THE CLASSIFICATION OF SIMPLY CONNECTED QF-3 ALGEBRAS

BY

MASAHISA SATO

THESIS

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Science in Mathematics at

The University of Tsukuba

June, 1985

THE CLASSIFICATION OF SIMPLY CONNECTED QF-3 ALGEBRAS

BY

MASAHISA SATO

THE CLASSIFICATION OF SIMPLY CONNECTED QF-3 ALGEBRAS

BY

MASAHISA SATO

1. INTRODUCTION. Throughout this paper, R denotes a finite dimensional algebra over an algebraic closed field K. We shall also assume R to be connected and basic and representation-finite.

In this paper we are concerened with simply connected algegras. Particularly the maximal length of Auslander-Reiten quivers of simply connected algebras having the same number of simple modules as well as the maximal grading of these algebras is discussed in the first half and simply connected QF-3 algebras are discussed in the latter half.

QF-3 algebras were introduced by Tachikawa [11] as a general notion of QF-algebras. simply connected algebras were introduced by Bongartz-Gabriel [4].

QF algebras are never simply connected, but Riedtmann pointed out that QF-algebras have universal coverings of simply connected algebras of Dynkin type. (cf. [6]) Further any QF-algebra is stable equivalent to some trivial extension algebra of simply connected algebra. (Also see [6]) Iwanaga pointed out that QF-3 algebra appears as the the covering of a trivial extension algebra. (see [2])

It is well known that any Auslander algebra is characterized as QF-3 algebra with a global dimension smaller than 2 and a dominant dimension larger than 2. (Auslander [1])

Under the influence of these back grounds, we would like to classify QF-3 algebras but not QF-algebras. This should be completed in section 6.

Here we give a summary of each section.

In secion 2 some notions and fundamental properties are presented.

Section 3 is devoted to the classification of the partially ordered set $\{{}_R M \mid \mathrm{Hom}_R(P,M) \neq 0\}$, here P is a projective module with a maximal grading.

In section 4 using the classification in section 3, we shall prove the maximal length and grading between all the simply connected algebras with the same number of simple modules.

In section 5 we study simply connected QF-3 algebras with an indecomposable projective injective faithful module. The quiver of these algebras are called elementary QF-3 quivers and there are 59 kinds of quivers listed at section 7. These algebras supply the list of algebras with sincere indecomposable modules due to Bongartz [5]. To determine these quivers, we prove the following facts.

(1) R has a matrix form as $R = \begin{pmatrix} A & {}_A M_K \\ 0 & K \end{pmatrix}$ such that M is A-K bimodule and an injective sincere A-module and M = rad R.

(2) Put $A = e_1 A \oplus \dots \oplus e_{n-1} A$ and $e_n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, here e_i 's are primitive idempotent. Then $e_n M e_i \neq 0$ and $e_i M e_1 \neq 0$ for any i.

The property (2) means that quivers of these algebras have unique minimal vertices and unique maximal vertices and there is no zero-relaion in these quivers.

In section 6 we study simply connected QF-3 algebras. It is proved that any quiver of a simply connected QF-3 algebras is some interlacing of elementary QF-3 quivers. This result is owing to the fact that any projective injective indecomposable module is a faithful injective projective module over its support algebra. i.e. The quivers of these kinds of support algebras are elementary QF-3 quivers.

We must remark any interlacing doesn't give a QF-3 algebra. They are not QF-3 nor simply connected in general. So we should discuss the properties under what conditions the algebra has a minimal faithful module and becomes simply connected. i.e. We should characterize simply connected QF-3 algebras. Of course, these conditions should be described by the way how to interlace elementary QF-3 quivers and how to give zero-relations after interlacing quivers because a given algebra must be verified concretely if this algebra is a QF-3 algebra or not.

The only result for a faithfulness is due to Happel-Ringel [7]. This is done for the case of indecomposable module using by the tilting theory. Their result is that any sincere indecomposable module is faithful. But in general this doesn't hold. Here we should give the concrete conditions for a projective injective module to be faithful.

Next we would like to show that relations which should be given in an interlaced quiver is uniquely determined depending on the way how to interlace the elementary QF-3 quivers.

After the above investigations, we introduce a notion of a QF-3 quiver with relations and we prove that the algebras constructed by QF-3 quivers with relations become QF-3 algebras. This enables us to construct any simply connected QF-3 algebra which has a direct sum of given indecomposable projective injective modules as a minimal faithful module.

2. DEFINITIONS AND FUNDAMENTAL RESULTS.

2.1. Throughout this paper modules mean right R-modules. R is called a QF-3 algebra if there are projective injective modules such that their direct sum is faithful. This is equivalent to the original definition that R has a minimal faithful module. This owes to Colby-Rutter. (cf. see [11]).

2.2 Let Q_R and Γ_R be a Gabriel quiver and an Auslander-Reiten quiver of R with a translation $\tau = DTr$ respectively. ${}_RM$ is called a successor of ${}_RN$ (and ${}_RN$ is a predecessor of ${}_RM$) if ${}_RM$ and ${}_RN$ are indecomposable and there is an chain of irreducible maps from ${}_RN$ to ${}_RM$.

If Q_R has no oriented cycle and no roop, then Q_R is partially ordered in a usual way. i.e. $a < b$ if tere is a chain of arrows from a to b. Further we put $[a,b] = \{c \mid a < c < b \text{ in } Q_R\}$.

We may identify R with KQ/I for some two sided ideal I of a path algebra KQ. For a vertex $a \in Q$, we denote by P(a), J(a) and k(a) an indecomposable projective module, injective module and simple module corresponding to a. i.e. $P(a) = [-,a]$, $J(a) = D[a,-]$ and $k(a) = P(a)/\text{rad } P(a)$, here $D = \text{Hom}_K(-,K)$ is a duality. Also we denote by [e] a corresponding vertex of Q_R to a primitive idempotent e.

2.3. By Bongartz-Gabriel [4] and Bautista-Larriòn-Salmeròn [3], algebras satisfying the following equivalent conditions are called simply connected algebras.

(1) The fundamental group of Γ_R is trivial. (See [4])

(2) Q_R with relation ideal I satisfies the separated condition. (See [3])

i.e. If rad $P(a) = M \oplus N$ and $P(b)M \neq 0$ and $P(c)N \neq 0$, then there exits no non-oriented path $b-d_1- \ldots -d_t-c$ in Q_R such that $a \nleq d_i$ for every i. Here - means any direction $\rightarrow$ or $\leftarrow$.

(3) Any different direct summands of a radical of an indecomposable projective module have no common predecessor.

(4) Γ_R is a translation quiver given by some graded tree. (See [4])

2.4. We would like to mention about important properties that simply connected algebras satisfy. Here we denote by $k(\Gamma_R)$ a mesh category of $k(\Gamma_R)$. These are mainly owing to [4].

(1) $k(\Gamma_R) \simeq$ ind-R. Here ind-R is a category of all finitely generated indecomposable modules. i.e. Any simply connected algebra is standard.

(2) R is isomorphic to a full subcategory consisting of projective vertices of Γ_R.

(3) Any indecomposable module M is determined by its dimension vector $\underline{\dim}$ M.

(4) $M(a) = \dim_K \mathrm{Hom}_R(P(a),M) = \dim_K K(\Gamma_R)(P(a),M)$.

(5) Γ_R has no oriented cycle and is partially ordered by the same way as Q_R.

(6) $\mathrm{Dim}_K\, KQ/I(a,b) \leq 1$ for any $a,b \in Q$.

(7) If $KQ/I(a,b) \neq 0$, then no path from a to b belongs to I.

2.5. Γ_R is constructed by a graded tree, we can define a length function L_x for $x \in \Gamma_R$ by

(1) $L_x(x)=1$.

(2) If $L_x(y)$ is defined, then $L_x(z) = \begin{cases} L_x(y)-1 & \text{if } z \to y \\ L_x(y)+1 & \text{if } y \to z. \end{cases}$

For any subset Y of Γ_R, we denote by $L_x(Y)$ the maximal number of $L_x(y)$ among $y \in Y$. Further we put $L(Y) = L_z(Y)$ for a projective z whose grading is 0. Clearly it is independent of the choice of x and it holds for $x \in \Gamma_R$ the property

$$(\#)\ L(Y)-L(x) = L_x(Y).$$

We define a starting function s_x at x by $s_x(y) = \dim_K K(\Gamma_R)(x,y)$ for $y \in \Gamma_R$ and we denote by S_x the support of s_x.

2.6. We denote simply by R^T and Γ_T an algebra and an Auslander-Reiten quiver of an addmissible graded tree (T,g).

We recall the definitions in [4]. Let m be a vertex of T and assume that a projective module (g(m),m) has no projective successor. i.e. The corresponding vertex in Q_R is maximal. We denote by $t_1, \ldots, t_r$ the neighbouring vertices of m in T, by $T^1, \ldots, T^r$ the corresponding connected components of $T\setminus\{m\}$, by μ_i the minimum of g on T^i, by g_i the grading $(g_i|T^i)-\mu_i$ on T^i.

The following theorem is fundamental.

Theorem (Bongartz-Gabriel [4]).

With the above situations, the following statements are equivalent.

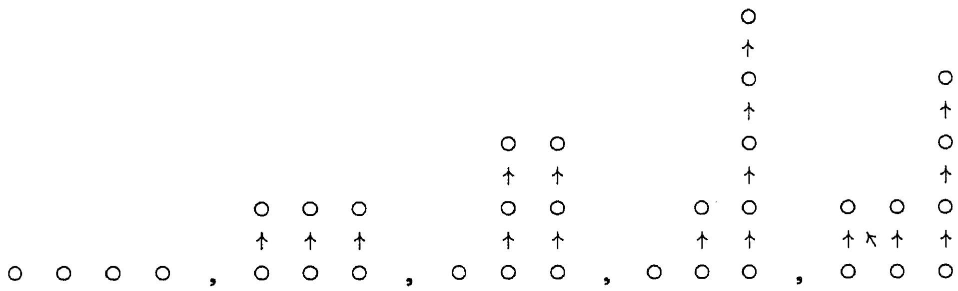
(1) (T,g) is representation-finite.

(2) (a) Each (T^i,g_i) is representation-finite.

(b) Each Γ_{T^i} contains $x_i = (g(m)-\mu_i-1,t_i)$ and the value of each $s_{x_i}^{T^i} \leq 1$.

(c) The partially ordered set $S_{x_1}^{T^1} \coprod \cdots \coprod S_{x_r}^{T^r}$ is representation-finite in the sense of Nazarova-Roiter [8].

Here a partially ordered set is representation-finite if it contains no subset of the following 5 forms;



[1,1,1,1] [2,2,2] [1,3,3] [1,2,5] [N,4] .

We notice that $r \leq 3$ in Theorem 2.6 (c), otherwise [1,1,1,1] appears.

