle S'_j. Then $\sum S'_j$ is the effective part of the Zariski decomposition of $K + A$, i.e., $N = \sum S'_j$.*

Lemma 2.11. *If (X, x) is not a rational double point, then $[N] = 0$.*

Proof. Let (X, x) be a quotient, simple elliptic or cusp singularity. Then f is the minimal resolution. Hence $-\pi(K)$ is effective. $\pi(K) = 0$ if and only if (X, x) is a rational double point, and otherwise $\mathrm{Supp}(\pi(K)) = A$. We know that $N = \pi(K + A)$. Hence if (X, x) is not a rational double point, then $[N] = 0$.

We assume that (X, x) is not as above. Let $S = \sum_{i=1}^{n} A_i$ be a maximal string and $S' = \sum_{i=1}^{n} a_i A_i$ the corresponding $\mathbb{Q}$-cycle. Let $S'' = \sum_{i=1}^{n} a'_i A_i$ be the $\mathbb{Q}$-cycle such that $S'' \cdot A_1 = -1$, $S'' \cdot A_i = 0$ $(i > 1)$. Then

$$(K + S - S' - S'') \cdot A_i = 0$$

for $i = 1, \dots, n$. Recall that S can be blown down to a quotient singularity. We see that $[S' + S''] = 0$ as above. Since a_i and a'_i are positive, We have $[S'] = 0$. By Theorem 2.10, we have $[N] = 0$. $\square$

Corollary 2.12. *If (X, x) is not a rational double point, then*

$$H^1(\mathcal{O}_M(2K + A)) = 0.$$

Proof. It is an immediate consequence of Theorem 2.8 and Lemma 2.11. $\square$

Corollary 2.13. *Let (X, x) be a singularity. Then*

$$\delta_2(X, x) = h^1_A(\mathcal{O}_M(2K + A)) = h^1(\mathcal{O}_M(-K - A)).$$

If V is a positive cycle, then

$$\delta_2(X,x) \geq V \cdot (K+A) - \chi(\mathcal{O}_V).$$

Proof. By the Serre duality, $h^1_A(\mathcal{O}_M(2K+A)) = h^1(\mathcal{O}_M(-K-A))$. We assume that (X,x) is not a rational double point. By Corollary 2.12, there exists an exact sequence

$$0 \to H^0(\mathcal{O}_M(2K+A)) \to H^0(\mathcal{O}_{M-A}(2K)) \to H^1_A(\mathcal{O}_M(2K+A)) \to 0.$$

Hence

$$\delta_2(X,x) = h^1_A(\mathcal{O}_M(2K+A))$$

by Proposition 1.3. Let (X,x) be a rational double point. Then $K=0$ and $H^1(\mathcal{O}_M(-A)) = 0$. Hence $H^1(\mathcal{O}_M(-K-A)) = 0$. Since a rational double point is a quotient singularity, $\delta_2(X,x) = 0$ (see Theorem 4.3). Hence we have the equation.

Let V be a positive cycle. Then

$$\delta_2(X,x) \geq h^1(\mathcal{O}_V(-K-A)) \geq -\chi(\mathcal{O}_V(-K-A)) = V \cdot (K+A) - \chi(\mathcal{O}_V). \qquad \square$$

3. $\mathbb{Q}$-Gorenstein singularities

(3.1) In this section, we study the plurigenera of $\mathbb{Q}$-Gorenstein singularities. Let $f\colon (M,A) \to (X,x)$ be the minimal good resolution and K the canonical divisor on M. Let $A = \bigcup A_i$ be the decomposition into irreducible components.

Definition 3.2. A singularity (X,x) is called a $\mathbb{Q}$-Gorenstein singularity if there exists a positive integer r such that $\mathcal{O}_X(rK_X)$ is invertible at x. It is well known that any rational singularity is a $\mathbb{Q}$-Gorenstein singularity. For a $\mathbb{Q}$-Gorenstein singularity (X,x), the minimal positive integer r which satisfies the condition above is called the index of (X,x), and denoted by $I(X,x)$. A $\mathbb{Q}$-Gorenstein singularity (X,x) with $I(X,x) = 1$ is called a Gorenstein singularity.

A $\mathbb{Q}$-Gorenstein singularity (X,x) is said to be log-canonical (resp. log-terminal) if the following condition is satisfied: We have, as $\mathbb{Q}$-divisor,

$$K_M = f^*K_X + \sum a_i A_i \text{ with } a_i \geq -1 \text{ (resp. } a_i > -1) \text{ for all } i.$$

(3.3) Let $L_m = m(K+A)$. By the Riemann-Roch theorem (2.6.1), we have

$$\chi_M(K+L_m) = -(K+L_m)\cdot L_m/2 + p_g(X,x) + \epsilon(K+L_m), \tag{3.3.1}$$

Let (X,x) be a $\mathbb{Q}$-Gorenstein singularity with the index $I(X,x)=s$. Set $m = sl+p$, $0 \le p < s$. Then $K + L_m - \ulcorner \pi(K+L_m) \urcorner = K + L_p - \ulcorner \pi(K+L_p) \urcorner$. By (2.4.1), we have that $K + L_m - \langle K+L_m \rangle = K + L_p - \langle K + L_p \rangle$. Thus $\epsilon(K+L_m) = \epsilon(K+L_p)$. Then $\chi_M(K+L_m)$ is cyclically expressed by finitely many polynomials of m as

$$\chi_M(K+L_m) = -(L_1\cdot L_1)m^2/2 - (K\cdot L_1)m/2 + \rho_p, \tag{3.3.2}$$

where $p = m - [m/s]s$ and $\rho_p = p_g(X,x) + \epsilon(K+L_p)$. If (X,x) is a Gorenstein singularity, then $\epsilon(K+L_m) = 0$ for any $m \ge 0$. If (X,x) is a rational singularity, then $\alpha(K+L_m - \langle K+L_m\rangle) = 0$ since $\pi(K+L_m) - \langle K+L_m\rangle$ is nef (cf. [M, 1.6]). Thus $\epsilon(K+L_m)$ is determined by the weighted dual graph for a rational singularity.

Theorem 3.4. *Let $F = \sum_{i=1}^n a_i A_i$ be a cycle with $a_i > 0$ $(i=1,\dots,n)$ such that $H^1(\mathcal{O}_F) = 0$. Then we have the following.*

(1) (Artin [A1, (1.7)]) The map φ: $\mathrm{Pic}(F) \to \mathbb{Z}^n$ defined by $\varphi(\mathcal{L}) = (\deg \mathcal{L}|_{A_i})$ is an isomorphism. Hence invertible sheaves on F are classified by their degree.

(2) (Lipman [Li, (11.1)]) If $\deg \mathcal{L}|_{A_i} \ge 0$ for $i=1,\dots,n$, then $H^1(\mathcal{L}) = 0$.

(3.5) Let $S = \sum S_i$ be the sum of the maximal strings S_i (cf. (2.1)) in A. Note that if $i \ne j$ then

$$\mathrm{Supp}(S_i) \bigcap \mathrm{Supp}(S_j) = \emptyset$$

and if F is an effective cycle such that $\mathrm{Supp}(F) \subset S$, then $H^1(\mathcal{O}_F) = 0$. We set $\mathcal{O}_F(L) = 0$ for any divisor L on M if $F = 0$. Let $S_i = \sum_{j=1}^{n_i} A_{i,j}$, where each $A_{i,j}$ is an irreducible component of A. The weighted dual graph $W(S_i)$ of S_i is the data on $A_{i,j}\cdot A_{i,j}$ and $A_{i,j}\cdot(A - A_{i,j})$ $(j = 1,\dots,n_i)$, and the weighted dual graph $W(S)$ of S is the sum of the data $W(S_i)$.

Lemma 3.6. *Suppose that (X,x) is not a quotient, simple elliptic, or cusp singularity. Then $h^1(\mathcal{O}_M(K+L_m))$ is determined by $W(S)$.*

Let r be a positive integer such that $rN \in A_{\mathbb{Z}}$. Set $m = rk+q$, $0 \le q < r$. Then

$$h^1(\mathcal{O}_M(K+L_m)) = -\chi(\mathcal{O}_{[mN]}(K+L_m)) + h^0(\mathcal{O}_{[qN]}(K+L_q)).$$

Proof. Consider the exact sequence

$$0 \to \mathcal{O}_M(K+L_m - [mN]) \to \mathcal{O}_M(K+L_m) \to \mathcal{O}_{[mN]}(K+L_m) \to 0.$$

By Theorem 2.8, we have

$$h^1(\mathcal{O}_M(K+L_m)) = h^1(\mathcal{O}_{[mN]}(K+L_m)). \tag{3.6.1}$$

By Theorem 3.4 and (3.5), $h^1(\mathcal{O}_{[mN]}(K+L_m))$ is determined by $W(S)$, since for a rational curve A_i,

$$(K+L_m)\cdot A_i = -2 - A_i\cdot A_i + m(-2+(A-A_i)\cdot A_i).$$

Let $m = rk+q = n+q$, where $0 \le q < r$. From Theorem 3.4, we have isomorphisms

$$\begin{aligned} \mathcal{O}_{[mN]}(K+L_m) &\cong \mathcal{O}_{nN+[qN]}(K+n(P+N)+L_q) \\ &\cong \mathcal{O}_{nN+[qN]}(K+nN+L_q), \end{aligned} \tag{3.6.2}$$

since $P\cdot A_i = 0$ for $A_i \subset \mathrm{Supp}(N)$. There exists an exact sequence (cf. (2.2.1))

$$0 \to \mathcal{O}_{[qN]}(K+L_q) \to \mathcal{O}_{nN+[qN]}(K+nN+L_q) \to \mathcal{O}_{nN}(K+nN+L_q) \to 0.$$

By the duality,

$$h^0(\mathcal{O}_{nN}(K+nN+L_q)) = h^1(\mathcal{O}_{nN}(-L_q)).$$

Since for $A_i \subset \mathrm{Supp}(N)$,

$$-L_q\cdot A_i = -qN\cdot A_i \ge 0$$

(cf. (2.9), Theorem 2.10), we get

$$h^0(\mathcal{O}_{nN}(K+nN+L_q)) = 0$$

by Theorem 3.4. Then we have

$$h^0(\mathcal{O}_{nN+[qN]}(K+nN+L_q)) = h^0(\mathcal{O}_{[qN]}(K+L_q)). \tag{3.6.3}$$

From (3.6.1), (3.6.2) and (3.6.3), we have

$$\begin{aligned} h^1(\mathcal{O}_M(K+L_m)) &= h^1(\mathcal{O}_{[mN]}(K+L_m)) \\ &= -\chi(\mathcal{O}_{[mN]}(K+L_m)) + h^0(\mathcal{O}_{[qN]}(K+L_q)). \qquad \square \end{aligned}$$

(3.7) In the situation above, we set $q = m - [m/r]r$ and $B_q = qN - [qN]$. Then we have

$$h^1(\mathcal{O}_M(K+L_m)) = -(N\cdot N)m^2/2 - (K\cdot N)m/2 + \sigma_q, \tag{3.7.1}$$

where $\sigma_q = B_q\cdot B_q/2 + K\cdot B_q/2 + h^0(\mathcal{O}_{[qN]}(K+Lq))$.

