ON THE NORMALITY OF PRODUCT SPACES

Keiko Chiba

THESIS

Submitted in partial fulfillment of the requirements for the degree of Doctor of Science in Doctoral Program in Mathematics in the

University of Tsukuba

June 1985

CONTENTS

0. Conventions

Throughout this paper all spaces are assumed to be Hausdorff spaces and maps continuous.

We shall use the Greek letters α, λ, κ,... to denote ordinal numbers. These same Greek letters will also stand for cardinal numbers. A cardinal is an initial ordinal and an ordinal is the set of its predecessors. Thus $\alpha \in \lambda$ and $\alpha < \lambda$ means the same thing, and ω denotes the first infinite ordinal and ω_1 the first uncountable ordinal.

For a set A, $|A|$ denotes the cardinal of A, $P(A)$ is the set of all subsets of A and A^f the collection of all finite subsets of A.

For a subset A in a space X, $\mathrm{Cl}A$ or $\mathrm{Cl}(A)$ denotes the closure of A in X.

For an ordinal λ, we denote by $W(\lambda)$ the space λ with the order topology.

CHAPTER 0

INTRODUCTION

The product of Hausdorff, regular or completely regular spaces are always Hausdorff, regular or completely regular, respectively. However, the product of two normal spaces need not be normal (cf. [M_3], [S_2]). Therefore it is important to decide whether $X \times Y$ is normal or not for given two normal spaces X and Y.

Further, in case $\mathcal{C}$ is a class of normal spaces, it is an interesting problem to find a necessary and sufficient condition of X to satisfy the following condition : $X \times Y$ is normal for any space Y of $\mathcal{C}$.

C. H. Dowker was a pioneer of this aspect. He proved in 1951 the following.

(i) (C. H. Dowker [D_2 , 1951]). A space X is countably paracompact and normal if and only if $X \times Y$ is normal for any compact metric space Y.

After C. H. Dowker's initial work, several remarkable results were obtained.

(ii) (H. Tamano [T_1 , 1962]). A space X is paracompact if and only if $X \times Y$ is normal for any compact space Y.

(iii) (K. Morita [M_7 , 1964]). A space X is a normal P-space if and only if $X \times Y$ is normal for any metrizable space Y.

(iv) (M. Atsuji [A_2 , 1976] and M. E. Rudin [R_4 , 1978]). A space X is discrete if and only if $X \times Y$ is normal for any

normal space Y.
Moreover, concerning (ii) Y. Katuta [K_2 , 1971] characterized the space X for which $X \times Y$ is normal for any paracompact space Y. Further, concerning (i) and (iii) K. Morita conjectured the following (cf. [M_8]).

Morita's Conjecture I. A space X is metrizable if and only if $X \times Y$ is normal for any normal P-space Y.

Morita's Conjecture II. A space X is metrizable and σ-locally compact if and only if $X \times Y$ is normal for any normal and countably paracompact space Y.

It was shown that the Morita's Conjectures I and II are true for every separable space X ([CC_2], [M_6] and [M_7]).

In chapter 1 of this paper we shall prove that the above Morita's Conjectures I and II are true for every M-space X.

We shall also investigate the space X whose product $X \times Y$ with any perfectly normal space Y is normal. Further, in § 3 in chapter 1, we shall study the normality of $X \times Y$ when X is a normal M-space. Every perfectly normal space and every normal M-space are normal P-spaces.

In chapter 2, we shall study the collectionwise normality of product spaces. When X and Y are collectionwise normal spaces, the product space $X \times Y$ is not necessarily collectionwise normal (even normal). In this paper we shall find a sufficient condition of X and Y to be $X \times Y$ collectionwise normal. And, under the assumption of " $X \times Y$ is normal " , we shall find some sufficient conditions of X and Y to be $X \times Y$ collectionwise normal.

In discussing the normality of product spaces, M. Atsuji $[A_2]$ defined the notion of *the property* $\mathcal{B}^*(\kappa)$ and M. E. Rudin $[R_4]$ defined *a* κ*-Dowker space* for each infinite cardinal κ. A normal space X is called a κ-Dowker space iff X has not the property $\mathcal{B}^*(\kappa)$. By considering these notions, they proved Theorem (iv).

On the other hand, Y. Yasui defined *the weak* $\mathcal{B}$*-property*. *A space* X *is said to have the weak* $\mathcal{B}$*-property iff* X *has the property* $\mathcal{B}^*(\kappa)$ *for every infinite cardinal* κ ($[Y_4]$).

M. E. Rudin proposed in $[R_7]$ that we call the weak $\mathcal{B}$-property, the $\mathcal{D}$-property. But we shall use the word " the weak $\mathcal{B}$-property " in this paper. In chapter 3, we shall obtain the fundamental properties of the weak $\mathcal{B}$-property.

In chapter 4, we shall study the Σ-products. H. H. Corson defined in $[C_{10}]$ an important and useful class of subspaces of product spaces, called Σ-products. Many topologists studied Σ -products. In particular, the following result is remarkable.

(v) Every Σ-product of metrizable spaces is normal (S. P. Gul'ko [G], or M. E. Rudin $[R_3]$).

However, let us consider some other properties except normality possessed by Σ-products. Every Σ-product which is not a product space cannot be paracompact ($[C_{10}]$). Similarly such a Σ-product cannot be metacompact, cannot be subparacompact (see Proposition 4.3 in chapter 4). Further, it is easy to see that such a

Σ-product cannot be perfectly normal. However, many Σ-products are countably paracompact (see §1 in chapter 4). Moreover, the following relations hold (cf. chapter 3).

normal metacompact ↘
normal subparacompact → weak $\mathcal{B}$ → countably paracompact
perfectly normal ↗

Therefore it seems interesting to investigate the weak $\mathcal{B}$-property of Σ-products. In chapter 4, concerning this, we shall obtain several results.

All the results in this paper except several results in chapter 3 (Lemma 3.4, Theorem 3.7 and Corollary 3.4) have been stated in [C_2], [C_4], [C_5], [C_6], [C_7] and [C_8].

The author wishes her gratitude to Prof. K. Morita and Prof. Y. Kodama for their kind advices and guidances on the occation of writting this paper.

CHAPTER 1

NORMALITY OF PRODUCT SPACES AND MORITA'S CONJECTURES

1. Morita's Conjectures

In [M_7] K. Morita introduced the notion of P-spaces[1)] and proved that a space X is a normal P-space if and only if $X \times Y$ is normal for any metrizable space Y.

He also conjectured in [M_8] that the converse of this result is true :

Morita's Conjecture I. A space X is metrizable if and only if $X \times Y$ is normal for any normal P-space Y.

In case X is separable the above conjecture holds ([CC_2], [M_7]).

A related result of K. Morita states that a metrizable space X is σ-locally compact if and only if $X \times Y$ is normal for any normal and countably paracompact space Y ([M_6, Theorem 3.2]).

---

1) A space X is called a P-space if for any open collection $\{G(\alpha_1,\dots,\alpha_i) \mid \alpha_1,\dots,\alpha_i \in \Omega,\ i = 1,2,\dots\}$ such that $G(\alpha_1,\dots,\alpha_i) \subset G(\alpha_1,\dots,\alpha_i,\alpha_{i+1})$ for $\alpha_1,\dots,\alpha_i,\alpha_{i+1} \in \Omega$; $i = 1,2,\dots,$ there is a closed collection $\{F(\alpha_1,\dots,\alpha_i) \mid \alpha_1,\dots,\alpha_i \in \Omega,\ i = 1, 2,\dots\}$ satisfying the two conditions below :

i) $F(\alpha_1,\dots,\alpha_i) \subset G(\alpha_1,\dots,\alpha_i)$ for $\alpha_1,\dots,\alpha_i \in \Omega$; $i = 1, 2,\dots$;

ii) for any sequence $\{\alpha_i \mid i = 1, 2,\dots\}$ such that $X = \bigcup_{i=1}^{\omega} G(\alpha_1,\dots,\alpha_i)$, $X = \bigcup_{i=1}^{\omega} F(\alpha_1,\dots,\alpha_i)$ holds.

Further, K. Morita conjectured that the assumption of metrizability of X can be omitted :

Motita's Conjecture II. A space X is metrizable and σ-locally compact if and only if $X \times Y$ is normal for any normal and countably paracompact space Y.

Since every normal P-space is countably paracompact [M_7], whenever the characterization in Conjecture I holds for a given space X, the corresponding characterization in Conjecture II holds for that space, too.

In this chapter we shall prove that the above Conjectures I and II are true for any M-space[2] X.

THEOREM 1.1. Let X be an M-space. Then the following are equivalent.

(a) X is metrizable.

(b) $X \times Y$ is normal for any normal P-space Y.

THEOREM 1.2. Let X be an M-space. Then the following are equivalent.

(a) X is metrizable and σ-locally compact.

(b) $X \times Y$ is normal for any normal and countably paracompact space Y.

To prove Theorems 1.1 and 1.2, we first prove the following.

---

2) A space X is called an M-space [M_7] if there exists a sequence $\{\mathcal{U}_n\}_{n=1}^{\omega}$ of locally finite open covers of X satisfying the following condition: If $\{K_n\}_{n=1}^{\omega}$ is a decreasing sequence of non-empty closed sets in X such that $K_n \subset \cap\{U \mid x \in U \in \mathcal{U}_n\}$ for each n = 1,2,... and some $x \in X$, then $\bigcap_{n=1}^{\omega} K_n \neq \emptyset$.

THEOREM 1.3. Let X be a space which is separable or compact or Fréchet. If $X \times Y$ is normal for any perfectly normal space Y, then X is perfectly normal.

A space X is called *a Fréchet space* if, whenever $x \in \mathrm{Cl}A$ in X, then there is a sequence $\{x_n \mid n = 1,2,\ldots\}$ in A which converges to x (A. V. Arhangel'skii [A_1]). The definition of M-spaces is due to Morita [M_7].

To prove Theorem 1.3, we need some lemmas.

Let λ be an infinite cardinal number. $\mathrm{cf}\lambda$ means the cofinality of λ.

A space X is called *a λ-Lindelöf space* if from every open covering $\{U_\alpha \mid \alpha \in \Omega\}$ of the space X with $|\Omega| \leqq \lambda$ a countable subcover $\{U_{\alpha_i} \mid i = 1,2,\ldots\}$ can be chosen.

LEMMA 1.1. Let λ be a cardinal number such that $\mathrm{cf}\lambda < \lambda$. If X is hereditarily μ-Lindelöf for each $\mu < \lambda$, then X is hereditarily λ-Lindelöf.

Proof. Of course we may assume $\lambda > \omega$. Let A be an arbitrary subspace of X. Let $\mathcal{U}$ be an arbitrary open cover of A such that $|\mathcal{U}| \leqq \lambda$. Let us put $\mathcal{U} = \{U_\alpha \mid \alpha < \lambda\}$. Put $W_\beta = \bigcup\{U_\alpha \mid \alpha < \beta\}$ for each $\beta < \lambda$. Since $\mathrm{cf}\lambda < \lambda$, there is a cofinal set Γ of λ such that $|\Gamma| < \lambda$. Then it is clear that $\bigcup\{W_\beta \mid \beta \in \Gamma\} = A$. If we put $|\beta| = \mu_\beta$, then $\mu_\beta < \lambda$ and therefore W_β is μ_β-Lindelöf. Hence the cover $\{U_\alpha \mid \alpha < \beta\}$ of W_β has a countable subcover $\{U_\alpha \mid \alpha \in \Omega_\beta\}$. Then $\bigcup\{U_\alpha \mid \alpha \in \bigcup_{\beta\in\Gamma}\Omega_\beta\} = A$. Let us put $|\bigcup_{\beta\in\Gamma} \Omega_\beta| = \mu$. Then A is

μ-Lindelöf, because $\mu \leqq \omega \cdot |\Gamma| < \lambda$. Therefore A has a countable subcover of $\{U_\alpha \mid \alpha \in \bigcup_{\beta \in \Gamma} \Omega_\beta\}$. Hence A is λ-Lindelöf. The proof of Lemma 1.1 is complete.

LEMMA 1.2. Let X be a space which is μ-Lindelöf for each $\mu < \lambda$ and not λ-Lindelöf. Then there exists a subset $\tilde{X}$ of X such that

(1) $|\tilde{X}| = \lambda$,

(2) there is an open neighborhood B_x of x in X such that $|B_x \cap \tilde{X}| < \lambda$ for each $x \in X$.

Proof. Let X be a space which is μ-Lindelöf for each $\mu < \lambda$ and not λ-Lindelöf. Let $\mathcal{U} = \{U_\alpha \mid \alpha < \lambda\}$ be an open cover of X which has not a countable subcover. Then clearly $|X - \bigcup\{U_\beta \mid \beta < \alpha\}| = \lambda$ for each $\alpha < \lambda$.

By transfinite induction, we can choose $x_\alpha \in X$ such that $x_\alpha \in X - (\bigcup\{U_\beta \mid \beta < \alpha\} \cup \{x_\beta \mid \beta < \alpha\})$ for each $\alpha < \lambda$. Let us put $\tilde{X} = \{x_\alpha \mid \alpha < \lambda\}$. Then $\tilde{X}$ has the required properties. To show this, let x be an arbitrary element of X. Then $x \in U_\alpha$ for some $\alpha < \lambda$. Since $U_\alpha \cap \tilde{X} \subset \{x_\beta \mid \beta \leqq \alpha\}$, $|U_\alpha \cap \tilde{X}| < \lambda$. Hence Lemma 1.2 has been proved.

The following lemma is well known.

LEMMA 1.3 (Šanin [J]). Let λ be an uncountable regular cardinal number. Let P be a set such that $|P| = \lambda$ and $\{r_p \mid p \in P\}$ be a collection of finite sets. Then there exists a subset $\tilde{P}$ of P such that

(1) $|\widetilde{P}| = \lambda$,

(2) $\{r_p - \widetilde{r} \mid p \in \widetilde{P}\}$ is disjoint where $\widetilde{r} = \cap\{r_p \mid p \in \widetilde{P}\}$.

PROOF OF THEOREM 1.3. We shall distinguish three cases.

(i) If X is separable, then X is metrizable by Theorem 1 in [CC_2] and so X is perfectly normal.

(ii) Let X be compact and not perfectly normal. Then X is not hereditarily Lindelöf. We shall prove that there exists a perfectly normal space Y such that $X \times Y$ is not normal. Let λ be the smallest cardinal such that X is not hereditarily λ-Lindelöf. Then there is an open subspace X^* of X such that X^* is hereditarily μ-Lindelöf for each $\mu < \lambda$ and not λ-Lindelöf. Let us put $A = X - X^*$. Then A is compact. Let Z be the quotient space obtained from X by contracting the set A to a point z_0. Let $f : X \longrightarrow Z$ be the identification map. Then f is a perfect map. Let us put $Z^* = f(X^*)$. Then $Z^* = Z - \{z_0\}$ is hereditarily μ-Lindelöf for each $\mu < \lambda$ and not λ-Lindelöf. Then by Lemma 1.1, we see $\mathrm{cf}\lambda = \lambda$. And by Lemma 1.2, there exists a subset $\widetilde{Z}$ of Z^* such that

(1) $|\widetilde{Z}| = \lambda$,

(2) we can choose an open neighborhood B_z of z in Z^* such that $|B_z \cap \widetilde{Z}| < \lambda$ for each $z \in Z^*$.

We shall prove that there exists a perfectly normal space Y such that $Z \times Y$ is not normal. Let Y be the Bing's space constructed by $P = \widetilde{Z}$ (cf. R. H. Bing [B_1 , Example H]). Since our notations are different from those of Bing [B_1], we restate the construction of Y. Let $Q = \mathcal{P}(P)$ (here $P = \tilde{Z}$). Let Y be the set of all functions $y : Q \longrightarrow \omega$. To each element p of P, we associate the function $y_p \in Y$ which is defined by

$$y_p(q) = \begin{cases} 1 & \text{if } p \in q \\ 0 & \text{if } p \notin q \end{cases}$$

where $q \in Q$. Putting $Y_P = \{y_p \mid p \in P\}$ we get a subset Y_P of Y. Let $R = Q^f$. Given an element y_p of Y_P and a point (r, n) of $R \times \omega$, we define a subset $V(y_p;r,n) = \{y \mid y \in Y,\ y(q) > n \text{ for } q \in Q,\ y(q) = y_p(q) \bmod 2 \text{ for } q \in r\} \cup \{y_p\}$. For each y in Y, we define an open neighborhood base $\mathcal{V}_y$ at y as follows :

$$\mathcal{V}_y = \begin{cases} \{y\} & \text{if } y \in Y - Y_P , \\ \{V(y_p;r,n) \mid (r,n) \in R \times \omega\} & \text{if } y = y_p \in Y_P. \end{cases}$$

Then Y is a perfectly normal space [B_1, Example H]. We shall prove that $Z \times Y$ is not normal.

Let us put $C = \{(p, y_p) \mid p \in P\}$ and $D = \{z_0\} \times Y_P$. Then C and D are disjoint closed subsets in $Z \times Y$. We shall prove that C and D can not be separated by open sets in $Z \times Y$.

Let O be an arbitrary open set in $Z \times Y$ such that $D \subset O$. For each p of P, there is an open neighborhood U_p of z_0 in Z and a member $V(y_p;r_p,n_p)$ of $\mathcal{V}_{y_p}$ such that

$$\bigcup_{p \in P} (U_p \times V(y_p;r_p,n_p)) \subset O.$$

Since λ is an uncountable regular cardinal, by Lemma 1.3, there is a subset $\widetilde{P}$ of P such that

(3) $|\widetilde{P}| = \lambda$,

(4) $\{r_p - \widetilde{r} \mid p \in \widetilde{P}\}$ is disjoint where $\widetilde{r} = \cap\{r_p \mid p \in \widetilde{P}\}$.

Let $\widetilde{R} = \mathcal{P}(\widetilde{r})$. For each $s \in \widetilde{R}$, we define an element q_s of Q by

$$q_s = \cap\{q \mid q \in s\} - \cup\{q \mid q \in \widetilde{r} - s\}.$$

Then $\{q_s \mid s \in \widetilde{R}\}$ is a finite disjoint covering of P. Therefore, we can choose a member $s_0 \in \widetilde{R}$ such that $|\widetilde{P} \cap q_{s_0}| = \lambda$. Let $P^* = \{p_i \mid i = 1, 2, \ldots\}$ be a sequence of mutually different point of $\widetilde{P} \cap q_{s_0}$. Since Z is compact, $E = \bigcup_{i=1}^{\omega}(Z - U_{p_i})$ is Lindelöf. Since $E \subset Z^*$, by the condition of (2), for each z of E, there is an open neighborhood B_z of z in Z^* such that $|B_z \cap \widetilde{Z}| < \lambda$. Then we have $|B_z \cap (\widetilde{P} \cap q_{s_0})| < \lambda$ because $\widetilde{P} \cap q_{s_0} \subset (P =)\ \widetilde{Z}$. Since E is Lindelöf, there is a sequence $\{z_i \mid i = 1, 2, \ldots\}$ of points in E such that $E \subset \bigcup_{i=1}^{\omega} B_{z_i}$. Since $|B_{z_i} \cap (\widetilde{P} \cap q_{s_0})| < \lambda$ for $i = 1, 2, \ldots$, and $cf\lambda > \omega$, we have $|E \cap (\widetilde{P} \cap q_{s_0})| \leqq |\bigcup_{i=1}^{\omega}(B_{z_i} \cap (\widetilde{P} \cap q_{s_0})| < \lambda$. On the other hand $|\widetilde{P} \cap q_{s_0}| = \lambda$. Therefore $\widetilde{P} \cap q_{s_0} \not\subset E$. Hence there is an element p^* of $\widetilde{P} \cap q_{s_0} - E$. Then $p^* \in U_{p_i}$ for $i = 1, 2, \ldots$. It is obvious that $(p^*, y_{p^*}) \in C$. To show $(p^*, y_{p^*}) \in ClO$, let U be an arbitrary open neighborhood of p^* and $V(y_{p^*}; r, n)$ be an arbitrary member of $\mathcal{V}_{y_{p^*}}$. Then $r \cap (r_{p_i} - \widetilde{r}) = \emptyset$ for some $p_i \in P^*$

because $\{r_{p_i} - \tilde{r} \mid i = 1,2,\ldots\}$ is disjoint by (4). Therefore, by the same reason as in [CC_2, Theorem 3], we have $V(y_{p^*}; r,n) \cap V(y_{p_i}; r_{p_i}, n_{p_i}) \neq \emptyset$. Since $p^* \in U_{p_i}$, we have $(U \times V(y_{p^*}; r,n)) \cap (U_{p_i} \times V(y_{p_i}; r_{p_i}, n_{p_i})) \neq \emptyset$ and thus $(p^*, y_{p^*}) \in \mathrm{Cl}\,O$. Hence $Z \times Y$ is not normal. Since $f : X \longrightarrow Z$ is a perfect map, $\psi = f \times 1_Y : X \times Y \longrightarrow Z \times Y$ is a closed onto map. Hence $X \times Y$ is not normal. The proof of (ii) is complete.

(iii) Let X be Fréchet and not perfectly normal. We shall prove that there exists a perfectly normal space Y such that $X \times Y$ is not normal. Of course we may assume that X is normal. Let A be a closed subset of X which is not a G_δ-set. Let us put $P = X - A$. Let Y be the Bing's space constructed by P. Let us put $C = \{(p, y_p) \mid p \in P\}$ and $D = A \times Y_P$. Then C and D are disjoint closed subsets in $X \times Y$. We shall prove that C and D can not be separated by open sets in $X \times Y$. Let O be an arbitrary open subset of $X \times Y$ such that $C \subset O$. For each element p of P, there are an open neighborhood U_p of p in X and an element $V(y_p; r_p, n_p)$ of $\mathcal{V}_{y_p}$ such that

$$\bigcup_{p \in P} (U_p \times V(y_p; r_p, n_p)) \subset O.$$

Let us put $P_i = \{ p \mid p \in P, \ |r_p| = i \}$ for $i = 0, 1, 2, \ldots.$ Since A is not a G_δ-set in X, there is an integer i such that $(\mathrm{Cl}P_i) \cap A \neq \emptyset$. Let x_0 be a point of $(\mathrm{Cl}P_i) \cap A$. Since

X is Fréchet, there is a sequence $\{p_n \mid n = 1, 2, \ldots\}$ in P_i which converges to x_0. Let us put $P' = \{p_n \mid n = 1, 2, \ldots\}$. We can choose a subsequence $\tilde{P} = \{p_{n_m} \mid m = 1, 2, \ldots\}$ of P' such that

(1) $\tilde{P}$ converges to x_0,

(2) $\{r_p - \tilde{r} \mid p \in \tilde{P}\}$ is disjoint where $\tilde{r} = \cap\{r_p \mid p \in \tilde{P}\}$.