3. THE CLASSFICATION OF PARTIALLY ORDERED SETS.

3.1. Let (T,g) be a representation finite graded tree and (T^i,g_i)'s $(i = 1, \ldots, r)$ graded trees stated in section 2. Put $p \in Q_R$ such that $P(p) = (g(m),m)$. Since p is maximal in Q_R, there is a slice in each Γ_{T^i} such that the full subquiver consisting of successors of rad $P(p)$ just coincides with the τ-orbit of this slise $\vec{s}(a_0)$. In this section we would like to classify these sections in the sense that successors of rad $P(p)$ are completely determined.

We notice that rad $P(p) = a_0 \oplus b_0 \oplus c_0$, maybe b_0 or $c_0 = 0$.

3.2. Assume a_0, b_0, $c_0 \neq 0$. Then $\vec{s}(a_0)$ is one of the following quivers.

(1) a_0 , b_0 , $c_0 \to \dots \to c_k$. ($k \geq 0$)

(2) a_0 , $b_0 \to c_0$, $c_0 \to \dots \to c_k$. ($k \geq 0$)

The above fact is proved easily since the other section with three components contains [2,2,2] or [1,3,3].

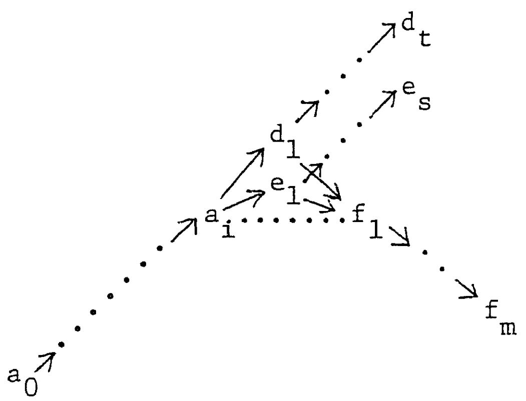
3.3. Assume $a_0, b_0 \neq 0$ and $c_0 = 0$.

We can get nineteen possible slices in this case.

The first csae is the case that quivers are linear.

(1) $a_0 \to \dots \to a_i$, $b_0 \to \dots \to b_j$. ($i,j \geq 0$)

For the other cases, consider a following subquiver in S_{a_0} .



, $b_0 \to \dots \to b_j$.

($i,j,m \geq 0$, $t \geq s \geq 1$)

We abbreviate injective to inj. throughout this section.

If a_i is injective, then the following five cases are possible.

(2) $s = 1$, $j = 0$.

(3) $s = 1$, $j = 1$, $1 \leq t \leq 4$ because of [1,2,5].

(4) $s = 1$, $j = 2,3$, $t = 1,2$ because of [1,3,3].

(5) $s = 1$, $j \leq 4$, $t = 1$ because of [1,2,5].

(6) $s = 2$, $j = 0$, $t = 2,3,4$ because of [1,3,3] and [1,2,5].

In the following we omit the description of partially ordered sets demanding in the each case to avoid the long explanations. These will be given by writing down the successor of a_0 concretely.

If a_i is not injective, then $s = 1$. Otherwise it appears [1,1,1,1]. The followinf six cases are possible.

(7) $t = 1$, $j, m \geq 0$.

(8) $j = 0$, $t = 5$, $m = 1$, d_1 is inj.

(9) $j = 0$, $t = 4$, $m = 1$.

(10) $j = 0$, $t = 3$, $m = 1$.

(11) $j = 0$, $t = 3$, $m = 2,3$, d_1 is inj.

(12) $j = 0$, $t = 2$, d_1 is inj.

For the case $j = 0$, $t = 2$ and d_1 is non-inj, the following four cases are possible.

(13) $m = 4,5$, f_1 is inj.

(14) $m = 4$, $\tau^{-1}e_1$ is inj.

(15) $m = 4$, $\tau^{-1}f_1$ and d_1 are inj.

(16) $1 \leq m \leq 3$.

The following three cases are $j \geq 1$.

(17) $j = 1$, $t = 2$, $m = 1,2$.

(18) $j = 1$, $t = 2$, $m = 3$, d_1 is inj.

(19) $j = 2$, $t = 2$, $m = 1$.

3.4. Assume rad $P(p) = a_0$. i.e. $b_0 = c_0 = 0$.

The following four cases are possible.

(i)

$$a_0 \to \ \cdots \ \to a_k. \qquad (k \geq 0)$$

(ii)

$$\begin{array}{ccccccccc}
 & & & d_1 & \to & \cdots & \to & d_s \\
 & & & \uparrow & & & & \\
a_0 & \to & \cdots\cdots \to & a_k & \to & c_1 \to & \cdots & \to & c_i \\
 & & & \downarrow & & & & \\
 & & & b_1 & & & &
\end{array}$$

(iii) $\qquad\qquad$ ($i = 1,2,\ s \geq i,\ k \geq 0$)

$$\begin{array}{ccccccccccc}
 & & & & & & & d_1 & \to & \cdots \to & d_t \\
 & & & & & & & \uparrow & & & \\
 & & & c_1 & \to & \cdots & \to & c_i & \to & e_1 \to \cdots \to & e_s \\
 & & & \uparrow & & & & & & & \\
a_0 & \to & \cdots\cdots \to & a_k & \to & b_1 & \to \cdots \to & b_j & & &
\end{array}$$

($t \geq s \geq 0,\ i,j \geq 1,\ k \geq 0$)

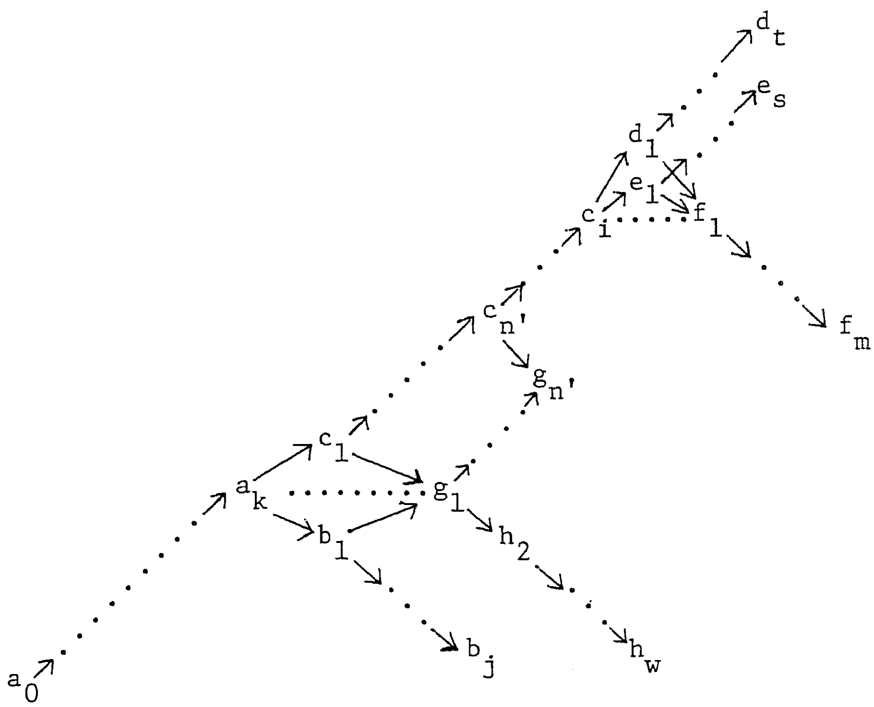
For (i) and (ii) the following three cases are possible.

(1) k is arbitraly for (i).

(2) a_k is inj., $i = 1$ for (ii).

(3) a_k is inj., $i = 1$, $s = 2,3,4$ for (ii).

To classify (iii) more detail, consider a subquiver in S_{a_0}.



Here n is the least number such that $c_{n'}$ is injective, m and w are the greatest numbers such that $f_m = \tau^{-m} c_{i-m+1}$ and $h_w = \tau^{-w} a_{k-w+1}$ exist.

3.5. Consider the case $k = 0$ and a_0 is injective. The following fourteen cases are possible.

(1) $t = s = 0$.

(2) $t = s = 1$.

(3) $j = 1$, c_i is inj., $s = 1$.

(4) $j = 1$, c_i is inj., $s = 2$, $t = 2,3,4$.

The following five cases are $j = 1$, c_i is non-inj. and $s = 1$.

(5) d_1 is non-inj., $t = 2$, $m \leq 5$.

(6) d_1 is non-inj., $t = 3,4$, $m = 1$.

(7) d_1 is inj., $t = 2$.

(8) d_1 is inj., $t = 3$, $m = 1,2,3$.

(9) d_1 is inj., $t = 4,5$, $m = 1$.

The rest cases are $j \geq 2$ and $s = 2$.

(10) $j = 2$, c_i is inj., $t = 2,3,4$.

(11) $j = 2$, c_i is non-inj., d_1 is inj., $t = 2$, $m = 1,2,3$.

(12) $j = 2$, c_i is non-inj., d_1 is non-inj., $m = 1,2$.

(13) $j = 3$, $t = 2$, $m = 0,1$.

(14) $j = 4$, $t = 2$, c_i is inj..

3.6. If $k = 0$ and a_0 is non-injective, the following sixteen cases are possible.

The following eight cases are $s = j = 1$.

(1) $t = 1$.

(2) $t = 2$, $n' = 3$, c_i is inj..

(3) $t = 2$, $n' = 2$, c_i is non-inj., $m = 1$.

(4) $t = 2$, $n' = 2$, c_i is inj..

(5) $t = 2$, $n' = 1$, c_i is non-inj., $m = 1,2$.

(6) $t = 2$, $n' = 1$, c_i is non-inj., $m = 3$, d_1 is inj..

(7) $t = 2$, c_i is inj..

(8) $t = 3,4$, $n = 1$, c_i is inj..

The rest cases are $\underline{s = t = 0.}$

(9) $j = 3$, $i = 3,4$, c_1 is inj..

(10) $j = 3$, $i = 5$, c_1 and b_1 are inj..

(11) $j = 2$, $n' = 2$, $i = 6$, b_1 is inj..

(12) $j = 2$, $n' \geq 2$, $i = 5$, b_1 is inj..

(13) $j = 2$, $n' = 2$, $i = 5$, b_1 is non-inj..

(14) $j = 2$, $n' \geq 2$, $i = 2,3,4$.

(15) $j = 2$, $n' = 1$.

(16) $j = 1$.

3.7. Assume $k \geq 1$. If $w = 0$ or 1, then S_{a_0} is same as 3.5 and 3.6. So we may assume $w \geq 2$.

The following three cases are possible and $\underline{s = j = n' = 1}$.