(3.8) It is well known that the plurigenera of a log-canonical singularity are simple (see Theorem 4.3 and 4.4). For remainder of this section, we assume that (X,x) is not a quotient, simple elliptic, or cusp singularity.

Theorem 3.9. *The plurigenera of* (X, x) *are represented as*

$$\delta_{m+1}(X, x) = -(P \cdot P)m^2/2 - (K \cdot P)m/2 + v(m),$$

where $v \colon \mathbb{Z} \to \mathbb{Q}$ *is a finite-valued function determined by the singularity* (X, x).

If (X, x) *is a* $\mathbb{Q}$*-Gorenstein singularity, then* v *is represented by* ρ *and* σ *defined in* (3.3) *and* (3.7), *respectively:* $v(m) = \rho_p - \sigma_q$.

Proof. We note that $L_1 \cdot L_1 = P \cdot P + N \cdot N$ and $K \cdot L_1 = K \cdot P + K \cdot N$. From (3.3) and (3.7), we have the results. □

Lemma 3.10. For a positive integer n, we define a function $\varphi_n \colon \mathbb{Z} \to \mathbb{Z}$ by $\varphi_n(a) = a - [a/n]n$. *Let* n_1 *and* n_2 *be positive integers and* n_3 *the least common multiple of* n_1 *and* n_2. *We define a map* $\varphi_{n_1,n_2} \colon \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ *by* $\varphi_{n_1,n_2}(a) = (\varphi_{n_1}(a), \varphi_{n_2}(a))$. *Then the map* φ_{n_1,n_2} *induces the bijection* $\{0, 1, \dots, n_3 - 1\} \to \varphi_{n_1,n_2}(\mathbb{Z})$.

Proof. Consider the bijection $\mathbb{Z}/(n_3) \to \mathbb{Z}/(n_1) \times \mathbb{Z}/(n_2)$. □

Corollary 3.11. *We write* $\delta_m = \delta_m(X, x)$ *for short. Let* (X, x) *be a* $\mathbb{Q}$*-Gorenstein singularity with* $I(X, x) = s$ *and* r *a positive integer such that* $rN \in A_{\mathbb{Z}}$. *Then the plurigenera* $\{\delta_m\}_{m \in \mathbb{N}}$ *are determined by* δ_m, $m = 1, 2, \dots, d + 2$, *where* d *is the least common multiple of* r *and* s.

Proof. We define polynomials $g_{(p,q)}(T)$ in $\mathbb{Q}[T]$ by

$$g_{(p,q)}(T) = -(P \cdot P)T^2/2 - (K \cdot P)T/2 + \rho_p - \sigma_q,$$

where $(p, q) \in \varphi_{s,r}(\mathbb{Z})$. By Theorem 3.9 and Lemma 3.10, the plurigenera are computed by $g_{(p,q)}(T)$'s: $\delta_{m+1} = g_{\varphi_{s,r}(m)}(m)$. If $\delta_1, \delta_2, \dots, \delta_{d+2}$ are given, then we have the values $P \cdot P$, $K \cdot P$ and $\rho_p - \sigma_q$, $(p, q) \in \varphi_{s,r}(\mathbb{Z})$ by solving the system of equations. □

Corollary 3.12. *We write* $\delta_m = \delta_m(X, x)$ *for short. Then we have the following.*

(1) *If* (X, x) *is a rational singularity, then* $\{\delta_m\}_{m \in \mathbb{N}}$ *are determined by the weighted dual graph of the singularity.*

(2) *Let* (X, x) *be a Gorenstein singularity. Then* $\{\delta_m\}_{m \in \mathbb{N}}$ *are determined by* p_g, δ_2, $\chi(\mathcal{O}_A)$ *and* $W(S)$. *In particular,* $\{\delta_m\}_{m \in \mathbb{N}}$ *are determined by* p_g *and the weighted dual graph of the singularity* (X, x).

Proof. (1) is obtained from (3.3) and Lemma 3.6. Assume that (X, x) is a Gorenstein singularity. By Corollary 2.12 and (3.3), we have

$$\delta_2 = -(K + L_1) \cdot L_1/2 + p_g = -K \cdot L_1 + \chi(\mathcal{O}_A) + p_g. \qquad (3.12.1)$$

Then

$$\begin{aligned}(K+L_m)\cdot L_m &= (m^2+m)K\cdot L_1 + m^2 A\cdot L_1\\ &= (m^2+m)(p_g-\delta_2+\chi(\mathcal{O}_A)) - 2m^2\chi(\mathcal{O}_A).\end{aligned}$$

Thus (2) follows from (3.3) and Lemma 3.6. □

(3.13) We note that quotient, simple elliptic or cusp singularities are characterized by the weighted dual graphs (cf. [Wt1, 3]).

Corollary 3.14. *Let (X,x) be a hypersurface (resp. complete intersection) singularity with $p_g(X,x)=1$. Then $\delta_2(X,x)\le 4$ (resp. ≤ 5).*

Proof. Note that a complete intersection singularity with $p_g=1$ is a minimally elliptic singularity (cf. Definition 5.5, Theorem 5.6). From (3.12.1), we have

$$\delta_2 = 1 - K\cdot L_1 + \chi(\mathcal{O}_A). \tag{3.14.1}$$

Since $p_g=1$, we have $0\le\chi(\mathcal{O}_A)\le 1$.

Assume that $\chi(\mathcal{O}_A)=0$. Then (X,x) is a simple elliptic or cusp singularity, and hence $L_1=0$. Then $\delta_2=1$.

Assume that $\chi(\mathcal{O}_A)=1$. If f is not minimal, then by [La1, Proposition 3.5], we have the star-shaped graph which consists of four rational curves such that the self-intersection number of the central curve A_1 is -1. In this case, $K=-2A_1-A_2-A_3-A_4$ and $K\cdot L_1=1$. Hence $\delta_2(X,x)=1$. If f is minimal, then (X,x) is a hypersurface (resp. complete intersection) singularity if and only if $-K\cdot K\le 3$ (resp. ≤ 4) by [La1, Theorem 3.13]. We have the assertion from the following

$$\delta_2 = 2 - K\cdot K - K\cdot A \le 1 - K\cdot K. \qquad \square$$

4. Log-canonical singularities, I

(4.1) In this section, we study a criterion, in terms of pluri-genera, for a singularity (X,x) to be a log-canonical singularity with $p_g(X,x)=0$.

Definition 4.2. For any singularity (X,x), the minimal positive integer m such that $\delta_m(X,x)\ne 0$ is called the δ-index of (X,x), and denoted by $I_\delta(X,x)$. If $\delta_m(X,x)=0$ for all $m\in\mathbb{N}$, we set $I_\delta(X,x)=\infty$.

Theorem 4.3 (Watanabe [Wt1, Theorem 3.9]). *A singularity (X, x) is a quotient singularity if and only if $I_\delta(X, x) = \infty$.*

Theorem 4.4 (Ishii [I5]). *Let (X, x) be a singularity such that $\{\delta_m(X, x)\}_{m \in \mathbb{N}}$ is bounded, i.e., there is an integer B such that $\delta_m(X, x) \le B$ for all $m \in \mathbb{N}$. We assume that (X, x) is not a quotient singularity. Then (X, x) is a log-canonical singularity with $I(X, x) = I_\delta(X, x)$, and $\delta_m(X, x) \le 1$ for all $m \in \mathbb{N}$. Let $I = I(X, x)$. Then we have the following.*

(1) *$\delta_m(X, x) = 1$ for $m \equiv 0 \pmod{I}$ and $\delta_m(X, x) = 0$ for $m \not\equiv 0 \pmod{I}$.*

(2) *$I = 1$ if and only if (X, x) is a simple elliptic or a cusp singularity.*

(3) *If $I > 1$, then (X, x) is the quotient with respect to a cyclic group of a simple elliptic or a cusp singularity.*

(4.5) We take the following characterization of du Bois singularities as its definition.

Proposition 4.6 (Steenbrink [St, (3.6)]). *A normal surface singularity (X, x) is a du Bois singularity if and only if the natural map $H^1(\mathcal{O}_M) \to H^1(\mathcal{O}_A)$ is an isomorphism, where $f\colon (M, A) \to (X, x)$ is a good resolution.*

Theorem 4.7 (Ishii [I2, Theorem 2.3]). *Every resolution of a du Bois singularity is a good resolution.*

(4.8) We note that a log-canonical singularity with $I(X, x) > 1$ is a rational singularity, and it is du Bois since $H^1(\mathcal{O}_M) = H^1(\mathcal{O}_A) = 0$. It is well known that the weighted dual graph of a rational singularity is a tree. By Theorem 4.7, the minimal good resolution of a rational singularity is minimal.

Throughout this section, $f\colon (M, A) \to (X, x)$ denote the minimal resolution, K the canonical divisor on M and $A = \bigcup A_i$ the decomposition into irreducible components. For any component A_i of A, we set $t_i = (A - A_i) \cdot A_i$, the cardinality of the intersection points on A_i.

Lemma 4.9. *If $\delta_2(X, x) = 0$, then the weighted dual graph of (X, x) is a chain (if (X, x) is a cyclic quotient singularity), or a star-shaped graph with three branches.*

Proof. By Lemma 1.10, (X, x) is a rational singularity. For any component A_i of A, $t_i \le 3$ by Corollary 2.13. If $t_i \le 2$ for all i, then A is a chain of curves.

We assume that $t_1 = 3$. Let A_n be any component of A. Let $\sum_{i=1}^{n} A_i$ be the minimal connected cycle containing A_1 and A_n. Then $t_i \ge 2$ for $i \le n - 1$. Applying

Corollary 2.13 to the positive cycle $\sum_{i=1}^{n-1} A_i$, we have $0 \geq \sum_{i=2}^{n-1}(t_i - 2)$. Hence $t_i = 2$ for $i = 2, \dots, n-1$. □

(4.10) For any rational singularity (X, x) with star-shaped graph, there exists a rational singularity of which the exceptional set of the minimal good resolution and the weighted dual graph are the same as those of (X, x) (cf. [P1], (6.8)). Let (X, x) be a rational singularity (X, x) with star-shaped graph. The notation A_0, D, $D_m^{(k)}$, P_i, e_i and d_i are defined as in (1.6). By Corollary 3.12, we can use the formula in Theorem 1.7 to compute the plurigenera of (X, x). We set

$$F_m^{(k)} = -2m - kb + \sum_{i=1}^{\beta}[(ke_i + m(d_i - 1))/d_i],$$

where $b = -A_0 \cdot A_0$. Since $A_0 \cong \mathbb{P}^1$, we have $b > 1$ and

$$\delta_m(X, x) = \sum_{k \geq 0} h^0(\mathcal{O}_{A_0}(F_m^{(k)})).$$

We always assume that $d_1 \leq \cdots \leq d_\beta$.