To show this, let k be the smallest integer such that there is not a subset P'_k of P' satisfying

3_k) P'_k converges to x_0,

4_k) $|\cap\{r_p \mid p \in P'_k\}| \geqq k$.

The existence of k is sure, because there is not a subset P'_{i+1} of P' satisfying 4_{i+1}) (cf. the definition of P' and P_i). If we put $P'_0 = P'$, then P'_0 satisfies 3_0) and 4_0). Hence we have $k > 0$. Let P'_{k-1} be a subset of P' satisfying 3_{k-1}) and 4_{k-1}). Put

$$r_{(k-1)} = \cap\{r_p \mid p \in P'_{k-1}\}.$$

We can choose a sequence $\{p_{n_m} \mid m = 1, 2, \ldots\}$ in P'_{k-1} such that

5_m) $p_{n_m} \notin \{p_{n_j} \mid j < m\}$,

6_m) $r_{p_{n_m}} \cap (\bigcup_{j<m} r_{p_{n_j}} - r_{(k-1)}) = \emptyset$.

Suppose that this is impossible. Let h be the smallest integer such that we can not choose $p_{n_h} \in P'_{k-1}$ satisfying 5_h) and 6_h). Then, by the choice of h we have

$r_p \cap (\bigcup_{j<h} r_{p_{n_j}} - r_{(k-1)}) \neq \emptyset$ for each $p \in P'_{k-1}$. Let us put $Q^* = \bigcup_{j<h} r_{p_{n_j}} - r_{(k-1)}$. Then Q^* is a finite set. If we put $P(q) = \{p \mid p \in P'_{k-1}, r_p \ni q\}$ for each $q \in Q^*$, then we have $P'_{k-1} = \bigcup\{P(q) \mid q \in Q^*\}$. Since Q^* is a finite set, by 3_{k-1}), $P(q)$ converges to x_0 for some $q \in Q^*$. Let us choose such q and put $P'_k = P(q)$. Then P'_k satisfies 3_k) and 4_k). This is contary to the choice of k. Accordingly, we can choose a sequence $\{p_{n_m} \mid m = 1, 2, \ldots\}$ in P'_{k-1} satisfying 5_m) and 6_m) for each $m= 1, 2, \ldots$. We put $\widetilde{P} = \{p_{n_m} \mid m = 1, 2, \ldots\}$. Then $\widetilde{P}$ satisfies the conditions (1) and (2) because $\widetilde{r} = \cap\{r_p \mid p \in \widetilde{P}\} \supset r_{(k-1)}$.

Let $\widetilde{R} = \mathcal{P}(\widetilde{r})$. For each $s \in \widetilde{R}$, we define $q_s = \cap\{q \mid q \in s\} - \cup\{q \mid q \in \widetilde{r} - s\}$. Then $\{q_s \mid s \in \widetilde{R}\}$ is a finite disjoint cover of P. Since $\widetilde{P}$ converges to x_0, we can choose an element $s_0 \in \widetilde{R}$ such that $\widetilde{P} \cap q_{s_0}$ converges to x_0. Let us put $P^* = \widetilde{P} \cap q_{s_0}$, then P^* is infinite because $x_0 \in \mathrm{Cl}P^* - P^*$. Let p^* be an element of P^*. Then $(x_0, y_{p^*}) \in A \times \{y_{p^*}\} \subset D$. To show $(x_0, y_{p^*}) \in \mathrm{Cl}\, O$, let U be an arbitrary neighborhood of x_0 and $V(y_{p^*}; r, n)$ an arbitrary member in $\mathcal{V}_{y_{p^*}}$. Then $U \cap P^*$ is infinite. Therefore there is an element p of $U \cap P^*$ such that $r \cap (r_p - \widetilde{r}) = \emptyset$. Then we have
$(U \times V(y_{p^*}; r, n)) \cap (U_p \times V(y_p; r_p, n_p)) \neq \emptyset$ (cf. (ii)

or [CC_2 , Theorem 3]). Hence $(x_0, y_{p*}) \in \mathrm{Cl}\ O$. The proof of Theorem 1.3 is complete.

REMARK 1.1. In Theorem 1.3 the condition " compact " of X can be replaced by " σ-compact " or " σ-locally compact paracompact ". This was pointed out by T. Ishii.

REMARK 1.2. M. E. Rudin proved independently the following theorem in [R_2]: If C is an infinite compact space and $X \times C$ is normal, then X is w(C)-collectionwise normal. Here w(C) is the weight of C.

Since the space in Bing's Example H is not ω_1-collectionwise normal, in case X is compact Theorem 1.3 follows from the Rudin's theorem, too.

PROOF OF THEOREM 1.1.

The implication (a) $\Longrightarrow$ (b) is in [M_7]. We shall prove the implication (b) $\Longrightarrow$ (a). Let X be an M-space which satisfies the condition (b). Then by [T_1], X is paracompact. By [M_7], there is a perfect map f from X onto a metrizable space T. For each point t of T, $f^{-1}(t)$ is first countable by Theorem 1.3. By Lemma 1.3 of [T_2], X is first countable. Hence $X \times X$ is first countable. Further it is easy to see that $X \times X$ satisfies the condition (b). Thus $X \times X$ is perfectly normal by Theorem 1.3. Therefore, by [O_2], X is metrizable.

Theorem 1.2 is proved from Theorem 3.2 of [M_6] and

Theorem 1.1.

REMARK 1.3. Recently, in [CPR] it has been proved that the above Morita's Conjectures I and II are true under the assumption of the Gödel's Axiom of Constructibility $V = L$.

2. Products with perfectly normal spaces

In this section we shall consider the following property and some related problems with that of §1.

*) $X \times Y$ is normal for any perfectly normal space Y.

We shall use the following notation. $w(X)$ is the weight of X, $d(X)$ is the density character of X (i.e., the smallest cardinal which arises as the cardinal of some dense subset of X), and $c(X)$ is the cellular number of X (i.e., the smallest cardinal number λ for which each pairwise disjoint family of nonvoid open subset of X has λ or fewer elements). Between these notions the relation $c(X) \leqq d(X) \leqq w(X)$ holds (W. W. Comfort, [C_9 , Theorem 1]).

In [CC_2] we have proved

THEOREM 1.4 ([CC_2]). If $X \times Y$ is normal for any perfectly normal space Y, then $d(X) = w(X)$.

As for $c(X)$, we obtain

THEOREM 1.5. Let X be a locally countably compact space. If $X \times Y$ is normal for any perfectly normal space Y, then $c(X) = w(X)$.

Using the following lemma, we can prove this theorem by the same way as that of [CC_2 , Theorem 3].

Lemma 1.4. Let X be a locally countably compact regular space such that $c(X) \geqq \omega$. If $\mathcal{U} = \{U_\alpha \mid \alpha \in \Omega\}$ is a collection of non-empty open subsets of X such that $|\Omega| > c(X)$, then $\mathcal{U}$ is not point-finite.

Proof. For each $\alpha \in \Omega$ we can choose a non-empty open subset V_α of X such that $\mathrm{Cl}\, V_\alpha \subset U_\alpha$ and $\mathrm{Cl}\, V_\alpha$ is countably compact. Let's put $\mathcal{V} = \{V_\alpha \mid \alpha \in \Omega\}$ and $\overline{\mathcal{V}} = \{\mathrm{Cl}\, V_\alpha \mid \alpha \in \Omega\}$. It is sufficient to show that $\overline{\mathcal{V}}$ is not point-finite. Assume that $\overline{\mathcal{V}}$ is point finite. Let's put $P_i = \{\{\alpha_1, \alpha_2, \ldots, \alpha_i\} \mid \alpha_j \in \Omega$ for $j = 1, 2, \ldots, i$; $\alpha_j \neq \alpha_k$ if $j \neq k$; $\bigcap_{1 \leqq j \leqq i} V_{\alpha_j} \neq \emptyset$, $\bigcap_{1 \leqq j \leqq i} V_{\alpha_j} \cap V_\alpha = \emptyset$ for each $\alpha \in \Omega - \{\alpha_1, \alpha_2, \ldots, \alpha_i\}\}$ for $i = 1, 2, \ldots$. Let $P = \bigoplus_{1 \leqq i < \omega} P_i$ be the disjoint sum of $\{P_i \mid i = 1, 2, \ldots\}$.

For each $\alpha \in \Omega$, there is an integer i and an element p of P_i such that $\alpha \in p$. To show this, let α be an arbitrary element of Ω. Assume that there is not an integer i and an element p of P_i such that $\alpha \in p$. Then $\alpha \notin P_1$. Therefore there is an element $\alpha_2 \in \Omega - \{\alpha\}$ such that $V_\alpha \cap V_{\alpha_2} \neq \emptyset$. Since $\{\alpha, \alpha_2\} \notin P_2$, there is an element $\alpha_3 \in \Omega - \{\alpha, \alpha_2\}$ such that $V_\alpha \cap V_{\alpha_2} \cap V_{\alpha_3} \neq \emptyset$. Repeating this process, we can choose a sequence $\{\alpha_i \mid i = 1, 2, \ldots\}$ of mutually distinct elements of Ω such that $\bigcap_{1 \leqq j \leqq i} V_{\alpha_j} \neq \emptyset$ for $i = 1, 2, \ldots$ where $\alpha_1 = \alpha$. Since $\mathrm{Cl}\, V_{\alpha_1}$ is countably compact, $\bigcap_{1 \leqq i < \omega} \mathrm{Cl}\, V_{\alpha_i} \neq \emptyset$. But, this contradicts the assumption that $\overline{\mathcal{V}}$ is point-finite. Hence there is an integer i and an element p of P_i such that $\alpha \in p$.

For each $\alpha \in \Omega$, we choose an element p of P such that $\alpha \in p$ and define a mapping $\psi: \Omega \longrightarrow P$ by $\psi(\alpha) = p$. Then ψ is finite to one. To show this, let α be an arbitrary element of Ω and put $\psi(\alpha) = \{\alpha_1, \alpha_2, \ldots, \alpha_i\}$. Then $\alpha \in \{\alpha_1, \alpha_2, \ldots, \alpha_i\}$. Hence $\psi^{-1}\psi(\alpha) = \{\alpha_1, \alpha_2, \ldots, \alpha_i\}$. Therefore ψ is finite to one.

Since $|\Omega| > \omega$ and ψ is finite to one, we have $|P| \geqq |\Omega|$. Since $|\Omega| > c(X) \geqq \omega$, there is an integer i such that $|P_i| > c(X)$.

Let us put $W_p = \bigcap\{V_\alpha \mid \alpha \in p\}$ for each element p of P_i. Then W_p is a non-empty open subset of X and $\{W_p \mid p \in P_i\}$ is disjoint. This contradicts the definition of $c(X)$. Therefore $\overline{\mathcal{V}}$ is not point-finite.

If X is a metrizable space, then X possesses the property *) (Morita [M_7]). But, even if X satisfies *), X need not be metrizable ([CC_2, Theorem 2]).

We don't know any necessary and sufficient condition for a space X to possess the property *). It has been shown in [C_3] and [CPR] that both the perfect normality of X and $c(X) = w(X)$ are the necessary condition for X to possess *). But they cannot be the sufficient condition. That is, we have the following.

EXAMPLE 1.1. There exists a perfectly normal space X such that $c(X) = w(X)$ but does not satisfy *).

Let X be $R^1 \times \{x \mid x \in R^1, x \geqq 0\}$ where R^1 is the real line. For each $x \in R^1$ we define

$B_{p,p',p''}(x) = (\{x' \mid p < x' < p' , x' \neq x\} \times \{x'' \mid 0 \leqq x'' < p''\})$
$\cup \{(x , 0)\}$ (cf., J. G. Ceder [C_1 , Example 9.1]). For each $x = (x_1 , x_2) \in X$ we define a neighborhood base $\mathcal{B}_x$ as follows :

$\mathcal{B}_x = \{x\}$ if $x_2 > 0$,

$\mathcal{B}_x = \mathcal{B}_{(x_1,0)} = \{B_{p,p',p''}(x_1) \mid p<x_1<p' , p''>0\}$ if $x_2 = 0$.

Then it is clear that X is perfectly normal and $c(X) = w(X)$. Let Y be the Bing's space constructed from $P = R^1$. Then $X \times Y$ is not normal. To show this, let us put

$$A = \{((p,0), y_p) \mid p \in P\} \text{ and}$$
$$B = \{((p,x), y_p) \mid p \in P , x > 0\}.$$

Then A and B are disjoint closed subsets of $X \times Y$. Let O be an arbitrary open set in $X \times Y$ such that $A \subset O$. Then, for each $p \in P$, there exist a member $U_p \in \mathcal{B}_{(p,0)}$ and a member $V(y_p ; r_p , n_p) \in \mathcal{V}_{y_p}$ such that $\bigcup_{p\in P}(U_p \times V(y_p ; r_p , n_p)) \subset O$. Let us put $U_p = B_{p',p'',p'''}(p)$ $p' < p < p''$ and $p''' > 0$. By Lemma 1.3, there exists an uncountable subset $\tilde{P}$ of P such that $\{r_p - \tilde{r} \mid p \in \tilde{P}\}$ is disjoint where $\tilde{r} = \cap\{r_p \mid p \in \tilde{P}\}$. There is an element $s_0 \in \tilde{R} = \mathcal{P}(\tilde{r})$ such that $|q_{s_0} \cap \tilde{P}| > \omega$. Since $R^1 \times \{0\}$ is separable, $\{U_p \mid p \in \tilde{P} \cap q_{s_0}\}$ is point uncountable at a point $(p_0 , 0)$. We may assume

$P' = \{p \mid p \in \tilde{P} \cap q_{s_0} , U_p \ni (p_0 , 0), p > p_0\}$ is uncountable.

Let $\{x_i \mid i = 1, 2,...\}$ be a sequence of real numbers such that

$0 < x_i < \frac{1}{i}$. Then for each $p \in P'$, $(R^1 \times \{x_i\}) \cap U_p \neq \emptyset$ for some i. Therefore we can choose i such that $P'' = \{p \mid p \in P',\ (R^1 \times \{x_i\}) \cap U_p \neq \emptyset\}$ is uncountable. Then $(\{p_0\} \times \{x \mid 0 \leqq x \leqq x_i\}) \cap U_p \neq \emptyset$ for each $p \in P''$. If we put $p_0' = \inf \{p \mid p \in P''\}$, then $p_0 \leqq p_0'$ because $p_0 < p$ for each $p \in P''$. Hence $(p_0', 0) \in U_p$ for each $p \in P''$. It is clear that $(p_0', x_i) \in U_p$ for each $p \in P'' - \{p_0'\}$. There is a sequence $\{p_j \mid j = 1, 2, \ldots\}$ of points of P'' such that $|p_j - p_0'| < \frac{1}{j}$. Then we have $(\{(p_j, x_i) \mid j = 1, 2, \ldots\}) \cap U_p \neq \emptyset$ for each p of $P'' - \{p_0'\}$. Put $P^* = P'' - \{p_0'\}$, then P^* is uncountable. Hence $\{U_p \mid p \in P^*\}$ is point uncountable at (p_j, x_i) for some p_j. Then $((p_j, x_i), y_{p_j}) \in B$. Let us put $P_0^* = \{p \mid p \in P^*,\ (p_j, x_i) \in U_p\}$. Then $y_{p_j} \in \mathrm{Cl}(\bigcup\{V(y_p ; r_p, n_p) \mid p \in P_0^*\})$ by the same argument as that of [CC_2, Theorem 3]. It is easy to see $((p_j, x_i), y_{p_j}) \in \mathrm{ClO}$.

EXAMPLE 1.2. There exists a normal σ-space[3)] X and a compact first countable space Y such that $X \times Y$ is not normal. Here the definition of σ-spaces is due to A. Okuyama [O_3].

Let Y be the " two arrow space " i.e., Let E be the unit square with lexicoghaphic order (cf. E. Michael [M_4, Example 10.4] or V. J. Mancuso [M_1, Example 4.8], etc.).

Let $Y = (\{y \mid 0 < y \leqq 1\} \times \{0\}) \cup (\{y \mid 0 \leqq y < 1\} \times \{1\})$ with the subspace topology of E. Then Y is a compact first countable space. Since Y is not perfectly normal, by the proof of Theorem 1.3, $X \times Y$ is not normal when X is the space in Bing's Example H constructed by a suitable set P. Then X is a normal σ-space.

---

3) A space X is called a σ-space [O_3] if X has a σ-discrete net.

3. Products with normal M-spaces

In this section we shall study the normality of product spaces with one factor is a normal M-space.

As for a normal countably compact space , the following is known.

THEOREM 1.6 (N. N. Noble [N_5 , Corollary 2.5], A. P. Kombarov [K_3 , Theorem 1.1]). Let X be a normal countably compact space and Y a paracompact sequencial space. Then $X \times Y$ is normal.

If we replace the word " countably compact " by " M-space ", the above theorem is not valid. In fact the Michael line Y (see [M_3]) is a paracompact sequential space such that $X \times Y$ is not normal for some metrizable space X.

As for normal M-spaces we have

THEOREM 1.7. Let X be a normal M-space and Y a paracompact first countable P-space. Then $X \times Y$ is normal.

Proof. By Morita [M_7 , Theorem 6.1], there is a quasi-perfect map f from X onto a metrizable space T. Then by [M_7 , Theorem 5.1], $T \times Y$ is paracompact. By A. Okuyama [O_3 , Lemma 4.3], the map ψ from $X \times Y$ onto $T \times Y$ defined as $\psi((x , y)) = (f(x) , y)$ is a closed map. Hence in order to prove the normality of $X \times Y$, it is sufficient to show that for any point (t , y) of $T \times Y$ and for any disjoint closed subsets A and B such that $A , B \subset \psi^{-1}((t , y)) =$

$f^{-1}(t) \times \{y\}$, there exist disjoint open subsets O_1 and O_2 of $X \times Y$ such that $A \subset O_1$ and $B \subset O_2$ (cf. S. Hanai [H , Theorem 5]). But this clearly holds because X is normal. Hence $X \times Y$ is normal.

Is it possible to replace the word " first countable " by " sequential " in Theorem 1.7 ?

Even if Y is the closed image of a metrizable space, I don't know whether $X \times Y$ is normal for any normal M-space X or not. It is easily seen from [CC_2 , Theorem 1] that the product space $X \times Y$ of a normal P-space X and the closed image Y of a metrizable space is not necessarily normal. Related with these we obtain

THEOREM 1.8. Let X be a locally countably compact normal P-space and Y the closed image of a metric space. Then $X \times Y$ is normal.

Proof. We shall proceed the proof by an analogous method with that in [CC_1 , Theorem]. Let X be a locally countably compact normal P-space, Z a metric space and $f : Z \longrightarrow Y$ be a closed onto map. Then by Morita [M_7], $X \times Z$ is normal. By Lašnev [L_1], $Y = \bigcup_{n=0}^{\omega} Y_n$ where Y_n is closed and discrete for each $n \geqq 1$ and $f^{-1}(y)$ is compact for each $y \in Y_0$ and $(\bigcup_{n=1}^{\omega} Y_n) \cap Y_0 = \emptyset$. Let E_i $i = 1, 2$ be disjoint closed subsets in $X \times Y$. Let us put $\psi = 1_X \times f : X \times Z \longrightarrow X \times Y$ and $F_i = \psi^{-1}(E_i)$ for $i = 1, 2$. Then F_1 and F_2 are disjoint closed subsets in $X \times Z$. For each $y \in \bigcup_{n=1}^{\omega} Y_n$, there are open sets $U_{1,y}$ and $U_{2,y}$ in X such that $\mathrm{Cl}U_{1,y} \cap \mathrm{Cl}U_{2,y} = \emptyset$ and $(X \times \{y\}) \cap E_i \subset U_{i,y} \times \{y\}$ for $i = 1, 2$. Since $\cup\{\mathrm{Cl}U_{1,y} \times f^{-1}(y) \mid y \in Y_n\} \cup F_1$ and $\cup\{\mathrm{Cl}U_{2,y} \times f^{-1}(y) \mid y \in Y_n\} \cup F_2$ is disjoint closed subsets of $X \times Z$, there are open sets $O_{1,n}$ and $O_{2,n}$ in $X \times Z$ such that $\cup\{\mathrm{Cl}U_{i,y} \times f^{-1}(y) \mid y \in Y_n\} \cup F_i \subset O_{i,n}$ and $O_{1,n} \cap O_{2,n} = \emptyset$ for $i = 1, 2$; $n = 1, 2, \ldots$ because $X \times Z$ is normal.

Now we can choose a collection $\{U_\alpha \mid \alpha \in \Omega_{i,y}\}$ of open sets in X and a collection $\{W_\alpha \mid \alpha \in \Omega_{i,y}\}$ of open sets in Z for each $y \in Y_n$ and for each $i \in \{1, 2\}$ such that

$$\begin{cases} f^{-1}(y) \subset W_\alpha & \text{for each } \alpha \in \Omega_{i,y}, \\ U_{i,y} \times f^{-1}(y) \subset \cup\{U_\alpha \times W_\alpha \mid \alpha \in \Omega_{i,y}\} \subset O_{i,n}, & \\ U_\alpha \subset U_{i,y} & \text{for each } \alpha \in \Omega_{i,y}, \\ f^{-1}f(W_\alpha) = W_\alpha & \text{for each } \alpha \in \Omega_{i,y}. \end{cases}$$

To show this let $x \in U_{1,y}$. Then there is an open neighborhood U_x of x in X such that $\mathrm{Cl}U_x$ is countably compact and $\mathrm{Cl}U_x \subset U_{1,y}$. Then $\mathrm{Cl}U_x \times f^{-1}(y) \subset O_{1,n}$. Let z be an arbitrary element of $f^{-1}(y)$. Since $\mathrm{Cl}U_x$ is countably compact and Z is first countable, there is an open neighborhood $W(z)$ of z in Z such that $\mathrm{Cl}U_x \times W(z) \subset O_{1,n}$. If we put $W'_x = \cup\{W(z) \mid z \in f^{-1}(y)\}$, then $f^{-1}(y) \subset W'_x$ and $U_x \times W'_x \subset O_{1,n}$. Since f is a closed map, we can take an open set W_x in Z such that $f^{-1}(y) \subset W_x \subset W'_x$ and $f^{-1}f(W_x) = W_x$. If we put $\Omega_{i,y} = U_{i,y}$, then

$\{U_\alpha \mid \alpha \in \Omega_{i,y}\}$ and $\{W_\alpha \mid \alpha \in \Omega_{i,y}\}$ are the required collections.

