COMPACTNESS OF COUNTABLY COMPACT SPACES

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THESIS

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Science
in Doctoral Program in Mathematics in the
University of Tsukuba

December 1985

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1. Introduction.

The notions of compactness and countable compactness are among the oldest in topology. Every compact space is countably compact, but the converse is not true. In fact, there is a wide gap between their notions. A space is said to be compact if every open cover has a finite subcover, on the other hand, a space is said to be countably compact if every countable open cover has a finite subcover. However, in spite of the gap, as described later, it is known that some weak conditions compel spaces to be compact under countable compactness. The purpose of this article is to study the following problem: What conditions will make a countably compact space compact ? This problem is important and interesting. Because, firstly, factorization of compactness is useful in proving compactness of spaces. Secondarily, the problem relates to many branches of general topology. In fact, there were many questios concerning this problem. For example,

(1) Is a countably compact T_2-space with a G_δ-diagonal compact ? [30]

(2) Is a countably compact regular perfect space compact ? , where a space is called perfect if each open set of X is a union of countably many closed subsets of X. [47]

(3) Is a countably compact T_1-space with a base of subinfinite rank(or, an ortho-base) compact ? [36]

Following Bacon[6], we call a sapce isocompact if each countably compact closed subset is compact. Since Bacon's paper, a lot of classes of isocompact spaces have been widely studied by many mathematicians. So, it should seem that a direction of study for the above problem is to find a large class of isocompact spaces which contains many known classes of isocompact spaces. For the purpose of finding such class, in the next section, we shall look over classes of isocompact spaces which has been extensively studied. Weakly $\delta\theta$-refinable spaces, $\mathfrak{F}$-spaces, spaces satisfying property θL, weakly $[\omega_1,\infty)^r$-refinable spaces, $\delta\theta$-penetrable spaces, pure spaces and so on are listed as a class of isocompact spaces. Other results relating to the above problem are also introduced.

In the third section, we note a connection with the above problem and closed-completeness. Some classes of spaces listed in the second section imply closed-completeness under a condition, and closed-completeness sometimes make proofs of compactness more simpler and systematic.

As a desired large class of isocompact spaces, the class of (k-) neat spaces are defined in the fourth section. This class contains neighborhood $\tilde{\mathfrak{F}}$-spaces, spaces satisfying property θL, weakly $[\omega_1,\infty)^r$-refinable spaces, $\delta\theta$-penetrable spaces, ultrapure^r spaces and pure spaces. It is proved that every (k-) neat space is isocompact and every ω_1-compact ω_1-neat T_1-space is closed-complete. These two theorems strengthen many results in this field. Other properties of (k-) neat spaces are also investigated, for instance, behavior of (k-) neat spaces by some maps.

A pseudocompact metacompact Tychonoff space is compact[45][51]. And a pseudocompact paraLindelöf Tychonoff space is compact[11]. These results and the above problem naturally raise the following question: When are spaces having a countably compact dense subset compact ? In the fifth section, this question will be examined. For this question, we shall give some answers.

Unless explicitly stated, no separation axioms will be required under consideration. A regular space means a regular T_2-space. All maps are assumed to be continuous. For a collection $\mathcal{U}$ of subsets of a space, we denote by $\mathcal{U}^*$ the union of elements of $\mathcal{U}$.

For later use, we define here, in the lump, some concepts relating realcompactness and cardinal functions. A space X is called closed-complete[19] (resp. realcompact[25], resp. Borel complete[27]) in case every closed (resp. z- , resp. Borel) ultrafilter on X with the countable intersection property (c.i.p.) is fixed. Realcompact spaces are required to be Tychonoff. Closed-complete spaces are called a-realcompact in [19]. A space X is called weakly Borel complete[40] if every Borel ultrafilter on X with c.i.p. has a cluster point. Borel completeness is equivalent to be hereditarily weakly Borel complete[40]. A collection $\mathcal{U}$ of subsets of a space is said to have the countable closure intersection property (c.c.i.p.) if for each countable subcollection $\mathcal{V}$ of $\mathcal{U}$, $\cap \overline{\mathcal{V}} \neq \phi$, where $\overline{\mathcal{V}} = \{ \overline{V} \mid V \in \mathcal{V} \}$. A space X is called almost realcompact[23] if every open ultrafilter on X with c.c.i.p. has a cluster point. The following chart summarizes the relationship of these notions.

chart 1.

closed-complete
↑
realcompact → almost realcompact → weakly Borel complete
↑
Borel complete

We define several cardinal functions for a space X.

(1) Density, $d(X) = \min\{ |S| \mid S \text{ is dense in } X.\}$.

(2) Hereditary density, $hd(X) = \sup\{ d(Y) \mid Y \subset X \}$.

(3) Lindelöf degree, $L(X) = \min\{ \alpha \mid X \text{ is } \alpha\text{-Lindelöf.}\}$, where a space is said to be α-Lindelöf if every open cover of X has a subcover of cardinality $\leq \alpha$.

(4) Hereditary Lindelöf degree, $hl(X) = \sup\{ L(Y) \mid Y \subset X\}$.

(5) Spread, $s(X) = \sup\{ |D| \mid D \text{ is a discrete subspace of } X.\}$.

(6) Tightness, $t(X) = \sup\{ t(x,X) \mid x \in X \}$, where $t(x,X)$ is defined in the following manner: $t(x,X) = \min\{ \alpha \mid \text{If } x \in \bar{A} \subset X, \text{ then there exists } B \subset A \text{ with } x \in \bar{B} \text{ and } |B| \leq \alpha \}$.

(7) Length of free sequences, $\partial(X) = \sup\{ \alpha \mid \text{there exists a free sequence of length } \alpha \}$, where a sequence $\{ x_\beta \mid \beta < \alpha \}$ of points of X is called free if $\{ x_\delta \mid \delta < \gamma \}$ and $\{ x_\eta \mid \gamma \leq \eta < \alpha \}$ have disjoint closures for every $\gamma < \alpha$.

For details, see [34].

2. Classes of isocompact spaces.

The study of isocompact spaces started from Bacon's paper[6]. As mentioned in the first section, a space is said to be isocompact if every countably compact closed subset is compact. In those days, it was known that a countably compact space is compact if it is either a Moore space or a paracompact space. And then, Bacon introduced a class of isocompact spaces(i.e. the class of spaces satisfying "property L") that includes all Moore spaces and all paracompact spaces. Since property L is a special case of property θL of Davis defined later, we don't define property L.

We denote by ω (ω_1) the first infinite(uncountable) cardinal and $\mathcal{P}(X)$ denotes the power set of a set X.

DEFINITION 2.1.[14][28] A space X is called an $\mathfrak{F}$-space if there is a function $B:\omega \times X \longrightarrow \mathcal{P}(X)$ such that the following are true:

(1) For each $n \in \omega$ and $x \in X$, $B(n+1, x) \subset B(n,x)$, and for each $x \in X$, $\bigcap_{n\in\omega} B(n,x) = \{x\}$.

(2) A subset U of X is open if and only if, for each $x \in U$, there exists $n_x \in \omega$ such that $B(n_x,x) \subset U$.

(3) If F is closed in X and $x \notin F$, then there exists $n \in \omega$ such that for each $y \in B(n,x) - \{x\}$, there exists $n_y \in \omega$ such that $\{x,y\} \not\subset \bigcup_{f\in F} B(n_y,f)$.

We say X is a neighborhood $\mathfrak{F}$-space if B(n,x) is an open neighborhood of x for each $n \in \omega$ and $x \in X$.

The class of $\mathfrak{F}$-spaces includes symmetrizable spaces.

A space is called ω_1-compact if the cardinality of each closed discrete subset is countable. Since every countably compact space is ω_1-compact and each closed subspace of an $\mathfrak{F}$-space is an $\mathfrak{F}$-space, isocompactness of $\mathfrak{F}$-spaces follows from the following theorem.

THEOREM 2.2.[28] Let X be an $\mathfrak{F}$-space. The following are equivalent.

(a) X is ω_1-compact.

(b) X is Lindelöf.

If $\mathcal{U}$ is a collection of sets, we define ord$(x,\mathcal{U})$ by $|\{U \in \mathcal{U} \mid x \in U\}|$.

DEFINITION 2.3.[53] A space X is said to be weakly $\delta\theta$-refinable if each open cover of X has an open refinement $\bigcup_{n \in \omega} \mathcal{U}_n$ satisfying that for each $x \in X$ there is $n \in \omega$ such that $0 < \text{ord}(x, \mathcal{U}_n) \leq \omega$.

Wicke and Worrell showed in [53] that weakly $\delta\theta$-refinable spaces are isocompact. More precisely, they proved the following.

THEOREM 2.4.[53] Suppose X is countably compact and $\mathcal{U} = \bigcup_{n\in\omega} \mathcal{U}_n$ is an open cover of X such that for each $x \in X$ there is n such that $0 < \mathrm{ord}(x, \mathcal{U}_n) \leq \omega$, then $\mathcal{U}$ has a finite subcover.