Theorem 4.11 (Okuma [O1]). *If $\delta_m(X, x) = 0$ for $m = 4, 6$, then (X, x) is a quotient singularity.*

Proof. By Lemma 1.10, $\delta_m(X, x) = 0$ for $m = 1, 2$. We assume that (X, x) is not a cyclic quotient singularity. By Lemma 4.9, the weighted dual graph of (X, x) is a star-shaped graph with three branches. Then

$$F_4^{(0)} = -8 + \sum_{i=1}^{3}[4 - 4/d_i] \text{ and } F_6^{(0)} = -12 + \sum_{i=1}^{3}[6 - 6/d_i].$$

Note that $[m - m/a_1] \leq [m - m/a_2]$ if $a_1 \leq a_2$.

Since $\delta_6(X, x) = 0$, we have $F_6^{(0)} \leq -1$. If $d_1 \geq 3$, then $F_6^{(0)} \geq 0$. Hence $d_1 = 2$. Since $\delta_4(X, x) = 0$, we have $F_4^{(0)} = -6 + [4 - 4/d_2] + [4 - 4/d_3] \leq -1$. Thus $d_2 \leq 3$.

If $d_1 = d_2 = 2$, then $\sum_{i=1}^{3}(d_i - 1)/d_i < 2$, and hence (X, x) is a quotient singularity by Theorem 1.8 and 4.3.

Assume $d_2 = 3$. Since $F_6^{(0)} = -5 + [6 - 6/d_3] \leq -1$, we have $d_3 \leq 5$. Again, we get $\sum_{i=1}^{3}(d_i - 1)/d_i < 2$, and hence (X, x) is a quotient singularity. □

Corollary 4.12. *Let (X, x) be any singularity. If (X, x) is not a quotient singularity, then $I_\delta(X, x) \leq 6$.*

Proof. The result is an immediate consequence of Theorem 4.3 and Theorem 4.11. □

Proposition 4.13. *Let (X,x) be a singularity with $I_\delta(X,x)=6$ and $\delta_{14}(X,x)=0$. Then (X,x) is a log-canonical singularity with $I(X,x)=6$.*

Proof. From the assumption, $\delta_m(X,x) = 0$ for $m = 1,2,3,4,5$. By Lemma 4.9, (X,x) has a star-shaped graph with three branches. Since $\delta_3(X,x) = 0$, we have $F_3^{(0)} = -6 + \sum_{i=1}^{3}[3 - 3/d_i] \leq -1$. Thus $d_1 = 2$. Similarly, we have $d_2 \leq 3$ by $d_1 = 2$ and $F_4^{(0)} \leq -1$. If $d_2 = 2$ or $d_3 \leq 5$, then $I_\delta(X,x) = \infty$ by the proof of Theorem 4.11. Hence we get $d_1 = 2$, $d_2 = 3$ and $d_3 \geq 6$. Since $\delta_{14}(X,x) = 0$, we have $F_{14}^{(0)} = -12 + [14 - 14/d_3] \leq -1$. Thus $d_3 = 6$. By Theorem 1.8 and 4.4, (X,x) is a log-canonical singularity with $I(X,x) = 6$. □

(4.14) We note that if $I_\delta(X,x) = 5$, then (X,x) is not a log-canonical singularity by Theorem 1.8 and 4.4 (cf. Theorem 4.17).

Proposition 4.15. *Let (X,x) be a singularity with $I_\delta(X,x)=4$ and $\delta_{14}(X,x)=0$. Then (X,x) is a log-canonical singularity with $I(X,x)=4$.*

Proof. As in the proof of the proposition above, we have $d_1 = 2$ and $d_2 \geq 3$. However $d_2 = 3$ implies the same result of the proposition above. Hence $d_2 \geq 4$. Then $d_2 = d_3 = 4$ by $F_{14}^{(0)} \leq -1$. By Theorem 1.8 and 4.4, (X,x) is a log-canonical singularity with $I(X,x) = 4$. □

Proposition 4.16. *Let (X,x) be a singularity with $I_\delta(X,x)=3$ and $\delta_{14}(X,x)=0$. Then (X,x) is a log-canonical singularity with $I(X,x)=3$.*

Proof. If $d_1 = 2$, we have the same result of the proposition above. Hence $d_1 \geq 3$. Then $d_1 = d_2 = d_3 = 3$ by $F_{14}^{(0)} \leq -1$. Again by Theorem 1.8 and 4.4, (X,x) is a log-canonical singularity with $I(X,x) = 3$. □

Theorem 4.17 (Okuma [O3]). *Let (X,x) be a singularity with $\delta_{14}(X,x)=0$. Then (X,x) is a log-canonical singularity.*

Proof. Since $\delta_{14}(X,x) = 0$, we have $\delta_1(X,x) = \delta_2(X,x) = 0$ by Lemma 1.10, and hence $I_\delta(X,x) \geq 3$.

If $I_\delta(X,x) = \infty$, then (X,x) is a quotient singularity, and it is log-canonical (more precisely, log-terminal). Assume that $I_\delta(X,x) \leq 6$ (cf. Corollary 4.12). If $I_\delta(X,x) \neq 5$, then we are done. By the proofs of the propositions above, there exists no singularity (X,x) with $I_\delta(X,x) = 5$ and $\delta_{14}(X,x) = 0$. □

Lemma 4.18. *Let (X,x) be a singularity with $\delta_1(X,x)=0$ and $\delta_2(X,x)=1$. Then we have one of the following.*

(1) *(X, x) has a star-shaped graph with three branches.*

(2) *(X, x) has a star-shaped graph with four branches.*

(3) *The exceptional divisor A is written as $\sum_{i=0}^{4} S_i$, where S_i, $i \geq 1$, are the maximal strings, and S_0 is a chain of curves.*

Proof. By Corollary 2.13, $t_i \leq 4$ for all A_i. Since (X, x) is not a cyclic quotient singularity, there exists a component A_j such that $t_j \geq 3$. Assume that (X, x) is not in the case (1). If $t_1 = 4$, then as in the proof of lemma 4.9, we have a star-shaped graph with four branches. If $t_i \leq 3$ for all A_i, then we may assume that $t_1 = t_2 = 3$. Then, as in the proof of lemma 4.9, we have $t_i \leq 2$ for $i \geq 3$. Thus $A - A_1 - A_2$ is a disjoint union of chains of curves. Since the weighted dual graph is a tree, there exists a unique minimal connected cycle S_0 containing A_1 and A_2. Since $t_1 = t_2 = 3$, a cycle $A - S_0$ is a disjoint union of four maximal strings in A. □

Lemma 4.19. *Let (X, x) be a singularity with $\delta_{14}(X, x) = 1$. If (X, x) has a star-shaped graph with three branches, then $\delta_2(X, x) = 0$.*

Proof. Assume that (X, x) has a star-shaped graph with three branches. Using the notation of (4.10), we have

$$F_m^{(k)} = m - kb + \sum_{i=1}^{3} [(ke_i - m)/d_i].$$

If $b \geq 3$, $F_2^{(k)} \leq F_2^{(k-1)} \leq \cdots \leq F_2^{(0)} < 0$, and hence $\delta_2(X, x) = 0$. If $\sum 1/d_i \geq 1$, then $\delta_2(X, x) = 0$ by Theorem 1.8. Assume that $b = 2$ and $\sum 1/d_i < 1$. We define a subset Δ^* of $\mathbb{N}^6$ as follows: $(e, d) = (e_1, e_2, e_3, d_1, d_2, d_3) \in \mathbb{N}^6$ is an element of Δ^* if and only if $d_1 \leq d_2 \leq d_3$, $\sum 1/d_i < 1$, $\sum e_i/d_i < 2$ (cf. [P1, p. 185]), $e_i < d_i$, and e_i and d_i are relatively prime for $i = 1, 2, 3$. We regard $F_m^{(k)}$ as a function of k, m and $(e, d) \in \Delta^*$, and write $F_m^{(k)}(e, d)$. Let

$$G^{(k)}(e, d) = k(\sum e_i/d_i - 2) + 2(1 - \sum 1/d_i).$$

Then

$$F_2^{(k)}(e, d) \leq 2 - 2k + \sum (ke_i - 2)/d_i = G^{(k)}(e, d).$$

Since $\sum e_i/d_i - 2 < 0$, we have $F_2^{(k)}(e, d) < 0$ for $k \geq 2$ (resp. $k \geq 3$) if $G^{(2)}(e, d) < 0$ (resp. $= 0$).

Let

$$\Delta = \{d \in \mathbb{N}^3 | (e, d) \in \Delta^* \text{ for some } e \in \mathbb{N}^3, \text{ and } F_{14}^{(0)} \leq 0\}.$$

Let $\Delta_1 = \{(2,3,d_3)|7 \le d_3 \le 13\}$ and $\Delta_2 = \{(2,4,5),(2,4,6)\}$. As in the proofs of the propositions above, we have $\Delta = \Delta_1 \bigcup \Delta_2 \bigcup\{(3,3,4)\}$.

We assume that $d \in \Delta_1$. Since $\delta_{14}(X,x) = 1$ and $F_{14}^{(0)} = 0$, we have

$$F_{14}^{(3)} = -3 + e_2 + [(3e_3 - 14)/d_3] \le -1.$$

Let $\Delta_1' = \{(e,d) \in \Delta^* | d \in \Delta_1, F_{14}^{(3)} \le -1\}$. We can easily get $F_2^{(k)}(e,d) < 0$ for $(e,d) \in \Delta_1'$ and $k = 0,1,2$. We will show

$$G^{(2)}(e,d) = 2(\sum (e_i - 1)/d_i - 1) \le 0$$

for $(e,d) \in \Delta_1'$. For $(e,d) \in \Delta_1'$ with $e_2 = 1$, we have $G^{(2)}(e,d) = 2((e_3-1)/d_3-1) < 0$. Let $e_2 = 2$. Then $3e_3 - 14 < d_3$, and $e_3/d_3 < 5/6$. The maximum of $\{(e_3-1)/d_3\}$ is $(7-1)/9 = 2/3$. Hence $G^{(2)}(e,d) = 2((e_3-1)/d_3 - 2/3) \le 0$. Then we have $F_2^{(k)} < 0$, for $k \ge 0$ and $(e,d) \in \Delta_1'$.