Let us put $L_{i,n} = \bigcup\{U_\alpha \times W_\alpha \mid \alpha \in \Omega_{i,y} ; y \in Y_n\}$ for $i = 1, 2 ; n = 1, 2, \ldots.$ Then $L_{i,n}$ are open sets in $X \times Z$ such that $L_{1,n} \cap L_{2,n} = \emptyset$, $F_i \cap (X \times f^{-1}(Y_n)) \subset L_{i,n}$, $(\mathrm{Cl}L_{1,n}) \cap F_2 = \emptyset$, $(\mathrm{Cl}L_{2,n}) \cap F_1 = \emptyset$, $\psi^{-1}\psi(L_{i,n}) = L_{i,n}$ for $i = 1, 2$ and $n = 1, 2, \ldots,$

If we consider the disjoint closed subsets $F_i \cap (X \times f^{-1}(Y_n))$ and $X \times Z - L_{i,n}$ of $X \times Z$ instead of F_1 and F_2, by the same argument as before we can find open subsets $J_{i,n}$ of $X \times Z$ such that $F_i \cap (X \times f^{-1}(Y_n)) \subset J_{i,n} \subset \mathrm{Cl}J_{i,n} \subset L_{i,n}$, $\psi^{-1}\psi(J_{i,n}) = J_{i,n}$. Here we have

$$\mathrm{Cl}(\psi(J_{i,n})) \subset \psi(\mathrm{Cl}L_{i,n}).$$

To show this, let $(x, y) \notin \psi(\mathrm{Cl}L_{i,n})$. Then $(\{x\} \times f^{-1}(y)) \cap \mathrm{Cl}L_{i,n} = \emptyset$. Therefore there is an open set M in $X \times Z$ such that $M \supset \{x\} \times f^{-1}(y)$ and $M \cap L_{i,n} = \emptyset$. Let $\pi : X \times Z \longrightarrow X$ be the projection. Then $\pi(M)$ is an open neighborhood of x. Since X is locally countably compact, there is an open neighborhood N of x such that $\mathrm{Cl}N \subset \pi(M)$ and $\mathrm{Cl}N$ is countably compact. Since $L_{i,n}$ is an inverse set, $(\pi(M) \times f^{-1}(y)) \cap L_{i,n} = \emptyset$. Hence $(\mathrm{Cl}N \times f^{-1}(y)) \cap L_{i,n} = \emptyset$. Since $\mathrm{Cl}N$ is countably compact and Z is first countable, there is an open set I in Z such that $I \supset f^{-1}(y)$ and $(N \times I) \cap \mathrm{Cl}J_{i,n} = \emptyset$. Since f is a closed map, we may assume $f^{-1}f(I) = I$. Here we have $\psi(N \times I) \cap \psi(J_{i,n}) = \psi(N \times I \cap J_{i,n}) = \emptyset$ because

$N \times I = \psi^{-1}\psi(N \times I)$. Since $(x, y) \in N \times f(I) = \psi(N \times I)$ and $\psi(N \times I)$ is open, $(x, y) \notin Cl(\psi(J_{i,n}))$. Thus $Cl(\psi(J_{i,n})) \subset \psi(ClL_{i,n}))$.

Then $\psi(J_{i,n})$ is an open set in $X \times Y$ such that $E_i \cap (X \times Y_n) \subset \psi(J_{i,n})$, $Cl(\psi(J_{1,n})) \cap E_2 = \emptyset$, $Cl(\psi(J_{2,n})) \cap E_1 = \emptyset$ for $i = 1, 2$; $n = 1, 2, \ldots$.

Now let us put $G_1 = \bigcup_{n=1}^{\omega}(\psi(J_{1,n}) - \bigcup_{k=1}^{n} Cl(\psi(J_{2,k})))$ and $G_2 = \bigcup_{n=1}^{\omega}(\psi(J_{2,n}) - \bigcup_{k=1}^{n} Cl(\psi(J_{1,k})))$. Then G_1 and G_2 are disjoint open subsets of $X \times Y$ such that $E_i \cap (X \times (\bigcup_{n=1}^{\omega} Y_n)) \subset G_i$ for $i = 1, 2$.

Let $O_{1,0}$ and $O_{2,0}$ be disjoint open sets in $X \times Z$ such that $F_i \subset O_{i,0}$ for $i = 1, 2$. For each $y \in Y_0$, since $f^{-1}(y)$ is compact, there are a collection $\{U_\alpha \mid \alpha \in \Omega_{i,y}\}$ of open sets in X and a collection $\{V_\alpha \mid \alpha \in \Omega_{i,y}\}$ of open sets in Z for each $i \in \{1, 2\}$ such that $f^{-1}(y) \subset V_\alpha$ for each $\alpha \in \Omega_{i,y}$, $F_i \cap (X \times f^{-1}(y)) \subset \bigcup\{U_\alpha \times V_\alpha \mid \alpha \in \Omega_{i,y}\} \subset O_{i,0}$, $f^{-1}f(V_\alpha) = V_\alpha$ for each $\alpha \in \Omega_{i,y}$. If we put

$$M_i = \bigcup\{U_\alpha \times V_\alpha \mid \alpha \in \Omega_{i,y} ; y \in Y_0\}$$

for $i = 1, 2$. Then we have $\psi^{-1}\psi(M_i) = M_i$ for $i = 1, 2$, $\psi(M_1) \cap \psi(M_2) = \emptyset$, $(ClM_1) \cap F_2 = \emptyset$, $(ClM_2) \cap F_1 = \emptyset$ and $F_i - \psi^{-1}(G_i) \subset M_i$ for $i = 1, 2$. By the same argument as above we can take open set N_i such that $\psi^{-1}\psi(N_i) = N_i$ and $F_i - \psi^{-1}(G_i) \subset N_i \subset ClN_i \subset M_i$ for $i = 1, 2$. As before we have $Cl(\psi(N_i)) \subset \psi(ClM_i)$.

Let us put $H_1 = \psi(N_1) \cup (G_1 - Cl(\psi(N_2)))$ and

$H_2 = \psi(N_2) \cup (G_2 - Cl(\psi(N_1)))$. Then it is easy to see that H_1 and H_2 are disjoint open subsets of $X \times Y$ such that $E_i \subset H_i$ for $i = 1, 2$. Thus $X \times Y$ is normal. Theorem 1.8 has been proved.

REMARK 1.4. In [N_3], K. Nagami posed the following problem " Let $X \times Y$ be normal and Y metrizable. Let Z be the image of Y under a closed map. Is $X \times Z$ normal ? "
The answer to this question is " no " ([CC_2 , Theorem 1]).
But from the proof of Theorem 1.8, we see that the answer is " yes " for the case X has an additional condition " locally countably compact ".

REMARK 1.5. In [N_1], K. Nagami proved the following theorem " Let X be a collectionwise normal[1)] P-space and Y a paracompact σ-space and $X \times Y$ normal. Then $X \times Y$ is collectionwise normal ".
And he questioned the following
" Let X be a normal P-space and Y a paracompact σ-space. Then is $X \times Y$ normal ? "
In [CC_2] we have shown that the above Nagami's question is negative. Here we shall give a simpler counter example to

---

1) A space X is collectionwise normal if for every discrete family $\{A_\lambda \mid \lambda \in \Lambda\}$ of closed sets of X there exists a disjoint family $\{G_\lambda \mid \lambda \in \Lambda\}$ of open sets of X such that $G_\lambda \supset A_\lambda$ for each $\lambda \in \Lambda$.

this problem.

EXAMPLE 1.3. There exist a normal countably compact space X and an M_1-space Y such that $X \times Y$ is not normal.

A regular space is called an M_1-space if it has a σ-closure preserving base (Ceder [C_1]). Every M_1-space is a paracompact σ-space ([C_1, Theorem 2.2], Siwiec - Nagata [SN, Theorem 1]).

Let $X = W(\omega_1)$. Then it is well known that X is a normal countably compact space. We denote by $\omega \times \omega_1$ the set $\{(n, \alpha) \mid n < \omega, \ \alpha < \omega_1\}$ of pairs of ordinals. Let us put $Y = (\omega \times \omega_1) \cup \{(\omega, \omega_1)\}$ and introduce the following topology in Y :

1) each point $y \in \omega \times \omega_1$ is isolated in Y, and

2) we take $\{([n, \omega] \times [\alpha, \omega_1]) \cap Y \mid n < \omega, \ \alpha < \omega_1\}$ as an open neighborhood base at (ω, ω_1) in Y.

Here $[\alpha, \omega_1]$ denotes the set $\{\beta \mid \alpha \leqq \beta \leqq \omega_1\}$ of ordinals. Then it is easy to see that Y is an M_1-space. Also $X \times Y$ is not normal. To show this, let us put

$$A = X \times \{(\omega, \omega_1)\} \text{ and}$$

$$B = \{(\alpha, n, \alpha) \mid \alpha < \omega_1 , \ n < \omega\}.$$

Then A and B are disjoint closed sets in $X \times Y$. Let O be an arbitrary open sets in $X \times Y$ such that $A \subset O$. For each point $(\alpha, \omega, \omega_1)$ of A, by the definition of a neighborhood base of (ω, ω_1) in Y, we can find a point $(n_\alpha , \gamma_\alpha)$ of $\omega \times \omega_1$ such that $\gamma_\alpha > \alpha$ and

$\{\alpha\} \times (([n_\alpha , \omega] \times [\gamma_\alpha , \omega_1]) \cap Y) \subset O$. Let us put $\Omega_n = \{\alpha \mid \alpha < \omega_1 , n_\alpha = n\}$ for each n in ω. Then Ω_{n_0} is cofinal in ω_1 for some n_0 in ω. Since Ω_{n_0} is cofinal in ω_1, we can choose a sequence $\{\alpha_i \mid i = 1, 2, \ldots\}$ in Ω_{n_0} such that $\gamma_{\alpha_i} < \alpha_{i+1}$ for each $i = 1, 2, \ldots$. Let us put $\alpha^* = \sup \{\alpha_i \mid i = 1, 2, \ldots\}$. Then $\alpha^* < \omega_1$ and we also have $\alpha^* = \sup \{\gamma_{\alpha_i} \mid i = 1, 2, \ldots\}$ because $\alpha_i < \gamma_{\alpha_i} < \alpha_{i+1}$ for each $i = 1, 2, \ldots$. Since $\{\alpha_i\} \times \{n_0\} \times \{\beta \mid \gamma_{\alpha_i} \leqq \beta < \omega_1\} \subset O$ and $\gamma_{\alpha_i} < \alpha^* < \omega_1$ for each $i = 1, 2, \ldots$, $\{(\alpha_i , n_0 , \alpha^*) \mid i = 1, 2, \ldots\} \subset O$. Therefore $(\alpha^*, n_0 , \alpha^*) \in (\mathrm{Cl}\, O) \cap B$. Thus A and B can not be separated by open sets in $X \times Y$. Hence $X \times Y$ is not normal.

REMARK 1.6. By Example 1.3, we see that Theorem 1.8 is not true if " the closed image Y of a metric space " is replaced by " an M_1-space Y ". Since every normal countably compact space is a collectionwise normal P-space ([M_7] and Theorem 3.3 in [O_3]), this example also shows that in Nagami's theorem of Remark 1.5 the condition " $X \times Y$ is normal " can not be dropped.

CHAPTER 2

COLLECTIONWISE NORMALITY OF PRODUCT SPACES

1. Main theorems

On collectionwise normality of a product space $X \times Y$, the following theorems are known.

[1] (Kombarov [K_3]). Let X be a normal countably compact space and Y a paracompact sequential space. Then $X \times Y$ is collectionwise normal.

[2] (Yajima [Y_1]). Let X be a collectionwise normal space which has a σ-closure preserving closed cover by countably compact sets and Y a paracompact first countable space. Then $X \times Y$ is collectionwise normal.

We shall consider another condition of X and Y such that $X \times Y$ is collectionwise normal. The following are known.

[3] (Nagami [N_2]). Let X be a paracompact Σ-space and Y a paracompact P-space. Then $X \times Y$ is paracompact.

[4] (Theorem 1.7 in chapter 1). Let X be a normal M-space and Y a paracompact first countable P-space. Then $X \times Y$ is normal.

[5] (Example 1.2 in chapter 1). There exists a normal σ-space X and a compact first countable space Y such that $X \times Y$ is not normal.

In this chapter we shall prove the following theorem which contains [4].

THEOREM 2.1. Let X be a collectionwise normal Σ-space and Y a paracompact first countable P-space. Then $X \times Y$ is collectionwise normal.

The definitions of Σ-spaces are due to Nagami [N_2] and P-spaces and M-spaces are due to Morita [M_7] and σ-spaces are due to Okuyama [O_3].

In general, when X and Y are collectionwise normal spaces, $X \times Y$ is not necessarily normal. Further, the following example is known.

[6] (Przumusiński [P_2]) (MA + $\neg$CH). There exists a paracompact first countable space X such that X^2 is normal but not collectionwise normal.

But, if we assume that $X \times Y$ is normal, then $X \times Y$ may be collectionwise normal. Concerning this, the following theorems are known.

[7] (Rudin and Starbird [RS], [S_3]). Let X be a collectionwise normal space and Y a paracompact M-space and $X \times Y$ normal. Then $X \times Y$ is collectionwise normal.

[8] (Nagami [N_1]). Let X be a collectionwise normal P-space and Y a paracompact σ-space and $X \times Y$ normal. Then $X \times Y$ is collectionwise normal.

In this chapter we shall prove the following results.

THEOREM 2.2. Let X be a closed image of a normal M-space and Y a paracompact first countable P-space and $X \times Y$ normal. Then $X \times Y$ is collectionwise normal.

THEOREM 2.3. Let X be a collectionwise normal space and Y a σ-locally compact paracompact space and $X \times Y$ normal. Then $X \times Y$ is collectionwise normal.

A space X is called a Σ-space [N_2] if there exists a sequence $\{\mathcal{F}_n \mid 1 \leqq n < \omega\}$ of locally finite closed covers of X satisfying the following condition : If $\{K_n \mid 1 \leqq n < \omega\}$ is a decreasing sequence of non-empty closed sets in X such that $K_n \subset \cap\{F \mid x \in F \in \mathcal{F}_n\}$ for each $n = 1,2,\ldots$ and some $x \in X$, then $\bigcap_{n=1}^{\omega} K_n \neq \emptyset$.

2. Proof of Theorem 2.1

For the proof, we shall use the following facts.

FACT 2.1. Let $\mathcal{A} = \{A_\gamma \mid \gamma \in \Gamma\}$ be a discrete collection of closed subsets of X. If there exists a normal open cover of X each of whose member meets at most one A_γ. Then there are open sets H_γ of X such that $H_\gamma \supset A_\gamma$ for each $\gamma \in \Gamma$ and $H_\gamma \cap H_\mu = \emptyset$ if $\gamma \neq \mu$.

Fact 2.1 is well known.

FACT 2.2 ([N_2, Lemma 1.4]). Let X be a Σ-space. Then X has a Σ-net $\{\mathcal{F}_n \mid n = 1,2,\ldots\}$ which satisfies the following conditions :

(N_1) $\mathcal{F}_n = \{F(\alpha_1, \ldots, \alpha_n) \mid \alpha_1, \ldots, \alpha_n \in \Xi\}$.

(N_2) Every $F(\alpha_1, \ldots, \alpha_n) = \cup\{F(\alpha_1, \ldots, \alpha_n, \alpha_{n+1}) \mid \alpha_{n+1} \in \Xi\}$.

(N_3) For every $x \in X$, there exists a sequence $\alpha_1, \alpha_2, \ldots$ such that $\{F(\alpha_1, \ldots, \alpha_n) \mid n = 1,2,\ldots\}$ is a net of $C(x)$.

Here $C(x) = \cap\{C(x, \mathcal{F}_n) \mid n = 1,2,\ldots\}$, $C(x, \mathcal{F}_n) = \cap\{F \mid x \in F \in \mathcal{F}_n\}$.

PROOF OF THEOREM 2.1.

This proof is a modification of that of [3] (Theorem 4.1 in [N_2]). Let X be a collectionwise normal Σ-space and Y a paracompact first countable P-space. Let $\{\mathcal{F}_n \mid n = 1,2,\ldots\}$ be a Σ-net of X satisfying the conditions (N_1) $\sim$ (N_3) in Fact 2.2. Since $\mathcal{F}_n$ is a locally finite closed cover of

X and X is strongly normal, by Katětov $[K_1]$, there exists a locally finite cozero-set cover $\mathcal{H}_n = \{H(\alpha_1, \dots, \alpha_n) \mid \alpha_1, \dots, \alpha_n \in \Xi\}$ such that

$F(\alpha_1, \dots, \alpha_n) \subset H(\alpha_1, \dots, \alpha_n)$ for each $\alpha_1, \dots, \alpha_n \in \Xi$.

Let $\mathcal{A}$ be a discrete family of closed subsets of $X \times Y$. Let $\mathcal{W}(\alpha_1, \dots, \alpha_n) = \{U_\lambda \times V_\lambda \ (\neq \emptyset) \mid \lambda \in \Lambda(\alpha_1, \dots, \alpha_n)\}$ be the maximal collection satisfying the following conditions :

(1) each U_λ is a finite union of cozero-sets $\{U_{\lambda,j} \mid 1 \leqq j \leqq m(\lambda)\}$ of X such that $F(\alpha_1, \dots, \alpha_n) \subset U_\lambda \subset H(\alpha_1, \dots, \alpha_n)$,

(2) each V_λ is an open set of Y,

(3) each member of $\mathcal{J}_\lambda = \{U_{\lambda,j} \times V_\lambda \mid 1 \leqq j \leqq m(\lambda)\}$ meets at most one member of $\mathcal{A}$.

Let us put $V(\alpha_1, \dots, \alpha_n) = \bigcup\{V_\lambda \mid \lambda \in \Lambda(\alpha_1, \dots, \alpha_n)\}$. Then $V(\alpha_1, \dots, \alpha_n) \subset V(\alpha_1, \dots, \alpha_n, \alpha_{n+1})$ for each $\alpha_1, \dots, \alpha_n, \alpha_{n+1} \in \Xi$. Since Y is a P-space, for each $\alpha_1, \dots, \alpha_n \in \Xi$, there exists a closed set $K(\alpha_1, \dots, \alpha_n)$ of Y such that

(4) $K(\alpha_1, \dots, \alpha_n) \subset V(\alpha_1, \dots, \alpha_n)$

(5) if $\bigcup_{n=1}^{\omega} V(\alpha_1, \dots, \alpha_n) = Y$, then $\bigcup_{n=1}^{\omega} K(\alpha_1, \dots, \alpha_n) = Y$.

Since Y is paracompact, for each $\alpha_1, \dots, \alpha_n \in \Xi$, there exists a locally finite collection $\{V'_\lambda \mid \lambda \in \Lambda(\alpha_1, \dots, \alpha_n)\}$ of cozero-sets in Y such that

(6) $V'_\lambda \subset V_\lambda$ for each $\lambda \in \Lambda(\alpha_1, \dots, \alpha_n)$,

(7) $K(\alpha_1, \dots, \alpha_n) \subset \bigcup\{V'_\lambda \mid \lambda \in \Lambda(\alpha_1, \dots, \alpha_n)\}$.

Let us put $\mathcal{G}_n = \{U_{\lambda,j} \times V'_\lambda \mid \lambda \in \Lambda(\alpha_1, \dots, \alpha_n),\ \alpha_1, \dots, \alpha_n \ \Xi,$

$1 \leqq j \leqq m(\lambda)\}$ for each $n = 1, 2, \ldots$ and put

$\mathcal{G} = \bigcup \{\mathcal{G}_n \mid n = 1,2,\ldots\}$.

Then we have

(8) each $\mathcal{G}_n$ is locally finite in $X \times Y$,

(9) each member of $\mathcal{G}$ meets at most one member of $\mathcal{A}$,

(10) each member of $\mathcal{G}$ is a cozero-set in $X \times Y$,

(11) $\mathcal{G}$ is a cover of $X \times Y$.

(8) $\sim$ (10) are clear.

Proof of (11). Let $(x,y) \in X \times Y$ be an arbitrary element. Let $\alpha_1, \ldots, \alpha_n, \ldots \in \Xi$ be elements such that $\{F(\alpha_1, \ldots, \alpha_n) \mid n = 1,2,\ldots\}$ be a net of $C(x)$. Then we have $\bigcup_{n=1}^{\omega} V(\alpha_1, \ldots, \alpha_n) = Y$. To show this, let y' be an arbitrary element of Y. Then, since $C(x)$ is countably compact ([N_2]) and the family $\mathcal{A}$ is discrete, $\{A_\xi \in \mathcal{A} \mid (C(x) \times \{y'\}) \cap A_\xi \neq \emptyset\}$ is finite. Therefore, by using the first countability of Y, there is a finite family $\{M_j \mid j = 1,2,\ldots,k\}$ of open sets in X and an open set G in Y such that

(12) $C(x) \subset \bigcup_{j=1}^{k} M_j$, $y' \in G$,

(13) each $M_j \times G$ meets at most one member of $\mathcal{A}$.