Since this theorem is important and interesting, and for convenience, we give a sketch of the proof of Wicke and Worrell.

Proof. For each $n \in \omega$, let $C_n = \{ x \in X \mid 0 < \mathrm{ord}(x, \mathcal{U}_n) \leq \omega \}$. Suppose that $\mathcal{U}$ has no countable subcover. Then we may assume that C_0 is not covered by any countable subfamily of $\mathcal{U}$. Let $E_0 = X - {}_{0}$. If E_0 is covered by a countable subfamily $\mathcal{V}$ of $\mathcal{U}$, then we can see $C_0 - \mathcal{V}^*$ is covered by a countable subfamily of $\mathcal{U}_0$. This is a contradiction. Nextly we take the first n_1 such that $E_0 \cap C_{n_1}$ is not covered by any countable subfamily of $\mathcal{U}$. We assume $C_{n_1} = C_1$. Let $E_1 = E_0 - \mathcal{U}_1^*$. By the same reason, E_1 is not covered by any countable subfamily of $\mathcal{U}$. By continuing this, we can obtain a decreasing sequence $\{ E_n \}_{n\in\omega}$ of nonempty closed sets with the empty intersection. Since X is countably compact, this is a contradiction.

Isocompactness for the paracompact case is due to [17], for metacompact spaces to [2], for metaLindelöf spaces to [1], for θ-refinable spaces to [57], and for $\delta\theta$-refinable spaces to [5].

For a collection $\mathcal{B}$ of subsets of a space X and $x \in X$, set $I(x, \mathcal{B}) = \cap \{B \mid B \in \mathcal{B}, x \in B\}$.

DEFINITION 2.5.[9] We say that an open cover $\cup_{n \in \omega} \mathcal{V}_n$ of a space X is a θ-penetration(resp. $\delta\theta$-penetration) of a cover $\mathcal{U}$ of X if, for every $x \in X$, $\cap \{I(x, \mathcal{V}_n) : n \in \omega$ and $0 < \mathrm{ord}(x, \mathcal{V}_n) < \omega\} \subset U$ for some $U \in \mathcal{U}$ (resp. $\cap \{ I(x, \mathcal{V}_n) : n \in \omega$ and $0 < \mathrm{ord}(x, \mathcal{V}_n) \leq \omega\} \subset U$ for some $U \in \mathcal{U}$), and that X is θ-penetrable(resp. $\delta\theta$-penetrable) if every open cover of X has a θ-penetration (resp. $\delta\theta$-penetration).

Spaces with a point countable separating open cover and weakly $\delta\theta$-refinable spaces are $\delta\theta$-penetrable[9, Remarks 2.1].

THEOREM 2.6.[12] Let $\mathcal{U}$ be an open cover of a countably compact space X. If there exists a $\delta\theta$-penetration of $\mathcal{U}$, then $\mathcal{U}$ has a finite subcover.

The above theorem shows that $\delta\theta$-penetrable spaces are iso-compact. And Chaber proved in [12] the following result as the answer for Heath's question mentioned in the first section.

THEOREM 2.7. Every space with a quasi-G_δ-diagonal is isocompact.

Let $[\omega]^{\omega}$ denote the set of all infinite subsets of ω. For $A, B \in [\omega]^{\omega}$, we write $A \subset^{*} B$ provided $|A - B| < \omega$. A family $\mathfrak{F} \subset [\omega]^{\omega}$ has the strong finite intersection property (s.f.i.p.) provided every intersection of finitely many elements of $\mathfrak{F}$ is an infinite set. Such a family $\mathfrak{F}$ is called maximal provided for no $A \in [\omega]^{\omega}$ is $A \subset^{*} F$ for all $F \in \mathfrak{F}$. Then p is defined to be the smallest cardinality of a maximal family with s.f.i.p.[18]. MA + $\neg$ CH implies $p > \omega_1$.

We recall Stephenson's question: Is a countably compact regular perfect space compact ? For this question, Weiss proved the following.

THEOREM 2.8.[52] ($p > \omega_1$) Every countably compact regular perfect space is compact.

We cannot delete the hypothesis $p > \omega_1$. In fact, under Jensen's Combinatorial Principle $\diamondsuit$ which is followed from Gödel's Axiom of Constructibility, Ostaszewski constructed a non-compact, hereditarily separable, locally compact, perfectly normal, countably compact space[38]. Another non-compact, perfectly normal, countably compact space is independently given in [21].

Now we define property θL of Davis which is a common generalization of property L and weak $\delta\theta$-refinability. Property θL is motivated by M. Michael's characterization of paracompactness in terms of cushioned refinements. We denote by Card the class of all infinite cardinals. For a collection $\mathcal{V}$ of subsets of a set, $\omega\mathcal{V}$ is the set of unions of countable subcollections of $\mathcal{V}$.

For $k \in$ Card, and $\mathcal{U}$ and $\mathcal{V}$ collections of subsets of a space X, we say $\mathcal{V}$ is k-weakly cushioned in $\mathcal{U}$ if and only if there exists a function f from $\mathcal{V}$ to $\mathcal{U}$ such that if $\mathcal{W} \subset \mathcal{V}$ with $|\mathcal{W}| \leq k$ and $x: \mathcal{W} \longrightarrow \cup \mathcal{W}$ with $x(G) \in G$ for each $G \in \mathcal{W}$, then $\overline{\{x(G):G \in \mathcal{W}\}} \subset \cup f(\mathcal{W})$.

DEFINITION 2.9.[15] For $k \in$ Card, we say a space X satisfies property θkL if and only if for every open cover $\mathcal{U}$ of X there exists a sequence $\langle \mathcal{D}_n : n \in \omega \rangle$ of collections of subsets of X and a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ of open refinements of $\mathcal{U}$ such that $\bigcup_{n \in \omega} \mathcal{D}_n$ covers X and for each $n \in \omega$, $\cup \mathcal{D}_n \subset \cup \mathcal{V}_n$ and $\mathcal{D}_n$ is k-weakly cushioned in $\omega \mathcal{V}_n$ in the space $\cup \mathcal{V}_n$.

We shall refer to property $\theta\omega$L as property θL.

Spaces satisfying property L and weakly $\delta\theta$-refinable spaces satisfy property θL[15, Theorem 2.2, 2.3].

THEOREM 2.10.[15] Every space satisfying property θL is isocompact.

DEFINITION 2.11.[58] A space X is said to be weakly $[\omega_1, \infty)^r$-refinable if for any open cover $\mathcal{U}$ of uncountable regular cardinality there exists an open refinement which can be expressed as $\bigcup_{\gamma \in \Gamma} \mathcal{V}_\gamma$, where $|\Gamma| < |\mathcal{U}|$ and if $x \in X$ there is some $\gamma \in \Gamma$ such that $0 < \text{ord}(x, \mathcal{V}_\gamma) < |\mathcal{U}|$.

Obviously weakly $\delta\theta$-refinable spaces are weakly $[\omega_1, \infty)^r$-refinable.

THEOREM 2.12.[58] Every weakly $[\omega_1, \infty)^r$-refinable space is isocompact.

A cover $\mathcal{E} = \bigcup_{n \in \omega} \mathcal{E}_n$ of a space X is called an interlacing if for each $n \in \omega$ and $U \in \mathcal{E}_n$, U is open in $\mathcal{E}^*_n$. Let $\mathcal{H}$ be a family of subsets of X. We say that an interlacing $\mathcal{E} = \bigcup \mathcal{E}_n$ is δ-suspended from $\mathcal{H}$ if for each $n \in \omega$ and $x \in \mathcal{E}^*_n$ there exists a countable subfamily $\mathcal{F}$ of $\mathcal{H}$ such that $St(x, \mathcal{E}_n) \cap (\bigcap \mathcal{F}) = \phi$.

DEFINITION 2.13.[3] A space X is said to be pure(ultrapure) if for each free closed ultrafilter(free closed family) $\mathcal{H}$ on X with c.i.p., there exists an interlacing on X that is δ-suspended from $\mathcal{H}$.

Every weakly $\delta\theta$-refinable space is ultrapure[3] and every ultrapure space is pure. Arhangel'skii defined astral spaces between ultrapure spaces and pure spaces, but in this article, we don't need to know what an astral space is. Spaces with a quasi-G_δ-diagonal and closed-complete spaces are pure. Spaces with a quasi- G_δ-diagonal are, in fact, astral[3].