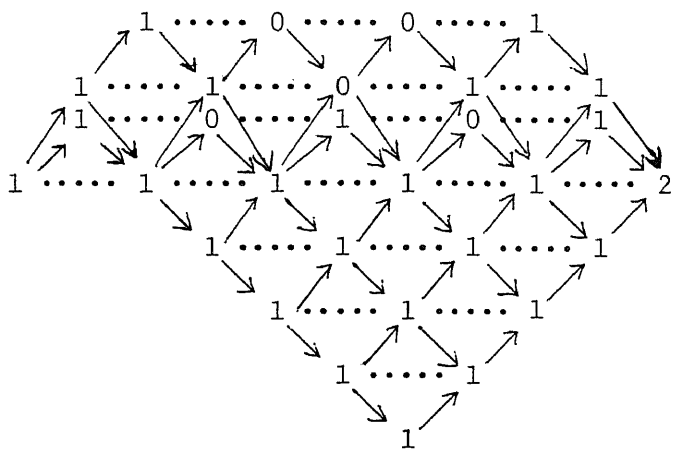
(1) $t = 1$.

(2) $m = 0$, $t = 2$, $w = 2,3$.

(3) $m = 1$, $t = 2$, $w = 2$.

3.8. The most complicated case is the case $\underline{s = t = 0.}$ By the same reason stated in 3.7, we may assume $\underline{a_k \text{ is not injective.}}$

In the case $j = 1,2$ or 3, consider a following subquiver of S_{a_0}. Here the numbers attached in the quiver are values of the starting function s_{a_0}.



The following twenty two cases are <u>$j = 1$</u>.

(1) $i = 1$.

(2) $i = 2,\ 1 \le w \le 4$.

The following seven cases are <u>$i = 2$ and $w = 5$.</u>

(3) $\tau^{-2}c_2$ is inj..

(4) $\tau^{-3}c_1$ is inj..

(5) $\tau^{-3}b_1$ is inj..

(6) $\tau^{-4}a_k$ is inj..

(7) $\tau^{-1}h_2$ is inj..

(8) h_3 is inj..

(9) h_4 is inj..

The following thirteen cases are <u>$i = 3$</u>.

(10) $w = 1,2$.

(11) $n' = 2,\ w = 3$.

(12) $n' = 2$, c_2 is inj., $w = 4,5$.

(13) $n' = 2$, g_2 and h_2 are inj., $w = 6$.

(14) $n' = 2$, $w = 3,4$, h_2 is inj..

(15) $n' = 1,\ w \ge 3$.

(16) c_2 is non-inj., g_2 is inj., $w = 3$.

(17) c_2 is non-inj., g_2 and $\tau^{-1}b_1$ are inj., $w = 4$.

(18) c_2 is non-inj., g_2 and $\tau^{-1}b_1$ are inj., $w = 5$, h_2 is inj..

(19) c_2 and $\tau^{-1}g_2$ are non-inj., $\tau^{-1}b_1$ is inj., $w = 3,4$.

(20) c_2 and g_2 are non-inj., $\tau^{-1}b_1$ is inj., $w = 5$, h_3 is inj..

(21) c_2, g_2 and $\tau^{-1}b_1$ are non-inj., $\tau^{-2}a_k$ is inj., $w = 3$.

(22) c_2, g_2, $\tau^{-1}b_1$ and $\tau^{-2}a_k$ are non-inj., h_2 is inj., $w = 3$.

3.9. Assume <u>$s = t = 0$, $j = 1$, a_k is non-inj. and $i \geq 4$.</u>

Then the following sixteen cases are possible.

(1) c_1 is inj..

The following five cases are <u>$n' = 2$.</u>

(2) $w = 1,2,3$.

(3) $i = 4$, g_1 is inj., $w = 4,5$.

(4) $w = 1,2$, $i \geq 5$.

(5) $w = 3$, $i = 5$, $\tau^{-1}b_1$ is inj..

(6) $w = 3$, $i = 6$, g_1 is inj..

The following four cases are <u>$n' \geq 3$, g_1 is inj..</u>

(7) $w = 1$.

(8) $w = 2$, $i = 4,5,6$.

(9) $w = 2$, $i = 7$, $n' = 3$.

(10) $w = 3,4$, $i = 4$.

The following cases are <u>c_1, c_2 and g_1 are non-inj..</u>

(11) $w = 1$, $n' \geq 4$.

(12) $w = 2$, $n' \geq 4$, $i = 4$.

(13) $w = 2$, $n' \geq 4$, $i = 5$, g_2 is inj..

(14) $w = 2$, $n' \geq 4$, $i = 5$, $\tau^{-1}b_1$ is inj..

(15) $w = 2$, $n' = 3$, $i = 4,5$.

(16) $w = 2$, $n' = 3$, $i = 6$, $\tau^{-1}b_1$ is inj..

3.10. Assume a_k is non-inj. and $j \geq 2$.

Then the following eleven cases are possible .

(1) $i = 2$, $w = 1$.

(2) $i = 2$, $n' = 1$, $w = 3,4,5$, b_1 is inj..

(3) $i = 2$, $n' \geq 2$, b_1 is inj., $w = 2,3$.

(4) $i = 2$, c_1 and b_1 are non-inj., $w = 2$.

(5) $n' = 1$, $w = 1$, $i \quad 3$.

(6) $n' = 1$, $w = 2$, $i = 3$.

(7) $n' = 1$, $w = 2$, $i = 4$, b_1 is inj..

(8) $n' \quad 2$, b_1 is inj., $w = 1$, $i = 3,4,5$.

(9) $n' = 2$, b_1 is inj., $w = 1$, $i = 6$.

(10) c_1 and b_1 are non-inj., $w = 1$, $i = 3,4$.

(11) $n' = 2$, b_1 is non-inj., $w = 1$, $i = 5$.

3.11. Assume a_k is non-inj. and $j = 3$.

Then $w = 1$ and the following three cases are possible.

(1) c_1 is inj., $i = 3,4$.

(2) c_1 is inj., $i = 5$, b_1 is inj..

(3) c_1 is non-inj., b_1 is inj., $i = 3$.

This completes the classification of S_{a_0}.

4. THE MAXIMAL GRADING AND LENGTH OF SIMPLY CONNECTED ALGEBRAS.

4.1. Let n be a natural number and S_n a set consisting of all the representation-finite graded trees with n vertices. We put

$$F(n) = \max \{ L(\Gamma_T) \mid (T,g) \in S_n \}$$

$$G(n) = \max \{ g(t) \mid (T,g) \in S_n \text{ and } t \in T \} .$$

In this section we shall prove the following theorem.

Theorem. Let n be a natural number. Then it holds that $G(2) = 1$, $G(3) = 3$, $G(4) = 5$, $G(5) = 7$, $G(6) = 11$, $G(7) = 15$, $G(8) = 25$ and $G(9) \leq 55$,

$$G(n) \leq \begin{cases} 60n-485 & (10 \leq n \leq 32) \\ n^2-6n+604 & (n \geq 33) \end{cases}$$

Also since $F(n) = G(n+1)-1$, we have $F(2) = 2$, $F(3) = 4$, $F(4) = 6$, $F(5) = 10$, $F(6) = 14$, $F(7) = 24$ and $F(8) \leq 54$,

$$F(n) \leq \begin{cases} 60n-426 & (9 \leq n \leq 31) \\ n^2-4n+598 & (n \geq 32) \end{cases}$$

We shall give the proof in 4.4 The graded trees which give F(4), F(5), F(6) and F(7) are as following.

$F(4) = 6$

```
                          3
                          |
0 - 1 - 2 - 3  ,  1 - 0 - 5
```

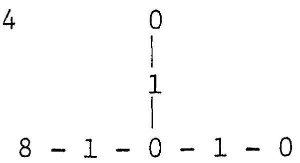
$F(5) = 10$

```
    1
    |
1 - 0 - 5
    |
    1
```

$F(6) = 14$

```
    1   6                  1                  7
    |   |                  |                  |
1 - 0 - 0 - 2  ,  0 - 1 - 2 - 3 - 4  ,  1 - 0 - 1 - 2 ,
                                              |
                                              1
```

$F(7) = 24$



4.2. Let (T_n, g_n) be a representation-finite graded tree such that $L(\Gamma_{T_n}) = F(n)$. Put p and q vertices in Γ_{T_n} such that $g_n(p) = 0$ and $L_p(q) = F(n)$. Then we can construct a representation-finite graded tree (T,g) such that $g(t) = F(n) + 1$ for some $t \in T$ by the following way.

$T = T_n \cup \{t\}$ ... the neighbou of t is a vertex corresponding to a τ-orbit of q.

$$g(x) = \begin{cases} g_n(x) & \text{if } x \in T_n \\ F(n)+1 & \text{if } x = t. \end{cases}$$

In fact $G(n+1) = F(n) + 1$ and the graded trees which gives $G(n+1)$ are always described as above by the following lemma.

Lemma. The following statements are true.

(1) $G(n+1) = F(n) + 1$ and $F(n+1) \geq F(n) + 2$.

(2) The vertex whose grading is $G(n)$ has only one neighbour.

(3) Let P_t be projective module corresponding to t whose grading is $G(n+1)$ for some graded tree (T^*, g^*) and (T,g) graded tree such that $T = T^* \setminus \{t\}$ and $g = g|T$. Then $q = \text{rad } P_t$ is simple injective as R_T-module.

Proof. First we prove (1). $G(n+1) \geq F(n)+1$ and $F(n+1) \geq F(n)+2$ are already shown just before. We show $G(n+1) \leq F(n)+1$. Let (T,g) be any representation-finite graded tree with n+1 vertices and let z be a vertex in T whose grading is maximal. Consider a connected component T_1 of $T\setminus\{z\}$ which contains a vertex whose grading is 0. By Theorem 2.6, $(T_1, g|T_1)$ is representation-finite, hence $L(\Gamma_{T_1}) \leq F(n)$.

Also $g(z) \le L(\Gamma_{T_1}) + 1 \le F(n) + 1$, hence $C(n+1) \le F(n) + 1$.

Next we prove (2). Assume contrary t has at least two neighbours, here t is a vertex whose grading is $G(n)$ in a graded tree (T^*, g^*). Let T be a connected component of $T \setminus \{t\}$ which contains a vertex whose grading is 0. Since $(T, g^*|T)$ is representation-finite and $|T| \le n-2$, we can construct two representation-finite graded tree (T_1, g_1) and (T_2, g_2) in the following way.

$$T_1 = T \cup \{t\} \qquad g_1 = g^*|T_1,$$

$$T_2 = T_1 \cup \{p\} \qquad g_2|T_1 = g_1 \text{ and } g(p) = L(\Gamma_{T_1})+1$$

Hence $G(n) \ge g_2(p) > L(\Gamma_{T_1}) \ge g_1(t) = g(t) = G(n)$, which is a contradiction.

Last we prove (3). Let L be a length function with respect to (T,g). By (2), rad P_t is indecomposable, hence the canonical inclusion map rad $P_t \to P_t$ is a irreducible map and $L(\text{rad } P_t) + 1 = L(P_t)$. On the other hand, $g^*(t) = G(n+1) = F(n) + 1$, thus $L(\text{rad } P_t) = F(n)$. This means there is no irreducible map starting from rad P_t in Γ_T, so rad P_t is a simple injective R_T-module.