By Lemma 2.1 in [Y_1], there are cozero-sets M_j' in X such that $M_j' \subset M_j$ and $C(x) \subset \bigcup_{j=1}^{k} M_j'$. Then $F(\alpha_1, \ldots, \alpha_i) \subset \bigcup_{j=1}^{k} M_j'$ for some i. Let us put $U_j = M_j' \cap H(\alpha_1, \ldots, \alpha_i)$. Then U_j are cozero-sets in X and $(\bigcup_{j=1}^{k} U_j) \times G \in \mathcal{W}(\alpha_1, \ldots, \alpha_i)$ by the maximality of

$\mathcal{W}(\alpha_1, \ldots, \alpha_i)$. Thus $y' \in V(\alpha_1, \ldots, \alpha_i)$.

Therefore we have $\bigcup_{n=1}^{\omega} K(\alpha_1, \ldots, \alpha_n) = Y$ by (5). Hence $y \in K(\alpha_1, \ldots, \alpha_n)$ for some n. By (7), $y \in V'_\lambda$ for some $\lambda \in \Lambda(\alpha_1, \ldots, \alpha_n)$. Then $(x,y) \in C(x) \times \{y\} \subset F(\alpha_1, \ldots, \alpha_n) \times V'_\lambda \subset U_\lambda \times V'_\lambda$. Since $x \in U_{\lambda,j}$ for some $j \leqq m(\lambda)$, $(x,y) \in U_{\lambda,j} \times V'_\lambda \in \mathcal{G}_n \subset \mathcal{G}$.

By (8) $\sim$ (11), $\mathcal{G}$ is a normal open cover of $X \times Y$, each of whose member meets at most one element of $\mathcal{A}$. By Fact 2.1, there exists a disjoint family $\{H_A \mid A \in \mathcal{A}\}$ of open sets in $X \times Y$ such that $H_A \supset A$ for each $A \in \mathcal{A}$. Hence $X \times Y$ is collectionwise normal. The proof of Theorem 2.1 is complete.

1) The author first proved Theorem 2.1 by another method. Y. Yajima pointed out that we can give a simpler proof by modifying the proof of [3].

REMARK 2.1. Theorem 2.1 is neither containd in [1] nor [2] in §1. In fact, let X be the space of irrationals of R with the euclidean topology where R is the real line and Y the Michael line $[M_3]$, then Y is a paracompact first countable space and $X \times Y$ is not normal ($[M_3]$). Therefore X does not satisfy the condition in [2]. Also X is not countably compact. But X is a collectionwise normal Σ-space.

Moreover, this example shows that the condition " Y is a P-space " can not be dropped in Theorem 2.1.

REMARK 2.2. We can not weaken the condition " Y is first countable " to the condition " for each $y \in Y$ is a G_δ-set ". In fact the following example exists.

EXAMPLE 2.1 (Example 1.3 in chapter 1). Let $X = W(\omega_1)$. Then it is well known that X is a normal countably compact space. Thus X is a collectionwise normal Σ-space. Let $Y = (\omega \times \omega_1) \cup \{(\omega, \omega_1)\}$ with the topology as follows : $\{([\alpha, \omega] \times [\beta, \omega_1]) \cap Y \mid \alpha < \omega, \beta < \omega_1\}$ is a neighborhood base of (ω, ω_1) and for each $y \in Y - \{(\omega, \omega_1)\}$, y is an isolated point of Y. Then Y is a paracompact perfectly normal space but Y is not first countable. Also $X \times Y$ is not normal.

REMARK 2.3. The paracompactness of Y can not be weaken the condition " collectionwise normal ". In fact the follwing example exists.

EXAMPLE 2.2. There exists a compact space X and a collectionwise normal perfectly normal first countable space Y such that $X \times Y$ is not normal. Let Y be the space constructed by R. Pol [P_1]. Then Y has the above properties, but Y is not paracompact. Therefore, by Theorem of Tamano [T_1], there exists a compact space X such that $X \times Y$ is not normal.

REMARK 2.4. The condition " X is a Σ-space " can not be replaced by the condition " X is a P-space ". In fact, let X be the Sorgenfrey line [S_2], then X is a paracompact first countable P-space such that $X^2 = X \times X$ is not normal.

REMARK 2.5. The author does not know whether we can generalize the condition " Y is first countable " to " Y is sequential " or not.

3. Proof of Theorem 2.2

To prove Theorem 2.2, we first prove the following lemmas. Lemma 2.1 is useful in proving collectionwise normality of a space X.

LEMMA 2.1. Let $\{A_\lambda | \lambda \in \Lambda\}$ be a discrete family of closed subsets of a space X. If there exists a family $\{G_{\lambda,n} | \lambda \in \Lambda, n<\omega\}$ of open subsets of X such that

(1) $\bigcup\{G_{\lambda,n} | n<\omega\} \supset A_\lambda$ for each $\lambda \in \Lambda$,

(2) $\mathrm{Cl}(G_{\lambda,n}) \cap A_\mu = \emptyset$ if $\lambda \neq \mu$,

(3) $\{G_{\lambda,n} | \lambda \in \Lambda\}$ is discrete in X,

then there exists a disjoint family $\{H_\lambda | \lambda \in \Lambda\}$ of open sets of X such that $H_\lambda \supset A_\lambda$ for each $\lambda \in \Lambda$.

Proof. Let us put $H_\lambda = \bigcup\{G_{\lambda,n} - \bigcup\{\mathrm{Cl}G_{\mu,i} | i \leqq n, \mu \in \Lambda, \mu \neq \lambda\} | n < \omega\}$. Then $\{H_\lambda | \lambda \in \Lambda\}$ satisfies the required conditions.

LEMMA 2.2. Let X and Y be collectionwise normal spaces and C a closed discrete subspace of X and assume $X \times Y$ is normal. Let $\{A_\lambda | \lambda \in \Lambda\}$ be a discrete family of closed subsets of $X \times Y$. Then there exists a discrete family $\{G_\lambda | \lambda \in \Lambda\}$ of open sets of $X \times Y$ such that $G_\lambda \supset A_\lambda \cap (C \times Y)$ and $\mathrm{Cl}(G_\lambda) \cap A_\mu = \emptyset$ if $\lambda \neq \mu$.

Proof. Since $X \times Y$ is normal, there are open sets H_λ in $X \times Y$ such that $A_\lambda \subset H_\lambda$, $\mathrm{Cl}(H_\lambda) \cap (\bigcup\{A_\mu | \mu \in \Lambda, \mu \neq \lambda\}) = \emptyset$. For each $x \in C$, there exists an open neighborhood $U(x)$ of x

such that $\{U(x) | x \in C\}$ is discrete in X. Let us put $A(\lambda, x) = \{y \in Y | (x, y) \in A_\lambda\}$ for each $x \in C$. Then $\{A(\lambda, x) | \lambda \in \Lambda\}$ is a discrete family of closed subsets of Y. Since Y is collectionwise normal, there exist open sets $O(\lambda, x)$ such that $O(\lambda, x) \supset A(\lambda, x)$ for each $\lambda \in \Lambda$ and $\{O(\lambda, x) | \lambda \in \Lambda\}$ is discrete in Y. Then $G_\lambda = H_\lambda \cap (\bigcup_{x \in C} (U_x \times O(\lambda, x))$ has the desired properties.

2) The referee of [C_7] has kindly pointed out this simple proof of Lemma 2.2.

THEOREM 2.4 (Hoshina and Chiba). Let X, Y and Z be spaces such that $X \times Y$ is normal and $Y \times Z$ is collectionwise normal. Suppose Y is first countable and there is a closed onto map $f : Z \longrightarrow X$ such that $X = \bigcup_{n<\omega} X_n$, X_n is closed discrete for $n \geqq 1$ and $f^{-1}(x)$ is countably compact for each $x \in X_0$. Then $X \times Y$ is collectionwise normal.

Proof. It is easy to see that X and Y are collectionwise normal. To prove collectionwise normality of $X \times Y$, let $\{A_\lambda | \lambda \in \Lambda\}$ be a discrete family of closed subsets of $X \times Y$. Then, by Lemma 2.2, for each $n \geqq 1$, there exists a discrete family $\{G_{\lambda,n} | \lambda \in \Lambda\}$ of open sets of $X \times Y$ such that $G_{\lambda,n} \supset A_\lambda \cap (X_n \times Y)$ and $Cl(G_{\lambda,n}) \cap A_\mu = \emptyset$ if $\lambda \neq \mu$.

Let us put $B_\lambda = A_\lambda - \bigcup_{n=1}^{\omega} G_{\lambda,n}$.

Then B_λ is closed in $X \times Y$ and $B_\lambda \subset X_0 \times Y$. Let us put $\phi = f \times 1_Y : Z \times Y \longrightarrow X \times Y$ and $F_\lambda = \phi^{-1}(A_\lambda)$ for each $\lambda \in \Lambda$.

Then $\{F_\lambda | \lambda \in \Lambda\}$ is a discrete family of closed subsets of $Z \times Y$. Since $Z \times Y$ is collectionwise normal, there are open sets H_λ in $Z \times Y$ such that $F_\lambda \subset H_\lambda$ and $H_\lambda \cap H_\mu = \emptyset$ if $\lambda \neq \mu$.

Let us put $A(\lambda,x) = \{y \in Y | (x,y) \in A_\lambda\}$ for each $x \in X_0$. For each $y \in A(\lambda,x)$, there are an open set $W_y(x)$ in Z and an open neighborhood $V_x(y)$ of y in Y such that $f^{-1}(x) \subset W_y(x)$ and $W_y(x) \times V_x(y) \subset H_\lambda$ because $f^{-1}(x)$ is countably compact and Y is first countable. Since f is a closed map, there is an open neighborhood $U_y(x)$ of x in X such that $f^{-1}(U_y(x)) \subset W_y(x)$.

Now let us put $K_{\lambda,x} = \bigcup\{U_y(x) \times V_x(y) | y \in A(\lambda,x)\}$ and $K_\lambda = \bigcup\{K_{\lambda,x} | x \in X_0\}$. Then we have

(1) K_λ is open in $X \times Y$, (2) $\phi^{-1}(K_\lambda) \subset H_\lambda$,

(3) $A_\lambda \cap (X_0 \times Y) \subset K_\lambda$.

Moreover it is easy to see that

(4) $K_\lambda \cap K_\mu = \emptyset$ if $\lambda \neq \mu$, (5) $K_\lambda \cap A_\mu = \emptyset$ if $\lambda \neq \mu$.

Since $B_\lambda \subset K_\lambda$ and $\{B_\lambda | \lambda \in \Lambda\}$ is a discrete family of closed subsets of $X \times Y$, there is a discrete family $\{G_{\lambda,0} | \lambda \in \Lambda\}$ of open sets of $X \times Y$ such that $B_\lambda \subset G_{\lambda,0}$, $\mathrm{Cl}G_{\lambda,0} \subset K_\lambda$ because $X \times Y$ is normal. Then we have $\mathrm{Cl}(G_{\lambda,0}) \cap A_\mu = \emptyset$ if $\lambda \neq \mu$ by (5). Let us consider $\{G_{\lambda,n} | 0 \leqq n < \omega\}$. Then, by Lemma 2.1, there are open sets P_λ in $X \times Y$ such that $A_\lambda \subset P_\lambda$ for each $\lambda \in \Lambda$ and $P_\lambda \cap P_\mu = \emptyset$ if $\lambda \neq \mu$. Hence the proof of collectionwise normality of $X \times Y$ is complete.

3) T. Hoshina and the author proved Theorem 2.4 independently.

REMARK 2.6. In Theorem 2.4, if we replace the condition " $f^{-1}(x)$ is countably compact " by " $f^{-1}(x)$ is compact ", then we can drop the first countability of Y.

PROOF OF THEOREM 2.2. Let X and Y be spaces in Theorem 2.2, i.e., Z be a normal M-space and $f : Z \longrightarrow X$ be a closed onto map and Y a paracompact first countable P-space such that $X \times Y$ is normal. Then $Z \times Y$ is collectionwise normal by Theorem 2.1 because every normal M-space is a collectionwise normal Σ-space ([N_2] and Theorem 3.3 in [O_3]). By Theorem 3.1 in [I_1], $X = \bigcup_{n<\omega} X_n$, X_n is closed discrete for $n \geqq 1$ and $f^{-1}(x)$ is countably compact for each $x \in X_0$. Therefore, by Theorem 2.4, $X \times Y$ is collectionwise normal.

THEOREM 2.5. Let X be a collectionwise normal Σ-space and Y a closed image of a paracompact first countable M-space and $X \times Y$ is normal. Then $X \times Y$ is collectionwise normal.

Proof. Let X be a collectionwise normal Σ-space and Z a paracompact first countable M-space and $f : Z \longrightarrow Y$ a closed onto map such that $X \times Y$ is normal. Then $X \times Z$ is collectionwise normal by Theorem 2.1 because every M-space is a P-space ([M_7]). By Fillippov [F], $Y = \bigcup_{n<\omega} Y_n$, Y_n is closed discrete for $n \geqq 1$ and $f^{-1}(x)$ is compact for each $y \in Y_0$.

Hence $X \times Y$ is collectionwise normal (see Remark 2.6).

We proved essentially the following in Theorem 1.8.

[9]. Let X be a locally countably compact space and Z a first countable space and $f : Z \longrightarrow Y$ a closed map onto a space Y such that $Y = \bigcup_{n<\omega} Y_n$, Y_n is closed discrete for $n \geqq 1$ and $f^{-1}(y)$ is compact for each $y \in Y_0$. If $X \times Z$ is normal, then $X \times Y$ is normal.

The following theorem can be proved in a way similar to the proof of Theorem 2.5 using [9].

THEOREM 2.6. Let X be a locally countably compact collectionwise normal Σ-space and Y a closed image of a paracompact first countable M-space. Then $X \times Y$ is collectionwise normal.

REMARK 2.7. Let X and Y be spaces in Theorem 2.2 or 2.5. It is not known whether $X \times Y$ is normal or not.

4. Proof of Theorem 2.3 and a related result

Even in case Y has only one non-isolated point, $X \times Y$ is not necessarily normal for a collectionwise normal space X. In fact let X be the Dowker space constructed by Rudin [R_1] and $Y = W(\omega + 1)$. Then X is collectionwise normal and $X \times Y$ is not normal [R_1]. However, in case Y has only one non-isolated point, the following holds. " For any collectionwise normal space X, if $X \times Y$ is normal, then $X \times Y$ is collectionwise normal ".

We shall prove the following theorem which contains both in case (i) Y is σ-locally compact paracompact, and in case (ii) Y has only one non-isolated point.

THEOREM 2.7. Let X be a collectionwise normal space and Y a paracompact space such that $Y = (\bigcup_{n<\omega} C_n) \cup D$ where C_n is a locally compact closed subspace of Y for each $n < \omega$ and for each $y \in D$ is an isolated point of Y. If $X \times Y$ is normal, then $X \times Y$ is collectionwise normal.

There exist spaces expressed as the form of Y in Theorem 2.7 but neither (i) not (ii). E. g., the Michael line [M_3] is so. Further, if Y is a paracompact subspace of F where F is the Bing's Example G or H [B_1], then Y satisfies the conditions in Theorem 2.7 but Y is necessarily neither (i) nor (ii). Therefore it seems good to consider the space Y satisfying the conditions in Theorem 2.7.

A subspace A of a space X is said to be P-embedded in X if every locally finite cozero-set cover of A has a refinement which can be extended to a locally finite cozero-set cover of X (Sapiro [S_1]).

LEMMA 2.3. Let X and Y be collectionwise normal spaces and Y is expressed as $Y = C \cup D$ such that $C \cap D = \emptyset$ and each $y \in D$ is an isolated point of Y. Let $X \times Y$ be normal and $\{A_\lambda | \lambda \in \Lambda\}$ a discrete family of closed subsets of $X \times Y$. If there exists a disjoint family $\{G_\lambda | \lambda \in \Lambda\}$ of open sets of $X \times Y$ such that $G_\lambda \supset A_\lambda \cap (X \times C)$. Then there exists a disjoint family $\{M_\lambda | \lambda \in \Lambda\}$ of open sets of $X \times Y$ such that $M_\lambda \supset A_\lambda$.

Proof. We can choose a discrete family $\{E_\lambda | \lambda \in \Lambda\}$ of open sets of $X \times Y$ such that $A_\lambda \cap (X \times C) \subset E_\lambda$, $\mathrm{Cl}E_\lambda \subset G_\lambda - \bigcup_{\mu \neq \lambda} A_\mu$. Let us put $D_\lambda = A_\lambda - E_\lambda$. Then $\{D_\lambda | \lambda \in \Lambda\}$ is a discrete family of closed subsets of $X \times D$. Since $X \times D$ is collectionwise normal, there is a disjoint family $\{J_\lambda | \lambda \in \Lambda\}$ of open sets of $X \times D$ such that $D_\lambda \subset J_\lambda$. We remark that J_λ is open in $X \times Y$. Since D_λ and $\bigcup_{\mu \neq \lambda} \mathrm{Cl}E_\mu$ are disjoint closed subsets of $X \times Y$ and $X \times Y$ is normal, there are open sets K_λ and L_λ in $X \times Y$ such that $D_\lambda \subset K_\lambda$, $\bigcup_{\mu \neq \lambda} \mathrm{Cl}E_\mu \subset L_\lambda$ and $K_\lambda \cap L_\lambda = \emptyset$. Let us put $M_\lambda = E_\lambda \cup (J_\lambda \cap K_\lambda)$. Then M_λ is open in $X \times Y$. Moreover, it is easy to see that $M_\lambda \supset A_\lambda$ and $\{M_\lambda | \lambda \in \Lambda\}$ is disjoint.

PROOF OF THEOREM 2.7. Let X and Y be spaces in Theorem 2.7. To prove that $X \times Y$ is collectionwise normal, let $\{A_\lambda \mid \lambda \in \Lambda\}$ be a discrete family of closed subsets of $X \times Y$. For each $n < \omega$, $X \times Y_n$ is collectionwise normal by Rudin and Starbird [7] in §1. Therefore, for each $n < \omega$, there is a discrete family $\{J_{\lambda,n} \mid \lambda \in \Lambda\}$ of open sets of $X \times C_n$ such that $A_\lambda \cap (X \times C_n) \subset J_{\lambda,n}$ for each $\lambda \in \Lambda$.

Also $X \times C_n$ is P-embedded in $X \times Y$ by Theorem 5.2 in [S_1] and Theorem 4 in [M_9]. Thus, by Theorem 2.4 in [MH], there is a locally finite family $\{H'_{\lambda,n} \mid \lambda \in \Lambda\}$ of cozero-sets of $X \times Y$ such that

$$A_\lambda \cap (X \times C_n) \subset H'_{\lambda,n} \cap (X \times C_n) \subset J_{\lambda,n} \quad \text{for each } \lambda \in \Lambda.$$

Let $H_{\lambda,n}$ be open sets of $X \times Y$ such that $A_\lambda \cap (X \times C_n) \subset H_{\lambda,n}$, $\mathrm{Cl}H_{\lambda,n} \subset H'_{\lambda,n}$. Let us put $L_{\lambda,n} = H_{\lambda,n} - \bigcup_{\mu \neq \lambda} \mathrm{Cl}H_{\mu,n}$. Then $L_{\lambda,n}$ is open in $X \times Y$, $A_\lambda \cap (X \times C_n) \subset L_{\lambda,n}$ for each $\lambda \in \Lambda$ and $\{L_{\lambda,n} \mid \lambda \in \Lambda\}$ is disjoint.

Since $X \times Y$ is normal, there is a discrete family $\{G_{\lambda,n} \mid \lambda \in \Lambda\}$ of open sets of $X \times Y$ such that $A_\lambda \cap (X \times C_n) \subset G_{\lambda,n}$, $\mathrm{Cl}G_{\lambda,n} \subset L_{\lambda,n} - \bigcup_{\mu \neq \lambda} A_\mu$. If we consider $\{G_{\lambda,n} \mid \lambda \in \Lambda,\ n < \omega\}$, by Lemma 2.1, there is a disjoint family $\{G_\lambda \mid \lambda \in \Lambda\}$ of open sets of $X \times Y$ such that $G_\lambda \supset A_\lambda \cap [X \times (\bigcup_{n<\omega} C_n)]$. By Lemma 2.3, there exists a disjoint family $\{M_\lambda \mid \lambda \in \Lambda\}$ of open sets of $X \times Y$ such that $M_\lambda \supset A_\lambda$. Therefore $X \times Y$ is collectionwise normal. The proof of Theorem 2.7 is complete.

4) The author first proved Theorem 2.7 by another method. T. Hoshina pointed out the proof can be made simpler by using the idea of P-embedding.

CHAPTER 3

THE WEAK β-PEOPERTY

1. The weak β-property

In discussing the normality of product spaces, M. Atsuji defined the notion of the property $\beta^*(\kappa)$ for each infinite cardinal κ. *A space* X *is said to have the property* $\beta^*(\kappa)$ *if for any decreasing family* $\{F_\alpha \mid \alpha < \kappa\}$ *of closed subsets of* X *with* $\bigcap\{F_\alpha \mid \alpha<\kappa\} = \emptyset$, *there exists a family* $\{G_\alpha \mid \alpha<\kappa\}$ *of open sets of* X *such that* $F_\alpha \subset G_\alpha$ *for each* $\alpha < \kappa$ *and* $\bigcap\{Cl G_\alpha \mid \alpha < \kappa\} = \emptyset$.

A space X has the property $\beta^*(\omega)$ iff X is countably paracompact (F. Ishikawa [I_2]).

For an accumulation point p of a space X, an accumulation degree of p is defined as $a(p) = \min\{|A| \mid A \subset X, p \in Cl(A - \{p\})\}$. Atsuji proved the following theorem.

THEOREM A (Atsuji [A_2]). Suppose Y contains an accumulation point p with $a(p) = \kappa$. If $X \times Y$ is normal, then X has the property $\beta^*(\kappa)$.

A normal space X is called a κ-Dowker space iff X has not the property $\beta^*(\kappa)$ ([A_2], [R_4]). A ω-Dowker space is a Dowker space [R_1]. M. E. Rudin has constructed a κ-Dowker space for each infinite cardinal κ. Thus the following theorem holds.

THEOREM B (Atsuji and Rudin). If $X \times Y$ is normal for every normal space X, then Y is discrete.

On the other hand, Y. Yasui defined *the weak B-property*. *A space X is said to have the weak B-property iff X has the property $B^*(\kappa)$ for every infinite cardinal* κ ([A_2], [Y_4]).

It is known that any normal metacompact space has the weak B-property (S. Lefschtz [L_2, p. 26]).