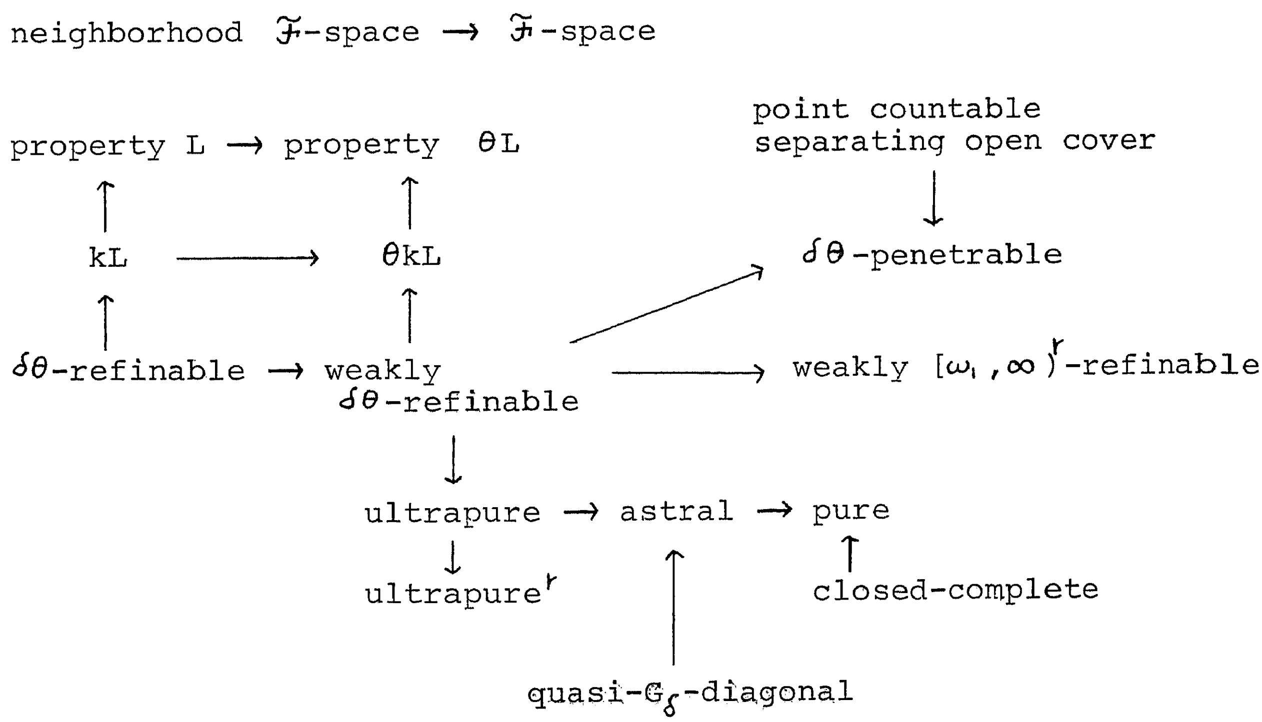
THEOREM 2.14.[3] Every pure space is isocompact.

An idea of weak $[\omega_1, \infty)^r$-refinability is to define the property by restricting the definition of weak $\delta\theta$-refinability to apply only to open covers of regular cardinality. Vaughan applied the same idea to ultrapure spaces, astral spaces and pure spaces.

He considered ultrapure^r spaces, astral^r spaces and pure^r spaces, for example, ultrapure^r spaces are defined in the following manner: a space is said to be ultrapure^r if for each free closed family $\mathcal{H}$ of regular cardinality with c.i.p., there exists an interlacing that is δ-suspended from $\mathcal{H}$. In [49], Vaughan showed that every ultrapure^r space is isocompact, and astral^r spaces and pure^r spaces are not always. In fact, he proved that the both statement "Every countably compact pure^r T_2-space is compact" and "Every countably compact astral^r T_2-space is compact" are consistent with ZFC.

The following chart summarizes the implications of notions appered in this section.

chart 2.



In the fourth section, we show that every neighborhood $\mathcal{F}$-space satisfies property θkL, every $\delta\theta$-penetrable space is pure and every ultrapure^r space is pure.

Though notions appered in the above are a kind of covering properties, there exist properties of a different type which force compactness under countable compactness. Gruenhage showed in [26], as answers for Lindgren and Nyikos's question[36] mentioned in the first section, that a countably compact space having either a base of subinfinite rank or an ortho-base is compact. A base $\mathcal{B}$ of a space is said to have subinfinite rank if for every $\mathcal{B}' \subset \mathcal{B}$ such that $\cap \mathcal{B}' \neq \phi$ and $\mathcal{B}'$ is infinite, at least two elements of $\mathcal{B}'$ are related by set inclusion, and $\mathcal{B}$ is said to be an ortho-base if for every $\mathcal{B}' \subset \mathcal{B}$, $\cap \mathcal{B}'$ is open or $\mathcal{B}'$ is a neighborhood base of some point. Such bases were introduced by Nyikos as natural generalizations of non-archimedean spaces[36][37]. All metric spaces have such bases.

THEOREM 2.15.[26] Every countably compact T_1-space is compact if it has either a base of subinfinite rank or an ortho-base.

Above base properties are hereditary, every T_1-space having either a base of subinfinite rank or an ortho-base is isocompact. After Gruenhage's paper, it was showed that in [22] that a T_1-space having a base of subinfinite rank is metacompact. Since metacompactness implies weak $\delta\theta$-refinability, isocompactness of T_1-space having a base of subinfinite rank easily follows from the result. But we don't know whether having an ortho-base implies a weak covering axiom(for example, weak $\delta\theta$-refinability and so on).

Other spaces which force compactness under countable compactness are left separated spaces, irreducible spaces and isoparacompact spaces and so on. For instance, see [32, Lemma 2.2][16, Theorem 2.1].

Let us denote the following statement by S : If $\mathfrak{F}$ is a family of less than 2^{ω} subsets of ω with s.c.i.p., then there exists an infinite $D \subset \omega$ such that for each $F \in \mathfrak{F}$ $D - F$ is finite.

Concerning compactness of separable countably compact spaces, the following are known.

THEOREM 2.16.[52] Under S, if X is a separable countably compact regular space with $L(X) < 2^{\omega}$, then X is compact.

THEOREM 2.17.[50] Every separable countably compact regular $[p,\infty]$-compact space is compact, where a space is called $[p,\infty]$-compact if every open cover has a subcover of cardinality strictly less than p.

Other informations of isocompact spaces can be obtained from [54][10][50]. Blair's papers [7][8][9] are also closely related to isocompact spaces. They are treated in the next section.

3. A relation with closed-completeness.

In this section we consider the problem from a different aspect. In proving isocompactness of spaces, there are two methods, one is, of course, to try a direct proof, and the other is to try to prove closed-completeness of spaces. The method of making use of closed-completeness has some better poins than direct proofs. In fact, the method of closed-completeness generalizes some results in former section and make proofs of isocompactness more simpler and systematic. For example, Blair showed the following theorem as a generalization of Theorem 2.7. We sketch the proof for the sake of seeing the essence of an idea.

THEOREM 3.1.[7] An ω_1-compact T_1-space with a quasi-G_δ-diagonal is closed-complete.

Proof. Let X be an ω_1-compact T_1-space with a quasi-G_δ-diagonal $\{\mathcal{G}_n\}_{n\in\omega}$ and suppose that there exists a free closed ultrafilter $\mathcal{H}$ on X with c.i.p.. Since $\mathcal{H}$ is free, we may assume that for each $x \in X$ there exists n such that $x \in \mathcal{G}_n^*$ and $X - St(x, \mathcal{G}_n) \in \mathcal{H}$. Let $A_n = \{ x \in \mathcal{G}_n^* \mid X - St(x, \mathcal{G}_n) \in \mathcal{H} \}$. Since $X = \cup A_n$, there exists n such that $H \cap A_n \neq \phi$ for any $H \in \mathcal{H}$. We take $H \in \mathcal{H}$ such that $H \subset \mathcal{G}_n^*$. Now, by Zorn's lemma, there exists a closed discrete subset D in X contained to $H \cap A_n$ such that, (1) $|G \cap D| \leq 1$ for $G \in \mathcal{G}_n$, (2) $H \cap A_n \subset \bigcup_{x\in D} St(x, \mathcal{G}_n)$. Since $|D|$ is countable, $\bigcap_{x\in D} X - St(x, \mathcal{G}_n) \in \mathcal{H}$. So, we get $H \cap (\bigcap_{x\in D} X - St(x, \mathcal{G}_n)) \cap A_n = \phi$. This is a contradiction.

A countably compact space is ω_1-compact and it is easy to see that a countably compact closed-complete space is compact. Since closed-completeness is closed hereditary, Theorem 2.7 easily follows from the above theorem.

We collect the same results as Theorem 3.1.

THEOREM 3.2. The following spaces are closed-complete if they are ω_1-compact T_1-spaces.

(1) Spaces satisfying property $\theta\omega_1 L$. [15, Theorem 2.5]

(2) weakly $[\omega_1, \infty)^r$-refinable spaces. [58, Corollary 3.6]

(3) $\delta\theta$-penetrable spaces. [9, Corollary 2.5]

Since an $\mathfrak{F}$-space is Lindelöf under ω_1-compactness, an $\tilde{\mathfrak{F}}$-space has also the same property. The case of weakly $\delta\theta$-refinable spaces is due to [8, Corollary 3.3]. In the next section, we generalize the above theorem.

4. k-neat spaces and related results.