4.3. By Lemma 4.2, it is sufficient to estimate the value F(n). The following lemma is useful to do this.

Lemma. q is successor of every projective module whose grading is maximal in (T_n, g_n).

Proof. Let v be vertex whose grading is maximal in (T_n, g_n). Assume there is no path from P_v to q. We consider full subtranslation quiver Γ (it may be non-connected) of Γ_{T_n} consisting of vertices which are not successors of P_v. So we put Γ^1 a connected component of Γ which contains q, further let u be a neighbouring vertex of v in T_n.

We can choose such u as P_u belongs to Γ^1. Let L_1 and L be length functions (2.5) with respect to Γ^1 and Γ respectively. $L-L_1$ has the constant value a for every vertex in Γ^1. We remark that

$$F(n) = G(n+1)-1 = L(q) = L_1(q)+a$$

since q belongs to Γ^1.

If a = 0, then as constracted in 4.3, there is a simply connected algebra whose maximal grading is larger than F(n).

So we may assume a > 0. Then must have two connected components. Let Γ^2 be another connected component of Γ which contains a vertex with zero grading and M a neighbour of P_v such that M is contained in Γ^2. We remark $L(\Gamma^2) \geq a$ since $L(\Gamma^2) \geq L(M) = g_n(v)-1 \geq L(P_u) = L_1(P_u)+a \geq a$.

Now we consider the following trees and their gradings.

$$T_n \backslash \{p\} = T_1 \cup T_2 \quad \text{(disjoint union of connected trees).}$$

We may assume that u is a vertex of T_1. Under this assumption, we define

$$g_1 = g_n - a|T_1 \quad \text{(a grading of } T_1\text{),}$$

$$g_2 = g_n|T_2 \quad \text{(a grading of } T_2\text{).}$$

We can check the facts that (T_1, g_2) and (T_2, g_2) are representation-finite graded tree and Γ^1 and Γ^2 are full subtranslation quiver of Γ_{T_1} and Γ_{T_2} respectively. Choose a simple injective module S_2 in Γ_{T_2} and $S_1 = P_z$ a simple projective module in Γ_{T_1}, here z is a vertex of T_1 such that $g_n(z) = a$. Then we can define a representation-finite translation quiver Q with n-1 vertices as follows.

$$Q_0 = (R_{T_1})_0 \cup \{P\} \cup (R_{T_2})_0 \quad \text{(set of vertices),}$$

$$Q_1 = (R_{T_1})_1 \cup (R_{T_2})_1 \cup \{S_2 \to P,\ P \to S_1\} \quad \text{(set of arrows),}$$

$$\tau^{-1} S_2 = S_1 \quad \text{(new translation).}$$

We put L_Q a length function with respect to Q, then we have $L_Q(q) = L_1(q)+2+L_Q(S_2) = L_1(q)+2+L_2(S_2) \geq L_1(q)+2+a = F(n)+2$, this is a contradiction.

4.4. We prove the theorem 4.1. Let t be a vertex of (T_n, g_n) whose grading is maximal. Let (T,g) be a graded tree given by a connected component of $T_n \setminus \{t\}$, which contains a vertex whose grading is 0. We put P_t a projective module corresponding to t and put $a = \mathrm{rad}\, P_t$. Further L, L' are denoted by length function on Γ_{T_n}, Γ_T respectively. Then $L(\Gamma_{T_n})-L'(\Gamma_T) = L(\Gamma_{T_n})-L(a) -\{L'(\Gamma_T)-L'(a)\} = L_a(\Gamma_{T_n})-L_a(\Gamma_T)$ since $L(a) = L'(a)$ and 2.5 (#).

On the other hand, $L(\Gamma_{T_n}) = F(n)$ and $L'(\Gamma_T) \leq F(n-1)$. Hence we get an inequation

$$F(n)-F(n-1) \leq L_a(\Gamma_{T_n})-L'_a(\Gamma_T) \qquad (*1)$$

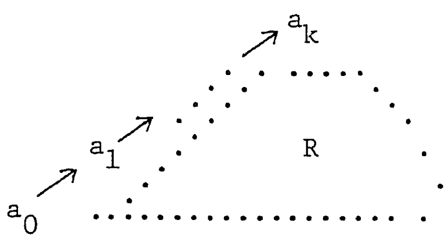
By using the classification of successor of a, the latter part of the inequation (*1) is caluculated concretely. Then

$$F(n)-F(n-1) \left\{ \begin{array}{lc} 30 & (n = 8) \\ 60 \quad \leq & (9 \leq n \leq 32) \\ 2n-5 & (n \geq 32) \end{array} \right. . \qquad (*2)$$

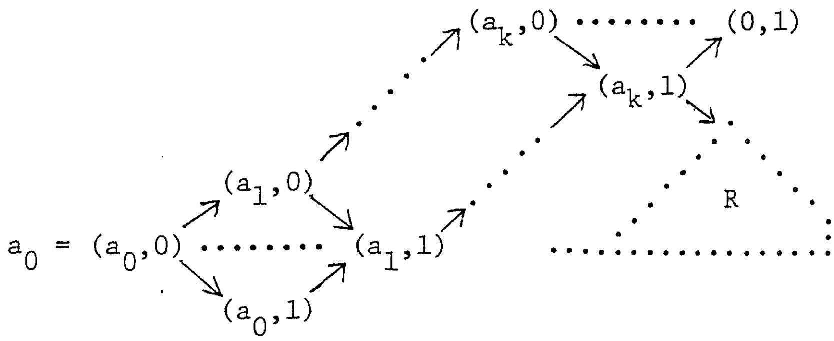
Hence we get the theorem 4.1.

4.5. In the proof of (*2), to avoid unnecessary lengthy, we only show the case of 3.4 (1).

The successor of $a = a_0$ in Γ_T forms the following quiver.



Then the successor of Γ_T form the following quiver.



Here (p,q) means a dimension vector ($\underline{\dim}$ p, s_{p_t}(z)) descrbed by a starting function s_{p_t} (2.5) and vertex z denoted by (p,q) in Γ_T.
Hence $L_a(\Gamma_{T_n}) - L'(\Gamma_T) \leq 2$.

By similar caluculation stated above, we get the estimations of F(n+1)-F(n) for each case in section 3. This list is presented in section 8.

5. ELEMENTARY QF-3 QUIVERS.

5.1. As the anoter application of the classification of quivers, we would like to determine all the simply connected QF-3 algebras with unique indecomposable faithful indecomposable projective injective module. We call these algebras elementary QF-3 algebras, also we call a quiver of an elementary QF-3 algebra an elementary QF-3 quiver.
We prove the following theorem.

Theorem. (1) There are 59 kinds of elementary QF-3 quivers listed at the end of this paper.

(2) The quivers have only possible commutative relations.

(3) The elementary QF-3 algebras are just simply connected QF-2 algebras.

We prove this theorem in 5.4.

5.2. Firat we prove the following lemma.

Lemma. Let $R \simeq KQ/I$ be an elementary QF-3 algebra. Then it holds

(1) Q has the smallest vertex and the largest vertex and any vertex is connected with both of them by non-zero path.

(2) I is generated by all the commutative relations. Particularly there is no zero-rlation in I.

Proof. First we show (2). Assume there is a non-zero path $x \overset{f_1}{\to} \cdots \overset{f_n}{\to} y$. Let a be the smallest vertex in Q. Then there are two paths $a \overset{g_1}{\to} \cdots \overset{g_m}{\to} x$ and $a \overset{h_1}{\to} \cdots \overset{h_t}{\to} y$ not belonging to I. Consider two paths $h_t \cdots h_1$ and $f_n \cdots f_1 g_m \cdots g_1$, then subquiver consisting of vertices between two common vertices on two paths has a commutative relation by 2.3 (2), hence $0 \neq \bar{h}_t \cdots \bar{h}_1 = k\bar{f}_n \cdots \bar{f}_1 \bar{g}_m \cdots \bar{g}_1$ for some non-zero k in K by 2.4 (7), here the symbol "$\bar{f}$" means a residue class of f in KQ/I. This is a contradiction.

Next we show (2). We choose $i = \sum_{x,y \in Q_0} \sum_{*z=x, z^*=y} k_z z$ in I, here $k_z \in K$ and z is a path from a vertex $^*z$ to a vertex z^*. In case of $KQ[x,y] \neq 0$, we choose a representative $z_{x,y}$ of a path from x to y. Since $\bar{z}_{x,y} \neq 0$ by the first part, then $\bar{z} = k_{z,z_{x,y}} \bar{z}_{x,y}$ with $k_{z,z_{x,y}} \in K$ for any z such that $^*z=x$ and $z^*=y$. Hence

$0 = \bar{i} = \sum_{x,y \in Q_0} (\sum_{*z=x, z^*=y} k_z k_{z,z_{x,y}}) \bar{z}_{x,y}$. The sum of the right hand term of the equation is a direct sum as K-space, hence $\sum_{*z=x, z^*=y} k_z k_{z,z_{x,y}} = 0$. Thus $i = i - 0 = \sum_{x,y \in Q_0} \sum_{*z=x, z^*=y} k_z (z - k_{z,z_{x,y}} z_{x,y})$.

From these $\{k_{z,z_{x,y}}\}$, we can rechoose $z_{x,y}$ such that $k_{z,z_{x,y}} = 1$ by [4]. Hence the assertion is valid.

5.3. Let R be an elementary QF-3 algebra of finite-representation type with a tree T and its grading g. The module P(t) corresponding to a vertex t in T whose grading is unique maximal is projective injective. Since P(t) is indecomposable injective, rad P(t) is indecomposable, hence t has only one neighbour. So we get a simply connected algebra B with a tree $T\setminus\{t\}$ and a grading $g|T\setminus\{t\}$. Clearly B has a sincere indecomposable module rad P(t).

Lemma. Let Q_R and Q_B be quivers of algebras R and B stated above respectively and x a vertex of Q_R such that the projective module R(-,x) appears in a τ-orbit described with t. Then

(1) Q_R is a quiver given by adding a vertex x to Q_B as unique maximal vertex.

(2) rad P(x) is a sincere injective B-module and isomorphic to DB(s,-), here s is a unique minimal vertex on Q_B.

(3) R and B are tilted algebras.

Proof. It is clear except that that rad P(x) is an injective B-module. Assume rad P(x) is non-injective over B, then $\underline{\dim}_R\, \tau^{-1}\text{rad } P(x) = (\underline{\dim}_B\, \tau^{-1}\text{rad } P(x) + \underline{\dim}_B \text{ rad } P(x), 1)$ is positive, hence rad P(x) is a non-injective R-module. Here $\underline{\dim}$ means a dimension vectors of modules.