We assume that $d \in \Delta_2$. If $e_2 = 1$, then $G^{(2)}(e,d) = 2((e_3-1)/d_3 - 1) < 0$. Let $e_2 = 3$. As above, we have $e_3 + d_3 < 7$ from $F_{14}^{(2)} \le -1$. Hence $e_3 = 1$. Then $G^{(2)}(e,d) = 2(1/2 - 1) < 0$. Clearly, $F_2^{(0)}$ and $F_2^{(1)}$ are negative. Hence $F_2^{(k)} < 0$ for $k \ge 0$.

If $d = (3,3,4)$, then $e = (e_1,e_2,e_3)$ $(e_1 \le e_2)$ such that $(e,d) \in \Delta^*$ is one of $(1,1,1)$, $(1,1,3)$, $(1,2,1)$, $(1,2,3)$ and $(2,2,1)$. Again, we have that $F_2^{(k)} < 0$ for $k \ge 0$.

Thus in any of the cases, we get $\delta_2(X,x) = 0$. □

Theorem 4.20 (Okuma [O3]). *Let (X,x) be a singularity with $I_\delta(X,x) = 2$ and $\delta_{14}(X,x) = 1$. Then (X,x) is a log-canonical singularity with $I(X,x) = 2$.*

Proof. Since $\delta_{14}(X,x) = 1$ and $\delta_2(X,x) \ne 0$, hence $\delta_2(X,x) = 1$ by Lemma 1.10. By the lemmas above, we have the weighted dual graph in (2) or (3) of Lemma 4.18.

Suppose (X,x) has a star-shaped graph. Then $d_1 = \cdots = d_4 = 2$ by $F_{14}^{(0)} \le 0$, and hence (X,x) is a log-canonical singularity with $I(X,x) = 2$ by Theorem 1.8.

Assume that $A = \sum_{i=0}^4 S_i$ as in (3) of lemma 4.18. By [Kr, Theorem 3.7], there exists a deformation $\pi\colon \overline{M} \to (\mathbb{C},0)$ of $M = \pi^{-1}(0)$ which induces a trivial deformation of S_i for $i = 1,2,3,4$, and for $c \ne 0$ near 0, $\pi^{-1}(c)$ has a connected component of the exceptional set $A_0 + \sum_{i=1}^4 S_i$, where A_0 is a rational curve. Note that π blows down to a deformation of (X,x). Let (Y,y) be a singularity obtained by contracting the exceptional divisor $A_0 + \sum_{i=1}^4 S_i$ above. By Theorem 1.5, we have $p_g(Y,y) = 0$, $\delta_2(Y,y) \le 1$ and $\delta_{14}(Y,y) \le 1$. Thus (Y,y) is a rational singularity which has a star-shaped graph with four branches. By Lemma 4.9, we have $\delta_2(Y,y) = \delta_{14}(Y,y) = 1$.

Applying the argument above to (Y, y), we have $d_1 = \cdots = d_4 = 2$. By the definition of d_i, we see that S_i is a curve with $S_i \cdot S_i = -2$, for $i \geq 1$. Recall that π induces a trivial deformation of S_i for $i \geq 1$. Let B be a cycle on M defined by $B = A + S_0$. Then $-B$ is numerically equivalent to $2K$. Since any rational singularity is a $\mathbb{Q}$-Gorenstein singularity, (X, x) is a log-canonical singularity with $I(X, x) = 2$ (cf. Definition 4.2 and Theorem 4.4). □

5. Log-canonical singularities, II

(5.1) In this section, we study a criterion for a singularity to be a log-canonical singularity with $I(X, x) = 1$. Recall that a log-canonical singularity with $I(X, x) = 1$ is a simple elliptic or cusp singularity. Let $f\colon (M, A) \to (X, x)$ be a resolution, K the canonical divisor on M and $A = \bigcup A_i$ the decomposition into irreducible components.

Definition 5.2. A positive cycle E is minimally elliptic if $\chi(\mathcal{O}_E) = 0$ and $\chi(\mathcal{O}_F) > 0$ for all cycles F such that $0 < F < E$.

(5.3) There exists a unique fundamental cycle Z on M (cf. [A2]) such that $Z > 0$, $A_i \cdot Z \leq 0$ for all i, and that Z is minimal with respect to those two properties. Note that $h^0(\mathcal{O}_Z) = 1$ (cf. [La1]).

Proposition 5.4 (Laufer [La1, Theorem 3.4]). *Let $f\colon (M, A) \to (X, x)$ be the minimal resolution of the singularity (X, x), Z the fundamental cycle and K the canonical divisor on M. Then the following are equivalent.*

(1) *Z is a minimally elliptic cycle.*

(2) *$A_i \cdot Z = -A_i \cdot K$ for all A_i.*

Definition 5.5. A singularity (X, x) is minimally elliptic if the minimal resolution of (X, x) satisfies the conditions of Proposition 5.4.

Theorem 5.6 (Laufer [La1, Theorem 3.10]). *A singularity (X, x) is minimally elliptic if and only if (X, x) is an elliptic Gorenstein singularity.*

(5.7) Let $f\colon (M, A) \to (X, x)$ be the minimal resolution of the singularity (X, x) and Z the fundamental cycle. By the natural surjective map $H^1(\mathcal{O}_M) \to H^1(\mathcal{O}_Z)$, we have $p_g(X, x) \geq h^1(\mathcal{O}_Z)$. Artin [A2] proved that $p_g(X, x) = 0$ if and only if $h^1(\mathcal{O}_Z) = 0$. If $p_g(X, x) = 1$, then $h^1(\mathcal{O}_Z) = 1$, and there exists a unique minimally elliptic cycle E by [La1, Proposition 3.1]. The support of E is the exceptional set of a minimally elliptic singularity by [La1, Lemma 3.3].

Lemma 5.8. *Let (X, x) be a minimally elliptic singularity which is not a du Bois singularity. Then $\delta_6(X, x) \geq 2$.*

Proof. First, we assume that the minimal resolution of the singularity (X, x) is a good resolution. Let $f\colon (M, A) \to (X, x)$ be the minimal resolution. Recall the equality (3.12.1)

$$\delta_2(X, x) = -K \cdot (K + A) + \chi(\mathcal{O}_A) + 1.$$

Since (X, x) is not a du Bois singularity, $H^1(\mathcal{O}_A) = 0$, and hence $\chi(\mathcal{O}_A) = 1$. Then we have $\delta_2(X, x) = -(K + A) \cdot K + 2$. Since f is minimal and $-(K + A) \geq 0$, we get $\delta_2(X, x) \geq 2$. By Lemma 1.9, we have $\delta_6(X, x) \geq 2$.

Now we assume that the minimal resolution of (X, x) is not good. Let $f\colon (M, A) \to (X, x)$ be the minimal good resolution of the singularity (X, x). By [La1, Proposition 3.5], (X, x) has a star-shaped graph with three branches, and the divisor A can be written as $A = \sum_{i=1}^{4} A_i$, where A_1 is the central curve with $-A_1 \cdot A_1 = -1$, and $A_2 \cdot A_2 \geq A_3 \cdot A_3 \geq A_4 \cdot A_4$. Then $-K = 2A_1 + \sum_{i=2}^{4} A_i$. Let $Z = \sum_{i=1}^{4} n_i A_i$ be the fundamental cycle on M. Then $(n_1, \dots, n_4)$ is one of $(6, 3, 2, 1)$, $(4, 2, 1, 1)$ or $(3, 1, 1, 1)$. Let $\mathcal{M}$ be the maximal ideal in $\mathcal{O}_X$ which defines the singular point x. By [La1, Theorem 3.13], there exists a function $g \in H^0(\mathcal{M})$ (under the assumption that X is sufficiently small) such that $f^*(g)$ has zero of order n_1 on A_1. Since (X, x) is minimally elliptic, we have $f_*\mathcal{O}_M(K) \cong \mathcal{M}$. On the other hand, we have

$$\mathcal{O}_M(6K + 5A) \cong \mathcal{O}_M(K - 5A) \cong \mathcal{O}_M(-7A_1 - \sum_{i=2}^{4} A_i).$$

Hence

$$f^*(g) \in H^0(\mathcal{O}_M(K)) \setminus H^0(\mathcal{O}_M(6K + 5A)).$$

Since $H^0(\mathcal{O}_M) \supsetneqq H^0(\mathcal{O}_M(K)) \supsetneqq H^0(\mathcal{O}_M(6K + 5A))$, we have $\delta_6(X, x) \geq 2$ by Proposition 1.3. □

Proposition 5.9. *Let (X, x) be an elliptic singularity which is not a du Bois singularity. Then $\delta_6(X, x) \geq 2$.*

Proof. (5.7), Theorem 1.4 and Lemma 5.8 implies the assertion. □

Example 5.10. There exists a singularity (X, x) with $\delta_m(X, x) = 1$ for $m = 1, \dots, 5$ which is not a du Bois singularity, but a minimally elliptic singularity.

Let (X, x) be a minimally elliptic singularity such that the minimal resolution of (X, x) is not good. Using the notation in the proof of Lemma 5.8, we assume that $A_2 \cdot A_2 = -2$, $A_3 \cdot A_3 = -3$ and $A_4 \cdot A_4 \leq -7$. Then

$$Z = 6A_1 + 3A_2 + 2A_3 + A_4 = -K + 4A_1 + 2A_2 + A_3.$$

Note that there exists such a minimally elliptic singularity. Since $Z > A$, we have $H^1(\mathcal{O}_A) = 0$ (cf. Definition 5.2). Thus (X, x) is not a du Bois singularity by Proposition 4.6. As in the proof of Lemma 5.8, we have

$$\begin{aligned}\delta_5(X, x) &= \dim_{\mathbb{C}} H^0(\mathcal{O}_M)/H^0(\mathcal{O}_M(K)) + \dim_{\mathbb{C}} H^0(\mathcal{O}_M(K))/H^0(\mathcal{O}_M(5K + 4A)) \\ &= 1 + \dim_{\mathbb{C}} H^0(\mathcal{O}_M(K))/H^0(\mathcal{O}_M(K - 4A_1)).\end{aligned}$$

From the exact sequence

$$0 \to \mathcal{O}_M(K - 4A_1) \to \mathcal{O}_M(K) \to \mathcal{O}_{4A_1}(K) \to 0,$$

we have

$$\dim_{\mathbb{C}} H^0(\mathcal{O}_M(K))/H^0(\mathcal{O}_M(K - 4A_1)) = 6 - h^1(\mathcal{O}_M(K - 4A_1)).$$

We will show that $h^1(\mathcal{O}_M(K - 4A_1)) = 6$. Since $H^1(\mathcal{O}_M) \cong H^1(\mathcal{O}_Z)$, we have $H^1(\mathcal{O}_M(-Z)) = 0$. From the exact sequence

$$0 \to \mathcal{O}_M(-Z) \to \mathcal{O}_M(K - 4A_1) \to \mathcal{O}_{2A_2+A_3}(K - 4A_1) \to 0,$$

we have $H^1(\mathcal{O}_M(K - 4A_1)) \cong H^1(\mathcal{O}_{2A_2+A_3}(K - 4A_1))$. Let $L = K - 4A_1$. Consider the exact sequences (cf. (2.2.1))

$$0 \to \mathcal{O}_{2A_2}(L - A_3) \to \mathcal{O}_{2A_2+A_3}(L) \to \mathcal{O}_{A_3}(L) \to 0;$$

$$0 \to \mathcal{O}_{A_2}(L - A_3 - A_2) \to \mathcal{O}_{2A_2}(L - A_3) \to \mathcal{O}_{A_2}(L - A_3) \to 0.$$

Then we get

$$\begin{aligned}h^1(\mathcal{O}_{2A_2+A_3}(K - 4A_1)) &= h^1(\mathcal{O}_{A_3}(L)) + h^1(\mathcal{O}_{A_2}(L - A_3)) + h^1(\mathcal{O}_{A_2}(L - A_3 - A_2)) \\ &= 2 + 3 + 1 = 6.\end{aligned}$$

Hence $\delta_5(X, x) = 1$. By Lemma 1.9, $\delta_m(X, x) = 1$ for $m = 1, \cdots, 5$.