In this section we define the class of (k-) neat spaces, and we shall generalize the results mentioned in the previous sections. We show that every neat space is isocompact and the class of neat spaces contains all of the following classes : neighborhood $\mathcal{F}$-spaces, spaces satisfying property θL, weakly $[\omega_1, \infty)^r$-refinable spaces, $\delta\theta$-penetrable spaces , ultrapure^r spaces and pure spaces. It is also showed that an ω_1-compact ω_1-neat T_1-space is closed-complete. By these results, we can neatly review many results in the area of isocompact spaces.

Define for each free closed ultrafilter $\mathcal{H}$ on X with c.i.p., $\lambda(\mathcal{H}) = \min\{|\mathcal{F}| : \mathcal{F} \subset \mathcal{H}, \cap \mathcal{F} = \phi\}$. $\lambda(\mathcal{H})$ is an uncountable regular cardinal.

DEFINITION 4.1.[44] Let $\mathcal{H}$ be a free closed ultrafilter on X with c.i.p. and $k \in$ Card. A system $\langle \{X_\gamma\}, \{\mathcal{V}_\gamma\}, \{f_\gamma\} \rangle_{\gamma\in\Gamma}$ is called a k-neat system for $\mathcal{H}$ if the following are satisfied:

(1) $|\Gamma| < \lambda(\mathcal{H})$.

(2) $\{X_\gamma\}_{\gamma\in\Gamma}$ is a cover of X and $\mathcal{V}_\gamma$ is an open collection of X such that $X_\gamma \subset \mathcal{V}_\gamma^*$ for each $\gamma\in\Gamma$.

(3) Each f_γ is a function from X_γ to $\mathcal{V}_\gamma$ such that if $A \subset X_\gamma$, $|A| \leq k$ and $f_\gamma | A$ is injective, then the closure of A in $\mathcal{V}_\gamma^*$ is contained in $\bigcup_{x\in A} f_\gamma(x)$.

(4) For each $\gamma\in\Gamma$ and $x \in X_\gamma$ there exists $H \in \mathcal{H}$ such that $f_\gamma(x) \cap X_\gamma \cap H = \phi$.

A space X is called a k-neat space if for each free closed ultrafilter $\mathcal{H}$ on X with c.i.p. there exists a k-neat system for $\mathcal{H}$. We shall refer to an ω-neat space as merely a neat space.

A k'-neat space is k-neat if $k' \geq k$. It is easily seen that for all $k \in$ Card, a space X with countable tightness is k-neat if and only if X is neat.

LEMMA 4.2. [9, Lemma 2.2] If $\mathcal{H}$ is a free closed ultrafilter on X with c.i.p. and if $\mathcal{V} = \cup_{n \in \omega} \mathcal{V}_n$ is a θ-penetration(resp. $\delta\theta$-penetration) of $\mathcal{U} = \{X - H : H \in \mathcal{H}\}$, then $\mathcal{V}$ has a subcover that is a weak θ-refinement(resp. weak $\delta\theta$-refinement) of $\mathcal{U}$.

THEOREM 4.3. The following spaces are (k-) neat. Moreover, the implications (a)$\rightarrow$(b), (d)$\rightarrow$(f) and (e)$\rightarrow$(f) hold.

(a) neighborhood $\mathcal{F}$-spaces.

(b) spaces satisfying property θL.

(c) weakly $[\omega_1, \infty)^r$-refinable spaces.

(d) $\delta\theta$-penetrable spaces.

(e) ultrapure^r spaces.

(f) pure spaces.

Proof. (a)$\rightarrow$(b). Let $\mathcal{U}$ be any open cover of X, and set $S = \{x \in X : x \in \cup_{z \in X-U} B(n,z)$ for each $n \in \omega$ and $x \in U \in \mathcal{U}\}$. We note that S is a discrete subset of X. Take $x \in S$, and select $U_x \in \mathcal{U}$ such that $x \in U_x$. For x and $X - U_x$ there exists n_x corresponding to (3) of Definition 2.1. We may assume $B(n_x, x) \subset U_x$. Since for $y \in B(n_x,x)-\{x\}$ there exists n_y such that $\{x,y\} \not\subset \bigcup_{z\in X-U_x} B(n_y,z)$, it follows from $x \in \bigcup_{z\in X-U_x} B(n_y,z)$ that $y \notin \bigcup_{z\in X-U_x} B(n_y,z)$. So $y \notin S$ for any $y \in B(n_x,x)-\{x\}$. Thus S is discrete in X. Set $\mathcal{D}_0 = \{\{x\} : x \in S\}$, $\mathcal{V}_0 = \{B(n_x,x) : x \in S\}$ and define a function f_0 from $\mathcal{D}_0$ to $\mathcal{V}_0$ such that $f_0(\{x\}) = B(n_x,x)$. $\mathcal{D}_0$ is k-weakly cushioned in $\mathcal{V}_0$ in the space $\mathcal{V}_0^*$ for any $k \in$ Card.

For $x \in X-S$ we can take $U_x \in \mathcal{U}$ and $n_x \in \omega - \{0\}$ such that $x \in U_x$ and $x \in \bigcup_{z\in X-U_x} B(n_x,z)$. Put $X_n = \{x \in X-S: n_x = n\}$. Obviously $X - S = \bigcup_{n=1}^{\infty} X_n$. Set $\mathcal{D}_n = \{\{x\} : x \in X_n\}$, $\mathcal{V}_n = \{U_x: x \in X_n\}$ and define a function f_n from $\mathcal{D}_n$ to $\mathcal{V}_n$ such that $f_n(\{x\}) = U_x$ for $n \geq 1$. It is easily proved that $\mathcal{D}_n$ is k-weakly cushioned in $\mathcal{V}_n$ in the space $\mathcal{V}_n^*$ for $k \in$ Card. Thus X satisfies property θkL.

(b). Let $\mathcal{H}$ be a free closed ultrafilter on X with c.i.p.. Then $\mathcal{U} = \{X - H: H \in \mathcal{H}\}$ is an open cover of X. For this $\mathcal{U}$ there exist sequences $\langle \mathcal{D}_n: n \in \omega \rangle$ and $\langle \mathcal{V}_n: n \in \omega \rangle$ of Definition 2.9. Let f_n be a function to be k-weakly cushioned from $\mathcal{D}_n$ to $\omega\mathcal{V}_n$. We may assume that each $\mathcal{D}_n$ is a disjoint collection and each f_n is injective. Put $X_n = \mathcal{D}_n^*$. For each $n \in \omega$ and $x \in X_n$ there exists uniquely $D_x \in \mathcal{D}_n$ such that $x \in D_x$. Put $W_x = f_n(D_x)$, $\mathcal{W}_n = \{W_x: x \in X_n\}$ and define a function g_n from X_n to $\mathcal{W}_n$ such that $g_n(x) = W_x$. $\langle \{X_n\}, \{\mathcal{W}_n\}, \{g_n\} \rangle_{n\in\omega}$ is a desired neat system for $\mathcal{H}$. Thus (b) implies neatness.

(c). Let $\mathcal{H}$ be a free closed ultrafilter on X with c.i.p.. We take a free subfamily $\mathcal{F}$ of $\mathcal{H}$ such that $|\mathcal{F}| = \lambda(\mathcal{H})$. Since the cardinality of the open cover $\mathcal{U} = \{X-F: F \in \mathcal{F}\}$ is uncountable regular, there exists an open refinement $\mathcal{G} = \bigcup_{\gamma\in\Gamma} \mathcal{G}_\gamma$ of $\mathcal{U}$ such that $|\Gamma| < \lambda(\mathcal{H})$ and for each $x \in X$ there exists $\gamma \in \Gamma$

such that $0 < \mathrm{ord}(x, \mathcal{G}_\gamma) < \lambda(\mathcal{H})$. Now for each $\gamma \in \Gamma$ we put $X_\gamma = \{x \in X: 0 < \mathrm{ord}(x, \mathcal{G}_\gamma) < \lambda(\mathcal{H})\}$, $\mathcal{V}_\gamma = \{\mathrm{St}(x, \mathcal{G}_\gamma): x \in X_\gamma\}$ and define a function f_γ from X_γ to $\mathcal{V}_\gamma$ such that $f_\gamma(x) = \mathrm{St}(x, \mathcal{G}_\gamma)$. $\langle \{X_\gamma\}, \{\mathcal{V}_\gamma\}, \{f_\gamma\} \rangle_{\gamma \in \Gamma}$ is a k-neat system for $\mathcal{H}$ for $k \in$ Card. Hence X is k-neat for $k \in$ Card.

(d)$\longrightarrow$(f). Let $\mathcal{H}$ be a free closed ultrafilter on X with c.i.p.. By Lemma 4.2 the open cover $\{X-H: H \in \mathcal{H}\}$ has a weak $\delta\theta$-refinement $\mathcal{U} = \bigcup_{n \in \omega} \mathcal{U}_n$. Put $X_n = \{x \in X: 0 < \mathrm{ord}(x, \mathcal{U}_n) \leq \omega\}$ for $n \in \omega$. If we set $\mathcal{E}_n = \{X_n \cap U: U \in \mathcal{U}_n\}$, then $\mathcal{E} = \bigcup_{n \in \omega} \mathcal{E}_n$ is obviously an interlacing on X that is δ-suspended from $\mathcal{H}$.