In this section we shall prove the following.

THEOREM 3.1. Every perfectly normal space has the weak B-property.

THEOREM 3.2. Every subparacompact normal space has the weak B-property.

A space X is called subparacompact if every open cover of X has a σ-discrete closed refinement [B_2].

To prove Theorems 3.1 and 3.2, we first prove the following Proposition.

PROPOSITION 3.1. Let X be a normal space satisfying the condition (*) : for any open cover $\{G_\alpha \mid \alpha \in \Omega\}$ of X there exists a cover $\{K_\alpha \mid \alpha \in \Omega\}$ of X by F_σ-subsets such that $K_\alpha \subset G_\alpha$ for each $\alpha \in \Omega$. Let Y be a space which has only one non-isolated point y_0 satisfying the following condition (#) : there is a local base $\mathcal{V}(y_0)$ of y_0 in Y which is point-finite at each $y \in Y - \{y_0\}$. Then $X \times Y$ is normal.

LEMMA 3.1 (Starbird [S_3, Lemma 2.9a]). Let X be a normal space and Y a space wich has only one non-isolated point y_0. Then $X \times Y$ is normal if and only if for any closed subset A of $X \times Y$ such that $A \subset X \times (Y - \{y_0\})$, A and $X \times \{y_0\}$ can be separated by open sets in $X \times Y$.

PROOF OF PROPOSITION 3.1. The proof is essentially in [CC_2, Theorem 2]. Let A be a closed subset of $X \times Y$ such that $A \cap (X \times \{y_0\}) = \emptyset$. We shall show that A and $X \times \{y_0\}$ can be separated by open sets in $X \times Y$. For each $x \in X$, there exists a neighborhood $U(x)$ of x in X and an element V_x of $\mathcal{V}(y_0)$ such that $\mathrm{Cl}(U(x) \times V_x) \cap A = \emptyset$. Let us put $U_V = \bigcup\{ U(x) \mid V_x = V \}$ for each $V \in \mathcal{V}(y_0)$. Then $(U_V \times \mathrm{Cl}V) \cap A = \emptyset$ and $\{U_V \mid V \in \mathcal{V}(y_0)\}$ is an open cover of X. By the assumption on X, there exists a family $\{F_{V,i} \mid V \in \mathcal{V}(y_0),\ i < \omega\}$ of closed sets of X such that $K_V = \bigcup\{F_{V,i} \mid i < \omega\} \subset U_V$ for each $V \in \mathcal{V}(y_0)$ and $\bigcup\{K_V \mid V \in \mathcal{V}(y_0)\} = X$. Since X is normal, there exist open sets $U_{V,i}$ such that $F_{V,i} \subset U_{V,i}$, $\mathrm{Cl}U_{V,i} \subset U_V$. Then we have $\mathrm{Cl}(U_{V,i} \times V) \cap A = \emptyset$.

Let us put $W_i = \bigcup\{U_{V,i} \times V \mid V \in \mathcal{V}(y_0)\}$. Then W_i are open subsets of $X \times Y$ and $X \times \{y_0\} \subset \bigcup\{W_i \mid i < \omega\}$. Further $\mathrm{Cl}(W_i) \cap A = \emptyset$ because $\mathcal{V}(y_0)$ is point-finite at each point $y \in Y - \{y_0\}$.

We remark that $Y - \{y_0\}$ is σ-discrete. To show this, let $H_n = \{y \in Y \mid y \in V$ for at most n elements $V \in \mathcal{V}(y_0)\}$. Then H_n is a discrete closed subspace of Y and $Y - \{y_0\} = \bigcup\{H_n \mid n < \omega\}$. Since $Y - \{y_0\}$ is a F_σ-set, there is a sequence $\{Z_i \mid i < \omega\}$ of open sets in $X \times Y$ such that $A \subset \bigcup\{Z_i \mid i < \omega\}$ and $\mathrm{Cl}(Z_i) \cap (X \times \{y_0\}) = \emptyset$. Let us put $W = \bigcup\{W_i - \bigcup\{\mathrm{Cl}Z_j \mid j \leqq i\} \mid i < \omega\}$ and $Z = \bigcup\{Z_i - \bigcup\{\mathrm{Cl}W_j \mid j \leqq i\} \mid i < \omega\}$. Then W and Z are clearly disjoint open sets in $X \times Y$ such that $A \subset Z$ and $X \times \{y_0\} \subset W$. By Lemma 3.1, $X \times Y$ is normal.

PROOF OF THEOREMS 3.1 and 3.2. Let X be a perfectly normal space or a subparacompact normal space. Then X satisfies the condition (*) in Proposition 3.1. Let κ be an arbitrary uncountable cardinal. We consider the following space Y_κ constructed by H. Ohta [O_1].

The space Y_κ : Let Z_κ be the set of mappings f from κ to $\{0, 1\}$ such that $f(\alpha) = 0$ for all but finitely many $\alpha \in \kappa$. Let $\tilde{f}_\kappa$ be the mapping which takes the constant value 1 on κ, and set $Y_\kappa = Z_\kappa \cup \{\tilde{f}_\kappa\}$. Let Q be the family of all finite subsets of κ. For each $q \in Q$, let

$$V(q) = \{\tilde{f}_\kappa\} \cup \{f \in Z_\kappa \mid f(\alpha) = 1 \text{ for all } \alpha \in q\}.$$

Topologize Y_κ by letting $\{V(q) \mid q \in Q\}$ be a neighborhood base of $\tilde{f}_\kappa$ and each point of Z_κ to be open. Then $\mathcal{V}(\tilde{f}_\kappa) = \{V(q) \mid q \in Q\}$ is point-finite at each $f \in Y_\kappa - \{\tilde{f}_\kappa\}$ and $Y_\kappa - \{\tilde{f}_\kappa\}$ is σ-discrete [O_1]. Furthermore $a(\tilde{f}_\kappa) = \kappa$. To show this, let $A \subset Y_\kappa$ with $|A| = \lambda < \kappa$. For each $f \in A - \{\tilde{f}_\kappa\}$, put $q_f = \{\alpha \in \kappa \mid f(\alpha) = 1\}$. Then q_f is finite and so $|\bigcup\{q_f \mid f \in A\}| < \omega \cdot \lambda < \kappa$. Hence there is an element $\alpha_0 \in \kappa - \bigcup\{q_f \mid f \in A\}$. Since

$V(\{\alpha_0\}) \cap (A - \{\tilde{f}_\kappa\}) = \emptyset$, we have $\tilde{f}_\kappa \notin \mathrm{Cl}(A - \{\tilde{f}_\kappa\})$.

The space Y_κ satisfies the condition (#) in Proposition 3.1. Therefore, $X \times Y_\kappa$ is normal by Proposition 3.1. Since $a(\tilde{f}_\kappa) = \kappa$, X has the property $\mathcal{B}^*(\kappa)$ by Theorem A.

On the other hand, since X is countably paracompact, X has the property $\mathcal{B}^*(\omega)$. Thus, X has the weak $\mathcal{B}$-property and our theorems have been proved.

REMARK 3.1. Let $\mathcal{U} = \{U_\alpha \mid \alpha \in \Omega\}$ be an open cover of a space X. If there is an open cover $\mathcal{V} = \{V_\alpha \mid \alpha \in \Omega\}$ of X such that $\mathrm{Cl}V_\alpha \subset U_\alpha$ for each $\alpha \in \Omega$, then $\mathcal{U}$ is said *shrinkable* [L_2 , p. 26]. A space X is said *shrinking* if every open cover of X is shrinkable [R_5].

Recently, in [CPR, § 4], it has been proved that every perfectly normal space is shrinking. Similarly it is shown that every subparacompact normal space is shrinking (see or [Y_6]).

Let Y_κ be the space in " Proof of Theorems 3.1 and 3.2 " for $\kappa > \omega$ and, for convinience, let $Y_\omega = W(\omega + 1)$. Then we have

THEOREM 3.3. Let X be a space. If $X \times Y_\kappa$ is normal, then X has the property $\mathcal{B}^*(\kappa)$ for each $\kappa \geqq \omega$.

THEOREM 3.4. Let X be a space. If $X \times Y_\kappa$ is normal for each $\kappa \geqq \omega$, then X has the weak $\mathcal{B}$-property.

A space X is called perfect if every closed set is a G_δ-set.

The following are direct consequences of Theorems 3.1 and 3.2.

COROLLARY 3.1. There is no perfect κ-Dowker space for every infinite cardinal κ.

COROLLARY 3.2. There is no subparacompact κ-Dowker space for every infinite cardinal κ.

It is well known that there is no perfect Dowker space (Theorem 2.1 in [R_6]) and Corollary 3.1 is a generalization of this theorem.

We denote by $nw(X)$ the net weight of a space X i.e., the minimum cardinal number of a net of X and by $\chi(y_0, Y)$ the character of a point y_0 in a space Y i.e., the minimum cardinal number of a local base of y_0 in Y [J_2]. Let τ, κ and λ denote infinite cardinals.

PROPOSITION 3.2. Let X be a normal space and let Y be a space satisfying the condition (#) of Proposition 3.1 such that $nw(X) \leqq \chi(y_0, Y)$. Then $X \times Y$ is normal.

Proof. This proof is a modification of [C_3, Proposition] and [O_1, Theorem 1]. Let us put $\chi(y_0, Y) = \tau$ and let us put $\mathcal{V}(y_0) = \{V_\beta \mid \beta < \tau\}$ and let $\mathcal{B} = \{B_\alpha \mid \alpha < \tau\}$ be a net of X. Let A be a closed subset of $X \times Y$ such that

$A \cap (X \times \{y_0\}) = \emptyset$. We shall prove that A and $X \times \{y_0\}$ can be separated by open sets in $X \times Y$. For each $x \in X$, let us choose an open neighborhood $U(x)$ of x in X and an element $V_{\beta(x)} \in \mathcal{V}(y_0)$ such that $\mathrm{Cl}(U(x) \times V_{\beta(x)}) \cap A = \emptyset$ and let us put

$$\Lambda = \{\alpha \mid \alpha < \tau, \ x \in B_\alpha \subset U(x) \ \text{and} \ \mathrm{Cl}(U(x) \times V_\beta) \cap A = \emptyset \ \text{for some} \ x \in X \ \text{and some} \ \beta < \tau\}.$$

For each $\alpha \in \Lambda$, choose $x_\alpha \in X$ and $\beta_\alpha < \tau$ such that $x_\alpha \in B_\alpha \subset U(x_\alpha)$ and $\mathrm{Cl}(U(x_\alpha) \times V_{\beta_\alpha}) \cap A = \emptyset$. Furthermore we can choose β_α so that $\beta_\alpha \neq \beta_{\alpha'}$ if $\alpha \neq \alpha'$, because $\{V_\beta \mid \beta \in \tau - \Omega\}$ is also a local base of y_0 for any subset Ω of τ with $|\Omega| < \tau$. Let us put $W = \bigcup\{U(x_\alpha) \times V_{\beta_\alpha} \mid \alpha \in \Lambda\}$. Then W is open in $X \times Y$ and $W \supset X \times \{y_0\}$. Further $(\mathrm{Cl}W) \cap A = \emptyset$. To show this, let $(x,y) \in A$. Then $y \neq y_0$. Therefore, if we put $\Lambda(y) = \{\alpha \mid \alpha \in \Lambda, \ y \in V_{\beta_\alpha}\}$, then $\Lambda(y)$ is a finite set because $\{V_\beta \mid \beta < \tau\}$ is point-finite at y. Hence we have $(x,y) \notin \mathrm{Cl}(\bigcup\{U(x_\alpha) \times V_{\beta_\alpha} \mid \alpha \in \Lambda(y)\})$. Since y is an isolated point, $(x,y) \notin \mathrm{Cl}(\bigcup\{U(x_\alpha) \times V_{\beta_\alpha} \mid \alpha \in \Lambda - \Lambda(y)\})$. Therefore $(x,y) \notin \mathrm{Cl}W$. Hence $(\mathrm{Cl}W) \cap A = \emptyset$. By Lemma 3.1, $X \times Y$ is normal.

THEOREM 3.5. If X is a normal space, then X has the property $\mathcal{B}^*(\kappa)$ for each $\kappa \geqq nw(X)$.

Proof. Let κ be an arbitrary cardinal $\geqq nw(X)$. Let Y_κ be the space used in the proof of Theorem 3.1, then it is easy to see $\chi(y_0, Y_\kappa) = \kappa$. By Proposition 3.2, $X \times Y_\kappa$ is normal. By Theorem 3.3, X has the property $\mathcal{B}^*(\kappa)$.

A space X is called τ-compact if any open cover $\mathcal{U} = \{U_\alpha \mid \alpha < \tau\}$ of X has a finite subcover.

The following lemma is obvious.

LEMMA 3.2. For each τ, if X is a τ-compact space, then X has the property $\mathcal{B}^*(\tau)$.

We have the following from Theorem 3.5 and Lemma 3.2.

COROLLARY 3.3. For each regular cardinal λ, $W(\lambda)$ has the weak $\mathcal{B}$-property.

The author does not know whether every normal scattered space (see [K_5 , p. 78]) is countably paracompact or not. As a special case of scattered spaces, we obtain

THEOREM 3.6. Let X be a normal space which is a disjoint union of C and D such that for each $x \in C$ is isolated in X and D is discrete. Then X has the weak $\mathcal{B}$-property.

In fact, if X is a space satisfying the condition of Theorem 3.6, then X is shrinking (or see " Proof that Karen Navy's space S is shrinking " in [R_7]). Here we shall prove the following which contains Theorem 3.6.

THEOREM 3.7. Let X be a space which is a disjoint union of C and D such that for each $x \in C$ is isolated in X and D is discrete. Let Y be a shrinking space (respectively, has the weak $\mathcal{B}$-property). If $X \times Y$ is normal, then $X \times Y$ is shrinking (resp., has the weak $\mathcal{B}$-property).

Proof. (i) In case " *shrinking* ". Let X be a space satisfying the condition of Theorem 3.7 and Y shrinking. Let $\mathcal{G} = \{G_\lambda \mid \lambda \in \Lambda\}$ be an arbitrary open cover of $X \times Y$. It is clear that $D \times Y$ is shrinking. Therefore, there are closed sets B_λ of $D \times Y$ such that $B_\lambda \subset G_\lambda$ for each $\lambda \in \Lambda$ and $\bigcup_{\lambda \in \Lambda} B_\lambda = D \times Y$. Since B_λ are closed sets of $X \times Y$ and $X \times Y$ is normal, there are open sets H_λ of $X \times Y$ such that $B_\lambda \subset H_\lambda$, $\mathrm{Cl} H_\lambda \subset G_\lambda$ for each $\lambda \in \Lambda$. Let us put $H = \bigcup_{\lambda \in \Lambda} H_\lambda$. Then H is open in $X \times Y$ such that $D \times Y \subset H$. Since $C \times Y$ is shrinking and $X \times Y - H$ is closed subset of $C \times Y$, $X \times Y - H$ is also shrinking. Thus, there are closed sets K_λ of $X \times Y - H$ such that $K_\lambda \subset (X \times Y - H) \cap G_\lambda$ for each $\lambda \in \Lambda$ and $\bigcup_{\lambda \in \Lambda} K_\lambda = X \times Y - H$.

Let us put $P_\lambda = (\mathrm{Cl} H_\lambda) \cup K_\lambda$ for each $\lambda \in \Lambda$. Then P_λ are closed sets of $X \times Y$, $P_\lambda \subset G_\lambda$ for each $\lambda \in \Lambda$ and $\bigcup_{\lambda \in \Lambda} P_\lambda = X \times Y$. Since $X \times Y$ is normal, $\mathcal{G}$ is shrinkable. Hence $X \times Y$ is shrinking.

(ii) In case " *the weak B-property* ". The proof is quite similar to that of (i).

The following is a direct consequence of Theorem 3.7.

COROLLARY 3.4. Let X be an arbitrary subspace of Bing's Example G or H [B_1] and Y a shrinking space. Then $X \times Y$ is normal if and only if $X \times Y$ is shrinking.

2. Closed maps

In this section we shall study the behavior of the weak $\mathcal{B}$-property under closed maps.

THEOREM 3.8. Let $f : X \longrightarrow Y$ be a perfect map. Then we have

(1) If X has the weak $\mathcal{B}$-property, then Y has the weak $\mathcal{B}$-property.

(2) If Y has the weak $\mathcal{B}$-property, then X has the weak $\mathcal{B}$-property.

Proof of (1). Let κ be an arbitrary infinite cardinal. Let $\mathcal{F} = \{F_\alpha \mid \alpha < \kappa\}$ be a decreasing family of closed subsets of Y with $\bigcap\{F_\alpha \mid \alpha < \kappa\} = \emptyset$. Let us put $A_\alpha = f^{-1}(F_\alpha)$. Then $\{A_\alpha \mid \alpha < \kappa\}$ is a decreasing family of closed subsets of X with $\bigcap\{A_\alpha \mid \alpha < \kappa\} = \emptyset$. Since X has the property $\mathcal{B}^*(\kappa)$, there exists a family $\{G_\alpha \mid \alpha < \kappa\}$ of open sets of X such that $A_\alpha \subset G_\alpha$ for each $\alpha < \kappa$ and $\bigcap\{\mathrm{Cl}G_\alpha \mid \alpha < \kappa\} = \emptyset$. Let us put $\kappa(f) = \{\lambda \mid \lambda \subset \kappa,\ \lambda \text{ is a finite set}\}$. Then, since $|\kappa(f)| = \kappa$, we can denote $\kappa(f) = \{\lambda_\alpha \mid \alpha < \kappa\}$. For each $\lambda \in \kappa(f)$, let's choose $g(\lambda) \in \kappa$ such that $g(\lambda) > \beta$ for each $\beta \in \lambda$ and $g(\lambda) \neq g(\lambda')$ for $\lambda \neq \lambda'$. We can choose such $g(\lambda)$ by transfinite induction.

For each $\alpha < \kappa$, we define V_α as follows :

$$V_\alpha = \begin{cases} G_{g(\lambda)} \cap (\bigcap\{G_\beta \mid \beta \in \lambda\}) & \text{if } \alpha = g(\lambda) \text{ for some } \lambda \in \kappa(f), \\ G_\alpha & \text{if } \alpha \neq g(\lambda) \text{ for every } \lambda \in \kappa(f). \end{cases}$$

Then V_α is open in X and it is easy to see $G_\alpha \supset V_\alpha \supset A_\alpha$.

Let us put $H_\alpha = Y - f(X - V_\alpha)$. Then H_α is open in Y, $F_\alpha \subset H_\alpha$ and $f^{-1}(H_\alpha) \subset V_\alpha$. Furthermore we have

$\bigcap\{ClH_\alpha \mid \alpha < \kappa\} = \emptyset$ ..... (a)

To prove (a), suppose that there exists an element y_0 of $\bigcap\{ClH_\alpha \mid \alpha < \kappa\}$. Then $y_0 \in f(Cl(f^{-1}(H_\alpha))$ for each $\alpha < \kappa$ because $ClH_\alpha = f(Cl(f^{-1}(H_\alpha))$. Hence $f^{-1}(y_0) \cap Cl(f^{-1}(H_\alpha)) \neq \emptyset$ and so $f^{-1}(y_0) \cap ClG_\alpha \neq \emptyset$ for each $\alpha < \kappa$. The family $\{f^{-1}(y_0) \cap ClG_\alpha \mid \alpha < \kappa\}$ has the finite intersection property because, for each $\lambda = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \in \kappa(f)$, $\bigcap\{ClG_{\alpha_i} \mid i = 1,2,\ldots,n\} \supset ClV_{g(\lambda)}$. Since $f^{-1}(y_0)$ is compact, we have $\bigcap\{f^{-1}(y_0) \cap ClG_\alpha \mid \alpha < \kappa\} \neq \emptyset$. But this is a contradiction. Thus, the proof of (a) is complete and we have proved that X has the property $\mathcal{B}^*(\kappa)$.

The proof of (2) is easy and so we omit this proof.

LEMMA 3.3. Let X be a regular space. Then X has the property $\mathcal{B}^*(\kappa)$ for each cardinal κ with $\kappa \geqq w(X)\cdot\omega$. Here $w(X)$ denotes the weight of a space X.

Lemma 3.3 follow from the following.

LEMMA 3.4. Let X be a regular space, κ be an infinite cardinal and $\mathcal{G} = \{G_\lambda \mid \lambda < \kappa\}$ an increasing open cover. Then the following are equivalent.

(1) there exists an open cover $\{U_{\lambda,\alpha} \mid \lambda, \alpha < \kappa\}$ of X such that $\bigcup_{\alpha<\kappa} ClU_{\lambda,\alpha} \subset G_\lambda$ for each $\lambda < \kappa$,

(2) there exists an open cover $\{V_\lambda \mid \lambda < \kappa\}$ of X such that $ClV_\lambda \subset G_\lambda$ for each $\lambda < \kappa$.

Proof. The implication (2) $\Rightarrow$ (1) is obvious. To prove (1) $\Rightarrow$ (2), let $\{U_{\lambda,\alpha} \mid \lambda, \alpha < \kappa\}$ be an open cover of X such that $\bigcup_{\alpha<\kappa} \mathrm{Cl}U_{\lambda,\alpha} \subset G_\lambda$ for each $\lambda < \kappa$.

For each $\alpha < \kappa$, there exists a cofinal subset Λ_α of κ such that $\kappa = \bigcup\{\Lambda_\alpha \mid \alpha < \kappa\}$ and $\Lambda_\alpha \cap \Lambda_\beta = \emptyset$ if $\alpha \neq \beta$. Let $\psi : \bigcup_{\lambda<\kappa}(\{\alpha\} \times \kappa) \longrightarrow \kappa$ be a one to one mapping such that $\psi(\alpha, \lambda) \in \Lambda_\alpha$ and $\psi(\alpha, \lambda) \geqq \lambda$ for each α, $\lambda < \kappa$. It is easy to see such ψ exists.