(5.11) Let (X, x) be an elliptic du Bois singularity and $f\colon (M, A) \to (X, x)$ the minimal resolution. Since $H^1(\mathcal{O}_A) = 1$, the divisor A is decomposed as $A = E_1 + E_2$, where E_1 is either a non-singular curve or a cycle of r rational curves with $r \geq 1$ (a cycle of one rational curve means a rational curve with an ordinary double point), and E_2 is void or a disjoint union of trees of non-singular rational curves. If $E_2 = 0$, then (X, x) is a simple elliptic or a cusp singularity.

We will use this notation in Lemma 5.12, 5.13 and Proposition 5.14 below.

Lemma 5.12. *If E_2 is a rational curve with $E_2 \cdot E_2 \leq -3$, then $\delta_3(X, x) \geq 2$.*

Proof. For any component A_i of A, we have $(2K+2A-E_2)\cdot A_i \geq 0$. By Theorem 2.8, $H^1(\mathcal{O}_M(3K+2A)) \cong H^1(\mathcal{O}_{E_2}(3K+2A))$. Since $(3K+2A)\cdot E_2 = K \cdot E_2 - 2 \geq -1$, we have $H^1(\mathcal{O}_{E_2}(3K+2A)) = 0$. Let $L = 3K+2A$. Then we get

$$0 \to H^0(\mathcal{O}_M(L)) \to H^0(\mathcal{O}_M(L+E_1)) \to H^0(\mathcal{O}_{E_1}(L+E_1)) \to 0,$$

and

$$\dim_{\mathbb{C}} H^0((\mathcal{O}_M(L+E_1))/H^0(\mathcal{O}_M(L)) = h^0(\mathcal{O}_{E_1}(L+E_1)) \geq \chi(\mathcal{O}_{E_1}(L+E_1)) = 2.$$

Since

$$\delta_3(X,x) = \dim_{\mathbb{C}} H^0(\mathcal{O}_{M-A}(3K))/H^0(\mathcal{O}_M(L))$$

and

$$H^0(\mathcal{O}_{M-A}(3K)) \supset H^0((\mathcal{O}_M(L+E_1)) \supset H^0(\mathcal{O}_M(L)),$$

we have $\delta_3(X,x) \geq 2$. $\square$

Lemma 5.13. *If E_2 is a rational curve with $E_2 \cdot E_2 = -2$, then $\delta_4(X, x) \geq 2$.*

Proof. As above, we have $H^1(\mathcal{O}_M(4K+3A)) \cong H^1(\mathcal{O}_{2E_2}(4K+3A))$. Let $L = 4K+3A$. From the exact sequence

$$0 \to \mathcal{O}_{E_2}(L-E_2) \to \mathcal{O}_{2E_2}(L) \to \mathcal{O}_{E_2}(L) \to 0,$$

we have $h^1(\mathcal{O}_{2E_2}(L)) = 2$. Consider the exact sequence

$$0 \to \mathcal{O}_M(L) \to \mathcal{O}_M(L+E_1) \to \mathcal{O}_{E_1}(L+E_1) \to 0.$$

As in the proof of Lemma 5.12,

$$\delta_4(X,x) \geq \dim_{\mathbb{C}} H^0(\mathcal{O}_M(L+E_1))/H^0(\mathcal{O}_M(L)) = 1 + h^1(\mathcal{O}_M(L+E_1)).$$

Since $h^1(\mathcal{O}_M(L+E_1)) \geq h^1(\mathcal{O}_{E_2}(L+E_1)) = 1$, we have $\delta_4(X,x) \geq 2$. $\square$

Proposition 5.14. *Let (X,x) be an elliptic du Bois singularity such that $E_2 \neq 0$. Then $\delta_3(X,x) \geq 2$ or $\delta_4(X,x) \geq 2$.*

Proof. Let A_1 be a curve in E_2 intersecting E_1. Then $h^1(\mathcal{O}_{E_1+A_1}) = 1$. Let (X', x') be the singularity obtained by contracting $E_1 + A_1$ in M. By Theorem 1.4, we have $p_g(X',x') \leq 1$. Hence $p_g(X',x') = h^1(\mathcal{O}_{E_1+A_1}) = 1$. By Proposition 4.6, the singularity (X',x') is an elliptic du Bois singularity. The result is an immediate consequence of Theorem 1.4, Lemma 5.12 and 5.13. $\square$

Theorem 5.15 (Okuma [O3]). *Let (X, x) be a singularity with $\delta_m(X, x) = 1$ for $m = 1, 4, 6$. Then (X, x) is a simple elliptic or a cusp singularity.*

Proof. By Lemma 1.10, $\delta_1(X, x) = \delta_6(X, x) = 1$ implies $\delta_3(X, x) = 1$. By Proposition 5.9, (X, x) is an elliptic du Bois singularity. Then Proposition 5.14 implies the assertion (cf.(5.11)). □

6. Equisingular deformations

(6.1) In this section, we discuss deformations. Let (X, x) be a singularity and $f\colon (M, A) \to (X, x)$ the minimal good resolution of (X, x). Let $A = \bigcup_{i=1}^{k} A_i$ be the decomposition into irreducible components. We denote by D_X the functor (cf. [Sc1]) of deformations of a singularity (X, x) (cf. [Wh2, 0]). In [Wh2], Wahl introduced the equisingular functor ES_M of deformations of (M, A) to which all A_i lift, and which blow down to deformations of (X, x). A deformation of the singularity (X, x) is called an equisingular deformation if it is obtained from an equisingular deformation of (M, A). It is well known that a deformations of M blows down if and only if $h^1(\mathcal{O}_M)$ does not jump (cf. [Wh2, (4.3)]). Hence equisingular deformations preserve the geometric genera and the weighted dual graphs of singularities, and so the plurigenera of Gorenstein and rational singularities by Theorem 3.12.

In [La2, La3, La4, La5], Laufer studied deformations of M in the analytic category. For a Gorenstein singularity (X, x), an equisingular deformation of (M, A) induces a topologically constant deformation of (X, x), and the converse holds, too (see [La5, V, VI]).

(6.2) Let $\Omega^1_M\langle A\rangle$ be the sheaf of 1-forms with logarithmic poles along A, and $\mathcal{S}$ its dual. Then there are exact sequences (cf. [Wh4]):

$$0 \to \Omega^1_M \to \Omega^1_M\langle A\rangle \to \bigoplus_{i=1}^{k} \mathcal{O}_{A_i} \to 0; \tag{6.2.1}$$

$$0 \to \mathcal{S} \to \Theta_M \to \bigoplus_{i=1}^{k} \mathcal{O}_{A_i}(A_i) \to 0; \tag{6.2.2}$$

$$0 \to \Theta_M(-A) \to \mathcal{S} \to \Theta_A \to 0. \tag{6.2.3}$$

By (6.2.2), We have the following exact sequence

$$0 \to H^1(\mathcal{S}) \to H^1(\Theta_M) \to H^1\left(\bigoplus_{i=1}^{k} \mathcal{O}_{A_i}(A_i)\right) \to 0.$$

There exists the versal deformation $\pi\colon \overline{M} \to (Q,o)$ of (M,A) with tangent space $T_{Q,o} \cong H^1(\Theta_M)$, and a submanifold (P,o) with tangent space $T_{P,o} \cong H^1(\mathcal{S})$ such that all of the A_i lift to above P and P is the maximal subspace of Q above which all of the A_i lift (cf. [La5, p. 26]).

Theorem 6.3 (Wahl [Wh2]). *(1) ES_M has a hull (in the sense of* [Scl]*) and the natural map $ES_M \to D_X$ is injective.*

(2) If any deformation of (M,A) to which all A_i lift blows down to a deformation of (X,x), then $T(ES_M) = H^1(\mathcal{S})$, where $T(ES_M)$ denotes the tangent space of ES_M. If $p_g(X,x) \le 1$, then this condition is satisfied.

(6.4) Let $B = \mathbb{C}\{z_1, \dots, z_n\}$. Let (X,x) be a q-h singularity defined by an ideal $I \subset B$. Let us recall that the tangent space T^1_X of D_X is given by the exact sequence

$$\mathrm{Hom}_R(\Omega^1_B \bigotimes R, R) \to \mathrm{Hom}_R(I/I^2, R) \to T^1_X \to 0,$$

where $R = B/I$. Since $\mathrm{Hom}_R(I/I^2, R)$ is graded, so is T^1_X: we write

$$T^1_X = \bigoplus_{i \in \mathbb{Z}} T^1_X(i).$$

Then we have the following.

Theorem 6.5 (Pinkham [P2, 4.6]). $T(ES_M) = \bigoplus_{i \ge 0} T^1_X(i)$.

Definition 6.6. A function $h \in \mathbb{C}\{z_0, z_1, z_2\} = \mathcal{O}_{\mathbb{C}^3,o}$ is called a quasi-homogeneous (q-h, for short) polynomial of degree d with weights $(\alpha_0, \alpha_1, \alpha_2) \in \mathbb{N}^3$, if

$$t^d h(z_0, z_1, z_2) = h(t^{\alpha_0} z_0, t^{\alpha_1} z_1, t^{\alpha_2} z_2)$$

for any $t \in \mathbb{C}$. We assume that α_0, α_1 and α_2 are relatively prime.

A function $h \in \mathcal{O}_{\mathbb{C}^3,o}$ is said to be semi-quasi-homogeneous (s-q-h, for short) of degree d with weights $(\alpha_0, \alpha_1, \alpha_2)$ if it is of the form $h = h_0 + h_1$, where h_0 is a q-h polynomial of degree d with weights $(\alpha_0, \alpha_1, \alpha_2)$ which defines an isolated singularity and all of the monomials of h_1 have degree strictly greater than d or $h_1 = 0$ (cf. [AGV, 12.1]). A singularity is said to be s-q-h if it is defined by a s-q-h function.