Further $\underline{\dim}_R\, P(x) = (\underline{\dim}_B \text{ rad } P(x), 1)$, hence $\underline{\dim}_R\, \tau^{-1}P(x) =$ $\underline{\dim}_R\, \tau^{-1}\text{rad } P(x) - \underline{\dim}_R\, P(x) = (\underline{\dim}_B\, \tau^{-1}\text{rad } P(x), 0)$.
Hence P(x) is a non-injective R-module, which is a contradiction.

5.4. Now we prove the Theorem 5.1. (2) is already proved. By 5.2, it is sufficient to get the quivers of B.

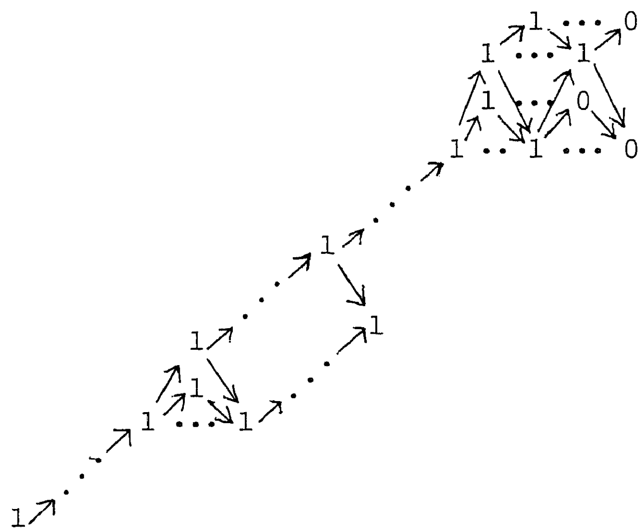
Let Γ_B an Auslander-Reiten quiver. Since B is isomorphic to subcategory of $K(\Gamma_B)$ consisting of injective vertices, we determine where the injective modules appear in Γ_B. As we show in proof of 5.2, $\mathrm{Hom}_B(\mathrm{rad}\ P_t, E) \neq 0$ for any injective B-module E. Hence E belongs to the support (2.5) S_{a_0} of a_0 by 2.6, here $a_0 = \mathrm{rad}\ P_t$.

So we look into the possible injective vertices of Γ_B using the classification of section 3.

It is remarkable that only the case of 3.4 happens since rad P_t is indecomposable and that a slice $\vec{s}_{a_0}$ which is one of the tpyes (i), (ii) or (iii) in 3.4 are a complete slice of B by 5.3 (3). So we can get all the possible quivers consisting of successors of a_0.

It is too bores to show all the cases, we only show on the case 3.5 (2) for m = 1.

The successors of a_0 described by the vertex of the value of the starting fuction s_{a_0} (2.5) are as following.



Hence we get a quiver Q_B as

```
o → o → ...... → o → o → ...... → o → o
                 ↓   ↓            ↓   ↓
                 o → o            o → o
```

But this becomes an infinite type. This doen't happen.

(3) is clear since a QF-2 algebra is an algebra that any indecomposable projective modules have same socle and any indecomposable injective modules have same top.

6. THE QUIVERS AND RELATIONS OF QF-3 ALGEBRAS.

6.1. We give a definition of a quiver embedding.

A quiver embedding $f: Q \to R$ between quivers Q and R means a quiver morphism described with a pair of functions f_0 and f_1 which satisfy the following properties (i) and (ii);

(i) $f_0: Q_0 \to R_0$ is an injection between sets of vertices.

(ii) $f_1: Q_1 \to R_1$ is a map between sets of arrows such that if $\alpha: a \to b$ is an arrow in Q, then $f_1(\alpha)$ is an arrow $f_0(a) \to f_0(b)$.

We write $Q \subseteq R$ if there is a quiver embedding $f: Q \to R$. Also we call of a support algebra of an indecomposable module M is embedded into Q_R since this algebra is a convex set in Q_R as proved by Bongartz. i.e. $a,b \in S(M)$ implie $[a,b] \subseteq S(M)$. Here S(M) is a support algebra of M.

The following fact for the interlacing of quivers of support algebras of injective projective modules (i.e. elementary QF-3 quivers) over simply connected algebras is remarkable.

Proposition. Let $R \simeq KQ/I$ be a simply connected algebra and P_1, P_2 non-isomorphic indecomposable projecitve injective modules.
We put canonical embedding $f_1 \colon Q_1 \to Q$ and $f_2 \colon Q_2 \to Q$ where $Q_1 = Q_{S(P_1)}$ and $Q_2 = Q_{S(P_2)}$. Then $f_1(Q_1) \cap f_2(Q_2)$ is empty or $[a,b]$ for some vertices a,b in Q.

Further a is a minimal vertex in $f_1(Q_1)$ iff b is a maximal vertex in $f_2(Q_2)$.

Proof. Assume $f_1(Q_1) \cap f_2(Q_2)$ is non-empty. Then this quiver has a unique minimal vertex. Otherwise there exist two vertices b and b' which are maximal vertices satisfying the property that b and b' belong to a $f_1(Q_1) \cap f_2(Q_2)$ and are incomparable, so we consider a subquiver $\alpha \colon b \to \dots \to q \leftarrow \dots \leftarrow b'$ through b, b' to some vertex q such that α has no common vertices and each arrow of α is in $f_1(Q_1) \setminus f_2(Q_2)$. Since there is a subquiver β for $f_2(Q_2)$ same as above and b and b' are maximal and incomparable, we can construct a quiver

$$Q' : \quad \begin{array}{ccccc} b & \to & \dots & \to & q \\ \downarrow & & & & \uparrow \\ \vdots & & & & \vdots \\ \downarrow & & & & \uparrow \\ q' & \leftarrow & \dots & \leftarrow & b' \end{array}$$

with no relations and no path connecting α and β by removing maximal or minimal vertices from a quiver Q. On the other hand, KQ' must be simply connected, but this is a contradiction. By the similar discussion, we can prove the existance of the unique minimal vertex.

Let a and b be a minimal and a maximal vertex in $f_1(Q_1) \cap f_2(Q_2)$ respectively. To show $f_1(Q_1) \cap f_2(Q_2) = [a,b]$, it suffices to prove that if $a < x < b$, then x is a vertex of $f_1(Q_1) \cap f_2(Q_2)$. Assume x is not in $f_1(Q_1)$, then there exist two paths $\alpha : a \to \ldots \to b \to \ldots \to d$ and $\beta : a \to \ldots \to x \to \ldots \to b \to \ldots \to d$, here d is a maximal vertex of $f_1(Q_1)$ and α is a path in $f_1(Q_1)$. Since x is not a vertex of $f_1(Q_1)$, $\beta \equiv 0 \pmod{I}$. On the other hand, $\alpha \not\equiv 0 \pmod{I}$ since α is a path in $f_1(Q_1)$. This contrdicts to 2.3 (3).

Next we show latter part. It suffices to prove that there are no two paths satisfying the property $a \to \ldots \to b \to c$ in $f_1(Q_1)$ and $a \to \ldots \to b \to d$ in $f_2(Q_2)$ but not in I. In this situation, $P_1 \simeq D((KQ/I)(a,-))$ since P_1 is indecomposable projective injective. Hence d is a vertex of $f_1(Q_1)$, which is a contradiction.

6.2. In general, it is difficult to investigate whether a give module is faithful or not.

In [7] Happel-Ringel proved that for an indecomposable module faithful modules coincide with sincere modules.

By the following theorem, we can determine all the faithful projective injective modules in terms of zero-relations and the properties of arrows in a Cabriel quiver.

Theorem . Let $R \simeq KQ_R/I$ be a simply connected algebra and $P_1 \oplus \ldots \oplus P_t$ a direct sum of indecomposable projective injective modules P_i's such that for any primitive idempotent f, there exists some P_i such that $P_i f \neq 0$. We put $Q_i = Q_{S(P_i)}$ and dente by f_i a canonical embedding $Q_i \to Q_R$ for $i = 1,\ldots,t$.

Then the following statements are equivalent.

(1) $P_1 \oplus \dots \oplus P_t$ is faithful.

(2) Any arrow in Q_R is an arrow in some Q_i and when $f_i(Q_i) \cap f_j(Q_j) = [a,b]$ and $a < a' < b' < b$, for an arrow $u \to a$ in $f_j(Q_j) \setminus f_i(Q_i)$ and a path $b \to \dots \to c$ in $f_i(Q_i) \setminus f_j(Q_j)$, if a path $u \to a' \to \dots \to b' \to \dots \to c$ belongs to no $f_s(Q_s)$, then it belongs to I.

Proof. We identify Q_i and $f_i(Q_i)$. Let $e_1, \dots, e_t$ be primitive idempotents such that $P_1 = e_1R, \dots, P_t = e_tR$. Here we denote by [g] a vertex of Q_R corresponding to gR for a primitive idempotent g.

First assume (2). It suffices to show if $(e_1R + \dots + e_tR)r = 0$ for some r in R, then fre = 0 for any primitive idempotents e and f.

Assume fre $\neq$ 0 for some f and e. Then there is a non-zero path $[e]=[g_0] \to [g_1] \to \dots \to [g_n]=[f]$ in KQ_R/I. We put m a minimum number such that $e_iRfRg_m \neq 0$ for some e_i. We remark $0 < m \leq n$ since $e_jRfRf \neq 0$ for some j by assumption and $e_iRfRe = e_iRfre = 0$ by 2.3 (4) and $e_iRr = 0$. By assumption, there is a path $[g_{m-1}] \to [g_m] \to \dots \to [e_j]$ in Q_j for some j. By Lemma 5.2, $[g_{m-1}] \to [g_m] \to \dots \to [e_j]$ doesn't belong to I. By Proposition 6.1, there are g and g' such that $Q_i \cap Q_j = [g,g']$ and $g_m, g_s \in [g,g']$. Since $[g_{m-1}] \to [g_m] \to \dots \to [g_s] \to \dots \to [f]$ belongs to no Q_p (p = 1, ... , t) and $[g_{m-1}] \to [g_m]$ is an arrow in $Q_j \setminus Q_i$ and $[g_s] \to \dots \to [f]$ is a path in $Q_i \setminus Q_j$, $[g_{m-1}] \to [g_m] \to \dots \to [g_s] \to \dots \to [f]$ belongs to I by assumption, hence $[e] \to \dots \to [f]$ belongs to I, which is a contradiction.

Next we prove (1) implies (2). Assume there is an arrow $[f] \to [e]$ in Q_R which is not an arrow in any Q_i. By 2.3 (4), we can put eRf = Kerf for some r in R. We show $(e_1R + \dots + e_tR)erf = 0$ and erf $\neq$ 0.

By assumption, we may assume $e_1Re \neq 0$. Hence $e_iReRf = 0$, otherwise $0 \neq e_iReeRf = e_iRf$ by 2.3 (4) and $[f] \to [e]$ is an arrow in Q_1.