For each $\beta < \kappa$, let us define V_β as follows :

$$V_\beta = \begin{cases} U_{\lambda,\alpha} & \text{if } \beta = \psi(\alpha, \lambda) \text{ for some } \alpha, \lambda < \kappa, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then, for each $\beta < \kappa$, V_β is defined uniquely because ψ is one to one. Let $\beta = \psi(\alpha, \lambda)$. Then

$$\mathrm{Cl}V_\beta = \mathrm{Cl}U_{\lambda,\alpha} \subset G_\lambda \subset G_\beta$$

because $\beta \geqq \lambda$ and $\{G_\lambda \mid \lambda < \kappa\}$ is increasing.

Further we have $\bigcup_{\beta<\kappa} V_\beta = X$. To show this, let x be an arbitrary element of X. Then $x \in U_{\lambda,\alpha}$ for some λ, $\alpha < \kappa$ because $\{U_{\lambda,\alpha} \mid \lambda, \alpha < \kappa\}$ is a cover of X. Let us put $\beta = \psi(\lambda, \alpha)$. Then $x \in V_\beta$.

EXAMPLE 3.1. There exist spaces X and Y and a quasi-perfect map $f : X \longrightarrow Y$ such that X has the weak $\mathcal{B}$-property and Y has not the weak $\mathcal{B}$-property. Let X' be the product space $W(\omega_1 + 1) \times W(\omega_1 + 1)$ and $X = X' - \{(\omega_1, \omega_1)\}$ with the subspace topology of X'. Then, since X is countably compact, X has the property $\mathcal{B}^*(\omega)$. Also X has the property $\mathcal{B}^*(\kappa)$ for each $\kappa \geqq \omega_1$ by Lemma 3.3 because $w(X) =$

ω_1. Thus X has the weak $\mathcal{B}$-property. Let Y be the quotient space obtained from X by collapsing the subset $\{\omega_1\} \times \omega_1$ to a point y_0 and $f : X \longrightarrow Y$ be a quotient map. Then it is easy to see that Y has not the property $\mathcal{B}^*(\omega_1)$ and the map f is a quasi-perfect map.

EXAMPLE 3.2. There exist spaces A and B and a quasi-perfect map $g : A \longrightarrow B$ such that B has the weak $\mathcal{B}$-property but A has not the weak $\mathcal{B}$-property. Let A be the space Y in Example 3.1, B the one point space y_0, and let $g : A \longrightarrow B$ be a natural map. Then g is a quasi-perfect map.

REMARK 3.2. Example 3.1 also shows that " the countable paracompactness and the weak $\mathcal{B}$-property are not equivalent " (cf. [C_4, p. 739]). In fact the space Y in Example 3.1 is countably compact. But Y is not normal.

Recently M. E. Rudin proved that in case $\mathrm{cf}\kappa > \omega$ the κ-Dowker space constructed by her in [R_4] is countably paracompact ([R_7]). This shows that the countable paracompactness and the weak $\mathcal{B}$-property are not equivalent even for normal spaces.

REMARK 3.3. The theorems 3.1 and 3.2 also follows from Lemma 3.4.

3. The relations among κ-paracompactness, the property $\mathcal{B}(\kappa)$ and the property $\mathcal{B}^*(\kappa)$

In this section we assume that all spaces are regular spaces. P. Zenor Z has introduced the property $\mathcal{B}$. It is known the following relations hold :

$$\text{paracompact} \underset{[Z]}{\longrightarrow} \mathcal{B} \longrightarrow \text{weak } \mathcal{B} \underset{[I_2]}{\longrightarrow} \text{countably paracompact}$$

The non-equivalence of the property $\mathcal{B}$ and the weak $\mathcal{B}$-property has been shown by Yasui [Y_4] and recently the non-equivalence of the paracompactness and the property $\mathcal{B}$ has been shown by M. E. Rudin in [R_7]. And the weak $\mathcal{B}$-property and the countable paracompactness are also not equivalent (cf. Remark 3.2 in § 2).

On the other hand, Yasui [Y_5] defined the property $\mathcal{B}(\kappa)$ for each infinite cardinal κ and proved a characterization theorem for the property $\mathcal{B}(\kappa)$ by the normality of product spaces (see § 4 Theorem C).

A space X is called κ-paracompact if any open cover $\mathcal{U} = \{U_\alpha \mid \alpha < \kappa\}$ of X has a locally finite open refinement ([M_5]).

A space X is said to have the property $\mathcal{B}(\kappa)$ if for any decreasing family $\{F_\alpha \mid \alpha < \kappa\}$ of closed subsets of X with $\bigcap\{F_\alpha \mid \alpha < \kappa\} = \emptyset$, there exists a decreasing family $\{G_\alpha \mid \alpha < \kappa\}$ of open subsets of X such that $G_\alpha \supset F_\alpha$ for each $\alpha < \kappa$ and $\bigcap\{\mathrm{Cl}G_\alpha \mid \alpha < \kappa\} = \emptyset$ ([Y_5]).

A space X is said to have the property $\mathcal{B}$ iff X has the property $\mathcal{B}(\kappa)$ for every infinite cardinal κ [Z].

It is easy to see that for each infinite cardinal κ,

(a) $\mathcal{B}(\kappa) \rightleftarrows \mathcal{B}(\mathrm{cf}\kappa)$,

(b) $\mathcal{B}^*(\kappa) \longleftarrow \mathcal{B}^*(\mathrm{cf}\kappa)$,

where $\mathrm{cf}\kappa$ is the cofinality of κ.
But the converse of (b) does not necessarily hold (see Proof of Theorem 3.9 (2), Part 3)).

Let us consider the following conditions (i), (ii) and (iii).

(i) κ-paracompactness,

(ii) the property $\mathcal{B}(\kappa)$,

(iii) the property $\mathcal{B}^*(\kappa)$.

It is known that the countable paracompactness, the property $\mathcal{B}(\omega)$ and the property $\mathcal{B}^*(\omega)$ are equivalent [I_2]. Also it is obvious that κ-paracompactness implies the property $\mathcal{B}(\kappa)$ for normal spaces and the property $\mathcal{B}(\kappa)$ implies the property $\mathcal{B}^*(\kappa)$ for each κ. But, even for normal spaces, the conditions (i), (ii) and (iii) are not equivalent in general. In fact, the following holds.

THEOREM 3.9. (1) There is a normal space X which shows non-equivalence of κ-paracompactness and the property $\mathcal{B}(\kappa)$ for each regular cardinal $\kappa \geqq \omega_2$.

(2) For each $\kappa \geqq \omega_1$, there exists a normal space X_κ which shows non-equivalence of the property $\mathcal{B}(\kappa)$ and the

property $\mathcal{B}^*(\kappa)$.

LEMMA 3.2'. For each κ, if X is a κ-compact spaces, then X has the property $\mathcal{B}(\kappa)$.

The minimum cardinal number κ such that every open cover of a space X has an open refinement of cardinality $\leqq \kappa$ is called the Lindelöf number of a space X and denoted by $L(X)$ [J].

LEMMA 3.5. A space X has the property $\mathcal{B}(\lambda)$ for each λ with $\mathrm{cf}\lambda \geqq L(X)^+$. Here α^+ denotes the cardinal successor of α.

Proof. Let $L(X) = \kappa$ and λ be a cardinal with $\mathrm{cf}\lambda \geqq \kappa^+$. Let $\{F_\alpha \mid \alpha < \kappa^+\}$ be a decreasing family of closed subsets of X with $\bigcap\{F_\alpha \mid \alpha < \kappa^+\} = \emptyset$. Let us put $G_\alpha = X - F_\alpha$. Then we have $\bigcup\{G_\alpha \mid \alpha < \kappa^+\} = X$. Since $L(X) = \kappa$, there is $\alpha_0 < \kappa^+$ such that $\bigcup\{G_\alpha \mid \alpha \leqq \alpha_0\} = X$. Then $F_\alpha = \emptyset$ for each $\alpha > \alpha_0$. Let us define

$$H_\alpha = \begin{cases} X & \text{if } \alpha \leqq \alpha_0, \\ \emptyset & \text{if } \alpha > \alpha_0. \end{cases}$$

Then $\{H_\alpha \mid \alpha < \kappa\}$ is a decreasing family of open sets in X such that $H_\alpha \supset F_\alpha$ and $\bigcap\{\mathrm{Cl}H_\alpha \mid \alpha < \kappa^+\} = \emptyset$. The proof of Lemma 3.5 is complete.

COROLLARY 3.5. A space X has the property $\mathcal{B}(\lambda)$ for each λ with $\mathrm{cf}\lambda \geqq w(X)^+$.

PROOF OF THEOREM 3.9 devided two parts. Proof of (1). Let us put $X = W(\omega_1)$. Then, by Corollary 3.5, X has the property $\mathcal{B}(\kappa)$ for every regular cardinal $\kappa \geqq \omega_2$. It is well known that X is not κ-paracompact for every $\kappa \geqq \omega_1$. Thus X shows non-equivalence of (i) and (ii) for every regular cardinal $\kappa \geqq \omega_2$.

Proof of (2). 1) In case κ is a regular cardinal such that $\kappa \geqq \omega_1$. Let us put $X_\kappa = W(\kappa)$. Then, by [Z, Collorary 2.2], X_κ has not the property $\mathcal{B}$ because X_κ is a countably compact regular space which is not compact. By Lemma 3.2' and Collorary 3.5, X_κ has not the property $\mathcal{B}(\kappa)$. On the other hand, by Corollary 3.3, X_κ has the property $\mathcal{B}^*(\kappa)$. Therefore X_κ shows non-equivalence of (ii) and (iii).

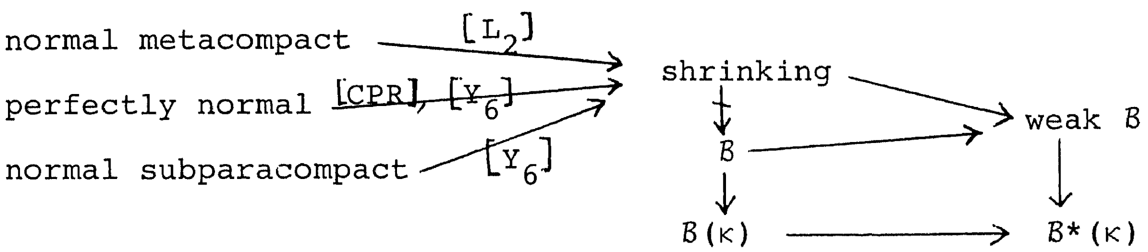
2) In case κ is an irregular cardinal with $\mathrm{cf}\kappa \geqq \omega_1$. Let us put $\lambda = \mathrm{cf}\kappa$. Then $\lambda \geqq \omega_1$ and λ is a regular cardinal. Let us put $X_\kappa = W(\lambda)$. Then X_κ has not the property $\mathcal{B}(\lambda)$. Since the property $\mathcal{B}(\kappa)$ is equivalent to the property $\mathcal{B}(\lambda)$, X_κ has not the property $\mathcal{B}(\kappa)$. On the other hand, by Corollary 3.3, X_κ has the property $\mathcal{B}^*(\kappa)$.

3) In case κ is an irregular cardinal with $\mathrm{cf}\kappa = \omega$. Then we have $\kappa \geqq \omega_\omega$. Let X be the Dowker space constructed by Rudin [R_1]. Then, since $|X| \leqq \omega_\omega$, by Theorem 3.5, X has the property $\mathcal{B}^*(\kappa)$. Since X is not countably paracompact, X has not the property $\mathcal{B}(\omega)$. Since the property $\mathcal{B}(\kappa)$ is equivalent to the property $\mathcal{B}(\omega)$, X has not the property $\mathcal{B}(\kappa)$.

REMARK 3.4. K. Navy constructed in [N_4] the space S which is normal and countably paracompact but not ω_1-paracompact. Recently Rudin proved in [R_7] that the space S has the property $\mathcal{B}$. This shows that " ω_1-paracompactness " and " the property $\mathcal{B}(\omega_1)$ " are not equivalent even for normal and countably paracompact spaces.

REMARK 3.5. It is easy to see that for any regular cardinals κ and λ with $\kappa \neq \lambda$, there exists a normal space X which has the property $\mathcal{B}(\kappa)$ but has not the property $\mathcal{B}(\lambda)$ (see Proof of Theorem 3.9 (2), part 1)).

The following relations hold :



These implications, it is shown that " shrinking " does not imply " the property $\mathcal{B}$ ". In fact, let X be a Σ-product of spaces $\{X_\alpha \mid \alpha \in \Omega\}$ where each X_α is a discrete space of two points and $|\Omega| \geqq \omega_1$. Then X is shrinking [R_5] but X has not the property $\mathcal{B}$ by [Z] because X is countably compact and not compact. But this space X is not metacompact, not subparacompact and not perfectly normal.

Next we shall show that " perfectly normal metacompact and subparacompact " does not imply " the property $\mathcal{B}$ ", more precisely, for each uncountable regular cardinal κ, " perfectly normal metacompact and subparacompact " does not imply " the property $\mathcal{B}(\kappa)$ ". That is, we shall prove

THEOREM 3.10. For each uncountable regular cardinal κ, there exists a space X_κ satisfying the following conditions :

(i) X_κ is perfectly normal metacompact and subparacompact,

(ii) X_κ has not the property $\mathcal{B}(\kappa)$.

To prove Theorem 3.10, we shall use the Bing's Example H [B_1]. We stated this space in chapter 1, but we restate it here again.

The Bing's Example H. Let P be an uncountable set, $Q = \mathcal{P}(P)$. Let F be the set of all functions $f : Q \longrightarrow \omega$. To each $p \in P$, associate the function $f_p \in F$ defined by $f_p(q) = 1$ iff $p \in q$. Let $F_P = \{f_p \mid p \in P\}$. Let $R = Q^f$. For each f_p and each $(r, n) \in R \times \omega$, we define a subset $V(f_p ; r,n) = \{f \mid f \in F,\ f(q) > n$ for $q \in Q$, $f(q) = f_p(q)$ mod 2 for $q \in r\} \cup \{f_p\}$. For each f in F, we define an open neifhborhood base $\mathcal{V}(f)$ at f as follows :

$$\mathcal{V}(f) = \begin{cases} \{f\} & \text{if } f \in F - F_P, \\ \{V(f_p ; r,n) \mid (r,n) \in R \times \omega\} & \text{if } f = f_p \in F_P. \end{cases}$$

F is a normal σ-space (in the sense of Okuyama [O_3]).

Michael [M_2] presented a subspace M of F which is metacompact. We recall the definition of M. $M = F_P \cup \{f : Q \longrightarrow \omega,\ f(q) = \text{odd for at most finitely many } q \in Q\}$.

Since the Bing's Example H constructed above depend upon only the cardinal number of a set P, we denote the space F by $H(\kappa)$ when $|P| = \kappa$; in this case we denote the subspace M of $H(\kappa)$ by $M(\kappa)$.

Let us put $I_\kappa = \kappa + 1$ and introduce the following topology in I_κ :

1) every point $\alpha < \kappa$ is isolated in I_κ , and

2) for each $\alpha < \kappa$, let us put $U_\alpha = (\alpha, \kappa]$ (the set $\{\gamma \mid \alpha < \gamma \leqq \kappa\}$ of ordinals) and we take $\{U_\alpha \mid \alpha < \kappa\}$ as an open neighborhood base at κ in I_κ ([Y5]).

To prove Theorem 3.10, we use the following theorem of Yasui.

THEOREM C(Yasui[Y5]). Let X be a normal space. Then X has the property $\mathcal{B}(\kappa)$ if and only if the topological product $X \times I_\kappa$ of X and I_κ is normal.

PROPOSITION 3.3. Let Y be a space which has only one non-isolated point y_0 and $|Y| = \kappa$ is an uncountable regular cardinal and has an open neighborhood base $\mathcal{U}(y_0) = \{U_y \mid y \in Y - \{y_0\}\}$ of y_0 in Y satisfying the condition (**) : if $Y' \subset Y - \{y_0\}$ and $|Y'| = \kappa$, then (1) $\{U_y \mid y \in Y'\}$ is not point-finite at some point $y' \in Y'$, (2) $\bigcap\{U_y \mid y \in Y'\} = \{y_0\}$. Then $H(\kappa) \times Y$ and $M(\kappa) \times Y$ are not normal.

Proof. We write H and M instead of $H(\kappa)$ and $M(\kappa)$ for the sake of brebity. Since M is closed in H, it is sufficient to prove that $M \times Y$ is not normal. Let us put

$P = Y - \{y_0\}$ and let us put $A = F_P \times \{y_0\}$ and $B = \bigcup\{\{f_p\} \times (Y - U_p) \mid p \in P\}$. Then A and B are disjoint closed subsets of $M \times Y$. We shall prove that A and B can not be separated by open sets in $M \times Y$.

Let G be an arbitrary open set in $M \times Y$ such that $A \subset G$. For each $p \in P$, we can choose $p' \in P$, $r_p \in R$ and $n_p < \omega$ such that

i) $\bigcup\{(V(f_p\,; r_p\,, n_p) \cap M) \times U_{p'} \mid p \in P\} \subset G$

and

ii) if $p_1 \neq p_2$, then $p_1' \neq p_2'$.

This choice is possible because $\{U_p \mid p \in P - P'\}$ is also a neighborhood base of y_0 for any subset P' of P with $|P'| < \kappa$.

By applying Šanin's lemma (Lemma 1.3 in chapter 1) for $\{r_p \mid p \in P\}$, we obtain a subset $\tilde{P}$ of P such that $|\tilde{P}| = \kappa$ and $\{r_p - \tilde{r} \mid p \in \tilde{P}\}$ is disjoint where $\tilde{r} = \bigcap\{r_p \mid p \in \tilde{P}\}$. Let us put $\tilde{R} = \mathcal{P}(\tilde{r})$. For each $s \in \tilde{R}$, we define an element q_s of Q by

$$q_s = \bigcap\{q \mid q \in s\} - \bigcup\{q \mid q \in \tilde{r} - s\}.$$

Here in case $s = \emptyset$, q_s means the element $P - \bigcup\{q \mid q \in \tilde{r}\}$ of Q. Then $\{q_s \mid s \in \tilde{R}\}$ is a finite cover of P because for each $p \in P$, if we put $s = \{q \in \tilde{r} \mid p \in q\}$, then $p \in q_s$. Therefore, we can choose a member s_0 of $\tilde{R}$ such that $|\tilde{P} \cap q_{s_0}| = \kappa$. Since $|\{p' \mid p \in \tilde{P} \cap q_{s_0}\}| = \kappa$, a collection $\{ U_{p'} \mid p \in \tilde{P} \cap q_{s_0}\}$ is not point-finite at some point

$p^* \in \{p' \mid p \in \tilde{P} \cap q_{s_0}\}$. Let us put $\tilde{\tilde{P}} = \{p \mid p \in \tilde{P} \cap q_{s_0}, p^* \in U_{p'}\}$. Then $\tilde{\tilde{P}}$ is an infinite set. Since $|\tilde{P} \cap q_{s_0}| = \kappa$, we have $\cap\{U_p \mid p \in \tilde{P} \cap q_{s_0}\} = \{y_0\}$. Therefore there is an element $p_1 \in \tilde{P} \cap q_{s_0}$ such that $p^* \notin U_{p_1}$. Then $(f_{p_1}, p^*) \in B$. Further we have $(f_{p_1}, p^*) \in \mathrm{Cl}G$. To show this, let $V(f_{p_1}; r, n)$ be an arbitrary basic neighborhood of f_{p_1} in H. Then there is an element $p_2 \in \tilde{\tilde{P}}$ such that $r \cap (r_{p_2} - \tilde{r}) = \emptyset$ because r is finite and a collection $\{r_p - \tilde{r} \mid p \in \tilde{\tilde{P}}\}$ is disjoint and $\tilde{\tilde{P}}$ is infinite. Let us define f as follows :

$$f(q) = \begin{cases} 2(n + n_{p_2}) + 1 & \text{if } p_1 \in q \in r \text{ or } p_2 \in q \in r_{p_2}, \\ 2(n + n_{p_2}) + 2 & \text{otherwise.} \end{cases}$$

Then, from the fact that $p_1, p_2 \in q_{s_0}$ and $r \cap (r_{p_2} - \tilde{r}) = \emptyset$, it is shown that for each $q \in r \cap r_{p_2}$, $p_1 \notin q$ iff $q \notin s_0$ iff $p_2 \notin q$. Therefore
$f \in V(f_{p_1}; r, n) \cap V(f_{p_2}; r_{p_2}, n_{p_2})$ (see [CC_2, p. 170 - 171]).
Also it is clear that $f \in M$. Since $p_2 \in \tilde{\tilde{P}}$, we have $p^* \in U_{(p_2)'}$. Hence
$((V(f_{p_1}; r, n) \cap M) \times \{p^*\}) \cap ((V(f_{p_2}; r_{p_2}, n_{p_2}) \cap M) \times U_{(p_2)'})$
$\neq \emptyset$. This proves that $(f_{p_1}, p^*) \in \mathrm{Cl}G$, because
$(V(f_{p_2}; r_{p_2}, n_{p_2}) \cap M) \times U_{(p_2)'} \subset G$.

Thus $B \cap \mathrm{Cl}G \neq \emptyset$. Hence $M \times Y$ is not normal.

PROOF OF THEOREM 3.10. Let κ be an arbitrary uncountable regular cardinal. Let us put $X_\kappa = M(\kappa)$. Then X_κ is a normal σ-space and so X_κ is perfectly normal and subparacompact. Further X_κ is metacompact. Since the space I_κ satisfies the conditions of Y in Proposition 3.3, $X_\kappa \times I_\kappa$ is not normal. Therefore X_κ has not the property $\mathcal{B}(\kappa)$ by Theorem C. Thus the proof of Theorem 3.10 is complete.

REMARK 3.6. Since $\{\kappa\}$ is not a G_δ-set in I_κ, it is shown that $M_\kappa \times I_\kappa$ is not normal by the analogous proof to that of [C_3, Theorem 1], too.

REMARK 3.7. Recently, A. Bešlagić and M. E. Rudin constructed in [BR] a normal space which has the weak $\mathcal{B}$-property but is not shrinking under the assumption of the Godel's Axiom of Constructibility $V = L$.

REMARK 3.8. The author does not know whether there exists a normal space which has the property $\mathcal{B}$ but is not shrinking or not.

4. The property $\mathcal{B}^*(\kappa)$

In this section, we shall show that for each pair of regular cardinals κ and λ with $\kappa \neq \lambda$, the property $\mathcal{B}^*(\kappa)$ does not imply the property $\mathcal{B}^*(\lambda)$ and for each regular cardinal κ, the property $\mathcal{B}^*(\kappa)$ is not preserved by product spaces.