(e)$\longrightarrow$(f). Let $\mathcal{H}$ be a free closed ultrafilter on X with c.i.p.. Let $\mathcal{F}$ be a free subfamily of $\mathcal{H}$ such that $|\mathcal{F}| = \lambda(\mathcal{H})$. Since $|\mathcal{F}|$ is regular, there exists an interlacing that is δ-suspended from $\mathcal{F}$. The interlacing is, of course, δ-suspended from $\mathcal{H}$. The proof is complete.

(f). Let $\mathcal{H}$ be a free closed ultrafilter on X with c.i.p.. Since X is pure, we can obtain an interlacing $\mathcal{E} = \bigcup_{n \in \omega} \mathcal{E}_n$ on X which is δ-suspended from $\mathcal{H}$. For each $n \in \omega$ and $E \in \mathcal{E}_n$ we take an open set U(E) of X such that $U = U(E) \cap \mathcal{E}_n^*$. Now for each $n \in \omega$ put $X_n = \mathcal{E}_n^*$, $\mathcal{V}_n = \{\mathrm{St}(x, \mathcal{F}_n): x \in X_n\}$, where $\mathcal{F}_n = \{U(E): E \in \mathcal{E}_n\}$, and define a function f_n from X_n to $\mathcal{V}_n$ such that $f_n(x) = \mathrm{St}(x, \mathcal{F}_n)$. $\langle \{X_n\}, \{\mathcal{V}_n\}, \{f_n\} \rangle_{n \in \omega}$ is a desired k-neat system for $\mathcal{H}$ for $k \in$ Card. Hence X is k-neat for $k \in$ Card.

Davis asked in [14, Question 4.2] whether every (neighborhood) $\mathfrak{F}$-space satisfies property θL. The above implication (a) $\longrightarrow$ (b) affirmatively answers the question in the case of neighborhood $\mathfrak{F}$-spaces. The following lemma is easy.

LEMMA 4.4. Let Y be a closed subspace of a space X, and $\mathfrak{F}$ be a free closed ultrafilter on Y with c.i.p.. Then $\mathcal{H} = \{ H : H$ is closed in X and $H \cap Y \in \mathfrak{F} \}$ is a free closed ultrafilter on X with c.i.p. and $\lambda(\mathfrak{F}) = \lambda(\mathcal{H})$ holds.

LEMMA 4.5. Every closed subspace of a k-neat space is k-neat.

Proof. Let Y be a closed subspace of a k-neat space X, and $\mathfrak{F}$ be a free closed ultrafilter on Y with c.i.p.. By Lemma 4.4, $\mathcal{H} = \{ H : H$ is closed in X and $H \cap Y \in \mathfrak{F} \}$ is a free closed ultrafilter on X with c.i.p. and $\lambda(\mathfrak{F}) = \lambda(\mathcal{H})$. We take a k-neat system for $\mathcal{H}$. We naturally restrict the system to Y. It is easily seen that the restricted system is a k-neat system for $\mathfrak{F}$.

THEOREM 4.6. A neat space is isocompact.

Proof. By the above lemma we show that a countably compact neat space is compact. Suppose that there exists a countably compact non-compact neat space X. Since X is not Lindelöf, X has an open cover $\mathcal{U}$ which has no countable subcover. We take a closed ultrafilter $\mathcal{H}$ on X containing $\{X-U: U \in \mathcal{U}\}$. Now $\mathcal{H}$ is a free closed ultrafilter on X with c.i.p.. There exists a neat system $\langle \{X_\gamma\}, \{\mathcal{V}_\gamma\}, \{f_\gamma\} \rangle_{\gamma \in \Gamma}$ for $\mathcal{H}$. By the fact $|\Gamma| < \lambda(\mathcal{H})$ we can get $\gamma_0 \in \Gamma$ such that $X_{\gamma_0} \cap H \neq \phi$ for any $H \in \mathcal{H}$. We fix this γ_0. There exists $F \in \mathcal{H}$ such that $F \subset \mathcal{V}_{\gamma_0}^*$ because $X - \mathcal{V}_{\gamma_0}^* \notin \mathcal{H}$. By the way of the selecting of γ_0 we can obtain a countable subset $A = \{x_n: n \in \omega\}$ of $F \cap X_{\gamma_0}$ which satisfies $f_{\gamma_0}(x_n) \cap \{x_j: j \geq n+1\} = \phi$ for any $n \in \omega$. Take an ω-limit point x of A(i.e. any neighborhood of x contains an infinite subset of A.). Since F is closed, $x \in F \subset \mathcal{V}_{\gamma_0}^*$. Hence $x \in \bigcup_{n \in \omega} f_{\gamma_0}(x_n)$. This contradicts the fact that x is an ω-limit point of A.

COROLLARY 4.7. The following spaces are isocompact.

(1) neighborhood $\mathcal{F}$-spaces. [28, Theorem 3.11]

(2) spaces satisfying property θL. [15, Theorem 2.4]

(3) weakly $[\omega_1, \infty)^r$-refinable spaces. [58, Corollary 3.3]

(4) $\delta\theta$-penetrable spaces. [12, Theorem 3.B]

(5) ultrapure^r spaces. [49]

(6) pure spaces. [3, Theorem 5]

Proof. Apply Theorem 4.3 and Theorem 4.6.

Though an $\mathfrak{F}$-space is isocompact[28, Theorem 3.11], the author does not know whether an $\mathfrak{F}$-space is neat. The method of the proof of Theorem 4.6 leads to the following theorem, whose proof is omitted.

THEOREM 4.8. An ω_1-compact ω_1-neat T_1-space is closed-complete.

COROLLARY 4.9. The following spaces are closed-complete if they are ω_1-compact T_1-spaces.

(1) spaces satisfying property $\theta\omega_1 L$. [15, Theorem 2.5]

(2) weakly $[\omega_1, \infty)^r$-refinable spaces. [58, Corollary 3.6]

(3) $\delta\theta$-penetrable spaces. [9, Corollary 2.5]

(4) ultrapure^r spaces.

(5) pure spaces.

Proof. Apply Theorem 4.3 and Theorem 4.8.

Corollary 4.2 in [54] is also a special case of Theorem 4.6 and Theorem 4.8. We shall give some mapping theorems and an example. The following lemma is easy.

LEMMA 4.10. Let f be a closed map from X onto Y with Lindelöf fibers and $\mathfrak{F}$ be a free closed ultrafilter on X with c.i.p.. Then $\mathcal{H} = \{ H: H \text{ is closed in } Y \text{ and } f^{-1}H \in \mathfrak{F} \}$ is a free closed ultrafilter on Y with c.i.p. such that $\lambda(\mathfrak{F}) = \lambda(\mathcal{H})$.

THEOREM 4.11. Let f be a closed map from X onto a k-neat space Y. If each fiber of f is Lindelöf, then X is k-neat.

Proof. Let $\mathcal{F}$ be a free closed ultrafilter on X with c.i.p.. Then by the above lemma $\mathcal{H} = \{ H: H$ is closed in Y and $f^{-1} H \in \mathcal{F} \}$ is a free closed ultrafilter on Y with c.i.p. such that $\lambda(\mathcal{F}) = \lambda(\mathcal{H})$. We get a k-neat system $\langle \{Y_\gamma\}, \{\mathcal{V}_\gamma\}, \{g_\gamma\} \rangle_{\gamma \in \Gamma}$ for $\mathcal{H}$. Put $X_\gamma = f^{-1} Y_\gamma$, $\mathcal{W}_\gamma = \{ W_x: x \in X_\gamma \}$, where $W_x = f^{-1}(g_\gamma(f(x)))$, and define a function h_γ from X_γ to $\mathcal{W}_\gamma$ such that $h_\gamma(x) = W_x$ for each $\gamma \in \Gamma$. It is easily seen that the system $\langle \{X_\gamma\}, \{\mathcal{W}_\gamma\}, \{h_\gamma\} \rangle_{\gamma \in \Gamma}$ is a desired k-neat system for $\mathcal{F}$.

COROLLARY 4.12. A perfect preimage of a k-neat space is k-neat.

LEMMA 4.13.[29] Let $\mathcal{H}$ be a free closed ultrafilter on X with c.i.p.. If B is a Borel set of X, and if B contains no member of $\mathcal{H}$, then there exists $H \in \mathcal{H}$ such that $H \cap B = \phi$.

THEOREM 4.14. Let f be a map from X onto a Borel complete T_1-space Y. If each fiber of f is k-neat, then X is k-neat.