(6.7) Let $h \in \mathbb{C}\{z_0, z_1, z_2\} = \mathcal{O}_{\mathbb{C}^3,o}$ define an isolated singularity (X,o) at the origin. Let J_h be an ideal of $\mathcal{O}_{\mathbb{C}^3,o}$ generated by $\partial h/\partial z_0, \partial h/\partial z_1$ and $\partial h/\partial z_2$. $Q_h = \mathcal{O}_{\mathbb{C}^3,o}/J_h$ is called Jacobian algebra. Then we have (cf. [Sc2, §1])

$$T^1_X \cong \mathcal{O}_{\mathbb{C}^3,o}/(h, J_h).$$

It is well known that $(h, J_h) = J_h$ if and only if h is q-h (after a change of coodinates).

We assume that h is a q-h polynomial of degree d with weights $\alpha = (\alpha_0, \alpha_1, \alpha_2)$. Then α induces a grading on $\mathcal{O}_{\mathbb{C}^3,o}$, and so on Q_h. Let $Q_h = \bigoplus_{i\geq 0} Q_h(i)$. Recall that a morphism of graded modules $\varphi \in \mathrm{Hom}_{\mathcal{O}_X}((h)/(h^2), \mathcal{O}_X)$ has degree n if $\varphi(h)$ has degree $d+n$. Hence we have $T^1_X(i) \cong Q_h(i+d)$ (cf. (6.4)), and $T(ES_M) \cong \bigoplus_{i\geq d} Q_h(i)$. We see that a s-q-h singularity is a fibre in an equisingular deformation of a q-h singularity by Theorem 6.5 (cf. [AGV, Theorem 12.1]).

(6.8) We assume that the weighted dual graph of (X, x) is a star-shaped graph. Let A_0, D, P_i, e_i and d_i be as in (1.6) (note that they are defined for star-shaped graphs). Let us introduce some results of [TW].

We define a $\mathbb{Q}$-divisor C on A_0 as follows: $C = D - \sum_{i=1}^{\beta} q_i P_i$, where $q_i = e_i/d_i$. Let

$$R = \bigoplus_{n\geq 0} H^0(\mathcal{O}_{A_0}(nC))T^n \subset \mathbb{C}(A_0)[T],$$

where $\mathbb{C}(A_0)$ is the field of rational functions of A_0, and T an indeterminate. Then $\mathrm{Spec}(R)$ is a q-h normal surface singularity, we denote by (Y, y), and the weighted dual graph of (Y, y) is the same as that of (X, x) (cf. [P1]).

By contracting the branches $S_1 \bigcup \cdots \bigcup S_\beta$, we get a normal surface M' with cyclic quotient singularities. Let $\Phi\colon (M', A') \to (X, x)$ be the morphism induced canonically, where A' is the image of A_0. We define a filtration on $\mathcal{O}_X$ by

$$F^n = \Phi_* \mathcal{O}_{M'}(-nA')$$

for $n \in \mathbb{Z}$. Note that $F^n = \mathcal{O}_X$ for $n \leq 0$. Let

$$\mathcal{R} = \bigoplus_{n\in\mathbb{Z}} F^n T^n \quad \text{and} \quad G = \bigoplus_{n\geq 0} (F^n/F^{n+1})T^n.$$

Then the natural map

$$\mathbb{C}[T^{-1}] \to \mathcal{R}$$

defines a deformation of $\mathrm{Spec}(G)$ with general fibre isomorphic to (X, x), since $G \cong \mathcal{R}/T^{-1}\mathcal{R}$ and $\mathcal{O}_X \cong \mathcal{R}/(T^{-1} - a)\mathcal{R}$ for $a \in \mathbb{C} - \{0\}$ (cf. [TW, (5.15)]). By [TW,(6.3)], R is the normalization of G, and $R = G$ if and only if $p_g(Y, y) = p_g(X, x)$. By [Wh5, (1.12), (3.4)], (X, x) is a fibre in an equisingular deformation of (Y, y) if $p_g(Y, y) = p_g(X, x)$.

Proposition 6.9. *Let (X, x) be a minimally elliptic singularity with a star-shaped graph. Then there exist a q-h minimally elliptic singularity (Y, y) and an equisingular deformation $\pi: \overline{Y} \to \mathbb{C}$ of (Y, y) such that $X = \pi^{-1}(a)$ for $a \in \mathbb{C} - \{0\}$.*

Proof. We use the notation in (6.8). Since the weighted dual graph of (Y, y) is the same as that of (X, x), we see that (Y, y) is a minimally elliptic singularity by Proposition 5.4 and Definition 5.5. By Theorem 5.6, a minimally elliptic singularity is a Gorenstein singularity with $p_g = 1$. From (6.8), we have the assertion. □

(6.10) Under the same notation as above, if (X, x) is a hypersurface minimally elliptic singularity, then so is (Y, y) by [La1, Theorem 3.13]. By Proposition 6.9 and (6.7), a hypersurface minimally elliptic singularity with star-shaped graph is a s-q-h singularity.

7. Complete intersections

(7.1) We use the same notation as in Section 6. Let (X, x) be a Gorenstein singularity with contractible X. Let Z be a cycle such that $\mathcal{O}_M(K) \cong \mathcal{O}_M(-Z)$. If (X, x) is not a rational double point, then $Z \geq A$.

Let $\mathcal{C}$ be a sheaf on M defined by an exact sequence

$$0 \to \mathcal{C} \to \mathbb{C}_M \to \mathbb{C}_E \to 0.$$

If $Z \geq A$, then the exterior differentiation gives an exact sequence (cf. [Wh4, (1.5), (1.6)])

$$0 \to \mathcal{C} \to \mathcal{O}_M(-Z) \xrightarrow{d} \Omega^1_M\langle A\rangle(-Z) \xrightarrow{d} \Omega^2_M(-Z + A) \to 0. \tag{7.1.1}$$

As X is contractible, $H^i(\mathcal{C}) = 0$ for all i. Hence $H^i(\mathcal{O}_M(-Z)) \cong H^i(d\mathcal{O}_M(-Z))$ for all i. In particular, $H^i(d\mathcal{O}_M(-Z)) \cong H^i(\mathcal{O}_M(K)) = 0$ for $i \geq 1$.

(7.2) In the rest of this section, we always assume that (X, x) is a complete intersection singularity which is not a rational double point. Let $\mu(X, x)$ and $\tau(X, x)$ denote Milnor number and Tjurina number of (X, x), respectively (cf. [LS]). We need the following results of Greuel [Gr1, Gr2] (cf. [LS]).

Proposition 7.3. *(1) $\mu(X, x) = h^1_{\{x\}}(d\Omega^1_X)$, and $\tau(X, x) = h^1_{\{x\}}(\Omega^1_X)$* [Gr2, p. 168].

(2) $H^q_{\{x\}}(\Omega^p_X) = 0$ for $p + q \leq 1$ [Gr2, Proposition 2.3].

(3) The following sequences are exact [Gr1, Satz 4.4]*:*

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \to d\mathcal{O}_X \to 0;$$
$$0 \to d\mathcal{O}_X \to \Omega^1_X \to d\Omega^1_X \to 0.$$

(4) $H^0_{\{x\}}(d\Omega^1_X) = 0$ [Gr1, Lemma 4.5].

(7.4) From (7.1.1), we have an exact sequence

$$0 \to H^1_A(d\mathcal{O}_M(-Z)) \to H^1_A(\Omega^1_M\langle A\rangle(K)) \to H^1_A(\mathcal{O}_M(2K+A)) \\ \to H^2_A(d\mathcal{O}_M(-Z)) \to H^2_A(\Omega^1_M\langle A\rangle(K)).$$

By Corollary 2.13, we have $h^1_A(\mathcal{O}_M(2K+A)) = \delta_2(X,x)$. By the Serre duality, we have $h^1_A(\Omega^1_M\langle A\rangle(K)) = h^1(\mathcal{S})$. If we set

$$\rho = \dim_{\mathbb{C}} \ker \left(H^2_A(d\mathcal{O}_M(-Z)) \to H^2_A(\Omega^1_M\langle A\rangle(K))\right),$$

then we have

$$\delta_2(X,x) = h^1(\mathcal{S}) + \rho - h^1_A(d\mathcal{O}_M(-Z)). \tag{7.4.1}$$

We note that $h^1_A(d\mathcal{O}_M(-Z)) \le h^1(\mathcal{S})$.

Let $U = M - A \cong X - \{x\}$.

Lemma 7.5. $h^1_A(d\mathcal{O}_M(-Z)) = h^1_{\{x\}}(d\mathcal{O}_X) + p_g(X,x) - 1$.

Proof. From the exact sequence

$$0 \to H^0(d\mathcal{O}_M(-Z)) \to H^0(d\mathcal{O}_U) \to H^1_A(d\mathcal{O}_M(-Z)) \to 0,$$

and isomorphisms

$$H^0(d\mathcal{O}_M(-Z)) \cong H^0(\mathcal{O}_M(K)) \cong H^0(f_*\mathcal{O}_M(K)),$$

we see that

$$H^1_A(d\mathcal{O}_M(-Z)) \cong H^0(d\mathcal{O}_U)/H^0(f_*\mathcal{O}_M(K)). \tag{7.5.1}$$

Using (2) and (3) of Proposition 7.3, we obtain $H^0_{\{x\}}(d\mathcal{O}_X) = 0$ and hence

$$H^1_{\{x\}}(d\mathcal{O}_X) \cong H^0(d\mathcal{O}_U)/H^0(d\mathcal{O}_X). \tag{7.5.2}$$

Let $\mathcal{M}$ be an ideal sheaf of $\mathcal{O}_X$ which defines the singular point x. We define a sheaf $\mathcal{C}'$ on X by the exact sequence

$$0 \to \mathcal{C}' \to \mathbb{C}_X \to \mathbb{C}_{\{x\}} \to 0.$$

Then we have

$$0 \to \mathcal{C}' \to \mathcal{M} \to d\mathcal{M} \to 0.$$

It is a subcomplex of

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \to d\mathcal{O}_X \to 0.$$

Note that $d\mathcal{O}_X \cong d\mathcal{M}$. Since X is contractible, we have

$$H^0(\mathcal{M}) \cong H^0(d\mathcal{M}) \cong H^0(d\mathcal{O}_X). \tag{7.5.3}$$

As (X, x) is a Gorenstein singularity with $p_g(X, x) \geq 1$, we have $f_*\mathcal{O}_M(K) \subset \mathcal{M}$. It is well known that

$$p_g(X, x) = \dim_{\mathbb{C}} H^0(\mathcal{O}_X)/H^0(f_*\mathcal{O}_M(K))$$

for a Gorenstein singularity (X, x). From (7.5.1), (7.5.2) and (7.5.3), we have the following

$$\begin{aligned} h^1_A(d\mathcal{O}_M(-Z)) - h^1_{\{x\}}(d\mathcal{O}_X) &= \dim_{\mathbb{C}} H^0(d\mathcal{O}_X)/H^0(f_*\mathcal{O}_M(K)) \\ &= \dim_{\mathbb{C}} H^0(\mathcal{M})/H^0(f_*\mathcal{O}_M(K)) = p_g(X, x) - 1. \qquad \square \end{aligned}$$

Lemma 7.6. $\rho = \mu(X, x) - \tau(X, x) + h^1_{\{x\}}(d\mathcal{O}_X)$.