THEOREM 3.11. For any regular cardinals κ and λ with $\kappa \neq \lambda$, there exists a space Z which has the property $\mathcal{B}^*(\kappa)$ but has not the property $\mathcal{B}^*(\lambda)$.

Proof. We distinguish two cases. (i) In case $\lambda < \kappa$. Let us put $X = W(\lambda^+)$. Let us denote by $\lambda \times \lambda^+$ the set $\{(\alpha, \beta) \mid \alpha < \lambda, \ \beta < \lambda^+\}$ of pairs of ordinals. Let us put $Y = (\lambda \times \lambda^+) \cup \{(\lambda, \lambda^+)\}$ and introduce the following topology in Y :

1) each point $y \in \lambda \times \lambda^+$ is isolated in Y, and

2) we take $\{([\alpha, \lambda] \times [\beta, \lambda^+]) \cap Y \mid \alpha < \lambda, \ \beta < \lambda^+\}$ as an open neighborhood base at (λ, λ^+) in Y.

Here $[\alpha, \lambda]$ denotes the set $\{\beta \mid \alpha \leqq \beta \leqq \lambda\}$ of ordinals. Let Z be the topological product $X \times Y$. Then Z is a completely regular space and has the property $\mathcal{B}^*(\kappa)$ by Lemma 3.3 because $w(Z) = \lambda^+ \leqq \kappa$. To prove that Z has not the property $\mathcal{B}^*(\lambda)$, we first prove the following 3).

3) Let us put $A = W(\lambda^+) \times \{(\lambda, \lambda^+)\}$ and $B = \{(\alpha, \beta, \alpha) \mid \alpha < \lambda^+, \ \beta < \lambda\}$. Then A and B are disjoint closed subsets of Z. And A and B are not separated by open sets in Z.

Proof of 3). Let G be an arbitrary open set in Z such that $A \subset G$. For each $\alpha < \lambda^+$, there exist $\beta_\alpha < \lambda$ and $\gamma_\alpha < \lambda^+$ such that $\{\alpha\} \times (([\beta_\alpha, \lambda] \times [\gamma_\alpha, \lambda^+]) \cap Y) \subset G$ by 2). Further we see that there exists $\beta^* < \lambda$ such that $|\{\alpha \mid \alpha < \lambda^+, \beta_\alpha = \beta^*\}| = \lambda^+$. Let us put $\Omega = \{\alpha \mid \alpha < \lambda^+, \beta_\alpha = \beta^*\}$. Then Ω is cofinal in λ^+. Let $\alpha_0 \in \Omega$ be an arbitrary element. Let us choose $\alpha_0' < \lambda^+$ such that $\alpha_0, \gamma_{\alpha_0} < \alpha_0'$. By induction, we can choose a sequence $\{\alpha_i \mid i < \omega\}$ in Ω and a sequence $\{\alpha_i' \mid i < \omega\}$ in λ^+ such that $\gamma_{\alpha_i} < \alpha_i' < \alpha_{i+1} < \alpha_{i+1}'$ for each $i < \omega$. Let us put $\alpha^* = \sup\{\alpha_i \mid i < \omega\}$. Then we also have $\alpha^* = \sup\{\alpha_i' \mid i<\omega\}$. Thus it is easy to see that $(\alpha^*, \beta^*, \alpha^*) \in B \cap \mathrm{Cl}G$ (cf. [C_2, Example 2], or see Example 1.3 in chapter 1).

4) For each $\beta < \lambda$, let us put $F_\beta = \{(\alpha, \gamma, \alpha) \mid \alpha < \lambda^+, \beta \leqq \gamma < \lambda\}$. Then, similarly to 3), we can prove that A and F_β can not be separated by open sets in Z. Therefore, for each open set G of Z such that $F_\beta \subset G$, we have $A \cap \mathrm{Cl}G \neq \emptyset$.

Now we shall prove

5) Z has not the property $\mathcal{B}^*(\lambda)$.

Proof of 5). Let F_β be the set defined in 4). Then $\mathcal{F} = \{F_\beta \mid \beta < \lambda\}$ is a decreasing family of closed subsets of Z with $\bigcap\{F_\beta \mid \beta < \lambda\} = \emptyset$. Let G_β be an arbitrary open set of Z such that $F_\beta \subset G_\beta$ for each $\beta < \lambda$. For the proof of 5), it is sufficient to prove the following relation

(a) $\bigcap\{\mathrm{Cl}G_\beta \mid \beta < \lambda\} \neq \emptyset$.

To prove (a), we denote by $\oplus\{\Gamma_\alpha \mid \alpha < \lambda\}$ the disjoint sum of Γ_α where Γ_α is a copy of λ for each $\alpha < \lambda$. Since $|\oplus\{\Gamma_\alpha \mid \alpha < \lambda\}| = \lambda$, there is an one-to-one onto mapping $f : \oplus\{\Gamma_\alpha \mid \alpha < \lambda\} \longrightarrow \lambda$. Let us put $\Lambda_\alpha = f(\Gamma_\alpha)$. Then Λ_α is cofinal in λ for each $\alpha < \lambda$ and $\lambda = \oplus\{\Lambda_\alpha \mid \alpha < \lambda\}$. For each $\beta < \lambda$ and each $\gamma \in \Lambda_\beta$, let us put

$$H_\gamma = \begin{cases} G_\beta \cap G_\gamma & \text{if } \gamma \geqq \beta, \\ G_\gamma & \text{if } \gamma < \beta. \end{cases}$$

For each $\alpha < \lambda$, there exists $\beta < \lambda$ uniquely with $\alpha \in \Lambda_\beta$. Hence H_α is defined for each $\alpha < \lambda$ and we have

(b) $F_\alpha \subset H_\alpha$ for $\alpha < \lambda$, and

(c) if $\alpha \in \Lambda_\beta$ and $\alpha \geqq \beta$, then $H_\alpha \subset G_\beta$.

By 4) and (b), we have $A \cap \mathrm{Cl}H_\alpha \neq \emptyset$. Thus there is a point $(\tau_\alpha, \lambda, \lambda^+) \in A \cap \mathrm{Cl}H_\alpha$ for each $\alpha < \lambda$. Let us put $\tau^* = \sup\{\tau_\alpha \mid \alpha < \lambda\}$. Then $\tau^* < \lambda^+$ and we see that

(d) $(\tau^*, \lambda, \lambda^+) \in \mathrm{Cl}G_\alpha$ for each $\alpha < \lambda$.

Proof of (d). Let α be an arbitrary ordinal with $\alpha < \lambda$ and O an arbitrary open neighborhood of $(\tau^*, \lambda, \lambda^+)$ in Z. Since $\tau^* = \sup\{\tau_\alpha \mid \alpha < \lambda\}$, there exists an element $\alpha_0 < \lambda$ such that $(\tau_\beta, \lambda, \lambda^+) \in O$ whenever $\beta \geqq \alpha_0$. Since Λ_α is cofinal in λ, there is an element $\gamma \in \Lambda_\alpha$ such that $\gamma \geqq \alpha_0$, α. Then $(\tau_\gamma, \lambda, \lambda^+) \in O$. Further, we have $(\tau_\gamma, \lambda, \lambda^+) \in \mathrm{Cl}G_\alpha$ because $(\tau_\gamma, \lambda, \lambda^+) \in \mathrm{Cl}H_\gamma$ and $H_\gamma \subset G_\alpha$ by (c). Hence $\mathrm{Cl}G_\alpha \cap O \neq \emptyset$ and so $(\tau^*, \lambda, \lambda^+) \in \mathrm{Cl}G_\alpha$. The proof of (d) is complete and hense (a) is proved.

Thus the proof of 5) is complete.

(ii) In case $\lambda > \kappa$. Let us put $X = W(\lambda+1)$. Let Y' be the product space $X \times X$ and $Y = Y' - \{(\lambda, \lambda)\}$ with the subspace topology of Y'. Let Z be the quotient space obtained from Y by collapsing the subset $\{\lambda\} \times \{\alpha \mid 0 \leqq \alpha < \lambda\}$ to a point z_0. Then it is easy to see that Z has not the property $\mathcal{B}^*(\lambda)$. On the other hand, since Z is κ-compact, Z has the property $\mathcal{B}^*(\kappa)$ by Lemma 3.2 in §1. The proof of Theorem 3.11 is complete.

EXAMPLE 3.3. For each regular cardinal κ, there are spaces X and Y both of which have the property $\mathcal{B}^*(\kappa)$ but $X \times Y$ has not the property $\mathcal{B}^*(\kappa)$.

Let $X = W(\kappa^+)$ and $Y = (\kappa \times \kappa^+) \cup \{(\kappa, \kappa^+)\}$ constructed in the proof of Theorem 3.11, case (i). Then both X and Y have the property $\mathcal{B}^*(\kappa)$ but $X \times Y$ has not the property $\mathcal{B}^*(\kappa)$. Therefore for each regular cardinal κ, the property $\mathcal{B}^*(\kappa)$ is not preserved by product spaces.

CHAPTER 4

Σ-PRODUCTS

All spaces in this chapter are assumed to be completely regular Hausdorff.

1. The weak $\mathcal{B}$-property and Σ-products

H. H. Corson defined in [C_{10}] an important and useful class of subspaces of product spaces, called Σ-products.

DEFINITION (Corson [C_{10}]). Let $x^* = (x^*_\alpha)_{\alpha \in A}$ be a specific point of a product $X = \Pi\{X_\alpha \mid \alpha \in A\}$. For $x \in X$ let us put $Q(x) = \{\alpha \in A \mid x_\alpha \neq x^*_\alpha\}$. The subspace $\Sigma\{X_\alpha \mid \alpha \in A\} = \{x \in X \mid |Q(x)| \leqq \omega\} \subset X$ is called a Σ-product of the spaces X_α, $\alpha \in A$. The point x^* is called a base point.

A Σ-product of spaces $\{X_\alpha \mid \alpha \in A\}$ is a proper subspace of the product space $\Pi_{\alpha \in A} X_\alpha$ if and only if uncountably many spaces X_α contain at least two elements. Such Σ-products we will call *proper*.

Every proper Σ-product can not be paracompact ([C_{10}]). Similarly we see that every proper Σ-product can not be metacompact, can not be subparacompact and has not the $\mathcal{B}$-property (see Proposition 4.3 below). It is easy to see that every proper Σ-product can not be perfectly normal. Further, a Σ-product of metrizable spaces need not be an M-space (see Proposition 4.4 below).

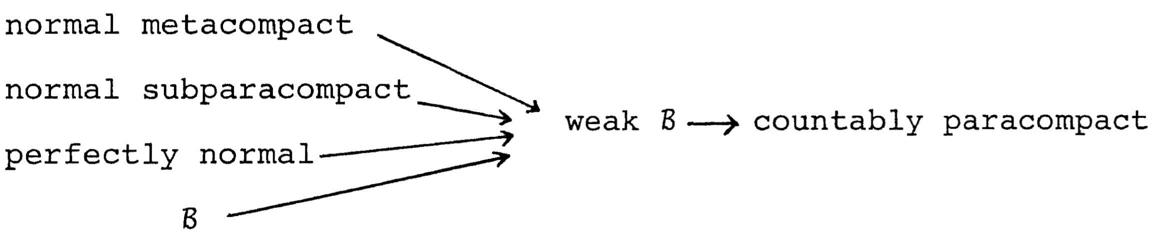
However, many Σ-products are countably paracompact. For example, the following are known.

(I) Every Σ-product of compact spaces is countably compact (H. H. Corson [C_{10}]).

(II) Every Σ-product of metrizable spaces is a collectionwise normal P-space (in the sense of K. Morita [M_7]) (S. P. Gul'ko [G] or M. E. Rudin [R_3] ; see also Proposition 4.2 below).

(III) Every normal Σ-product of strong Σ-spaces (in the sense of K. Nagami [N_2]) is countably paracompact (Y. Yajima [Y_2]).

Also the following relations are known (see chapter 3, § 3).



Therefore, it seems interesting to investigate the weak $\mathcal{B}$-property of Σ-products.

In the remainder of this section, we shall give several propositions as preliminaries and in the next section we shall state the results concerning the weak $\mathcal{B}$-property of Σ-products.

THEOREM A (Yajima [Y_3]). Every Σ-product of $\{X_\alpha \mid \alpha \in A\}$ where each X_α is a paracompact Σ-space of countable tightness, is normal.

This imply

PROPOSITION 4.1. Every Σ-product of $\{X_\alpha \mid \alpha \in A\}$ where each X_α is a paracompact Σ-space of countable tightness, is a normal P-space.

Proof. Let $X = \Sigma\{X_\alpha \mid \alpha \in A\}$ where each X_α is a paracompact Σ-space of countable tightness. Let Y be an arbitrary metrizable space. Then $X \times Y$ is the Σ-product of $\{X_\alpha \mid \alpha \in A\} \cup \{Y\}$. By Theorem of Yajima, $X \times Y$ is normal. Hence, by Theorem of Morita [M_7], X is a normal P-space.

PROPOSITION 4.2. Every Σ-product of metrizable spaces is a normal P-space.

PROPOSITION 4.3. Every proper Σ-product can not be metacompact, can not be subparacompact and has not the $\mathcal{B}$-property.

Proof. Let Σ_0 be the Σ-product of $\{X_\alpha \mid \alpha < \omega_1\}$ where each X_α is a discrete space of two points. Then Σ_0 is not metacompact by Theorem 5.3.2 in [E] because Σ_0 is countably compact and not compact. Similarly Σ_0 is not subparacompact by [B_2] and Σ_0 has not the $\mathcal{B}$-property by [Z]. Every proper Σ -product contains Σ_0 as a closed subset. Therefore, such a Σ-product is never metacompact, never subparacompact and never has the $\mathcal{B}$-property.

A Σ-product of metrizable spaces need not be an M-space. In fact we have

PROPOSITION 4.4. Let X be a Σ-product of $\{X_\alpha \mid \alpha \in A\}$ where each X_α is metrizable. Then X is an M-space if and only if $\{\alpha \mid \alpha \in A,\ X_\alpha$ is not compact$\}$ is countable.

Proof. We may assume $A = \tau$ where τ is an uncountable cardinal. Let $X = \Sigma\{X_\alpha \mid \alpha < \tau\}$ where each X_α is metrizable. Put $\Lambda = \{\alpha \mid \alpha < \tau,\ X_\alpha$ is not compact$\}$. If Λ is countable, put $Y = \Pi\{X_\alpha \mid \alpha \in \Lambda\}$ and $Z = \Sigma\{X_\alpha \mid \alpha \in \tau - \Lambda\}$. Then Y is metrizable, Z is countably compact and X is homeomorphic to $Y \times Z$. Hence X is an M-space.

If Λ is uncountable, we put $Y = \Sigma\{X_\alpha \mid \alpha \in \Lambda\}$, then Y is a closed subset of X. Let $x^* = (x^*_\alpha)_{\alpha<\tau}$ be the base point of X and denote $x^* = (y^*, z^*)$, $y^* \in Y$. We shall show that Y is not an M-space, in fact Y is not a *q-space*. Let $\{U_n \mid n < \omega\}$ be an arbitrary sequence of open neighborhoods of y^* in Y. Let V_n be an open set in $\Pi\{X_\alpha \mid \alpha \in \Lambda\}$ such that $V_n \cap Y = U_n$. Put $\Omega_n = \{\alpha \mid \alpha \in \Lambda,\ \pi_\alpha(V_n) \neq X_\alpha\}$ where $\pi_\alpha : \Pi\{X_\beta \mid \beta \in \Lambda\} \longrightarrow X_\alpha$ is the projection. Then Ω_n is finite. Therefore $\Omega = \bigcup\{\Omega_n \mid n < \omega\}$ is countable. Hence there is $\alpha_0 \in \Lambda - \Omega$. Let us put
$C = \Pi\{X_\alpha \mid \alpha \in \Omega \cup \{\alpha_0\}\} \times \{b^*\}$ where $y^* = (a^*, b^*)$,
$b^* \in \Pi\{X_\alpha \mid \alpha \in \Lambda - (\Omega \cup \{\alpha_0\})\}$. Then $C \subset Y$.

Here we choose a sequence $\{x^n_{\alpha_0} \mid n < \omega\}$ in X_{α_0} which has no accumulation point and define

$$y^n_\alpha = \begin{cases} x^n_{\alpha_0} & \text{if } \alpha = \alpha_0, \\ x^*_\alpha & \text{if } \alpha \neq \alpha_0,\ \alpha \in \Lambda. \end{cases}$$

Then $y^n = (y^n_\alpha)_{\alpha \in \Lambda} \in U_n$ for each $n < \omega$ but $\{y^n \mid n < \omega\}$ has no accumulation point in Y.

2. Theorems

THEOREM 4.1. Every Σ-product of compact spaces has the weak $\mathcal{B}$-property.

This follows from Theorem 4.2 below and the theorem (I) of Corson.

THEOREM 4.2. Suppose every countable product of spaces $\{X_\alpha \mid \alpha < \tau\}$ is Lindelöf. Let X be a Σ-product of $\{X_\alpha \mid \alpha<\tau\}$ Then X is countably paracompact iff X has the weak $\mathcal{B}$-property.

Proof. Let us assume that X is countably paracompact. We shall show that X has the weak $\mathcal{B}$-property. Let κ be an arbitrary infinite cardinal. Let us prove that X has the property $\mathcal{B}^*(\kappa)$. We may assume that κ is an uncountable regular cardinal because the property $\mathcal{B}^*(\mathrm{cf}\kappa)$ imply the property $\mathcal{B}^*(\kappa)$ (cf. chapter 3, §3, (b)). Let $\mathcal{F} = \{ F_\lambda \mid \lambda < \kappa\}$ be a decreasing family of closed sets of X such that $\bigcap\{F_\lambda \mid \lambda < \kappa\} = \emptyset$.

By transfinite induction, for each $\alpha < \omega_1$, we choose $\lambda(\alpha)$, Ω_α, $U_{\lambda(\alpha)}$ satisfying the following conditions (1) $\sim$ (4) :

(1) $\lambda(\alpha) < \kappa$, $\Omega_\alpha \subset \tau$, $|\Omega_\alpha| \leqq \omega$.

(2) $U_{\lambda(\alpha)}$ is open in X, $U_{\lambda(\alpha)} \supset F_{\lambda(\alpha)}$.

(3) For each $x \in \mathrm{Cl}U_{\lambda(\alpha)}$, $x_\delta \neq x^*_\delta$ for some $\delta \in \Omega_\alpha$ where $x^* = (x^*_\delta)_{\delta<\tau}$ is the base point of X.

(4) If $\alpha \neq \beta$, then $\lambda(\alpha) \neq \lambda(\beta)$ and $\Omega_\alpha \cap \Omega_\beta = \emptyset$.

For arbitrary $\alpha < \omega_1$, let us assume that we have choosen

$\lambda(\beta)$, Ω_β, $U_{\lambda(\beta)}$ for each β with $\beta < \alpha$. Put $Y_\alpha = \Pi\{X_\gamma \mid \gamma \in \bigcup_{\beta<\alpha}\Omega_\beta\}$, $Z_\alpha = \Sigma\{X_\gamma \mid \gamma \in \tau - \bigcup_{\beta<\alpha}\Omega_\beta\}$. Then $X = Y_\alpha \times Z_\alpha$. Let $x^* = (y^*, z^*)$, $y^* \in Y_\alpha$, $z^* \in Z_\alpha$. Let us put $K_\alpha = Y_\alpha \times \{z^*\}$. Then K_α is Lindelöf and $K_\alpha \subset X$. Since K_α is Lindelöf and $\{F_\lambda \mid \lambda < \kappa\}$ is a decreasing family with $\cap\{F_\lambda \mid \lambda < \kappa\} = \phi$ and $\mathrm{cf}\kappa > \omega$, we can choose $\lambda(\alpha)$ such that $F_{\lambda(\alpha)} \cap K_\alpha = \phi$. Here we can choose $\lambda(\alpha) \neq \lambda(\beta)$ for every $\beta < \alpha$.

For an arbitrary element (y, z^*) of K_α, there are an open neighborhood U_y of y and an open neighborhood V_y of z^* such that $(\mathrm{Cl}(U_y \times V_y)) \cap F_{\lambda(\alpha)} = \phi$. Since $\{U_y \mid y \in Y_\alpha\}$ is an open cover of Y_α and Y_α is Lindelöf, there is a locally finite countable open cover $\{W_i \mid i = 1,2,\ldots\}$ of Y_α such that $W_i \subset U_{y_i}$ for some $y_i \in Y_\alpha$ for $i = 1,2,\ldots$. Since $z^* \in V_{y_i}$, there is a finite subset Γ_i of $\tau - \bigcup_{\beta<\alpha}\Omega_\beta$ such that $V_{y_i} \supset \cap\{p_\gamma^{-1}(x_\gamma^*) \mid \gamma \in \Gamma_i\}$ where $p_\gamma : Z_\alpha \longrightarrow X_\gamma$ is the projection. Let us put $\Omega_\alpha = \bigcup_{i=1}^{\omega}\Gamma_i$ and $U_{\lambda(\alpha)} = X - \bigcup_{i=1}^{\omega}\mathrm{Cl}(W_i \times V_{y_i})$. Then $\lambda(\alpha)$, Ω_α, $U_{\lambda(\alpha)}$ satisfies (1) $\sim$ (4).

(1) and (4) are obvious. Since $\{W_i \times V_{y_i} \mid i = 1,2,\ldots\}$ is locally finite, $U_{\lambda(\alpha)}$ is open. Since $\mathrm{Cl}(W_i \times V_{y_i}) \subset \mathrm{Cl}(U_{y_i} \times V_{y_i})$, $(\mathrm{Cl}(W_i \times V_{y_i})) \cap F_{\lambda(\alpha)} = \phi$. Hence $U_{\lambda(\alpha)} \supset F_{\lambda(\alpha)}$.

To show (3), let $x \in \mathrm{Cl}U_{\lambda(\alpha)}$. Then $x \in X - \bigcup_{i=1}^{\omega}(W_i \times V_{y_i})$.