Next in a situation of a latter part of (2), assume a path $\alpha : u \to a' \to \dots \to b' \to \dots \to c$ is not in I, then the element in eRf corresponding to x (here [e]=c and [f]=u) is not zero and $(e_1R + \dots + e_tR)x = 0$ since any path $u \to a' \to \dots \to b' \to \dots \to c \to \dots \to e_i$ belongs to I for every $i = 1, \dots, t$, this is a contradiction.

6.3. We call a connected quiver with a relation ideal $\overline{I}$ a QF-3 quiver if it satisfies the following conditions (1) - (7);

(1) There are elementary QF-3 quivers $Q_1, \dots, Q_n$ and their embeddings $f_1, \dots, f_n$ into Q.

(2) $f_1(Q_1) \cup \dots \cup f_n(Q_n) = Q$ and Q has no oriented cycles with the partial order <.

(3) All the maximal vertices (resp. minimal vertices) are mapped to different vertices each other.

(4) For any pair of quivers Q_i and Q_j, $f_i(Q_i) \cap f_j(Q_j)$ is empty or [a,b] for some vertices a and b, which satisfies the property (*);

(*) b is maximal in $f_j(Q_j)$ iff a is minimal in $f_i(Q_i)$.

(5) The generators of $\overline{I}$ are as follows;

(i) The commutative relation of minimal rectangles.

(ii) For quivers Q_i, Q_j such that $f_i(Q_i) \cap f_j(Q_j) = [b,c]$ and vertices b' and c' such that $b < b' < c' < c$, the zero-relation of a path $a \to b' \to \dots \to c' \to \dots \to d \to e$ such that $a \to b'$ is an arrow in $f_i(Q_i) \setminus f_j(Q_j)$ and $b' \to \dots \to d \to e$ is a path in $f_j(Q_j) \setminus f_i(Q_i)$ whose composition with $a \to b'$ belongs to no $f_s(Q_s)$, but a path

$a \to b' \to \dots \to c' \to \dots . \to d$ belongs to some $f_s(Q_s)$.

(6) Assume $f_i(Q_i) \cap f_j(Q_j) = [c,d]$ and there exist arrows $a \to c$ in Q_i and $b \to c$ in Q_j such that there are non-zero path $y \to \dots \to a \to c$ and $y \to \dots \to b \to c$. Then there exist no non-oriented path $a - x_1 - \dots - x_t - b$ with such that $c \not\leq x_i$ for $i=1, \dots, t$.

(7) $KQ/\overline{I}$ is representation-finite.

6.4. We prove the main theorem.

Theorem. Let Q be a quiver with the relation ideal $\overline{I}$. Then $KQ/\overline{I}$ is simply connected QF-3 algebra iff Q is a QF-3 quivers.

Proof. First assume $KQ/\overline{I}$ is a simply connected QF-3 algebra. In this case, there are projective injective modules $P_1, \dots, P_n$ such that their direct sum is faithful and they are non-isomorphic each other. We put Q_i a quiver of a support algebre $S(P_i)$ for each $i = 1, \dots, n$. Then Q_i is an elementary QF-3 since P_i is a faithful projective injective module over $S(P_i)$ and clearly there is a canonical embedding . Hence (1) holds.

We identify Q_i and $f_i(Q_i)$ in the following if there is no confusion.

Since $P_1 \oplus \dots \oplus P_n$ is faithful, $f_1(Q_1) \cup \dots \cup f_n(Q_n) = Q$ by Theorem 6.2. Of course, Q has no oriented cycle.

(3) holds since $P_1/\text{rad } P_1, \dots, P_n/\text{rad } P_n$ (resp. $\text{Soc}(P_1), \dots, \text{Soc}(P_n)$) are pairwise non-isomorphic.

(4), (5)-(i) and (5)-(ii) are already proved in Proposition 6.1, Theorem 6.3 and 2.4 (7).

(6) is a special case of 2.3 (2).

Next we show that if Q is a QF-3 quiver with relation I induced from (5), then $KQ/\overline{I}$ is a simply connected QF-3 algebra.

Let a_i, b_i be a maximal and a minimal vertex of Q_i respectively 1, ..., n. First by the remark (iii) stated before, we get $Q_{S(P_i)} = Q_i$.

Consider the opposite case, the quiver of support algebra of $\mathrm{Hom}_K(KQ/\overline{I}(b_i,-),K)$ is also Q_i by (5), hence P_i is injective. Thus $P_1 \oplus \dots \oplus P_n$ is projective injective. If $KQ/\overline{I}$ is simply connected, then $P_1 \oplus \dots \oplus P_n$ is faithful by the properties (2), (5)-(ii) and Theorem 4.3.

Last we prove $KQ/\overline{I}$ is simply connected using 2.3 (2).

Assume $KQ/\overline{I}$ is not simply connected. Then there is a vertex c such that rad P(c) has not separated radicals. That is, there are a path $a \to c \leftarrow b$ and non-oriented path $a - x_1 - \dots - x_s - b$ such that a and b belong to supports of different direct summands of rad P(c) each other and $x_1, \dots, x_t$ are not larger than c.

By (2), $a \to c$ and $b \to c$ belong to some Q_s and Q_t respectively. Here Q_s and Q_t are different, otherwise there are non-oriented paths $\overline{[e_s]} \to \dots \to a \to c$ and $\overline{[e_s]} \to \dots \to b \to c$ from a minimal vertex $\overline{[e_s]}$ in Q_s. Since $c \in Q_s \cap Q_t$, there are vertices c',d such that $Q_s \cap Q_t = [c', d']$. It must be c'=c, otherwise, there is a subquiver;

$$
\begin{array}{ccc}
\overline{[e_s]} & \to \dots \to & a \\
\downarrow & & \\
\vdots & & \downarrow \\
\downarrow & & \\
c' & \to \dots \to & c \\
\uparrow & & \\
\vdots & & \uparrow \\
\uparrow & & \\
[e_t] & \to \dots \to & b
\end{array}
$$

such that the upper square is in Q_s and the lower in Q_t, which contradics that a and b are in different supports of rad P(c).

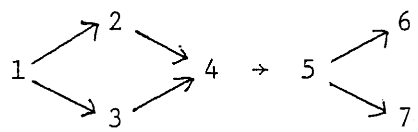
If $\{s,t\} \neq \{i,j\}$, then Q contains subquiver ;

wih no relations. But these subquivers never exist by assumption (7). Hence So $KQ/\overline{I}$ is simply connected by [3] Theorem 2.2.

This completes the proof of the theorem.

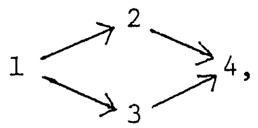
6.5 We give some exaamples in this section.

1. The QF-3 quiver with relations;



$$[1,5] = [2,7] = [3,6] = 0$$

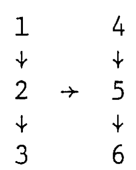
is an interlacing of the following three elementary QF-3 quivers.



$2 \to 4 \to 5 \to 6$, $3 \to 4 \to 5 \to 7$

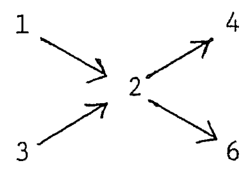
2. Consider two quivers with relations;

(i)



$[1,5] = [2,6] = 0$,

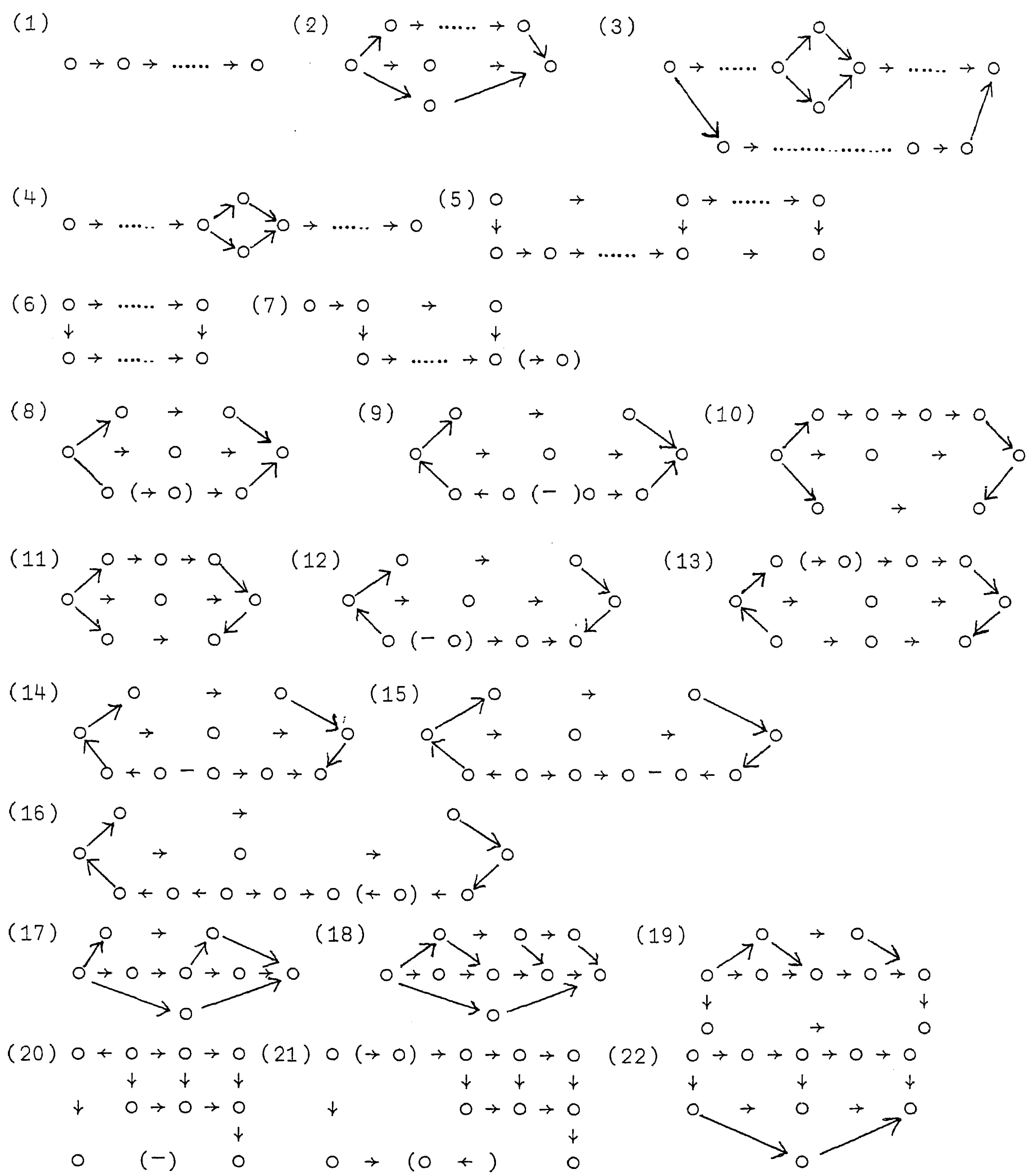
(ii)