Proof. Since $H^1(d\mathcal{O}_M(-Z)) = H^2(d\mathcal{O}_M(-Z)) = 0$, we have

$$H^2_A(d\mathcal{O}_M(-Z)) \cong H^1(d\mathcal{O}_U) \cong H^2_{\{x\}}(d\mathcal{O}_X).$$

By the vanishing theorem of Wahl [Whl], $H^1(\Omega^1_M\langle A\rangle(K)) = 0$. Similarly, we get

$$H^2_A(\Omega^1_M\langle A\rangle(K)) \cong H^2_{\{x\}}(\Omega^1_X).$$

Then

$$\rho = \dim_{\mathbb{C}} \ker\left(H^2_{\{x\}}(d\mathcal{O}_X) \to H^2_{\{x\}}(\Omega^1_X)\right).$$

From Proposition 7.3, $H^0_{\{x\}}(d\Omega^1_X) = 0$ and we have an exact sequence

$$\begin{aligned} 0 \to H^1_{\{x\}}(d\mathcal{O}_X) \to H^1_{\{x\}}(\Omega^1_X) \to H^1_{\{x\}}(d\Omega^1_X) \\ \to H^2_{\{x\}}(d\mathcal{O}_X) \to H^2_{\{x\}}(\Omega^1_X), \end{aligned}$$

and hence $\rho = \mu(X, x) - \tau(X, x) + h^1_{\{x\}}(d\mathcal{O}_X)$. $\square$

Theorem 7.7 (Okuma [O2]). $\delta_2(X,x) = h^1(\mathcal{S}) + \mu(X,x) - \tau(X,x) - p_g(X,x) + 1$.

Proof. The theorem is immediately obtained from (7.4.1), Lemma 7.5 and Lemma 7.6. □

Corollary 7.8. *Let $\pi\colon \overline{X} \to T$ be an equisingular deformation of (X,x). We set $X_t = \pi^{-1}(t)$ for $t \in T$. Then*

$$\tau(X_t) \geq \mu(X,x) - \delta_2(X,x) \text{ for any } t \in T. \tag{7.8.1}$$

In particular, if $p_g(X,x) = 1$, then $\tau(X_t) \geq \mu(X,x) - 5$.

Proof. We note that X_t is a complete intersection isolated singularity for any $t \in T$ (cf. [KS]). From (7.4) and Lemma 7.5, $h^1(\mathcal{S}) \geq p_g - 1$. By Theorem 7.7, we have that $\delta_2(X_t) \geq \mu(X_t) - \tau(X_t)$. By Theorem 3.12, δ_2 is determined by p_g and the weighted dual graph of the singularity, and so is μ by [St, (2.26)]. The property of the equisingular deformations implies that

$$\delta_2(X_t) = \delta_2(X,x) \quad \text{and} \quad \mu(X_t) = \mu(X,x).$$

Then we get (7.8.1). If $p_g(X,x) = 1$, then $\delta_2(X,x) \leq 5$ by Corollary 3.14. □

(7.9) For the remainder of this section, (X,o) denotes a hypersurface singularity defined by a function $h \in \mathbb{C}\{z_0, z_1, z_2\} = \mathcal{O}_{\mathbb{C}^3,o}$. It is well known that

$$\mu(X,o) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^3,o}/J_h \text{ and } \tau(X,o) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^3,o}/(J_h, h),$$

and that $\mu(X,o) = \tau(X,o)$ if and only if h is q-h (after a change of coordinates).

We set $\mu = \mu(X,o)$. Let $\varphi_1, \dots, \varphi_\mu$ be functions in $\mathcal{O}_{\mathbb{C}^3,o}$ which induce $\mathbb{C}$-basis of $\mathcal{O}_{\mathbb{C}^3,o}/J_h$. Then we define a function $H(z,t) \in \mathbb{C}\{z_0, z_1, z_2, t_1, \dots, t_\mu\} = \mathcal{O}_{\mathbb{C}^3 \times \mathbb{C}^\mu, o}$ as following

$$H(z,t) = h + \sum_{i=1}^{\mu} t_i \varphi_i,$$

and we set

$$Y(X,o) = \{ (t_0) \in (\mathbb{C}^\mu, o) \mid \mu(H(z,t_0)) = \mu \},$$

where $\mu(H(z,t_0))$ denotes Milnor number of the singularity defined by $H(z,t_0)$. Then $Y(X,o)$ is an analytic subset of $(\mathbb{C}^\mu, o)$.

Definition 7.10. The modality $m(X,o)$ of the singularity (X,o) is the dimension of $Y(X,o)$ (cf. [Ga]). If (X,o) is defined by a quasi-homogeneous polynomial h of degree d, then the inner modality $m_0(X,o)$ of the singularity (X,o) is defined as the dimension of the vector space $\bigoplus_{i\geq d} Q_h(i)$ (cf. [YW]). Note that $m_0(X,o) \leq m(X,o)$ if (X,o) is a q-h singularity (see the proof of the follow).

Proposition 7.11 (Okuma [O2]). *If $p_g(X,o)=1$, then $\delta_2(X,o) \leq m(X,o)$.*
If (X,o) is a q-h singularity, then $\delta_2(X,o) = m_0(X,o) \leq 4$.

Proof. Let $(\mathbb{C}^{\tau(X,o)}, o)$ be the versal deformation space of the singularity (X,o) and

$$p\colon (\mathbb{C}^{\mu(X,o)}, o) \to (\mathbb{C}^{\tau(X,o)}, o)$$

be a projection corresponding to the natural map of the tangent spaces

$$\mathcal{O}_{\mathbb{C}^3,o}/J_h \to \mathcal{O}_{\mathbb{C}^3,o}/(J_h, h).$$

There is a submanifold P of $(\mathbb{C}^{\tau(X,o)}, o)$ which represents ES_M. By the property of the equisingular deformations, $p^{-1}(P) \subset Y(X,o)$. By Theorem 6.3, we see that the dimension of $p^{-1}(P)$ is $h^1(\mathcal{S}) + \mu(X,o) - \tau(X,o)$. Hence

$$h^1(\mathcal{S}) + \mu(X,o) - \tau(X,o) \leq m(X,o).$$

From Theorem 7.7, we get $\delta_2(X,o) \leq m(X,o)$.

We assume that h is a q-h polynomial of degree d. Then Theorem 7.7 and 6.3, and (6.7) implies that

$$\delta_2(X,o) = h^1(\mathcal{S}) = \dim_{\mathbb{C}} \bigoplus_{i\geq d} Q_h(i) = m_0(X,o).$$

By Corollary 3.14, $\delta_2(X,o) \leq 4$. $\square$

Remark 7.12. If the invariance of Milnor number implies the invariance of the topological type for two dimensional hypersurface singularities (cf. [LR]), then, in the proof above, we have $p^{-1}(P) = Y(X,o)$ (cf. (6.1)). In this case, $Y(X,o)$ is nonsingular, and $\delta_2(X,o) = m(X,o)$ holds.

It is known that for any q-h hypersurface singularity (X,o), an inequality $\delta_2(X,o) \geq m_0(X,o)$ holds (see [YW]).

Proposition 7.13 (Okuma [O2]). *Let (X, o) be a singularity defined by a s-q-h function $h \in \mathcal{O}_{\mathbb{C}^3,o}$ with weights $(1,1,1)$. Then $\delta_2(X, o) \geq m(X, o)$.*

Proof. We write $h = h_0 + h_1$ as in Definition 6.6. Let (X_0, o) be a singularity defined by h_0. Then by [GK], $m_0(X_0, o) = m(X_0, o)$. Hence we have that $\delta_2(X_0, o) \geq m(X_0, o)$ by [YW]. On the other hand, (X, o) is a fibre in an equisingular deformation of (X_0, o) by (6.7). Thus $\delta_2(X, o) = \delta_2(X_0, o)$. Since the modality is upper semi-continuous by [Ga], we have

$$\delta_2(X, o) = \delta_2(X_0, o) \geq m(X_0, o) \geq m(X, o). \qquad \square$$

Proposition 7.14 (Okuma [O2]). *Let (X, o) be a hypersurface singularity with $p_g(X, o) = 1$ such that the weighted dual graph of it is a star-shaped graph. Then (X, o) is defined by a s-q-h function of which the quasi-homogeneous part defines a singularity (X_0, o) with $m_0(X_0, o) = \delta_2(X, o)$.*

In particular, for such a singularity with $\delta_2(X, o) \leq 2$, we have $\delta_2(X, o) = m(X, o)$.

Proof. We know that (X, o) is a s-q-h singularity by (6.10). Q-h hypersurface singularities with $p_g = 1$ and $m_0 \leq 4$ are listed in [YW]. The lists of all the singularities for which $m \leq 2$ are given in [AGV, 15.1]. Then we see that for a s-q-h function of which the q-h part has inner modality $m_0 \leq 2$, we have $m = m_0$. By Proposition 7.11, using the notation of Proposition 7.13, if $\delta_2(X, o) \leq 2$ then we have

$$m(X, o) = m_0(X_0, o) = \delta_2(X_0, o) = \delta_2(X, o). \qquad \square$$

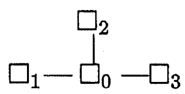
(7.15) From Proposition 6.9 and 7.14, we see that a minimally elliptic singularity with a star-shaped graph is easy to deal. However there exist minimally elliptic singularities of which the weighted dual graphs are not star-shaped.

We classify the weighted dual graphs of minimally elliptic singularities with $\delta_2 \leq 2$. In the following, the symbol " ◯ " corresponds to a component with self-intersection number -2 and " $\square_i$ " corresponds to a component A_i. We set $b_i = -A_i \cdot A_i$ and $t_i = (A - A_i) \cdot A_i$.

Proposition 7.16. *Let (X, x) be a minimally elliptic singularity with $\delta_2(X, x) \leq 2$.*

(1) *If $\delta_2(X, x) = 1$ if and only if (X, x) is a simple elliptic, cusp singularity or a singularity with the weighted dual graph*

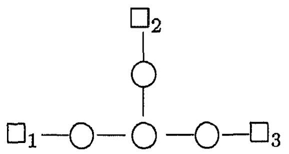
D_{b_1,b_2,b_3} :



Where $b_0 = 1 < b_1 \leq b_2 \leq b_3$ *and* $1/b_1 + 1/b_2 + 1/b_3 < 1$.