Proof. Let $\mathcal{H}$ be a free closed ultrafilter on X with c.i.p.. Set $\mathcal{B} = \{ B: B$ is a Borel set of Y and $f^{-1} B \supset H$ for some $H \in \mathcal{H} \}$. It follows from Lemma 4.13 that $\mathcal{B}$ is a Borel ultrafilter on Y with c.i.p.. So $\cap \mathcal{B} = \{ y \}$ for some $y \in Y$(i.e. $f^{-1} y \in \mathcal{H}$). Put $E = f^{-1} y$ and $\mathcal{H}|E = \{ E \cap H: H \in \mathcal{H} \}$. We can easily see that $\mathcal{H}|E$ is a free closed ultrafilter on E with c.i.p. such that $\lambda(\mathcal{H}) = \lambda(\mathcal{H}|E)$. Let $\langle \{E_\gamma\}, \{\mathcal{V}_\gamma\}, \{g_\gamma\} \rangle_{\gamma \in \Gamma}$ be a k-neat system for

$\mathcal{H}|E$. We extend this system in the following manner. Set $\mathcal{W}_\gamma$ = $\{ V \cup X-E: V \in \mathcal{V}_\gamma \}$ and define a function h_γ from E_γ to $\mathcal{W}_\gamma$ such that $h_\gamma(x) = g_\gamma(x) \cup X-E$ for each $\gamma \in \Gamma$. We get a system consisting of $\{ E_\gamma, X-E: \gamma \in \Gamma \}$, $\{ \mathcal{W}_\gamma, \{X-E\} : \gamma \in \Gamma \}$ and $\{ h_\gamma, j : \gamma \in \Gamma \}$, where j is the trivial function from X-E to $\{ X-E \}$. This system is a desired one for $\mathcal{H}$.

COROLLARY 4.15. A product of a Borel complete T_1-space and a k-neat space is k-neat.

The same method of the proof of Theorem 4.14 leads to the following theorem, whose proof is omitted.

THEOREM 4.16. Let f be a closed map from X onto a closed-complete T_1-space. If each fiber of f is k-neat, then X is k-neat.

EXAMPLE 4.17. We give a neat space that is not an $\mathcal{F}$-space, not a pure space, not a weakly $[\omega_1, \infty)^r$-refinable space and not satisfying property θL. Let X be an hereditarily separable non-Lindelöf space constructed in [35] under the continuum hypothesis. X is a θ-penetrable space, hence a neat space, that is not weakly $\delta\theta$-refinable[9, Remarks 2.1.(b)]. X is not even weakly $[\omega_1, \infty)^r$-refinable because the cardinality of X is ω_1. Moreover X does not satisfy property θL by [15, Theorem 2.8], and X is not an $\mathcal{F}$-space by [28, Theorem 3.3]. Let Y be the Tychonoff space mentioned in [25, 9L]. Since Y is a P-space(i.e. G_δ-sets are open.), it is neat. Since Y is an ω_1-compact non-closed-complete space [9, Remarks 2.7], by Corollary 4.9.(5), Y is not pure. By Corollary 4.15, $X \times Y$ is neat because X is hereditarily realcompact (hence Borel complete[27, Theorem 3.6]). Obviously $X \times Y$ is not an $\mathfrak{F}$-space, not a pure space, not a weakly $[\omega_1, \infty)^r$-refinable space and not satisfying property θL.

REMARK 4.18. The above space Y answers some questions in [15] and [16]. Since a P-space satisfies property L, Y affirmatively answers Question 3.3, 3.4 and 3.5 in [15], because Y is an ω_1-compact non-closed-complete P-space(as mentioned in Example 4.17) which is not weakly $\delta\theta$-refinable by [8, Corollary 3.3]. Question 3.3 in [15] was already answered in [16, Example 2.2], but the space is not regular though it is T_2. The space negatively answers Question 4.2 in [16]. Because an ω_1-compact P-space must be preparacompact.

5. Compactness of spaces having a countably compact dense subset.

We recall the main problem of this article: What conditions will make a countably compact space compact ? An interesting variation of the problem is to take a property $\mathcal{P}$ which is weaker than countable compactness, and find a property $\mathcal{Q}$ such that $\mathcal{P}$ and $\mathcal{Q}$ imply compactness. For example, let $\mathcal{P}$ be pseudocompactness[20], then the variation of the above problem is what conditions will make pseudocompact spaces compact ? Some answers are known.

THEOREM 5.1.

(1) Every pseudocompact Tychonoff metacompact space is compact[45][51].

(2) Every pseudocompact Tychonoff paraLindelöf space is compact[11].

More generally, Uspenskii showed, applying a method of Watson, that every pseudocompact Tychonoff σ-metacompact space is compact [48], where a space is called σ-metacompact if each open cover of the space has a σ-point finite open refinement. It is natural to ask whether every pseudocompact Tychonoff metaLindelöf space is compact. But the answer is negative, in fact, Scott constructed under CH a pseudocompact Tychonoff metaLindelöf space which is not compact[45].

Now, we try to consider a medium condition between pseudo-compactness and countable compactness. That is to have a countably compact dense subset. We propose the following question: When is a space having a countably compact dense subset compact ? From Theorem 5.1, a Tychonoff space having a countably compact dense subset is compact if it is either metacompact or para-Lindelöf. But, more generally, from Theorem 2.4, weakly $\delta\theta$-refinable regular spaces also have the same property.

PROPOSITION 5.2. A weakly $\delta\theta$-refinable regular space X is compact if it has a countably compact dense subset.

Proof. Let $\mathcal{U}$ be any open cover of X and $\mathcal{V}$ be an open cover of X such that for each $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $\bar{V} \subset U$. We take a weak $\delta\theta$-refinement $\mathcal{W}$ of $\mathcal{V}$. Let Y be a countably compact dense subset of X, then from Theorem 2.4, $\{W \cap Y : W \in \mathcal{W}\}$ has a finite subcover $\{W_1 \cap Y, \cdots\cdots, W_n \cap Y\}$ of Y. Each $W_i \cap Y$ is contained in some $V_i \in \mathcal{V}$. Since Y is dense, $X = \cup_{i=1}^{m} \overline{V}_i$, so $\mathcal{U}$ has a finite subcover of X. Thus X is compact.

The next theorem which generalizes Proposition 5.2 is proved the same way as Proposition 5.2.

LEMMA 5.3. Let $\mathcal{U}$ be an open cover of a countably compact space X. If there exists an interlacing $\mathcal{E} = \cup_{n\in\omega} \mathcal{E}_n$ on X such that, for each $n\in\omega$ and $x\in \mathcal{E}_n^*$, $St(x, \mathcal{E}_n) \subset \mathcal{V}^*$ for some countable subfamily $\mathcal{V} \subset \mathcal{U}$, then $\mathcal{U}$ has a finite subcover of X.

Proof. This lemma is proved the same way as [53, Theorem 1.1].

THEOREM 5.4. The following spaces are compact if they are regular spaces having a countably compact dense subset.

(1) spaces satisfying property θL.

(2) $\delta\theta$-penetrable spaces.

(3) ultrapure spaces.

Proof. (1). Let $\mathcal{U}$ be any open cover of X and $\mathcal{V}$ be an open cover of X such that for each $V\in\mathcal{V}$ there exists $U\in\mathcal{U}$ such that $\overline{V}\subset U$. For this $\mathcal{V}$, since X satisfies property θL, we can take sequences $\langle \mathcal{D}_n : n\in\omega\rangle$ and $\langle \mathcal{V}_n : n\in\omega\rangle$ in Definition 2.9. Let Y be a countably compact dense subset of X. If we restrict the discussion of [15, Theorem 2.4] to Y, we can obtain a countable subfamily $\mathcal{W}$ of $\mathcal{V}$ which covers Y. So Y is covered by a finite subfamily of $\mathcal{W}$. Since Y is dense in X, we can take a finite subcover of X from $\mathcal{U}$. (2) and (3) are similarly proved by [12, Theorem 3.B] and Lemma 5.3 respectively.

Being motivated the definition of isocompactness, we call a space CL-isocompact[42] if the closure of each countably compact subset is compact. Obviously CL-isocompact spaces are isocompact. Since each property of Theorem 5.4 is closed hereditary, we get the following corollary.

COROLLARY 5.5. The following spaces are CL-isocompact if they are regular.

(1) spaces satisfying property θL.

(2) $\delta\theta$-penetrable spaces.

(3) ultrapure spaces.

Now we recall weakly Borel complete spaces. Since a weakly Borel complete space is closed-complete, it is isocompact. There is a pseudocompact weakly Borel complete space which is not compact, see the example of 5I in [25]. Nextly we shall prove that a weakly Borel complete space is CL-isocompact.

THEOREM 5.6. A regular weakly Borel complete space is CL-isocompact.

Proof. Weak Borel completeness is closed hereditary[40]. So, we show that a weakly Borel complete space which has a dense countably compact subset is compact. Let X be weakly Borel complete, and Y be a dense countably compact subset of X.