Let $x = (y, z)$, $y \in Y_\alpha$, $z \in Z_\alpha$. Then $y \in W_i$ for some i. Then $z \notin V_{y_i}$. Therefore x_δ (= z_δ) $\neq x_\delta^*$ for some $\delta \in \Gamma_i \subset \Omega_\alpha$.

Hence, by transfinite induction, for each $\alpha < \omega_1$, we can choose $\lambda(\alpha)$, Ω_α, $U_{\lambda(\alpha)}$ satisfying the conditions (1) $\sim$ (4).

For each $x \in \bigcap\{\mathrm{Cl}U_{\lambda(\alpha)} \mid \alpha < \omega_1\}$, $\{\delta \mid \delta < \tau,\ x_\delta \neq x_\delta^*\}$ is uncountable. But, this is a contradiction. Hence $\bigcap\{\mathrm{Cl}U_{\lambda(\alpha)} \mid \alpha < \omega_1\} = \emptyset$. For each $\lambda \in \kappa - \{\lambda(\alpha) \mid \alpha < \omega_1\}$, we put $U_\lambda = X$. Then $U_\lambda \supset F_\lambda$ for each $\lambda < \kappa$ and $\bigcap\{\mathrm{Cl}U_\lambda \mid \lambda < \kappa\} = \emptyset$. Hence X has the weak $\mathcal{B}$-property.

COROLLARY 4.1. Every Σ-product of Lindelöf Σ-spaces is countably paracompact iff it has the weak $\mathcal{B}$-property.

COROLLARY 4.2. Every Σ-product of Lindelöf Σ-spaces of countable tightness has the weak $\mathcal{B}$-property.

Proof. Let $X = \Sigma\{X_\alpha \mid \alpha < \tau\}$ where each X_α is a Lindelöf Σ-space of countable tightness. By Theorem A of Yajima, X is normal and by Proposition 4.1, X is countably paracompact. Since the countable product of Lindelöf Σ-spaces is Lindelöf, X has the weak $\mathcal{B}$-property by Theorem 4.2.

THEOREM 4.3. Suppose every countable product of spaces $\{X_\alpha \mid \alpha < \omega_1\}$ is paracompact. Let X be a Σ-product of $\{X_\alpha \mid \alpha < \omega_1\}$. Then X is countably paracompact iff X has the weak $\mathcal{B}$-property.

This follows from the following.

PROPOSITION 4.5. Suppose every countable product of spaces $\{X_\alpha \mid \alpha < \kappa\}$ is paracompact. Let X be a Σ-product of $\{X_\alpha \mid \alpha < \kappa\}$. Then X has the $\mathcal{B}^*(\mu)$-property for every $\mu \geqq \kappa$. Here $\kappa > \omega$.

Proof. Let $\mathcal{F} = \{F_\lambda \mid \lambda < \kappa\}$ be a decreasing family of closed subsets of X with $\bigcap\{F_\lambda \mid \lambda < \mu\} = \emptyset$ where $\mu \geqq \kappa$.

Let us denote $\{A \mid A \subset \kappa, \ |A| \leqq \omega\}$ by $\{A_\alpha \mid \alpha < \kappa\}$. For each $\alpha < \kappa$, there exists a cofinal set Λ_α of μ such that $\mu = \bigcup\{\Lambda_\alpha \mid \alpha < \kappa\}$ and $\Lambda_\alpha \cap \Lambda_\beta = \emptyset$ if $\alpha \neq \beta$.

For each $\alpha < \kappa$, let us put $Y_\alpha = \Pi\{X_\beta \mid \beta \in A_\alpha\}$ and $Z_\alpha = \Sigma\{X_\beta \mid \beta \in \kappa - A_\alpha\}$. Then $X = Y_\alpha \times Z_\alpha$. Let $x^* = (y^*, z^*)$, $y^* \in Y_\alpha$, $z^* \in Z_\alpha$ where x^* is the base point of X and put $K_\alpha = Y_\alpha \times \{z^*\}$. Then $K_\alpha \subset X$ and $\bigcap\{F_\lambda \mid \lambda \in \Lambda_\alpha\} = \emptyset$.

For each $\alpha < \kappa$ and each $\lambda \in \Lambda_\alpha$, let us put $B_{\lambda,\alpha} = F_\lambda \cap K_\alpha$ and let $B'_{\lambda,\alpha}$ be a subset of Y_α such that $B'_{\lambda,\alpha} \times \{z^*\} = B_{\lambda,\alpha}$. Then $\{B'_{\lambda,\alpha} \mid \lambda \in \Lambda_\alpha\}$ is a decreasing family of closed subsets of Y_α with $\bigcap\{B'_{\lambda,\alpha} \mid \lambda \in \Lambda_\alpha\} = \emptyset$ because $\bigcap\{F_\lambda \mid \lambda \in \Lambda_\alpha\} = \emptyset$. Since Y_α has the weak $\mathcal{B}$-property, there are open sets $U'_{\lambda,\alpha}$ in Y_α such that $B'_{\lambda,\alpha} \subset U'_{\lambda,\alpha}$ for each $\lambda \in \Lambda_\alpha$ and $\bigcap\{\mathrm{Cl}_{Y_\alpha} U'_{\lambda,\alpha} \mid \lambda \in \Lambda_\alpha\} = \emptyset$. Let us put $C_{\lambda,\alpha} = (Y_\alpha - U'_{\lambda,\alpha}) \times \{z^*\}$. Then $C_{\lambda,\alpha}$ is a closed set of K_α and $C_{\lambda,\alpha} \cap F_\lambda = \emptyset$.

Claim. There is an open set $W_{\lambda,\alpha}$ in X such that $C_{\lambda,\alpha} \subset W_{\lambda,\alpha}$, $(\mathrm{Cl} W_{\lambda,\alpha}) \cap F_\lambda = \emptyset$.

Proof. For each $y \in C'_{\lambda,\alpha} = Y_\alpha - U'_{\lambda,\alpha}$, there exist an open neighborhood M_y of y in Y_α and an open neighborhood N_y of z^* in Z_α such that $(\mathrm{Cl}(M_y \times N_y)) \cap F_\lambda = \emptyset$. Then $\{M_y \mid y \in C'_{\lambda,\alpha}\}$ is an open cover of $C'_{\lambda,\alpha}$. Since Y_α is paracompact and $C'_{\lambda,\alpha}$ is closed in Y_α, there exists a locally finite open family $\mathcal{V}$ in Y_α such that $\mathcal{V}$ covers $C'_{\lambda,\alpha}$ and $\mathcal{V}$ is a refinement of $\{M_y \mid y \in C'_{\lambda,\alpha}\}$. For each $V \in \mathcal{V}$, let's choose a point $y_V \in C'_{\lambda,\alpha}$ such that $V \subset M_{y_V}$ and put $W_{\lambda,\alpha} = \bigcup\{V \times N_{y_V} \mid V \in \mathcal{V}\}$. Then $W_{\lambda,\alpha}$ satisfies the conditions in Claim.

Let us put $U_{\lambda,\alpha} = X - \mathrm{Cl}W_{\lambda,\alpha}$ for each $\lambda \in \Lambda_\alpha$ and each $\alpha < \kappa$. Then $U_{\lambda,\alpha}$ is open in X and $U_{\lambda,\alpha} \supset F_\lambda$. Further $\bigcap_{\alpha<\kappa} \bigcap_{\lambda\in\Lambda_\alpha} \mathrm{Cl}U_{\lambda,\alpha} = \emptyset$ holds.

To show this, let x be an arbitrary element of X. Then $x \in K_\alpha$ for some $\alpha < \kappa$. Since, for each $\lambda \in \Lambda_\alpha$, $K_\alpha \subset (U'_{\lambda,\alpha} \times \{z^*\}) \cup W_{\lambda,\alpha}$ and $\bigcap_{\lambda \in \Lambda_\alpha} U'_{\lambda,\alpha} = \emptyset$, $x \in W_{\lambda,\alpha}$ for some $\lambda \in \Lambda_\alpha$. Then $x \notin \mathrm{Cl}U_{\lambda,\alpha}$ because $\mathrm{Cl}U_{\lambda,\alpha} \subset X - W_{\lambda,\alpha}$.

For each $\lambda < \mu$, there exists $\alpha < \kappa$ uniquely with $\lambda \in \Lambda_\alpha$. Let us put $P_\lambda = U_{\lambda,\alpha}$ where $\lambda \in \Lambda_\alpha$. Then P_λ is defined for each $\lambda < \mu$ and P_λ is open in X such that $F_\lambda \subset P_\lambda$ for each $\lambda < \mu$ and $\bigcap_{\lambda<\mu} \mathrm{Cl}P_\lambda = \emptyset$. The proof of Proposition 4.5 is complete.

COROLLARY 4.3. Let X be a Σ-product of paracompact Σ-spaces $\{X_\alpha \mid \alpha < \omega_1\}$. Then X is countably paracompact iff X has the weak $\mathcal{B}$-property.

COROLLARY 4.4. Let X be a Σ-product of paracompact Σ-spaces $\{X_\alpha \mid \alpha < \omega_1\}$. If X is normal, then X has the weak $\mathcal{B}$-property.

Proof. Since every countable product of paracompact Σ-spaces is a paracompact Σ-space ([N_2]), this corollary follows from Theorem A of Yajima and Theorem 4.3.

COROLLARY 4.5. Let X be a Σ-product of paracompact Σ-spaces of countable tightness $\{X_\alpha \mid \alpha < \omega_1\}$. Then X has the weak $\mathcal{B}$-property.

3. Remarks

REMARK 4.1. Let Σ_0 be the space in the proof of Proposition 4.3. Then Σ_0 is normal. But Σ_0 is not metacompact, not subparacompact, not perfectly normal and has not the $\mathcal{B}$-property. Moreover Σ_0 has the weak $\mathcal{B}$-property by Theorem 4.1. Therefore, Σ_0 gives a gap between each of the metacompactness, the subparacompactness, the perfect normality, the $\mathcal{B}$-property and the weak $\mathcal{B}$-property for normal spaces.

REMARK 4.2. It is not known whether we can replace " ω_1 " by " τ " for arbitrary uncountable cardinal τ in Theorem 4.3, Corollaries 4.3, 4.4 and 4.5 or not.

EXAMPLE 4.1. There exists a non-countably paracompact Σ-product of M_1-spaces.

Let X_0 be the space described in Example 1.3 in chapter 1. In fact, $X_0 = (\omega \times \omega_1) \cup \{(\omega, \omega_1)\}$ with the topology as follows :
$\{([\alpha, \omega] \times [\beta, \omega_1]) \cap X_0 \mid \alpha < \omega, \beta < \omega_1\}$ is a neighborhood base of (ω, ω_1) and for each $x \in X_0 - \{(\omega, \omega_1)\}$, x is an isolated point of X.

For each λ with $1 \leqq \lambda < \omega_1$, let X_λ be a discrete space of two points. Then $W(\omega_1)$ can be embedded as a closed subspace of $\Sigma\{X_\lambda \mid 1 \leqq \lambda < \omega_1\}$.

However, $X_0 \times W(\omega_1)$ is not countably paracompact (chapter 3, Proof of Theorem 3.11, (i)). Since Σ-product of $\{X_\lambda \mid 0 \leqq \lambda < \omega_1\}$ contains a non-countably paracompact closed subspace, it is not countably paracompact.

Let's remember the following relations :

$$\begin{array}{ccc} \text{shrinking} & \longrightarrow & \text{normal} \\ \downarrow & & \\ \text{weak } \mathcal{B} & \longrightarrow & \text{countably paracompact} \end{array}$$

Concerning the shrinking property of Σ-products, M. E. Rudin proved in [R_5] the following.

(IV) Every Σ-product of metrizable spaces is shrinking.

After that, A. L. Donne [D_1] obtained a nice generalization of (IV) as follows :

(V) Every Σ-product of paracompact M-spaces of countable tightness is shrinking.

To be a Σ-product of paracompact M-spaces $\{X_\alpha \mid \alpha \in A\}$ shrinking, the condition " each X_α has countable tightness " is necessary by Theorem of A. P. Kombarov [K_4]. But, in Corollaries 4.1 and 4.3 in this paper, the condition " countable tightness " is not necessary.

Thus, the following question seems natural from the results in this paper.

Question. Has every Σ-product of paracompact M-spaces the weak $\mathcal{B}$-property ?

REFERENCES

[A_1] A. V. Arhangel'skii : Some types of factormappings and the relations between classes of topological spaces, Soviet Math. Dokl. 4 (1963), 1726 - 1729.

[A_2] M. Atsuji : On normality of the product of two spaces, General topology and its relations to modern analysis and algebra IV, proceedings of the fourth Prague Topological Symposium, 1976.

[B_1] R. H. Bing : Metrization of topological spaces, Canad. J. Math., 3 (1951), 175 - 186.

[B_2] D. K. Burke : On subparacompact spaces, Proc. Amer. Math. Soc., 24 (1969), 655 - 663.

[BR] A. Bešlagić and M. E. Rudin : Set theoretic constructions of nonshrinking open covers, Preprint.

[C_1] J. G. Ceder : Some generalizations of metric spaces, Pacific J. Math., 11 (1961), 105 - 125.

[C_2] K. Chiba : On products of normal spaces, Rep. Fac. Sci. Shizuoka Univ., 9 (1974), 1 - 11.

[C_3] K. Chiba : Two remarks on the normality of product spaces, Rep. Fac. Sci. Shizuoka Univ., 11 (1976), 17 - 22.

[C_4] K. Chiba : On the weak $\mathcal{B}$-property of Σ-products, Math. Japonica, 27, No. 6 (1982), 737 - 746.

[C_5] K. Chiba : On the weak $\mathcal{B}$-property, Math. Japonica, 29, No. 4 (1984), 551 - 567.

[C_6] K. Chiba : On collectionwise normality of product spaces. I, Proc. Amer. Math. Soc., 91, No. 4 (1984), 649 - 652

[C_7] K. Chiba : On collectionwise normality of product spaces. II, Proc. Amer. Math. Soc., 91, No. 4 (1984), 653 - 657.

[C_8] K. Chiba : Remarks on the weak β-property of Σ-products, to appear in Q & A in General topology, vol. 3, No. 1.

[C_9] W. W. Comfort : A survey of cardinal invariants, Gen. Top. and Appl., 1 (1971), 163 - 199.

[C_{10}] H. H. Corson : Normality in subsets of product spaces, Amer. J. Math., 81 (1959), 785 - 796.

[CC_1] T. Chiba and K. Chiba : Q-paracompactness and closed mappings, Sci. Rep. Tokyo Kyoiku Daigaku, 11 (1972), 159 - 162.

[CC_2] T. Chiba and K. Chiba : A note on normality of product spaces, Sci. Rep. Tokyo Kyoiku Daigaku, 12 (1974), 55 - 63.

[CPR] K. Chiba, T. C. Przumusiński and M. E. Rudin : Normality of product spaces and Morita's conjectures, accepted for publication in Top. and Appl.

[D_1] A. L. Donne : Shrinking property in Σ-products of paracompact p-spaces, to appear in Top. and Appl..

[D_2] C. H. Dowker : On countably paracompact spaces, Canad. J. Math., 3 (1951), 219 - 224.

[E] R. Engelking : General topology, Warszawa (1977).

[F] V. V. Fillippov : On feathered paracompacts, Soviet Math. Dokl., 9 (1968), 161 - 164.

[G] S. P. Gul'ko : On properties of subsets of Σ-products, Soviet Math. Dokl., 18 (1977), 1438 - 1442.

[H] S. Hanai : Inverse Images of Closed Mappings. II, Proc. Japan Acad., 37 (1961), 302 - 304.

[I_1] T. Ishii : wM-spaces and closed maps, Proc. Japan Acad., 46 (1970), 16 - 21.

[I_2] F. Ishikawa : On countably paracompact spaces, Proc. Japan Acad., 31 (1955), 686 - 687.

[J] I. Juhász : Cardinal functions in topology, Amsterdam, 1971.

[K_1] M. Katětov : On expantion of locally finite coverings, Colloq. Math., 6 (1958), 145 - 151.

[K_2] Y. Katuta : On the normality of the product of a normal space with a paracompact space, Gen. Top. and Appl., 1 (1971), 295 - 319.

[K_3] A. P. Kombarov : On the product of normal spaces, Uniformities on Σ-products, Soviet Math. Dokl., 13 (1972), 1068 - 1071.

[K_4] A. P. Kombarov, On tightness and normality of Σ-products, Soviet Math. Dokl., 19 (1978), 403 - 407.

[K_5] K. Kuratowski : Topology, vol. I, New Yoke - London - Warszawa, 1966.

[L_1] N. Lašnev : Continuous decompositions and closed mappings of metric spaces, Soviet Math. Dokl., 6 (1965), 1504 - 1506.

[L_2] S. Lefshetz : Algebraic topology, Princeton Univ. Press, Princeton, New Jersey, 1942.

[M_1] V. J. Mancuso : Inverse images and first countability, Gen. Top. and Appl., 2 (1972), 29 - 44.

[M_2] E. Michael : Point-finite and locally finite coverings, Canad. J. Math., 7 (1955), 275 - 279.

[M_3] E. Michael : The product of a normal space and a metric space need not be normal, Bull. Amer. Math. Soc., 69 (1963), 375 - 376.

[M_4] E. Michael : A Quintuple Quotient Quest, Gen. Top. and Appl., 2 (1972), 91 - 138.

[M_5] K. Morita : Paracompactness and product spaces, Fund. Math., 50 (1962), 223 - 236.

[M_6] K. Morita : Products of normal spaces with metric spaces II, Tokyo Kyoiku Daigaku, 8 (1963), 87 - 92.

[M_7] K. Morita : Products of normal spaces with metric spaces, Math. Ann., 154 (1964), 365 - 382.

[M_8] K. Morita : Some problems on normality of product spaces, General topology and its relations to modern analysis and algebra IV, proceedings of the fourth Prague topological symposium, 1976, Part B, Prague (1977), 296 - 297.

[M_9] K. Morita : On the dimension of the product of topological spaces, Tsukuba J. Math., 1 (1977), 1 - 6.

[N_1] K. Nagami : σ-Spaces and Product Spaces, Math. Ann., 181 (1969), 109 - 118.

[N_2] K. Nagami : Σ-spaces, Fund. Math., LXV (1969), 169 - 192.

[N_3] K. Nagami : Normality of products, Actes du Congres Intern. Math., 2 (1970), 33 - 37.

[N_4] K. Navy : Paracompactness in para-Lindelöf spaces, Thesis, Univ. of Wisconsin, Madison, Wisconsin (1981).

[N_5] N. Noble : Products with closed projections. II, Trans. of The Amer. Math. Soc., 160 (1971), 169 - 183.

[O_1] H. Ohta : On normal, non-rectangular products, Quart. J. Math., (2), 32 (1981), 339 - 344.

[O_2] A. Okuyama : On metrizability of M-spaces, Proc. Japan Acad., 40 (1964), 176 - 179.

[O_3] A. Okuyama : Some generalizations of metric spaces, their metrization theorems and product spaces, Sci. Rep. Tokyo Kyoiku Daigaku, 9 (1967), 236 - 254.

[P_1] R. Pol : A perfectly normal locally metrizable non-paracompact space, Fund. Math., XCVII (1977), 37 - 42.

[P_2] T. Przumusiński : A Lindelöf space X such that X^2 is normal but not paracompact, Fund. Math., 78 (1973), 291 - 296.

[R_1] M. E. Rudin : A normal space X for which $X \times I$ is not normal, Fund. Math., LXXIII (1971), 179 - 186.

[R_2] M. E. Rudin : The normality of products with a compact factor, Gen. Top. and Appl., 5 (1975), 45 - 59.

[R_3] M. E. Rudin : Σ-products of metric spaces are normal, Preprint, (1977).

[R_4] M. E. Rudin : κ-Dowker spaces, Czech. Math. J., 28 (103) (1978), Praha, 324 - 326.

[R_5] M. E. Rudin : The shrinking property, Canad. Math. Bull., 26 (1983), 385 - 388.

[R_6] M. E. Rudin : Dowker spaces, in : The Handbook of Set Theoretic Topology, K. Kunen and J. Vaughan, editors, North Holland, Amsterdam (1984).

[R_7] M. E. Rudin : κ-Dowker spaces, Dowker Memorial Volume, London Mathematical Society, (1984).

[RS] M. E. Rudin and M. Starbird : Products with a metric factor, Gen. Top. and Appl., 5 (1985), 235 - 248.

[S_1] H. L. Shapiro : Extensions of pseudometrics, Canad. J. Math., 18 (1966), 981 - 998.

[S_2] R. H. Sorgenfrey : On the topological product of paracompact, Bull. Amer. Math. Soc., 53 (1947), 631 - 632.

[S_3] M. Starbird : The normality of product with a compact or a metric factor, Ph. D. Thesis at Univ. of Wisconsin, (1974).

[SN] F. Siwiec and J. Nagata : A note on nets and metrization, Proc. Japan Acad., 44 (1968), 623 - 627.

[T_1] H. Tamano : On compactifications, J. Math. Kyoto Univ., 1 - 2 (1962), 162 - 193.

[T_2] Y. Tanaka : On sequential spaces, Sci. Rep. Tokyo Kyoiku Daigaku, 11 (1971), 68 - 72.

[Y_1] Y. Yajima : Topological games and products I, Fund. Math., CXIII (1981), 141 - 153.

[Y_2] Y. Yajima : The normality of Σ-products and the perfect κ-normality of Cartesian products, J. Math. Soc. Japan, vol. 36, No. 4 (1984), 689 - 699.

[Y_3] Y. Yajima : Σ-products of Σ-spaces, Fund. Math., CXXIII (1984), 29 - 37.

[Y_4] Y. Yasui : On the gaps between the refinements of the increasing open coverings, Proc. Japan Acad., 48 (1972), 86 - 90.

[Y_5] Y. Yasui : On the characterizations of the $\mathcal{B}$-property by the normality of product spaces, Top. and Appl., 15 (1983), 323 - 326.

[Y_6] Y. Yasui : A remark on shrinkable open covers, Q & A in General topology, vol. 2, No. 2 (1985).

[Z] P. Zenor : A class of countably paracompact spaces, Proc. Amer. Math. Soc., 24 (1970), 258 - 262.

INDEX

(*Terms*)

(Symbols)