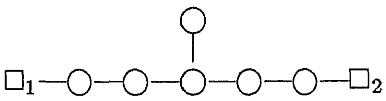
(2) *If* $\delta_2(X, x) = 2$ *if and only if the weighted dual graph of* (X, x) *is one of the following.*

$\tilde{E}_6$:



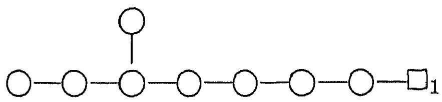
$2 \leq b_1 \leq b_2 \leq b_3, 2 < b_3$

$\tilde{E}_7$:



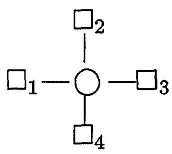
$2 \leq b_1 \leq b_2, 2 < b_2$

$\tilde{E}_8$:



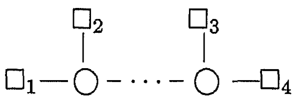
$2 < b_1$

$\tilde{D}_4$:



$2 \leq b_1 \leq b_2 \leq b_3 \leq b_4, 2 < b_4$

$\tilde{D}_{i+4}$ $(i \geq 1)$:



$2 \leq b_1 \leq b_2, 2 \leq b_3 \leq b_4, 2 < b_4$
The number of " ○ " *is* $i + 1$.

(3) *The list of the* (b_i) *corresponding to a hypersurface is the following.*

type	(b_i)
D_{b_1,b_2,b_3}	(2.3.7), (2.3.8), (2.3.9), (2.4.5), (2.4.6), (2.4.7), (2.5.5),(2.5.6) (3.3.4), (3.3.5), (3.3.6), (3.4.4), (3.4.5), (4.4.4)
$\tilde{E}_6$	(2.2.3), (2.2.4), (2.2.5), (2.3.3), (2.3.4), (3.3.3),
$\tilde{E}_7$	(2.3), (2.4), (2.5), (3.3), (3.4)
$\tilde{E}_8$	(3), (4), (5)
$\tilde{D}_4$	(2.2.2.3), (2.2.2.4), (2.2.2.5), (2.2.3.3) (2.2.3.4), (2.3.3.3)
$\tilde{D}_{i+4}$ $(i \geq 1)$	(2.2.2.3), (2.2.2.4), (2.2.2.5), (2.2.3.3) (2.3.2.3), (2.2.3.4), (2.3.2.4), (2.3.3.3)

Proof. From the proof of Corollary 3.14 and Lemma 5.8, we have (1). In (1), the inequality $1/b_1 + 1/b_2 + 1/b_3 < 1$ is the condition for the intersection matrix to be negative definite.

If $\delta_2(X,x) \neq 1$, then the minimal good resolution is minimal and weighted dual graph is a tree of rational curves (cf. [La1, Proposition 3.5]). Assume that $\delta_2 = 2$. From the equality (3.14.1), $\delta_2 = 2$ if and only if $K \cdot L_1 = 0$. Let Z be a fundamental cycle. Note that $K = -Z$. We set $Z = \sum a_i A_i$ and $D = Z - A$. Then we have $K \cdot (K + A) = Z \cdot D$ and

(7.16.1) $$\sum (a_i - 1)(b_i - 2) = 0.$$

Since K is nef and not trivial, $b_i \geq 2$ for all i and $b_j \geq 3$ for some j. Note that $a_i \geq 1$ for all i. By (7.16.1), if $b_j \geq 3$ then $a_j = 1$ and if $a_i \geq 2$ then $b_i = 2$. Assume $b_1 \geq 3$. Then $a_1 = 1$. Let C be the minimal chain of curves containing A_1 such that C intersects D and any curve in C is not contained in D. There exists such a chain since A is connected. For any curve A_i in C, we have $a_i = 1$. Let A_j be a curve in C intersecting a curve A_{j+1} in D. Then we have $K \cdot A_j = -A_j \cdot A_j - 2 = b_j - 2$ and $Z \cdot A_j = (D + A) \cdot A_j = D \cdot A_j + t_j - b_j$. Since $K = -Z$, we have $D \cdot A_j + t_j = 2$. Since $D \cdot A_j$ and t_j are positive, we have $t_j = 1$ and hence $A_j = A_1$. Then

$$-b_1 + 2 = Z \cdot A_1 = (a_2 A_2 + A_1) \cdot A_1 = a_2 - b_1.$$

Hence $a_2 = b_2 = 2$. Thus we see that if a curve A_j with $a_j = 1$ intersects a curve A_i then $t_j = 1$ and $a_i = 2$. Let A_1, $A_3, \dots, A_n$ be all of curves intersecting A_2. Then we have

$$0 = Z \cdot A_2 = a_1 - a_2 b_2 + \sum_{i=3}^{n} a_i = -3 + \sum_{i=3}^{n} a_i.$$

If $a_3 = a_4 = a_5 = 1$, then we have the type $\tilde{D}_4$ (after a change of suffices). Assume $a_3 = 1$ and $a_4 = 2$. Then $t_3 = 1$ and $b_4 = 2$. Let $A_2, A_5, \dots, A_m$ be all of curves itersecting A_4. Then

$$0 = Z \cdot A_4 = a_2 - a_4 b_4 + \sum_{i=5}^{m} a_i = -2 + \sum_{i=5}^{m} a_i.$$

If $a_5 = a_6 = 1$, then we have $\tilde{D}_5$. If $a_5 = 2$, then we apply the same methods as above. After all, we have the type $\tilde{D}_{i+4}$. Assume $a_3 = 3$ and that $A_2, A_4, \dots, A_n$ are all of curves intersecting A_3. As above, we have $\sum_{i=4}^{n} a_i = 4$. Since a curve A_i with

$a_i = 1$ must intersect a curve A_j with $a_j = 2$, we have $a_4 = a_5 = 2$ or $a_4 = 4$. If $a_4 = a_5 = 2$, then we have the type $\tilde{E}_6$. If $a_4 = 4$, then we go on as above. After all, we have the types $\tilde{E}_7$ and $\tilde{E}_8$.

A singularity whose weighted dual graph is described in (2) is a minimally elliptic singularity (cf. Definition 5.5). We have $\delta_2(X, x) = 2$ for these singularities by (3.14.1).

The list of (3) is computed from the condition [La1, Theorem 3.13]

$$-3 \leq Z \cdot Z \leq -1. \qquad \square$$

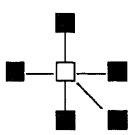
Corollary 7.17. *Let (X, o) be a hypersurface singularity of type D_{b_1,b_2,b_3}, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$ or $\tilde{D}_4$. Then $m(X, o) = \delta_2(X, o)$.*

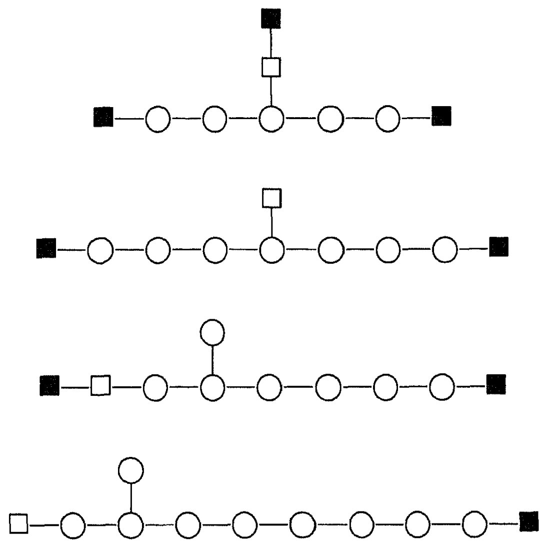
Proof. It follows from Proposition 7.14 and 7.16. □

(7.18) Hypersurface singularities with modality equal to 1 and 2 are said to be unimodal and bimodal, respectively. For hypersurface singularities, D_{b_1,b_2,b_3} correspond to 14 exceptional families of unimodal singularities, and $\tilde{E}_6$, $\tilde{E}_7$ and $\tilde{E}_8$ correspond to 14 exceptional families of bimodal singularities, and $\tilde{D}_4$ correspond to the functions listed in [AGV, 15.1] of type $J_{3,0}$, $Z_{1,0}$, $W_{1,0}$, $Q_{2,0}$, $S_{1,0}$ and $U_{1,0}$ in 8 infinite series of bimodal singularities. The weighted dual graphs of 8 infinite series of bimodal singularities are of type $\tilde{D}_4$ or $\tilde{D}_{i+4}$ (cf. [La2, IV]). We note that minimally elliptic singularities with $\delta_2 \leq 2$ are Kodaira singularities (cf. [Kr]).

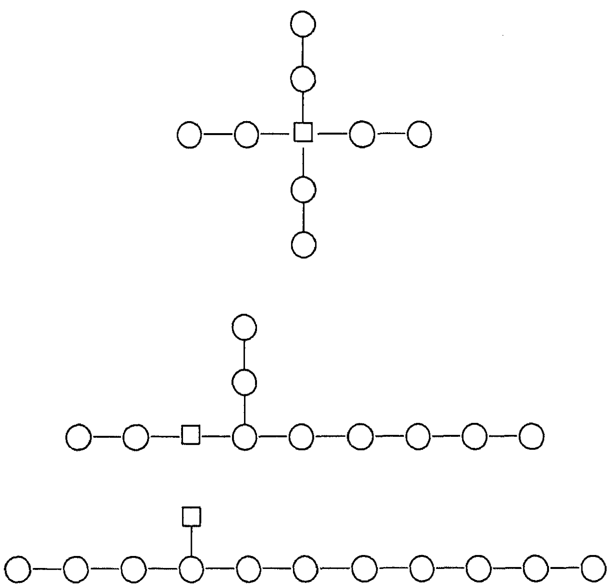
As in the proof of Proposition 7.16, we can list the weighted dual graphs of minimally elliptic singularities with a given second plurigenus (if we make exertion). The following are the list of weighted dual graphs of hypersurface minimally elliptic singularities with star-shaped graphs and $\delta_2 \geq 3$. Recall that $\delta_2 \leq 4$ (see Corollary 3.14). In the following, the symbol " □ " corresponds to a component with self-intersection number -3 and " ■ " corresponds to a component with self-intersection number -2 or -3 and the number of components with self-intersection number -3 corresponding to ■ is at most one in each weighted dual graph.

The case of $\delta_2 = 3$.





The case of $\delta_2 = 4$.



(7.19) As [P1, 6], we can get a q-h polynomial from a weighted dual graph of a hypersurface singularity listed in Proposition 7.16 or (7.18). Those polynomials are the same as those of [YW, §5]. The parameters in the polynomial correspond to the moduli of the intersection points on the central curve. Note that all of the functions which define hypersurface minimally elliptic singularities with star-shaped graphs are gainable as (6.10).

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