We may assume that X is Tychonoff, because, if we consider the absolute EX of X[56], then EX is a Tychonoff weakly Borel complete space having a countably compact dense subset. Since X is a continuous image of EX, we may prove compactness of EX. So, we assume X is Tychonoff.

Suppose that X is not compact. Since X is pseudocompact, X is not realcompact. We take a free zero ultrafilter $\mathcal{Z}$ on X with c.i.p.. Each element of $\mathcal{Z}$ must intersect with Y. Put $\mathcal{A} = \{\mathcal{H} \mid \mathcal{H}$ is a closed family such that (1) $\mathcal{Z} \subset \mathcal{H}$. (2) If $H \in \mathcal{H}$, then $H \cap Y \neq \phi$. (3) $\mathcal{H}$ is closed under the finite intersections. $\}$. Let $\mathcal{H}$ be a maximal element of $\mathcal{A}$. It is easily showed that $\mathcal{H}$ is closed under the countable intersections, and $X \in \mathcal{H}$ by the maximality.

Put $\mathcal{D} = \{ B \in Bo(X) \mid B \supset H \cap Y$ for some $H \in \mathcal{H} \}$. Here Bo(X) is the set of Borel sets of X. We take a Borel ultrafilter $\mathcal{B}$ on X containing $\mathcal{D}$. Put $\mathcal{E} = \{ B \in Bo(X) \mid$ If $B \not\supset H \cap Y$ for any $H \in \mathcal{H}$, then $B \cap H \cap Y = \phi$ for some $H \in \mathcal{H}$.$\}$.

Now, $\mathcal{E}$ satisfies the following conditions.

(a) If F is closed in X, then $F \in \mathcal{E}$.

(b) If $B \in \mathcal{E}$, then $X-B \in \mathcal{E}$.

(c) If $\mathcal{E} \supset \{B_i\}_{i=1}^{\infty}$, then $\cap B_i \in \mathcal{E}$.

Firstly we show (a). Let F be a closed subset of X, and suppose that $F \not\supset H \cap Y$ for any $H \in \mathcal{H}$. Obviously $F \notin \mathcal{H}$. Put $\mathcal{L} = \mathcal{H} \cup \{ F \cap H \mid H \in \mathcal{H} \}$. $\mathcal{L}$ satisfies (1) and (3) of $\mathcal{A}$, and $\mathcal{H} \neq \mathcal{L}$, because $F \in \mathcal{L}$. By the maximality of $\mathcal{H}$, there exists

$H \in \mathcal{H}$ such that $F \cap H \cap Y = \phi$. This shows that $F \in \mathcal{E}$. The proof of (b) and (c) is a routine matter. We omit the proof.

Since Bo(X) is the smallest σ-field containing the set of closed subsets of X, we get $\mathcal{E}$ = Bo(X).

Suppose that $B \in \mathcal{B}$, and $B \cap H \cap Y = \phi$ for some $H \in \mathcal{H}$. Then $X-B \in \mathcal{D} \subset \mathcal{B}$. It is a contradiction that $\mathcal{B}$ is a filter. Therefore for each $B \in \mathcal{B}$, $B \cap H \cap Y \neq \phi$ for any $H \in \mathcal{H}$. It follows from $\mathcal{E}$ = Bo(X) that for each $B \in \mathcal{B}$ there exists some $H(B) \in \mathcal{H}$ such that $B \supset H(B) \cap Y$. This fact gives that $\mathcal{B}$ has c.i.p.. Since $\mathcal{Z} \subset \mathcal{B}$, we obtain that $\cap \{ Z \mid Z \in \mathcal{B} \cap \mathcal{Z}(X) \} = \phi$. Here $\mathcal{Z}(X)$ is the set of zero-sets of X. This is a contradiction that X is weakly Borel complete.

For a general case we shall prove the next theorem.

THEOREM 5.7. Let X be a regular isocompact space. If X is represented as the union of a countably compact dense subset X_1 and an almost realcompact dense subset X_2, then X is compact.

Proof. Firstly we show that X is almost realcompact. Let $\mathcal{U}$ be an open ultrafilter on X with c.c.i.p.. Put $\mathcal{U}|X_2 = \{ U \cap X_2 : U \in \mathcal{U} \}$. Then it is easily seen that $\mathcal{U}|X_2$ is an open ultrafilter on X_2. If $\mathcal{U}|X_2$ has c.c.i.p. in X_2, then $\mathcal{U}|X_2$ has a cluster point in X_2 by almost realcompactness of X_2. Hence $\mathcal{U}$ has a cluster point in X. If $\mathcal{U}|X_2$ has not c.c.i.p. in X_2, then there exists a countable subfamily $\mathcal{V} \subset \mathcal{U}$ such that $(\cap \overline{\mathcal{V}}) \cap X_2 = \phi$. Since $\cap \overline{\mathcal{V}}$ is countably compact closed in X, it is compact. So we get $\cap \overline{\mathcal{U}} \neq \phi$. Thus X is almost realcompact.

Now we consider the absolute EX of X[56]. Since EX is realcompact[56, Theorem 4.6] and pseudocompact, it is compact. We conclude that X is compact.

COROLLARY 5.8. Let X be a regular neat space. If X is represented as the union of a countably compact dense subset and an almost realcompact dense subset, then X is compact.

EXAMPLE 5.9. We cannot omit the regularity of Theorem 5.7. Let X be Tychonoff plank[25] (i.e. $X = \omega_1 +1 \times \omega +1 - \{(\omega_1, \omega)\}$) and Y be the space obtained from X by contracting $\omega_1 \times \{\omega\}$ to the one point. Though this T_2-space Y satisfies all conditions of Theorem 5.7 except the regularity, Y is not compact.

We note that Theorem 2.16 is generalized in the following manner: If X is a regular $[p,\infty]$-compact space, then the closure of each separable countably compact subset is compact. The proof is quite similar to the proof of Theorem 2.16.

The rest of this section is devoted to some investigations of the class itself of CL-isocompact spaces. The class of CL-isocompact spaces behaves well with respect to topological operations.

PROPOSITION 5.10. The following facts hold.

(a) Let f be a perfect map from X onto Y. Then, X is CL-isocompact if and only if Y is CL-isocompact.

(b) Let X be a regular CL-isocompact space, and Y be an F_σ-subset of X. Then Y is CL-isocompact.

(c) If $X = \prod_\alpha X_\alpha$, with X_α CL-isocompact for $\alpha \in A$, then X is CL-isocompact.

(d) If $X = \bigoplus_\alpha X_\alpha$, with X_α CL-isocompact for $\alpha \in A$, then X is CL-isocompact.

(e) If each X_α is a CL-isocompact subset of X, then $\bigcap_\alpha X_\alpha$ is CL-isocompact.

(f) The following (1), (2) and (3) are equivalent.

(1) X is hereditarily CL-isocompact.

(2) X is hereditarily isocompact.

(3) For each $x \in X$, $X-\{x\}$ is CL-isocompact.

Proof. (a). Compactness and countable compactness are preserved by perfect maps. From this fact, it is easy to show (a). (b). We set $Y = \bigcup_{i=1}^{\infty} Y_i$, each Y_i is closed in X. Let E be any countably compact subset of Y. Since each Y_i is CL-isocompact, $Cl(E \cap Y_i)$ is compact. $\cup Cl(E \cap Y_i)$ contains E as a dense subset. Since $\cup Cl(E \cap Y_i)$ is a σ-compact space having a countably compact dense subset, it is compact. We get $Cl_Y E = \bigcup_{i=1}^{\infty} Cl(E \cap Y_i)$. (c). Let E be any countably compact subset of X. Since each $Pr_\alpha E$ is countably compact, $Cl(Pr_\alpha E)$ is compact. Here Pr_α is the projection of X onto X_α. The closure of E in X is contained in the compact space $\prod_\alpha Cl(Pr_\alpha E)$. The closure of E must be compact. (d) is trivial. (e). $\bigcap_\alpha X_\alpha$ can be naturally embedded as a closed subspace into $\prod_\alpha X_\alpha$. By (b) and (c), $\bigcap_\alpha X_\alpha$ is CL-isocompact. (f). The equivalence of (1) and (2) is obvious. We assume (3). Let Y be any subspace of X. Since $Y = \cap \{ X-\{x\}: x \in X-Y \}$, Y is CL-isocompact by (e).

Bacon proved in [6] that the product of an isocompact space and an hereditarily isocompact space is isocompact. The following result generalizes it.

PROPOSITION 5.11. Let X be CL-isocompact, and Y be isocompact. Then $X \times Y$ is isocompact.

Proof. Let E be any countably compact closed subset of $X \times Y$. Since $Pr_X E$ is countably compact, $Cl(Pr_X E)$ is compact. Therefore $Pr_Y E$ is closed countably compact in Y. So, $Pr_Y E$ must be compact. E is contained in the compact space $Cl(Pr_X E) \times Pr_Y E$. The proof is complete.

PROPOSITION 5.12. The following (a) and (b) hold.

(a) For each Tychonoff space X, there exists a CL-isocompact space pX with the following properties.