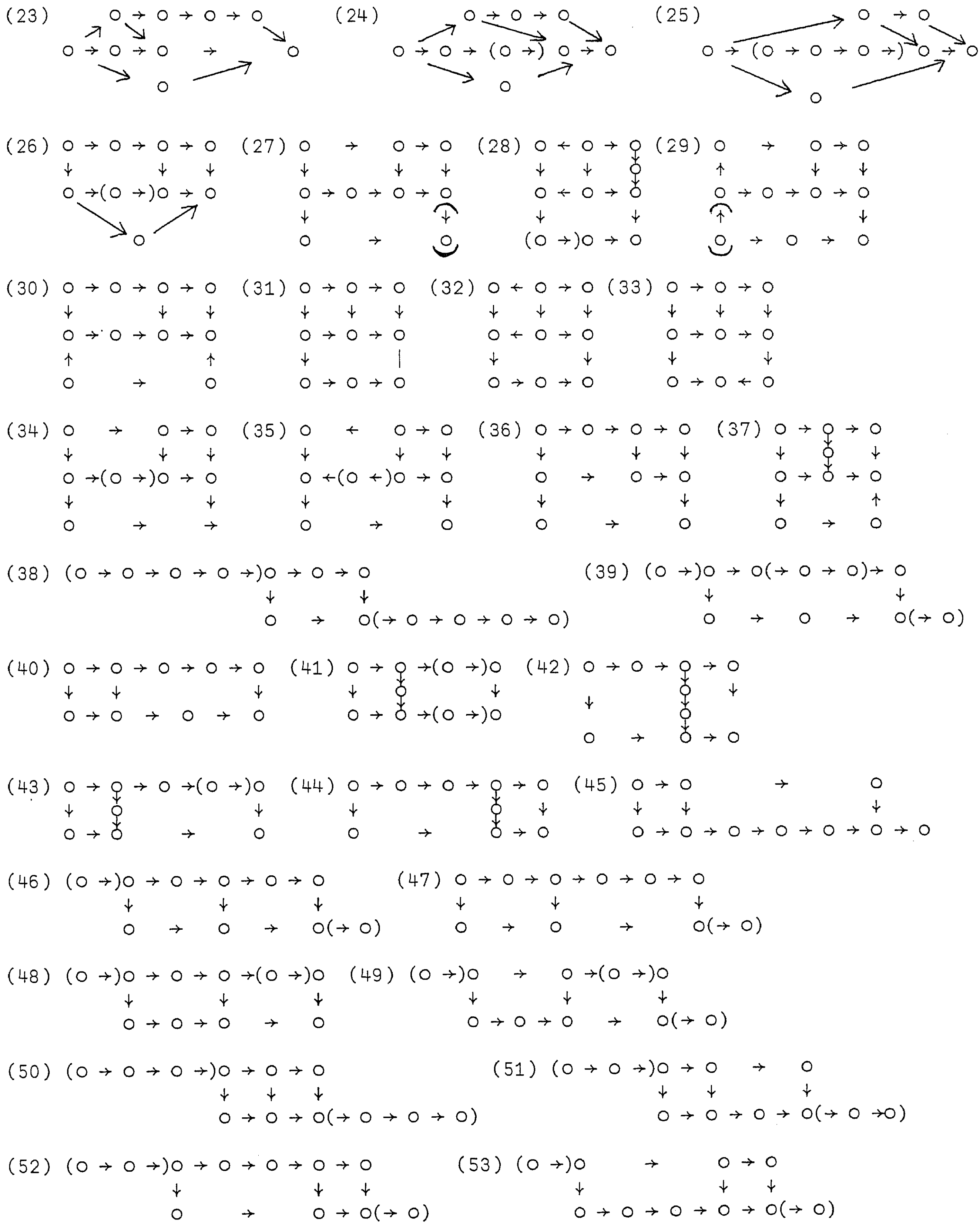


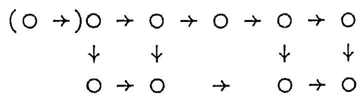
$[1,6] = [4,3] = 0$.

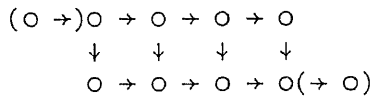
7. THE LIST OF ELEMENTARY QF-3 QUIVERS.

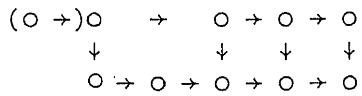
7.1. In a following list, the symbol — means an arrow → or ← and ( ) means that any number of arrows in a parenthesis can be removed in a quiver. Any squares have commutative relations and there are no other relations.

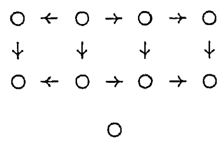


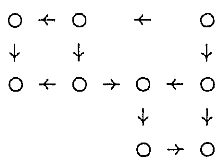

(23)
(24)
(25)
(26)
(27)
(28)
(29)
(30)
(31)
(32)
(33)
(34)
(35)
(36)
(37)
(38)
(39)
(40)
(41)
(42)
(43)
(44)
(45)
(46)
(47)
(48)
(49)
(50)
(51)
(52)
(53)

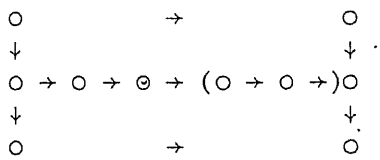
(54) 

(55) 

(56) 

(57) 

(58) 

(59) 