(1) $X \subset pX \subset \beta X$. Here βX is the Stone-Čech compactification of X.

(2) If f is a map from X onto a CL-isocompact space Y, then f has a continuous extension f^p that maps pX onto Y.

(b) If a Tychonoff space X has a dense countably compact subspace, then $pX = \beta X$. Conversely, if $pX = \beta X$, then X is pseudocompact.

Proof. (a) is obtained from (b) and (c) of Proposition 5.10 and Theorem 2.1 in [55]. (b) is trivial. Note that $pX \subset \upsilon X$, where υX is the Hewitt's realcompactification.

6. Miscellaneous notes.

We shall give several notes in this section.

Some classes of isocompact spaces are closely related to Lindelöf property. For example, the following results are known.

Theorem 6.1.

(1) An ω_1-compact $\mathfrak{F}$-space is Lindelöf. [28, Theorem 3.3]

(2) Let $k \geq \omega_1$, then an ω_1-compact T_1-space satisfying property kL is Lindelöf. [13, Theorem 3.3]

(3) An hereditarily ω_1-compact space satisfying property θL is Lindelöf. [15, Theorem 2.8]

To be hereditarily ω_1-compact is equivalent to be countable spread. Gruenhage showed in [26] that for each regular space X having a base of subinfinite rank, $d(X) = hd(X) \geq hl(X) = s(X)$ holds. Comparing the result, we show that $hd(X) \geq s(X) = hl(X)$ holds for each T_1-space X having an ortho-base[43].

We need two lemmas. For convenience, for a cardinal τ, we say a space X to be τ-developable if there exist τ open covers $\{\mathcal{H}_\alpha\}_{\alpha<\tau}$ such that for each $x \in X$ $\{St(x, \mathcal{H}_\alpha)\}_{\alpha<\tau}$ is a neighborhood base of x.

LEMMA 6.2. Let X be a space having an ortho-base $\mathcal{B}$ and D be the set of isolated points of X. If D is dense in X, then X is $|D|$-developable.

Proof. Set $D = \{ d_\alpha : \alpha < \tau \}$, where τ is a cardinal. For each $x \in X-D$ and $\alpha < \tau$, we take $B_\alpha(x) \in \mathcal{B}$ such that $x \in B_\alpha(x)$ and $d_\alpha \notin B_\alpha(x)$. Put $\mathcal{H}_\alpha = \{ \{d_\alpha\} : \alpha < \tau \} \cup \{ B_\alpha(x) : x \in X-D \}$. $\mathcal{H}_\alpha$ is obviously an open cover of X. Let x be a point of X and W be a neighborhood of x. If $x \in D$, then $St(x, \mathcal{H}_\alpha) = \{x\} \subset W$ for some α. So, we assume $x \in X-D$. Suppose that $St(x, \mathcal{H}_\alpha) \not\subset W$ for any $\alpha < \tau$. Then for each α, we can take $H_\alpha \in \mathcal{H}_\alpha$ such that $x \in H_\alpha$ and $H \not\subset W$. Since $\{ H_\alpha \}_{\alpha < \tau}$ cannot be a neighborhood base of x, $H = \bigcap_\alpha H_\alpha$ must be open. But $H \cap D = \phi$, because $H_\alpha \not\ni d_\alpha$. Since D is dense in X, this is a contradiction.

The following lemma is well known in the countable case and can be easily carried over to the general case. So we omit the proof.

LEMMA 6.3. Let X be a τ-developable T_1-space. If the cardinality of each closed discrete subset is at most τ, then X is τ-Lindelöf(i.e. every open cover has a subcover of the cardinality τ.)

THEOREM 6.4. Let X be a T_1-space having an ortho-base. Then $hd(X) \geq s(X) = hl(X)$ holds.

Proof. Since $hd(X) \geq s(X)$ and $hl(X) \geq s(X)$ are obvious, we show $s(X) \geq hl(X)$. Let $s(X) = \tau$. Since for each subspace Y of X, $s(X) \leq \tau$ and Y has an ortho-base, the proof is complete if we show that X is τ-Lindelöf. Suppose that there exists an open cover $\mathcal{U}$ of X which has not a subcover of the cardinality τ. Firstly we take $x_0 \in X$ and $U_0 \in \mathcal{U}$ such that $x_0 \in U_0$. Put $V_0 = U_0$. Let $\gamma < \tau^+$. We assume that for each $\beta < \gamma$ we could take $x_\beta \in X$ and an open set V_β such that the following (*) is satisfied.

$$(*) \quad \begin{cases} V_\beta \cap \{x_\alpha : \alpha < \gamma\} = \{x_\beta\} & \text{for each } \beta < \gamma . \\ \text{There exists } U_\beta \in \mathcal{U} \text{ such that } V_\beta \subset U_\beta & \text{for each } \beta < \gamma . \end{cases}$$

Then, if we set $A = \{x_\alpha : \alpha < \gamma\}$, since $|A| \leq \tau$, Cl A is τ-Lindelöf by Lemma 6.2 and 6.3. Thus $\mathrm{Cl}\, A \cup (\bigcup_{\beta<\gamma} V_\beta)$ is covered by τ-elements of $\mathcal{U}$. So we can take $x_\gamma \in X - \mathrm{Cl}\, A \cup (\bigcup_{\beta<\gamma} V_\beta)$. We take $U_\gamma \in \mathcal{U}$ and an open set V_γ such that $x_\gamma \in V_\gamma \subset U_\gamma$ and $V_\gamma \cap A = \phi$. Now by the induction we get the discrete space $\{x_\alpha : \alpha < \tau^+\}$. This is a contradiction to $s(X) = \tau$.

There exists a space having an ortho-base such that $hd(X) \neq d(X)$. In fact, the space in [20, 3.6.I] is such a space.

Concerning SH(Souslin's hypothesis), we note the following theorem.

THEOREM 6.5. The following (a), (b) and (c) are equivalent.

(a) SH is false.

(b) There exists a non-metrizable non-archimedean space such that $s(X)$ is countable.

(c) There exists a non-metrizable regular space having an ortho-base such that $s(X)$ is countable.

Proof. The equivalence of (a) and (b) is due to [4]. Also, refer [37, Theorem 1.7]. (b)$\rightarrow$(c) is trivial. We show (c)$\rightarrow$(b). Let X be a space of (c). Since by Theorem 6.4 X is regular Lindelöf, it is paracompact. Therefore X is a proto-metrizable space(i.e. paracompact space with an ortho-base). It follows from Fuller's result[24, Theorem 6] that X is the perfect irreducible image of a non-archimedean space Y. Since metrizability is an invariant of perfect maps, Y is not metrizable. Since the spread of a non-archimedean space is equal to the cellularity, by the irreducibility of the map, $s(Y)$ must be countable. Thus Y is the desired space.

COROLLARY 6.6. The following (a) and (b) are equivalent.

(a) SH.

(b) Each regular space having an ortho-base is metrizable if the spread is countable.

REMARK 6.7. The proof of Theorem 6.4 essentially shows that $\partial(X) = t(X)\cdot L(X)$ holds for any T_1-space X having an ortho-base. If we see again the proof of Theorem 6.4, we can see that the proof claims $L(X) \leq \theta(X)$. Since , in general, $\partial(X) \leq t(X)\cdot L(X)$ holds[33], we obtain $L(X) \leq \partial(X) \leq t(X)\cdot L(X)$. So, $t(X)\cdot L(X) \leq t(X)\cdot \partial(X) \leq t(X)\cdot L(X)$ is obtained. Here $t(X)\cdot \partial(X) = \partial(X)$ holds, because it is easily showed that $t(X) \leq \partial(X)$ holds for a space X having an orhto-base. Thus $\partial(X) = t(X)\cdot L(X)$.

θ-penetrability implies weak Borel completeness in the presence of a suitable nonmeasurability hypothesis. We omit the proof.

THEOREM 6.8.[42] Every θ-penetrable space of non-measurable cardinal is weakly Borel complete.

COROLLARY 6.9.[42] A quasi-developable space of non-measurable cardinal is Borel complete.

The Rudin's Dowker space in [41] is known as a closed-complete space which is not weakly Borel complete[46]. By the above theorem, the Dowker space is not θ-penetrable.

A space is called feebly compact if every locally finite family of open sets is finite. If a space is Tychonoff, then feeble compactness coincides with pseudocompactness. Porter and Woods studied in [39], motivated by Stephenson's question in the first section and Weiss's paper[52], compactness of feebly compact regular RC-perfect spaces, where a space X is called RC-perfect if each open subset of X can be written as a union of countably many regular closed subsets of X. They showed in [39] that, under MA+$\neg$CH, every feebly compact RC-perfect separable regular space is compact. However, under $\diamondsuit$, there exists a feebly compact, locally compact, RC-perfect zero-dimensional, separable T_2-space that is not countably compact.

Ismail and Nyikos defined the class of C-closed spaces[32], where a space is called C-closed if every countably compact subset is closed. For the study of C-closed spaces, refer to [31][32].

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