7.2. The graded trees which corresponding to the quivers in 7.1 are as following. In a list the sequence of numbers $i_1, \ldots, i_k$ means that the grading attached with the vertex named number j in a tree written just above is i_j.

```
1 - 2 - ... - n                           1 2 ... n .

      n+2
       |
n+4 -  1 - 2 - ... - n+1                  0 1 2 ... n 1 1 2n+2.
       |
      n+3
```

```
    1                                       n+m+k+4
    |                                          |
2 - 3 - ... - n+2 - n+3 - n+4 - ...  n+k+3 - ... - n+m+4 - ... - n+m+k+4

        n+1 n+1 n ... 1 0 1 ... m 2n+m+3 ... 2n+m+k+2 2(n+m)+k+1
```

```
                      n+2
                       |
1 - 2 - ... - n - n+1 - n+4 - ... - n+m+3
                       |
                      n+3

                                         1 ... n n+1 n+1 2n+3 ... 2n+m+2
```

```
                                     m+n+3     m+n+2
                                       |         |
1 - ... - m - m+1 - m+2 - ... n+3 - ... m+n - m+n+1

                                         m ... 1 0 1 ... n n+2 n+m+2
```

```
                                 n+m+2
                                   |
1 - ... - n - n+1 - n+2 - ... - m+1 - ... - n+m+1

                                         n ... 1 0 1 ... m n+m+1
```

```
    1         n+3
    |          |
2 - 3 - ... - n+2 - n+4(- n+5)

                                        0 2 1 2 ... n+1 (n+4)
```

```
        3
        |
1 - 2 - 5 - 6 - 7                        2 1 1 11 0 1 2
        |
        4
```

```
        3   7
        |   |                           1 0 2 1 2 3 11
1 - 2 - 4 - 5 - 6                       4 1 1 0 1 2 10
```

```
        3                               5 0 2 1 2 3 10
        |                               4 3 1 0 1 2 9
1 - 2 - 4 - 5 - 6 - 7                   1 0 2 1 4 5 6
```

```
        3
        |                               3 2 3 2 11 0 1
        4                               1 0 5 2  9 2 1
        |                               4 1 4 1  8 1 0
1 - 2 - 7 - 6 - 5
```

```
        3
        |
1 - 2 - 5 - 6 - 7 - 8                   2 1 1 17 0 1 2 3
        |
        4
```

```
        3       8                       4 3 3 2 1 0 1 17
        |       |                       1 0 2 1 2 3 10 12
1 - 2 - 4 - 5 - 6 - 7                   6 1 1 0 1 2 3 13
                                        4 1 1 0 1 2 9 11
```

```
        3   5
        |   |
1 - 2 - 4 - 6 - 7 - 8                    2 1 17 0 2 1 2 3

    2   4
    |   |                                4 16 1 1 0 1 2 3
1 - 3 - 5 - 6 - 7 - 8                    8 10 1 1 0 1 2 3

            4
            |                            4 3 2 2 1 0 5 16
1 - 2 - 3 - 5 - 6 - 7 - 8                8 3 2 0 1 2 3 14
                                         4 3 2 2 1 0 9 10
                                         2 1 0 4 1 2 3 14
                       7 2 1 3 0 1 2 13, 6 5 2 2 1 0 1 12
                       3 2 1 5 0 1 2 9 , 1 0 1 3 2 5 6 7
```

```
        3
        |
1 - 2 - 4 - 5 - 6 - 7 - 8

  4 3 3 2 1 0 7 16,  0 1 3 2 3 4 11 12,  3 2 2 1 0 5 6 15
  2 1 1 0 3 4 5 14,  7 2 0 1 2 3 4  13,  1 0 4 1 2 3 4 13
  5 0 2 1 2 3 10 11, 6 1 3 0 1 2 3  12,  1 0 2 1 4 5 6 7
  2 1 3 2 1 0 7 8
```

```
        3
        |                                3 2 17 0 1 2 3 4
        4                                1 0 15 2 1 2 3 6
        |                                5 2 15 2 1 0 1 2
1 - 2 - 5 - 6 - 7 - 8                    3 2 13 2 1 0 1 8
  6 1 10 1 0 1 2 5,  3 2 3 2 1 0 11 12,  4 1 14 1 0 1 2 5
```

```
            4
            |
            5
            |
1 - 2 - 3 - 6 - 7 - 8 - 9                18 17 0 3 2 1 2 3 4
```

```
        3               9                 5 4 4 3 2 1 0 1 29
        |               |                 3 2 2 1 0 1 6 7 27
1 - 2 - 4 - 5 - 6 - 7 - 8                 4 3 3 2 1 0 1 16 18
                                          6 1 1 0 1 2 3 4 24
   3 2 2 1 0 1 6 15 17, 4 1 1 0 1 2 3 10 22,  2 1 1 0 1 4 5 14 16
   8 1 1 0 1 2 3 4  16, 4 1 1 0 1 2 9 10 12
```

```
        3           6                     4 3 3 2 1 29 0 1 2
        |           |                     3 2 2 1 0 18 1 2 15
1 - 2 - 4 - 5 - 7 - 8 - 9                 2 1 1 0 1 23 2 3 10
                                          2 1 1 0 1 17 2 9 14
```

```
        3   5
        |   |
1 - 2 - 4 - 6 - 7 - 8 - 9                3 2 2 1 29 0 1 2 3
```

```
                    5   7
                    |   |                4 3 2 1 1 0 4 1 28
1  - 2 - 3 - 4 - 6 - 8 - 9               8 3 2 1 1 0 2 1 26
                                         4 3 2 1 1 0 10 1 12
```

```
            4
            |
1 - 2 - 3 - 5 - 6 - 7 - 8 - 9
```

```
28 5 0 2 1 2 3 4 5,   26 3 2 0 1 2 3 4 9,  12 11 2 0 1 2 3 4 5,
20 3 2 0 1 2 3 4 13,  27 4 3 1 0 1 2 3 4,  25 2 1 3 0 1 2 3 8,
24 1 0 2 1 2 3 6 7 ,  25 6 3 3 2 1 0 1 2,  23 4 3 3 2 1 0 1 10,
24 5 2 2 1 0 1 2 7 ,  23 4 1 1 0 1 2 5 6,  19 2 1 3 0 1 2 3 12,
20 7 2 2 1 0 1 2 3 ,  18 5 2 2 1 0 1 2 11, 13 6 1 1 0 1 2 5 6 ,
4 3 2 2 1 0 5 16 17,  14 3 2 0 1 2 3 4 21, 4 3 2 2 1 0 9 10 11,
```

```
        3
        |
1 - 2 - 4 - 5 - 6 - 7 - 8 - 9
```

```
5 4 4 3 2 1 0 17 18,  5 4 4 3 2 1 0 9 28,     4 3 3 2 1 0 7 8 27,
2 1 1 0 3 4 5 6  25,  3 2 2 1 0 5 6 7 26,     0 1 3 2 3 4 5 12 23,
7 2 0 1 2 3 4 5  24,  4 3 3 2 1 0 7 16 17,    3 2 2 1 0 5 6 15 16,
2 1 1 0 3 4 5 14 15,  6 1 3 0 1 2 3 4 23,     5 0 2 1 2 3 4 11 22,
4 3 1 0 1 2 3 10 21,  0 1 3 2 3 4 11 12 13,   3 2 0 1 2 3 8 9 20,
5 4 2 1 0 1 2 3  22,  7 4 4 3 2 1 0 1 20,     6 3 3 2 1 0 1 8 19,
5 2 2 1 0 1 6 7  18,  4 1 1 0 1 4 5 6 17,     6 5 3 2 1 0 1 2 13,
1 0 2 1 4 5 6 7  8 ,
```

```
        3
        |
        4
        |
1 - 2 - 5 - 6 - 7 - 8 - 9
```

```
3 2 29 0 1 2 3 4 5,   1 0 27 2 1 2 3 4 7,   5 2 27 2 1 0 1 2 3,
3 2 25 2 1 0 1 2 9,   1 0 19 2 1 2 3 4 13,  4 1 26 1 0 1 2 3 6,
3 2 21 2 1 0 1 2 11,  2 1 24 1 0 1 2 5 8,   8 1 12 1 0 1 2 3 6,
```

```
        3       6
        |       |
1 - 2 - 4 - 5 - 7 - 8 - 9 - 10           3 2 2 1 0 30 1 2 3 26
```

```
        3         7
        |         |                     4 3 3 2 1 0 30 1 2 27
1 - 2 - 4 - 5 - 6 - 8 - 9 - 10          3 2 2 1 0 1 29 2 7 26


        3             8
        |             |
1 - 2 - 4 - 5 - 6 - 7 - 9 - 10
                                        4 3 3 2 1 0 1 17 16 19
    5 4 4 3 2 1 0 28 1 30,  4 3 3 2 1 0 1 27 8 29,
    1 0 2 1 2 3 4 22 11 24, 1 0 2 1 2 3 4 22 5 36,


            4
            |                           32 3 2 0 1 2 3 4 5 24
1 - 2 - 3 - 5 - 6 - 7 - 8 - 9 - 10      5  4 3 3 2 1 0 17 18 19


        3
        |
1 - 2 - 4 - 5 - 6 - 7 - 8 - 9 - 10

6 5 5 4 3 2 1 0 29 30,  5 4 4 3 2 1 0 9 28 29,  4 3 3 2 1 0 7 8 27 28,
2 1 1 0 3 4 5 6 25 26,  3 2 2 1 0 5 6 7 26 27,  5 4 4 3 2 1 0 17 18 19,
0 1 3 2 3 4 5 6 23 36,  0 1 3 2 3 4 5 12 23 24, 7 2 0 1 2 3 4 5 24 25,
1 0 4 1 2 3 4 5 24 25,  5 0 2 1 2 3 4 5 22 35,  3 2 0 1 2 3 4 5 20 45,
5 0 2 1 2 3 4 11 22 23, 3 2 0 1 2 3 4 9 20 33,  4 3 1 0 1 2 3 4 21 34,


        3
        |
        4
        |
1 - 2 - 5 - 6 - 7 -8 - 9 - 10           1 0 31 2 1 2 3 4 5 24


        3             8
        |             |
1 - 2 - 4 - 5 - 6 - 7 - 9 - 10 - 11     4 3 3 2 1 0 1 31 2 27 28


        3                   9
        |                   |           5 4 4 3 2 1 0 1 29 28 31
1 - 2 - 4 - 5 - 6 - 7 - 8 - 10 - 11     4 3 3 2 1 0 1 8 28 27 30


        3
        |
1 - 2 - 4 - 5 - 6 - 7 - 8 - 9 - 10 -11
                                        4 3 3 2 1 0 7 8 27 28 29
6 5 5 4 3 2 1 0 29 30 31,  5 4 4 3 2 1 0 17 18 19 20,
0 1 3 2 3 4 5 6 23 36 37,  4 3 3 2 1 0 7 16 17 18 19,
```

```
        3                       10
        |                        |
1 - 2 - 4 - 5 - 6 - 7 - 8 - 9 - 11 - 12     5 4 4 3 2 1 0 1 28 30 29 32
```

```
        3
        |
1 - 2 - 4 - 5 - 6 - 7 - 8 - 9 - 10 - 11 - 12
```

6 5 5 4 3 2 1 0 29 31 30 32, 5 4 4 3 2 1 0 17 18 20 19 21

```
            4       7
            |       |
1 - 2 - 3 - 5 - 6 - 8 - 9                    4 3 2 2 1 0 16 1 18
```

```
    2       5
    |       |
1 - 3 - 4 - 6 - 7 - 8 - 9 - 10               28 30 1 0 2 1 2 3 4 5
```

```
                5
                |                            27 26 3 2 0 1 2 3 4 9
1 - 2 - 3 - 4 - 6 - 7 - 8 - 9 - 10           21 20 3 2 0 1 2 3 4 13
```

```
        3
        |                                    1 0 2 1 4 5
1 - 2 - 4 - 5 - 6                            2 1 3 0 1 6
```

```
        3
        |
1 - 2 - 4 - 5 - 6 - 7 - 8 - 9 - 10 - 11 - 12 - 13

                          6 5 5 4 3 2 1 0 29 30 31 32 33
```

```
            4
            |
            5
            |
1 - 2 - 3 - 6 - 7 - 8 - 9 -10                30 29 0 3 2 1 2 3 4 5
```

8. THE ESTIMATION OF F(N+1)-F(N).

8.1. The following list is a estimation of F(n+1)-F(n) for each case in section 3. Here in a list the value $\boxed{a_1, \ldots, a_k, \mid m}$ means if $n = m - k + i$ ($i = 1, \ldots, k$), then $F(n+1)-F(n) \le a_i$.

	3.3.		3.5.		3.6.		3.8.		3.9.	
(1)	3		n	3	7,2n-2	6	n	4	n-1	4
(2)	n-2	5	2n-5	5	30	8	2,3,5,11	8	4,8,24	9
(3)	2,6,11	9	n-1	5	28	8	20	9	17,28	11
(4)	3,12,22	9	10,15,26	9	17	7	40	9	n+2	5
(5)	n-7	9	10,18,36	9	13,22	8	30	9	27	10
(6)	8,13,24	9	12,23	8	28	9	60	9	28	11
(7)	n-2	5	n+1	6	10	6	50	9	n-2	7
(8)	25	9	10,17,21	9	15,26	8	37	9	9,14,25	10
(9)	23	8	16,27	9	16,28	8	24	9	29	11
(10)	9	7	9,15,26	9	27	9	4,8	7	15,27	10
(11)	15,27	9	14,19,30	9	28	9	12	8	n-2	7
(12)	n+1	6	13,22	8	26	8	14,25	10	18	8
(13)	32	10	28	8	29	8	29	11	32	9
(14)	28	9	30	9	6,15,26	7	12,19	9	23	9
(15)	36	10			n+2	4	4	5	14,26	9
(16)	2,8,16	8			n-1	3	19	8	27	10
(17)	9,23	8					20	9		
(18)	27	9					28	10		
(19)	22	8					12,23	9		
(20)							30	10		
(21)							24	8		
(22)							16	8		

	3.2.		3.4.		3.7.		3.10		3.11.	
(1)	n-2	3	2		2n-6	6	6	6	14,26	6
(2)	5,6,20	7	n	5	16,28	9	8,9,10	10	27	10
(3)			10,13,16	9	15,28	9	8,13	8	14	8
(4)							12	7		
(5)							n+1	7		
(6)							14	8		
(7)							16	9		
(8)							8,13,24	9		
(9)							28	10		
(10)							15,26	8		
(11)							29	9		

REFERENCES

[1] M. Auslander: Representation theory of artin algebra II, Comm. in algebra 1, (1974) 269-310.

[2] I. Assem, Y. Iwanaga: On a class of representation-finite QF-3 algebra, Proc. the 4-th ICRA, Careton-Ottawa Lecture Note 1985-vol. 2, 3.01-3.22.

[3] R. Bautista, F. Larrion, L. Salmerón: On simply connected algebras, J. London Math. Soc. 27 (2) 1983, 212-220.

[4] K. Bongartz, P. Gabriel: Covering space in representation-theory, Invent. Math. 65 (1981), 331-378.

[5] K. Bongartz: Treue einfach zusammenhängende Algebren I, Comment. Math. Helvetci 57 (1982), 282-330.

[6] O. Bretscher, C. Läser, C. Riedtmann: Selfinjective and simply connected algebra, Manu. Math., 36 (1981), 253-357.

[7] D. Happel, C.M. Ringel: Tilted algebra, Trans. Amer. Math. Soc., 274 (2) 1982, 399-443.

[8] L.A. Nazarova, A.V. Roiter: Representations of partially ordered set, Zap. Naucn. Sem. LOMI 28, (1973) 5-31. J. Soviet Math 23, (1975), 585-606.

[9] M. Sato: On maximal gradings of simply connected algebras, Tsukuba J. Math., 8 (2) 1984, 319-331.

[10] M. Sato: On simply connected QF-3 algebras and their construction, to appear in J. Algebra.

[11] H. Tachikawa: Quasi-Frobenius Rings and Generalized QF-3 and QF-1 Rings, Lecture notes in Math. 351, Springer-Verlag 1982.

[12] H. Tachikawa, T. Wakamatsu: Refrection functors for self-injective algebras, preprint.